Decomposable Approximations Revisited

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Abstract Nuclear C*-algebras enjoy a number of approximation properties, most famously the completely positive approximation property. This was recently sharpened to arrange for the incoming maps to be sums of order-zero maps. We show that, in addition, the outgoing maps can be chosen to be asymptotically order-zero. Further these maps can be chosen to be asymptotically multiplicative if and only if the C*-algebra and all its traces are quasidiagonal.

1 Introduction

Approximation properties are ubiquitous in operator algebras, characterizing many key notions and providing essential tools. In particular, and central to this note, a foundational result of Choi-Effros [10] and Kirchberg [18] describes nuclearity of a C*-algebra in terms of completely positive approximations. Precisely, A is nuclear if and only if there exist finite dimensional algebras (F_i) and completely positive contractive (c.p.c.) maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{1.1}$$

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that approximate the identity in the point-norm topology, i.e.

$$\lim_i \|\phi_i(\psi_i(x)) - x\| = 0, \quad x \in A. \tag{1.2}$$

Some 30 years later, via Connes' celebrated work on injective von Neumann algebras [9], this approximation property was shown to imply a stronger version of itself: one can always take every ϕ_i to be a convex combination of contractive order-zero maps [16]. This has proved crucial to applications to near inclusions (for example, [16, Theorem 2.3]). In this note we observe a further improvement: every ψ_i can be taken to be asymptotically order zero, meaning that if $a, b \in A$ are self-adjoint and $ab = 0$, then

$$\lim_i \|\psi_i(a)\psi_i(b)\| = 0. \tag{1.3}$$

It was known that (1.3) could be arranged under the stronger hypothesis of finite nuclear dimension [29, Proposition 3.2] and this proved vital to various applications (cf. [7, 21, 26, 27]).

Our proof follows the strategy in [16] by obtaining suitable factorizations of the canonical inclusion $A \hookrightarrow A^{**}$ with respect to the weak* topology; then adjusting these to take values in A; and finally applying a Hahn-Banach argument to get asymptotic factorizations in the point-norm topology. To do this in general, however, we require some quasidiagonal ideas. Indeed, the main technical hurdle is showing that if A is quasidiagonal and all traces on A are quasidiagonal in the sense of [5], then one can take every ψ_i to be asymptotically multiplicative (see Theorem 2.2), while retaining the decomposition of ϕ_i as a convex combination of contractive order zero maps. This should be compared with Blackadar and Kirchberg's characterization of nuclear quasidiagonal C*-algebras in [3] as those with approximations (1.1) and (1.2) with ψ_i asymptotically multiplicative.

Since all traces on nuclear quasidiagonal C*-algebras in the UCT class are quasidiagonal [24], our result improves the Blackadar-Kirchberg characterization in this case. Cones over nuclear C*-algebras are quasidiagonal [25] and satisfy the UCT, so all their traces are quasidiagonal (though we show how Gabe's work [14] gives a simpler proof of this fact in Proposition 3.2). Thus we obtain our main theorem for general nuclear A by taking an order-zero splitting $A \to CA$, applying the improved approximation maps on CA, then using the quotient map $CA \to A$ to get back to A (see the proof of Theorem 3.1 for details).

2 Quasidiagonal Traces

In this note, a *trace* on a C^*-algebra means a tracial state. Write $T(A)$ for the collection of all traces on A. Various approximation properties for traces were studied in [5]; of particular relevance here is the notion of quasidiagonality for traces.

Definition 2.1 A trace τ on a C^*-algebra A is *quasidiagonal* if there exist finite dimensional algebras F_i, tracial states τ_i on F_i and c.p.c. maps $\theta_i\colon A \to F_i$ such that $\mathrm{tr}_i \circ \theta_i \to \tau$ in the weak* topology and

$$\lim_i \|\theta_i(ab) - \theta_i(a)\theta_i(b)\| = 0 \tag{2.1}$$

for all $a, b \in A$. Write $T_{\mathrm{qd}}(A)$ for the set of quasidiagonal traces of A.

When A is unital the maps θ_i can be taken to be unital and completely positive (u.c.p.). Theorem 3.1.6 of [5] lists several other characterizations of amenable traces.

The main technical result of this note is the following.

Theorem 2.2 *Let A be a separable and nuclear C^*-algebra. Then A is quasidiagonal and $T(A) = T_{\mathrm{qd}}(A)$ if and only if there exist a sequence of finite-dimensional C^*-algebras (F_n) and c.p.c. maps*

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi_n} A \tag{2.2}$$

such that

1. $\|(\phi_n \circ \psi_n)(a) - a\| \to 0$ *for all* $a \in A$;
2. *every* ϕ_n *is a convex combination of finitely many contractive order zero maps; and*
3. $\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0$ *for all* $a, b \in A$.

Only one implication of this theorem requires much work. Indeed, if A has approximations with properties 1–3, then A is quasidiagonal (this is an easy implication in [3, Theorem 5.2.2]; the maps ψ_i are approximately multiplicative by 3, and 1 ensures that they are approximately isometric). It is equally routine to check that all traces are quasidiagonal. Indeed, since a trace composed with an order-zero map is a trace by [28, Corollary 4.4], and each ϕ_n is a convex combination of order zero maps, given a trace $\tau_A \in T(A)$, it follows that $\tau_A \circ \phi_n$ defines a trace on F_n. Then condition 1 ensures that $\tau_A \circ \phi_n \to \tau_A$ weak*.

In order to prove the reverse implication it will suffice to prove a σ-weak version for the canonical inclusion $\iota\colon A \hookrightarrow A^{**}$. Namely, we prove the following proposition in the remainder of this section.

Proposition 2.3 *Let A be a separable nuclear, and quasidiagonal C^*-algebra with $T(A) = T_{\mathrm{qd}}(A)$. Then there are nets of finite-dimensional C^*-algebras (F_i) and of c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A^{**} \tag{2.3}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \iota(a)$ *in the σ-weak topology for every* $a \in A$;

2. *ϕ_i is an order zero map;*
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ *for every* $a, b \in A$.

With this proposition in hand we prove Theorem 2.2 by following the same steps used to prove [16, Theorem 1.4] from the preparatory lemma [16, Lemma 1.3]. Indeed, using the notation of Proposition 2.3, first apply Lemma 1.1 of [16] to see that for every i there is a net of contractive order zero maps $(\phi_{i,\lambda}: F_i \to A)_\lambda$ such that $\phi_{i,\lambda}(x)$ converges σ-weakly to $\phi_i(x)$ for every $x \in F_i$. We may therefore assume that the image of ϕ_i is contained in A for every i. The argument now ends with a familiar Hahn-Banach argument, similar to the one used to prove the completely positive approximation property of a C^*-algebra from the assumption that its enveloping von Neumann algebra is semidiscrete (see [6, Proposition 2.3.8]). Briefly, given a finite subset $\mathscr{F}$ of A and $\epsilon > 0$, let $K_0 \subset \mathscr{B}(A)$ be the collection of all c.p.c maps $\theta: A \to A$ which factorize as $A \overset{\psi}{\to} F \overset{\phi}{\to} A$, where ψ is a c.p.c. map with $\|\psi(ab) - \psi(a)\psi(b)\| \leq \epsilon$ for all $a, b \in \mathscr{F}$, and ϕ is a contractive order zero map. Since the identity map on A lies in the point-weak closure of K_0, it lies in the point norm closure of the convex hull of K_0. As a convex combination of maps in K_0 can be factorized in the form $A \overset{\psi}{\to} F \overset{\phi}{\to} A$, where ψ is a c.p.c. map with $\|\psi(ab) - \psi(a)\psi(b)\| \leq \epsilon$ for all $a, b \in \mathscr{F}$ and ϕ a convex combination of contractive order zero maps, we can find such ψ and ϕ additionally satisfying $\|\phi(\psi(a)) - a\| < \epsilon$ for $a \in \mathscr{F}$. Theorem 2.2 follows by using a countable dense subset of A to produce the required sequence of maps.

The proof of Proposition 2.3 requires some lemmas and will first be carried out in the case when A is unital. We will split A^{**} into two pieces, the finite and properly infinite summands, and then handle each piece separately.[1] The properly infinite case is handled by a combination of Blackadar and Kirchberg's characterization of NF-algebras in [3] and Haagerup's very short proof that semidiscreteness implies hyperfiniteness for properly infinite von Neumann algebras [15, Section 2].

Recall that if ρ is a normal state on a von Neumann algebra M, the seminorm $\|\cdot\|_\rho^\sharp$ is given by

$$\|x\|_\rho^\sharp = \rho\left(\frac{xx^* + x^*x}{2}\right)^{1/2}, \quad x \in M. \tag{2.4}$$

It is a standard fact (see e.g. [2, III.2.2.19]) that if $\{\rho_i\}$ is a separating family of normal states on M, then the topology generated by $\{\|\cdot\|_{\rho_i}^\sharp\}$ agrees with the σ-strong* topology on any bounded subset of M. This will be used in both of the following lemmas.

Lemma 2.4 *Let A be a unital, quasidiagonal and nuclear* C*-*algebra. Let* $\pi_\infty :$ *$A \to M$ be the properly infinite summand of the universal representation of A. Then*

[1]Recall that a von Neumann algebra is finite if it admits a separating family of tracial states, and properly infinite if it has no finite summand.

there are nets of finite-dimensional C^*-algebras F_i and nets of c.p.c. maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} M \tag{2.5}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \pi_\infty(a)$ *in the σ-strong* topology (and hence also in the σ-weak topology) for every $a \in A$;*
2. ϕ_i *is a *-homomorphism for every i; and*
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ *for every $a, b \in A$.*

Proof Fix $\epsilon > 0$, a finite subset $\mathscr{F}$ of unitaries in A, and finitely many normal states $\rho_1, \dots, \rho_m$ on M. We will produce a factorization

$$A \xrightarrow{\psi} F \xrightarrow{\phi} M \tag{2.6}$$

where F is a matrix algebra, ϕ is a *-homomorphism and ψ is a u.c.p. map, such that

$$\|\phi(\psi(u)) - u\|_{\rho_i}^{\sharp} < 2\epsilon^{\frac{1}{2}} \tag{2.7}$$

and

$$\|\psi(uv) - \psi(u)\psi(v)\| < \epsilon, \tag{2.8}$$

for all $u, v \in \mathscr{F}$ and $i = 1, \dots, m$. In this way we obtain the desired net indexed by finite subsets of unitaries, finite subsets of normal states and tolerances ϵ. By working with $\rho = \frac{1}{m}\sum_{i=1}^{m} \rho_i$, and replacing ϵ by ϵ/m, it suffices to obtain the single estimate

$$\|\phi(\psi(u)) - u\|_{\rho}^{\sharp} < 2\epsilon^{\frac{1}{2}}, \quad u \in \mathscr{F}, \tag{2.9}$$

in place of (2.7).

Since A is nuclear and quasidiagonal, it is NF by [3, Theorem 5.2.2] and so, by part (vi) of this theorem, there exists a matrix algebra F and u.c.p. maps

$$A \xrightarrow{\psi} F \xrightarrow{\theta} A \tag{2.10}$$

such that

$$\|(\theta \circ \psi)(u) - u\| < \epsilon \tag{2.11}$$

and

$$\|\psi(uv) - \psi(u)\psi(v)\| < \epsilon, \tag{2.12}$$

for all $u, v \in \mathscr{F}$. The estimate in (2.11) gives

$$\|\pi_\infty(\theta(\psi(u)) - u)\|_\rho^\sharp < \epsilon, \tag{2.13}$$

for all $u \in \mathscr{F}$.

We now follow the proof of [15, Theorem 2.2]. As M is properly infinite, we can fix a unital embedding $\iota : F \to M$. Then by [15, Proposition 2.1] there exists an isometry $v \in M$ such that $\theta(x) = v^*\iota(x)v$ for all $x \in F$. If v is a unitary (which is impossible, in general), then we're done because $\mathrm{Ad}(v) \circ \iota$ is the desired $*$-homomorphism. Since the σ-strong closure of unitaries in any von Neumann algebra is the set of all isometries (cf. [23, Lemma XVI.1.1]), the remainder of the proof (which follows the estimates on page 167 of [15]) amounts to approximating v with a suitable unitary.

We may assume that M is concretely represented on some Hilbert space $\mathscr{H}$ so that ρ is a vector state, given by a unit vector $\xi \in \mathscr{H}$. Using the identity $\|x\xi\|^2 + \|x^*\xi\|^2 = 2(\|x\|_\rho^\sharp)^2$, which is valid for all $x \in M$, and Eq. (2.13) we have

$$\|(v^*\iota(\psi(u))v - \pi_\infty(u))\xi\| < 2^{\frac{1}{2}}\epsilon \tag{2.14}$$

and

$$\|(v^*\iota(\psi(u)^*)v - \pi_\infty(u^*))\xi\| < 2^{\frac{1}{2}}\epsilon. \tag{2.15}$$

This implies

$$\Re\langle\iota(\psi(u))v\xi, v\pi_\infty(u)\xi\rangle > 1 - 2^{\frac{1}{2}}\epsilon \tag{2.16}$$

and

$$\Re\langle\iota(\psi(u)^*)v\xi, v\pi_\infty(u^*)\xi\rangle > 1 - 2^{\frac{1}{2}}\epsilon. \tag{2.17}$$

Now choose a unitary $w \in M$ such that, for all $u \in \mathscr{F}$,

$$\Re\langle\iota(\psi(u))w\xi, w\pi_\infty(u)\xi\rangle > 1 - 2\epsilon \tag{2.18}$$

and

$$\Re\langle\iota(\psi(u)^*)w\xi, w\pi_\infty(u^*)\xi\rangle > 1 - 2\epsilon. \tag{2.19}$$

Then, since $\|\iota(\psi(u))w\xi\| \leq 1$ and $\|\iota(\psi(u^*))w\xi\| \leq 1$, we have

$$\|\iota(\psi(u))w\xi - w\pi_\infty(u)\xi\|^2 \leq 2 - 2\Re\langle\iota(\psi(u))w\xi, w\pi_\infty(u)\xi\rangle < 4\epsilon \tag{2.20}$$

and

$$\|\iota(\psi(u^*))w\xi - w\pi_\infty(u^*)\xi\|^2 \leq 2 - 2\Re\langle\iota(\psi(u^*))w\xi, w\pi_\infty(u^*)\xi\rangle < 4\epsilon, \tag{2.21}$$

for all $u \in \mathscr{F}$. Then $\phi = \mathrm{Ad}(w^*) \circ \iota : F \to M$ is a *-homomorphism with

$$\|\phi(\psi(u)) - \pi_\infty(u)\|_\rho^\sharp < (4\epsilon)^{\frac{1}{2}}, \quad u \in \mathscr{F}, \tag{2.22}$$

as required. □

Next we deal with the finite part of A^{**}. We need the following standard uniqueness fact. Let A be a separable nuclear C^*-algebra, and N a finite von Neumann algebra. Then it is well known, though most often stated when N is a factor (see [17] and [1] which give converse statements), or when N has separable predual (see [11, Theorem 5]) that two *-homomorphisms $\phi_1, \phi_2 : A \to N$ are σ-strong* approximately unitarily equivalent in that there is a net of unitaries u_i such that $u_i\phi_1(a)u_i^* \to \phi_2(a)$ in the σ-strong* topology for all $a \in A$ if and only if $\tau \circ \phi_1 = \tau \circ \phi_2$ for all normal traces τ on N. Indeed, ϕ_1 and ϕ_2 extend to normal representations $\phi_1^{**}, \phi_2^{**} : A^{**} \to N$ that agree on traces. Since A^{**} is injective, it is hyperfinite,[2] so there is an increasing net of finite dimensional subalgebras (F_λ) that is σ-strong* dense in A^{**}. For each λ, the condition that $\tau \circ \phi_1^{**}|_{F_\lambda} = \tau \circ \phi_2|_{F_\lambda}^{**}$ for all normal traces τ on N gives a unitary u_λ with $\mathrm{Ad}(u_\lambda) \circ \phi_1^{**}|_{F_\lambda} = \phi_2^{**}|_{F_\lambda}$. The net of unitaries (u_λ) witnesses the σ-strong* approximate unitary equivalence of ϕ_1^{**} and ϕ_2^{**} and hence also of ϕ_1 and ϕ_2.

Lemma 2.5 *Let A be a separable, unital and nuclear C^*-algebra and assume $T(A) = T_{\mathrm{qd}}(A)$. Let $\pi_{\mathrm{fin}}: A \to M$ be the finite summand of the universal representation of A. Then there are nets of finite dimensional C^*-algebras F_i and of c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} M \tag{2.23}$$

such that

1. *$(\phi_i \circ \psi_i)(a) \to \pi_{\mathrm{fin}}(a)$ in the σ-strong* topology (and therefore also in the σ-weak topology) for every $a \in A$;*
2. *ϕ_i is a *-homomorphism for every n; and*
3. *$\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ for every $a, b \in A$.*

[2] See [12] for the extension of Connes' theorem to the non-separable predual case used here.

Proof Recall that M has a separating family of normal tracial states. As pointed out in the remarks preceding Lemma 2.4, on any bounded subset of M the σ-strong* topology agrees with the topology generated by the family of seminorms $\{\|\cdot\|_{2,\tau}\}$ (where τ runs through all normal tracial states of M). As in the proof of Lemma 2.4, the required nets of finite dimensional C^*-algebras and c.p.c. maps will ultimately be indexed by finite subsets $\mathscr{F}$ of A, positive numbers ϵ, and finite subsets $\{\tau_1, \ldots, \tau_m\}$ of normal tracial states of M. Moreover, the same argument found in the proof of Lemma 2.4 shows that it suffices to consider a single normal trace τ (by considering $\tau = \frac{1}{m}\sum_{i=1}^{m} \tau_i$), which we fix for the remainder of the proof.

Write N for $\pi_\tau(A)''$. We claim it is enough to obtain finite dimensional algebras F_i and maps $\psi_i\colon A \to F_i$ and $\phi_i\colon F_i \to N$ (as opposed to $\phi_i\colon F_i \to M$) satisfying (2), (3), and

$$\|(\phi_i \circ \psi_i)(a) - \pi_\tau(a)\|_{2,\tau} \to 0. \tag{2.24}$$

For this, first note that $J = \{x \in M : \tau(x^*x) = 0\}$ is a (closed, two-sided) ideal of M, and therefore of the form Mp for some central projection $p \in M$. Using the fact that τ is a faithful trace on both N and $M(1-p)$, we get that $N \cong M(1-p)$ (extending the identity on $A/J \cap A$). Identifying N with this direct summand, it follows that $\|\pi_\tau(a) - \pi_{\text{fin}}(a)\|_{2,\tau} = 0$, which proves the claim.

Being finite, N is the direct sum of a (finite) type I von Neumann algebra and type II_1 von Neuman algebra. We can therefore deal with each summand separately, and combine the two approximations to prove the lemma. To ease the notation, we may as well assume that N itself is type I or type II_1.

First assume N is finite type I, so of the form $N \cong \oplus_i L^\infty(X_i) \otimes M_{n_i}$ for some $n_i \in \mathbb{N}$ and measure spaces X_i. Write $\pi_\tau(a) = \oplus_i \pi_\tau^{(i)}(a)$. If the direct sum is infinite then, by normality of τ, $\pi_\tau(a)$ is the limit in $\|\cdot\|_{2,\tau}$ of the finite sums $\oplus_{i=1}^n \pi_\tau^{(i)}(a)$, and so it suffices to prove the result when the sum $N \cong \oplus_i L^\infty(X_i) \otimes M_{n_i}$ is finite. In this case N is a (non-separable) AF C*-algebra, so given a finite subset $\mathscr{F}$ of the unit ball of N and $\epsilon > 0$ there exists some finite dimensional C*-subalgebra $F \subset N$ such that for each $x \in \mathscr{F}$, there exists a contraction $y_x \in F$ with $\|x - y_x\| < \epsilon$. Fix any conditional expectation $\psi : N \to F$ (an expectation exists by Arveson's Extension Theorem) and note that for $x_1, x_2 \in \mathscr{F}$

$$\begin{aligned}\|\psi(x_1x_2) - \psi(x_1)\psi(x_2)\| &\le \|x_1x_2 - y_{x_1}y_{x_2}\| + \|x_1 - y_{x_1}\| + \|x_2 - y_2\| \\ &\le 4\epsilon.\end{aligned} \tag{2.25}$$

Also, ψ composed with the inclusion map $\phi\colon F \hookrightarrow N$ is the identity on F, so that $\|\phi(\psi(x)) - x\| \le 2\epsilon$ for $x \in \mathscr{F}$. Thus the required approximations exist in the finite type I case.

Assume now that N is type II_1. The center $Z(N)$ of N is an abelian von Neumann algebra with faithful normal state τ, so of the form $L^\infty(X, \mu)$, where μ is induced by τ. Let $E\colon N \to L^\infty(X, \mu)$ denote the center valued trace. Let $(a_j)_{j=1}^\infty$ be a sequence of positive contractions in A that is dense in the unit ball of A_+ and such that $\|a_j\| < 1$ for all j.

Fix $k \in \mathbb{N}$. Given a k-tuple $i = (i_1, \ldots, i_k) \in \{1, \ldots, k\}^k$, let p_i be the projection in $L^\infty(X, \mu)$, whose characteristic function is the set

$$\{x \in X : \frac{i_j - 1}{k} \leq E(\pi_\tau(a_j))(x) < \frac{i_j}{k}, \; j = 1, \ldots, k\}. \tag{2.26}$$

These are pairwise orthogonal and $\sum_i p_i = 1_N$. Some of the p_i may be zero; in what follows we only work with and sum over those indices i for which $p_i \neq 0$. Note that

$$\|E(\pi_\tau(a_j)) - \sum_i \frac{i_j}{k} p_i\|_{L^\infty(X,\mu)} \leq \frac{1}{k}, \; j = 1, \ldots, k. \tag{2.27}$$

Now, any normal trace on N is of the form $\tau(f \cdot)$ for some $f \in L^1(X, \mu)_+$ with $\|f\|_{L^1(X,\mu)} = 1$. For such an f,

$$\tau(f \pi_\tau(a_j)) = \tau\big(fE(\pi_\tau(a_j))\big) \approx_{\frac{1}{k}} \sum_i \tfrac{i_j}{k} \tau(fp_i), \quad j = 1, \ldots, k. \tag{2.28}$$

Also, for each index i,

$$|\tau(p_i \pi_\tau(a_j)) - \tau(p_i) \tfrac{i_j}{k}| \leq \frac{1}{k} \tau(p_i), \quad j = 1, \ldots, k. \tag{2.29}$$

Now, for each $i = (i_1, \ldots, i_k)$, the map $\frac{1}{\tau(p_i)} \tau(\pi_\tau(\cdot) p_i)$ is a tracial state on A. Because all traces on A are quasidiagonal, there exist matrix algebras $F_{k,i}$ and u.c.p. maps $\psi_{k,i} \colon A \to F_{k,i}$ such that

$$\left| \mathrm{tr}_{F_{k,i}}\big(\psi_{k,i}(a_j)\big) - \frac{1}{\tau(p_i)} \tau\big(p_i \pi_\tau(a_j)\big) \right| < \frac{1}{k}, \quad j = 1, \ldots, k \tag{2.30}$$

and

$$\|\psi_{k,i}(a_{j_1} a_{j_2}) - \psi_{k,i}(a_{j_1}) \psi_{k,i}(a_{j_2})\| < \epsilon, \quad j_1, j_2 = 1, \ldots, k. \tag{2.31}$$

Combining (2.30) and (2.29) gives

$$\left| \mathrm{tr}_{F_{k,i}}\big(\psi_{k,i}(a_j)\big) - \frac{i_j}{k} \right| \leq \frac{2}{k}. \tag{2.32}$$

Define $F_k := \bigoplus_i F_{k,i}$ and $\psi_k := \oplus \psi_{k,i}$ so that (3) holds. Since each $p_i N p_i$ is type II_1, there exists a unital *-homomorphism $\phi_{k,i} : F_{k,i} \to p_i N p_i$ (see e.g.

[6, Lemma 2.4.8]). Define $\phi_k : F_k \to N$ by $\phi_k = \oplus_i \phi_{k,i}$. This is a unital *-homomorphism. Further, for each $f \in L^1(X,\mu)_+$ with $\|f\|_{L^1(X,\mu)} = 1$, we have

$$\begin{aligned}\tau\big(f\phi_k(\psi_k(a_j))\big) &= \sum_i \tau(fp_i)\mathrm{tr}_{F_{k,i}}(\psi(a_j))\\ &\overset{(2.32)}{\approx_{\frac{2}{k}}} \sum_i \tau(fp_i)\tfrac{i_j}{k}\\ &\overset{(2.28)}{\approx_{\frac{1}{k}}} \tau(f\pi_\tau(a_j)), \quad j = 1,\ldots,k. \end{aligned} \tag{2.33}$$

Thus the sequence of maps $(\phi_k \circ \psi_k)$ satisfies

$$\lim_{k\to\infty} \sup_{\substack{f\in L^1(X,\mu)_+\\ \|f\|_{L^1(X,\mu)}=1}} \left|\tau\big(f\phi_k(\psi_k(a_j))\big) - \tau\big(f\pi_\tau(a_j)\big)\right| = 0, \quad j \in \mathbb{N}. \tag{2.34}$$

Write N^ω for the ultraproduct of N with respect to some fixed free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ (defined with respect to τ). We claim that the sequence $(\phi_k \circ \psi_k)$ induces a *-homomorphism, call it $\theta: A \to N^\omega$, that agrees on traces with π_τ^ω (the composition of π_τ with the canonical embedding of N into N^ω). Indeed, this follows as in the proof of Lemma 3.21 of [4]: fix j and write $x_{j,k} = \phi_k(\psi_k(a_j)) - \pi_\tau(a_j)$. As $E(x_{j,k} - E(x_{j,k})) = 0$, [13, Theorem 3.2] gives $y_{j,k,l}$ and $z_{j,k,l}$ in N for $l = 1,\ldots,10$ such that

$$x_{j,k} - E(x_{j,k}) = \sum_{l=1}^{10} [y_{j,k,l}, z_{j,k,l}] \tag{2.35}$$

with $\|y_{j,k,l}\| \le 12\|x_{j,k} - E(x_{j,k})\|$ and $\|z_{j,k,l}\| \le 12$. These estimates ensure that $(y_{j,k,l})_k$ and $(z_{j,k,l})_k$ represent elements $y_{j,l}$ and $z_{j,l}$ in N^ω. Since

$$\|E(x_{j,k})\| = \sup_{\substack{f\in L^1(X,\mu)_+\\ \|f\|_{L^1(X,\mu)}=1}} \left|\tau\big(f\phi_k(\psi_k(a_j))\big) - \tau\big(f\pi_\tau(a_j)\big)\right|, \tag{2.36}$$

it follows that $(E(x_{j,k}))_k$ represents $0 \in N^\omega$ and so $(x_{j,k})_k$ represents the finite sum of commutators $\sum_{l=1}^{10}[y_{j,l}, z_{j,l}]$ in N^ω and hence is zero in all traces on N^ω.

By the remark preceding the lemma, θ and π_τ^ω are σ-strong* approximately unitarily equivalent. Because A is separable and we work in an ultrapower, a standard reindexing argument (using Kirchberg's ϵ-test from [20, Appendix A]) shows that θ and π_τ^ω are actually unitarily equivalent. That is, there exists a sequence (u_k) of unitaries in N such that

$$\lim_{k\to\omega} \|u_k(\phi_k \circ \psi_k)(a)u_k^* - \pi_\tau(a)\|_{2,\tau} = 0, \quad a \in A. \tag{2.37}$$

Let $\tilde{\phi}_k = \operatorname{Ad} u_k \circ \phi_k$. Passing to a subsequence, if necessary, we obtain

$$\lim_{k\to\infty} \|(\tilde{\phi}_k \circ \psi_k)(a) - \pi_\tau(a)\|_{2,\tau} = 0, \quad a \in A, \tag{2.38}$$

as was to be proved. □

Proof of Proposition 2.3 For unital C^*-algebras, one just takes direct sums of the maps provided by Lemmas 2.4 and 2.5. The non-unital case follows from the unital case as follows.

Assume A is non-unital and $T(A) = T_{\mathrm{qd}}(A)$. Then by [5, Proposition 3.5.10] we have $T(\tilde{A}) = T_{\mathrm{qd}}(\tilde{A})$, too, where $\tilde{A}$ is the unitization of A. Hence we can find nets of finite-dimensional C^*-algebras (F_i) and c.p.c. maps

$$\tilde{A} \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} (\tilde{A})^{**} \tag{2.39}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \iota_{\tilde{A}}(a)$, in the σ-weak topology for all $a \in \tilde{A}$
2. every ϕ_i is a convex combination of finitely many contractive order zero maps; and
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in \tilde{A}$.

The short exact sequence $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ induces a canonical isomorphism $(\tilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$. The desired maps are now gotten by restricting each ψ_i to A and using the σ-weakly continuous projection $(\tilde{A})^{**} \to A^{**}$ to push the ϕ_i's back into A^{**}. □

3 The Main Theorem

Theorem 3.1 *Let A be a nuclear C^*-algebra. Then there exist nets of finite-dimensional C^*-algebras (F_i) and c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{3.1}$$

such that

1. *$\|(\phi_i \circ \psi_i)(a) - a\| \to 0$ for all $a \in A$;*
2. *every ϕ_i is a convex combination of finitely many contractive order zero maps;*
3. *$\|\psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in A_+$ that satisfy $ab = 0$.*

To prove Theorem 3.1 we will apply Theorem 2.2 to the cone $CA = C_0(0, 1] \otimes A$ of A. We will need to know that all traces on CA are quasidiagonal for nuclear

A. While this follows from [24, Corollary 6.1],[3] it is really the case that the required statement is a recasting of the "order zero quasidiagonality result" of [22, Proposition 3.2] used as the starting point in [24]. More generally, Gabe's "order zero quasidiagonality" of amenable traces [14, Proposition 3.6] can also be expressed in this language, as set out below.

Proposition 3.2 (Gabe, c.f. [14, Proposition 3.6]) *Let A be a C^*-algebra. Then every amenable trace on CA is quasidiagonal. In particular if A is nuclear, then all traces on CA are quasidiagonal.*

Proof It is well known that traces of the form $\delta_t \otimes \tau_A$, where δ_t is evaluation at some $t \in (0,1]$ and τ_A is a trace on A, generate the Choquet simplex of traces on the cone CA.[4] Since the amenable traces on CA form a face ([19, Lemma 3.4], see also [6, Proposition 6.3.7]) and the set of quasidiagonal traces is a weak*-closed, convex subset of $T(A)$ [5, Proposition 3.5.1], it suffices to show that any amenable trace on CA of the form $\delta_t \otimes \tau_A$ for some $t \in (0,1]$ and some trace τ_A on A is quasidiagonal.

Note too that if $\delta_t \otimes \tau_A$ is an amenable trace on CA, then τ_A is amenable on A. This follows from [5, Theorem 3.1.6] by checking that the tensor product functional μ_{τ_A} on the algebraic tensor product $A \odot A^{\mathrm{op}}$ given by $\mu_{\tau_A}(a \otimes b^{\mathrm{op}}) = \tau_A(ab)$ is continuous with respect to the minimal tensor product. Let $g \in C_0(0,1]$ be a positive contraction with $g(t) = 1$. Then μ_{τ_A} factorizes as

$$A \odot A^{\mathrm{op}} \xrightarrow{a \otimes b^{\mathrm{op}} \mapsto (g \otimes a) \otimes (g \otimes b)^{\mathrm{op}}} CA \odot (CA)^{\mathrm{op}} \xrightarrow{\mu_{\delta_t \otimes \tau_A}} \mathbb{C}; \tag{3.2}$$

the first of these maps is the tensor product of two c.p.c. maps, so contractive with respect to the minimal tensor product, while contractivity of $\mu_{\delta_t \otimes \tau_A}$ follows from the assumption that $\delta_t \otimes \tau_A$ is amenable.

At this point, if A is not unital, then we can unitize A, and τ_A (since the unitization of an amenable trace remains amenable). As a final reduction, by considering the map $CA \to C_0((0,t],A)$ given by restriction, and then identifying $C_0((0,t],A)$ with CA (by rescaling), we may as well assume that $t = 1$. Then [14, Proposition 3.6] gives a c.p.c. order zero map $\Psi : A \to \mathcal{Q}_\omega$ (where $\mathcal{Q}$ denotes the universal UHF algebra and $\mathcal{Q}_\omega$ its ultrapower) such that

$$\tau_{\mathcal{Q}_\omega}\big(\Psi(a)\Psi(1_A)^{n-1}\big) = \tau_A(a), \quad a \in A,\ n \in \mathbb{N}. \tag{3.3}$$

By the correspondence between order zero maps from A and *-homomorphisms from CA (see [28, Corollary 4.1]) we obtain a *-homomorphism $\psi : CA \to \mathcal{Q}_\omega$ such that $\psi(\mathrm{id}_{(0,1]} \otimes a) = \Psi(a)$ for every $a \in A$. Then for every $a \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned} \tau_{\mathcal{Q}_\omega}\big(\psi(\mathrm{id}^n_{(0,1]} \otimes a)\big) &= \tau_{\mathcal{Q}_\omega}\big(\Psi(a)\Psi(1_A)^{n-1}\big) \\ &= \tau_A(a) = (\delta_1 \otimes \tau_A)(\mathrm{id}^n_{(0,1]} \otimes a). \end{aligned} \tag{3.4}$$

[3] The cone CA is quasidiagonal by [25] and satisfies the UCT, since it is contractible.

[4] That is, any trace on CA lies in the weak*-closed convex hull of the specified traces.

Thus ψ witnesses the quasidiagonality of the trace $\delta_1 \otimes \tau_A$.[5] □

Proof of Theorem 3.1 Let $\mathscr{F} \subset A$ be finite and $\epsilon > 0$. Then there is a separable nuclear subalgebra B of A containing $\mathscr{F}$. Write $\iota : B \to A$ for the canonical inclusion map.

Let $\theta\colon B \to CB$ be the c.p.c. order zero map $b \mapsto \mathrm{id}_{(0,1]} \otimes b$. Notice that CB satisfies the hypotheses of Theorem 2.2: it is certainly separable and nuclear, it is quasidiagonal by a theorem of Voiculescu [25], and all of its traces are quasidiagonal by Proposition 3.2 (as CB is nuclear, all traces are amenable). Then there are a finite dimensional algebra F and c.p.c maps $\psi\colon CB \to F$ and $\phi\colon F \to CB$ such that

1. $\|(\phi \circ \psi)(\theta(x)) - \theta(x)\| < \epsilon$;
2. ϕ is a convex combination of finitely many contractive order zero maps; and
3. $\|\psi\big(\theta(x)\theta(y)\big) - \psi(\theta(x))\psi(\theta(y))\| < \epsilon$;

for all $x, y \in \mathscr{F}$. Let $\eta\colon CB \to B$ be given by the point evaluation at 1 so that $\eta \circ \theta = \mathrm{id}_B$.

Define a c.p.c. map $\overline{\psi}\colon A \to F$ by extending $\psi \circ \theta$ to A (using Arveson's extension theorem) and set $\overline{\phi} = \iota \circ \eta \circ \phi\colon F \to A$. Then $\overline{\phi}$ is a convex combination of contractive order zero maps (because $\iota \circ \eta$ is a *-homomorphism), $\|(\overline{\phi} \circ \overline{\psi})(x) - x\| < \epsilon$ for every $x \in \mathscr{F}$, and $\|\overline{\psi}(x)\overline{\psi}(y)\| < \epsilon$ if $x, y \in \mathscr{F}$ are orthogonal positive elements. □

Remark 3.3 As with the approximations in [16], attempting to merge the approximations of Theorem 3.1 with the nuclear dimension by additionally asking for a uniform bound on the number of summands in the decompositions of Φ_i as a convex combination of order zero maps is very restrictive. By the main result of [8], such approximations only exist for *AF* C^*-algebras.

4 The Proof of Proposition 3.2, Revisited

This section was added in the revision submitted to the editors on 24 August, 2017. The only changes made to the earlier sections are the addition of footnote 5, and the updating of references to precise locations in [14] to reflect the final published version. We have also updated the publication information in the bibliography for [1, 4, 8, 14, 24].

In Proposition 3.2 we claimed that any amenable trace τ on a cone CA over a C*-algebra is quasidiagonal. In our proof we produced a *-homomorphism ψ from CA into the ultraproduct $\mathscr{Q}_\omega$ of the universal UHF-algebra so that $\tau = \tau_{\mathscr{Q}_\omega} \circ \psi$. When A is nuclear, the Choi-Effros lifting theorem immediately gives a c.p.c. lift of ψ to $\ell^\infty(\mathscr{Q})$. In the general case, such a c.p.c. lift must be produced in order to obtain quasidiagonality of τ, and we are sorry that we neglected to do this in the

[5] In the case of non-nuclear A, this uses Lemma 4.1 in the next section to show that ψ lifts to a c.p.c. map from CA to $\ell^\infty(\mathscr{Q})$.

first place. Thus for the purposes of all the other statements in the original version of this article, which refer only to nuclear C*-algebras, no additional ingredients are needed. We complete the proof of Proposition 3.2 in this section, using Lemma 4.1 to give the required lift in the case of general A.

We're very grateful to Jamie Gabe for pointing out this omission and to Wilhelm Winter—the wizard of functional calculus—for suggesting the functional calculus trickery used below. We're relieved that all statements in the preceding sections remain correct as stated.

In the initial stages of the proof of Proposition 3.2, we reduced to the situation where A is a unital C*-algebra, τ_A is an amenable trace on A, and we need to show that the trace $\tau = \delta_1 \otimes \tau_A$ on the cone $CA = C_0((0,1]) \otimes A$ is quasidiagonal. We also (somewhat implicitly) assume that A is separable. As both amenability and quasidiagonality are local properties it suffices to treat the case of separable A. Using [14, Proposition 3.6] we obtain a c.p.c. order zero map $\Psi : A \to \mathcal{Q}_\omega$ satisfying (3.3). By construction this c.p.c. order zero map comes with a c.p.c. lift to $\ell^\infty(\mathcal{Q})$. This is not recorded explicitly in [14], but is readily seen from the proof: Ψ is of the form $(1_{\mathcal{Q}_\omega} - e)\psi_0(\cdot)(1_{\mathcal{Q}_\omega} - e)$ for a c.p.c. map $\psi_0 : A \to \mathcal{Q}_\omega$ with a c.p.c. lift to $\ell^\infty(\mathcal{Q})$ and a positive contraction $e \in \mathcal{Q}_\omega$ (and so e lifts to a representative sequence of positive contractions in $\ell^\infty(\mathcal{Q})$). We then use the duality between c.p.c. order zero maps and *-homomorphisms from cones [28, Corollary 4.1] to obtain a *-homomorphism $\psi : CA \to \mathcal{Q}_\omega$ with $\psi(\mathrm{id}_{(0,1]} \otimes a) = \Psi(a)$ for all $a \in A$; checking in (3.4) that ψ has $\tau_{\mathcal{Q}_\omega} \circ \psi = \delta_1 \otimes \tau_A$. To complete the proof of Proposition 3.2 we must show that ψ (which is uniquely determined by Ψ) has a c.p.c. lift to $\ell^\infty(\mathcal{Q})$. We do this in the following lemma, which may well be of use in other situations.

Lemma 4.1 *Let A be a separable unital* C*-*algebra, and let B be a unital* C*-*algebra. Suppose that $(\psi_n)_{n=1}^\infty$ is a sequence of c.p.c. maps from A into B inducing an order zero map $\psi : A \to B_\omega$. Then there exists a sequence of cpc maps $(\phi_n)_{n=1}^\infty$ from $C_0((0,1]) \otimes A$ into B inducing a *-homomorphism $\phi : C_0((0,1]) \otimes A \to B_\omega$ such that $\phi(\mathrm{id} \otimes x) = \psi(x)$ for $x \in A$.*

Before giving the proof of Lemma 4.1 we record the following fact which will be used repeatedly.

Lemma 4.2 *Let A be a separable unital* C*-*algebra, and let B be a unital* C*-*algebra. Suppose that $(\psi_n)_{n=1}^\infty$ is a sequence of c.p.c. maps from A into B inducing an order zero map $\psi : A \to B_\omega$. For a contraction $a \in A$, there exists a sequence $(b_n)_{n=1}^\infty$ of contractions in B so that $(\psi_n(1_A)b_n)_{n=1}^\infty$ and $(b_n\psi_n(1_A))_{n=1}^\infty$ both represent $\psi(a)$.*

Proof By Kirchberg's ϵ-test [20, Lemma A.1] to prove the lemma it suffices to take $\epsilon > 0$ and find a sequence $(b_n)_{n=1}^\infty$ of contractions so that

$$\lim_{n\to\omega} \|\psi_n(1_A)b_n - \psi_n(a)\| \quad \text{and} \quad \lim_{n\to\omega} \|b_n\psi_n(1_A) - \psi_n(a)\| < \epsilon. \tag{4.1}$$

Recall from [28, Theorem 3.3] that the supporting *-homomorphism of ψ is a *-homomorphism $\pi : A \to \mathcal{M}(C^*(\psi(A)))$ satisfying $\psi(x) = \pi(x)\psi(1_A)$

$= \psi(1_A)\pi(x)$, for $x \in A$. Then, for $f \in C_0((0,1])_+$, $f(\psi)$ is defined by $f(\psi)(x) = \pi(x)f(\psi(1_A))$, for $x \in A$; this is a c.p. order zero map (which is contractive when f is). Taking $f(t) = t^r$, where $0 < r < 1$, gives

$$\psi(x) = \pi(x)\psi(1_A)^r\psi(1_A)^{1-r} = \psi^r(a)\psi(1_A)^{1-r} = \psi(1_A)^{1-r}\psi^r(x), \quad x \in A. \tag{4.2}$$

Fix $0 < r < 1$ small enough so that $\sup_{t\in[0,1]} |t^{1-r} - t| < \epsilon$. For a contraction $a \in A$, let $(b_n)_{n=1}^{\infty}$ be a sequence of positive contractions in B representing the contraction $\psi^r(a)$. Then

$$\begin{aligned}\lim_{n\to\omega} \|\psi_n(1_A)b_n - \psi_n(a)\| &= \lim_{n\to\omega} \|\psi_n(1_A)b_n - \psi_n(1_A)^{1-r}b_n\| \\ &\leq \lim_{n\to\omega} \|\psi_n(1_A) - \psi_n(1_A)^{1-r}\| < \epsilon. \end{aligned} \tag{4.3}$$

Likewise

$$\lim_{n\to\omega} \|b_n\psi_n(1_A) - \psi_n(a)\| < \epsilon, \tag{4.4}$$

as required. □

Proof of Lemma 4.1 We use Kirchberg's ϵ-test [20, Lemma A.1] to handle the reindexing and allow us to prove a slightly weaker statement. To set this up, for each n, let[6] X_n be the set of c.p.c. maps $C_0((0,1]) \otimes A \to B$, let $(a_i)_{i=1}^{\infty}$ be a sequence which is dense in the positive contractions of A and $(f_i)_{i=1}^{\infty}$ a sequence which is dense in the positive contractions of $C_0((0,1])$. For $i, i_1, i_2, j_1, j_2 \in \mathbb{N}$, define $r_n^{(i_1,i_2,j_1,j_2)}, s_n^{(i)} : X_n \to [0,\infty)$ by

$$r_n^{(i_1,i_2,j_1,j_2)}(\phi_n) = \|\phi_n(f_{i_1} \otimes a_{i_2})\phi_n(f_{j_1} \otimes a_{j_2}) - \phi_n(f_{i_1}f_{j_1} \otimes a_{i_2}a_{j_2})\| \tag{4.5}$$

and

$$s_n^{(i)}(\phi_n) = \|\phi_n(\mathrm{id} \otimes a_i) - \psi_n(a_i)\|. \tag{4.6}$$

Then a sequence $(\phi_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$ induces a *-homomorphism $\phi : C_0((0,1]) \otimes A \to B_\omega$ if and only if $\lim_{n\to\omega} r^{(i_1,i_2,j_1,j_2)}(\phi_n) = 0$ for all $i_1, i_2, j_1, j_2 \in \mathbb{N}$ and has $\phi(\mathrm{id} \otimes x) = \psi(x)$ for all $x \in A$ if and only if $\lim_{n\to\omega} s_n^{(i)}(\phi_n) = 0$ for all $i \in \mathbb{N}$. Thus, by Kirchberg's ϵ-test we can fix $\epsilon > 0$ and $i_0 \in \mathbb{N}$, and it suffices to find a sequence $(\phi_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$ such that

$$\lim_{n\to\omega} r_n^{(i_1,i_2,j_1,j_2)}(\phi_n) \leq \epsilon \quad \text{and} \quad \lim_{n\to\omega} s_n^{(i)}(\phi_n) \leq \epsilon \tag{4.7}$$

for $i, i_1, i_2, j_1, j_2 = 1, \ldots, i_0$.

[6] There is no dependence on n in the definition of X_n. We use this notation for consistency with the usual formulation of Kirchberg's ϵ-test.

For $\delta > 0$ to be chosen later, define $g_\delta \in C_0((0,1])$ by

$$g_\delta(t) = \begin{cases} \delta^{-2}t, & 0 \le t \le \delta \\ t^{-1}, & t > \delta \end{cases}. \tag{4.8}$$

Note that

$$0 \le g_\delta(t)t \le 1, \quad t \in [0,\infty), \quad \text{and} \quad \sup_{t\in[0,1]} |(g_\delta(t)t - 1)t| \le \delta. \tag{4.9}$$

Fix $m \in \mathbb{N}$ with the property that

$$|f_i(x) - f_i(y)| \le \epsilon/3 \text{ whenever } |x - y| \le 1/m \tag{4.10}$$

for all for $i \le i_0$. Let $h_0, \dots, h_m$ be standard 'saw-tooth' partition of unity corresponding to the division of $[0,1]$ into m intervals. That is, h_i is (the restriction to $[0,1]$ of) the piecewise affine function satisfying $h_i((i-1)/m) = 0$, $h_i(i/m) = 1$, $h_i((i+1)/m) = 0$, and defined to be affine on the intervals $[(i-1)/m, i/m]$ and $[i/m, (i+1)/m]$ and 0 outside the interval $[(i-1)/m, (i+1)/m]$. In this way, for any scalars $\alpha_0, \dots, \alpha_m$, the element $f = \sum_{l=0}^m \alpha_l h_l$ is the piecewise affine function in $C([0,1])$ with $f(j/m) = \alpha_j$ and affine on the intervals $[j/n, (j+1)/n]$.

To simplify notation we will write e_n for $\psi_n(1_A)$. For each n, we define a map $\phi_n : C_0((0,1]) \otimes A \to B$ by

$$\phi_n(f \otimes a) = \sum_{l=0}^m f(l/m) h_l(e_n)^{1/2} g_\delta(e_n)^{1/2} \psi_n(a) g_\delta(e_n)^{1/2} h_l(e_n)^{1/2} \tag{4.11}$$

for $f \in C_0((0,1])$ and $a \in A$. This certainly extends to a linear map, and as ψ_n is completely positive, so too is each map in the sum defining ϕ_n, and so ϕ_n is completely positive. Now for a positive contraction $f \in C_0((0,1])$ we have

$$\begin{aligned} \phi_n(f \otimes 1_A) &= \sum_{l=0}^m f(l/m) g_\delta(e_n) h_l(e_n) e_n \\ &\le g_\delta(e_n) e_n \sum_{l=0}^m h_l(e_n) \le 1_B, \end{aligned} \tag{4.12}$$

as $0 \le g_\delta(t)t \le 1$ and $\sum_{l=0}^m h_l(t) = 1$ for all $t \in [0,1]$. Since f is an arbitrary positive contraction, it follows that ϕ_n is contractive (without any condition on δ). It remains to show that we can choose δ so that the sequence $(\phi_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$ verifies the two conditions of (4.7).

Since the sequence $(\psi_n)_{n=1}^\infty$ induces an order zero map, we have

$$\lim_{n\to\omega} \|[e_n, \psi_n(a)]\| = 0, \quad a \in A. \tag{4.13}$$

Therefore

$$\lim_{n\to\omega} \left\| \phi_n(\mathrm{id}\otimes a) - \psi_n(a) g_\delta(e_n) \sum_{l=0}^{m} \frac{l}{m} h_l(e_n) \right\| = 0. \tag{4.14}$$

Now $\sum_{l=0}^{m} \frac{l}{m} h_l(t) = t$, so

$$\lim_{n\to\omega} \|\phi_n(\mathrm{id}\otimes a) - \psi_n(a) g_\delta(e_n) e_n\| = 0. \tag{4.15}$$

Fix a contraction $a \in A$, and by Lemma 4.2, let $(b_n)_{n=1}^\infty$ be a sequence of contractions in B so that $(b_n e_n)_{n=1}^\infty$ represents $\psi(a)$. Using (4.9) for the second estimate below, we have

$$\lim_{n\to\omega} \|\psi_n(a) g_\delta(e_n) e_n - \psi_n(a)\| \le \lim_{n\to\omega} \|b_n\| \|e_n g_\delta(e_n) e_n - e_n\| \le \delta. \tag{4.16}$$

In this way

$$\lim_{n\to\omega} \|\phi_n(\mathrm{id}\otimes a) - \psi_n(a)\| \le \delta \tag{4.17}$$

for any contraction $a \in A$. Then, provided we ensure $\delta < \epsilon$, we get $\lim_{n\to\omega} s_n^{(i)}(\phi_n) \le \epsilon$ for all i.

Now we show ‘almost multiplicativity’. For this, fix $i_1, i_2, j_1, j_2 \le i_0$. Using (4.13) and the fact that $h_k h_l = 0$ for $|k-l| \ge 2$ we have

$$\begin{aligned} &\lim_{n\to\omega} \|\phi_n(f_{i_1}\otimes a_{i_2})\phi_n(f_{j_1}\otimes a_{j_2}) - \phi_n(f_{i_1}f_{j_1}\otimes a_{i_2}a_{j_2})\| \\ &= \lim_{n\to\omega} \Bigg\| \sum_{|k-l|\le 1} f_{i_1}(k/m) f_{j_1}(l/m) h_k(e_n) h_l(e_n) g_\delta(e_n)\psi_n(a_{i_2}) g_\delta(e_n)\psi_n(a_{j_2}) \\ &\qquad - \sum_{l=0}^{m} f_{i_1}(l/m) f_{j_1}(l/m) h_l(e_n) g_\delta(e_n) \psi_n(a_{i_2}a_{j_2}) \Bigg\|. \end{aligned} \tag{4.18}$$

Using Lemma 4.2, we can find a sequence $(b_n)_{n=1}^\infty$ of contractions in B so that $(e_n b_n)_{n=1}^\infty$ represents $\psi(a_{i_2})$. In this way (4.9) gives

$$\lim_{n\to\omega} \|g_\delta(e_n)\psi_n(a_{i_2})\| = \lim_{n\to\omega} \|g_\delta(e_n) e_n b_n\| \le 1. \tag{4.19}$$

Likewise $\lim_{n\to\omega}\|g_\delta(e_n)\psi_n(a_{j_2})\|\leq 1$. As, for each l, $\sum_{k=l-1}^{l+1}h_k$ acts as a unit on h_l,[7] we may use the estimates above to obtain

$$\begin{aligned}\lim_{n\to\omega}\Big\|\sum_{|k-l|\leq 1} & f_{i_1}(k/m)f_{j_1}(l/m)h_k(e_n)h_l(e_n)g_\delta(e_n)\psi_n(a_{i_2})g_\delta(e_n)\psi_n(a_{j_2}) \\ & -\sum_{l=0}^{m}f_{i_1}(l/m)f_{j_1}(l/m)h_l(e_n)g_\delta(e_n)\psi_n(a_{i_2})g_\delta(e_n)\psi_n(a_{j_2})\Big\| \\ \leq \lim_{n\to\omega}\Big\|\sum_{l=0}^{m}f_{j_1}(l/m)h_l(e_n)\sum_{k=l-1}^{l+1}&\big(f_{i_1}(k/m)-f_{i_1}(l/m)\big)h_k(e_n)\Big\|.\end{aligned}\tag{4.20}$$

Using the choice of m in (4.10), observe that $|\sum_{k=l-1}^{l+1}\big(f_{i_1}(k/m)-f_{i_1}(l/m)\big)h_k|\leq(\epsilon/3)\sum_{k=l-1}^{l+1}h_k$ (in $C([0,1])$) for every $l=0,\ldots,m$, and therefore

$$\Big|\sum_{l=0}^{m}f_{j_1}(l/m)h_l\sum_{k=l-1}^{l+1}\big(f_{i_1}(k/m)-f_{i_1}(l/m)\big)h_k\Big|\leq(\epsilon/3)\sum_{l=0}^{m}f_{j_1}(l/m)h_l\leq\epsilon/3,\tag{4.21}$$

as f_{j_1} is a contraction. This shows that the last limit in (4.20) is bounded above by $\epsilon/3$.

Using this, the order zero identity $\psi(1_A)\psi(a_{i_2}a_{j_2})=\psi(a_{i_1})\psi(a_{j_2})$ (see [4, (1.3)], for example) and (4.13) again, we have

$$\begin{aligned}&\lim_{n\to\omega}\|\phi_n(f_{i_1}\otimes a_{i_2})\phi_n(f_{j_1}\otimes a_{j_2})-\phi_n(f_{i_1}f_{j_1}\otimes a_{i_2}a_{j_2})\| \\ &\leq\lim_{n\to\omega}\Big\|\sum_{l=0}^{m}f_{i_1}(l/m)f_{j_1}(l/m)h_l(e_n)g_\delta(e_n)^2e_n\psi_n(a_{i_2}a_{j_2}) \\ &\qquad-\sum_{l=0}^{m}f_{i_1}(l/m)f_{j_1}(l/m)h_l(e_n)g_\delta(e_n)\psi_n(a_{i_2}a_{j_2})\Big\|+\epsilon/3.\end{aligned}\tag{4.22}$$

Let $f_{i_1,j_1}(t)=\sum_{l=0}^{m}f_{i_1}(l/m)f_{j_1}(l/m)h_l(t)$, so that f_{i_1,j_1} is a contraction in $C_0((0,1])$. We aim to control

$$\lim_{n\to\omega}\big\|f_{i_1,j_1}(e_n)\big(g_\delta(e_n)^2e_n-g_\delta(e_n)\big)\psi_n(a_{i_2}a_{j_2})\big\|.\tag{4.23}$$

Use Lemma 4.2 to find a sequence $(b_n)_{n=1}^{\infty}$ of contractions in B so that $(e_nb_n)_n$ represents $\psi(a_{i_2}a_{j_2})$. As (4.9) gives $0\leq g_\delta(e_n)e_n\leq 1$ in B, so $\|g_\delta(e_n)^2e_n^2-$

[7] When $l=0$ or $l=m$ there are only two terms in this sum, but the result still holds.

$g_\delta(e_n)e_n\| \le \frac{1}{\sqrt{2}} - \frac{1}{2}$, giving

$$\lim_{n\to\omega} \left\| \big(g_\delta(e_n)^2 e_n - g_\delta(e_n)\big)\psi_n(a_{i_2}a_{j_2})\right\| = \lim_{n\to\omega} \left\| \big(g_\delta(e_n)^2 e_n^2 - g_\delta(e_n)e_n\big)b_n\right\| \tag{4.24}$$

$$\le \frac{1}{\sqrt{2}} - \frac{1}{2},$$

independently of the value of δ used in the definition of g_δ. Fix a polynomial function $p_{i_1,j_1} \in C_0((0,1])$ (so with no constant term) so that $\|p_{i_1,j_1} - f_{i_1,j_1}\| \le \epsilon/3(\frac{1}{\sqrt{2}} - \frac{1}{2})$. Therefore, using the fact that $(e_n b_n)_{n=1}^\infty$ represents the same sequence as $(\psi_n(a_{i_2}a_{j_2}))_{n=1}^\infty$, we have

$$\lim_{n\to\omega} \left\|f_{i_1,j_1}(e_n)\big(g_\delta(e_n)^2 e_n - g_\delta(e_n)\big)\psi_n(a_{i_2}a_{j_2})\right\| \tag{4.25}$$

$$\le \lim_{n\to\omega} \left\|p_{i_1,j_1}(e_n)\big(g_\delta(e_n)^2 e_n^2 - g_\delta(e_n)e_n\big)b_n\right\| + \epsilon/3,$$

again independent of the choice of δ. Finally, as $\sup_{t\in[0,1]} \left|t\big(g_\delta(t)^2t^2 - g_\delta(t)t\big)\right| \le \delta$ (from (4.9)) and $p_{i_1,j_1}(t)$ factors as $q_{i_1,j_1}(t)t$ for some polynomial $q_{i_1,j_1}(t)$ (this time with a possible constant term) we can now choose $\delta \le \epsilon$ so that

$$\lim_{n\to\omega} \left\|p_{i_1,j_1}(e_n)\big(g_\delta(e_n)^2 e_n^2 - g_\delta(e_n)e_n\big)\right\| \le \epsilon/3 \tag{4.26}$$

for all $i_1, j_1 \le i_0$. Putting (4.26) together with (4.22) and (4.25), we obtain

$$\lim_{n\to\omega} r_n^{(i_1,i_2,j_1,j_2)}(\phi_n) \le \epsilon, \tag{4.27}$$

for all $i_1, i_2, j_1, j_2 \le i_0$, as required. □

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