Problem Set 2: Recursion

EG1. Find a formula for the $n$th term of the Fibonacci sequence, given by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.

EG2. Find a formula for $a_n$ if $a_0 = 1$, $a_1 = 6$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for $n \geq 1$.

EG3. Find an explicit formula for the sequence defined by $c_0 = 1$, $c_n = 3c_{n-1} + 2$ for $n > 0$.

EG4. A derangement of $1, 2, \ldots, n$ is a permutation that leaves no element fixed. Show that the number of derangements is

$$n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

1. If a sequence $(a_n)$ satisfies $a_{n+1} = a_n + a_{n-1}$, prove that $a_{n+2} = 3a_n - a_{n-2}$.

2. Suppose a sequence is given recursively by $a_0 = b$, $a_1 = c$, $a_{n+2} = \frac{n+1+n}{2}$. Find $\lim_{n \to \infty} a_n$.

3. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1, \quad \text{for } n = 1, 2, 3, \ldots .$$

Prove there exists a real number $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

4. In tennis, once a game reaches “deuce,” it continues until one player is ahead by two points. Suppose the server has probability $p$ of winning any particular point. What is the probability the server wins a deuce game?

5. Pairs of gray socks come in $n$ increasingly dark shades. Coming out of the dryer, they are paired at random, What is the probability every sock is paired with one that is at most one shade away?

6. Let $a$ and $b$ be positive numbers and define a sequence $(x_n)$ by $x_0 = a$, $x_1 = b$, and

$$x_{n+1} = \frac{1}{2} \left( \frac{1}{x_n} + x_{n-1} \right).$$

For what $a$ and $b$ is $(x_n)$ periodic?

7. The sequence $(x_n)$ is defined by $x_1 = 2$, $x_2 = 3$, and

$$x_{2m+1} = x_{2m} + x_{2m-1}, \quad m \geq 1,$$

$$x_{2m+2} = x_{2m+1} + 2x_{2m}, \quad m \geq 1.$$

Determine $x_n$ as a function of $n$.

8. How many regions are formed when $n$ lines are drawn in the plane with no two parallel and no three concurrent?

9. We flip a coin and score 1 point for each head and 2 points for each tail. What is the probability our score is exactly $n$ at some point?

10. A coin is tossed $n$ times. What is the probability that two heads will turn up in succession somewhere in the sequence of tosses?

11. For how many subsets $S$ of $\{1, 2, \ldots, n\}$ does $S \cup (S + 1) \supset \{1, 2, \ldots, n\}$?

12. Let $A$ and $E$ be opposite vertices of an octagon. From $A$ the frog jumps to one of the two adjacent vertices. When the frog first reaches $E$, it stops and stays there. If $a_n$ is the number of distinct paths of exactly $n$ jumps ending at $E$, prove that

$$a_{2n+1} = 0$$

and

$$a_{2n} = \frac{((2 + \sqrt{2})^{n-1} - (2 - \sqrt{2})^{n-1})}{\sqrt{2}}.$$

13. For what real numbers $a$ does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?