Problem Set 11: Functional Equations

EG1 Find all complex-valued functions for which \( f(z) + zf(-z) = 1 + z \) for all \( z \in \mathbb{C} \).

EG2. Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(x^2) - f(y^2) = (x + y)[f(x) - f(y)] \).

EG3. Suppose \( f \) and \( g \) are nonconstant, differentiable, real-valued functions on \( \mathbb{R} \). Furthermore, suppose that for each pair of real numbers \( x \) and \( y \),

\[
\begin{align*}
f(x + y) &= f(x)f(y) - g(x)g(y), \\
g(x + y) &= f(x)g(y) + g(x)f(y).
\end{align*}
\]

If \( f'(0) = 0 \), prove that \( (f(x))^2 + (g(x))^2 = 1 \) for all \( x \).

1. Given a constant \( C \), find all functions \( f \) such that \( f(x) + C f(2 - x) = (x - 1)^3 \) for all \( x \).

2. Given

\[
\begin{align*}
f(x) &= \frac{u(x + 1) + u(x - 1)}{2}, \\
g(x) &= \frac{u(x + 4) + u(x - 4)}{2},
\end{align*}
\]

express \( u(x) \) in terms of \( f \) and \( g \).

3. Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(x^2 - y^2) = (x - y)[f(x) + f(y)] \).

4. Establish necessary and sufficient conditions on the constant \( k \) for the existence of a continuous real-valued function \( f(x) \) satisfying \( f(f(x)) = kx^9 \) for all real \( x \).

5. Determine \( f(x) \), if, for all real \( x \) and \( y \), \( f(xy) = f(x)f(y) - x - y \).

6. Find all real functions \( f \) such that \( f(x + 2) = f(x) \) and \( f'(x) = f(x + 1) - 2 \) for all real \( x \).