Dr. Friedman’s Calculus III Notes, or
I’m looking at an integral - so what the heck do I do now?

Integrals on curves (one dimensional integrals)

1. I’m integrating a function: $\int_C f(x, y, z) \, dr$.

   (a) There’s only one option: parameterize and evaluate:
   $$\int_C f(x, y, z) \, dr = \int_a^b f(x(t), y(t), z(t)) \left| \frac{dr}{dt} \right| \, dt$$

2. I’m integrating a vector field: $\int_C F \cdot dr$ or $\int_C Mdx + Ndy + Pdz$.

   (a) $C$ is not a closed curve
   i. $F$ is not conservative
      A. Your only option is to parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt} \, dt$
   ii. $F$ is conservative: $F = \nabla f$
      A. Option 1: You can parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt} \, dt$
      B. Option 2: You can find the potential and use the fundamental theorem of line integrals: $\int_C F \cdot dr = f(r(b)) - f(r(a))$
      C. Option 3: You can parametrize and integrate but using a different path with the same endpoints.

   (b) $C$ is a closed curve
   i. $F$ is not conservative
      A. Option 1: Parameterize and integrate: $\int_C F \cdot dr = \int_a^b F \cdot \frac{dr}{dt} \, dt$
      B. Option 2: If $C$ is a simple closed curve in the plane and $F$ is a vector field in the plane, you could use Green’s theorem: $\int_C Mdx + Ndy = \int_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA$ if $C$ is the boundary of $R$ and $C$ is oriented counterclockwise
      C. Option 3: If $C$ is a closed curve in space and it is the boundary of a surface $S$, then you can use Stokes’s theorem: $\int_C F \cdot dr = \oint_S (\nabla \times F) \cdot N \, dS$, where $N$ satisfies the right hand rule with respect to the orientation of $C$.
   ii. $F$ is conservative
      A. The integral is 0.

Integrals on surfaces (two dimensional integrals)

1. I’m integrating a function: $\iint_S f(x, y, z) \, dS$. 

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(a) There’s only one option: parameterize and evaluate:

\[ \iint_S f(x, y, z) \, dS = \iint_R f(x(u, v), y(u, v), z(u, v)) \left| \frac{dr}{du} \times \frac{dr}{dv} \right| \, dA, \]

where \( R \) is the parametrizing region in the \( u-v \) plane.

2. I’m integrating a flux integral of a vector field \( \iint F \cdot N \, dS \)

   (a) \( S \) is not orientable
      i. The integral cannot be well-defined

   (b) \( S \) is orientable but not the boundary of anything
      i. Parametrize and evaluate:

\[ \iint_S F(x, y, z) \cdot N \, dS = \iint_R F \cdot \left( \frac{dr}{du} \times \frac{dr}{dv} \right) \, dA, \]

      ii. In the rare case that \( F \) is the curl of another vector field, say \( F = \nabla \times W \), then you could use Stokes’s theorem, but this doesn’t come up very often.

   (c) \( S \) is the boundary of a solid \( Q \).
      i. Option 1: Parametrize and evaluate:

\[ \iint_S F \cdot N \, dS = \iint_R F \cdot \left( \frac{dr}{du} \times \frac{dr}{dv} \right) \, dA, \]

      ii. Option 2: Use the divergence theorem

\[ \iint_S F \cdot N \, dS = \iiint_Q \nabla \cdot F \, dV, \]

where \( N \) is the normal pointing out of the solid. If you want to compute flux into the solid, change the sign of the answer.

**Integrals on solids (three dimensional integrals)**

1. We only integrate functions on solids. Use chapter 14 methods to evaluate \( \iiint_Q f(x, y, z) \, dV \)

2. If you happen to know that the function \( f \) is a divergence \( f = \nabla \cdot F \), then you could use the divergence theorem \( \iiint_Q \nabla \cdot F \, dV = \iint_S F \cdot N \, dS \), where \( S \) is the boundary of \( Q \) and \( N \) is the outward pointing normal, but you’d almost never do this unless it really simplifies nicely for some reason.