Spinning constructions for higher dimensional knots [DRAFT]

Greg Friedman
Yale University

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1 Introduction

As the introduction to any book on knot theory will tell you, if you have ever tied your own shoe laces, then you are already somewhat familiar with knot theory. Of course, your shoe laces aren't really knotted, in a mathematical sense, since you can untie them. A more appropriate mental image would be that of a circular string, one with no free ends, looping around itself in three-dimensional space. Defined more precisely as differentiable embeddings of S^1 into $\mathbb{R}^3$ or S^3 (the three dimensional sphere), knots, as mathematical objects, have been the focus of intense study for over a century. Originally studied mainly for its inherent interest and for providing some of the simplest examples of topological embeddings, knot theory today remains a field of active research and maintains strong and growing ties with many other disciplines, not only in mathematics but also in physics, chemistry, and biology (see, e.g., [1, Chapter 7]). The concept has also evolved to include knots in higher dimensional spaces: knottings of $n-2$ dimensional spheres inside n-dimensional space.

This exposition is intended to provide some introduction to this last concept. Once our shoe laces have been taken away (or turned into spheres!), how can we construct and visualize concrete examples of higher dimensional knots. One important theoretical method has been *surgery theory* (see [11] for a survey of the confluence of surgery and knot theories). Surgery is both a powerful and a complex tool, but construction of knots by surgical methods often does not allow one to "see" the knot. Knots can be obtained by surgery and their properties studied, but often these knots can be described only in terms of their algebraic properties. In this paper, we want to be able to visualize our knots, at least as far as it is possible to do so with our three-dimensional brains. This brings us to a series of constructions known as *knot spinnings*. Various contributions and refinements have been made to this theory, dating back as far as 1925, but the various spinning constructions all have the appeal of being geometric in nature and highly visual. On top of providing a myriad of examples of importance in knot theory, these constructions provide an excellent introduction to thinking about higher-dimensional knots and higher-dimensional topology in general.

There exist by now a nearly uncountable number of books and papers in the literature concerning knot theory. We mention just a few for the reader who would like to pursue knot theory further: Colin Adams's *The Knot Book*[1] is an introduction to knot theory for non-mathematicians but provides an expansive and readable overview; Charles Livingston's *Knot Theory* [14] is an excellent basic introduction to classical knot theory that assumes no background beyond linear algebra; Dale Rolfsen's *Knots and Links* [15] is the by-now classic introductory text assuming some background in algebraic topology; and W.B. Raymond Lickorish's *An Introduction to Knot Theory* [12] also assumes some background in topology but contains some more recent topics such as the exciting new field of quantum invariants of knots. Of these, only Adams's [1] and Rolfsen's [15] books touch upon higher-dimensional knot theory. We would be remiss not to mention also the plethora of knot resources now available on the World Wide Web. We refer the reader to Peter Suber's "Knots on the We" [17] as a comprehensive index.

I thank Joan Doran for drawing all figures.

2 Some basics

We begin with the precise definition of a knot.

Let S^n be the n-dimensional sphere. Without further comment, we will be free to think of S^n in several ways: as an abstract manifold, as the set of points in $\mathbb{R}^{n+1}$ unit distance from the origin, or as $\mathbb{R}^n$ *compactified* by adding a *point at infinity*. More generally, we will use S^n to denote any object *diffeomorphic* to the sphere, i.e. any object that admits a differentiable bijection to S^n with differentiable inverse. Similarly, we will use B^n to denote any object diffeomorphic to the unit ball in $\mathbb{R}^n$, the set of points within unit distance of the origin. Note that the boundary of B^n, denoted ∂B^n, is S^{n-1}.

Figure 1: Some differentiable knots $S^1 \subset \mathbb{R}^3$.

With these conventions, a knot of dimension n is a differentiable embedding $K : S^{n-2} \hookrightarrow S^n$ or $K : S^{n-2} \hookrightarrow \mathbb{R}^n$, where a differentiable embedding is a smooth injective map whose derivative matrix is also injective at all points. There is no real theoretical difference between letting S^n or $\mathbb{R}^n$ serve as the codomain since if we think of S^n as the compactified $\mathbb{R}^n$, we are free to push a knot off the point at infinity so that it will lie in $\mathbb{R}^n$. Sticking with spheres has some technical advantages, and we will principally use spheres as the ambient space, though occasionally it will suit us to use $\mathbb{R}^n$ instead.

The requirement that a knot be differentiable is a common restriction; it is designed to avoid "infinite knottedness", which can occur if we only require the embedding to be continuous (Figure 2). There are other assumptions that can be made to prevent these problems, but we adhere to the differentiability criterion as the one with which the reader is most likely to be familiar.

Figure 2: A "wild" knot that is not differentiable at the point of infinite "knottedness".

Now, if you have a knotted string lying on your desk and you pick it up and move it someplace else, we would like to think of it as the same knot. This is taken care of mathematically by considering *equivalence classes* of knots. We call two knots $K_0, K_1 : S^{n-2} \hookrightarrow S^n$ *equivalent* if there is an orientation-preserving diffeomorphism $f : S^n \to S^n$ such that $fK_0 = K_1$. In particular, this will be true if there is an ambient diffeotopy of S^n taking K_0 to K_1, i.e. a map $H : S^n \times [0,1] \to S^n$ that is a diffeomorphism for each fixed time $t \in [0,1]$ and that moves the knot K_0 at time 0 to the knot K_1 at time 1. In fact, this stronger condition is sometimes used as the definition of knot equivalence; for *classical knots*, the case $n = 3$, these two conditions are equivalent.

It is a standard abuse, in which we shall engage freely, to use the word "knot" and the same symbol, K, to refer to the equivalence class of the knot K or even to the image of K. We also sometimes speak of the knotted pair of spaces (S^n, S^{n-2}) or (S^n, K), where we follow the standard convention in topology by which the symbol (X, Y) refers to two spaces with $Y \subset X$. We refer to K or (S^n, K) as n-dimensional or an n-knot. This is also not a universal notation; it is perhaps slightly more standard to refer to such a knot as an $n-2$ knot.

The *unknot* in dimension n is the equivalence class of the "standard embedding" $S^{n-2} \subset S^n$. In other words, if $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1\}$, then the unknot can be represented as $\{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1, x_{n+1} = x_n = 0\}$. The 3-dimensional unknot is equivalent to the unit circle in the x-y plane in $\mathbb{R}^3$. Note that the unknot is the boundary of the ball $B^{n-1} \subset B^{n+1}$ obtained by letting $B^{n-1} = \{\vec{x} \in B^{n+1} \mid x_n = x_{n+1} = 0\}$. We will refer to this particular pair (B^{n+1}, B^{n-1}) as the unknotted ball pair or the standard ball pair.

We conclude this introductory section with a construction that will be used repeatedly. Consider an n-dimensional knot K, and choose any image point $K(x) \in S^n$. Since S^n is a manifold, this point has an open neighborhood diffeomorphic to $\mathbb{R}^n$ that we shall denote B^n_- (see Figure 4). If this neighborhood is chosen small enough, the intersection $B^n_- \cap K$ will be a ball B^{n-2}_- of dimension $n-2$, and together the pair (B^n_-, B^{n-2}_-) is unknotted, i.e. it is diffeomorphic to the standard ball pair. This follows from the general theory of differentiable embeddings of manifolds. Since the complement of an open smoothly embedded ball in a sphere is a closed ball (Figure 3), the complements $S^n - B^n_-$ and $K - B^{n-2}_-$ will each be balls, and we label this complementary pair by (B^n_K, B^{n-2}_K). The ball B^{n-2}_K may be knotted in B^n_K.

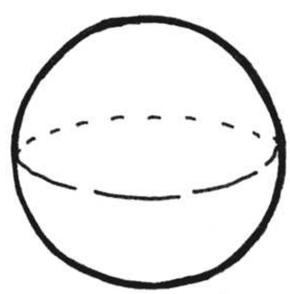

Figure 3: Any sphere S^n is the union of two balls B^n along the equator S^{n-1}. This diagram illustrates the case $n = 2$.

We also observe that the common boundary of the pairs (B^n_-, B^{n-2}_-) and (B^n_K, B^{n-2}_K) is the *unknotted* pair of spheres (S^{n-1}, S^{n-3}). In what follows, it will often be convenient to identify this with the *standard unknot*, which we have already discussed.

You might be asking, what if we choose the neighborhood of a different point to remove in this construction. It turns out that we get the same pair (B^n_K, B^{n-2}_K) up to diffeomorphism. To see this, consider the ball neighborhoods of two different points. We can simply slide one ball to the other along the knot, which complementarily takes the complement of one neighborhood to the complement of the other. Note that while this idea has nice intuitive appeal, it does require some technical checking to ensure that such sliding is always allowed. However, this theory is well-established, and we avoid going far afield to visit the details here.

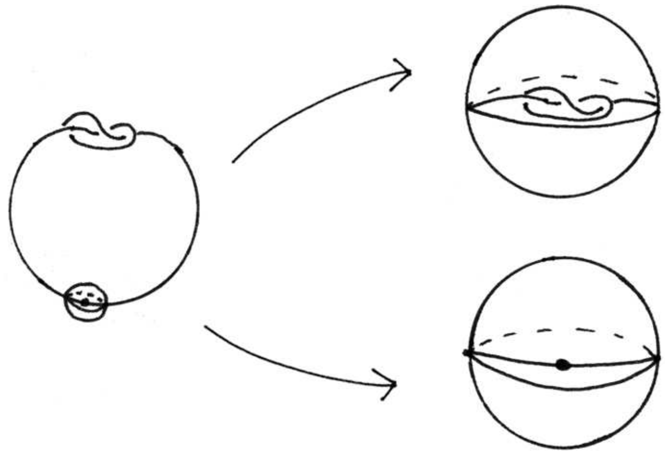

Figure 4: Removing a trivial neighborhood from a knotted circle to obtain a knotted arc

3 Basic spinnings

3.1 Simple spinning

Examples of knots $S^1 \hookrightarrow S^3$ are numerous. Just play with a length of wire and then solder the ends together. These knots also can be easily drawn on paper as *knot diagrams* (Figure 1). However, to get knots of higher dimensions requires a little bit more ingenuity.

The earliest spinning construction is due to Emil Artin in 1925 [2]. Artin used spinning to construct 4-dimensional knots from 3-dimensional knots, but the same idea can be used to create an $n+1$ dimensional knot from any n dimensional knot. This construction is generally referred to as "spinning," but we will call it *simple spinning* to differentiate it from the more general constructions to follow.

In this section, it will be most convenient to consider knots in $\mathbb{R}^n$ instead of S^n (see Section 2).

To see the basic idea, consider the upper half plane $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and choose a point $(x_0, y_0) \in H^2$ with $y_0 > 0$. Now rotate H^2 around the x-axis in $\mathbb{R}^3$ (Figure 5). The point will sweep out a circle. Analytically, the circle will be parametrized in $\mathbb{R}^3$ by the set of points $(x_0, y_0 \cos\theta, y_0 \sin\theta)$, as θ runs from 0 to 2π (assuming that we rotate counterclockwise as see from the positive x-axis looking in the negative x direction).

To see how this applies to knots, let us consider a knot K in $\mathbb{R}^3$. Up to equivalence, we can arrange for the image of K to lie in the upper half space $H^3 = \{(x, y, z) \mid z \geq 0\}$ except for an unknotted arc that dips below the x-y

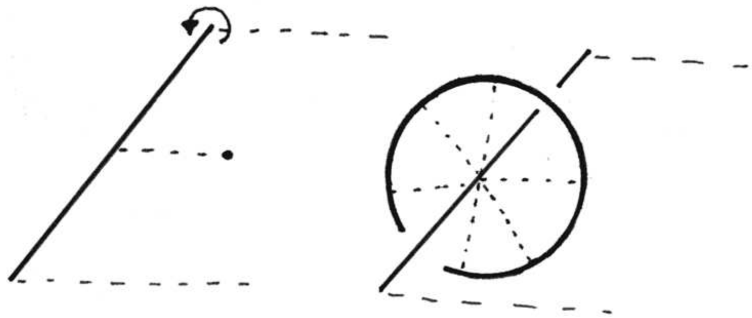

Figure 5: Spinning a point in the half-plane around the axis.

plane $\mathbb{R}^2 = \{(x, y, z) \mid z = 0\}$ (Figure 6). Let us remove the interior of this unknotted arc; what remains is a knotted arc in H^3 with its endpoints in $\mathbb{R}^2$. We can now rotate H^3 in $\mathbb{R}^4$ just as we rotated H^2 in $\mathbb{R}^3$. Analytically, we parametrize by θ, and each point (x, y, z) in the upper half space sweeps out the circle $(x, y, z\cos\theta, z\sin\theta)$. Note that $\mathbb{R}^2$ remains fixed. By thinking about how the longitude lines swing around the globe with the north and south poles remaining fixed, you can imagine how the the knotted arc gets spun into the image of a 2-sphere S^2. Thus, by spinning, we obtain a knotted S^2 in $\mathbb{R}^4$.

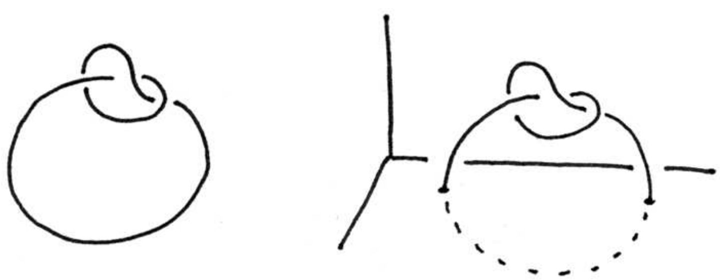

Figure 6: Turning a knotted circle into a knotted arc in the upper half space in order to spin it about the plane. We must remove the extra arc so that we spin into a sphere, not a torus!

You might be worried that our new spun knot will not be a differentiable embedding if the knotted arc does not meet $\mathbb{R}^2$ perpendicularly. This turns out not to be a problem for several reasons: 1) it is easy to make the arc meet $\mathbb{R}^2$ perpendicularly; 2) even if not, we could modify our newly constructed knot slightly to make it differentiable; and 3) most generally, there is a method called “smoothing” of modifying the differential structure of $\mathbb{R}^4$ (within its diffeomorphism class) to make things work out. This will be possible in all of our constructions, and we will avoid mentioning it explicitly.

You might also be asking, what if we had chosen a different way to split our original knot into a knotted arc? It turns out that we get the same spun knot, essentially for the same reason by which we noted in Section 2 that B^n_K is independent of such choices. In fact, notice that if we start with our knot in S^3, then our construction to get a knotted arc in the upper half plane is completely equivalent to the construction of B^n_K by removing a small ball neighborhood of a point on the knot.

This simple spinning construction already has several important ramifications. For example, it can be shown very easily (see, e.g., [15]) that the fundamental group of the complement of this spun knot in $\mathbb{R}^4$ is isomorphic to the fundamental group of the complement of our original knot in $\mathbb{R}^3$. Based on known results about knots in $\mathbb{R}^3$, this implies the existence of an infinite number of inequivalent knots in $\mathbb{R}^4$.

The construction for higher dimensions is similar. We begin with a knot $K : S^{n-2} \to \mathbb{R}^n$. Again, we can manipulate the knot within its equivalence class so that it lies mostly in the upper half space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and so that the intersection of the knot with the lower half space is an unknotted ball. We then remove the interior of this unknotted ball to obtain the complementary knotted ball B^{n-2} in H^n. Its intersection with $\mathbb{R}^{n-1}$ is unknotted. Now we spin H^n into R^{n+1} so that each point $(x_1, \cdots, x_n)$ sweeps out the circle $(x_1, \cdots, x_{n-1}, x_n \cos\theta, x_n \sin\theta)$.

It is a little harder now to see that our knotted ball in the upper half plane gets spun into a sphere S^{n-1}, but the idea of pivoting a longitude around its poles extends to higher dimensions. To see this, consider S^{n-1} as the unit sphere in $\mathbb{R}^n$, $S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| = 1\}$, and consider $\mathbb{R}^n$ as $\mathbb{R}^{n-2} \times \mathbb{R}^2$. Then we can define the latitude for a point $y \in S^{n-1}$ as its projection onto the $\mathbb{R}^{n-2}$ factor and its longitude as the angular polar coordinate of the projection of y onto the $\mathbb{R}^2$ factor. Hence the latitude is always well-defined, while the longitude is either undefined or a unique angle, dependent on whether or not y lies in the sphere S^{n-3} that is the intersection of S^{n-1} with $\mathbb{R}^{n-2} \times 0$. Notice that in the case where the longitude in undefined, the point on the sphere is uniquely determined by its latitude (just as on a globe). To simplify the notation in abstract cases, we will simply refer to the latitude-longitude coordinates (z, θ) whether θ is defined or not. Then the point (z, θ) in these coordinates on S^{n-1} corresponds to the point in $\mathbb{R}^n$ determined by the rectangular coordinates $(z, r\cos\theta, r\sin\theta)$ for $z \in \mathbb{R}^{n-2}$, $r \geq 0$, $\theta \in [0, 2\pi)$, and such that $|z|^2 + r^2 = 1$.

Now, consider the set of points in S^{n-1} with a fixed longitude $\theta = 0$. These points can be written in rectangular coordinates as $(z, r, 0)$, where $r \geq 0$ is chosen so that $|z|^2 + r^2 = 1$. In other words, this longitude is the graph of $r = +\sqrt{1 - |z|^2}$ in $\mathbb{R}^{n-2} \times \mathbb{R}$, which is diffeomorphic to a ball B^{n-2}. Its boundary is the $n-3$ sphere with $|z|^2 = 1$ and $r = 0$, and this corresponds to the *generalize pole* of S^{n-1}. Now for each point $(z, r, 0)$ in rectangular coordinates, we can spin to get the set of points $(z, r\cos\theta, r\sin\theta)$ as θ runs from 0 to 2π. The points of the generalized pole remain fixed as $(z, 0, 0)$, and the rest of the chosen longitude sweeps out the rest of the sphere. Analogously, as we spin a knot, the knotted ball sweeps out a knotted sphere.

The results about fundamental groups continue to hold in this higher-dimensional setting, and by iterating the spin construction, we establish the existence of an infinite number of inequivalent knots in any dimension $n \geq 3$.

3.2 Superspinning

Having spun knots around circles, how about spinning around higher dimensional spheres? *Superspinning* of classical knots was initially used by D.B.A. Epstein [4] in 1960 to show that two $n-2$ spheres can be embedded in euclidean n-space ($n \geq 3$) such that neither can be shrunk to a point in the complement of the other. The construction was generalized by Sylvain Cappell in 1970 [3] as a way to construct an $n+p$ dimensional knot from any n-dimensional knot by spinning it around a p-sphere S^p, $p \geq 1$. Cappell utilized superspinning to demonstrate the existence of knots whose complements are homotopy equivalent but not homeomorphic.

This time let us jump straight to the general construction. We start with a knot $K : S^{n-2} \hookrightarrow S^n$ and construct the knotted ball pair (B^n_K, B^{n-2}_K) as in Section 2.

Next, we consider S^{n+p} as ∂B^{n+p+1}. Since $B^{n+p+1} \cong B^{p+1} \times B^n$, ∂B^{n+p+1} can be written as $[S^p \times B^n] \bigcup_{S^p \times S^{n-1}} [B^{p+1} \times S^{n-1}]$. Here $\bigcup_{S^p \times S^{n-1}}$ indicates that we are gluing the two spaces along their common boundary $S^p \times S^{n-1}$. If you are not familiar with the fact that $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$, try thinking of some low-dimensional examples. For $n = 3$, $p = 1$, our example yields a decomposition of S^3 into two unknotted solid tori $S^1 \times Bs^2$. Try to picture the decomposition of the unknotted S^2 in S^4, taking $p = 1$, $n = 3$ and recalling that S^0 is a pair of points.

Furthermore, if we consider the standard ball pair (B^{n+p+1}, B^{n+p-1}) and write it as $(B^{p+1} \times B^n, B^{p+1} \times B^{n-2})$, we can take boundaries to obtain the pair of spaces $([S^p \times B^n] \bigcup [B^{p+1} \times S^{n-1}], [S^p \times B^{n-2}] \bigcup [B^{p+1} \times S^{n-3}]$. We can compress the notation a little and write this as $S^p \times (B^n, B^{n-2}) \bigcup B^{p+1} \times (S^{n-1}, S^{n-3})$. Since the boundary of the standard ball pair is an unknotted sphere, we have so far just found a complicated way to write the unknot.

But now, within this construction, we can replace each standard (B^n, B^{n-2}) in the product $S^p \times (B^n, B^{n-2})$ with the knotted pair (B^n_K, B^{n-2}_K) to get the product $S^p \times (B^n_K, B^{n-2}_K)$. The tricky part here is getting the loose ends $(S^p \times \partial B^{n-2}_K)$ to close up to form a legitimate sphere, but this is accomplished by the set $B^{p+1} \times S^{n-3}$ that closed up the unknot in the last paragraph. So, we define the superspun knot K^* to be the subset of S^{n+p} given by $[S^p \times B^{n-2}_K] \bigcup_{S^p \times S^{n-3}} [B^{p+1} \times S^{n-3}]$. In effect, we have taken all of the unknotted cross-sections of $S^p \times (B^n, B^{n-2})$ and replaced them with knotted cross-sections. As K^* is diffeomorphic to the standard decomposition of S^{n+p-2}, we see that K^* is a sphere of dimension $n+p-2$ knotted in S^{n+p}.

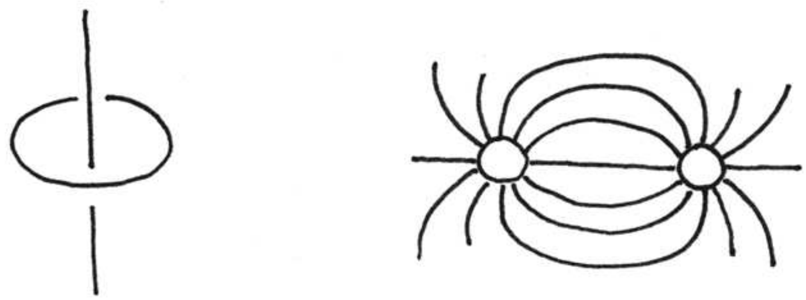

Figure 7: The decomposition of S^3 (thought of as $\mathbb{R}^3$ plus a "point at infinity") into two solid tori $S^1 \times B^2$ and $B^2 \times S^1$. The left picture shows the circular cores of the tori (the vertical line becomes a circle as it wraps through the point at infinity). The right picture shows a slice through the y-z plane: The two disks are a slice of one solid torus (cut a donut in half and then view it on end), while the arcs represent slices of such meridional disks of the other solid torus.

If $p = 1$, superspinning K gives us the same simple spun knot that we obtained in the previous section (why?).

It turns out that the fundamental group of the complement of a superspun knot is also the same as the fundamental group of the complement of the original knot, but, in general, superspinning does not create the same knots as does iterated simple spinning.

3.3 Frame spinning

Even more general than superspinning is *frame spinning*: why limit ourselves to spinning about spheres? How about other manifolds? Frame spinning was introduced by Dennis Roseman in 1989 [16], though the name is due to Alexander Suciu [18], who used frame spinning to construct inequivalent knots that have the same complement (classifying which knots are determined by their complements has been one of the main themes of knot theory).

To describe frame spinning, let us once again begin with an n-dimensional knot K and construct (B^n_K, B^{n-2}_K). This time, however, our additional data comes in the form of an m-dimensional manifold M^m differentiably embedded in S^{n+m-2} with a framing ϕ. This last condition means that we in fact consider an embedding $\phi : M^m \times B^{n-2} \hookrightarrow S^{n+m-2}$. Furthermore, we assume that S^{n+m-2} is embedded in the standard, unknotted way into S^{n+m} with the standard framing (so that $S^{n+m} = \partial B^{n+m+1} = S^{n+m-2} \times B^2 \bigcup B^{n+m-1} \times S^1$). Putting these framings together, we get a pair of tubular neighborhoods of M^m in (S^{n+m}, S^{n+m-2}) of the form $N = M^m \times (B^n, B^{n-2})$, where this (B^n, B^{n-2}) is an *unknotted* ball pair (although the exact embedding of N into S^{n+m} depends on our choice of framing ϕ).

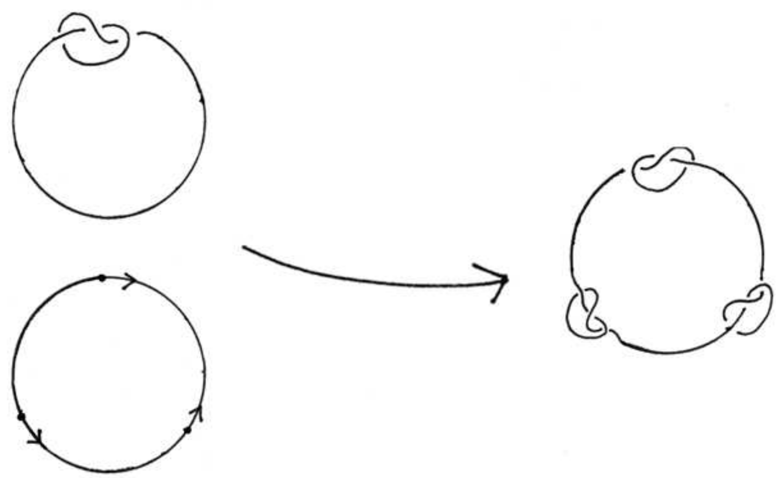

Figure 8: The trefoil knot spun about the manifold M consisting of three disjoint points in S^1. Note that the framing at each point (indicated by an arrow that depicts the orientation of the framing) determines how to attach the knot.

The idea now is to take all of those unknotted ball pairs and replace them with our knotted ball pair (B^n_K, B^{n-2}_K) as we did for superspinning. In other words, having used the framing to identify the neighborhood pair N as $M^m \times (B^n, B^{n-2})$, we remove it, and then replace it with $M^m \times (B^n_K, B^{n-2}_K)$, glued in along the same framing. Thus, our frame spun knot will be

$$(S^{n+m-2} - M^m \times B^{n-2}) \bigcup_{M^m \times S^{n-3}} M^m \times B^{n-2}_K$$

sitting inside the $n + m$ sphere

$$(S^{n+m} - M^m \times B^n) \bigcup_{M^m \times S^{n-1}} M^m \times B^n_K.$$

In the special case where M^m is the sphere S^m embedded in the standard way into S^{n+m-2}, we recover superspinning (why?).

If the manifold M has multiple components, or even components of different dimensions, then we can spin different knots (also possibly of different dimensions) around each component. It is possible to generalize this construction even further, but first we should study some other types of spinning.

4 Spinning with a twist

4.1 Twist spinning

Twist spinning, introduced by E.C. Zeeman in 1965 [20], was an early generalization of Artin's construction. Again, we begin with an n-dimensional knot and obtain an $n+1$-dimensional knot, but the difference between simple spinning and twist spinning can be illustrated celestially. As the moon orbits the Earth, it always keeps the same face towards the Earth. This is analogous to simple spinning in which the knot is rotated around the plane of rotation but always keeps "the same face" toward this plane. By comparison, twist spinning is like the Earth orbiting the sun: as the earth orbits, it also rotates around its own axis.

Before giving a general formula, let us consider heuristically the case of twist spinning a knot of dimension 3. As in the simple spinning constructions, we replace the knot with a knotted arc in the upper half space whose endpoints lie in the x-y plane. We can also assume that the knotted part of the arc is contained within a ball whose intersection with the arc is its north and south poles. Now, as we rotate half-space around the plane, we simultaneous spin this ball on its axis (Figure 9). It is only necessary that the end result lines up with the starting position, so we are free to spin the ball on its axis any integral number k times as we rotate H^3.

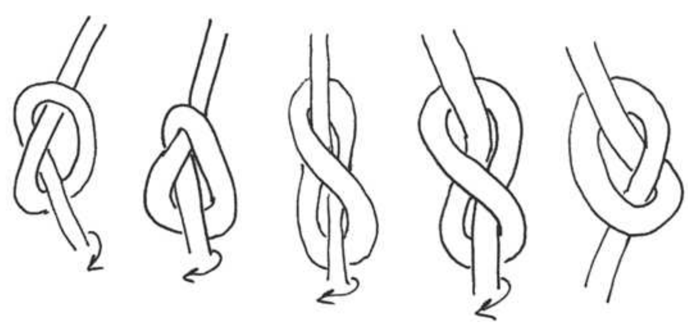

Figure 9: A 180 degree twist of the trefoil knot.

Let us be more specific. Given an n-knot K, then just as for superspinning about S^1 (which is equivalent to simple spinning), we can consider the two space pairs $S^1 \times (B^n_K, B^{n-2}_K)$ and $B^2 \times (S^{n-1}, S^{n-3})$. To superspin, we simply glued these pairs together along their common boundary $S^1 \times (S^{n-1}, S^{n-3})$. In order to create the k-twist spin, however, we glue these pairs together in the following way: we represent points in $S^1 \times S^{n-1}$ by (η, z, θ), where $\eta \in S^1$ and (z, θ) are latitude/longitude coordinates for S^{n-1} (see Section 3.1). If (η, z, θ) is such a point in the boundary of $B^2 \times S^{n-1}$, we attach that point to the boundary of $S^1 \times B^n_K$ by $(\eta, z, \theta) \to (\eta, z, \theta + k\eta)$. The addition here is standard angle addition in the circle, which we can think of as $\mathbb{R}/2\pi\mathbb{Z}$, applied to the longitude coordinate. In this way, as we glue the pieces together, we introduce a k-fold twisting by rotations of the longitude coordinate.

Zeeman showed that a twist spun knot depends only on K and $|k|$, i.e. k-twist spinning and $-k$-twist spinning yield the same knot. Furthermore, he proved the surprising fact that any 1-twist spun knot (and hence also any -1-twist spun knot) is actually unknotted.

4.2 Frame twist spinning

Now that we have seen how to add twisting to Artin's basic spinning construction, can we add some kind of twisting to our other spins? For superspinning about spheres of dimension greater than 1, the answer is no! This is related to the fact that for $n > 1$, all maps $S^n \to S^1$ that can be deformed to maps to a single point. This fact ensures that any attempt at twisting can be deformed to give back ordinary superspinning. On the other hand, there are infinitely many maps $S^1 \to S^1$ that cannot be so deformed. Roughly, these are the maps that run around the circle k times ($e^{i\theta} \to e^{ik\theta}$), and their existence allows for the non-triviality of the twist spin construction. A rigorous formulation and proof of these facts about maps can be found in any book that deals with introductory *homotopy theory*, for example [8].

However, where superspinning fails to be twistable, frame spinning does allow a twist if the manifold M^m admits a map $M^m \to S^1$ that cannot be deformed into the trivial map to a point. Just as for twist spinning, this map provides us with enough data to alter the gluing map of the construction by twisting the longitude coordinate of B^n_K as we glue. The gluing of the latitude coordinates is once again controlled by the framing ϕ of M.

So let us be specific. Recall that, in frame spinning, we used the framing of M^m in S^{n+m-2} along with the trivial framing of S^{n+m-2} in S^{n+m} to identify a neighborhood N of M^m in (S^{n+m}, S^{n+m-2}) with the product $M^m \times (B^n, B^{n-2})$. Then we replaced (B^n, B^{n-2}) with the knotted ball pair (B^n_K, B^{n-2}_K) and glued it back in *along the same framing*. Suppose, however, that we are given a map $\tau : M^m \to S^1$. Then we can use this map to augment the gluing with a twist along the longitude. This is done as follows: we use the framings to assign coordinates (x, z, θ) to the boundary $M \times S^{n-1}$ of the neighborhood N in S^{n+m}. Here $x \in M$ and (z, θ) are latitude/longitude coordinates on $S^{n-1} = \partial B^n$. The boundary of the complement of N in S^{n+m} possesses the same coordinates, as these two boundaries agree. Again, we cut out $N \cong M \times (B^n, B^{n-2})$ and replace it with $M^m \times (B^n_K, B^{n-2}_K)$, which we glue back in, but instead of gluing the point $(x, z, \theta) \in \partial N$ right back to its counterpart in $\partial(S^{n+m} - N)$, we glue it by the attaching map $f : (x, z, \theta) \to (x, z, \theta + \tau(x))$, where again the addition in S^1 is standard angle addition.

In other words, we form

$$[(S^{n+m}, S^{n+m-2}) - M^m \times (B^n, B^{n-2})] \bigcup_f [M^m \times (B^n_K, B^{n-2}_K)],$$

where $\bigcup_f$ indicates a gluing via the attaching map given above.

This construction was introduced by the author in his dissertation. He goes on to calculate various algebraic invariants of frame twist-spun knots; see [6] and [7]. If the map τ is homotopic to the trivial map, we recover Roseman's frame spinning. If $M = S^1$ and τ is the map that wraps the circle around itself k times, we recover Zeeman's k-twist spinning (why?).

5 More general spinnings

5.1 Deform spinning

An even more general class of spinning constructions is known to exist. The first example, roll spinning, was introduced in a short paper by Ralph Fox in 1966 [5]. This example was formalized and generalized by R.A. Litherland in 1979 [13], who introduced *deform spinning* and showed that both Fox's roll spinning and Zeeman's twist spinning are special cases. This is another construction that takes n-knots to $n+1$ knots.

The tersest description of deform spinning comes from once again thinking of a simple spin as

$$S^1 \times (B^n_K, B^{n-2}_K) \bigcup_{\partial} B^2 \times (S^{n-1}, S^{n-3}),$$

where $\bigcup_\partial$ indicates gluing along the common boundary in the obvious (untwisted) fashion. Suppose now that we have a 1-parameter family f_ψ of *deformations* rel $\partial(B^n_K \times I)$. In other words, each f_ψ, $\psi \in [0, 2\pi]$, is a diffeomorphism $B^n_K \to B^n_K$ such f_ψ restricted to the boundary ∂B^n_K is the identity for all ψ and such that $f_0 = f_{2\pi}$ is the identity map of B^n_K. The family f_ψ should also depend differentiably on the parameter ψ. Litherland then describes the deform spin of K as

$$(S^1 \times B^n_K, \cup_{\psi \in S^1} \psi \times f_\psi(B^{n-2}_K)) \cup_\partial B^2 \times (S^{n-1}, S^{n-3}).$$

In other words, as we spin, we deform the knotted arc according to f_ψ. Note that in this description $S^1 \times B^n_K$ is the ordinary undeformed product, but we equally well could have used the deformation of the pair; Litherland demonstrates the equivalence of the two approaches and uses it to redefine the deform spin in terms of *crossed products* of spaces. However, for our purposes it is perhaps easier to maintain the original viewpoint. In this language, it is easy to observe that simple spinning corresponds to setting f_ψ equal to the identity for each ψ, while k-twist spinning corresponds to setting f_ψ equal to rotation of the longitude coordinate of B^n_K by $k\psi$ (technically, to get around the fact that we need to keep the boundary of B^n_K fixed, we rotate a smaller interior ball allowing the region between the two boundaries to become stretched around; however, it is clear that so long as the ball being rotated encompasses the knotted part of B^{n-2}_K, this does not affect the final construction of the deform spun knot).

Litherland also shows that, thinking of the collection f_ψ as a differentiable map f: $B^n_K \times I \to B^n_K \times I$, the type of the deform spun knot is dependent only upon the (pseudo-)isotopy class of f rel $\partial(B^n_K \times I)$.

With this definition of deform spinning, we can define roll spinning of a classical knot $K : S^1 \hookrightarrow S^3$ as follows: Recall our definition of (B^n_K, B^{n-2}_K) by removing an unknotted ball neighborhood of a point on the knot K. Since we are dealing with a classical knot, we can parametrize S^1 by angles ψ and consider $(B^n_{K,\psi}, B^{n-2}_{K,\psi})$ built as the complement of the neighborhood of the point $K(\psi)$. We have already noted that for different choices of ψ, the pairs $(B^n_{K,\psi}, B^{n-2}_{K,\psi})$ are all diffeomorphic. Nevertheless, starting from a fixed base point, say $\psi = 0$, we can view the collection $f_\psi : (B^n_{K,0}, B^{n-2}_{K,0}) \to (B^n_{K,\psi}, B^{n-2}_{K,\psi})$ as a one parameter family of deformations and use this to deform spin. We obtain the roll spin of K. k-roll spinning can be created by rolling the basepoint around the knot k times. A more technical formulation is given in [13] (see also [19]). Note that this construction depends on a choice of framing of K in order to control the twist of the ball (the "roll" in the aeronautical sense) as it traverses the knot; we can define rolling with respect to some fixed framing (for example, that in which K and a longitude of the boundary torus of the framed neighborhood of K do not link), but if we use a different framing, we will twist as we roll. This leads to twist roll-spun knots or, generally, l-twist k-roll spun knots.

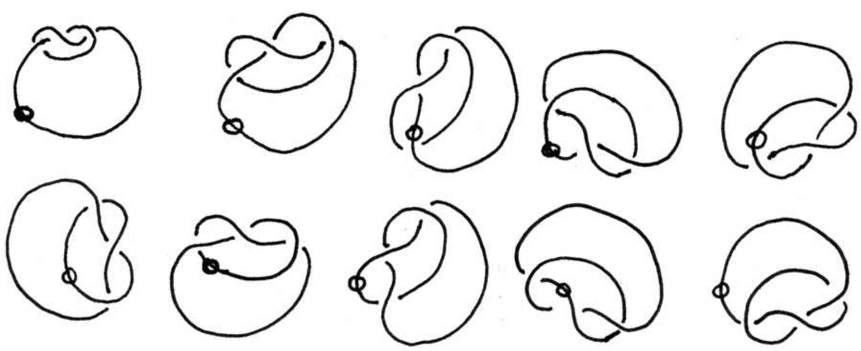

Figure 10: Rolling the trefoil. The circle in these pictures represents B_-. Rather than moving B_-, it is more illustrative to hold it fixed and roll the knot around it!

Unfortunately, untwisted roll spinning cannot be generalized to higher dimensional knots, again because any path in the sphere S^n, $n > 1$, can be shrunk to a point. Hence, any path along which we would roll B^n_- is equivalent to the stationary path.

Litherland also treats another example of deform spinning that applies only to knots which possess symmetries, i.e. periodic homeomorphisms $S^n \to S^n$ that take the knot to itself. This construction is called *symmetry spinning.* However, the construction is slightly technical, involving certain branched covers of the sphere, and so we omit a detailed description. The interested reader is referred to [13] or [9], in which Taizo Kanenobu utilizes symmetry spinning to obtain some results about commutator subgroups of the fundamental groups of the complements of knots in dimension 4.

5.2 Frame deform spinning

Putting together frame spinning and deform spinning, we can introduce a new knot construction, *frame deform spinning* (this construction is, perhaps, implicit in a remark in [16, §3]). By now the method should be obvious: we begin with an n-knot K and an m-manifold M^m embedded with framing in S^{n+m-2}, which itself sits unknotted and with the standard framing in S^{n+m}. We also posit a map f from M^m into the space of diffeomorphisms of B^n_K rel ∂B^n_K. The map f takes $x \in M^m$ to the diffeomorphism $f_x : B^n_K \to B^n_K$ such that the map $M^m \times B^n_K \to B^n_K$ given by $(x, y) \to f_x(y)$ is differentiable. Then we can define the frame deform spin of K as

$$[(S^{n+m}, S^{n+m-2}) - M^m \times (B^n, B^{n-2})] \bigcup_{M^m \times S^{n-3}} [M^m \times (B^n_K, \cup_{x \in M^m} f_x(B^{n-2}_K)].$$

If K is a classical knot in S^3 and there is a non-trivial map $g : M^m \to S^1$, we can compose g with the 1-parameter families of deformations used to define roll spinning and twist roll spinning to create *frame roll spinning* and *frame twist roll spinning*.

6 Other constructions

We close by briefly mentioning two other known constructions of knots related to spinning.

The first, due to John Klein and Alexander Suciu in 1991 [10], is called *diff-spinning*. It is a modified version of frame spinning in which the manifold M^m is altered by a diffeomorphism in the process of spinning. Note that the complement of a knot frame spun about M^m is diffeomorphic to the union of $B^{n+m-1} \times S^1$ with $M^m \times X$, where X is the complement of the knot K that is being spun. Suppose now that we are given a self-diffeomorphism Φ of M^m that extends to a diffeomorphism $\bar{\Phi}$ of $B^{n+m-1} \supset S^{n+m-2} \supset M^m$. Then, roughly speaking, the diff-spin is formed by removing this complement and replacing it with the twisted product $(B^{n+m-1} \times_{\bar{\Phi}} S^1) \bigcup (M^m \times_{\Phi} X)$. If Φ satisfies a certain algebraic condition (see [10, §5]), this space will also be the complement of a knot, the diff-spun knot.

Another type of spinning, also introduced by Roseman in [16], is what he calls "spinning a knot about a projection". We shall refer to this as *projection spinning*. This clever construction involves many technical details, but, very roughly, the idea is to spin about an immersed manifold M, rather than an embedded one as we did in frame spinning. (Recall that a map of manifolds $M \to N$ is an *immersion* if it restricts to an embedding in a neighborhood of each point in M, but an immersion need not be globally 1-1.) Away from the points of M at which the immersion is 1-1 (locally an embedding), the construction is the same as for frame spinning, i.e. each cross section is replaced with a knot. Where the immersion fails to be 1-1, neighborhoods are replaced by *multi-knots* in which some knot K is blended together with itself in multiple directions. If M is embedded, we recover frame-spinning. We refer the reader to [16] both for the technical definitions of projection spinning and for some nice graphical illustrations of the process. In Remark 7 of [16], Roseman notes that it is further possible to deform projection spin, perhaps the ultimate in knot spinning constructions.

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