Additivity and non-additivity for perverse signatures

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Abstract

A well-known property of the signature of closed oriented 4n-dimensional manifolds is Novikov additivity, which states that if a manifold is split into two manifolds with boundary along an oriented smooth hypersurface, then the signature of the original manifold equals the sum of the signatures of the resulting manifolds with boundary. Wall showed that this property is not true of signatures on manifolds with boundary and that the difference from additivity could be described as a certain Maslov triple index. Perverse signatures are signatures defined for any stratified pseudomanifold, using the intersection homology groups of Goresky and MacPherson. In the case of Witt spaces, the middle perverse signature is the same as the Witt signature. This paper proves a generalization to perverse signatures of Wall's non-additivity theorem for signatures of manifolds with boundary. Under certain topological conditions on the dividing hypersurface,Novikov additivity for perverse signatures may be deduced as a corollary. In particular, Siegel's version of Novikov additivity for Witt signatures is a special case of this corollary.

1 Introduction

The signature of compact $4n$-dimensional manifolds is an interesting and important manifold invariant. It satisfies a number of remarkable properties commonly referred to as the 'signature package'. These include cobordism invariance [43], equality to the index of the signature operator and to the L-genus [30], and Novikov additivity [3]. Signature has been used to prove various obstruction theorems. For instance, Rokhlin's theorem ([39]) shows that for a $4n$-dimensional smooth compact manifold to carry a spin-structure, its signature must be

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divisible by 16. Following on the heels of the successes in topology and geometric topology of smooth compact manifolds in the 1950s and 1960s, including this work on the signature package, mathematicians began to explore which of the results from the smooth compact setting might be generalized to the setting of singular spaces. In the years since then, there have been a number of interesting developments, although there is still no general signature package for singular manifolds. In [31], the second author of this paper defined a family of 'perverse signatures', based on the intersection homology groups of Goresky and MacPherson, that may be defined for any stratified pseudomanifold. In this paper, we identify when a generalization of Novikov additivity holds for these signatures, as well as identifying the additivity defect in the case that it does not. In future papers, we will explore further aspects of the signature package for perverse signatures.

More details will be given below, but to explain briefly our main results, Theorem 4.1 and Corollary 4.13, recall that for a closed oriented n-dimensional pseudomanifold X, field coefficients F, and perversity parameters[1] $\bar{p}, \bar{q}$ such that $\bar{p}+\bar{q}=\bar{t}$, there is a duality isomorphism of intersection homology groups

$$I^{\bar{p}}H_i(X;F) \cong I^{\bar{q}}H_{n-i}(X;F),$$

determined by the intersection pairing

$$\pitchfork\colon I^{\bar{p}}H_i(X;F) \otimes I^{\bar{q}}H_{n-i}(X;F) \to F.$$

In the most well-known case, if X is a $4n$-dimensional Witt space, which implies that $I^{\bar{m}}H_*(X;F) \cong I^{\bar{n}}H_*(X;F)$ for the lower-middle and upper-middle perversities $\bar{m}$ and $\bar{n}$, then one obtains a symmetric middle-dimensional self-pairing $I^{\bar{m}}H_{2n}(X;F) \otimes I^{\bar{n}}H_{2n}(X;F) \to F$, and hence a signature invariant. This is the well-known Witt signature. More generally, though, it is possible to define signatures on any closed oriented $4n$-dimensional pseudomanifold as follows: If $\bar{p}+\bar{q}=\bar{t}$ and $\bar{p}(k) \leq \bar{q}(k)$ for all k, then there is a map $I^{\bar{p}}H_*(X;F) \to I^{\bar{q}}H_*(X;F)$, and this induces a nonsingular symmetric pairing on $\operatorname{im}(I^{\bar{p}}H_{2n}(X;F) \to I^{\bar{q}}H_{2n}(X;F))$. We refer to signatures of such pairings as *perverse signatures* $\sigma_{\bar{p}\to\bar{q}}(X)$ and note that the Witt space signature is a special case. Similarly, in analogy with the case for manifolds, there is also a signature on compact oriented pseudomanifolds with boundary.

Our main results are to extend to this setting the famous Novikov additivity and Wall non-additivity theorems. In particular we have the following (which occurs below as Theorem 4.1):

Theorem 1.1. *Let $Z \subset X$ be a bicollared codimension one subpseudomanifold of the closed oriented $4n$-pseudomanifold X such that $X = Y_1 \cup_Z Y_2$ and $\partial Y_1 = Z = -\partial Y_2$, accounting for orientations. Then*

$$\sigma_{\bar{p}\to\bar{q}}(X) = \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_1) + \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_2) + \sigma(V;A,B,C).$$

[1]We will assume in the introduction that $\bar{p}$ and $\bar{q}$ satisfy the conditions of Goresky and MacPherson [26], but this will be generalized below.

Here, the term $\sigma(V; A, B, C)$ is a certain Maslov index that generalizes Wall's correction term to Novikov additivity for manifolds with boundary. It will be explained more fully below. However, we do note one significant corollary:

Corollary 1.2. *With the hypotheses of the preceding theorem, suppose in addition that* $I^{\bar p}H_{2n}(Z; F) \to I^{\bar q}H_{2n}(Z; F)$ *is surjective and* $I^{\bar p}H_{2n-1}(Z; F) \to I^{\bar q}H_{2n-1}(Z; F)$ *is injective (for example if* $I^{\bar p}H_*(Z; F) \cong I^{\bar q}H_*(Z; F)$*). Then*

$$\sigma_{\bar p \to \bar q}(X) = \sigma_{\bar p \twoheadrightarrow \bar q}(Y_1) + \sigma_{\bar p \twoheadrightarrow \bar q}(Y_2).$$

We will also generalize these results to pseudomanifolds glued along partial boundaries in Corollary 4.13, and we will show that Wall's theorem for manifolds with boundary follows as a consequence of Theorem 1.1 in Corollary 5.3.

Motivation. There are several motivations for this work, aside from the general motivation of extending the signature package to singular spaces. One of these motivations comes from Sen's conjecture and related conjectures arising in string theory. These are conjectures about the signatures of certain $4n$-dimensional noncompact manifolds arising as moduli spaces of particles, such as (n+1)-monopoles in the case of Sen's original conjecture. In the 4- dimensional cases, for which the conjecture has been proved, the signature turns out to be the perverse signature of a compactification of the moduli space as a stratified space [29]. From analytic considerations, it seems likely that this will be true more generally, which leaves still the question of how to calculate these perverse signatures to resolve the conjecture. Our additivity and non-additivity results give a tool for this. It would also be interesting to compare the topological obstruction to additivity for perverse signatures in this paper to the analytic obstruction to the Mayer-Vietoris techniques for reduced cohomology, which were also motivated by Sen's conjecture and are related to perverse signatures, developed in some of the same settings by Carron in [12], [14], [13].

A second motivation comes from global analysis and PDEs. For manifolds with boundary, the Maslov triple index term in Wall's non-additivity formula has been interpreted analytically in the context of analytic signature theorems for manifolds with corners of codimension two in [28] and in terms of a gluing formula for the η-invariant and the spectral flow for operators with varying boundary conditions in [33]. It seems very likely, therefore that our non-additivity formula will also turn out to relate to analytic signature theorems for pseudomanifolds with boundary and signature gluing theorems for pseudomanifolds. In particular, although a signature theorem has been proved for manifolds with cusp-bundle ends in [44], and has been interpreted in terms of perverse signatures for pseudomanifold compactifications of these spaces in [29], there is as yet no analytic signature or signature gluing theorem for manifolds with cusp-edge corners. This is an interesting analytic case to tackle, and having a sense of what should arise from the topology is helpful in doing this.

A third motivation comes from spectral sequences of perverse sheaves. In [16] and [31], the difference between various perverse signatures in the case of a pseudomanifold with only two strata was interpreted in terms of a signature on the pages of the Leray spectral sequence of the fibration on the unit normal bundle of the singular stratum. It should be possible to interpret the difference between perverse signatures for a general pseudomanifold in terms of the pages of the hypercohomology spectral sequence for perverse sheaves near the lower strata.

Finally, a fourth motivation is a Wall-type non-additivity result for Witt spaces and possibly also for the new more general signature theory introduced by Banagl in [4]. Intersection homology of pseudomanifolds was developed in the late 1970s and early 1980s, through the work of McCrory [36], Cheeger [15], and Goresky and MacPherson [26]. Intersection homology groups for a pseudomanifold are parametrized by a function called a perversity. There is a subclass of stratified spaces, called Witt spaces, for which there is a Poincaré dual 'middle perversity' intersection homology, and for $4n$-dimensional Witt spaces, it is therefore possible to define a 'middle perversity signature'. Most of the signature package has been generalized to Witt spaces. In particular, the Witt cobordism group and the invariance of signature under Witt cobordism was proved by Siegel [42] in 1983. In the same paper, he proved a version of Novikov additivity for Witt spaces where the dividing hypersurface is again Witt. In a very recent paper, [1], progress has also been made on the analytic side of the signature package for Witt spaces. In particular, the authors prove that the topological middle perversity signature for Witt spaces is the signature of the unique extension of the signature operator for the spaces endowed with iterated cone metrics. The signature on Witt spaces is a particular case of a perverse signature, so our theorem generalizes Siegel's additivity theorem to a Wall-type nonadditivity theorem for these spaces.

Banagl has extended signature theory further to a class of "non-Witt" spaces (despite the terminology, this class of spaces includes all Witt spaces); these spaces are defined in terms of certain signature conditions on the neighborhoods of odd codimensional strata. If a non-Witt space is actually Witt, Banagl's signature agrees with the Witt space signature. It seems possible that Banagl's signature may in fact always be a perverse signature. Our non-additivity result may help determine if this is true, and, if so, gives an additivity and non-additivity result for Banagl's signatures and also would imply stratification independence of the Banagl signatures.

Outline. In order to generalize Wall's theorem to perverse signatures, we first need to review past results and make some new definitions. In the next section, we review signatures for manifolds, and in the following section we review intersection homology and make some new constructions. In Section 4, we prove our non-additivity result, obtaining as a corollary our additivity theorem. We prove it first for stratified pseudomanifolds without boundary, then generalize to those with boundary. In Section 5, we discuss the relationships of our work to Wall's original theorem and give two examples of calculations. Finally, in an appendix we carefully establish some conventions regarding orientation and intersection numbers that we use in the paper.

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2 Background on signatures and (non-)additivity

In this section and the following, we recall known results concerning signatures and provide a crash course on the relevant version of intersection homology.

2.1 Additivity and non-additivity

Recall that the signature of a closed connected oriented $4n$-manifold is the signature $\sigma(M)$ of the nondegenerate symmetric intersection pairing

$$H_{2n}(M;\mathbb{Q}) \otimes H_{2n}(M;\mathbb{Q}) \to \mathbb{Q},$$

i.e. $\sigma(M)$ is the dimension of the largest positive definite subspace of this pairing minus the dimension of the largest negative definite subspace. Alternatively, this is the same as the signature of cup product pairing $H^{2n}(M;\mathbb{Q}) \otimes H^{2n}(M;\mathbb{Q}) \to H^{4n}(M;\mathbb{Q}) \cong \mathbb{Q}$ or the signature of the pairing given by exterior product of forms in de Rham cohomology $H^{2n}(M;\mathbb{R}) \otimes H^{2n}(M;\mathbb{R}) \to H^{4n}(M;\mathbb{R}) \cong \mathbb{R}$.

If N is a manifold with boundary, we instead have a nondegenerate intersection pairing

$$H_{2n}(N;\mathbb{Q}) \otimes H_{2n}(N,\partial N;\mathbb{Q}) \to \mathbb{Q}.$$

This descends to a nondegenerate symmetric pairing on

$$\operatorname{im}(H_{2n}(N;\mathbb{Q}) \to H_{2n}(N,\partial N;\mathbb{Q})),$$

where the arrow is induced by inclusion. The signature of this pairing is the signature $\sigma(N)$.

Suppose now that M is a closed, oriented $4n$-manifold, and that $M = M_1 \cup_Z M_2$, where M_1, M_2 are manifolds-with-boundary oriented compatibly with M and $Z = \partial M_1 = -\partial M_2$. The Novikov additivity theorem for the signature of compact $4n$-manifolds is:

Theorem 2.1 (Novikov)**.**

$$\sigma(M) = \sigma(M_1) + \sigma(M_2).$$

Since signature theory of compact manifolds is nontrivial (i.e. there exist manifolds with non-zero signature), the theory of signatures of manifolds with boundary must also, by Novikov additivity, be nontrivial. It also turns out to be more subtle. The Atiyah-Patodi-Singer index theorem, [2], showed that the signature of a manifold with boundary may be realized as the index of the signature operator with a certain global boundary condition [5], but that it differs from the L-genus of the manifold by a spectral invariant of the boundary called the η-invariant. It is also clear that signature for manifolds with boundary cannot satisfy a general Novikov additivity, as any manifold may be broken up into pieces that are homeomorphic to a disk, which has trivial signature. In [45], Wall identified the defect in additivity for signatures of manifolds with boundary in terms of the Maslov triple index:

Theorem 2.2 (Wall [45])**.** *Suppose M^{4n} is a compact oriented manifold with boundary such that $M = M_1 \cup M_2$, where M_1, M_2 are compact oriented manifolds with boundary. Let*

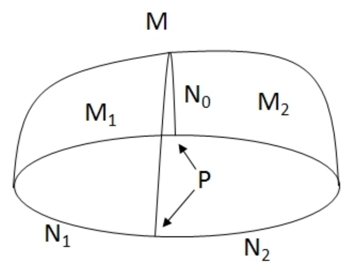

Figure 1: A schematic of the hypothesis of Wall's theorem: The manifold-with-boundary M is split into the pieces M_1 and M_2 along the hypersurface N_0. The boundary of M is split into N_1 and N_2 along P.

$N_1 = \partial M \cap M_1$ and $N_2 = \partial M \cap M_2$. Suppose $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ is a manifold with boundary N_0 such that $\partial M_1 = N_0 - N_1$, $\partial M_2 = N_2 - N_0$, and $\partial N_1 = \partial N_2 = \partial N_0 = P$ (see Figure 2.1). Then

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) - \sigma(V; A, B, C), \tag{1}$$

where $\sigma(V; A, B, C)$ is the Maslov triple index for the symplectic vector space $H_{2n-1}(P; \mathbb{Q})$ with respect to the three Lagrangian subspaces $A = Kernel\ (H_{2n-1}(P; \mathbb{Q}) \to H_{2n-1}(N_1; \mathbb{Q}))$, $B = Kernel\ (H_{2n-1}(P; \mathbb{Q}) \to H_{2n-1}(N_0; \mathbb{Q}))$ and $C = Kernel\ (H_{2n-1}(P; \mathbb{Q}) \to H_{2n-1}(N_2; \mathbb{Q}))$.

We recall the definition of $\sigma(V; A, B, C)$ in the next subsection.

2.2 Wall's Maslov index and some care with signs

Here, we briefly review the algebraic version of the Maslov index presented by Wall in [45]. We also make some observations regarding a couple of sign issues that are not completely clear in Wall's original paper. For an expository account containing many viewpoints on the Maslov index, we refer the reader to [10].

Suppose V is a vector space over F, $\Phi : V \times V \to F$ is a bilinear map, and $A, B, C \subset V$ are such that $\Phi(A \times A) = \Phi(B \times B) = \Phi(C \times C) = 0$. Wall considers the space $W = \frac{A \cap (B+C)}{A \cap B + A \cap C}$ (which is isomorphic to the spaces formed by permuting A, B, and C). Given $a, a' \in A$ representing elements of W, then $a = -b - c$ and $a' = -b' - c'$ for some $b, b' \in B$ and $c, c' \in C$. It is easy to show using these relations that we must have

$$\Phi(b, a') = -\Phi(c, a') = \Phi(c, b') = -\Phi(a, b') = \Phi(a, c') = -\Phi(b, c'). \tag{2}$$

From here, one obtains a well-defined pairing Ψ on W by setting $\Psi(a, a') = \Phi(a, b')$. This pairing is unaltered by even permutation of A, B, C and is altered by a sign for odd permutations. If Φ is skew-symmetric, Ψ is symmetric, and its signature is denote $\sigma(V; A, B, C)$. In the statement of Wall's theorem above, V is the vector space $H_{2n-1}(P; \mathbb{Q})$, and A, B, C are the kernels of the various maps induced by including P in N_1, N_0, and N_2.

In Wall's ensuing topological arguments, there are some sign issues with which one needs to take care. In the proof of his non-additivity theorem, Wall instead uses the formulation

$W = \frac{B\cap(C+A)}{B\cap C+B\cap A}$ (for appropriate choices of A, B, C). This cyclic permutation should not affect signs. However, Wall ultimately encounters an intersection pairing $(\partial\eta) \pitchfork (\partial\xi')$ representing $\Phi(\partial\eta, \partial\xi')$, where $\partial\eta \in B$ and $\partial\xi' \in C$. Taking B, C, A in that order, this is then a pairing between an element from the first subspace and an element from the second subspace. By definition, this is Ψ (whereas Wall states that this intersection pairing represents $-\Psi$). However, the intersection number $(\partial\eta) \pitchfork (\partial\xi')$ is not itself quite correct. This intersection pairing is in Wall's space Z (our P), which is the boundary of a space X_0 (our N_0), which is itself *the negative of* part of the boundary of Y_+ (our M_2). By "negative of", we mean with the reversed orientation. Wall at first encounters the intersection $\eta \pitchfork_{Y_+} \xi'$ and states this is equal to $\eta \pitchfork_{X_0} \partial\xi'$ in X_0. However, with the conventions we establish below in the Appendix, since the degree of η is even, $\eta \pitchfork_{Y_+} \xi'$ will be the negative of the intersection number $\eta \pitchfork_{X_0} \partial\xi'$ because X_0 has its orientation reversed as it appears in the boundary of Y_+. Then from here, we do have that $\eta \pitchfork_{X_0} \partial\xi' = \partial\eta \pitchfork_Z \partial\xi'$. Putting these sign issues together, it is correct that Wall arrives at the pairing $-\Psi$, and the statement of Wall's non-additivity theorem is correct in [45].

3 Background and preliminaries on intersection homology

In this section, we review intersection homology and make the necessary definitions to allow us to state our generalized non-additivity theorem. We begin with a basic review of pseudomanifolds and intersection homology; the experts might want to skim this section, as we use some recent generalizations with which they might not be familiar. Then in the second subsection below we define perverse signatures. We also will need some symplectic vector space that plays the role of $H_{2n-1}(P;\mathbb{Q})$ from Wall's theorem. We define this space in the third subsection.

3.1 Review of intersection homology

We begin with a brief review of basic definitions. For further reference, we refer the reader to [22, 21] as the background resources most suited to the brand of intersection homology treated here: singular intersection homology with general perversities and stratified coefficient systems. Other standard sources for more classical versions of intersection homology include [26, 27, 6, 34, 4, 32, 23]. Although we will not pursue them in detail here, various analytic approaches to intersection homology can be found in, e.g [15, 17, 9, 7]; these are particularly useful for relating intersection homology to L^2-cohomology and harmonic forms, as in [15], [29], [40], and others, and for relating perverse signatures to L^2-signatures, as in [19], [31].

Stratified pseudomanifolds. The definition of *stratified pseudomanifolds* is given inductively. A 0-dimensional *stratified pseudomanifold* is just a discrete set of points. Let $c(Z)$ denote the open cone on the space Z, and let $c(\emptyset)$ be a point. An n-dimensional *(topological)* *stratified pseudomanifold* X (see [27] or [11]) is a paracompact Hausdorff space together with a filtration

$$X = X^n \supseteq X^{n-1} \supseteq X^{n-2} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset$$

such that

1. $X - X^{n-1}$ is dense in X, and
2. for each point $x \in X^i - X^{i-1}$, there exists a *distinguished neighborhood* U of x, a *compact* $n-i-1$ dimensional stratified pseudomanifold L, and a homeomorphism

$$\phi : \mathbb{R}^i \times c(L) \to U$$

that takes $\mathbb{R}^i \times c(L^{j-1})$ onto $X^{i+j} \cap U$.

A space is a *(topological) pseudomanifold* if it can be given the structure of a stratified pseudomanifold by some choice of filtration.

The X^i are called *skeleta*. We denote $X_i = X^i - X^{i-1}$; this is an i-manifold that may be empty. We refer to the connected components of the various X_i as *strata*[2]. If a stratum Z is a subset of $X_n = X - X^{n-1}$ it is called a *regular stratum*; otherwise it is called a *singular stratum*. The space L occuring in the distinguished neighborhood U of a point y is the *link* of y or of the stratum containing y. A stratified pseudomanifold is defined to be orientable (respectively oriented) if $X - X^{n-1}$ is.

We note that this definition is slightly more general than the one in common usage [26], as it is usual to assume that $X^{n-1} = X^{n-2}$. We will not make that assumption here, but when we do assume $X^{n-1} = X^{n-2}$, intersection homology with Goresky-MacPherson perversities is known to be a topological invariant; in particular, it is invariant under choice of stratification (see [27], [6], [32]). Examples of pseudomanifolds include complex algebraic and analytic varieties (see [6, Section IV]).

A piecewise linear (PL) pseudomanifold is a pseudomanifold with a PL structure compatible with the filtration, meaning that each skeleton is a PL subspace, and such that each link is a PL stratified pseudomanifold and the distinguished neighborhood homeomorphisms $U \cong \mathbb{R}^{n-k} \times cL$ are PL homeomorphisms. In this paper, we will restrict ourselves entirely to the PL setting. This is sufficient for the purpose of analysts or algebraic geometers wishing to consider Thom-Mather or Whitney stratified spaces, which are PL pseudomanifolds. Our results should also hold for the class of topological pseudomanifolds, but we wish to avoid the technical details we would need to pursue, such as topological general position or, alternatively, some extremely careful sheaf theory.

N.B. We will usually refer simply to "a pseudomanifold X" and assume that we have tacitly chosen a stratification unless explicitly stated otherwise.

[2]This definition agrees with some sources, but is slightly different from others, including our own past work, which would refer to Y_i as the stratum and what we call strata as "stratum components."

Pseudomanifolds with boundary. A space X is a *stratified pseudomanifold with boundary* if it satisfies the definition to be a stratified pseudomanifold, except that points may have neighborhoods that are stratified homeomorphic either to $\mathbb{R}^k \times cL$ or to $\mathbb{R}^k_+ \times cL$, where $\mathbb{R}^k_+$ is the euclidean half-space but L is still a compact stratified pseudomanifold without boundary in either case. The *boundary* ∂X of a stratified pseudomanifold with boundary consists of those points $x \in X$ that only have neighborhoods of the latter type. The boundary ∂X inherits the structure of a stratified pseudomanifold (without boundary) as a subspace of X: each point has a neighborhood $\mathbb{R}^{k-1} \times cL \subset \partial X$ obtained by neglecting the last coordinate of $\mathbb{R}^k_+$. We will also assume as part of the definition of a stratified pseudomanifold with boundary that ∂X has a stratified collar, i.e. there is a neighborhood N of ∂X in X that is stratified homeomorphic to $\partial X \times [0, 1)$, where $[0, 1)$ is given the trivial stratification and $\partial X \times [0, 1)$ the product stratification.

A *closed* stratified pseudomanifold of dimension n is a compact stratified pseudomanifold of dimension n without boundary. As with pseudomanifolds, we will often refer simply to "pseudomanifolds with boundary," leaving the dimension and the stratification tacit.

Remark 3.1. The boundary of a stratified pseudomanifold depends quite strongly on the choice of stratification. For example, let M be a manifold with boundary B (in the usual manifold sense). If we consider M to be a pseudomanifold with a trivial stratification, then B is also the pseudomanifold boundary of M in the sense just defined. However, if we let $\hat{M}$ be the space M with the stratification $M \supset B$, then $\hat{M}$ is *not* a stratified pseudomanifold with boundary. This is because points of B do not have neighborhoods stratified homeomorphic to $\mathbb{R}^k_+ \times cL$, i.e. the local stratified collaring requirement for the boundary of $\hat{M}$ fails. Instead, we just have for every point on the stratum $\hat{M} - B$ a neighborhood homeomorphic to $\mathbb{R}^n$ and for every point on the stratum B, a neighborhood in B homeomorphic to $\mathbb{R}^{n-1}$ and a neighborhood in $\hat{M}$ stratified homeomorphic to $\mathbb{R}^{n-1} \times c(\text{pt})$. This makes $\hat{M}$ instead a stratified pseudomanifold *without* boundary. This distinction will be important later, because it will allow us to treat non-additivity for stratified pseudomanifolds with boundary by restratifying them into stratified pseudomanifolds without boundary.

Intersection homology. We will work mostly with PL chain intersection homology theory with general perversities and stratified coefficient systems. General perversities (those not necessarily satisfying the axioms of Goresky and MacPherson [26]) are indispensable for certain results, such as the intersection homology Künneth theorem of [24]. Similarly, stratified coefficients are necessary in order to properly formulate the most useful version of intersection homology with general perversities. More detailed overviews of this version of the theory can be found in [22, 21].

General perversities. A *general perversity on a stratified pseudomanifold* X is any function $\bar{p}$: {singular strata of X} $\to \mathbb{Z}$. It is technically convenient also to define $\bar{p}(Z) = 0$ if Z is a regular stratum of X.

Stratified coefficient systems. In order to formulate the chain version of intersection homology for general perversities that seems best to fit with the classical sheaf-theoretic versions of intersection homology, we must use "stratified coefficients," as introduced in [23] (see also [22] for an exposition). Since the situation simplifies somewhat in the PL category (and since we will not be working with local coefficient systems), we present a simpler definition here than is found elsewhere.

First, recall that the PL chain complex $C_*(X;G)$ of a PL space X is defined to be $\varinjlim_{T\in\mathcal{T}} C^T_*(X;G)$, where each T is a triangulation of X compatible with the PL structure and $C^T_*(X;G)$ is the corresponding simplicial chain complex with coefficients in the abelian group G. The limit is taken over all triangulations compatible with the PL structure[3]. In other words, elements of $C_*(X;G)$ are represented by sums of chains, each of which is taken from some fixed triangulation of X. In particular, any $\xi \in C_j(X;G)$ can be represented as a finite sum $\xi = \sum g_i\sigma_i$, where $g_i \in G$ and σ_i is a j-simplex in some triangulation of X. Furthermore, $\partial\xi = \sum g_i\partial\sigma_i$.

Now, suppose X is a stratified pseudomanifold. We define $C_j(X;G_0)$ to be the subgroup of $\xi \in C_j(X;G)$ such that when we write ξ as $\sum g_i\sigma_i$, no σ_i is contained in X^{n-1}. It is easy to check that this is a G-module. In order to define $C_*(X;G_0)$ as a chain complex, we define $\partial\xi$ to be $\sum g_i\partial\sigma_i - \sum_{\sigma_i\subset X^{n-1}} g_i\sigma_i$. In other words, to obtain $\partial\xi \in C_*(X;G_0)$, we remove from $\partial\xi \in C_*(X;G)$ those simplices contained in X^{n-1}. This is a chain complex, and we denote its homology $H_*(X;G_0)$. Some readers will notice that $C_*(X;G_0)$ is isomorphic to $C_*(X,X^{n-1};G)$, but for the upcoming definition of intersection homology, we require this formulation. We refer to $H_*(X;G_0)$ as homology with *stratified coefficients* G_0. Note that there is no group G_0 - the subscript 0 simply signifies that we are using the construction defined in this paragraph; this notation originates from a more general situation in [23] in which the first author originally defined these groups using different systems of local coefficients on different strata.

For more details of this construction (and more general cases), see [22, 23, 21]. N.B. If $\bar{p}(S) \leq \text{codim}(S) - 2$ for all singular strata S, in particular if $\bar{p}$ satisfies the Goresky-MacPherson conditions on a perversity, then $I^{\bar{p}}C_*(X;G_0) \cong I^{\bar{p}}C_*(X;G)$, so stratified coefficients are unnecessary in this case.

Intersection homology. Given a stratified pseudomanifold $X = X^n$, a general perversity $\bar{p}$, and an abelian group G, one defines the *intersection chain complex* $I^{\bar{p}}C_*(X;G_0)$ as a subcomplex of $C_*(X;G_0)$ as follows: An i-simplex σ in X is *$\bar{p}$-allowable* if

$$\dim(\sigma \cap Z) \leq i - \text{codim}(Z) + \bar{p}(Z)$$

for any singular stratum Z of X. The chain $\xi \in C_i(X;G_0)$ is $\bar{p}$-allowable if each simplex with non-zero coefficient in ξ or in $\partial\xi$ is allowable. Notice that simplices that disappear from the boundary because of the coefficient system G_0 do not need to be checked for allowability.

[3]It is technically necessary to work with such chains in discussing intersection homology, since degenerate cases can occur if a triangulation is not sufficiently fine. However, it is also possible to work with any sufficiently fine fixed triangulation; see [35].

Notice that this is also why it is not sufficient to work in $C_*(X, X^{n-1}; G)$, where we have no control over simplices that live in X^{n-1}. Let $I^{\bar{p}}C_*(X; G_0)$ be the complex of $\bar{p}$-allowable chains. The associated homology theory is denoted $I^{\bar{p}}H_*(X; \mathcal{G}_0)$.

Relative intersection homology is defined similarly, in the obvious way, though we note that the filtration on a subspace will always be that inherited from the larger space by restriction, i.e. if $Y \subset X$, then $Y^k = Y \cap X^k$, regardless of the actual dimensions involved. We also assume that Y inherits the formal dimension of X, regardless of actual geometric dimension, so that if Z is a stratum of codimension k in X, then we consider $Z \cap Y$ to have the same codimension k in Y. Thus a chain in Y is defined to be allowable if and only if it is allowable in X.

If $\bar{p}$ is a perversity in the sense of Goresky-MacPherson [26] and X has no strata of codimension one, then $I^{\bar{p}}H_*(X; G_0)$ is isomorphic to the intersection homology groups $I^{\bar{p}}H_*(X; G)$ of Goresky-MacPherson [26, 27]. If $\bar{p}$ is not a Goresky-MacPherson perversity, then we need stratified coefficients in order for some of the main properties of intersection homology, such as duality and the cone formula, to hold - see [21, 22]. General perversities are useful because, among other things, they allow us to talk about relative and absolute cohomologies in the same framework as the Goresky-MacPherson intersection homologies. Suppose that $Z \subset X$ are smooth manifolds. Then if $\bar{p}(Z) > \text{codim}(Z) - 2$ we get $I^{\bar{p}}H_*(X; G_0) \cong H_*(X, Z; G)$, and if $\bar{p}(Z) < 0$, we get $I^{\bar{p}}H_*(X; G_0) \cong H_*(X - Z; G)$. Note that if Z is the boundary of M, then also $H_*(X - Z; G) \cong H_*(X; G)$.

Intersection homology with general perversities can also be formulated sheaf theoretically [21, 22] or analytically [41]. In these languages, it is more customary to use cohomological indexing and refer to intersection *cohomology* but these are really the same theories (up to various indexing issues).

Even with general perversities and G_0 coefficients, the basic properties of $I^{\bar{p}}H_*(X; G_0)$ established in [32] and [23] hold with little or no change to the proofs, such as stratum-preserving homotopy equivalence, excision, the Künneth theorem for which one term is an unstratified manifold, Mayer-Vietoris sequences, etc.

Duality. Finally, we recall the intersection homology version of Poincaré duality, due initially to Goresky and MacPherson [26] and later extended to the more general cases considered here [21]. Suppose X is a closed oriented n-pseudomanifold and that F is a field. Suppose that $\bar{p}$ and $\bar{q}$ are perversities such that $\bar{p} + \bar{q} \leq \bar{r}$, i.e. $\bar{p}(Z) + \bar{q}(Z) \leq \bar{r}(Z)$ for all singular strata Z. Then there is an intersection pairing $\pitchfork$: $I^{\bar{p}}H_i(X; F_0) \otimes I^{\bar{q}}H_j(X; F_0) \to I^{\bar{r}}H_{i+j-n}(X; F_0)$. If $\bar{p} + \bar{q} = \bar{t}$, i.e. $\bar{p}(Z) + \bar{q}(Z) = \text{codim}(Z) - 2$ for all singular strata Z, then this induces a duality isomorphism pairing $I^{\bar{p}}H_i(X; F_0) \cong I^{\bar{q}}H_{n-i}(X; F_0)$. If X is a compact pseudomanifold with boundary, we obtain an analogous Lefschetz-type duality $I^{\bar{p}}H_i(X; F_0) \cong I^{\bar{q}}H_{n-i}(X, \partial X; F_0)$.

3.2 Perverse signatures

Now we can define perverse signatures. If X is a PL stratified pseudomanifold and $\bar{p} \leq \bar{q}$, let $I^{\bar{p}\to\bar{q}}H_*(X; \mathbb{Q}_0) = \text{im}(I^{\bar{p}}H_*(X; \mathbb{Q}_0) \to I^{\bar{q}}H_*(X; \mathbb{Q}_0))$, where the map is induced by inclusion of chain complexes. If (Y,Z) is a pair of a pseudomanifold and any subspace Z, let $I^{\bar p\twoheadrightarrow\bar q}H_*(Y,Z;\mathbb{Q}_0)=\mathrm{im}(I^{\bar p}H_*(Y;\mathbb{Q}_0)\to I^{\bar q}H_*(Y,Z;\mathbb{Q}_0))$. The reason for the double arrow in the second notation is that $I^{\bar p\twoheadrightarrow\bar q}H_*(Y,Z;\mathbb{Q}_0)$ is really the image of a composition of two maps taking the perversity from $\bar p$ to $\bar q$ and the space from Y_i to (Y,Z).

By duality, if X is closed, oriented, and $4n$-dimensional, and if Y is a compact, oriented $4n$-pseudomanifold with boundary, then there are nonsingular intersection pairings between $I^{\bar p}H_{2n}(X;\mathbb{Q}_0)$ and $I^{\bar q}H_{2n}(X;\mathbb{Q}_0)$ if $\bar p+\bar q=\bar t$, and similarly between $I^{\bar p}H_{2n}(Y;\mathbb{Q}_0)$ and $I^{\bar q}H_{2n}(Y,\partial Y;\mathbb{Q}_0)$. If also $\bar p\leq\bar q$, this induces nonsingular symmetric pairings on $I^{\bar p\to\bar q}H_{2n}(X;\mathbb{Q}_0)$ and $I^{\bar p\twoheadrightarrow\bar q}H_{2n}(Y,\partial Y;\mathbb{Q}_0)$. The symmetry comes from the symmetry of the chain level pairing (see the footnote on page 16). To see that these pairings are nonsingular, we simply note that if $[x]\in I^{\bar p\to\bar q}H_{2n}(X;\mathbb{Q}_0)\subset I^{\bar q}H_{2n}(X;\mathbb{Q}_0)$, then there must be a $[y]$ in $I^{\bar p}H_{2n}(X;\mathbb{Q}_0)$ such that $[x]\pitchfork[y]\neq 0$. But then the image of $[y]$ in $I^{\bar p\to\bar q}H_{2n}(X;\mathbb{Q}_0)$ must be non-zero, since we could instead consider $[x]$ as represented by a cycle in $I^{\bar p}C_{2n}(X;\mathbb{Q}_0)$ and $[y]$ by a cycle in $I^{\bar q}C_{2n}(X;\mathbb{Q}_0)$, but the intersection numbers would be the same. So $[y]$ cannot be 0 in $I^{\bar p\to\bar q}H_{2n}(X;\mathbb{Q}_0)\subset I^{\bar q}H_{2n}(X;\mathbb{Q}_0)$. The same argument works for $I^{\bar p\twoheadrightarrow\bar q}H_{2n}(Y,Z;\mathbb{Q}_0)$.

The respective signatures of these intersection forms, denoted $\sigma_{\bar p\to\bar q}(X)$ and $\sigma_{\bar p\twoheadrightarrow\bar q}(Y)$, are the $\bar p\to\bar q$ *perverse signatures* of X and Y.

3.3 A new intersection homology pairing

In this subsection, we will define the homology groups that provide the symplectic pairing for our non-additivity theorem. The relationship between this pairing and the usual Goresky-MacPherson intersection pairing is akin to the relationship between the intersection pairing on the boundary of a manifold and the pairing in its interior. In fact, this will be made precise in Section 5.1, where we will show it reduces to the intersection pairing on the boundary of a manifold in the appropriate context.

Motivation. To motivate how we define the groups we will need, let M be a manifold with boundary and G an abelian group, and consider the long exact sequence

$$\to H_i(M;G)\to H_i(M,\partial M;G)\to H_{i-1}(\partial M;G)\to .$$

On the other hand, for a stratified space X and $\bar p\leq\bar q$, we will see that there is a long exact sequence

$$\to I^{\bar p}H_i(X;G_0)\to I^{\bar q}H_i(X;G_0)\to I^{\bar q/\bar p}H_i(X;G_0)\to,$$

where $I^{\bar q/\bar p}H_i(X;G_0)$ is defined to be the homology of the quotient $I^{\bar q}C_i(X;G_0)/I^{\bar p}C_i(X;G_0)$.

But if M is a manifold with boundary, stratified as $M\supset\partial M$, and we choose perversities $\bar p,\bar q$ such that $\bar p(\partial M)<0$ and $\bar q(\partial M)>\bar t(\partial M)=-1$, then an easy computation (see [21]), shows that $I^{\bar p}H_*(M;G_0)\cong H_*(M;G)$ and $I^{\bar q}H_*(M;G_0)\cong H_*(M,\partial M;G)$. Thus we expect $I^{\bar q/\bar p}H_i(X;G_0)$ to play a role analogous to that classically played by the boundary of a manifold, though with a dimension shift. We will make this connection with boundaries even more precise in Section 5.1.

Two $\bar q/\bar p$ long exact sequences. Let X be an n-dimensional PL stratified pseudomanifold; let G be an abelian group; and let $\bar p, \bar q$ be general perversities such that $\bar p(Z) \le \bar q(Z)$ for all singular strata $Z \subset X$. Then $I^{\bar p}C_*(X;G_0)$ is a subgroup of $I^{\bar q}C_*(X;G_0)$, and we denote the quotient group by $I^{\bar q/\bar p}C_*(X;G_0)$ and its homology by $I^{\bar q/\bar p}H_*(X;G_0)$. Note that a cycle x in $I^{\bar q/\bar p}C_i(X;G_0)$ is a $\bar q$-allowable chain such that ∂x is $\bar p$-allowable. A homology between cycles x_1 and x_2 is provided by a $\bar q$-allowable chain y such that $\partial y = x_1 - x_2 + p$, where p is $\bar p$-allowable.

Suppose $Y \subset X$ and that Y is given the subspace filtration, i.e. $Y^k = Y \cap X^k$ regardless of actual geometric (co)dimensions; in particular a chain is $\bar p$-allowable in Y if and only if it is $\bar p$-allowable as a chain in X. Then we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{\bar p}C_*(Y;G_0) & \xrightarrow{i_{\bar p\to\bar q}} & I^{\bar q}C_*(Y;G_0) & \xrightarrow{\pi_{\bar q/\bar p}} & I^{\bar q/\bar p}C_*(Y;G_0) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i_{Y\subset X}} & & \big\downarrow{\scriptstyle i_{Y\subset X}} & & \big\downarrow{\scriptstyle i_{Y\subset X}} & & \\
0 & \longrightarrow & I^{\bar p}C_*(X;G_0) & \xrightarrow{i_{\bar p\to\bar q}} & I^{\bar q}C_*(X;G_0) & \xrightarrow{\pi_{\bar q/\bar p}} & I^{\bar q/\bar p}C_*(X;G_0) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \pi_{X,Y}} & & \big\downarrow{\scriptstyle \pi_{X,Y}} & & \big\downarrow{\scriptstyle \pi_{X,Y}} & & \\
0 & \longrightarrow & I^{\bar p}C_*(X,Y;G_0) & \xrightarrow{i_{\bar p\to\bar q}} & I^{\bar q}C_*(X,Y;G_0) & \xrightarrow{\pi_{\bar q/\bar p}} & I^{\bar q/\bar p}C_*(X,Y;G_0) & \longrightarrow & 0,
\end{array} \tag{3}$$

where $I^{\bar q/\bar p}C_*(X,Y;G_0)$ is defined to be the quotient of $I^{\bar q/\bar p}C_*(X;G_0)$ by $I^{\bar q/\bar p}C_*(Y;G_0)$. The righthand map in each of the first two rows is an injection because any chain supported in Y that is $\bar p$-allowable in X (and hence 0 in $I^{\bar q/\bar p}C_*(X;G_0)$) will already be $\bar p$-allowable in Y. Therefore, by the serpent lemma, the last row is also a short exact sequence. In particular, we have long exact sequences associated to the third row of this diagram:

$$\longrightarrow I^{\bar p}H_i(X,Y;G_0) \xrightarrow{(i_{\bar p\to\bar q})_*} I^{\bar q}H_i(X,Y;G_0) \xrightarrow{(\pi_{\bar q/\bar p})_*} I^{\bar q/\bar p}H_i(X,Y;G_0) \xrightarrow{d} I^{\bar p}H_{i-1}(X,Y;G_0) \longrightarrow \tag{4}$$

and to the third column:

$$\longrightarrow I^{\bar q/\bar p}H_i(Y;G_0) \xrightarrow{(i_{Y\subset X})_*} I^{\bar q/\bar p}H_i(X;G_0) \xrightarrow{(\pi_{X,Y})_*} I^{\bar q/\bar p}H_i(X,Y;G_0) \xrightarrow{\delta} I^{\bar q/\bar p}H_{i-1}(Y;G_0) \longrightarrow \tag{5}$$

Observe that the "connecting maps" d can be described as acting on a representative chain x by taking x to its boundary ∂x. Meanwhile, an element of $I^{\bar q/\bar p}H_*(X,Y;G_0))$ can be represented by a chain x such that $\partial x = a + b$ with b a $\bar q$-allowable chain in Y and a a $\bar p$-allowable chain in X, and δx is represented by b. Note also that a representative x of a class in $I^{\bar q/\bar p}H_*(X,Y;G_0)$ is a $\bar q$-allowable chain on X such that ∂x is the sum of a $\bar p$-allowable chain on X and a $\bar q$-allowable chain on Y. We will use this fact in the proof of our main theorem.

A pairing on $\bar q/\bar p$ intersection homology. Now we want to define intersection pairings on our new groups. Suppose $\bar p + \bar q \le \bar r$ and R is a ring. We define a pairing $\Phi : I^{\bar q/\bar p}H_i(X;R_0)\otimes$

$I^{\bar q/\bar p}H_j(X,\partial X;R_0)\to I^{\bar r}H_{n-i-j+1}(X;R_0)$ as follows. Let $\pitchfork$ denote the Goresky-MacPherson intersection pairing on intersection chains. If x,y are chains in stratified general position representing respective elements of $I^{\bar q/\bar p}H_i(X;R_0)$ and $I^{\bar q/\bar p}H_j(X,\partial X;R_0)$, let $\tilde\Phi(x,y)=x\pitchfork\partial y+(-1)^{n-|x|}(\partial x)\pitchfork y$, where $|x|$ denotes the degree of x. We need to see that $\tilde\Phi$ makes sense as a map on chains, and then we want to show it descends to a well-defined pairing $\Phi([x],[y])$ on homology.

To make sense on chains, we need to know that $x\pitchfork\partial y+(-1)^{n-|x|}(\partial x)\pitchfork y$ is an $\bar r$-allowable chain. It follows from the standard stratified general position arguments [37, 26, 25] that we can choose x and y in stratified general position (which includes boundaries being in stratified general position with respect to x, y, and each other), and we may also assume that x does not intersect ∂X. The Goresky-MacPherson intersection pairing extends to the relevant chains in this setting by [25]. Note that each intersection $x\pitchfork\partial y$ or $(\partial x)\pitchfork y$ is between a $\bar p$-allowable chain and a $\bar q$-allowable chain; to see this in the case of $x\pitchfork\partial y$, we should observe that ∂y is the sum of a $\bar p$-allowable chain in X and a $\bar q$-allowable chain in ∂X, but the part in ∂X does not intersect x, which can be assumed to lie in the interior of X, so we have the intersection of a $\bar q$-admissible chain with a $\bar p$-admissible chain, which will be $\bar r$-admissible. Furthermore[4],

$$\begin{aligned}\partial\tilde\Phi(x,y)&=(\partial x)\pitchfork(\partial y)+(-1)^{n-|x|+n-|x|-1}(\partial x)\pitchfork(\partial y)\\&=(\partial x)\pitchfork(\partial y)-(\partial x)\pitchfork(\partial y)\\&=0,\end{aligned}$$

so indeed we obtain an admissible $\bar r$-cycle. It is important to note that, despite appearances, this cycle is not necessarily the boundary $(-1)^{n-|x|}\partial(x\pitchfork y)$ as $x\pitchfork y$ is the intersection of two $\bar q$ admissible chains, thus is not necessarily well-defined in $I^{\bar r}C_*(X;R_0)$ unless $\bar q+\bar q\le r$.

To show that this pairing is well-defined on homology, suppose that z is another chain representing the same class as x and in stratified general position with respect to y. Then from the definitions, $z-x=\partial Q+P$, where Q is another $\bar q$-allowable chain whose boundary is $\bar p$-allowable and P is $\bar p$-allowable. Again, we can assume everything in stratified general position and that P and Q do not intersect ∂X. Then

$$\begin{aligned}\tilde\Phi(z,y)&=z\pitchfork\partial y+(-1)^{n-|z|}(\partial z)\pitchfork y\\&=(x+\partial Q+P)\pitchfork\partial y+(-1)^{n-|z|}(\partial(x+\partial Q+P))\pitchfork y\\&=x\pitchfork\partial y+(-1)^{n-|z|}(\partial x)\pitchfork y+(\partial Q+P)\pitchfork\partial y+(-1)^{n-|z|}(\partial P)\pitchfork y\\&=x\pitchfork\partial y+(-1)^{n-|z|}(\partial x)\pitchfork y+P\pitchfork\partial y+(-1)^{n-|z|}(\partial P)\pitchfork y+\partial Q\pitchfork\partial y.\end{aligned}$$

Note that each intersection is of a $\bar p$-allowable chain with a $\bar q$-allowable chain since $\bar p$-allowable chains are also $\bar q$-allowable. Now, since $|z|=|P|$, we see that $P\pitchfork\partial y+(-1)^{n-|z|}(\partial P)\pitchfork y=(-1)^{n-|z|}\partial(P\pitchfork y)$, which is well defined because P is $\bar p$-allowable and y is $\bar q$-allowable. Similarly, $\partial Q\pitchfork\partial y=\partial(Q\pitchfork\partial y)$, using again that only the $\bar p$-allowable part of ∂y can intersect

[4] We use the sign conventions of Dold [20] or Goresky-MacPherson [26] so that $\partial(a\pitchfork b)=(\partial a)\pitchfork b+(-1)^{n-|a|}a\pitchfork(\partial b)$.

Q. Thus $\tilde{\Phi}(z, y) = \tilde{\Phi}(x, y)$. A similar argument shows that the pairing is independent of the choice of chain representing $[y]$, so $\Phi([x], [y])$ is well-defined in $I^{\bar{r}}H_{n-i-j+1}(X; R_0)$.

Now, let Z be a compact oriented $4n - 1$ PL stratified pseudomanifold with (possibly empty) boundary. Suppose $\bar{p} + \bar{q} = \bar{t}$. Then the composition of $\Phi : I^{\bar{q}/\bar{p}}H_i(Z; \mathbb{Q}_0) \otimes I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z; \mathbb{Q}_0) \to I^{\bar{t}}H_0(Z; \mathbb{Q}_0)$ with the augmentation $I^{\bar{t}}H_0(Z; \mathbb{Q}_0) \to \mathbb{Q}$ gives us a bilinear form $I^{\bar{q}/\bar{p}}H_i(Z; \mathbb{Q}_0) \otimes I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z; \mathbb{Q}_0) \to \mathbb{Q}$. We shall abuse notation and also refer to this bilinear form as Φ in what follows.

Lemma 3.2. *$\Phi : I^{\bar{q}/\bar{p}}H_i(Z; \mathbb{Q}_0) \otimes I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z; \mathbb{Q}_0) \to \mathbb{Q}$ is nonsingular, and if $\partial Z = \emptyset$, it is skew-symmetric on $I^{\bar{q}/\bar{p}}H_{2n}(Z; \mathbb{Q}_0)$.*

Proof. Consider the following diagram with coefficients in $\mathbb{Q}_0$:

$$\begin{array}{ccccccc}
\longrightarrow & I^{\bar{q}}H_i(Z) & \longrightarrow & I^{\bar{q}/\bar{p}}H_i(Z) & \longrightarrow & I^{\bar{p}}H_{i-1}(Z) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \operatorname{Hom}(I^{\bar{p}}H_{4n-1-i}(Z, \partial Z), \mathbb{Q}) & \longrightarrow & \operatorname{Hom}(I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z), \mathbb{Q}) & \longrightarrow & \operatorname{Hom}(I^{\bar{q}}H_{4n-i}(Z, \partial Z), \mathbb{Q}) & \cdots\rightarrow
\end{array}$$

The top is the long exact sequence induced by the short exact sequence

$$0 \longrightarrow I^{\bar{p}}C_*(Z; \mathbb{Q}_0) \longrightarrow I^{\bar{q}}C_*(Z; \mathbb{Q}_0) \longrightarrow I^{\bar{q}/\bar{p}}C_*(Z; \mathbb{Q}_0) \longrightarrow 0.$$

The bottom is the $\operatorname{Hom}(\cdot, \mathbb{Q})$ dual of the same long exact sequence for $(Z, \partial Z)$; it is also exact because $\mathbb{Q}$ is a field. The first and third vertical morphisms take a class $[x]$ to $[x] \pitchfork \cdot$. By Goresky-MacPherson [26, 27] (and [21] for general perversities and pseudomanifolds with boundaries), these are isomorphisms. The second vertical map takes $[x]$ to $\Phi([x], \cdot)$ composed with the augmentation $I^{\bar{t}}H_0(X; \mathbb{Q}_0) \to \mathbb{Q}$.

We claim that the diagram commutes up to sign. It is standard that the square with top corners $I^{\bar{p}}H_i(Z)$ and $I^{\bar{q}}H_i(Z)$ (not shown as a square on the diagram) commutes. To see that the first square commutes, let $[x] \in I^{\bar{q}}H_i(Z)$, $[z]$ be the image of $[x]$ in $I^{\bar{p}/\bar{q}}H_i(Z)$, and $[y] \in I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z)$. Then $\Phi([z], [y]) = [z \pitchfork \partial y + (-1)^{4n-1-|x|}\partial z \pitchfork y] = [z \pitchfork \partial y] = [x \pitchfork \partial y] = [x] \pitchfork [y]$ because $\partial x = \partial z = 0$ and $[z]$ and $[x]$ are represented by the same chain. On the other hand, going down then right takes $[x]$ to a map that acts on $[y]$ by first applying the map to the left on homology that gives $[\partial y]$ and then applying $[x] \pitchfork \cdot$. So this square also commutes.

To see that the second square commutes up to sign, suppose $[x] \in I^{\bar{q}/\bar{p}}H_i(Z)$. Then going right then down takes $[x]$ first to $[\partial x]$, then to the map that acts on $[y] \in I^{\bar{q}}H_{4n-i+1}(Z, \partial Z)$ by $[\partial x] \pitchfork [y]$ (note that ∂y is supported in ∂Z and cannot intersect x, which can be assumed to have support in the interior of Z). On the other hand, going down then right takes $[x]$ to the map that acts on $[y] \in I^{\bar{q}}H_{4n-i+1}(Z, \partial Z)$ by first taking it to $[y] \in I^{\bar{q}/\bar{p}}H_{4n-i}(Z, \partial Z)$ and then applying $\Phi([x], \cdot)$ to obtain $[x \pitchfork \partial y + (-1)^{4n-1-|x|}(\partial x) \pitchfork y]$. But again ∂y must lie in ∂Z, so this is just $[(-1)^{4n-1-|x|}(\partial x) \pitchfork y] = (-1)^{4n-1-|x|}[\partial x] \pitchfork [y]$.

We can now apply the five lemma to conclude that Φ determines a nonsingular pairing. Even though the diagram does not commute on the nose, commuting up to sign implies that it is possible to change signs of some of the maps to obtain a commuting diagram. Changing signs does not affect exactness of the horizontal sequences.

To show Φ is anti-symmetric when $i = 2n$ and $\partial Z = \emptyset$, we calculate [5],

$$\begin{aligned}
\Phi([x],[y]) &= [x \pitchfork (\partial y) + (-1)^{4n-1-|x|}(\partial x) \pitchfork y] \\
&= [x \pitchfork (\partial y) - (\partial x) \pitchfork y] \\
&= [(-1)^{(4n-1-|x|)(4n-1-(|y|-1))}(\partial y) \pitchfork x - (-1)^{(4n-1-|y|)(4n-1-(|x|-1))} y \pitchfork (\partial x)] \\
&= [(-1)^{(4n-1-2n)(4n-1-(2n-1))}(\partial y) \pitchfork x - (-1)^{(4n-1-2n)(4n-1-(2n-1))} y \pitchfork (\partial x)] \\
&= [(\partial y) \pitchfork x - y \pitchfork (\partial x)] \\
&= -\Phi([y],[x]).
\end{aligned}$$

□

In Section 5.1, below, we will show that if X is a $4n-1$-manifold with boundary, appropriately stratified and with an appropriate choice of perversities, then Φ becomes the classical intersection pairing on ∂X.

4 Non-additivity of perverse signatures

This section contains our non-additivity theorems. We prove our first main result, on non-additivity of perverse signatures for pseudomanifolds without boundary, in the first subsection, then obtain our second main result, for pseudomanifolds with boundary, as a corollary in the second subsection.

4.1 Non-additivity of perverse signatures for pseudomanifolds without boundary

In this section, we prove a generalization of the Wall non-additivity theorem for the perverse pairings of intersection homology theory. The general outline of the proof is the same as that in [45], but there are some subtleties and generalizations that need to be addressed.

Throughout this section, let X be a $\mathbb{Q}$-oriented stratified $4n$-pseudomanifold without boundary (though it may possess codimension one strata). Let $Z \subset X$ be a bicollared codimension one subpseudomanifold such that $X = Y_1 \cup_Z Y_2$ and $\partial Y_1 = Z = -\partial Y_2$, accounting for orientations.

Let $V = I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$ equipped with the anti-symmetric pairing Φ defined in Section 3.3. Let $A = \ker(i_{Z\subset Y_1} : I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0))$, $C = \ker(i_{Z\subset Y_2} : I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0))$, and let $B = \ker(d : I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{p}}H_{2n-1}(Z;\mathbb{Q}_0))$.

Theorem 4.1. *With the assumptions and definitions considered above,*

$$\sigma_{\bar{p}\to\bar{q}}(X) = \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_1) + \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_2) + \sigma(V;A,B,C),$$

where σ is Wall's Maslov triple index.

[5]Recall that on an m-pseudomanifold $a \pitchfork b = (-1)^{(m-|a|)(m-|b|)} b \pitchfork a$; see [20, 26].

Before presenting the proof, we provide some easy but interesting corollaries. We will consider the case where Z is not the entire boundary of each Y_i (as in Wall's original theorem) below in Section 4.2.

Corollary 4.2. *With the hypotheses of Theorem 4.1, suppose* $I^{\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar q}H_{2n}(Z;\mathbb{Q}_0)$ *is surjective and* $I^{\bar p}H_{2n-1}(Z;\mathbb{Q}_0) \to I^{\bar q}H_{2n-1}(Z;\mathbb{Q}_0)$ *is injective. Then*

$$\sigma_{\bar p\to\bar q}(X) = \sigma_{\bar p\twoheadrightarrow\bar q}(Y_1) + \sigma_{\bar p\twoheadrightarrow\bar q}(Y_2),$$

as in Novikov's additivity theorem.

Proof. In this case, $I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) = 0$ by the long exact sequence relating $\bar p$ and $\bar q$ intersection homology. Thus V and hence $\sigma(V;A,B,C)$ are trivial. □

This corollary allows us to recover Siegel's theorem regarding Novikov additivity of Witt spaces [42]. Indeed, when X, Y_i, Z are all Witt-spaces, $I^{\bar m}H_* \cong I^{\bar n}H_*$ for each, and thus $\sigma_{\bar m\to\bar n}(X) = \sigma_{\bar m\twoheadrightarrow\bar n}(Y_1) + \sigma_{\bar m\twoheadrightarrow\bar n}(Y_2)$. These signatures $\sigma_{\bar m\to\bar n}(X)$ and $\sigma_{\bar m\to\bar n}(Y_1)$ are just the signatures of the middle-perversity middle-dimension intersection pairings on these Witt spaces [42]. Of course, when $\bar p = \bar q = \bar m$, Theorem 4.1 also gives a non-additivity result that applies, for instance, when a Witt space is decomposed along a codimension one subpseudomanifold that is not Witt.

In terms of another important property in the signature package, note that this corollary also implies a weak cobordism invariance, namely cobordism rel neighborhoods of the singular strata. Of course, the resulting cobordism group is infinite dimensional, and for this reason, not very useful. We are hopeful that it may be possible in the future to define a set of spaces for which various perverse signatures satisfy a better cobordism invariance.

The following corollaries are particular examples that are standard for signatures.

Corollary 4.3. *If* (Y_1, Z) *is homeomorphic to* $(-Y_2, Z)$ *rel* Z *(i.e. by an isomorphism that fixes* Z *pointwise), then* $\sigma_{\bar p\to\bar q}(X) = 0$.

Proof. Since $Y_1 \cong -Y_2$, their signatures are the negatives of each other. In addition, with the hypotheses, it is clear that the inclusions $I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar q/\bar p}H_{2n}(Y_1;\mathbb{Q}_0)$ and $I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar q/\bar p}H_{2n}(Y_2;\mathbb{Q}_0)$ are isomorphic maps with identical kernels (this is why we require Z to be fixed by the homeomorphism). Therefore $A = C$. But an odd permutation of the subspaces of A, B, C alters $\sigma(V;A,B,C)$ by a sign. Hence $\sigma(V;A,B,C) = 0$. □

Corollary 4.4. *If* X *is the suspension of the compact pseudomanifold* Z*, then* $\sigma_{\bar p\to\bar q}(X) = 0$.

Proof. The hypotheses of the preceding corollary apply taking Y_1 and Y_2 to be the two cones on Z. □

Remark 4.5. Corollary 4.4 can be obtained with less machinery by observing that if $\bar p \le \bar q$ and $\bar p + \bar q = \bar t$, then in fact $\bar p \le \bar m \le \bar n \le \bar q$, where $\bar m, \bar n$ are the lower- and upper-middle perversities. Thus $I^{\bar p}H_*(X;\mathbb{Q}_0) \to I^{\bar p}H_*(X;\mathbb{Q}_0)$ factors through $I^{\bar m}H_*(X;\mathbb{Q}_0)$ and $I^{\bar n}H_*(X;\mathbb{Q}_0)$. But for a $4k$-dimensional suspension X, $I^{\bar m}H_{2k}(X;\mathbb{Q}_0) = I^{\bar n}H_{2k}(X;\mathbb{Q}_0) = 0$.

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. First we show that $I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$ decomposes as a direct sum of $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)$, $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$, and a third piece, and that the signature pairing is diagonal with respect to this decomposition. Consider the morphisms (induced by inclusion)

$$\begin{aligned} I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0)\oplus I^{\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0) &\to I^{\bar{p}}H_{2n}(X;\mathbb{Q}_0)\\ &\twoheadrightarrow I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)\\ &\hookrightarrow I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0)\\ &\to I^{\bar{q}}H_{2n}(X,Z;\mathbb{Q}_0)\\ &\cong I^{\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\oplus I^{\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0). \end{aligned}$$

The image of the composition is $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$, so we have an induced surjection

$$\operatorname{im}(I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0)\oplus I^{\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0)\to I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0))\to I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0).$$

Since all groups are really $\mathbb{Q}$-vector spaces, there is a (non-unique) splitting of this map. We claim that this splitting is isometric in that it preserves the intersection pairing (where the intersection pairing on $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$ is given by the orthogonal sum). Indeed, suppose that $[x]=[x_1]+[x_2]$ and $[y]=[y_1]+[y_2]$ are elements of $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$ and that $[\tilde{x}]=[\tilde{x}_1]+[\tilde{x}_2]$ and $[\tilde{y}]=[\tilde{y}_1]+[\tilde{y}_2]\in \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0)\oplus I^{\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0)\to I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0))$ are their images under a given splitting. Each $[\tilde{x}_i]$ and $[\tilde{y}_i]$ may be represented by $\bar{p}$-allowable cycles with support in the interior of Y_i, and the same cycles represent the $[x_i]$ and $[y_i]$. Furthermore, we can always assume that the representing cycles are in stratified general position. Then, by definition, both intersection pairings are given by counting the intersection numbers of the the representative chains, and it is clear that chains in Y_1 do not intersect those in Y_2. So $x\pitchfork y=x_1\pitchfork x_2+y_1\pitchfork y_2=\tilde{x}_1\pitchfork\tilde{x}_2+\tilde{y}_1\pitchfork\tilde{y}_2=\tilde{x}\pitchfork\tilde{y}$, and the pairing is preserved. Notice that if we choose a different splitting that, say, takes $[x]$ to $[\tilde{x}']$, then $[\tilde{x}-\tilde{x}']$ must map to 0 in $I^{\bar{q}}H_{2n}(X,Z;\mathbb{Q}_0)$, i.e. it is $\bar{q}$-homologous in X to a chain in Z. Such a chain clearly does not intersect any $\bar{p}$-allowable chain in the interior of either Y_i, and this explains why the choice of splitting does not affect the isometry type of the pairing.

Now, continuing with the proof of Theorem 4.1, we fix a splitting, and by an abuse of notation, let $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_i,Z;\mathbb{Q}_0)$ also denote its image under the splitting in $I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$. It is geometrically clear that a chain from $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)$ does not intersect a chain from $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$ in $I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$. Thus $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)$ and $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$ are orthogonal in $I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$. We also know that the restriction of the intersection pairing to each subspace is nonsingular, and it follows that they must be disjoint. Thus, choosing an appropriate basis, we can write $I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$ as a direct sum with respect to which the intersection pairing is block diagonal, as desired (we will use $\perp$ to indicate direct sums of orthogonal subspaces):

$$I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)\cong I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0)\perp I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)\perp K.$$

It follows that

$$\sigma_{\bar{p}\to\bar{q}}(X) = \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_1) + \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(Y_2) + \sigma(K),$$

so we must show that $\sigma(K) = \sigma(V; A, B, C)$. To do this, we will first decompose K as a direct sum $L \oplus S \oplus M$ and show that $\sigma(K) = \sigma(L)$. Then we will show that we can identify $\sigma(L)$ with the desired Maslov triple index. Let

$$S = K \cap \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_2; \mathbb{Q}_0) \to I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0)),$$

and let $S^\perp$ denote the annihilator of S in K under $\pitchfork$.

Lemma 4.6. *Under the nonsingular pairing between $I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0)$ and $I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0)$. The annihilator of*

$$im(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$$

is

$$im(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0)).$$

Proof. The orthogonality of the two spaces is geometrically evident as $\bar{p}$-allowable chains in Y_1 or Y_2 can be assumed to have support in the interior of the space (i.e. away from the boundary Z), due to the bicollar on Z.

On the other hand, suppose $[x]$ is in $I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0)$ but not in $\operatorname{im}(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0))$. Then, by the long exact sequence of the pair, $[x]$ has a nontrivial image in $I^{\bar{q}}H_{2n}(X, Z; \mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(Y_1, Z; \mathbb{Q}_0) \oplus I^{\bar{q}}H_{2n}(Y_2, Z; \mathbb{Q}_0)$. But since $I^{\bar{q}}H_{2n}(Y_i, Z; \mathbb{Q}_0)$ is dual to $I^{\bar{p}}H_{2n}(Y_i; \mathbb{Q}_0)$, this implies there must be some $[y] \in I^{\bar{p}}H_{2n}(Y_i; \mathbb{Q}_0)$ for $i = 0$ or $i = 1$ such that y and x intersect with non-zero intersection number. Therefore $[x]$ cannot be orthogonal to $\operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$.

Thus the annihilator of $\operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$ must be exactly $\operatorname{im}(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0))$. ☐

Corollary 4.7. $S^\perp = I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0) \cap$ *im*$(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0))$.

Proof. First, observe that any $[x] \in I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0) \cap \operatorname{im}(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0))$ is in K because any cycle that has a $\bar{q}$-allowable representative with support in Z must have trivial intersection with any $\bar{p}$-allowable chain with support in the interior of Y_1 or Y_2, and any chain in either $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_i, Z; \mathbb{Q}_0)$ can be so represented. For the same reason, $[x] \in S^\perp$ because any element of S can be so represented. Thus

$$I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0) \cap \operatorname{im}(I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X; \mathbb{Q}_0)) \subset S^\perp.$$

On the other hand, anything in $S^\perp$ lies in $K \subset I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0)$ by definition, and so by the preceding lemma, to complete the proof we need only show that $S^\perp$ annihilates $\operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$. Since $S^\perp \subset K$, it automatically annihilates each $I^{\bar{p}\twoheadrightarrow q}H_{2n}(Y_i, Z; \mathbb{Q}_0)$, so we only have to worry about showing that $S^\perp$ annihilates $K \cap \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$, which is nearly the definition of S. Suppose $[x] \in S^\perp \subset I^{\bar{q}}H_{2n}(X; Q_0)$, $[y] \in K \cap \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1; \mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0))$, and $x \pitchfork y \neq 0$. Since $[x] \in I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0)$, we can choose a $\bar{p}$-allowable representative for $[x]$, and we can think of y as $\bar{q}$-allowable without changing $x \pitchfork y \neq 0$. But now y is representing an element of $K \cap \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \to I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)) = S$, which contradicts the assumption that $[x] \in S^\perp$. So $S^\perp$ indeed annihilates $\operatorname{im}(I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \to I^{\bar{p}}H_{2n}(X;\mathbb{Q}_0))$. □

Lemma 4.8. *$S \subset S^\perp$.*

Proof. $S \subset K \subset I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0)$ by definition, so it suffices, by the preceding corollary, to show that $S \subset \operatorname{im}(I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0))$.

Recall that S is the intersection of the image of $I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0)$ in $I^{\bar{p}\to\bar{q}}H_{2n}(;\mathbb{Q}_0 X)$ with K, which is the annihilator and additive complement of $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0) \oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0)$. Thus if $[x] \in S$, then x can be written as the sum of two $\bar{p}$-allowable cycles, one each in Y_1 and Y_2. These cycles have well-defined images respectively in $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0) \subset I^{\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(X,Y_2;\mathbb{Q}_0)$ and $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0) \subset I^{\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(X,Y_1;\mathbb{Q}_0)$. But each of these images must be 0 since the pairing on each $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_i,Z;\mathbb{Q}_0)$ is nonsingular and $[x]$ is orthogonal to everything in these spaces. Thus $[x]$ must be 0 in $I^{\bar{q}}H_{2n}(Y_1,Z;\mathbb{Q}_0) \oplus I^{\bar{q}}H_{2n}(Y_2,Z;\mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(X,Z;\mathbb{Q}_0)$. Therefore by the relative sequence, $[x]$ must be in the image of $I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0)$.

□

We can now proceed as in Wall [45]:

Since the intersection form restricted to K is nonsingular, we have[6] $(S^\perp)^\perp = S$. Therefore the radical of the restriction of the intersection form to $S^\perp$ is S. This implies that the form is nonsingular when restricted to any additive complement L of S in $S^\perp$. We can thus complete the multiplication table for the intersection pairing on K as follows:

	L	S	M
L	A	0	0
S	0	0	B
M	0	B^t	D

Here $L \oplus S = S^\perp$, M is an additive complement of $S^\perp$, chosen so that L and M are mutually annihilating under $\pitchfork$. This can be done by an appropriate change of basis if necessary, because the pairing is nonsingular on L. The letters A, B, D represent matrices of intersection numbers, and B^t is the transpose of B. It appears because the intersection pairing is symmetric. Furthermore, the form on $S \oplus M$ must be nonsingular, because the entire pairing on K is nonsingular, and the annihilator of S in $S \oplus M$ must be $(S \oplus M) \cap S^\perp = S$. So S is self-annihilating on $S \oplus M$, which implies that the signature of the pairing on $S \oplus M$ is[7] 0. It readily follows that $\sigma(K) = \sigma(L)$.

[6] Clearly $S \subset (S^\perp)^\perp$, and then we can apply dimension counting $(\dim(S^\perp = \dim(K) - \dim(S))$ since K is finite-dimensional.

[7] It is a standard fact about nonsingular bilinear forms that their signatures are 0 if they possess a subspace U such that $U = U^\perp$, but it is harder than expected to find a clear, concise proof in the expository topology literature. Thus we include a brief proof here, owing largely to the treatment in [8].

Suppose we have a nonsingular symmetric form $(\cdot,\cdot)$ on the finite dimensional vector space V. Let

Finally, we want to identify $\sigma(L)$ with the indicated Maslov triple index. To do this, we first will identify L with a space W defined using the spaces A, B, and C that occur in the Maslov index. For this part of the proof, will refer to the following commutative diagram of long exact sequences derived from the diagram (3).

$$\begin{array}{ccccccccc}
\longrightarrow & I^{\bar p}H_{2n+1}(X,Z;\mathbb{Q}_0) & \xrightarrow{\delta} & I^{\bar p}H_{2n}(Z;\mathbb{Q}_0) & \xrightarrow{(i_{Z\subset X})_*} & I^{\bar p}H_{2n}(X;\mathbb{Q}_0) & \longrightarrow \\
& \downarrow{\scriptstyle (i_{\bar p\to\bar q})_*} & & \downarrow{\scriptstyle (i_{\bar p\to\bar q})_*} & & \downarrow{\scriptstyle (i_{\bar p\to\bar q})_*} & \\
\longrightarrow & I^{\bar q}H_{2n+1}(X,Z;\mathbb{Q}_0) & \xrightarrow{\delta} & I^{\bar q}H_{2n}(Z;\mathbb{Q}_0) & \xrightarrow{(i_{Z\subset X})_*} & I^{\bar q}H_{2n}(X;\mathbb{Q}_0) & \longrightarrow \\
& \downarrow{\scriptstyle (\pi_{\bar q/\bar p})_*} & & \downarrow{\scriptstyle (\pi_{\bar q/\bar p})_*} & & \downarrow{\scriptstyle (\pi_{\bar q/\bar p})_*} & \\
\longrightarrow & I^{\bar q/\bar p}H_{2n+1}(X,Z;\mathbb{Q}_0) & \xrightarrow{\delta} & I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) & \xrightarrow{(i_{Z\subset X})_*} & I^{\bar q/\bar p}H_{2n}(X;\mathbb{Q}_0) & \longrightarrow \\
& \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & \\
\longrightarrow & I^{\bar p}H_{2n}(X,Z;\mathbb{Q}_0) & \xrightarrow{\delta} & I^{\bar p}H_{2n-1}(Z;\mathbb{Q}_0) & \xrightarrow{(i_{Z\subset X})_*} & I^{\bar p}H_{2n-1}(X;\mathbb{Q}_0) & \longrightarrow .
\end{array}$$

Recall again that the "connecting map" d can be described as acting on a representative chain x by taking x to its boundary ∂x, that an element of $I^{\bar q/\bar p}H_*(X,Y;G_0))$ can be represented by a chain x such that $\partial x = a + b$ with b a $\bar q$-allowable chain in Y and a a $\bar p$-allowable chain in X, and, in this case, δx is represented by b.

As above, let

$$A = \ker((i_{Z\subset Y_1})_* : I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar q/\bar p}H_{2n}(Y_1;\mathbb{Q}_0)), \tag{6}$$

$$C = \ker((i_{Z\subset Y_2})_* : I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar q/\bar p}H_{2n}(Y_2;\mathbb{Q}_0)), \tag{7}$$

and

$$B = \ker(d : I^{\bar q/\bar p}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar p}H_{2n-1}(Z;\mathbb{Q}_0)), \tag{8}$$

and define

$$W = \frac{B\cap(C+A)}{B\cap C + B\cap A}. \tag{9}$$

The reader may want to refer back to Section 2.2 to recall how W comes into Wall's Maslov index.

V_+, V_- be the maximal positive definite, respectively negative definite, subspaces of V. Then by definition, the signature of the form is $\sigma = \dim(V_+) - \dim(V_-)$. Let U be a subspace such that $U^\perp = U$. Since $\dim(U) + \dim(U^\perp) = \dim(V)$, $\dim(U) = \frac{1}{2}\dim(V)$. Clearly also $U \cap V_+ = U \cap V_- = 0$. But then $\dim(V) \geq \dim(U) + \dim(V_+) - \dim(U\cap V_+) = \dim(U) + \dim(V_+)$ and $\dim(V) \geq \dim(U) + \dim(V_-) - \dim(U\cap V_-) = \dim(U) + \dim(V_-)$ by the inclusion/exclusion formula. Thus $\frac{1}{2}\dim(V) \geq \dim(V_+)$ and $\frac{1}{2}\dim(V) \geq \dim(V_-)$. But by diagonalizing the form, it follows easily from non-singularity that $\dim(V_+) + \dim(V_-) = \dim(V)$. This forces $\dim(V_+) = \dim(V_-) = \frac{1}{2}\dim(V)$. Thus the signature must be 0.

We seek to define a map $f : S^\perp \to W$. To begin, for $[z_1] \in S^\perp$, define an element $\tilde{f}([z_1]) \in I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$ as follows. If $[z_1] \in S^\perp \cong I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0) \cap \operatorname{im}(I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0))$, then $[z_1]$ is an element of $I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0)$, and we can lift $[z_1]$ (non-uniquely) to $[z_2] \in I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0)$. Let $\tilde{f}([z_1]) = (\pi_{\bar{q}/\bar{p}})_*([z_2]) \in I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$.

Proposition 4.9. *For every $[z_1] \in S^\perp$, $\tilde{f}([z_1]) \in B \cap (C + A)$. Further, up to elements of $B\cap A + B\cap C$, $\tilde{f}([z_1])$ is independent of the choice of $[z_2]$ made in the definition, so $\tilde{f}$ defines a map $f : S^\perp \to W$. The map f is a homomorphism.*

Proof. We begin by demonstrating that $\tilde{f}([z_1]) \in B\cap(C+A)$. First, $\tilde{f}([z_1])$ is by construction in $\operatorname{im}((\pi_{\bar{q}/\bar{p}})_*)$, so by exactness of the $\bar{q}/\bar{p}$ sequence, it lies in B. To see that $\tilde{f}([z_1])$ also lies in $A + C$, we note that

$$A \cong \operatorname{im}(\delta : I^{\bar{q}/\bar{p}}H_{2n+1}(Y_1, Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0))$$

and

$$C \cong \operatorname{im}(\delta : I^{\bar{q}/\bar{p}}H_{2n+1}(Y_2, Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0))$$

by the long exact sequence (5). So

$$\begin{aligned} A + C &= \operatorname{im}(\delta + \delta : I^{\bar{q}/\bar{p}}H_{2n+1}(Y_1, Z;\mathbb{Q}_0) \oplus I^{\bar{q}/\bar{p}}H_{2n+1}(Y_2, Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)) \\ &\cong \operatorname{im}(\delta : I^{\bar{q}/\bar{p}}H_{2n+1}(X, Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)). \end{aligned} \tag{10}$$

Now let us go through the definition of $\tilde{f}$ again carefully, referring to the diagram above. If $[z_1] \in S^\perp$, then by Corollary 4.7, z_1 can be represented by a $\bar{p}$-allowable cycle, a, in X (representing a class in $I^{\bar{p}}H_{2n}(X;\mathbb{Q}_0)$) that is $\bar{q}$-homologous to a $\bar{q}$-allowable cycle, b, in Z which represents the class $[z_2]$ in the definition of $\tilde{f}$. So $(i_{\bar{q}/\bar{p}})_*([a]) = (i_{Z\subset X})_*([b])$ in $I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0)$. Let ξ be a $\bar{q}$-allowable $2n+1$ chain in X realizing such a homology, i.e. $\partial\xi = b - a$. Since a is $\bar{p}$-allowable in X and b is $\bar{q}$-allowable in Z, ξ represents an element of $I^{\bar{q}/\bar{p}}H_{2n+1}(X, Z;\mathbb{Q}_0)$. By definition of δ, the image $\delta(\xi)$ in $I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$ is also represented by b, and so is $(\pi_{\bar{q}/\bar{p}})_*[b] = \tilde{f}([z_1])$. Thus $\tilde{f}([z_1]) \in \operatorname{im}(\delta) = A + C$.

Next, suppose that $[z_2], [z_2'] \in I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0)$ are two choices of lifts for the same $[z_1] \in S^\perp \subset \operatorname{im}(I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0))$, i.e., $(i_{Z\subset X})_*([z_2]) = (i_{Z\subset X})_*([z_2']) = [z_1]$. Let $\tilde{f}([z_1]) := (\pi_{\bar{q}/\bar{p}})([z_2])$ and $\tilde{f}'([z_1]) := (\pi_{\bar{q}/\bar{p}})([z_2'])$. We need to show that $\tilde{f}([z_1]) - \tilde{f}'([z_1]) \in B \cap A + B \cap C$.

We have

$$\begin{aligned} [z_2] - [z_2'] &\in \operatorname{im}(\delta : I^{\bar{q}}H_{2n+1}(X, Z;\mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0)) \\ &\cong \operatorname{im}(\delta + \delta : I^{\bar{q}}H_{2n+1}(Y_1, Z;\mathbb{Q}_0) \oplus I^{\bar{q}}H_{2n+1}(Y_2, Z;\mathbb{Q}_0) \to I^{\bar{q}}H_{2n}(Z;\mathbb{Q}_0)). \end{aligned}$$

Let $[x_1], [x_2]$ be elements, respectively, of $I^{\bar{q}}H_{2n+1}(Y_1, Z;\mathbb{Q}_0)$, $I^{\bar{q}}H_{2n+1}(Y_2, Z;\mathbb{Q}_0)$ such that $\delta[x_1] + \delta[x_2] = [z_2] - [z_2']$. Then $\delta \circ (\pi_{\bar{q}/\bar{p}})_*([x_1]) \in B \cap A$ and $\delta \circ (\pi_{\bar{q}/\bar{p}})_*([x_2]) \in B \cap C$. So $(\pi_{\bar{q}/\bar{p}})([z_2] - [z_2']) = (\pi_{\bar{q}/\bar{p}}) \circ \delta([x_1] + [x_2]) \in B \cap A + B \cap C$ as required.

Putting our arguments so far together, we see that f is well-defined as a function from $S^\perp$ to W. But f is then also a homomorphism since for a sum $[z_1] + [z_1']$, we can certainly find a lift of the form $[z_2] + [z_2']$, we have just shown that this choice is acceptable and does not affect the image, and the other maps are all homomorphisms. ∎

Now we can let $f([z_1]) = [z_3]$, where by an abuse of notation, $[z_3]$ is also taken to be the class that z_3 represents in W as $\tilde{f}([z_1])$.

Proposition 4.10. *The map f surjects onto W with kernel S, hence $L \cong W$.*

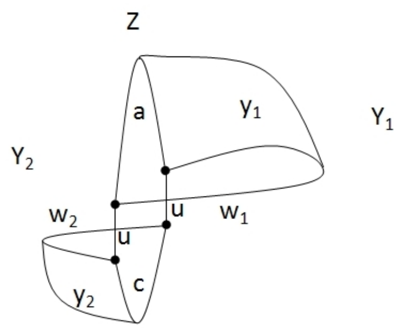

Figure 2: A schematic for the argument that f is surjective.

Proof. First we show that $f : S^{\perp} \to W$ is surjective. Suppose $[x]$ is a class in $B \cap (A + C)$. Since $[x] \in B$, there exists an $[x_2] \in I^{\bar{q}}H_{2n}(Z; \mathbb{Q}_0)$ with $(\pi_{\bar{q}/\bar{p}})_*[x_2] = [x]$. To get surjectivity of f, we need to show there is a $[v] \in I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0)$ such that $(i_{\bar{p}\to\bar{q}})_*[v] = (i_{Z\subset X})_*[x_2] := [x_1]$. Then $f[x_1] = [x]$.

It might aid the reader to refer to the schematic in Diagram (4.1) during the following argument.

Since $[x] \in A + C$, we can write $[x] = [a] + [c]$, where $[a] \in A$ and $[c] \in C$. Since $[a] \in A$, it is the image under δ of some $[y_1]$, which may be represented by a $\bar{q}$-allowable chain y_1 with support in Y_1 such that $\partial y_1 = a + w_1$ and w_1 is a $\bar{p}$-allowable relative $2n$ chain in Y_1. Similarly, there is a $\bar{q}$-allowable chain y_2 with support in Y_2 such that $\partial y_2 = c + w_2$ and w_2 is a $\bar{p}$-allowable $2n$ chain in Y_2. Then $d[y_1] = [w_1]$ and $d[y_2] = [w_2]$. Since x is also in B, $d[x] = [\partial x] = [0] \in I^{\bar{p}}H_{2n-1}(Z; \mathbb{Q}_0)$, so there is a $\bar{p}$-allowable chain u in Z such that $\partial u = \partial x = \partial a + \partial c$. Now consider the $\bar{p}$-allowable $2n$-chain $v = w_1 + u + w_2$. We have

$$\begin{aligned}\partial(w_1 + u + w_2) &= \partial w_1 + \partial u + \partial w_2 \\ &= -\partial a + \partial a + \partial c - \partial c \\ &= 0,\end{aligned}$$

so v represents a class $[v] \in I^{\bar{p}}H_{2n}(X; \mathbb{Q}_0)$, and

$$\begin{aligned}\partial(y_1 + y_2) &= \partial y_1 + \partial y_2 \\ &= a + w_1 + c + w_2 \\ &= a + c - u + w_1 + u + w_2.\end{aligned}$$

In other words, $y_1 + y_2$ provides a $\bar{q}$-allowable homology from the $\bar{p}$-allowable cycle $-(w_1+u+w_2)$ in X to the $\bar{q}$-allowable cycle $a+c-u$ in Z. This means that $(i_{\bar{p}\to\bar{q}})_*[v] = (i_{Z\subset X})_*[a+c-u]$, so we can set $[x_2] = [a + c - u]$ and we get that $[x_1] = (i_{Z \subset X})_*[a + c - u] \in S^\perp$ by Corollary 4.7. Finally, since u is $\bar{p}$-allowable, we have that $(\pi_{\bar{q}/\bar{p}})_*[x_2] = (\pi_{\bar{q}/\bar{p}})_*[a+c-u] = [a+c] = [x]$ as desired.

Lastly, we must show that $\ker f = S$, which will suffice, as L is an additive complement of S in $S^\perp$. Suppose $[x] \in S = K \cap \operatorname{im}(I^{\bar{p}}H_{2n}(Y_1;\mathbb{Q}_0) \oplus I^{\bar{p}}H_{2n}(Y_2;\mathbb{Q}_0) \to I^{\bar{p}\to\bar{q}}H_{2n}(X;\mathbb{Q}_0))$. Then we can write $[x] = (i_{\bar{p}\to\bar{q}})_*[x_1 + x_2]$, where each x_i is a $\bar{p}$-allowable cycle with support in Y_i. Since $[x] \in K$, we get $(\pi_{X,Z})_*[x] = [0] \in I^{\bar{q}}H_{2n}(Y_1, Z;\mathbb{Q}_0) \oplus I^{\bar{q}}H_{2n}(Y_2, Z;\mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(X, Z;\mathbb{Q}_0)$ (or else its image would be nontrivial in $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1, Z;\mathbb{Q}_0) \oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2, Z;\mathbb{Q}_0)$ and thus it could not be orthogonal to this group, on which the intersection pairing is nonsingular). So the cycle $x_1 + x_2$ which we can take to represent $[x]$ is $\bar{q}$-homologous to a cycle in Z. But using that $I^{\bar{q}}H_{2n}(X, Z;\mathbb{Q}_0) \cong I^{\bar{q}}H_{2n}(Y_1, Z;\mathbb{Q}_0) \oplus I^{\bar{q}}H_{2n}(Y_2, Z;\mathbb{Q}_0)$, it follows that each of x_1 and x_2 is individually $\bar{q}$-homologous to a $\bar{q}$-allowable cycle in Z. Thus $[x_1]$ and $[x_2]$ are each individually elements of K and so are individually elements of S. We claim that $f([x_1]) \in B \cap A$ and $f([x_2]) \in B \cap C$; the proofs are the same so we will just show the first. Let y_1 be a $\bar{q}$-homology in Y_1 from x_1 to a $\bar{q}$-chain x_1' with support in Z. Then x_1' represents $f([x_1])$, and it is clear that $f([x_1]) \in B$ since x_1' is a cycle. But it is also clear that $[x_1'] \in A$, since $\partial y_1 = x_1' - x_1$, which is the sum of a $\bar{q}$-allowable chain on Z and a $\bar{p}$-allowable chain on Y_1, thus y_1 is a cycle in $I^{\bar{q}/\bar{p}}H_{2n+1}(X, Z;\mathbb{Q}_0)$ and $\delta[y_1] = [x_1']$. It follows now that $f(S) = 0 \in W$.

Conversely, suppose $[x] \in S^\perp$ and that $f([x]) = 0 \in W$, i.e. $f([x]) \in B \cap A + B \cap C$. We will show that $[x] \in S$; the reader may want to refer to Diagram (4.1) for a schematic of the construction. Since $[x] \in S^\perp$, we can assume by Corollary 4.7 that $[x]$ is represented by a $\bar{q}$-allowable cycle x supported in Z, and this same chain represents $f([x])$. Since $f([x]) \in B \cap A + B \cap C \subset I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$, there is a $\bar{q}$-allowable chain z supported in Z such that $\partial z = x - (x_1 + x_2) - u$, where $[x_1] \in B \cap A$, $[x_2] \in B \cap C$, and u is $\bar{p}$ allowable in Z. Note that x_1 and $x_1 + u$ represent the same element of $B \cap A$, so we can represent $f([x])$ as $[x_1 + u] + [x_2]$ in $B \cap A + B \cap C$. Since $[x_1 + u] \in B$, $0 = d[x_1 + u] = [\partial(x_1 + u)] \in I^{\bar{p}}H_{2n-1}(Z;\mathbb{Q}_0)$, so there is a $2n$-dimensional $\bar{p}$-chain w in Z such that $\partial w = \partial(x_1 + u)$, which are both $\bar{p}$-allowable. Notice also that $x_1 + u - w$ is a cycle in the usual sense (its boundary is identically 0), and $u - w$ is $\bar{p}$-allowable, so $[x_1 + u - w] = [x_1] \in B \cap A$. Because this class is also in A, there is a $2n + 1$ dimensional $\bar{q}$-chain y_1 in Y_1 such that $\partial y_1 = x_1 + u - w - p_1$, where p_1 is $\bar{p}$-allowable. Notice that $0 = \partial\partial y_1 = \partial(x_1 + u - w) + \partial p_1 = \partial p_1$, so $\partial p_1 = 0$, and p_1 is a cycle in Y_1. But now we observe that $[p_1]$ is in S: it is represented by a $\bar{p}$-cycle in Y_1, and since it is $\bar{q}$-homologous by y_1 to a cycle supported in Z, it is orthogonal to $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_1, Z) \oplus I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(Y_2, Z)$. Now observe that $\partial\partial z = 0$ and $\partial x = 0$, so that $\partial(x_1 + u) = -\partial x_2$. Thus by a similar argument, there is a $[p_2] \in S$ represented by a cycle p_2 in Y_2 that is $\bar{q}$-homologous by some y_2 to the cycle $x_2 + w$. Putting these together, $p_1 + p_2$ is $\bar{q}$-homologous to $x_1 + u - w + x_2 + w = x_1 + u + x_2$. But we have already seen that $x_1 + u + x_2$ is $\bar{q}$-homologous to x, and so $[p_1 + p_2] = [x] \in I^{\bar{q}}H_{2n}(X;\mathbb{Q}_0)$. Thus $[x] \in S$.

□

Finally, we must show that, under our isomorphism $L \cong W$, the signature of L becomes the Maslov index $\sigma(V; A, B, C)$ associated with the pairing Φ on $V = I^{\bar{q}/\bar{p}}H_{2n}(Z;\mathbb{Q}_0)$. For

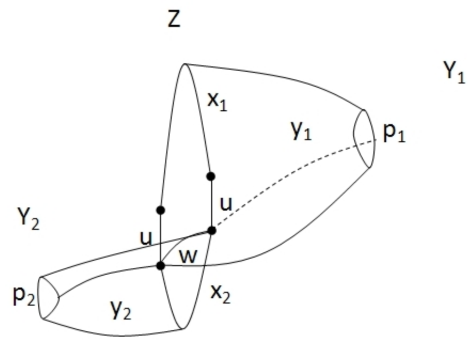

Figure 3: A schematic for the argument that $\ker(f) \subset S$.

this Maslov triple index to make sense, we need the spaces A, B, and C to be self-annihilating subspaces of V under the pairing $\Phi([x],[y]) := x \pitchfork \partial y + (-1)^{n-|x|}(\partial x) \pitchfork y$, so we need the following lemma.

Lemma 4.11. $\Phi(A \times A) = \Phi(B \times B) = \Phi(C \times C) = 0.$

Proof. It is clear that $\Phi(B \times B) = 0$, for if $[x] \in B$, then $[x] \in im(I^{\bar{q}}H_i(Z;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_i(Z;\mathbb{Q}_0))$. So $[x]$ can be represented as a $\bar{q}$-cycle, i.e. $\partial x = 0$. Thus if $[x],[y] \in B$, certainly $\Phi([x],[y]) = 0$ from the definition.

Now suppose $[x],[y] \in A$ are represented by $\bar{q}$-allowable chains x, y in stratified general position in Z. The fact that $[x],[y] \in A$ means that there exist $\bar{q}$-allowable $2n+1$-chains ξ, η in Y_1 such that $\partial\xi = x+u$, $\partial\eta = y+v$, and u, v are $\bar{p}$-allowable chains in Y_1. We can assume that ξ and η are in stratified general position rel Z and that in a collared neighborhood of Z, ξ looks like $[0,1] \times x$ and η looks like $[0,1] \times y$. Consider the chain $\xi \pitchfork v - u \pitchfork \eta$ in Y_1. Since u and v are $\bar{p}$-allowable and η and ξ are $\bar{q}$-allowable, this is a well-defined $\bar{t}$-allowable 1-chain. Next we compute, using $\pitchfork_{Y_1}$ to denote intersection numbers in Y_1 and $\pitchfork_Z$ to denote those in Z:

$$\begin{aligned}
\partial(\xi \pitchfork_{Y_1} v - u \pitchfork_{Y_1} \eta) &= (\partial\xi) \pitchfork_{Y_1} v + (-1)^{4n-|\xi|}\xi \pitchfork_{Y_1} \partial v - (\partial u) \pitchfork_{Y_1} \eta - (-1)^{4n-|u|}u \pitchfork_{Y_1} \partial\eta \\
&= (\partial\xi) \pitchfork_{Y_1} v - \xi \pitchfork_{Y_1} \partial v - (\partial u) \pitchfork_{Y_1} \eta - u \pitchfork_{Y_1} \partial\eta \\
&= (x+u) \pitchfork_{Y_1} v + \xi \pitchfork_{Y_1} \partial y + \partial x \pitchfork_{Y_1} \eta - u \pitchfork_{Y_1} (y+v) \\
&= x \pitchfork_{Y_1} v + u \pitchfork_{Y_1} v + \xi \pitchfork_{Y_1} \partial y + \partial x \pitchfork_{Y_1} \eta - u \pitchfork_{Y_1} y - u \pitchfork_{Y_1} v \\
&= \xi \pitchfork_{Y_1} \partial y + \partial x \pitchfork_{Y_1} \eta \\
&= x \pitchfork_Z \partial y + (-1)^{|\partial x|}\partial x \pitchfork_Z y \\
&= x \pitchfork_Z \partial y - \partial x \pitchfork_Z y \\
&= \Phi([x],[y]).
\end{aligned}$$

Here was have used that Z is $4n-1$ dimensional, Y_1 is $4n$-dimensional, x, y, u, v are $2n$-dimensional, and ξ, η are $2n+1$ dimensional. We have also used the geometrically clear fact that x does not intersect v and y does not intersect u, which follows from x and y being in stratified general position and our collar assumptions on ξ and η. For the sign conventions relating intersection numbers in Y_1 with those in Z, see the Appendix. We conclude from this argument that the intersection number $\Phi(x, y)$ must be 0, as it represents the boundary of a 1-chain in Y_1. Thus $\Phi(A \times A) = 0$. An analogous argument shows that $\Phi(C \times C) = 0$. □

Now we can relate the intersection pairing on $L \subset I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0)$ to the pairing Φ on $V = I^{\bar{q}/\bar{p}}H_{2n}(Z; \mathbb{Q}_0)$. Suppose that $[x], [y] \in L \subset S^{\perp}$. Then $[x]$ and $[y]$ can be represented by $\bar{q}$-allowable cycles x and y in Z that are homologous via $\bar{q}$-allowable chains χ and γ in X to $\bar{p}$ allowable cycles $\tilde{x}$ and $\tilde{y}$ in X. By definition, $[x] \pitchfork [y] = \tilde{x} \pitchfork_X y$.

The representatives x and y descend also to represent classes $[x]$ and $[y]$ in $B \cap (A+C) \subset I^{\bar{q}/\bar{p}}H_{2n}(Z; \mathbb{Q}_0)$, which in turn represent $f([x])$ and $f([y])$ in W. Since $[y] \in A + C$, we can write $[y] = [a] + [c] \in I^{\bar{q}/\bar{p}}H_{2n}(Z; \mathbb{Q}_0)$, where $[a] \in A$, $[c] \in C$ are represented by $\bar{q}-$allowable chains in Z, and y is $\bar{q}$ homologous to $a + c + w$ for some $\bar{p}$-allowable chain w on Z. Since $[a] \in A$ and $[c] = [c + w] \in C$, there exist $\bar{q}$-allowable chains $\xi \in Y_1$ and $\eta \in Y_2$ such that $\partial\xi = a + u$, $\partial\eta = c + v + w$, and u, v are $\bar{p}$-allowable chains in Y_1 and Y_2, respectively with boundaries in Z. We can further assume that in the collar neighborhood of Z, ξ and η have a product structure $[-1, 0] \times a$ and $[0, 1] \times -v-w$ and that all chains are in stratified general position. Observe that $a + w + c$ is $\bar{q}$-homologous to the $\bar{p}$-cycle $-u - v$ via $\xi + \eta$.

Now again consider the pairing $[x] \pitchfork [y]$ in $I^{\bar{p}\to\bar{q}}H_{2n}(X; \mathbb{Q}_0)$. We have $\tilde{x} \pitchfork_X y = \tilde{x} \pitchfork_X (a+c+w) = \tilde{x} \pitchfork_X -u-v$. Now we have a $\bar{p}$ allowable chain on the right, so we can replace $\tilde{x}$ with the $\bar{q}$ allowable chain, $x \subset Z$ to which it is $\bar{q}$-homotopic to obtain $x \pitchfork_X -u-v$. By pushing x into Y_1 along the collar and using the product structure of u near Z, we get this is equal to the intersection $x \pitchfork_X -u = x \pitchfork_Z -\partial u = x \pitchfork_Z \partial a$. But this is precisely the pairing Φ on $I^{\bar{q}/\bar{p}}H_{2n}(Z; \mathbb{Q}_0)$, $\Phi([x], [a]) := x \pitchfork_Z \partial a + (-1)^{n-|x|}(\partial x) \pitchfork_Z a$ because x is a cycle, and by definition this is in turn equal to $\Psi(f([x]), f([y]))$ on W. Thus the intersection pairing on L may be identified with the pairing Ψ on W as desired.

To check the sign in this last equality, we must be careful about which roles A and C are playing. Certainly we have $\frac{B\cap(A+C)}{B\cap A+B\cap C} \cong \frac{B\cap(C+A)}{B\cap C+B\cap A}$ as spaces, but A and C play different roles in the pairing. In Wall, the choice of which plays which role is determined so that A is associated to the half of the space whose boundary orientation agrees with the orientation of the intersection and C is associated with the space whose boundary has the opposite orientation of that assigned to the intersection. Thus we can let Wall's A correspond to ours and Wall's C corresponds to ours, and we can also use the order B, C, A for these subspaces. So, since $[x]$ represents an element of B, our $\Phi([x], [a])$ corresponds to Wall's

$$-\Phi(\text{element of first subspace}, \text{element of last subspace}),$$

where the negative sign comes from our choice of $y = a+c$ rather than the $y+a+c = 0$ that Wall uses. So, using (2), this is $\Psi([x], [y])$. Thus the intersection pairing restricted to $S^{\perp}$ is taken to Wall's pairing Ψ determined from Φ on W, and we conclude by Wall's definition that $\sigma(L) = \sigma(W; A, B, C)$.

This completes the proof of the Theorem 4.1. □

4.2 Pseudomanifolds with boundary

If we start with a pseudomanifold X without boundary and decompose it as $Y_1 \cup_Z Y_2$ along a pseudomanifold Z which is not a stratum of X, then Y_1 and Y_2 with the subspace stratification induced from X are pseudomanifolds with boundary. We would like to be able to further decompose X by cutting the Y_i into pieces, but to do this, we need a version of our non-additivity theorem for a stratified pseudomanifold X (here, our Y_i's) with boundary. To get this result as a corollary of Theorem 4.1, we use the restratification trick we discussed in Section 3.1.

For intuition, consider Figure 4.2 below, where X^{4k} is a compact oriented stratified pseudomanifold with boundary $\partial X = W$ such that $X = Y_1 \cup_Z Y_2$. Assume that Y_1, Y_2 are compact oriented stratified pseudomanifolds with boundary such that $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2 = Z$ is a bicollared (in X) pseudomanifold with boundary such that $\partial Y_1 = Z - W_1$, $\partial Y_2 = W_2 - Z$, and $\partial W_1 = \partial W_2 = \partial Z$. Also assume that $(W_1, \partial W_1) \subset (Y_1, Z)$ and $(W_2, \partial W_2) \subset (Y_2, Z)$, $(Z, \partial Z) \subset (Y_1, W_1)$, and $(Z, \partial Z) \subset (Y_2, W_2)$ are collared as pairs. Note that $\partial X = W_2 - W_1$. The orientations are chosen to agree with Wall's conventions in [45] (see also Section 5.1, below).

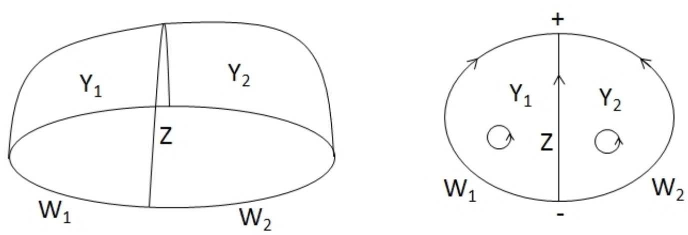

Figure 4: A schematic of a splitting of a pseudomanifold with boundary (left) and a flatter schematic of the relative orientations (right).

Note that since X is a pseudomanifold with boundary W, W is *not* a union of strata of X. We begin by restratifying X so that W becomes a union of strata, and we obtain a stratified pseudomanifold without boundary which we will denote by $\hat{X}$ (remember, though, that $X = \hat{X}$ as topological spaces). The strata of $\hat{X}$ are defined as follows:

1. for each stratum S of X, then $S \cap X - W$ is a stratum of $\hat{X}$,

2. for each stratum S of X such that $S \cap W \neq \emptyset$, then $S \cap W$ is a stratum of $\hat{X}$.

It is not hard to see that $\hat{X}$ is a PL stratified pseudomanifold. In fact, certainly X and $\hat{X}$ agree off W, and if $N \cong W \times [0,1] \subset \hat{X}$ is a collared neighborhood of W with $W = W \times \{1\}$, then under the subspace stratification from $\hat{X}$, N is stratified as the product of W, with its stratification inherited from X, and $[0,1]$ with the stratification $[0,1] \supset \{1\}$. Let $\hat{Y}_i$, $\hat{Z}$ and $\hat{W}_i$ be the restratifications of Y_i, Z and W_i as subspaces of $\hat{X}$. Note that, with these stratifications, $\hat{X}$ and $\hat{Z}$ are PL stratified pseudomanifolds *without boundary*, while $\partial\hat{Y}_1 = \hat{Z}$ and $\partial\hat{Y}_2 = -\hat{Z}$.

Suppose $\bar p, \bar q$ are perversities on X, and induced also on the subspaces Y_i. Let $\hat p$ be the perversity on $\hat Y_i$ that agrees with $\bar p$ on $Y_i - W_i$ and is such that $\hat p(S) < 0$ for all $S \subset \hat W_i$. Let $\bar q = \bar t - \bar p$. Note that then $\hat q(S) > \bar t(S)$ for all $S \subset \hat W_i$. Then we get the following isomorphisms of intersection homology groups.

Lemma 4.12. *1. $I^{\hat p}H_*(\hat X; G_0) \cong I^{\bar p}H_*(X; G_0)$ and $I^{\hat q}H_*(\hat X; G_0) \cong I^{\bar q}H_*(X, \partial X; G_0)$,*

2. $I^{\hat p}H_*(\hat Z; G_0) \cong I^{\bar p}H_*(Z; G_0)$ and $I^{\hat q}H_*(\hat Z; G_0) \cong I^{\bar q}H_*(Z, \partial Z; G_0)$,

3. $I^{\hat p}H_*(\hat Y_i; G_0) \cong I^{\bar p}H_*(Y_i; G_0)$ and $I^{\hat q}H_*(\hat Y_i, \hat Z; G_0) \cong I^{\bar q}H_*(Y_i, \partial Y_i; G_0)$.
Therefore

1. $I^{\hat p \to \hat q}H_*(\hat X; G_0) \cong I^{\bar p \twoheadrightarrow \bar q}H_*(X, \partial X; G_0)$,

2. $I^{\hat p \to \hat q}H_*(\hat Z; G_0) \cong I^{\bar p \twoheadrightarrow \bar q}H_*(Z, \partial Z; G_0)$,

3. $I^{\hat p \twoheadrightarrow \hat q}H_*(\hat Y_i, \hat Z; G_0) \cong I^{\bar p \twoheadrightarrow \bar q}H_*(Y_i, \partial Y_i; G_0)$.

Furthermore, these last isomorphisms preserve the intersection pairing when G is a ring.

Proof. We will show the proof for $\hat Y_i$; the others are the same (though easier without the extra boundary component).

By [21, Lemma 2.4], we may assume $\hat p$ to be arbitrarily negative on $\hat W_i$. Therefore, it follows from the definition that no $\hat p$-allowable simplex can intersect $\hat W_i$. Thus $I^{\hat p}H_*(\hat Y_i; G_0) \cong I^{\hat p}H_*(\hat Y_i - \hat W_i; G_0) \cong I^{\bar p}H_*(Y_i - W_i; G_0) \cong I^{\bar p}H_*(Y_i; G_0)$, the last isomorphism by stratum-preserving homotopy equivalence.

Next, by [21, Lemma 2.4], we might assume $\hat q$ to be arbitrarily large on $\hat W_i$. Thus there is no impediment to chains intersecting W_i. Thus in the neighborhood N of $\hat W_i$ that is the product of W_i with $(0, 1] \supset \{1\}$, all allowable chains are homologous by product homologies to chains in $\hat W_i$. But as $\hat W_i$ consists entirely of singular strata, the coefficient system is 0 there, and so $I^{\hat q}H_*(N; G_0) = 0$ and similarly $I^{\hat q}H_*(N, N \cap Z_0; G_0) = 0$. The isomorphism $I^{\hat q}H_*(\hat Y_i, \hat Z; G_0) \cong I^{\bar q}H_*(Y_i, \partial Y_i; G_0)$ now follows by some easy arguments from the Mayer-Vietoris sequence for the pair consisting of $(N, N \cap Z_0)$ and $(\hat Y_i - \hat W_i, \hat Z - \hat W_i \cap \hat Z)$.

The rest of the lemma follows easily. □

Let $\hat V = I^{\hat q/\hat p}H_{2n}(\hat Z; \mathbb{Q}_0)$ equipped with the anti-symmetric pairing Φ. Let

$$\hat A = \ker(I^{\hat q/\hat p}H_{2n}(\hat Z; \mathbb{Q}_0) \to I^{\hat q/\hat p}H_{2n}(\hat Y_1; \mathbb{Q}_0)),$$

$$\hat C = \ker(I^{\hat q/\hat p}H_{2n}(\hat Z; \mathbb{Q}_0) \to I^{\hat q/\hat p}H_{2n}(\hat Y_2; \mathbb{Q}_0)),$$

and

$$\hat B = \ker(d : I^{\hat q/\hat p}H_{2n}(\hat Z; \mathbb{Q}_0) \to I^{\hat p}H_{2n-1}(\hat Z; \mathbb{Q}_0)).$$

Corollary 4.13.

$$\sigma_{\bar p \twoheadrightarrow \bar q}(X) = \sigma_{\bar p \twoheadrightarrow \bar q}(Y_1) + \sigma_{\bar p \twoheadrightarrow \bar q}(Y_2) + \sigma(\hat V, \hat A, \hat B, \hat C).$$

Proof. The corollary follows from Theorem 4.1 and the preceding lemma. □

It is reasonable to ask the following question: Is it possible to identify $\sigma(\hat{V};\hat{A},\hat{B},\hat{C})$ as an invariant of a pairing involving only subspaces of intersection homology groups associated with Z? Unfortunately, the obvious choices do not seem to be correct. For example, $\sigma(\hat{V},\hat{A},\hat{B},\hat{C})$ cannot be the signature of the pairing Φ on $\mathrm{im}(I^{\bar{q}/\bar{p}}H_{2n}(Z) \to I^{\bar{q}/\bar{p}}H_{2n}(Z,\partial Z))$, which would be a natural guess. To see this, suppose all spaces are manifolds with boundary as in Wall's original non-additivity theorem. In this case, Z is a manifold and $I^{\bar{q}/\bar{p}}H_{2n}(Z) = 0$. Thus this term would always have to be 0, which is certainly not true. Another natural guess would be that $\sigma(\hat{V};\hat{A},\hat{B},\hat{C})$ would be expressible in terms of a pairing on the intersection homology of ∂Z. However, this cannot be, as Theorem 4.1 should be a special case of Corollary 4.13 in which all boundaries (except for the intersection Z itself) are empty. In such a case, any groups associated with ∂Z would vanish, and this would violate the existence of the Maslov index term in Theorem 4.1.

Remark 4.14. Rather than restratifying as we have done, it is tempting to do "the usual thing" and treat pseudomanifolds with boundary by simply adding a cone on the boundary and working with the resulting space. However, that will not quite do here, as $Z \cup_{\partial Z} c(\partial Z)$ will not generally be bicollared in $X \cup_{\partial X} c(\partial X)$.

One alternative would be the following construction. Beginning with the bi-collared $Z \times [0,1] \subset X$, consider $X' = X \cup_{\partial Z \times [0,1]} (c(\partial Z) \times [0,1])$. Then X' has boundaries homeomorphic to $W_1 \cup_{\partial Z} c(\partial Z)$ and $W_2 \cup_{\partial Z} c(\partial Z)$. By separately coning off these boundary components we get a space X'' that possesses strata $[0,1] \supset \{0,1\}$ and such that $X'' - [0,1]$ is homeomorphic to the interior of X. If the perversities $\bar{p}$ and $\bar{q}$ are extended so that $\bar{p}$ takes values < 0 on the new strata and $\bar{q}$ takes values $> \bar{t}$ on the new strata, then the intersection homology of X'' with these perversities is homeomorphic to that of $\hat{X}$ with respect to $\hat{p}$ and $\hat{q}$.

5 Relationship to previous work

In this section, we relate perverse signatures and our non-additivity theorem to other signatures and to Wall non-additivity.

5.1 Relation to Wall's non-additivity theorem

If we take our pseudomanifolds to be manifolds then, as expected, we recover Wall's non-additivity theorem. The relationship between our Maslov index and Wall's is not completely obvious, however, since the pairing Φ at first glance seems mysterious from the manifold point of view. We will show that, in fact, that if M is a manifold with boundary and X is the pseudomanifold obtained by coning off ∂M, then $I^{\bar{q}/\bar{p}}H_*(X)$ is just $H_{*-1}(\partial M)$ and the pairing Φ is the intersection pairing on ∂M, up to sign.

Suppose M^m is a compact manifold with boundary, ∂M. Let X denote $M \cup_{\partial M} c(\partial M)$. We suppose X is stratified as $X \supset v$, where v is the cone vertex. Let $\bar{p}, \bar{q}$ be perversities for which $\bar{p}(v) < 0$ and $\bar{q}(v) > m-2$. Then $I^{\bar{p}}H_*(X;G_0) \cong H_*(M;G)$ and $I^{\bar{q}}H_*(X;G_0) \cong$

$H_*(M, \partial M; G)$. We would then expect from the long exact sequences of the pairs that $I^{\bar{q}/\bar{p}}H_*(X; G_0) \cong H_{*-1}(\partial M; G)$. We will make this isomorphism explicit.

Suppose ξ is an j-chain in ∂M. Let $c\xi$ denote the chain obtained by coning off ξ in $c(\partial M) \subset X$. In other words, if $\xi = \sum a_i \sigma_i$, then $c\xi = \sum a_i c(\sigma_i)$, where for a simplex σ, $c(\sigma) : \Delta^{j+1} \to X$ is the cone on the map $\sigma : \Delta^j \to \partial M$ obtained by extending σ linearly to the cone vertex. We take the convention that the new vertex is the *first* vertex in $c\sigma$. With this convention, $\partial(c\xi) = \xi + c(\partial\xi)$. This coning c determines a homomorphism $c : C_{*-1}(\partial M; G) \to I^{\bar{q}/\bar{p}}C_*(X; G_0)$, since every $c\xi$ is $\bar{q}$-allowable, as can be confirmed from the definition of allowability as $(c(\sigma))^{-1}(v) \subset$ the 0-skeleton of Δ^{j+1} for every singular simplex σ in ∂M. Furthermore, c is a chain map, as $\partial(c\xi) = \xi + c(\partial\xi) = c\partial\xi \in I^{\bar{q}/\bar{p}}C_*(X; G_0)$, since ξ is $\bar{p}$-allowable. As a chain map, c induces a homomorphism on homology, which we claim is an isomorphism.

Lemma 5.1. *The homomorphism c induces an isomorphism $H_{*-1}(\partial M; G) \to I^{\bar{q}/\bar{p}}H_*(X; G_0)$.*

Proof. Consider the diagram (with coefficients suppressed)

$$\begin{array}{ccccccccc}
\longrightarrow & H_{i+1}(M, \partial M) & \longrightarrow & H_i(\partial M) & \longrightarrow & H_i(M) & \longrightarrow & H_i(M, \partial M) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & I^{\bar{q}}H_{i+1}(X) & \longrightarrow & I^{\bar{q}/\bar{p}}H_{i+1}(X) & \longrightarrow & I^{\bar{p}}H_i(X) & \longrightarrow & I^{\bar{q}}H_i(X) & \longrightarrow
\end{array}$$

It will suffice to show this diagram commutes (up to sign). The map from $H_i(M)$ to $I^{\bar{p}}H_i(X)$ is given by inclusion, the map $H_i(M, \partial M)$ to $I^{\bar{q}}H_i(X)$ is given by taking a representative ξ to $\xi - c(\partial\xi)$.

It is easy to check the the squares on the right and in the middle commute. For the square on the left, note that if ξ is a chain representing an element of $H_{i+1}(M, \partial M)$, then the image of $\xi - c(\partial\xi)$ in $I^{\bar{q}/\bar{p}}H_{i+1}(X)$ is simply $-c(\partial\xi)$ as ξ is $\bar{p}$-allowable. This is enough to establish that the left square commutes up to sign. Thus the diagram commutes up to sign and has exact rows, which is enough to establish the isomorphism via the five-lemma. □

Proposition 5.2. *If R is a ring, M is a $4n-1$ manifold the isomorphism c of the preceding lemma takes the intersection pairing $H_{2n-1}(\partial M; R) \otimes H_{2n-1}(\partial M; R) \to R$ to the pairing $-\Phi : I^{\bar{q}/\bar{p}}H_{2n}(X; R_0) \otimes I^{\bar{q}/\bar{p}}H_{2n}(X; R_0) \to R$, i.e. $x \pitchfork_{\partial M} y = -\Phi(cx, cy)$.*

Proof. Suppose $x \in H_i(\partial M; R)$, $y \in H_j(\partial M; R)$, represented by cycles in general position. Let cx be the cone on x as described above, and let cy be the cone on y except assuming that y has first been pushed outward slightly into $c\partial M \subset X$ along the cone line so that x does not intersect cy. In fact, we observe geometrically that $x \pitchfork_X cy = (\partial(cx)) \pitchfork_X cy = 0$, while $(cx) \pitchfork_X y = (cx) \pitchfork_X \partial(cy)$ must equal $x \pitchfork_{\partial M} y$ up to sign. This will establish the claim once we work out the sign.

We write out the argument in simplicial notation, which of course is not quite the actual situation, but it provides the correct intuition and reasoning. With this abuse of notation, simplices of cx have the form $[v,\sigma]=[v,v_0,\dots,v_i]$, where v is the singular point of X and the v_i are the vertices of σ, a simplex of x. The orientation here corresponds to a basis of vectors $[v,v_0],\dots,[v,v_i]$. To compare with the orientation of σ, though, it is best to note that $[v,v_0,\dots,v_i]=-[v_0,v,v_1,\dots,v_i]$. Here $[v_0,v,v_1,\dots,v_i]$ has an orientation corresponding to a basis of vectors $[v_0,v],[v_0,v_1],\dots,[v,v_i]$, which is a basis for σ with a vector from v_0 to v, which corresponds to an outward pointing normal from M, adjoined at the beginning. Thus, using our conventions from the Appendix, $\Phi(cx,cy)=cx\pitchfork_X\partial(cy)=cx\pitchfork_X y=-x\pitchfork_{\partial M} y$. $\square$

There is an alternative way to formulate the above correspondences using codimension one strata. In particular, instead of forming X, we can stratify M as $M\supset\partial M$, where ∂M is treated as a codimension one stratum of the stratified space M. If we then choose perversities $\bar p,\bar q$ such that $\bar p(\partial M)<0$, $\bar q(\partial M)\geq 0$, then we will again have $I^{\bar p}H_*(M;G_0)=H_*(M;G)$ and $I^{\bar q}H_*(M;G_0)=H_*(M,\partial M;G)$. This follows from [21, Lemma 2.4], which says that this is equivalent to choosing $\bar p(\partial M)$ arbitrarily negative and $\bar q(\partial M)$ arbitrarily large, and then simple arguments taking into account with the stratified coefficient system.

Using this alternative correspondence, we can recover Wall's non-additivity theorem [45]. In Wall's situation, we suppose M^{4n} is a compact oriented manifold with boundary W such that $M=M_1\cup M_2$, where M_1,M_2 are compact oriented manifolds with boundary and $M_1\cap M_2=\partial M_1\cap\partial M_2$ is a manifold with boundary N such that $\partial M_1=N-W_1$, $\partial M_2=B_2-N$, and $\partial N_1=\partial N_2=\partial N=P$.

Let $V=H_{2n-1}(P;\mathbb{Q})$, and let A,B,C be the respective kernels of the maps induced by inclusion from V to $H_{2n-1}(W_1;\mathbb{Q})$, $H_{2n-1}(N;\mathbb{Q})$, $H_{2n-1}(W_2;\mathbb{Q})$. For a $4n$ manifold with boundary, $\sigma(M)$ denotes the signature of the pairing on $\operatorname{im}(H_{2n}(M;\mathbb{Q})\to H_{2n}(M,\partial M;\mathbb{Q}))$.

Corollary 5.3 (Wall)**.**

$$\sigma(M)=\sigma(M_1)+\sigma(M_2)-\sigma(V;A,B,C).$$

Proof. Wall's theorem follows from our Theorem 4.1 as follows. Restratify M as $M\supset\partial M$ and choose $\bar p(\partial M)$ arbitrarily negative and $\bar q(\partial M)=\bar t(\partial M)-\bar p(\partial M)=-1-\bar p(\partial M)$. Then, $I^{\bar p}H_*(M;G_0)=H_*(M;G)$, $I^{\bar q}H_*(M;G_0)=H_*(M,\partial M;G)$, $I^{\bar p}H_*(M_i;G_0)=H_*(M_i;G)$, and $I^{\bar q}H_*(M_i,N;G_0)=H_*(M_i,\partial M_i;G)$. In particular, $I^{\bar p\to\bar q}H_*(M;\mathbb{Q}_0)\cong\operatorname{im}(H_*(M;\mathbb{Q})\to H_*(M,\partial M;\mathbb{Q}))$, and $I^{\bar p\twoheadrightarrow\bar q}H_*(M_i,N;\mathbb{Q}_0)\cong\operatorname{im}(H_*(M_i;\mathbb{Q})\to H_*(M_i,\partial M_i;\mathbb{Q}))$. Furthermore, by Lemma 5.1 and Proposition 5.2, $I^{\bar q/\bar p}H_*(N;\mathbb{Q}_0)\cong H_{*-1}(P;R_0)$ with Φ corresponding to the negative of the intersection pairing on P. The corollary thus follows from Theorem 4.1. $\square$

5.2 Examples

In this section, we provide some simple sample computations applying our (non-)additivity theorem.

First example. Suppose W is a compact oriented $4k$-dimensional $\mathbb{Q}$-Witt space with non-zero Witt signature. Let M be a $4m$-dimensional connected compact oriented PL manifold with boundary ∂M. Consider the space $X = M \times W \cup_{\partial M\times W} \partial M \times \bar{c}W$, i.e. the space obtained from $M \times W$ by coning off the boundary fiberwise. Then X is not a Witt space, as W is the link of the stratum $\partial M \times v$, where v is the cone point of the closed cone $\bar{c}W$, and by assumption, W has non-vanishing middle-dimensional middle-perversity intersection homology. Furthermore, because the signature of W does not vanish, X cannot be a Banagl "non-Witt" space. Nonetheless, the signature $\sigma_{\bar{m}\to\bar{n}}(X)$ is defined, and we will show that if $\partial M \cong S^{4m-1}$, then $\sigma_{\bar{m}\to\bar{n}}(X) = \sigma(M)\sigma(W)$, where $\sigma(M)$ is the usual manifold signature of M and $\sigma(W) = \sigma_{\bar{m}\to\bar{n}}(W)$ is the Witt signature of W.

Lemma 5.4. *Suppose Y is a compact oriented $4k$-dimensional pseudomanifold and that N is a closed oriented $4n$-dimensional manifold. Then for perversities $\bar{p} \leq \bar{q}$, $\bar{p} + \bar{q} = \bar{t}$, we have $\sigma_{\bar{p}\to\bar{q}}(N \times Y) = \sigma(N)\sigma_{\bar{p}\to\bar{q}}(Y)$.*

Proof. By the Künneth theorem for intersection homology in which one term is a manifold (see [32], which extends to more general perversities and stratified coefficients), $I^{\bar{p}}H_*(N \times Y;\mathbb{Q}_0) \cong H_*(M;\mathbb{Q}) \otimes I^{\bar{p}}H_*(Y;\mathbb{Q}_0)$, and similarly for $\bar{q}$. Thus, by the naturality of the Künneth theorem, $I^{\bar{p}\to\bar{q}}H_*(N\times Y;\mathbb{Q}_0) \cong H_*(N;\mathbb{Q}) \otimes I^{\bar{p}\to\bar{q}}H_*(Y;\mathbb{Q}_0)$. The lemma now follows just as it does for manifolds (e.g. [30, 18]), using stratified general position arguments to see that the intersection pairing of the product behaves as one expects. □

Now suppose W and M as above with ∂M PL homeomorphic to S^{4m-1}. Let $\hat{M}$ be the closed manifold $M \cup_{S^{4m-1}} D^{4m}$. By the lemma, $\sigma_{\bar{m}\to\bar{n}}(\hat{M} \times W) = \sigma(M)\sigma_{\bar{m}\to\bar{n}}(W) = \sigma(M)\sigma(W)$, the last equality because W is Witt. Notice that $S^{4m-1} \times W$ is also a Witt space and so $I^{\bar{m}}H_*(W;\mathbb{Q}) \cong I^{\bar{n}}H_*(W;\mathbb{Q})$. Thus, by Corollary 4.2, $\sigma_{\bar{m}\to\bar{n}}(\hat{M} \times W) = \sigma_{\bar{m}\twoheadrightarrow\bar{n}}(M \times W) + \sigma_{\bar{m}\twoheadrightarrow\bar{n}}(D^{4m} \times W)$. But $D^{4m} \times W$ possesses an orientation-reversing self-homeomorphism, so $\sigma_{\bar{m}\twoheadrightarrow\bar{n}}(D^{4m} \times W) = 0$, and $\sigma_{\bar{m}\twoheadrightarrow\bar{n}}(M \times W)$ is the Witt signature $\sigma_{\bar{m}\twoheadrightarrow\bar{n}}(M \times W) = \sigma(M \times W)$. Thus $\sigma(M \times W) = \sigma(M)\sigma(W)$.

Returning now to our space X, obtained by coning off the boundary of $M \times W$ fiberwise, we see by a second application of Corollary 4.2 that $\sigma_{\bar{m}\to\bar{n}}(X) = \sigma(M \times W) + \sigma_{\bar{m}\to\bar{n}}(S^{4n-1} \times \bar{c}W)$. But $S^{4n-1} \times \bar{c}W$ again possesses an orientation-reversing self-homeomorphism, so its perverse signature is 0. Putting the preceding arguments together, we obtain $\sigma_{\bar{m}\to\bar{n}}(X) = \sigma(M \times W) = \sigma(M)\sigma(W)$.

A second example. In this example, we show that coning off a boundary does not change perverse signatures, i.e. that $\sigma_{\bar{p}\to\bar{q}}(X \cup_{\partial X} \bar{c}(\partial X)) = \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(X)$. In this example, X may be neither Witt nor Banagl non-Witt. This example utilizes an explicit computation of the Maslov index term of Theorem 4.1.

We first need a lemma that is clearly useful in its own right as a computational tool.

Lemma 5.5. *Let $\bar{p} \leq \bar{q}$, $\bar{p} + \bar{q} = \bar{t}$, and suppose Y is a closed, oriented $4n-1$ dimensional pseudomanifold with closed cone $\bar{c}Y$. Then $\sigma_{\bar{p}\twoheadrightarrow\bar{q}}(\bar{c}Y) = 0$.*

Proof. Notice that $\bar{c}Y$ is a pseudomanifold with boundary $\partial(\bar{c}Y) = Y$. By the standard cone formula for intersection homology, for any perversity $\bar{r}$, $I^{\bar{r}}H_*(\bar{c}Y;\mathbb{Q}_0)$ is either 0 or isomorphic to $I^{\bar{r}}H_{2n}(Y;\mathbb{Q}_0)$, with the isomorphism determined by inclusion $Y \hookrightarrow \bar{c}Y$. Thus $I^{\bar{p}\twoheadrightarrow\bar{q}}H_{2n}(\bar{c}Y, Y;\mathbb{Q}_0) = 0$, because any possible non-zero element $[x] \in I^{\bar{p}}H_{2n}(\bar{c}Y;\mathbb{Q}_0)$ can be written with the support of x in Y, and so the image of $[x]$ is 0 in $I^{\bar{q}}H_{2n}(\bar{c}Y, Y;\mathbb{Q}_0)$. Thus certainly $\sigma_{\bar{p}\twoheadrightarrow\bar{q}}(\bar{c}Y) = 0$. □

Proposition 5.6. *Suppose $\hat{X}$ is a closed oriented $4n$-pseudomanifold of the form $\hat{X} = X \cup_{\partial X} \bar{c}(\partial X)$. Let $\bar{p} \leq \bar{q}$, $\bar{p} + \bar{q} = \bar{t}$ be two perversities on $\hat{X}$. Then $\sigma_{\bar{p}\to\bar{q}}(\hat{X}) = \sigma_{\bar{p}\twoheadrightarrow\bar{q}}(X)$.*

Proof. We apply Theorem 4.1. By the preceding lemma, $\sigma_{\bar{p}\twoheadrightarrow\bar{q}}(\bar{c}(\partial X)) = 0$, so we need only show that the Maslov index term vanishes.

Consider $\ker(d : I^{\bar{q}/\bar{p}}H_{2n}(\partial X;\mathbb{Q}_0) \to I^{\bar{p}}H_{2n-1}(\partial X;\mathbb{Q}_0))$, which is the group B in the index term. By the exact sequence (5), $B = \operatorname{im}(I^{\bar{q}}H_{2n}(\partial X;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(\partial X;\mathbb{Q}_0))$, and so every class $[x] \in B$ can be represented by a $\bar{q}$-allowable $2n$-cycle in ∂X. Consider now $(i_{\partial X\subset\bar{c}(\partial X)})_*[x] \in I^{\bar{q}/\bar{p}}H_{2n}(\bar{c}(\partial X);\mathbb{Q}_0)$. This is also represented by the same chain x. In $\bar{c}(\partial X)$, we have that $\partial(\bar{c}x) = \pm x$, so if we can show $\bar{c}x$ is $\bar{q}$ allowable, then we have $(i_{\partial X\subset\bar{c}(\partial X)})_*[x] = 0$, which implies $B \subset A$, where $A = \ker((i_{\partial X\subset\bar{c}(\partial X)})_* : I^{\bar{q}/\bar{p}}H_{2n}(\partial X;\mathbb{Q}_0) \to I^{\bar{q}/\bar{p}}H_{2n}(\bar{c}(\partial X);\mathbb{Q}_0))$. Thus $B \cap (A + C) = B \cap A$, and so $W = \frac{B\cap(A+C)}{B\cap A + B\cap C} = 0$, which implies that the index term must vanish.

To show $\bar{c}x$ is $\bar{q}$-allowable, first note that the conditions on $\bar{p}, \bar{q}$ imply that $\bar{q} \geq \bar{n}$, where $\bar{n}$ is the upper middle perversity. For any simplex of $\bar{c}x$, we only need to check allowability at the cone vertex v (the allowability of $\bar{c}x$ otherwise comes for free - see the arguments in [23]). For simplices σ of $\bar{c}x$ that intersect the cone vertex, we know that $\sigma^{-1}(v)$ is in the 0-skeleton of the model Δ^{2n+1}. So by definition of allowability, we only need to check that $0 \leq 2n + 1 - 4n + \bar{q}(v) = 1 - 2n + \bar{q}(v)$. But $\bar{q}(v) \geq \bar{n}(v) = 2n - 1$. So $1 - 2n + \bar{q}(v) \geq 0$, and $\bar{q}$-allowability is confirmed. □

A Orientations and intersection numbers

In this appendix we establish conventions for orientation and intersection numbers. This is not meant to be a thorough treatise on every possible case that can occur in the stratified world, but rather the working through of the simplest manifold cases in order to establish compatibility of convention choices.

Let M be an m-dimensional oriented manifold with boundary. We choose the orientation of ∂M by adjoining an outward-pointing normal in the first component, i.e. if $x \in \partial M$, $e_1, \ldots, e_{m-1}$ is a basis for $T_x\partial M$, and $n \in T_xM$ is an "outward pointing" vector, then the ordered collection $\langle e_1, \ldots, e_{m-1}\rangle$ agrees with the orientation for ∂M if and only if $\langle n, e_1, \ldots, e_{m-1}\rangle$ agrees with the orientation for M. This convention seems to agree with the standard conventions for simplices.

Suppose ξ, η are cycles of complementary dimension in ∂M in general position and intersecting generically at the point x. Then the contribution to the intersection number $\xi \pitchfork \eta$ of the intersection at x is ± 1 according to whether a local basis for ξ concatenated with a local basis for η agrees or disagrees with the orientation at $T_x\partial M$. It makes sense to talk about local bases for ξ and η as generic intersections will occur in the interiors of oriented simplices.

Suppose now that there is a chain Ξ in M with $\partial\Xi = \xi$ contained in ∂M. We may assume that in a neighborhood of ∂M, Ξ looks like the chain $[0,1] \times \xi$ with the "1" end of the cylinder on the boundary (suitably simplicialized). Note that this gives the proper boundary $\partial([0,1] \times \xi) = 1 \times \xi - 0 \times \xi$ with $1 \times \xi = \xi \subset \partial M$. Note also that the $[0,1]$ component points in the direction of an outward pointing normal. Thus if ξ and η intersect at x, the intersection number contribution at x of $\xi \pitchfork \eta$ in ∂M is equal to the intersection number contribution at x of Ξ and η in M. This is because the intersection number of Ξ with η in M is determined by using the basis of $[0,1] \times \xi$ (i.e. the outward normal and then the basis of ξ) and then the basis for η. Since the normal comes at the beginning, there is agreement with how we expect to compare orientations in M with those in ∂M. On the other hand, suppose H is a chain in M with $\partial H = \eta$ and that H looks like $[0,1] \times \eta$ in a neighborhood of ∂M. Then the intersection at x of ξ with H compares with the basis for M the basis obtained from ξ then from the outward normal, then from η. So to compare properly with the intersection number of ξ and η in ∂M, we must move the normal to the front. This changes the orientation number by $(-1)^{|\xi|}$. So the intersection number of ξ with η in ∂M is $(-1)^{|\xi|}$ times the intersection number of ξ with H in M.

Summarizing, we have:

$$\partial\Xi \pitchfork_{\partial M} \eta = \Xi \pitchfork_M \eta$$
$$\xi \pitchfork_{\partial M} \partial H = (-1)^{|\xi|}\xi \pitchfork_M \eta.$$

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