

“All mathematicians live in two different worlds. They live in a crystalline world of perfect platonic forms. An ice palace. But they also live in the common world where things are transient, ambiguous, subject to vicissitudes. Mathematicians go backward and forward from one world to another. They’re adults in the crystalline world, infants in the real one.”

- Sylvain Cappell

Extending Poincaré Duality to Homotopically Stratified Spaces

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Poincaré Duality

If M is an n -dimensional compact oriented closed manifold, then

$$H_i(M; \mathbb{Q}) \cong \text{Hom}(H_{n-i}(M; \mathbb{Q}); \mathbb{Q})$$

Goal

- ▶ Poincaré Duality extends to certain *singular spaces* using

INTERSECTION HOMOLOGY

due to Goresky-MacPherson

- ▶ Goal: Extend Poincaré Duality to

MANIFOLD HOMOTOPICALLY STRATIFIED SPACES

Manifold Stratified Spaces

Definition

A *Manifold Stratified Space* is a filtered space

$$X = X^n \supset X^{n-2} \supset X^{n-3} \supset \dots \supset X^0 \supset X^{-1} = \emptyset$$

such that

- ▶ $S_k = X^k - X^{k-1}$ is a k -manifold (or empty)
 - ▶ S_k is called the *k-stratum*
- ▶ $X - X^{n-2}$ is dense in X
- ▶ local normality conditions

Local Normality Conditions

- ▶ pseudomanifolds: cone bundle neighborhoods (each point has a neighborhood homeomorphic to $\mathbb{R}^k \times cL$)
 - ▶ Algebraic varieties
 - ▶ Simplicial pseudomanifolds
- ▶ homotopically stratified spaces: local homotopy conditions [Quinn]
 - ▶ Quotients of manifolds by topological locally-linear group actions
 - ▶ Mapping cylinders of algebraic varieties (even under fairly nice maps) [Cappell-Shaneson]

Intersection Homology on PL Pseudomanifolds I: Perversities

A *perversity* is a function

$$\bar{p} : \{2, 3, \dots\} \rightarrow \mathbb{N}$$

such that

- ▶ $\bar{p}(2) = 0$
- ▶ $\bar{p}(k) \leq \bar{p}(k + 1) \leq \bar{p}(k) + 1$

Idea: assign numbers to strata

These numbers will determine the allowable degeneracy of intersections of chains with strata

Intersection Homology on PL Pseudomanifolds II

Intersection chain complex

$$I^{\bar{p}}C_*(X) \subset C_*(X)$$

$\xi \in I^{\bar{p}}C_i(X)$ if for each k ,

- ▶ $\dim |\xi \cap S_{n-k}| \leq i - k + \bar{p}(k)$
- ▶ $\dim |\partial\xi \cap S_{n-k}| \leq i - 1 - k + \bar{p}(k)$

Then $I^{\bar{p}}H_*(X) = H_*(I^{\bar{p}}C_*(X))$.

Properties of Intersection Homology on Pseudomanifolds

- ▶ Rational Poincaré duality [G-M]:

$$I^{\bar{p}}H_*(X; \mathbb{Q}) \cong \text{Hom}(I^{\bar{q}}H_{n-*}(X; \mathbb{Q}), \mathbb{Q})$$

when $\bar{p}(k) + \bar{q}(k) = k - 2$ for all k

- ▶ If X has only strata of even codimension (e.g. complex algebraic varieties), then

$$I^{\bar{m}}H_*(X; \mathbb{Q}) \cong \text{Hom}(I^{\bar{m}}H_{n-*}(X; \mathbb{Q}), \mathbb{Q})$$

- ▶ Topological invariance (independence of stratification)
- ▶ Applications:
 - ▶ Signatures and Characteristic classes
 - ▶ Generalizations to singular algebraic varieties of Kaehler package: Lefschetz hyperplane theorem, hard Lefschetz theorem, Hodge theory

Proof of Poincaré Duality [Goresky-MacPherson]

- ▶ Express intersection homology as sheaf theory

$$I^{\bar{p}}H_*(X) = \mathbb{H}^{n-*}(\mathcal{I}^{\bar{p}}\mathcal{C}^*)$$

- ▶ The intersection chain sheaf $\mathcal{I}^{\bar{p}}\mathcal{C}^*$ has an axiomatic characterization
- ▶ Verdier Duality

$$\begin{aligned}\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{C}^*)[-n]) &\cong \text{Hom}(\mathbb{H}^*(\mathcal{I}^{\bar{p}}\mathcal{C}^*); \mathbb{Q}) \\ &\cong \text{Hom}(I^{\bar{p}}H_{n-*}(X); \mathbb{Q})\end{aligned}$$

- ▶ $\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{C}^*)$ satisfies the axioms for $\mathcal{I}^{\bar{q}}\mathcal{C}^*[n]$
 - ▶ So $\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{C}^*)[-n]) \cong I^{\bar{q}}H_*(X; \mathbb{Q})$

From PL Chains to Sheaves

$\mathcal{I}^{\bar{p}}\mathcal{C}^*$ is the sheaf of germs of PL chains.

The sections of $\mathcal{I}^{\bar{p}}\mathcal{C}^*$ are given by

$$\Gamma(U; \mathcal{I}^{\bar{p}}\mathcal{C}^*) = I^{\bar{p}}C_{n-*}^{\infty}(U)$$

We need (locally-finite) infinite chains since we need groups that behave well under restrictions.

Nice Facts About $\mathcal{I}^{\bar{p}}\mathcal{C}^*$

- ▶ $\mathcal{I}^{\bar{p}}\mathcal{C}^*$ is a complex of soft sheaves, which implies that

$$\mathbb{H}^*(X; \mathcal{I}^{\bar{p}}\mathcal{C}^*) = H^*(\Gamma(X; \mathcal{I}^{\bar{p}}\mathcal{C}^*)) = I^{\bar{p}}H_{n-*}^\infty(X).$$

- ▶ Can get the previous IH via compact supports

$$\mathbb{H}_c^*(X; \mathcal{I}^{\bar{p}}\mathcal{C}^*) = H^*(\Gamma_c(X; \mathcal{I}^{\bar{p}}\mathcal{C}^*)) = I^{\bar{p}}H_{n-*}(X).$$

- ▶ $\mathcal{I}^{\bar{p}}\mathcal{C}^*$ is quasi-isomorphic to the Deligne sheaf

$$\mathcal{P}^* = \tau_{\leq \bar{p}(n)} Ri_{n*} \cdots \tau_{\leq \bar{p}(2)} Ri_{2*} \mathbb{Q},$$

which can be described axiomatically by axioms independent of the stratification.

- ▶ This axiomatic characterization implies topological invariance ($IH_*(X)$ is independent of the stratification of X)
- ▶ The Deligne sheaf can be defined on *any* filtered space

Axioms

The Deligne sheaf axioms for perversity \bar{p}

- ▶ \mathcal{S}^* is bounded, $\mathcal{S}^i = 0$ for $i < 0$, and $\mathcal{S}^*|_{X-X^{n-2}} = \mathbb{Q}$
- ▶ For $x \in X^{n-k} - X^{n-k-1}$, $H^j(\mathcal{S}_x^*) = 0$ for $j > \bar{p}(k)$
- ▶ For $x \in X^{n-k} - X^{n-k-1}$, the canonical attaching map

$$\mathcal{S}^*|_{X-X^{n-k-1}} \rightarrow Ri_{k*}\mathcal{S}^*|_{X-X^{n-k}}$$

is a quasi-isomorphism for $j \leq \bar{p}(k)$

These axioms are essentially equivalent to the fact that

$$I^{\bar{p}}H_j^c(\mathbb{R}^k \times cL) \cong \begin{cases} 0, & j \geq \dim(L) - \bar{p}(\dim(L) + 1) \\ IH_{j-1}(L), & j < \dim(L) - \bar{p}(\dim(L) + 1), \end{cases}$$

Manifold Homotopically Stratified Spaces (MHSSs) [Quinn]

A filtered space X is a *Manifold Homotopically Stratified Space (MHSS)* [Quinn] if

- ▶ X is locally-compact, separable, and metric.
- ▶ $X = X^n \supset X^{n-2} \supset X^{n-3} \supset \dots \supset X^0 \supset X^{-1} = \emptyset$
- ▶ $S_k = X^k - X^{k-1}$ is a k -manifold (or empty) and is locally closed in X
- ▶ For each $k > i$, X_i is *forward tame* in $X_i \cup X_k$.
- ▶ For each $k > i$, the *holink evaluation*

$$\text{holink}_s(X_i \cup X_k, X_i) \rightarrow X_i$$

is a fibration.

- ▶ For each x , there is a stratum-preserving homotopy

$$\text{holink}(X, x) \times I \rightarrow \text{holink}(X, x)$$

from the identity into a compact subset of $\text{holink}(X, x)$.

Examples of MHSSs

Quinn's intention - a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds.

- ▶ Quotients of manifolds by topological locally linear group actions [Beshears, Quinn, Weinberger, Yan]
- ▶ Mapping cylinders and cones of stratified maps of algebraic varieties [Cappell-Shaneson]

What's Different for MHSSs?

For pseudomanifolds:

- ▶ Proof of PD relies strongly on local computations involving the distinguished neighborhoods $U \cong \mathbb{R}^k \times cL$
- ▶ PD can be proven purely sheaf-theoretically - don't really need to understand the geometric interpretation with chains

For MHSSs

- ▶ No distinguished neighborhoods - must use local homotopy properties
- ▶ But pure sheaf theory doesn't work well under homotopies (or in path spaces, which are not locally compact)
- ▶ Solution: use interplay between singular chains and sheaves
 - ▶ Use the chain interpretation of intersection homology to prove things about Deligne sheaves

Poincaré Duality for MHSSs

- ▶ Use singular chain intersection homology [King]
- ▶ Express singular intersection homology as a sheaf theory
 - ▶ $I^{\bar{p}}H_*^{\text{sing}}(X) = \mathbb{H}^{n-*}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$
- ▶ Show $\mathcal{I}^{\bar{p}}\mathcal{S}^*$ satisfies the Deligne axioms
- ▶ Show $\mathcal{D}(\mathcal{I}^{\bar{q}}\mathcal{S}^*)$ satisfies the axioms to be $\mathcal{I}^{\bar{q}}\mathcal{S}^*[n]$
 - ▶ $\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)[-n]) \cong I^{\bar{q}}H_*(X)$
- ▶ Apply Verdier duality
 - ▶ $\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)[-n]) \cong \text{Hom}(\mathbb{H}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*); \mathbb{Q})$

Conclude $I^{\bar{q}}H_*(X; \mathbb{Q}) \cong \text{Hom}(I^{\bar{p}}H_{n-*}(X), \mathbb{Q})$

Singular Chain $I\bar{H}$ [King]

Define $I\bar{P}S_*(X) \subset S_*(X)$ by

$$\xi \in I\bar{P}S_i(X)$$

if each singular simplex $\sigma \in \xi$ satisfies

$$\sigma^{-1}(S_{n-k}) \subset i - k + \bar{p}(k) \text{ skeleton of } \Delta^i$$

and each $i - 1$ simplex τ in $\partial\xi$ satisfies

$$\tau^{-1}(S_{n-k}) \subset i - 1 - k + \bar{p}(k) \text{ skeleton of } \Delta^{i-1}.$$

Then $I\bar{P}H_*(X) = H_*(I\bar{P}S_*(X))$.

Properties of Singular Chain IH

- ▶ Defined for all filtered spaces.
- ▶ Topological invariance on compact topological pseudomanifolds [King] and MHSSs [Quinn]
- ▶ IH_*^c is a *stratum-preserving homotopy invariant*
 - ▶ (careful! IH_*^∞ is not)

But

- ▶ Not clear that this agrees with the sheaf-theoretic definitions

The Singular Intersection Chain Sheaf

- ▶ Consider the presheaf

$$IS^* : U \rightarrow I^{\bar{p}}S_{n-*}^{\infty}(X, X - \bar{U})$$

with the natural restriction

$$I^{\bar{p}}S_{n-*}^{\infty}(X, X - \bar{U}) \rightarrow I^{\bar{p}}S_{n-*}^{\infty}(X, X - \bar{V})$$

for $V \subset U$.

- ▶ This isn't a sheaf, but it does generate a sheaf \mathcal{IS}^* .
- ▶ \mathcal{IS}^* is homotopically fine, so:

$$IH_{n-*}(X) \cong \mathbb{H}^*(\mathcal{IS}^*) \cong H^*(\Gamma(X; \mathcal{IS}^*))$$

- ▶ On *pseudomanifolds*, this agrees with Deligne sheaf intersection homology [GBF]

How to Avoid Distinguished Neighborhoods

- ▶ *Pure subsets* of MHSSs have *Approximate Tubular Neighborhoods*¹ [Hughes - extending Hughes-Taylor-Weinberger-Williams]
- ▶ It follows that points have *local approximate tubular neighborhoods*
- ▶ These are *teardrops* of *stratified approximate fibrations*
 - ▶ *teardrops* - generalize mapping cylinders
 - ▶ *approximate fibrations* - generalize fibrations
 - ▶ They have *approximate liftings*

¹subject to minor dimension restrictions

Idea of the Rest of the Proof

- ▶ Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- ▶ Show that \mathcal{IS}^* satisfies the Deligne axioms
- ▶ Show that $\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$ satisfies the axioms for $\mathcal{I}^{\bar{q}}\mathcal{S}^*$

Key steps - Let U be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- ▶ $I^{\bar{p}}H_{n-j}^\infty(U) = 0$ for $j > \bar{p}(k)$
 - ▶ Spectral sequence for IH_*^∞ of approx. tubular nbhds
- ▶ Restriction isomorphisms
 $I^{\bar{p}}H_{n-j}^\infty(U) \rightarrow I^{\bar{p}}H_{n-j}^\infty(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
 - ▶ Chain arguments using properties of stratified approximate fibrations