“All mathematicians live in two different worlds. They live in a crystalline world of perfect platonic forms. An ice palace. But they also live in the common world where things are transient, ambiguous, subject to vicissitudes. Mathematicians go backward and forward from one world to another. They’re adults in the crystalline world, infants in the real one.”

- Sylvain Cappell

Extending Poincaré Duality to Homotopically Stratified Spaces

Greg Friedman

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Fort Worth,TX

March 17, 2008

Poincaré Duality

If M is an n-dimensional compact oriented closed manifold, then

$$H_i(M;\mathbb{Q}) \cong \mathrm{Hom}(H_{n-i}(M;\mathbb{Q});\mathbb{Q})$$

Goal

- Poincaré Duality extends to certain *singular spaces* using

  INTERSECTION HOMOLOGY

  due to Goresky-MacPherson

- Goal: Extend Poincaré Duality to

  MANIFOLD HOMOTOPICALLY STRATIFIED SPACES

Manifold Stratified Spaces

Definition

A *Manifold Stratified Space* is a filtered space

$$X = X^n \supset X^{n-2} \supset X^{n-3} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$$

such that

- $S_k = X^k - X^{k-1}$ is a k-manifold (or empty)
  - S_k is called the *k-stratum*
- $X - X^{n-2}$ is dense in X
- local normality conditions

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Local Normality Conditions

- pseudomanifolds: cone bundle neighborhoods (each point has a neighborhood homeomorphic to $\mathbb{R}^k \times cL$)
  - Algebraic varieties
  - Simplicial pseudomanifolds
- homotopically stratified spaces: local homotopy conditions [Quinn]
  - Quotients of manifolds by topological locally-linear group actions
  - Mapping cylinders of algebraic varieties (even under fairly nice maps) [Cappell-Shaneson]

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Intersection Homology on PL Pseudomanifolds I: Perversities

A *perversity* is a function

$$\bar{p} : \{2, 3, \ldots\} \to \mathbb{N}$$

such that

- $\bar{p}(2) = 0$
- $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$

Idea: assign numbers to strata
These numbers will determine the allowable degeneracy of intersections of chains with strata

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Intersection Homology on PL Pseudomanifolds II

Intersection chain complex

$$I^{\bar{p}}C_*(X) \subset C_*(X)$$

$\xi \in I^{\bar{p}}C_i(X)$ if for each k,

- $\dim |\xi \cap S_{n-k}| \leq i - k + \bar{p}(k)$
- $\dim |\partial\xi \cap S_{n-k}| \leq i - 1 - k + \bar{p}(k)$

Then $I^{\bar{p}}H_*(X) = H_*(I^{\bar{p}}C_*(X))$.

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Properties of Intersection Homology on Pseudomanifolds

- Rational Poincaré duality [G-M]:

  $$I^{\bar{p}}H_*(X;\mathbb{Q}) \cong \mathrm{Hom}(I^{\bar{q}}H_{n-*}(X;\mathbb{Q}),\mathbb{Q})$$

  when $\bar{p}(k) + \bar{q}(k) = k - 2$ for all k
  - If X has only strata of even codimension (e.g. complex algebraic varieties), then

    $$I^{\bar{m}}H_*(X;\mathbb{Q}) \cong \mathrm{Hom}(I^{\bar{m}}H_{n-*}(X;\mathbb{Q}),\mathbb{Q})$$

- Topological invariance (independence of stratification)
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  - Signatures and Characteristic classes
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Proof of Poincaré Duality [Goresky-MacPherson]

- Express intersection homology as sheaf theory

$$I^{\bar p}H_*(X) = \mathbb{H}^{n-*}(\mathcal{I}^{\bar p}\mathcal{C}^*)$$

- The intersection chain sheaf $\mathcal{I}^{\bar p}\mathcal{C}^*$ has an axiomatic characterization
- Verdier Duality

$$\begin{aligned}\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar p}\mathcal{C}^*)[-n]) &\cong \mathrm{Hom}(\mathbb{H}^*(\mathcal{I}^{\bar p}\mathcal{C}^*);\mathbb{Q})\\ &\cong \mathrm{Hom}(I^{\bar p}H_{n-*}(X);\mathbb{Q})\end{aligned}$$

- $\mathcal{D}(\mathcal{I}^{\bar p}\mathcal{C}^*)$ satisfies the axioms for $\mathcal{I}^{\bar q}\mathcal{C}^*[n]$
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From PL Chains to Sheaves

$\mathcal{I}^{\bar{p}}\mathcal{C}^*$ is the sheaf of germs of PL chains.

The sections of $\mathcal{I}^{\bar{p}}\mathcal{C}^*$ are given by

$$\Gamma(U; \mathcal{I}^{\bar{p}}\mathcal{C}^*) = I^{\bar{p}}C^{\infty}_{n-*}(U)$$

We need (locally-finite) infinite chains since we need groups that behave well under restrictions.

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Nice Facts About $\mathcal{I}^{\bar p}\mathcal{C}^*$

- $\mathcal{I}^{\bar p}\mathcal{C}^*$ is a complex of soft sheaves, which implies that
$$\mathbb{H}^*(X;\mathcal{I}^{\bar p}\mathcal{C}^*) = H^*(\Gamma(X;\mathcal{I}^{\bar p}\mathcal{C}^*)) = I^{\bar p}H^{\infty}_{n-*}(X).$$
- Can get the previous IH via compact supports
$$\mathbb{H}^*_c(X;\mathcal{I}^{\bar p}\mathcal{C}^*) = H^*(\Gamma_c(X;\mathcal{I}^{\bar p}\mathcal{C}^*)) = I^{\bar p}H_{n-*}(X).$$
- $\mathcal{I}^{\bar p}\mathcal{C}^*$ is quasi-isomorphic to the Deligne sheaf
$$\mathcal{P}^* = \tau_{\leq \bar p(n)}Ri_{n*}\cdots\tau_{\leq \bar p(2)}Ri_{2*}\mathbb{Q},$$
which can be described axiomatically by axioms independent of the stratification.
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Axioms

The Deligne sheaf axioms for perversity $\bar{p}$

- $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$, and $\mathcal{S}^*|_{X - X^{n-2}} = \mathbb{Q}$
- For $x \in X^{n-k} - X^{n-k-1}$, $H^j(\mathcal{S}^*_x) = 0$ for $j > \bar{p}(k)$
- For $x \in X^{n-k} - X^{n-k-1}$, the canonical attaching map
$$\mathcal{S}^*|_{X - X^{n-k-1}} \to Ri_{k*}\mathcal{S}^*|_{X - X^{n-k}}$$
is a quasi-isomorphism for $j \leq \bar{p}(k)$

These axioms are essentially equivalent to the fact that

$$I^{\bar{p}}H^c_j(\mathbb{R}^k \times cL) \cong \begin{cases} 0, & j \geq \dim(L) - \bar{p}(\dim(L)+1) \\ IH_{j-1}(L), & j < \dim(L) - \bar{p}(\dim(L)+1), \end{cases}$$

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- $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$, and $\mathcal{S}^*|_{X - X^{n-2}} = \mathbb{Q}$
- For $x \in X^{n-k} - X^{n-k-1}$, $H^j(\mathcal{S}^*_x) = 0$ for $j > \bar{p}(k)$
- For $x \in X^{n-k} - X^{n-k-1}$, the canonical attaching map
$$\mathcal{S}^*|_{X - X^{n-k-1}} \to Ri_{k*}\mathcal{S}^*|_{X - X^{n-k}}$$
is a quasi-isomorphism for $j \leq \bar{p}(k)$

These axioms are essentially equivalent to the fact that

$$I^{\bar{p}}H^c_j(\mathbb{R}^k \times cL) \cong \begin{cases} 0, & j \geq \dim(L) - \bar{p}(\dim(L)+1) \\ IH_{j-1}(L), & j < \dim(L) - \bar{p}(\dim(L)+1), \end{cases}$$

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Manifold Homotopically Stratified Spaces (MHSSs) [Quinn]

A filtered space X is a *Manifold Homotopically Stratified Space (MHSS)* [Quinn] if

- X is locally-compact, separable, and metric.
- $X = X^n \supset X^{n-2} \supset X^{n-3} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$
- $S_k = X^k - X^{k-1}$ is a k-manifold (or empty) and is locally closed in X
- For each $k > i$, X_i is *forward tame* in $X_i \cup X_k$.
- For each $k > i$, the *holink evaluation*

  $$\text{holink}_s(X_i \cup X_k, X_i) \to X_i$$

  is a fibration.
- For each x, there is a stratum-preserving homotopy

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Examples of MHSSs

Quinn's intention - a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds.

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For pseudomanifolds:

- Proof of PD relies strongly on local computations involving the distinguished neighborhoods $U \cong \mathbb{R}^k \times cL$
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Poincaré Duality for MHSSs - Outline of proof

- Use singular chain intersection homology [King]
- Express singular intersection homology as a sheaf theory
  - $I^{\bar{p}}H_*^{\text{sing}}(X) = \mathbb{H}^{n-*}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$
- Show $\mathcal{I}^{\bar{p}}\mathcal{S}^* =$ Deligne sheaf (by axioms)
- Show $\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$ satisfies the axioms to be $\mathcal{I}^{\bar{q}}\mathcal{S}^*[n]$
  - $\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)[-n]) \cong I^{\bar{q}}H_*(X)$
- Apply Verdier duality
  - $\mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)[-n]) \cong \text{Hom}(\mathbb{H}^*(\mathcal{I}^{\bar{p}}\mathcal{S}^*);\mathbb{Q})$

Conclude $I^{\bar{q}}H_*(X;\mathbb{Q}) \cong \text{Hom}(I^{\bar{p}}H_{n-*}(X),\mathbb{Q})$

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Singular Chain IH [King]

Define $I^{\bar{p}}S_*(X) \subset S_*(X)$ by

$$\xi \in I^{\bar{p}}S_i(X)$$

if each singular simplex $\sigma \in \xi$ satisfies

$$\sigma^{-1}(S_{n-k}) \subset i - k + \bar{p}(k) \text{ skeleton of } \Delta^i$$

and each $i-1$ simplex τ in $\partial\xi$ satisfies

$$\tau^{-1}(S_{n-k}) \subset i - 1 - k + \bar{p}(k) \text{ skeleton of } \Delta^{i-1}.$$

Then $I^{\bar{p}}H_*(X) = H_*(I^{\bar{p}}S_*(X))$.

Properties of Singular Chain IH

- Defined for all filtered spaces.
- Topological invariance on compact topological pseudomanifolds [King] and MHSSs [Quinn]
- IH^c_* is a *stratum-preserving homotopy invariant*
  - (careful! IH^∞_* is not)

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The Singular Intersection Chain Sheaf

- Consider the presheaf

$$IS^* : U \to I^{\bar{p}}S^{\infty}_{n-*}(X, X - \bar{U})$$

with the natural restriction

$$I^{\bar{p}}S^{\infty}_{n-*}(X, X - \bar{U}) \to I^{\bar{p}}S^{\infty}_{n-*}(X, X - \bar{V})$$

for $V \subset U$.

- This isn't a sheaf, but it does generate a sheaf $\mathcal{IS}^*$.
- $\mathcal{IS}^*$ is homotopically fine, so:

$$IH_{n-*}(X) \cong \mathbb{H}^*(\mathcal{IS}^*) \cong H^*(\Gamma(X; \mathcal{IS}^*))$$

- On *pseudomanifolds*, this agrees with Deligne sheaf intersection homology [GBF]

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How to Avoid Distinguished Neighborhoods

- *Pure subsets* of MHSSs have *Approximate Tubular Neighborhoods*[1] [Hughes - extending Hughes-Taylor-Weinberger-Williams]
- It follows that points have *local approximate tubular neighborhoods*
- These are *teardrops* of *stratified approximate fibrations*
  - *teardrops* - generalize mapping cylinders
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    - They have *approximate liftings*

[1] subject to minor dimension restrictions

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Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{IS}^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$ satisfies the axioms for $\mathcal{I}^{\bar{q}}\mathcal{S}^*$

Key steps - Let U be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I^{\bar{p}}H^{\infty}_{n-j}(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for IH^{∞}_* of approx. tubular nghbds
- Restriction isomorphisms $I^{\bar{p}}H^{\infty}_{n-j}(U) \to I^{\bar{p}}H^{\infty}_{n-j}(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations

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Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{IS}^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}^{\bar{p}}\mathcal{S}^*)$ satisfies the axioms for $\mathcal{I}^{\bar{q}}\mathcal{S}^*$

Key steps - Let U be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

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