“All mathematicians live in two different worlds. They live in a crystalline world of perfect platonic forms. An ice palace. But they also live in the common world where things are transient, ambiguous, subject to vicissitudes. Mathematicians go backward and forward from one world to another. They’re adults in the crystalline world, infants in the real one.”
- Sylvain Cappell
Extending Poincaré Duality to Homotopically Stratified Spaces

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If $M$ is an $n$-dimensional compact oriented closed manifold, then

$$H_i(M; \mathbb{Q}) \cong \text{Hom}(H_{n-i}(M; \mathbb{Q}); \mathbb{Q})$$
Goal

- Poincaré Duality extends to certain *singular spaces* using
  
  **INTERSECTION HOMOLOGY**
  
  due to Goresky-MacPherson

- Goal: Extend Poincaré Duality to
  
  **MANIFOLD HOMOTOPICALLY STRATIFIED SPACES**
**Definition**

A *Manifold Stratified Space* is a filtered space

\[ X = X^n \supset X^{n-2} \supset X^{n-3} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset \]

such that

- \( S_k = X^k - X^{k-1} \) is a \( k \)-manifold (or empty)
  - \( S_k \) is called the \( k \)-stratum
- \( X - X^{n-2} \) is dense in \( X \)
- local normality conditions
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Local Normality Conditions

- pseudomanifolds: cone bundle neighborhoods
  (each point has a neighborhood homeomorphic to \( \mathbb{R}^k \times cL \))
  - Algebraic varieties
  - Simplicial pseudomanifolds

- homotopically stratified spaces: local homotopy conditions [Quinn]
  - Quotients of manifolds by topological locally-linear group actions
  - Mapping cylinders of algebraic varieties (even under fairly nice maps) [Cappell-Shaneson]
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\[ \tilde{p} : \{2, 3, \ldots \} \rightarrow \mathbb{N} \]

such that

- \( \tilde{p}(2) = 0 \)
- \( \tilde{p}(k) \leq \tilde{p}(k + 1) \leq \tilde{p}(k) + 1 \)

Idea: assign numbers to strata
These numbers will determine the allowable degeneracy of intersections of chains with strata
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Intersection Homology on PL Pseudomanifolds II

Intersection chain complex

\[ I^{\bar{p}}C_*(X) \subset C_*(X) \]

\[ \xi \in I^{\bar{p}}C_i(X) \text{ if for each } k, \]
\[ \dim |\xi \cap S_{n-k}| \leq i - k + \bar{p}(k) \]
\[ \dim |\partial \xi \cap S_{n-k}| \leq i - 1 - k + \bar{p}(k) \]

Then \[ I^{\bar{p}}H_*(X) = H_*(I^{\bar{p}}C_*(X)) \].
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Properties of Intersection Homology on Pseudomanifolds

- Rational Poincaré duality [G-M]:
  \[ I^{\bar{p}} H_\ast(X; \mathbb{Q}) \cong \text{Hom}(I^{\bar{q}} H_{n-\ast}(X; \mathbb{Q}), \mathbb{Q}) \]
  when \( \bar{p}(k) + \bar{q}(k) = k - 2 \) for all \( k \)
  - If \( X \) has only strata of even codimension (e.g. complex algebraic varieties), then
    \[ I^{\bar{m}} H_\ast(X; \mathbb{Q}) \cong \text{Hom}(I^{\bar{m}} H_{n-\ast}(X; \mathbb{Q}), \mathbb{Q}) \]

- Topological invariance (independence of stratification)

- Applications:
  - Signatures and Characteristic classes
  - Generalizations to singular algebraic varieties of Kähler package: Lefschetz hyperplane theorem, hard Lefschetz theorem, Hodge theory
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Proof of Poincaré Duality
[Goersky-MacPherson]

- Express intersection homology as sheaf theory
  \[ I^\bar{p} H_*(X) = \mathbb{H}^{n-*}(\mathcal{I}^\bar{p} C^*) \]

- The intersection chain sheaf \( \mathcal{I}^\bar{p} C^* \) has an axiomatic characterization

- Verdier Duality
  \[ \mathbb{H}^{-*}(\mathcal{D}(\mathcal{I}^\bar{p} C^*)[-n]) \cong \text{Hom}(\mathbb{H}^*(\mathcal{I}^\bar{p} C^*); \mathbb{Q}) \]
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\( \mathcal{I}^p C^* \) is the sheaf of germs of PL chains.

The sections of \( \mathcal{I}^p C^* \) are given by

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Nice Facts About $\mathcal{I}^\bar{p}C^*$

- $\mathcal{I}^\bar{p}C^*$ is a complex of soft sheaves, which implies that
  \[ H^*(X; \mathcal{I}^\bar{p}C^*) = H^*(\Gamma(X; \mathcal{I}^\bar{p}C^*)) = I^\bar{p}H^*_n(X). \]

- Can get the previous $IH$ via compact supports
  \[ H^*_c(X; \mathcal{I}^\bar{p}C^*) = H^*(\Gamma_c(X; \mathcal{I}^\bar{p}C^*)) = I^\bar{p}H^*_n(X). \]

- $\mathcal{I}^\bar{p}C^*$ is quasi-isomorphic to the Deligne sheaf
  \[ \mathcal{P}^* = \tau_{\leq \bar{p}(n)} Ri_n^* \cdots \tau_{\leq \bar{p}(2)} Ri_2^* \mathbb{Q}, \]
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Axioms

The Deligne sheaf axioms for perversity $\bar{p}$

- $S^*$ is bounded, $S^i = 0$ for $i < 0$, and $S^*|_{X - X^{n-2}} = \mathbb{Q}$
- For $x \in X^{n-k} - X^{n-k-1}$, $H^j(S_x^*) = 0$ for $j > \bar{p}(k)$
- For $x \in X^{n-k} - X^{n-k-1}$, the canonical attaching map

$$S^*|_{X - X^{n-k-1}} \rightarrow R\iota_{k*}S^*|_{X - X^{n-k}}$$

is a quasi-isomorphism for $j \leq \bar{p}(k)$

These axioms are essentially equivalent to the fact that

$$I^\bar{p}H^c_j(\mathbb{R}^k \times cL) \cong \begin{cases} 0, & j \geq \text{dim}(L) - \bar{p}(\text{dim}(L) + 1) \\ IH_{j-1}(L), & j < \text{dim}(L) - \bar{p}(\text{dim}(L) + 1), \end{cases}$$
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IH_{j-1}(L), & j < \text{dim}(L) - \bar{p}(\text{dim}(L) + 1), 
\end{cases}
\]
Axioms

The Deligne sheaf axioms for perversity $\bar{p}$

- $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$, and $\mathcal{S}^*|_{X - X^{n-2}} = \mathbb{Q}$
- For $x \in X^{n-k} - X^{n-k-1}$, $H^j(\mathcal{S}^*_x) = 0$ for $j > \bar{p}(k)$
- For $x \in X^{n-k} - X^{n-k-1}$, the canonical attaching map $\mathcal{S}^*|_{X - X^{n-k-1}} \to R\iota_{k*}\mathcal{S}^*|_{X - X^{n-k}}$ is a quasi-isomorphism for $j \leq \bar{p}(k)$

These axioms are essentially equivalent to the fact that

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Manifold Homotopically Stratified Spaces (MHSSs) [Quinn]

A filtered space $X$ is a Manifold Homotopically Stratified Space (MHSS) [Quinn] if

- $X$ is locally-compact, separable, and metric.
- $X = X^n \supset X^{n-2} \supset X^{n-3} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$
- $S_k = X^k - X^{k-1}$ is a $k$-manifold (or empty) and is locally closed in $X$
- For each $k > i$, $X_i$ is forward tame in $X_i \cup X_k$.
- For each $k > i$, the holink evaluation

$$\text{holink}_s(X_i \cup X_k, X_i) \to X_i$$

is a fibration.
- For each $x$, there is a stratum-preserving homotopy

$$\text{holink}(X, x) \times I \to \text{holink}(X, x)$$

from the identity into a compact subset of $\text{holink}(X, x)$.  

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Extending Poincaré Duality to MHSSs

Greg Friedman

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Intersection Homology

Poincaré Duality on Pseudomanifolds

Homotopically Stratified Spaces

Poincaré Duality for MHSS
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**Manifold Homotopically Stratified Spaces (MHSSs) [Quinn]**

Greg Friedman
Examples of MHSSs

Quinn’s intention - a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds.

- Quotients of manifolds by topological locally linear group actions [Beshears, Quinn, Weinberger, Yan]
- Mapping cylinders and cones of stratified maps of algebraic varieties [Cappell-Shaneson]
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What’s Different for MHSSs?

For pseudomanifolds:
- Proof of PD relies strongly on local computations involving the distinguished neighborhoods $U \cong \mathbb{R}^k \times cL$
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For MHSSs
- No distinguished neighborhoods - must use local homotopy properties
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- Use singular chain intersection homology [King]
- Express singular intersection homology as a sheaf theory
  - \( I^\bar{p} H^\text{sing}_*(X) = \mathbb{H}^{n-*}(I^\bar{p} S^*) \)
- Show \( I^\bar{p} S^* = \) Deligne sheaf (by axioms)
- Show \( D(I^\bar{p} S^*) \) satisfies the axioms to be \( I^q S^*[n] \)
  - \( \mathbb{H}^{-*}(D(I^\bar{p} S^*)[-n]) \cong I^q H_*(X) \)
- Apply Verdier duality
  - \( \mathbb{H}^{-*}(D(I^\bar{p} S^*)[-n]) \cong \text{Hom}(\mathbb{H}^*(I^\bar{p} S^*); \mathbb{Q}) \)

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Conclude \( I \bar{q} H_*(X; \mathbb{Q}) \cong \text{Hom}(I \bar{p} H_{n-*}(X), \mathbb{Q}) \)
Define $\bar{I}^p S_*(X) \subset S_*(X)$ by

$$\xi \in \bar{I}^p S_i(X)$$

if each singular simplex $\sigma \in \xi$ satisfies

$$\sigma^{-1}(S_{n-k}) \subset i-k + \bar{p}(k) \text{ skeleton of } \Delta^i$$

and each $i-1$ simplex $\tau$ in $\partial \xi$ satisfies

$$\tau^{-1}(S_{n-k}) \subset i - 1 - k + \bar{p}(k) \text{ skeleton of } \Delta^{i-1}.$$ 

Then $\bar{I}^p H_*(X) = H_*(\bar{I}^p S_*(X))$. 
Properties of Singular Chain $IH$

- Defined for all filtered spaces.
- Topological invariance on compact topological pseudomanifolds [King] and MHSSs [Quinn]
  - $IH^c_\ast$ is a *stratum-preserving homotopy invariant*
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The Singular Intersection Chain Sheaf

Consider the presheaf

\[ IS^*: U \to I\bar{p}S_{n-*}^{\infty}(X, X - \bar{U}) \]

with the natural restriction

\[ I\bar{p}S_{n-*}^{\infty}(X, X - \bar{U}) \to I\bar{p}S_{n-*}^{\infty}(X, X - \bar{V}) \]

for \( V \subset U \).

This isn’t a sheaf, but it does generate a sheaf \( IS^* \).

\( IS^* \) is homotopically fine, so:

\[ IH_{n-*}(X) \cong \mathbb{H}^*(IS^*) \cong H^*(\Gamma(X; IS^*)) \]

On pseudomanifolds, this agrees with Deligne sheaf intersection homology [GBF].
Consider the presheaf

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The Singular Intersection Chain Sheaf

- Consider the presheaf
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How to Avoid Distinguished Neighborhoods

- Pure subsets of MHSSs have Approximate Tubular Neighborhoods\(^1\) [Hughes - extending Hughes-Taylor-Weinberger-Williams]
- It follows that points have local approximate tubular neighborhoods
- These are teardrops of stratified approximate fibrations
  - teardrops - generalize mapping cylinders
  - approximate fibrations - generalize fibrations
    - They have approximate liftings

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Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}^p S^*)$ satisfies the axioms for $\mathcal{I}^q S^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I^p H_{n-j}^\infty (U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH_{*}^\infty$ of approx. tubular nghbds

- Restriction isomorphisms
  $I^p H_{n-j}^\infty (U) \rightarrow I^p H_{n-j}^\infty (U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
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- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}pS^*)$ satisfies the axioms for $\mathcal{I}qS^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I\bar{p}H_{n-j}^\infty(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_*$ of approx. tubular neighborhoods
- Restriction isomorphisms
  $I\bar{p}H_{n-j}^\infty(U) \rightarrow I\bar{p}H_{n-j}^\infty(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}\bar{p}S^*)$ satisfies the axioms for $\mathcal{I}\bar{q}S^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $\bar{p}H^\infty_{n-j}(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_*$ of approx. tubular nghbdns
- Restriction isomorphisms
  $\bar{p}H^\infty_{n-j}(U) \to \bar{p}H^\infty_{n-j}(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}^pS^*)$ satisfies the axioms for $\mathcal{I}^qS^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I^pH^\infty_{n-j}(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_*$ of approx. tubular nghbds

- Restriction isomorphisms
  $I^pH^\infty_{n-j}(U) \to I^pH^\infty_{n-j}(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
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  - Spectral sequence for $IH^\infty_*$ of approx. tubular nghbds
- Restriction isomorphisms
  $I^pH_{n-j}^\infty(U) \rightarrow I^pH_{n-j}^\infty(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}^pS^*)$ satisfies the axioms for $\mathcal{I}^qS^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I^pH^\infty_{n-j}(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_*$ of approx. tubular nghbds
- Restriction isomorphisms
  $I^pH^\infty_{n-j}(U) \rightarrow I^pH^\infty_{n-j}(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $D(\mathcal{I}\tilde{p}S^*)$ satisfies the axioms for $\mathcal{I}\tilde{q}S^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I\tilde{p}H_\infty^{n-j}(U) = 0$ for $j > \tilde{p}(k)$
  - Spectral sequence for $IH_\infty^*$ of approx. tubular nghbds

- Restriction isomorphisms
  $I\tilde{p}H_\infty^{n-j}(U) \rightarrow I\tilde{p}H_\infty^{n-j}(U - U \cap X^{n-k})$ for $j \leq \tilde{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $\mathcal{D}(\mathcal{I}\bar{p}S^*)$ satisfies the axioms for $\mathcal{I}\bar{q}S^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I\bar{p}H^\infty_{n-j}(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_\ast$ of approx. tubular nghbds

- Restriction isomorphisms
  $I\bar{p}H^\infty_{n-j}(U) \rightarrow I\bar{p}H^\infty_{n-j}(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations
Idea of the Rest of the Proof

- Compute with singular chains (both compact and closed supports) in local approximate tubular neighborhoods
- Show that $\mathcal{I}S^*$ satisfies the Deligne axioms
- Show that $D(\mathcal{I}^pS^*)$ satisfies the axioms for $\mathcal{I}^qS^*$

Key steps - Let $U$ be a local approximate tubular neighborhood of a point $x \in X^{n-k} - X^{n-k-1}$

- $I^pH_{n-j}^\infty(U) = 0$ for $j > \bar{p}(k)$
  - Spectral sequence for $IH^\infty_*$ of approx. tubular nghbds

- Restriction isomorphisms
  $I^pH_{n-j}^\infty(U) \rightarrow I^pH_{n-j}^\infty(U - U \cap X^{n-k})$ for $j \leq \bar{p}(k)$
  - Chain arguments using properties of stratified approximate fibrations