An introduction to intersection homology with general perversity functions

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June 26, 2009

2000 Mathematics Subject Classification: Primary: 55N33, 57N80; Secondary: 55N45, 55N30, 57P10

Keywords: intersection homology, perversity, pseudomanifold,Poincaré duality, Deligne sheaf, intersection pairing

Abstract

We provide an expository survey of the different notions of perversity in intersection homology and how different perversities require different definitions of intersection homology theory itself. We trace the key ideas from the introduction of intersection homology by Goresky and MacPherson through to the recent and ongoing work of the author and others.

Contents

1 Introduction

When Goresky and MacPherson first introduced intersection homology in [32], they required its perversity parameters to satisfy a fairly rigid set of constraints. Their perversities were functions on the codimensions of strata, $\bar{p} : \mathbb{Z}^{\geq 2} \to \mathbb{Z}$, satisfying

$$\bar{p}(2) = 0 \text{ and } \bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1.$$

These strict requirements were necessary for Goresky and MacPherson to achieve their initial goals for intersection homology: that the intersection homology groups $I^{\bar{p}}H_*(X)$ should satisfy a generalized form of Poincaré duality for stratified pseudomanifolds and that they should be topological invariants, i.e they should be independent of the choice of stratification of X.

In the ensuing years, perversity parameters have evolved as the applications of intersection homology have evolved, and in many cases the basic definitions of intersection homology itself have had to evolve as well. Today, there are important results that utilize the most general possible notion of a perversity as a function

$$\bar{p} : \{\text{components of singular strata of a stratified pseudomanifold}\} \to \mathbb{Z}.$$

In this setting, one usually loses topological invariance of intersection homology (though this should be seen not as a loss but as an opportunity to study stratification data), but duality results remain, at least if one chooses the right generalizations of intersection homology.

Complicating this choice is the fact that there are a variety of approaches to intersection homology to begin with, even using Goresky and MacPherson's perversities. These include (at the least) the original simplicial chain definition [32]; Goresky and MacPherson's Deligne sheaves [33, 6]; King's singular chain intersection homology [32]; Cheeger's L^2 cohomology and L^2 Hodge theory [16]; perverse differential forms on Thom-Mather stratified spaces (and, later, on unfoldable spaces [7]), first published by Brylinski [8] but attributed to Goresky and MacPherson; and the theory of perverse sheaves [4]. Work to find the "correct" versions of these theories when general perversities are allowed has been performed by the author, using stratified coefficients for simplicial and singular intersection chains [26]; by Saralegi, using "relative" intersection homology and perverse differential forms in [54]; and by the author, generalizing the Deligne sheaf in [22]. Special cases of non-Goresky-MacPherson perversities in the L^2 Hodge theory setting have also been considered by Hausel, Hunsicker, and Mazzeo [37]; Hunsicker and Mazzeo [39]; and Hunsicker [38]. And arbitrary perversities have been available from the start in the theory of perverse sheaves!

This paper is intended to serve as something of a guidebook to the different notions of perversities and as an introduction to some new and exciting work in this area. Each stage of development of the idea of perversities was accompanied by a flurry of re-examinings of what it means to have an intersection homology theory and what spaces such a theory can handle as input, and each such re-examining had to happen within one or more of the contexts listed above. In many cases, the outcome of this re-examination led to a modification or expansion of the basic definitions. This has resulted in a, quite justified, parade of papers consumed with working through all the technical details. However, technicalities often have the unintended effect of obscuring the few key main ideas. Our goal then is to present these key ideas and their consequences in an expository fashion, referring the reader to the relevant papers for further technical developments and results. We hope that such a survey will provide something of an introduction to and overview of the recent and ongoing work of the author, but we also hope to provide a readable (and hopefully accurate!) historical account of this particular chain of ideas and an overview of the work of the many researchers who have contributed to it. We additionally hope that such an overview might constitute a suitable introduction for those wishing to learn about the basics of intersection homology and as preparation for those wishing to pursue the many intriguing new applications that general perversities bring to the theory.

This exposition is not meant to provide a comprehensive historical account but merely to cover one particular line of development. We will focus primarily on the approaches to intersection homology by simplicial and singular chains and by sheaf theory. We will touch only tangentially upon perverse differential forms when we consider Saralegi's work in Section 10; we advise the reader to consult [54] for the state of the art, as well as references to prior work, in this area. Also, we will not discuss L^2-cohomology. This is a very active field of research, as is well-demonstrated elsewhere in this volume [30], but the study of L^2-cohomology and L^2 Hodge theories that yield intersection homology with general perversities remains under development. The reader should consult the papers cited above for the work that has been done so far. We will briefly discuss perverse sheaves in Section 8.2, but the reader should consult [4] or any of the variety of fine surveys on perverse sheaves that have appeared since for more details.

We will not go into many of the myriad results and applications of intersection homology theory, especially those beyond topology proper in analysis, algebraic geometry, and representation theory. For broader references on intersection homology, the reader might start with [6, 42, 2]. These are also excellent sources for the material we will be assuming regarding sheaf theory and derived categories and functors.

We proceed roughly in historical order as follows: Section 2 provides the original Goresky-MacPherson definitions of PL pseudomanifolds and PL chain intersection homology. We also begin to look closely at the cone formula for intersection homology, which will have an important role to play throughout. In Section 3, we discuss the reasons for the original Goresky-MacPherson conditions on perversities and examine some consequences, and we introduce King's singular intersection chains. In Section 4, we turn to the sheaf-theoretic definition of intersection homology and introduce the Deligne sheaf. We discuss the intersection homology version of Poincaré duality, then we look at our first example of an intersection homology result that utilizes a non-Goresky-MacPherson perversity, the Cappell-Shaneson superduality theorem.

In Section 5, we discuss "subperversities" and "superperversities". Here we first observe the schism that occurs between chain-theoretic and sheaf-theoretic intersection homology when perversities do not satisfy the Goresky-MacPherson conditions. Section 6 introduces *stratified coefficients*, which were developed by the author in order to correct the chain version of intersection homology for it to conform with the Deligne sheaf version.

In Section 7, we discuss the further evolution of the chain theory to the most general possible perversities and the ensuing results and applications. Section 8 contains the further generalization of the Deligne sheaf to general perversities, as well as a brief discussion of perverse sheaves and how general perversity intersection homology arises in that setting. Some indications of recent work and work-in-progress with these general perversities is provided in Section 9.

Finally, Sections 10 and 11 discuss some alternative approaches to intersection homology with general perversities. In Section 10, we discuss Saralegi's "relative intersection chains", which are equivalent to the author's stratified coefficients when both are defined. In Section 11, we present the work of Habegger and Saper from [35]. This work encompasses another option to correcting the schism presented in Section 5 by providing a sheaf theory that agrees with King's singular chains, rather than the other way around; however, the Habegger-Saper theory remains rather restrictive with respect to acceptable perversities.

Acknowledgment. I sincerely thank my co-organizers and co-editors – Eugénie Hunsicker, Anatoly Libgober, and Laurentiu Maxim – for making an MSRI workshop and this accompanying volume possible. I thank my collaborators Jim McClure and Eugénie Hunsicker as the impetus and encouragement for some of the work that is discussed here and for their comments on earlier drafts of this paper. And I thank an anonymous referee for a number of helpful suggestions that have greatly improved this exposition.

2 The original definition of intersection homology

We begin by recalling the original definition of intersection homology as given by Goresky and MacPherson in [32]. We must start with the spaces that intersection homology is intended to study.

2.1 Piecewise linear stratified pseudomanifolds

The spaces considered by Goresky and MacPherson in [32] were *piecewise linear (PL) stratified pseudomanifolds*. An n-dimensional PL stratified pseudomanifold X is a piecewise linear space (meaning it is endowed with a compatible family of triangulations) that also possesses a filtration by closed PL subspaces (the stratification)

$$X = X^n \supset X^{n-2} \supset X^{n-3} \supset \cdots \supset X^1 \supset X^0 \supset X^{-1} = \emptyset$$

satisfying the following properties:

1. $X - X^{n-2}$ is dense in X,
2. for each $k \geq 2$, $X^{n-k} - X^{n-k-1}$ is either empty or is an $n-k$ dimensional PL manifold,
3. if $x \in X^{n-k} - X^{n-k-1}$, then x has a *distinguished neighborhood* N that is PL homeomorphic to $\mathbb{R}^{n-k} \times cL$, where cL is the open cone on a compact $k-1$ dimensional stratified PL pseudomanifold L. Also, the stratification of L must be compatible with the stratification of X.

   A PL stratified pseudomanifold is oriented (respectively, orientable) if $X - X^{n-2}$ is.

A few aspects of this definition deserve comment. Firstly, the definition is inductive: to define an n-dimensional PL stratified pseudomanifold, we must already know what a $k-1$ dimensional PL stratified pseudomanifold is for $k - 1 < n$. The base case occurs for $n = 0$; a 0-pseudomanifold is a discrete set of points. Secondly, there is a gap from n to $n-2$ in the filtration indices. This is more-or-less intended to avoid issues of pseudomanifolds with boundary, although there are now established ways of dealing with these issues that we will return to below in Section 5.

The sets X^i are called *skeleta*, and we can verify from condition (2) that each has dimension $\leq i$ as a PL complex. The sets $X_i := X^i - X^{i-1}$ are traditionally called *strata*, though it will be more useful for us to use this term for the connected components of X_i, and we will favor this latter usage rather than speaking of "stratum components."[1] The strata of $X^n - X^{n-2}$ are called *regular strata*, and the other strata are called *singular strata*. The space L is called the link of x or of the stratum containing x. For a PL stratified pseudomanifold L is uniquely determined up to PL homeomorphism by the stratum containing x. The cone cL obtains a natural stratification from that of L: $(cL)^0$ is the cone point and for $i > 0$, $(cL)^i = L^{i-1} \times (0,1) \subset cL$, where we think of cL as $\frac{L\times[0,1)}{(x,0)\sim(y,0)}$. The compatibility condition

[1] It is perhaps worth noting here that the notation we employ throughout mostly will be consistent with the author's own work, though not necessarily with all historical sources.

of item (3) of the definition means that the PL homeomorphism should take $X^i \cap N$ to $\mathbb{R}^{n-k} \times (cL)^{i-(n-k)}$.

Roughly, the definition tells us the following. An n-dimensional PL stratified pseudomanifold X is mostly the n-manifold $X - X^{n-2}$, which is dense in X. (In much of the literature, X^{n-2} is also referred to as Σ, the singular locus of X.) The rest of X is made up of manifolds of various dimensions, and these must fit together nicely, in the sense that each point in each stratum should have a neighborhood that is a trivial fiber bundle, whose fibers are cones on lower-dimensional stratified spaces.

We should note that examples of such spaces are copious. Any complex analytic or algebraic variety can be given such a structure (see [6, Section IV]), as can certain quotient spaces of manifolds by group actions. PL pseudomanifolds occur classically as spaces that can be obtained from a pile of n-simplices by gluing in such a way that each $n-1$ face of an n-simplex is glued to exactly one $n-1$ face of one other n-simplex. (Another classical condition is that we should be able to move from any simplex to any other, passing only through interiors of $n-1$ faces. This translates to say that $X - X^{n-2}$ is path connected, but we will not concern ourselves with this condition.) Other simple examples arise by taking open cones on manifolds (naturally, given the definition), by suspending manifolds (or by repeated suspensions), by gluing manifolds and pseudomanifolds together in allowable ways, etc. One can construct many useful examples by such procedures as "start with this manifold, suspend it, cross that with a circle, suspend again,..." For more detailed examples, the reader might consult [6, 2, 44].

More general notions of stratified spaces have co-evolved with the various approaches to intersection homology, mostly by dropping or weakening requirements. We shall attempt to indicate this evolution as we progress.

2.2 Perversities

Besides the spaces on which one is to define intersection homology, the other input is the perversity parameter. In the original Goresky-MacPherson definition, a perversity $\bar{p}$ is a function from the integers ≥ 2 to the non-negative integers satisfying the following properties

1. $\bar{p}(2) = 0$,
2. $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.

These conditions say that a perversity is something like a sub-step function. It starts at 0, and then each time the input increases by one, the output either stays the same or increases by one. Some of the most commonly used perversities include the zero perversity $\bar{0}(k) = 0$, the top perversity $\bar{t}(k) = k-2$, the lower-middle perversity $\bar{m}(k) = \lfloor \frac{k-2}{2} \rfloor$, and the upper middle perversity $\bar{n}(k) = \lfloor \frac{k-1}{2} \rfloor$.

The idea of the perversity is that the input number k represents the codimension of a stratum $X_{n-k} = X^{n-k} - X^{n-k-1}$ of an n-dimensional PL stratified pseudomanifold, while the output will control the extent to which the PL chains in our homology computations will be permitted to interact with these strata.

The reason for the arcane restrictions on $\bar{p}$ will be made clear below in Section 3. We will call any perversity satisfying conditions (1) and (2) a *Goresky-MacPherson perversity*, or a *GM perversity.*

2.3 Intersection homology

At last, we are ready to discuss intersection homology.

Let X be an n-dimensional PL stratified pseudomanifold, and let $C_*^T(X)$ denote the simplicial chain complex of X with respect to the triangulation T. The PL chain complex $C_*(X)$ is defined to be $\varinjlim_T C_*^T(X)$, where the limit is taken with respect to the directed set of compatible triangulations. This PL chain complex is utilized by Goresky and MacPherson in [32] (see also [6]), and it is useful in a variety of other contexts (see, e.g. [46]). However, it turns out that this is somewhat technical overkill for the basic definition of intersection homology, as what follows can also be performed in $C_*^T(X)$, assuming T is sufficiently refined with respect to the the stratification of X (for example, pick any T, take two barycentric subdivisions, and you're set to go - see [45]).

We now define the perversity $\bar{p}$ intersection chain complex $I^{\bar{p}}C_*(X) \subset C_*(X)$. We say that a PL *$j$-simplex* σ is *$\bar{p}$-allowable* provided

$$\dim(\sigma \cap X_{n-k}) \leq j - k + \bar{p}(k)$$

for all $k \geq 2$. We say that a PL *i-chain* $\xi \in C_i(X)$ is *$\bar{p}$-allowable* if each i-simplex occurring with non-zero coefficient in ξ is $\bar{p}$-allowable and if each $i-1$ simplex occurring with non-zero coefficient in $\partial\xi$ is $\bar{p}$-allowable. Notice that the simplices in ξ must satisfy the simplex allowability condition with $j = i$ while the simplices of $\partial\xi$ must satisfy the condition with $j = i - 1$.

Then $I^{\bar{p}}C_*(X)$ is defined to be the complex of allowable chains. It follows immediately from the definition that this is indeed a chain complex. The intersection homology groups are $I^{\bar{p}}H_*(X) = H_*(I^{\bar{p}}C_*(X))$.

Some remarks are in order.

Remark 2.1. The allowability condition at first seems rather mysterious. However, the condition $\dim(\sigma \cap X_{n-k}) \leq j - k$ would be precisely the requirement that σ and X_{n-k} intersect in general position if X_{n-k} were a submanifold of X. Thus introducing a perversity can be seen as allowing deviation from general position to a degree determined by the perversity. This seems to be the origin of the nomenclature.

Remark 2.2. It is a key observation that if ξ is an i-chain, then it is not every $i-1$ face of every i-simplex of ξ that must be checked for its allowability, but only those that survive in $\partial\xi$. Boundary pieces that cancel out do not need to be checked for allowability. This seemingly minor point accounts for many subtle phenomena, including the next remark.

Remark 2.3. Intersection homology with coefficients $I^{\bar{p}}H_*(X;G)$ can be defined readily enough beginning with $C_*(X;G)$ instead of $C_*(X)$. However, $I^{\bar{p}}C_*(X;G)$ is generally *NOT* the same as $I^{\bar{p}}C_*(X)\otimes G$. This is precisely due to the boundary cancellation behavior: extra boundary cancellation in chains may occur when G is a group with torsion, leading to allowable chains in $I^{\bar{p}}C_*(X;G)$ that do not come from any G-linear combinations of allowable chains in $I^{\bar{p}}C_*(X;\mathbb{Z})$. For more details on this issue, including many examples, the reader might consult [29].

Remark 2.4. In [32], Goresky and MacPherson stated the allowability condition in terms of skeleta, not strata. In other words, they define a j-simplex to be allowable if

$$\dim(\sigma \cap X^{n-k}) \leq j - k + \bar{p}(k)$$

for all $k \geq 2$. However, it is not difficult to check that the two conditions are equivalent for the perversities we are presently considering. When we move on to more general perversities, below, it becomes necessary to state the condition in terms of strata rather than in terms of skeleta.

2.4 Cones

It turns out that understanding cones plays a crucial role in almost all else in intersection homology theory, which perhaps should not be too surprising, as pseudomanifolds are all locally products of cones with euclidean space. Most of the deepest proofs concerning intersection homology can be reduced in some way to what happens in these distinguished neighborhoods. The euclidean part turns out not to cause too much trouble, but cones possess interesting and important behavior.

So let L be a compact $k-1$ dimensional PL stratified pseudomanifold, and let cL be the open cone on L. Checking allowability of a j-simplex σ with respect to the cone vertex $\{v\} = (cL)^0$ is a simple matter, since the dimension of $\sigma \cap \{v\}$ can be at most 0. Thus σ can allowably intersect v if and only if $0 \leq j-k+\bar{p}(k)$, i.e. if $j \geq k-\bar{p}(k)$. Now, suppose ξ is an allowable i-cycle in L. We can form the chain $\bar{c}\xi \in I^{\bar{p}}C_{i+1}(cL)$ by taking the cone on each simplex in the chain (by extending each simplex linearly to the cone point). We can check using the above computation (and a little more work that we'll suppress) that $\bar{c}\xi$ is allowable if $i+1 \geq k-\bar{p}(k)$, and thus $\xi = \partial\bar{c}\xi$ is a boundary; see [6, Chapters I and II]. Similar, though slightly more complicated, computations show that any allowable cycle in cL is a boundary. Thus $I^{\bar{p}}H_i(cL) = 0$ if $i \geq k-1-\bar{p}(k)$. On the other hand, if $i < k-1-\bar{p}(k)$, then no i-chain ξ can intersect v nor can any chain of which it might be a boundary. Thus ξ is left to its own devices in $cL - v$, i.e. $I^{\bar{p}}H_i(cL) = I^{\bar{p}}H_i(cL-v) \cong I^{\bar{p}}H_i(L \times (0,1))$. It turns out that intersection homology satisfies the Künneth theorem when one factor is euclidean space and we take the obvious product stratification (see [6, Chapter I]), or alternatively we can use the invariance of intersection homology under stratum-preserving homotopy equivalences (see [23]), and so in this range $I^{\bar{p}}H_i(cL) \cong I^{\bar{p}}H_i(L)$.

Altogether then, we have

$$I^{\bar{p}}H_i(cL^{k-1}) \cong \begin{cases} 0, & i \geq k-1-\bar{p}(k), \\ I^{\bar{p}}H_i(L), & i < k-1-\bar{p}(k). \end{cases} \tag{1}$$

We will return to this formula many times.

3 Goresky-MacPherson perversities

The reasons for the original Goresky-MacPherson conditions on perversities, as enumerated in Section 2.2, are far from obvious. Ultimately, they come down to the two initially most important properties of intersection homology: its topological invariance and its Poincaré duality.

The topological invariance property of traditional intersection homology says that when $\bar{p}$ is a Goresky-MacPherson perversity and X is a stratified pseudomanifold (PL or topological, as we'll get to soon) then $I^{\bar{p}}H_*(X)$ depends only on X and not on the choice of stratification (among those allowed by the definition). This is somewhat surprising considering how the intersection chain complex depends on the strata.

The desire for $I^{\bar{p}}H_*(X)$ to be a topological invariant leads fairly quickly to the condition that we should not allow $\bar{p}(k)$ to be negative. This will be more evident once we get to the sheaf theoretic formulation of intersection homology, but for now, consider the cone formula (1) for cL^{k-1}, and suppose that $\bar{p}(k) < 0$. Then we can check that no allowable PL chain may intersect v. Thus we see that the intersection homology of cL is the same as if we removed the cone point altogether. A little more work (see [22, Corollary 2.5]) leads more generally to the conclusion that if $\bar{p}(k) < 0$, then $I^{\bar{p}}H_*(X) \cong I^{\bar{p}}H_*(X - X_k)$. This would violate the topological invariance since, for example, topological invariance tells us that if M^n is a manifold then $I^{\bar{p}}H_*(M) \cong H_*(M)$, no matter how we stratify it[2]. But if we now allow, say, a locally-flat PL submanifold N^{n-k} and stratify by $M \supset N$, then if $\bar{p}(k) < 0$ we would have $H_*(M) \cong I^{\bar{p}}H_*(M) \cong I^{\bar{p}}H_*(M - N) \cong H_*(M - N)$. This presents a clear violation of topological invariance.

The second Goresky-MacPherson condition, that $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1$, also derives from topological invariance considerations. The following example is provided by King [41, p. 155]. We first note that, letting SX denote the suspension of X, we have $cSX \cong \mathbb{R} \times cX$ (ignoring the stratifications). This is not hard to see topologically (recall that cX is the *open* cone on X). But now if we assume X is $k-1$ dimensional and that we take the obvious stratifications of $\mathbb{R} \times cX$ (assuming some initial stratification on X), then

$$I^{\bar{p}}H_i(\mathbb{R} \times cX) \cong \begin{cases} 0, & i \geq k-1-\bar{p}(k), \\ I^{\bar{p}}H_i(X), & i < k-1-\bar{p}(k). \end{cases} \tag{2}$$

This follows from the cone formula (1) and the intersection homology Künneth theorem for which one term is unstratified [41] (or stratum-preserving homotopy equivalence [23]).

But now it also follows by an easy argument, using (1) and the Mayer-Vietoris sequence, that

$$I^{\bar{p}}H_i(SX) \cong \begin{cases} I^{\bar{p}}H_{i-1}(X), & i > k-1-\bar{p}(k), \\ 0, & i = k-1-\bar{p}(k), \\ I^{\bar{p}}H_i(X), & i < k-1-\bar{p}(k), \end{cases} \tag{3}$$

and, since SX has dimension k,

[2] Note that one choice of stratification is the trivial one containing a single regular stratum, in which case it is clear from the definition that $I^{\bar{p}}H_*(M) \cong H_*(M)$.

$$I^{\bar{p}}H_i(cSX) \cong \begin{cases} 0, & i \geq k - \bar{p}(k+1), \\ I^{\bar{p}}H_i(SX), & i < k - \bar{p}(k+1). \end{cases} \tag{4}$$

So, $I^{\bar{p}}H_i(\mathbb{R} \times cX)$ is 0 for $i \geq k - 1 - \bar{p}(k)$, while $I^{\bar{p}}H_i(cSX)$ must be 0 for $i \geq k - \bar{p}(k+1)$ and also for $i = k - 1 - \bar{p}(k)$ even if $k - 1 - \bar{p}(k) < k - \bar{p}(k+1)$. Also, it is not hard to come up with examples in which the terms that are not forced to be zero are, in fact, non-zero. If $k - 1 - \bar{p}(k) \geq k - \bar{p}(k+1)$ (i.e. $1 + \bar{p}(k) \leq \bar{p}(k+1)$), so that the special case $i = k - 1 - \bar{p}(k)$ is already in the zero range for $I^{\bar{p}}H_*(cSX)$, then topological invariance would require $k - 1 - \bar{p}(k) = k - \bar{p}(k+1)$, i.e. $\bar{p}(k+1) = \bar{p}(k) + 1$. So if we want topological invariance, $\bar{p}(k+1)$ cannot be greater than $\bar{p}(k) + 1$.

On the other hand, if $k - 1 - \bar{p}(k) < k - \bar{p}(k+1)$, then the 0 at $I^{\bar{p}}H_{k-1-\bar{p}(k)}(cSX)$ forced by the suspension formula drops below the truncation dimension cutoff at $k - \bar{p}(k+1)$ that arises from the cone formula. If $k - 1 - \bar{p}(k) = k - 1 - \bar{p}(k+1)$ (i.e. $\bar{p}(k) = \bar{p}(k+1)$), no contradiction occurs. But if $k - 1 - \bar{p}(k) < k - 1 - \bar{p}(k+1)$ (i.e. $\bar{p}(k+1) < \bar{p}(k)$), then $I^{\bar{p}}H_{k-1-\bar{p}(k+1)}(cSX)$ could be non-zero, which means, using the formula for $I^{\bar{p}}H_*(\mathbb{R} \times cX)$, that we must have $k - 1 - \bar{p}(k+1) < k - 1 - \bar{p}(k)$ (i.e. $\bar{p}(k+1) > \bar{p}(k)$), yielding a contradiction.

Hence the only viable possibilities for topological invariance are that $\bar{p}(k+1) = \bar{p}(k)$ or $\bar{p}(k+1) = \bar{p}(k) + 1$.

It turns out that both possibilities work out, and in [33], Goresky and MacPherson showed using sheaf theory that any perversity satisfying the two Goresky-MacPherson conditions yields a topologically invariant intersection homology theory. King [41] later gave a non-sheaf proof that holds even when $\bar{p}(2) > 0$.

Why, then, did Goresky and MacPherson limit consideration to perversities for which $\bar{p}(2) = 0$? For one thing, they were primarily concerned with the Poincaré duality theorem for intersection homology, which states that if X is a compact oriented n-dimensional PL stratified pseudomanifold, then there is a non-degenerate pairing

$$I^{\bar{p}}H_i(X; \mathbb{Q}) \otimes I^{\bar{q}}H_{n-i}(X; \mathbb{Q}) \to \mathbb{Q}$$

if $\bar{p}$ and $\bar{q}$ satisfy the Goresky-MacPherson conditions *and* $\bar{p} + \bar{q} = \bar{t}$, or, in other words, $\bar{p}(k) + \bar{q}(k) = k - 2$. If we were to try to allow $\bar{p}(2) > 0$, then we would have to have $\bar{q}(2) < 0$, and we have already seen that this causes trouble with topological invariance. So if we want both duality and invariance, we must have $\bar{p}(2) = \bar{q}(2) = 0$. Without this condition we might possibly have one or the other, but not both. In fact, King's invariance results for $\bar{p}(2) > 0$ implies that duality cannot hold in general when we pair a perversity with $\bar{p}(2) > 0$ with one with $\bar{q}(2) < 0$, at least not without modifying the definition of intersection homology, which we do below.

But there is another interesting reason that Goresky and MacPherson did not obtain King's invariance result for $\bar{p}(2) > 0$. When intersection homology was first introduced in [32], Goresky and MacPherson were unable initially to prove topological invariance. They eventually succeeded by reformulating intersection homology in terms of sheaf theory. But, as it turns out, when $\bar{p}(2) \neq 0$ the original sheaf theory version of intersection homology does not agree with the chain version of intersection homology we have been discussing and for which King proved topological invariance. Furthermore, the sheaf version is not a topological invariant when $\bar{p}(2) > 0$ (some examples can be found in [24]). Due to the powerful tools that sheaf theory brings to intersection homology, the sheaf theoretic point of view has largely overshadowed the chain theory. However, this discrepancy between sheaf theory and chain theory for non-GM perversities turns out to be very interesting in its own right, as we shall see.

3.1 Some consequences of the Goresky-MacPherson conditions

The Goresky-MacPherson perversity conditions have a variety of interesting consequences beyond turning out to be the right conditions to yield both topological invariance and Poincaré duality.

Recall that the allowability condition for an i-simplex σ is that $\dim(\sigma \cap X_k) \leq i-k+\bar{p}(k)$. The GM perversity conditions ensure that $\bar{p}(k) \leq k-2$, and so for any perversity we must have $i-k+\bar{p}(k) \leq i-2$. Thus no i-simplex in an allowable chain can intersect any singular stratum in the interiors of its i or its $i-1$ faces. One simple consequence of this is that no 0- or 1-simplices may intersect X^{n-2}, and so $I^{\bar{p}}H_0(X) \cong H_0(X - X^{n-2})$.

Another consequence is the following fantastic idea, also due to Goresky and MacPherson. Suppose we have a local coefficient system of groups (i.e. a locally constant sheaf) defined on $X - X^{n-2}$, even perhaps one that cannot be extended to all of X. If one looks back at early treatments of homology with local coefficient systems, for example in Steenrod [56], it is sufficient to assign a coefficient group to each simplex of a triangulation (we can think of the group as being located at the barycenter of the simplex) and then to assign to each boundary face map a homomorphism between the group on the simplex and the group on the boundary face. This turns out to be sufficient to define homology with coefficients - what happens on lower dimensional faces does not matter (roughly, everything on lower faces cancels out because we still have $\partial^2 = 0$). Since the intersection i-chains with the GM perversities have the barycenters of their simplices and of their top $i-1$ faces outside of X^{n-2}, a local coefficient system $\mathcal{G}$ on X^{n-2} is sufficient to define the intersection chain complex $I^{\bar{p}}C_*(X;\mathcal{G})$ and the resulting homology groups. For more details on this construction, see, e.g. [26].

Of course now the stratification does matter to some extent since it determines where the coefficient system is defined. However, see [6, Section V.4] for a discussion of stratifications adapted to a given coefficient system defined on an open dense set of X of codimension ≥ 2.

One powerful application of this local coefficient version of intersection homology occurs in [12], in which Cappell and Shaneson study singular knots by considering the knots in their ambient spaces as stratified spaces. They employ a local coefficient system that wraps around the knot to mimic the covering space arguments of classical knot theory. This work also contains one of the first useful applications of intersection homology with non-GM perversities. In order to explain this work, though, we first need to discuss the sheaf formulation of intersection homology, which we pick up in Section 4, below.

3.2 Singular chain intersection homology

Before moving on to discuss the sheaf-theoretic formulation of intersection homology, we jump ahead in the chronology a bit to King's introduction of singular chain intersection homology in [41]. As one would expect, singular chains are a bit more flexible than PL chains (pun somewhat intended), and the singular intersection chain complex can be defined on any filtered space $X \supset X^{n-1} \supset X^{n-2} \supset \cdots$, with no further restrictions. In fact, the "dimension" indices of the skeleta X^k need no longer have a geometric meaning. These spaces include both PL stratified pseudomanifolds and *topological stratified pseudomanifolds*, the definition of which is the same as of PL pseudomanifolds but with all requirements of piecewise linearity dropped. We also extend the previous definition now to allow an $n-1$ skeleton, and we must extend perversities accordingly to be functions $\bar{p} : \mathbb{Z}^{\geq 1} \to \mathbb{Z}$. King defines *loose perversities*, which are arbitrary functions of this type. We will return to these more general perversities in greater detail as we go on.

To define the singular intersection chain complex, which we will denote $I^{\bar{p}}S_*(X)$, we can no longer use dimension of intersection as a criterion (especially if the index of a skeleton no longer has a dimensional meaning). Instead, the natural generalization of the allowability condition is that a singular i-simplex $\sigma : \Delta^i \to X$ is allowable if

$$\sigma^{-1}(X_{n-k}) \subset \{i - k + \bar{p}(k) \text{ skeleton of } \Delta^i\}.$$

Once allowability has been defined for simplices, allowability of chains is defined as in the PL case, and we obtain the chain complex $I^{\bar{p}}S_*(X)$ and the homology groups $I^{\bar{p}}H_*(X)$. If X is a PL stratified pseudomanifold, the notation $I^{\bar{p}}H_*(X)$ for singular chain intersection homology causes no confusion; as King observes, the PL and singular intersection homology theories agree on such spaces. Also as for PL chains, and by essentially the same arguments, if X has no codimension one stratum and $\bar{p}$ is a GM perversity, singular intersection homology can take local coefficients on $X - X^{n-2}$.

From here on, when we refer to chain-theoretic intersection homology, we will mean both the singular version (in any context) and the PL version (on PL spaces).

4 Sheaf theoretic intersection homology

Although intersection homology was developed originally utilizing PL chain complexes, this approach was soon largely supplanted by the techniques of sheaf theory. Sheaf theory was brought to bear by Goresky and MacPherson in [33], originally as a means to demonstrate the topological invariance (stratification independence) of intersection homology with GM perversities; this was before King's proof of this fact using singular chains. However, it quickly became evident that sheaf theory brought many powerful tools along with it, including a Verdier duality approach to the Poincaré duality problem on pseudomanifolds. Furthermore, the sheaf theory was able to accommodate topological pseudomanifolds. This sheaf-theoretic perspective has largely dominated intersection homology theory ever since.

The Deligne sheaf. We recall that if X^n is a stratified topological pseudomanifold[3], then a primary object of interest is the so-called *Deligne sheaf*. For notation, we let $U_k = X - X^{n-k}$ for $k \geq 2$, and we let $i_k : U_k \hookrightarrow U_{k+1}$ denote the inclusion. Suppose that $\bar{p}$ is a GM perversity. We have seen that intersection homology should allow a local system of coefficients defined only on $X - X^{n-2}$; let $\mathcal{G}$ be such a local system. The Deligne sheaf complex $\mathcal{P}^*$ (or, more precisely $\mathcal{P}^*_{\bar{p},\mathcal{G}}$) is defined by an inductive process. It is[4]

$$\mathcal{P}^* = \tau_{\leq \bar{p}(n)} Ri_{n*} \dots \tau_{\leq \bar{p}(2)} Ri_{2*}(\mathcal{G} \otimes \mathcal{O}),$$

where $\mathcal{O}$ is the orientation sheaf on $X - X^{n-2}$, Ri_{k*} is the right derived functor of the pushforward functor i_{k*}, and $\tau_{\leq m}$ is the sheaf complex truncation functor that takes the sheaf complex $\mathcal{S}^*$ to $\tau_{\leq m}\mathcal{S}^*$ defined by

$$(\tau_{\leq m}\mathcal{S}^*)^i = \begin{cases} 0, & i > m, \\ \ker(d_i), & i = m, \\ \mathcal{S}^i, & i < m. \end{cases}$$

Here d_i is the differential of the sheaf complex. Recall that $\tau_{\leq m}\mathcal{S}^*$ is quasi-isomorphic to $\mathcal{S}^*$ in degrees $\leq m$ and is quasi-isomorphic to 0 in higher degrees.

Remark 4.1. Actually, the orientation sheaf $\mathcal{O}$ is not usually included here as part of the definition of $\mathcal{P}^*$, or it would be only if we were discussing $\mathcal{P}^*_{\bar{p},\mathcal{G}\otimes\mathcal{O}}$. However, it seems best to include this here so as to eliminate having to continually mess with orientation sheaves when discussing the equivalence of sheaf and chain theoretic intersection homology, which, without this convention, would read that $\mathbb{H}^*(X; \mathcal{P}^*_{\bar{p},\mathcal{G}\otimes\mathcal{O}}) \cong I^{\bar{p}}H_{n-*}(X; \mathcal{G})$; see below. Putting $\mathcal{O}$ into the definition of $\mathcal{P}^*$ as we have done here allows us to leave this nuisance tacit in what follows.

The connection between the Deligne sheaf complex (also called simply the "Deligne sheaf") and intersection homology is that it can be shown that, on an n-dimensional PL pseudomanifold, $\mathcal{P}^*$ is quasi-isomorphic to the sheaf $U \to I^{\bar{p}}C^{\infty}_{n-*}(U; \mathcal{G})$. Here the ∞ indicates that we are now working with Borel-Moore PL chain complexes, in which chains may contain an infinite number of simplices with non-zero coefficients, so long as the collection of such simplices in any chain is locally-finite. This is by contrast to the PL chain complex discussed above for which each chain can contain only finitely many simplices with non-zero coefficient. This sheaf of intersection chains is also soft, and it follows via sheaf theory that the hypercohomology of the Deligne sheaf is isomorphic to the Borel-Moore intersection homology

$$\mathbb{H}^*(X; \mathcal{P}^*) \cong I^{\bar{p}}H^{\infty}_{n-*}(X; \mathcal{G}).$$

[3] For the moment, we again make the historical assumption that there are no codimension one strata.

[4] There are several other indexing conventions. For example, it is common to shift this complex so that the coefficients $\mathcal{G}$ live in degree $-n$ and the truncations become $\tau_{\leq \bar{p}(k)-n}$. There are other conventions that make the cohomologically nontrivial degrees of the complex symmetric about 0 when n is even. We will stick with the convention that $\mathcal{G}$ lives in degree 0 throughout. For more details on other conventions, see e.g. [33].

It is also possible to recover the intersection homology we introduced initially by using compact supports:

$$\mathbb{H}^*_c(X;\mathcal{P}^*) \cong I^{\bar p}H^c_{n-*}(X;\mathcal{G}).$$

Now that we have introduced Borel-Moore chains, we will use "c" to indicate the more familiar compact (finite number of simplices) supports. If the results we discuss hold in both contexts (in particular if X is compact) we will forgo either decoration. More background and details on all of this can be found in [33, 6].

It was shown later, in [26], that a similar connection exists between the Deligne sheaf and singular chain intersection homology on *topological* pseudomanifolds. Continuing to assume GM perversities, one can also define a sheaf via the sheafification of the presheaf of *singular chains*[5] $U \to I^{\bar p}S_{n-*}(X, X-\bar U;\mathcal{G})$. This sheaf turns out to be homotopically fine, and it is again quasi-isomorphic to the Deligne sheaf. Thus once again, $\mathbb{H}^*_c(X;\mathcal{P}^*) \cong I^{\bar p}H^c_{n-*}(X;\mathcal{G})$ and $\mathbb{H}^*(X;\mathcal{P}^*) \cong I^{\bar p}H^\infty_{n-*}(X;\mathcal{G})$, which is the homology of the chain complex $I^{\bar p}S^\infty_*(X;\mathcal{G})$ consisting of chains that can involve an infinite, though locally-finite, number of simplices with non-zero coefficient.

The Goresky-MacPherson proof of topological invariance follows by showing that the Deligne sheaf is uniquely defined up to quasi-isomorphism via a set of axioms that do not depend on the stratification of the space. This proof is given in [33]. However, we would here like to focus attention on what the Deligne sheaf accomplishes locally, particularly in mind of the maxim that a sheaf theory (and sheaf cohomology) is a machine for assembling local information into global. So let's look at the local cohomology (i.e. the stalk cohomology) of the sheaf $\mathcal{P}^*$ at $x \in X_{n-k}$. This is $\mathcal{H}^*(\mathcal{P}^*)_x = H^*(\mathcal{P}^*_x) \cong \varinjlim_{x\in U} \mathbb{H}^*(U;\mathcal{P}^*) \cong \varinjlim_{x\in U} I^{\bar p}H^\infty_{n-*}(U;\mathcal{G})$, and we may assume that the limit is taken over the cofinal system of distinguished neighborhoods $N \cong \mathbb{R}^{n-k} \times cL^{k-1}$ containing x. It is not hard to see that $\mathcal{P}^*$ at $x \in X_{n-k}$ depends only on the stages of the iterative Deligne construction up through $\tau_{\le \bar p(k)} Ri_{k*}$ (at least so long as we assume that $\bar p$ is nondecreasing[6], as it will be for a GM perversity). Then it follows immediately from the definition of τ that $\mathcal{H}^*(\mathcal{P}^*)_x = 0$ for $* > \bar p(k)$. On the other hand, the pushforward construction, together with a Künneth computation and an appropriate induction step (see [6, Theorem V.2.5]), shows that for $* \le \bar p(k)$ we have

$$\begin{aligned}\mathcal{H}^*(\mathcal{P}^*)_x &\cong \mathbb{H}^*(N - N\cap X^{n-k};\mathcal{P}^*)\\ &\cong \mathbb{H}^*(\mathbb{R}^{n-k}\times(cL-v);\mathcal{P}^*)\\ &\cong \mathbb{H}^*(\mathbb{R}^{n-k+1}\times L;\mathcal{P}^*)\\ &\cong \mathbb{H}^*(L;\mathcal{P}^*|_L).\end{aligned}$$

It can also be shown that $\mathcal{P}^*|_L$ is quasi-isomorphic to the Deligne sheaf on L, so $\mathbb{H}^*(L;\mathcal{P}^*|_L) \cong I^{\bar p}H_{k-1-*}(L)$.

For future reference, we record the formula

$$\mathcal{H}^i(\mathcal{P}^*)_x \cong \begin{cases} 0, & i > \bar p(k),\\ \mathbb{H}^i(L;\mathcal{P}^*), & i \le \bar p(k),\end{cases} \tag{5}$$

[5]Since X is locally compact, we may use either c or ∞ to obtain the same sheaf.

[6]If $\bar p$ ever decreases, say at k, then the truncation $\tau_{\le \bar p(k)}$ might kill local cohomology in other strata of lower codimension.

for $x \in X_{n-k}$ and L the link of x. Once one accounts for the shift in indexing between intersection homology and Deligne sheaf hypercohomology and for the fact that we are now working with Borel-Moore chains, these computations work out to be equivalent to the cone formula (1). In fact,

$$\begin{aligned} H^*(\mathcal{P}^*_x) &\cong I^{\bar{p}}H^{\infty}_{n-*}(\mathbb{R}^{n-k} \times cL; \mathcal{G}) \\ &\cong I^{\bar{p}}H^{\infty}_{k-*}(cL; \mathcal{G}) && \text{(by the Künneth theorem)} \\ &\cong I^{\bar{p}}H_{k-*}(cL, L \times (0,1); \mathcal{G}), \end{aligned}$$

and the cone formula (1) translates directly, via the long exact sequence of the pair $(cL, L \times (0,1))$, to this being 0 for $* > \bar{p}(k)$ and $I^{\bar{p}}H_{k-1-*}(L; \mathcal{G})$ otherwise.

So the Deligne sheaf recovers the local cone formula, and one would be hard pressed to find a more direct or natural way to "sheafify" the local cone condition than the Deligne sheaf construction. This reinforces our notion that the cone formula is really at the heart of intersection homology. In fact, the axiomatic characterization of the Deligne sheaf alluded to above is strongly based upon the sheaf version of the cone formula. There are several equivalent sets of characterizing axioms. The first, $AX1_{\bar{p},\mathcal{G}}$, is satisfied by a sheaf complex $\mathcal{S}^*$ if

1. $\mathcal{S}^*$ is bounded and $\mathcal{S}^* = 0$ for $i < 0$,
2. $\mathcal{S}^*|_{X - X^{n-2}} \cong \mathcal{G} \otimes \mathcal{O}$,[7]
3. for $x \in X_{n-k}$, $H^i(\mathcal{S}^*_x) = 0$ if $i > \bar{p}(k)$, and
4. for each inclusion $i_k : U_k \to U_{k+1}$, the "attaching map" α_k given my the composition of natural morphisms $\mathcal{S}^*|_{U_{k+1}} \to i_{k*}i_k^*\mathcal{S}^* \to Ri_{k*}i_k^*\mathcal{S}^*$ is a quasi-isomorphism in degrees $\leq \bar{p}(k)$.

These axioms should technically be thought of as applying in the derived category of sheaves on X, in which case all equalities and isomorphisms should be thought of as quasi-isomorphisms of sheaf complexes. The first axiom acts as something of a normalization and ensures that $\mathcal{S}^*$ lives in the bounded derived category. The second axiom fixes the coefficients on $X - X^{n-2}$. The third and fourth axioms are equivalent to the cone formula (5); see [6, Sections V.1 and V.2]. In fact, it is again not difficult to see that the Deligne sheaf construction is designed precisely to satisfy these axioms. It turns out that these axioms completely characterize a sheaf up to quasi-isomorphism (see [6, Section V.2]), and in fact it is by showing that the sheafification of $U \to I^{\bar{p}}S_*(X, X - \bar{U}; \mathcal{G})$ satisfies these axioms that one makes the connection between the sheaf of singular intersection chains and the Deligne sheaf.

Goresky and MacPherson [33] proved the stratification independence of intersection homology by showing that the axioms $AX1$ are equivalent to other sets of axioms, including one that does not depend on the stratification of X. See [6, 33] for more details.

[7] See Remark 4.1 on page 13.

4.1 Duality

It would take us too far afield to engage in a thorough discussion of how sheaf theory and, in particular, Verdier duality lead to proofs of the intersection homology version of Poincaré duality. However, we sketch some of the main ideas, highlighting the role that the perversity functions play in the theory. For complete accounts, we refer the reader to the excellent expository sources [6, 2].

The key to sheaf-theoretic duality is the Verdier dualizing function $\mathcal{D}$. Very roughly, $\mathcal{D}$ functions as a fancy sheaf-theoretic version of the functor $\mathrm{Hom}(\cdot, R)$. In fact, $\mathcal{D}$ takes a sheaf complex $\mathcal{S}^*$ to a sheaf complex $\mathcal{H}om^*(\mathcal{S}^*, \mathbb{D}^*_X)$, where $\mathbb{D}^*_X$ is the Verdier dualizing sheaf on the space X. In reasonable situations, the dualizing sheaf $\mathbb{D}^*_X$ is quasi-isomorphic (after reindexing) to the sheaf of singular chains on X; see [6, Section V.7.2.]. For us, the most important property of the functor $\mathcal{D}$ is that it satisfies a version of the universal coefficient theorem. In particular, if $\mathcal{S}^*$ is a sheaf complex over the Dedekind domain R, then for any open $U \subset X$,

$$\mathbb{H}^i(U; \mathcal{D}\mathcal{S}^*) \cong \mathrm{Hom}(\mathbb{H}^{-i}_c(U; \mathcal{S}^*); R) \oplus \mathrm{Ext}(\mathbb{H}^{-i+1}_c(U; \mathcal{S}^*); R).$$

The key, now, to proving a duality statement in intersection homology is to show that if X is orientable over a ground field F and $\bar{p}$ and $\bar{q}$ are dual perversities, meaning $\bar{p}(k)+\bar{q}(k) = k-2$ for all $k \geq 2$, then $\mathcal{D}\mathcal{P}^*_{\bar{p}}[-n]$ is quasi-isomorphic to $\mathcal{P}^*_{\bar{q}}$. Here $[-n]$ is the degree shift by $-n$ degrees, i.e. $(\mathcal{S}^*[-n])^i = \mathcal{S}^{i-n}$, and this shift is applied to $\mathcal{D}\mathcal{P}^*$ (it is not a shifted $\mathcal{P}^*$ being dualized). It then follows from the universal coefficient theorem with field coefficients F that

$$I^{\bar{q}}H^{\infty}_{n-i}(X; F) \cong \mathrm{Hom}(I^{\bar{p}}H^c_i(X; F), F),$$

which is intersection homology Poincaré duality for pseudomanifolds.

To show that $\mathcal{D}\mathcal{P}^*_{\bar{p}}[-n]$ is quasi-isomorphic to $\mathcal{P}^*_{\bar{q}}$, it suffices to show that $\mathcal{D}\mathcal{P}^*_{\bar{p}}[-n]$ satisfies the axioms $AX1_{\bar{q}}$. Again, we will not go into full detail, but we remark the following main ideas, referring the reader to the axioms $AX1$ outlined above:

1. On $X - X^{n-2}$, $\mathcal{D}\mathcal{P}^*_{\bar{p}}[-n]$ restricts to the dual of the coefficient system $\mathcal{P}^*_{\bar{p}}|_{X-X^{n-2}}$, which is again a local coefficient system. If X is orientable and the coefficient system is trivial, then so is its dual.

2. Recall that the third and fourth axioms for the Deligne sheaf concern what happens at a point x in the stratum X_{n-k}. To compute $H^*(\mathcal{S}^*_x)$, we may compute $\varinjlim_{x\in U} \mathbb{H}^*(U; \mathcal{S}^*)$. In particular, if we let each U be a distinguished neighborhood $U \cong \mathbb{R}^{n-k} \times cL$ of x

and apply the universal coefficient theorem, we obtain

$$
\begin{aligned}
H^i(U; \mathcal{DP}^*_{\bar{p}}[-n]_x) &\cong \varinjlim_{x\in U} \mathbb{H}^i(\mathcal{DP}^*_{\bar{p}}[-n]) \\
&\cong \varinjlim_{x\in U} \mathbb{H}^{i-n}(U; \mathcal{DP}^*_{\bar{p}}) \\
&\cong \varinjlim_{x\in U} \mathrm{Hom}(\mathbb{H}^{n-i}_c(U; \mathcal{P}^*_{\bar{p}}), F) \\
&\cong \varinjlim_{x\in U} \mathrm{Hom}(I^{\bar{p}}H^c_i(\mathbb{R}^{n-k} \times cL; F), F) \\
&\cong \varinjlim_{x\in U} \mathrm{Hom}(I^{\bar{p}}H^c_i(cL; F), F).
\end{aligned} \tag{6}
$$

The last equality is from the Künneth theorem with compact supports. From the cone formula, we know that this will vanish if $i \geq k-1-\bar{p}(k)$, i.e. if $i > k-2-\bar{p}(k) = \bar{q}(k)$. This is the third item of $AX1_{\bar{q}}$.

3. The fourth item of $AX1_{\bar{q}}$ is only slightly more difficult, but the basic idea is the same. By the computations (6), $H^i(\mathcal{DP}^*_{\bar{p}}[-n]_x)$ comes down to computing $I^{\bar{p}}H^c_i(cL; F)$, which we know is isomorphic to $I^{\bar{p}}H^c_i(L; F)$ when $i < k-1-\bar{p}(k)$, i.e. $i \leq \bar{q}(k)$. It is then an easy argument to show that in fact the attaching map condition of $AX1_{\bar{q}}$ holds in this range.

4. The first axiom also follows from these computations; one checks that the vanishing of $H^i(\bar{P}^*_{\bar{p},x})$ for $i < 0$ and for $i > \bar{p}(k)$ for $x \in X_{n-k}$ is sufficient to imply that $H^i(\mathcal{DP}^*_{\bar{p}}[-n]_x)$ also vanishes for $i < 0$ or i sufficiently large.

We see quite clearly from these arguments precisely why the dual perversity condition $\bar{p}(k) + \bar{q}(k) = k-2$ is necessary in order for duality to hold.

A more general duality statement, valid over principal ideal domains, was provided by Goresky and Siegel in [34]. However, there is an added requirement that the space X be *locally* $(\bar{p}, R)$*-torsion free*. This means that for each $x \in X_{n-k}$, $I^{\bar{p}}H^c_{k-2-\bar{p}(k)}(L_x)$ is R-torsion free, where L_x is the link of x in X. The necessity of this condition is that when working over a principal ideal domain R, the Ext terms of the universal coefficient theorem for Verdier duals must be taken into account. If these link intersection homology groups had torsion, then there would be a possibly non-zero Ext term in the computation (6) when $i = \bar{q}(k)+1$, due to the degree shift in the Ext term of the universal coefficient theorem. This would prevent the proof that $\mathcal{DP}^*_{\bar{p}}[-n]$ satisfies $AX1_{\bar{q}}$, so this possibility is eliminated by hypothesis. With these assumption, there result duality pairings analogous to those that occur for manifolds using ordinary homology with $\mathbb{Z}$ coefficients. In particular, one obtains a nondegenerate intersection pairing on homology mod torsion and a nondegenerate torsion linking pairing on torsion subgroups. See [34] and [22] for more details.

This circle of ideas is critical in leading to the need for superperversities in the Cappell-Shaneson superduality theorem, which we shall now discuss.

4.2 Cappell-Shaneson superduality

The first serious application (of which the author is aware) of a non-GM perversity in sheaf theoretic intersection homology occurs in Cappell and Shaneson's [12], where they develop a generalization of the Blanchfield duality pairing of knot theory to study L-classes of certain codimension 2 subpseudomanifolds of manifolds. Their pairing is a perfect Hermitian pairing between the perversity $\bar{p}$ intersection homology $\mathbb{H}^*(X;\mathcal{P}^*_{\bar{p},\mathcal{G}})$ (with $\bar{p}$ a GM perversity) and $\mathbb{H}^{n-1-*}(X;\mathcal{P}^*_{\bar{q},\mathcal{G}^*})$, where $\mathcal{G}^*$ is a Hermitian dual system to $\mathcal{G}$ and $\bar{q}$ satisfies $\bar{p}(k)+\bar{q}(k)=k-1$. This assures that $\bar{q}$ satisfies the GM perversity condition $\bar{q}(k)\leq\bar{q}(k+1)\leq\bar{q}(k)+1$, but it also forces $\bar{q}(2)=1$. In [26], we referred to such perversities as *superperversities*, though this term was later expanded by the author to include larger classes of perversities $\bar{q}$ for which $\bar{q}(k)$ may be greater than $\bar{t}(k)=k-2$ for some k.

Cappell and Shaneson worked with the sheaf version of intersection homology throughout. Notice that the Deligne sheaf remains perfectly well-defined despite $\bar{q}$ being a non-GM perversity; the truncation process just starts at a higher degree. Let us sketch how these more general perversities come into play in the Cappell-Shaneson theory.

The Cappell-Shaneson superduality theorem holds in topological settings that generalize those in which one studies the Blanchfield pairing of Alexander modules in knot theory; see [12] for more details. The Alexander modules are the homology groups of infinite cyclic covers of knot complements, and one of the key features of these modules is that they are torsion modules over the principal ideal domain $\mathbb{Q}[t,t^{-1}]$. In fact, the Alexander polynomials are just the products of the torsion coefficients of these modules. Similarly, the Cappell-Shaneson intersection homology groups $\mathbb{H}^*(X;\mathcal{P}^*_{\bar{p},\mathcal{G}})$ are torsion modules over $\mathbb{Q}[t,t^{-1}]$ (in fact, $\mathcal{G}$ is a coefficient system with stalks equal to $\mathbb{Q}[t,t^{-1}]$ and with monodromy action determined by the linking number of a closed path with the singular locus in X). Now, what happens if we try to recreate the Poincaré duality argument from Section 4.1 in this context? For one thing, the dual of the coefficient system over $X-X^{n-2}$ becomes the dual system $\mathcal{G}^*$. More importantly, all of the Hom terms in the universal coefficient theorem for Verdier duality vanish, because all modules are torsion, but the Ext terms remain. From here, it is possible to finish the argument, replacing all Homs with Exts, but there is one critical difference. Thanks to the degree shift in Ext terms in the universal coefficient theorem, at a point $x\in X_{n-k}$, $H^i(\mathcal{D}\mathcal{P}^*_{\bar{p},\mathcal{G}}[-n]_x)$ vanishes not for $i>k-2-\bar{p}(k)$ but for $i>k-1-\bar{p}(k)$, while the attaching isomorphism holds for $i\leq k-1-\bar{p}(k)$. It follows that $\mathcal{D}\mathcal{P}^*_{\bar{p},\mathcal{G}}[-n]_x$ is quasi-isomorphic to $\mathcal{P}^*_{\bar{q},\mathcal{G}^*}$, but now $\bar{q}$ must satisfy $\bar{p}(k)+\bar{q}(k)=k-1$.

The final duality statement that arises has the form

$$I^{\bar{p}}H_i(X;\mathcal{G})^*\cong \mathrm{Ext}(I^{\bar{q}}H_{n-i-1}(X;\mathcal{G}),\mathbb{Q}[t,t^{-1}])\cong \mathrm{Hom}(I^{\bar{q}}H_{n-i-1}(X;\mathcal{G});\mathbb{Q}(t,t^{-1})/\mathbb{Q}[t,t^{-1}]),$$

where $\bar{p}(k)+\bar{q}(k)=k-1$, X is compact and orientable, and the last isomorphism is from routine homological algebra. We refer the reader to [12] for the remaining technical details.

Note that this is somewhat related to our brief discussion of the Goresky-Siegel duality theorem. In that theorem, a special condition was added to ensure the vanishing of the extra Ext term. In the Cappell-Shaneson duality theorem, the extra Ext term is accounted for by the change in perversity requirements, but it is important that all Hom terms vanish, otherwise there would still be a mismatch between the degrees in which the Hom terms survive truncation and the degrees in which the Ext terms survive truncation. It might be an enlightening exercise for the reader to work through the details.

While the Cappell-Shaneson superduality theorem generalizes the Blanchfield pairing in knot theory, the author has identified an intersection homology generalization of the Farber-Levine $\mathbb{Z}$-torsion pairing in knot theory [21]. In this case, the duality statement involves Ext^2 terms and requires perversities satisfying the duality condition $\bar{p}(k) + \bar{q}(k) = k$.

5 Subperversities and superperversities

We have already noted that King considered singular chain intersection homology for perversities satisfying $\bar{p}(2) > 0$, and, more generally, he defined in [41] a *loose* perversity to be an arbitrary function from $\{2, 3, \dots\}$ to $\mathbb{Z}$. It is not hard to see that the PL and singular chain definitions of intersection homology (with constant coefficients) go through perfectly well with loose perversities, though we have seen that we would expect to forfeit topological invariance (and perhaps Poincaré duality) with such choices. On the sheaf side, Cappell and Shaneson [12] used a perversity with $\bar{p}(2) > 0$ in their superduality theorem. Somewhat surprisingly, however, once we have broken into the realm of non-GM perversities, the sheaf and chain theoretic versions of intersection homology no longer necessarily agree.

A very basic example comes by taking $\bar{p}(k) < 0$ for some k; we will call such a perversity a *subperversity*. In the Deligne sheaf construction, a subperversity will truncate everything away and wind up with the trivial sheaf complex, whose hypercohomology groups are all 0. In the chain construction, however, we have only made it more difficult for a chain to be allowable with respect to the kth stratum. In fact, it is shown in [22, Corollary 2.5] that the condition $\bar{p}(k) < 0$ is homologically equivalent to declaring that allowable chains cannot intersect the kth stratum at all. So, for example, if $\bar{p}(k) < 0$ for all k, then $I^{\bar{p}}H^c_*(X) \cong H^c_*(X - X^{n-2})$.

The discrepancy between sheaf theoretic and chain theoretic intersection homology also occurs when perversities exceed the top perversity $\bar{t}(k) = k - 2$ for some k; we call such perversities *superperversities*. To see what the issue is, let us return once again to the cone formula, which we have seen plays the defining local (and hence global) role in intersection homology. So long as $\bar{p}$ is non-decreasing (and non-negative), the arguments of the preceding section again yield the sheaf-theoretic cone formula (5) from the Deligne construction. However, the cone formula can fail in the chain version of superperverse intersection homology.

To understand why, suppose L is a compact $k-1$ pseudomanifold, so that $(cL)_k = v$, the cone point. Recall from Section 2.4 that the cone formula comes by considering cones on allowable cycles and checking whether or not they are allowable with respect to v. In the dimensions where such cones are allowable, this kills the homology. In the dimensions where the cones are not allowable, we also cannot have any cycles intersecting the cone vertex, and the intersection homology reduces to $I^{\bar{p}}H^c_i(cL - v) \cong I^{\bar{p}}H^c_i(L \times \mathbb{R}) \cong I^{\bar{p}}H^c_i(L)$, the first isomorphism because $cL - v$ is homeomorphic to $L \times \mathbb{R}$ and the second using the Künneth theorem with the unstratified $\mathbb{R}$ (see [41]) or stratum-preserving homotopy equivalence (see [23]). These arguments hold in both the PL and singular chain settings. However, there is a subtle point these arguments overlook when perversities exceed $\bar{t}$.

If $\bar{p}$ is a GM perversity, then $\bar{p}(k) \leq k-2$ and so $k-1-\bar{p}(k) > 0$ and the cone formula guarantees that $I^{\bar{p}}H_0^c(cL)$ is always isomorphic to $I^{\bar{p}}H_0^c(L)$. In fact, we have already observed, in Section 3.1, that 0- and 1-simplices cannot intersect the singular strata. Now suppose that $\bar{p}(k) = k-1$. Then extending the cone formula should predict that $I^{\bar{p}}H_0^c(cL) = 0$. But, in these dimensions, the argument breaks down. For if x is a point in $cL - (cL)^{k-2}$ representing a cycle in $I^{\bar{p}}S_0^c(cL)$, then $\bar{c}x$ is a 1-simplex, and a quick perversity computation shows that it is now an allowable 1-simplex. However, it is not allowable as a chain since $\partial(\bar{c}x)$ has two 0-simplices, one supported at the cone vertex. This cone vertex is not allowable. The difference between this case and the prior ones is that when $i > 0$ the boundary of a cone on an i-cycle is (up to sign) that i-cycle. But when $i = 0$, there is a new boundary component. In the previous computations, this was not an issue because the 1-simplex would not have been allowable either. But now this ruins the cone formula.

In general, a careful computation shows that if L is a compact $k-1$ filtered space and $\bar{p}$ is any loose perversity, then the singular intersection homology cone formula becomes [41]

$$I^{\bar{p}}H_i^c(cL) \cong \begin{cases} 0, & i \geq k-1-\bar{p}(k), i \neq 0, \\ \mathbb{Z}, & i = 0 \text{ and } \bar{p}(k) \geq k-1, \\ I^{\bar{p}}H_i(L), & i < k-1-\bar{p}(k). \end{cases} \tag{7}$$

Which is the right cone formula? So when we allow superperversities with $\bar{p}(k) > \bar{t}(k) = k-2$, the cone formula (1) no longer holds for singular intersection homology, and there is a disagreement with the sheaf theory, for which the sheaf version (5) of (1) always holds by the construction of the Deligne sheaf (at least so long as $\bar{p}$ is non-decreasing). What, then, is the "correct" version of intersection homology for superperversities (and even more general perversities)? Sheaf theoretic intersection homology allows the use of tools such as Verdier duality, and the superperverse sheaf intersection homology plays a key role in the Cappell-Shaneson superduality theorem. On the other hand, singular intersection homology is well-defined on more general spaces and allows much more easily for homotopy arguments, such as those used in [41, 23, 25, 27].

In [35], Habegger and Saper created a sheaf theoretic generalization of King's singular chain intersection homology provided $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1$ and $\bar{p}(2) \geq 0$. This theory satisfies a version of Poincaré duality but is somewhat complicated. We will return to this below in Section 11.

Alternatively, a modification of the chain theory whose homology agrees with the hypercohomology of the Deligne sheaf even for superperversities (up to the appropriate reindexing) was introduced independently by the author in [26] and by Saralegi in [54]. This chain theory has the satisfying property of maintaining the cone formula (1) for completely general perversities, even those that are not necessarily non-decreasing, while yielding the usual intersection homology groups for GM perversities. Recently, the author has found also a generalization of the Deligne sheaf construction that yields sheaf complexes whose hypercohomology groups agree with the homology groups of this chain theory and are the usual ones for GM perversities. Of course these groups generally will not be independent of the stratification, but they do possess Poincaré duality for pseudomanifolds. Thus this theory seems to be a reasonable candidate for the most general possible intersection homology theory. We will describe this theory and its characteristics in the following sections.

Superperversities and codimension one strata. It is a remarkable point of interest that the perversity issues we have been discussing provide some additional insight into why codimension one strata needed to be left out of the definition of stratified pseudomanifolds used by Goresky and MacPherson (though I do not know if it was clear that this was the issue at the time). On the one hand, if we assume that X has a codimension one stratum and let $\bar{p}(1)=0$, then $\bar{p}(1)$ is greater than $\bar{t}(1)$, which we would expect to be $1-2=-1$, and so we run into the trouble with the cone formula described earlier in this section. On the other hand, if we let $\bar{q}(1)=\bar{t}(1)=-1$, then we run into the trouble with negative perversities described prior to that. In this latter case, the Deligne sheaf is always trivial, yielding only trivial sheaf intersection homology, so there can be no non-trivial Poincaré duality via the sheaf route (note that $\bar{p}(1)=0$ and $\bar{q}(1)=-1$ are dual perversities at $k=1$, so any consideration of duality involving the one perversity would necessarily involve the other). Similarly, there is no duality in the chain version since, for example, if $X^n\supset X^{n-1}$ is $S^1\supset \text{pt}$ then easy computations shows that $I^{\bar{q}}H_1(X)\cong H_1^c(S^1-\text{pt})=0$, while $I^{\bar{p}}H_0(X)\cong\mathbb{Z}$. Note that the first computation shows that we have also voided the stratification independence of intersection homology.

One of the nice benefits of our (and Saralegi's) "correction" to chain-theoretic intersection homology is that it allows one to include codimension one strata and still obtain Poincaré duality results. In general, though, the stratification independence does need to be sacrificed. One might argue that this is the preferred trade-off, since one might wish to use duality as a tool to study spaces *together with* their stratifications.

6 "Correcting" the definition of intersection chains

As we observed in the previous section, if $\bar{p}$ is a superperversity (i.e. $\bar{p}(k)>k-2$ for some k), then the Deligne sheaf version of intersection homology and the chain version of intersection homology need no longer agree. Modifications of the chain theory to correct this anomaly were introduced by the author in [26] and by Saralegi in [54], and these have turned out to provide a platform for the extension of other useful properties of intersection homology, including Poincaré duality. These modifications turn out to be equivalent, as proven in [28]. We first present the author's version, which is slightly more general in that it allows for the use of local coefficient systems on $X-X^{n-1}$.

As we saw in Section 5, the discrepancy between the sheaf cone formula and the chain cone formulas arises because the boundary of a 1-chain that is the cone on a 0-chain has a 0-simplex at the cone point. So to fix the cone formula, it is necessary to find a way to make the extra 0 simplex go away. This is precisely what both the author's and Saralegi's corrections do, though how they do it is described in different ways.

Motivated by the fact that Goresky-MacPherson perversity intersection chains need only have their coefficients well-defined on $X - X^{n-2}$, the author's idea was to extend the coefficients $\mathcal{G}$ on $X - X^{n-1}$ (now allowing codimension one strata) to a *stratified coefficient system* by including a "zero coefficient system" on X^{n-1}. Together these are denote $\mathcal{G}_0$. Then a coefficient on a singular simplex $\sigma : \Delta^i \to X$ is defined by a lift of $\sigma|_{\sigma^{-1}(X-X^{n-1})}$ to the bundle $\mathcal{G}$ on $X - X^{n-1}$ and by a "lift" of $\sigma|_{\sigma^{-1}(X^{n-1})}$ to the 0 coefficient system over X^{n-1}. Boundary faces then inherit their coefficients from the simplices they are boundaries of by restriction. A simplex has coefficient 0 if its coefficient lift is to the zero section over all of Δ^i. In the PL setting, coefficients of PL simplices are defined similarly. In principle, there is no reason the coefficient system over X^{n-1} must be trivial, and one could extend this definition by allowing different coefficient systems on all the strata of X; however, this idea has yet to be investigated.

With this coefficient system $\mathcal{G}_0$, the intersection chain complex $I^{\bar{p}}S_*(X; \mathcal{G}_0)$ is defined exactly as it is with ordinary coefficients - allowability of simplices is determined by the same formula, and chains are allowable if each simplex with a non-zero coefficient in the chain is allowable. So what has changed? The subtle difference is that if a simplex that is in the boundary of a chain has support in X^{n-1}, then that boundary simplex must now have coefficient 0, since that is the only possible coefficient for simplices in X^{n-1}; thus such boundary simplices vanish and need not be tested for allowability. This simple idea turns out to be enough to fix the cone formula.

Indeed, let us reconsider the example of a point x in $cL - (cL)^{k-2}$, together with a coefficient lift to $\mathcal{G}$, representing a cycle in $I^{\bar{p}}S_0^c(cL; \mathcal{G}_0)$, where $\bar{p}(k) = k-1$. As before, $\bar{c}x$ is a 1-simplex, and it is allowable. Previously, $\bar{c}x$ was not, however, allowable *as a chain* since the component of $\partial(\bar{c}x)$ in the cone vertex was not allowable. However, if we consider the boundary of $\bar{c}x$ in $I^{\bar{p}}S_0^c(cL; \mathcal{G}_0)$, then the simplex at the cone point vanishes because it must have a zero coefficient there. Thus allowability is not violated by $\bar{c}x$; it is now an allowable *chain*.

A slightly more detailed computation (see [26]) shows that, in fact,

$$I^{\bar{p}}H_i^c(cL^{k-1}; \mathcal{G}_0) \cong \begin{cases} 0, & i \geq k-1-\bar{p}(k), \\ I^{\bar{p}}H_i^c(L; \mathcal{G}_0), & i < k-1-\bar{p}(k), \end{cases} \tag{8}$$

i.e. we recover the cone formula, even if $\bar{p}(k) > k-2$.

Another pleasant feature of $I^{\bar{p}}H_*(X; \mathcal{G}_0)$ is that if $\bar{p}$ does happen to be a GM perversity and X has no codimension one strata, then $I^{\bar{p}}H_*(X; \mathcal{G}_0) \cong I^{\bar{p}}H_*(X; \mathcal{G})$, the usual intersection homology. In fact, this follows from our discussion in Section 3.1, where we noted that if $\bar{p}$ is a GM perversity then no allowable i-simplices intersect X^{n-2} in either the interiors of their i faces or the interiors of their $i-1$ faces. Thus no boundary simplices can lie entirely in X^{n-2} and canceling of boundary simplices due to the stratified coefficient system does not occur. Thus $I^{\bar{p}}H_*(X; \mathcal{G}_0)$ legitimately extends the original Goresky-MacPherson theory. Furthermore, working with this "corrected" cone formula, one can show that the resulting intersection homology groups $I^{\bar{p}}H_*^{\infty}(X;\mathcal{G}_0)$ agree on topological stratified pseudomanifolds (modulo the usual reindexing issues) with the Deligne sheaf hypercohomology groups (and similarly with compact supports), assuming that $\bar{p}(2) \geq 0$ and that $\bar{p}$ is non-decreasing. This was proven in [26] under the assumption that $\bar{p}(2) = 0$ or 1 and that $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1$, but the more general case follows from [22].

Thus, in summary, $I^{\bar{p}}H_*(X;\mathcal{G}_0)$ satisfies the cone formula, generalizes intersection homology with GM perversities, admits codimension one strata, and agrees with the Deligne sheaf for the superperversities we have considered up to this point. It turns out that stratified coefficients also permit useful results for even more general contexts.

Remark 6.1. A similar idea for modifying the definition of intersection homology for non-GM perversities occurs in the unpublished notes of MacPherson [44]. There, only locally-finite chains in $X - X^{n-1}$ are considered, but their closures in X are used to determine allowability.

7 General perversities

We have now seen that stratified coefficients $\mathcal{G}_0$ allow us to recover the cone formula (8) both when $\bar{p}$ is a GM perversity and when it is a non-decreasing superperversity. How far can we push this? The answer turns out to be "quite far!" In fact, the cone formula will hold if $\bar{p}$ is completely arbitrary. Recall that we have defined a stratum of X to be a connected component of any $X_k = X^k - X^{k-1}$. For a stratified pseudomanifold, possibly with codimension one strata, we define a *general perversity* $\bar{p}$ on X to be a function

$$\bar{p} : \{\text{singular strata of X}\} \to \mathbb{Z}.$$

Then a singular simplex $\sigma : \Delta^i \to X$ is $\bar{p}$-allowable if

$$\sigma^{-1}(Z) \subset \{i - \text{codim}(Z) + \bar{p}(Z) \text{ skeleton of } \Delta^i\}$$

for each singular stratum Z of X. Even in this generality, the cone formula (8) holds for $I^{\bar{p}}H_*^c(cL^{k-1};\mathcal{G}_0)$, replacing $\bar{p}(k)$ with $\bar{p}(v)$, where v is the cone vertex.

Such general perversities were considered in [44], following their appearance in the realm of perverse sheaves (see [4] and Section 8.2, below), and they appear in the work of Saralegi on intersection differential forms [53, 54]. They also play an important role in the intersection homology Künneth theorem of [28], which utilizes "biperversities" in which the set $X_k \times Y_l \subset X \times Y$ is given a perversity value depending on $\bar{p}(k)$ and $\bar{q}(l)$ for two perversities $\bar{p}, \bar{q}$ on X and Y, respectively; see Section 9, below.

In this section, we discuss some of the basic results on intersection homology with general perversities, most of which generalize the known theorems for GM perversities. We continue, for the most part, with the chain theory point of view. In Section 8, we will return to sheaf theory and discuss sheaf-theoretic techniques for handling general perversities.

Remark 7.1. One thing that we can continue to avoid in defining general perversities is assigning perversity values to regular strata (those in $X - X^{n-1}$) and including this as part of the data to check for allowability. The reason is as follows: If Z is a regular stratum, the allowability conditions for a singular i-simplex σ would include the condition that $\sigma^{-1}(Z)$ lie in the $i+\bar{p}(Z)$ skeleton of Δ^i. If $\bar{p}(Z)\geq 0$, then this is true of any singular i-simplex, and if $\bar{p}(Z)<0$, then this would imply that the singular simplex must not intersect Z at all, since X^{n-1} is a closed subset of X. Thus there are essentially only two possibilities. The case $\bar{p}(Z)\geq 0$ is the default that we work with already (without explicitly checking the condition that would always be satisfied on regular strata). On the other hand, the case $\bar{p}(Z)<-1$ is something of a degeneration. If $\bar{p}(Z)<0$ for all regular strata, then all singular chains must be supported in X^{n-1} and so $I^{\bar{p}}S_*(X;\mathcal{G}_0)=0$. If there are only some regular strata such that $\bar{p}(Z)<0$, then, letting X^+ denote the pseudomanifold that is the closure of the union of the regular strata Z of X such that $\bar{p}(Z)\geq 0$, we have $I^{\bar{p}}H_*(X;\mathcal{G}_0)\cong I^{\bar{p}}H_*(X^+;\mathcal{G}_0|_{X^+})$. We could have simply studied intersection homology on X^+ in the first place, so we get nothing new. Thus it is reasonable to concern ourselves only with singular strata in defining allowability of simplices.

This being said, there are occasional situations where it is useful in technical formulae to assume that $\bar{p}(Z)$ is defined for all strata. This comes up, for example, in [28], where we define perversities on product strata $Z_1\times Z_2\subset X_1\times X_2$ using formulas such as $Q_{\{\bar{p},\bar{q}\}}(Z_1\times Z_2)=\bar{p}(Z_1)+\bar{q}(Z_2)$ for perversities $\bar{p},\bar{q}$. Here $Z_1\times Z_2$ may be a singular stratum, for example, even if Z_1 is regular but Z_2 is singular. The formula has the desired consequence in [28] by setting $\bar{p}(Z_1)=0$ for Z_1 regular, and this avoids having to write out several cases.

Efficient perversities. It turns out that such generality contains a bit of overkill. In [22], we define a general perversity $\bar{p}$ to be *efficient* if $-1\leq\bar{p}(Z)\leq\operatorname{codim}(Z)-1$ for each singular stratum $Z\subset X$. Given a general $\bar{p}$, we define its *efficientization* $\check{p}$ as

$$\check{p}(Z)=\begin{cases}\operatorname{codim}(Z)-1, & \text{if } \bar{p}(Z)\geq\operatorname{codim}(Z)-1,\\ \bar{p}(Z), & \text{if } 0\leq\bar{p}(Z)\leq\operatorname{codim}(Z)-2,\\ -1, & \text{if } \bar{p}(Z)\leq -1.\end{cases}$$

It is shown in [22, Section 2] that $I^{\bar{p}}H_*(X;\mathcal{G}_0)\cong I^{\check{p}}H_*(X;\mathcal{G}_0)$. Thus it is always sufficient to restrict attention to the efficient perversities.

Efficient perversities and interiors of simplices. Efficient perversities have a nice feature that makes them technically better behaved than the more general perversities. If $\bar{p}$ is a perversity for which $\bar{p}(Z)\geq\operatorname{codim}(Z)$ for some singular stratum Z, then any i-simplex σ will be $\bar{p}$-allowable with respect to Z. In particular, Z will be allowed to intersect the image under σ of the interior of Δ^i. As such, $\sigma^{-1}(X-X^{n-1})$ could potentially have an infinite number of connected components, and a coefficient of σ might lift each component to a different branch of $\mathcal{G}$, even if $\mathcal{G}$ is a constant system. This could potentially lead to some pathologies, especially when considering intersection chains from the sheaf point of view. However, if $\bar{p}$ is efficient, then for a $\bar{p}$-allowable σ we must have $\sigma^{-1}(X-X^{n-1})$ within the $i-1$ skeleton of Δ^i. Hence assigning a coefficient lift value to one point of the interior of Δ^i determines the coefficient value at all points (on $\sigma^{-1}(X-X^{n-1})$ by the unique extension of the lift and on $\sigma^{-1}(X^{n-1})$, where it is 0). This is technically much simpler and makes the complex of chains in some sense smaller.

In [28], the complex $I^{\bar p}S_*(X;\mathcal{G}_0)$ was defined with the assumption that this "unique coefficient" property holds, meaning that a coefficient should be determined by its lift at a single point. However, as noted in [28, Appendix], even for inefficient perversities, this does not change the intersection homology. So we are free to assume all perversities are efficient, without loss of any information (at least at the level of quasi-isomorphism), and this provides a reasonable way to avoid the issue entirely.

7.1 Properties of intersection homology with general perversities and stratified coefficients

One major property that we lose in working with general perversities and stratified coefficients is independence of stratification. However, most of the other basic properties of intersection homology survive, including Poincaré duality, some of them in even a stronger form than GM perversities allow.

Basic properties. Suppose X^n is a topological stratified pseudomanifold, possibly with codimension one strata, let $\mathcal{G}$ be a coefficient system on $X - X^{n-1}$, and let $\bar p$ be a general perversity. What properties does $I^{\bar p}H_*(X;\mathcal{G}_0)$ possess?

For one thing, the most basic properties of intersection homology remain intact. It is invariant under stratum-preserving homotopy equivalences, and it possesses an excision property, long exact sequences of the pair, and Mayer-Vietoris sequences. The Künneth theorem when one term is an unstratified manifold M holds true (i.e. $I^{\bar p}S^c_*(X\times M;(\mathcal{G}\times\mathcal{G}')_0)$ is quasi-isomorphic to $I^{\bar p}S^c_*(X;\mathcal{G}_0)\otimes S^c_*(M;\mathcal{G}'_0)$). There are versions of this intersection homology with compact supports and with closed supports. And $U\to I^{\bar p}S_*(X, X-\bar U;\mathcal{G}_0)$ sheafifies to a homotopically fine sheaf whose hypercohomology groups recover the intersection homology groups, up to reindexing. It is also possible to work with PL chains on PL pseudomanifolds. For more details, see [26, 22].

Duality. Let us now discuss Poincaré duality in our present context.

Theorem 7.2 (Poincaré duality). *If F is a field*[8]*, X is an F-oriented n-dimensional stratified pseudomanifold, and $\bar p+\bar q=\bar t$ (meaning that $\bar p(Z)+\bar q(Z) = codim(Z)-2$ for all singular strata Z), then*

$$I^{\bar p}H^{\infty}_i(X;F_0)\cong Hom(I^{\bar q}H^c_{n-i}(X;F_0),F).$$

For compact orientable PL pseudomanifolds without codimension one strata and with GM perversities, this was initially proven in [32] via a combinatorial argument; a proof extending to the topological setting using the axiomatics of the Deligne sheaf and Verdier duality was obtained in [33]. This Verdier duality proof was extended to the current setting in [22] using a generalization of the Deligne sheaf that we will discuss in the following section.

[8] Recall that even in the Goresky-MacPherson setting, duality only holds, in general, with field coefficients.

It also follows from the theory of perverse sheaves [4]. Recent work of the author and Jim McClure in [31] shows that intersection homology Poincaré duality can be proven using a cap product with an intersection homology orientation class by analogy to the usual proof of Poincaré duality on manifolds (see, e.g. [36]). A slightly more restrictive statement (without proof) of duality for general perversities appears in the unpublished lecture notes of MacPherson [44] as far back as 1990.

As is the case for classical intersection homology, more general duality statements hold. These can involve local-coefficient systems, non-orientable pseudomanifolds, and, if X is *locally* $(\bar{p}, R)$*-torsion free* for the principal ideal domain R, then there are torsion linking and mod torsion intersection dualities over R. For complete details, see [22].

Pseudomanifolds with boundary and Lefschetz duality. General perversities and stratified coefficients can also be used to give an easy proof of a Lefschetz version of the duality pairing, one for which X is a pseudomanifold with boundary:

Definition 7.3. An n-dimensional *stratified pseudomanifold with boundary* is a pair $(X, \partial X)$ such that $X - \partial X$ is an n-dimensional stratified pseudomanifold and the *boundary* ∂X is an $n-1$ dimensional stratified pseudomanifold possessing a neighborhood in X that is stratified homeomorphic to $\partial X \times [0,1)$, where $[0,1)$ is unstratified and $\partial X \times [0,1)$ is given the product stratification.

Remark 7.4. A pseudomanifold may have codimension one strata that are not part of a boundary, even if they would be considered part of a boundary otherwise. For example, let M be a manifold with boundary ∂M (in the usual sense). If we consider M to be unstratified, then ∂M is the boundary of M. However, if we stratify M by the stratification $M \supset \partial M$, then ∂M is *not* a boundary of M as a stratified pseudomanifold, and in this case M is a stratified pseudomanifold without boundary.

We can now state a Lefschetz duality theorem for intersection homology of pseudomanifolds with boundary.

Theorem 7.5 (Lefschetz duality). *If F is a field, X is a compact F-oriented n-dimensional stratified pseudomanifold, and $\bar{p} + \bar{q} = \bar{t}$ (meaning that $\bar{p}(Z) + \bar{q}(Z) = codim(Z) - 2$ for all singular strata Z), then*

$$I^{\bar{p}}H_i(X; F_0) \cong Hom(I^{\bar{q}}H_{n-i}(X, \partial X; F_0), F).$$

This duality also can be extended to include local-coefficient systems, non-compact or non-orientable pseudomanifolds, and, if X is *locally* $(\bar{p}, R)$*-torsion free* for the principal ideal domain R, then there are torsion linking and mod torsion intersection dualities over R.

In fact, in the setting of intersection homology with general perversities, this Lefschetz duality follows easily from Poincaré duality. To see this, let $\hat{X} = X \cup_{\partial X} \bar{c}\partial X$, the space obtained by adjoining to X a cone on the boundary (or, equivalently, pinching the boundary to a point). Let v denote the vertex of the cone point. Let $\bar{p}_-$, $\bar{q}_+$ be the dual perversities on $\hat{X}$ such that $\bar{p}_-(Z) = \bar{p}(Z)$ and $\bar{q}_+(Z) = \bar{q}(Z)$ for each stratum Z of X, $\bar{p}_-(v) = -2$, and

$\bar{q}_+(v) = n$. Poincaré duality gives a duality isomorphism between $I^{\bar{p}_-}H_*(\hat{X})$ and $I^{\bar{q}_+}H_*(\hat{X})$. But now we simply observe that $I^{\bar{p}_-}H_*(\hat{X}) \cong I^{\bar{p}_-}H_*(\hat{X} - v) \cong I^{\bar{p}}H_*(X)$, because the perversity condition at v ensures that no singular simplex may intersect v. On the other hand, since $I^{\bar{q}_+}H_*(c\partial X) = 0$ by the cone formula, $I^{\bar{q}_+}H_*(\hat{X}) \cong I^{\bar{q}_+}H_*(\hat{X}, \bar{c}\partial X)$ by the long exact sequence of the pair, but $I^{\bar{q}_+}H_*(\hat{X}, \bar{c}\partial X) \cong I^{\bar{q}_+}H_*(X, \partial X) \cong I^{\bar{q}}H_*(X, \partial X)$ by excision.

Notice that general perversities are used in this argument even if $\bar{p}$ and $\bar{q}$ are GM perversities.

PL intersection Pairings. As in the classical PL manifold situation, the duality isomorphism of intersection homology arises out of a more general pairing of chains. In [32], Goresky and MacPherson defined the intersection pairing of PL intersection chains in a PL pseudomanifold as a generalization of the classical manifold intersection pairing. For manifolds, the intersection pairing is dual to the cup product pairing in cohomology. Given a ring R and GM perversities $\bar{p}, \bar{q}, \bar{r}$ such that $\bar{p} + \bar{q} \leq \bar{r}$, Goresky and MacPherson constructed an intersection pairing

$$I^{\bar{p}}H^c_i(X; R) \otimes I^{\bar{q}}H^c_j(X; R) \to I^{\bar{r}}H^c_{i+j-n}(X; R).$$

This pairing arises by pushing cycles into a stratified version of general position due to McCrory [47] and then taking chain-theoretic intersections.

The Goresky-MacPherson pairing is limited in that a $\bar{p}$-allowable chain and a $\bar{q}$-allowable chain can be intersected only if there is a GM perversity $\bar{r}$ such that $\bar{p}+\bar{q} \leq \bar{r}$. In particular, we must have $\bar{p} + \bar{q} \leq t$. This is more than simply a failure of the intersection of the chains to be allowable with respect to a GM perversity — if $\bar{p}+\bar{q} \not\leq \bar{t}$, then there are even technical difficulties with defining the intersection product in the first place. See [22, Section 5] for an in depth discussion of the details.

If we work with stratified coefficients, however, the problems mentioned in the preceding paragraphs can be circumvented, and we obtain pairings

$$I^{\bar{p}}H_i(X; R_0) \otimes I^{\bar{q}}H_j(X; R_0) \to I^{\bar{r}}H_{i+j-n}(X; R_0)$$

for *any* general perversities such that $\bar{p} + \bar{q} \leq \bar{r}$.

Goresky and MacPherson extended their intersection pairing to topological pseudomanifolds using sheaf theory [33]. This can also be done for general perversities and stratified coefficients, but first we must revisit the Deligne sheaf construction. We do so in the next section.

A new approach to the intersection pairing via intersection cohomology cup products is presently being pursued by the author and McClure in [31].

Further applications. Some further applications of general perversity intersection homology will be discussed below in Section 9.

8 Back to sheaf theory

8.1 A generalization of the Deligne construction

Intersection chains with stratified coefficients were introduced to provide a chain theory whose homology agrees with the hypercohomology of the Deligne sheaf when $\bar p$ is a superperversity, in particular when $\bar p(2) > 0$ or when X has codimension one strata. However, when $\bar p$ is a general perversity, our new chain formulation no long agrees with the Deligne construction. For one thing, we know that if $\bar p$ is ever negative, the Deligne sheaf is trivial. The classical Deligne construction also has no mechanism for handling perversities that assign different values to strata of the same codimension, and, even if we restrict to less general perversities, any decrease in perversity value at a later stage of the Deligne process will truncate away what might have been vital information coming from an earlier stage. Thus, we need a generalization of the Deligne process that incorporates general perversities and stratified coefficients. One method was provided by the author in [22], and we describe this now.

The first step is to modify the truncation functor to be a bit more picky. Rather than truncating a sheaf complex in the same degree at all stalks, we truncate more locally. This new truncation functor is a further generalization of the "truncation over a closed subset" functor presented in [33, Section 1.14] and attributed to Deligne; that functor is used in [33, Section 9] to study extensions of Verdier duality pairings in the context of intersection homology with GM perversities. Our construction is also related to the "intermediate extension" functor in the theory of perverse sheaves; we will discuss this in the next subsection.

Definition 8.1. Let $\mathcal{A}^*$ be a sheaf complex on X, and let $\mathfrak{F}$ be a locally-finite collection of subsets of X. Let $|\mathfrak{F}| = \cup_{V\in\mathfrak{F}} V$. Let P be a function $\mathfrak{F} \to \mathbb{Z}$. Define the presheaf $T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$ as follows. If U is an open set of X, let

$$T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*(U) = \begin{cases} \Gamma(U;\mathcal{A}^*), & U\cap|\mathfrak{F}| = \emptyset,\\ \Gamma(U;\tau_{\leq \inf\{P(V)|V\in\mathfrak{F}, U\cap V\neq\emptyset\}}\mathcal{A}^*), & U\cap|\mathfrak{F}|\neq\emptyset.\end{cases}$$

Restriction is well-defined because if $m < n$ there is a natural inclusion $\tau_{\leq m}\mathcal{A}^* \hookrightarrow \tau_{\leq n}\mathcal{A}^*$.

Let the *generalized truncation sheaf* $\tau^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$ be the sheafification of $T^{\mathfrak{F}}_{\leq P}\mathcal{A}^*$.

For maps $f : \mathcal{A}^* \to \mathcal{B}^*$ of sheaf complexes over X, we can define $\tau^{\mathfrak{F}}_{\leq P} f$ in the obvious way. In fact, $T^{\mathfrak{F}}_{\leq P} f$ is well-defined by applying the ordinary truncation functors on the appropriate subsets, and we obtain $\tau^{\mathfrak{F}}_{\leq P} f$ again by passing to limits in the sheafification process.

Using this truncation, we can modify the Deligne sheaf.

Definition 8.2. Let X be an n-dimensional stratified pseudomanifold, possibly with codimension one strata, let $\bar p$ be a general perversity, let $\mathcal{G}$ be a coefficient system on $X - X^{n-1}$, and let $\mathcal{O}$ be the orientation sheaf on $X - X^{n-1}$. Let X_k stand also for the set of strata of dimension k. Then we define the *generalized Deligne sheaf* as[9]

$$\mathcal{Q}^*_{\bar p,\mathcal{G}} = \tau^{X_0}_{\leq\bar p} Ri_{n*}\dots\tau^{X_{n-1}}_{\leq\bar p} Ri_{1*}(\mathcal{G}\otimes\mathcal{O}).$$

[9]This definition differs from that in [22] by the orientation sheaf $\mathcal{O}$ - see Remark 4.1 on page 13. For consistency, we also change notation slightly to include $\mathcal{G}$ as a subscript rather than as an argument.

If $\bar{p}$ is a GM perversity, then it is not hard to show directly that $\mathcal{Q}^*_{\bar{p},\mathcal{G}}$ is quasi-isomorphic to the usual Deligne sheaf $\mathcal{P}^*_{\bar{p},\mathcal{G}}$. Furthermore, it is shown in [22] that $\mathcal{Q}^*_{\bar{p},\mathcal{G}}$ is quasi-isomorphic to the sheaf generated by the presheaf $U \to I^{\bar{p}}S_{n-*}(X, X - \bar{U}; \mathcal{G}_0)$, and so $\mathbb{H}^*(\mathcal{Q}^*_{\bar{p},\mathcal{G}}) \cong I^{\bar{p}}H^{\infty}_{n-*}(X; \mathcal{G}_0)$ and similarly for compact supports. It is also true, generalizing the Goresky-MacPherson case, that if $\bar{p} + \bar{q} = \bar{t}$, then $\mathcal{Q}^*_{\bar{p}}$ and $\mathcal{Q}^*_{\bar{q}}$ are appropriately Verdier dual, leading to the expected Poincaré and Lefschetz duality theorems. Furthermore, for any general perversities such that $\bar{p} + \bar{q} \leq \bar{r}$, there are sheaf pairings $\mathcal{Q}^*_{\bar{p}} \otimes \mathcal{Q}^*_{\bar{q}} \to \mathcal{Q}^*_{\bar{r}}$ that generalize the PL intersection pairing. If $\bar{p} + \bar{q} \leq \bar{t}$, there is also a pairing $\mathcal{Q}^*_{\bar{p}} \otimes \mathcal{Q}^*_{\bar{q}} \to \mathbb{D}^*_X[-n]$, where $\mathbb{D}^*_X[-n]$ is the shifted Verdier dualizing complex on X. See [22] for the precise statements of these results.

8.2 Perverse sheaves

The theory of perverse sheaves provided, as far back as the early 1980s, a context for the treatment of general perversities. To quote Banagl's introduction to [2, Chapter 7]:

> In discussing the proof of the Kazhdan-Lusztig conjecture, Beilinson, Bernstein and Deligne discovered that the essential image of the category of regular holonomic $\mathcal{D}$-modules under the Riemann-Hilbert correspondence gives a natural abelian subcategory of the nonabelian bounded constructible derived category [of sheaves] on a smooth complex algebraic variety. An intrinsic characterization of this abelian subcategory was obtained by Deligne (based on discussions with Beilinson, Bernstein, and MacPherson), and independently by Kashiwara. It was then realized that one still gets an abelian subcategory if the axioms of the characterization are modified to accommodate an arbitrary perversity function, with the original axioms corresponding to the middle perversity. The objects of these abelian categories were termed *perverse sheaves*...

Thus, the phrase "perverse sheaves" refers to certain subcategories, indexed by various kinds of perversity functions, of the derived category of bounded constructible sheaf complexes on a space X. The general theory of perverse sheaves can handle general perversities, though the middle perversities are far-and-away those most commonly encountered in the literature (and, unfortunately, many expositions restrict themselves solely to this case). The remarkable thing about these categories of perverse sheaves is that they are abelian, which the derived category is not (it is only "triangulated").[10] The Deligne sheaf complexes on the various strata of X (and with appropriate coefficients systems) turn out to be the simple objects of these subcategories.

The construction of perverse sheaves is largely axiomatic, grounded in a number of quite general categorical structures. It would take us well too far afield to provide all the details. Rather, we provide an extremely rough sketch of the ideas and refer the reader to the

[10] There is an old joke in the literature that perverse sheaves are neither perverse nor sheaves. The first claim reflects the fact that perverse sheaves form abelian categories, which are much less "perverse" than triangulated categories. The second reflects simply the fact that perverse sheaves are actually complexes of sheaves.

following excellent sources: [4], [40, Chapter X], [2, Chapter 7], [5], and [20, Chapter 5]. For a more historical account, the reader should see [43].

The starting point for any discussion of perverse sheaves is the notion of T-*structures*. Very roughly, a T-structure on a triangulated category D is a pair of subcategories $(D^{\leq 0}, D^{\geq 0})$ that are complementary, in the sense that for any S in D, there is a distinguished triangle

$$S_1 \to S \to S_2,$$

with $S_1 \in D^{\leq 0}$ and S_2 in $D^{\geq 0}$. Of course there are a number of axioms that must be satisfied and that we will not discuss here. The notation reflects the canonical T-structure that occurs on the derived category of sheaves on a space X: $D^{\leq 0}(X)$ is defined to be those sheaf complexes $\mathcal{S}^*$ such that $\mathcal{H}^j(\mathcal{S}^*) = 0$ for $j > 0$, and $D^{\geq 0}(X)$ is defined to be those sheaf complexes $\mathcal{S}^*$ such that $\mathcal{H}^j(\mathcal{S}^*) = 0$ for $j < 0$. Here $\mathcal{H}^*(\mathcal{S}^*)$ denotes the derived cohomology sheaf of the sheaf complex $\mathcal{S}^*$, such that $\mathcal{H}^*(\mathcal{S}^*)_x = H^*(\mathcal{S}^*_x)$.

The *heart* (or *core*) of a T-structure is the intersection $D^{\leq 0} \cap D^{\geq 0}$. It is always an abelian category. In our canonical example, the heart consists of the sheaf complexes with nonvanishing cohomology only in degree 0. In this case, the heart is equivalent to the abelian category of sheaves on X. Already from this example, we see how truncation might play a role in providing perverse sheaves - in fact, for the sheaf complex $\mathcal{S}^*$, the distinguished triangle in this example is provided by

$$\tau_{\leq 0}\mathcal{S}^* \to \mathcal{S}^* \to \tau_{\geq 0}\mathcal{S}^*.$$

Furthermore, this example can be modified easily by shifting the truncation degree from 0 to any other integer k. This T-structure is denoted $(D^{\leq k}(X), D^{\geq k}(X))$.

The next important fact about T-structures is that if X is a space, U is an open subspace, $F = X - U$, and T-structures satisfying sufficient axioms on the derived categories of sheaves on U and F are given, they can be "glued" to provide a T-structure on the derived category of sheaves on X. The idea the reader should have in mind now is that of gluing together sheaves truncated at a certain dimension on U and at another dimension on F. This then starts to look a bit like the Deligne process. In fact, let P be a perversity[11] on the two stratum space $X \supset F$, and let $(D^{\leq P(U)}(U), D^{\geq P(U)}(U))$ and $(D^{\leq P(F)}(F), D^{\geq P(F)}(F))$ be T-structure on U and F. Then these T-structures can be glued to form a T-structure on X, denoted $({}^P D^{\leq 0}, {}^P D^{\geq 0})$.

It turns out that the subcategories ${}^P D^{\leq 0}$ and ${}^P D^{\geq 0}$ can be described quite explicitly. If $i : U \hookrightarrow X$ and $j : F \hookrightarrow X$ are the inclusions, then

$$\begin{aligned} {}^P D^{\leq 0} &= \{\mathcal{S}^* \in D^+(X) \mid \mathcal{H}^k(i^*\mathcal{S}^*) = 0 \text{ for } k > P(U) \text{ and } \mathcal{H}^k(j^*\mathcal{S}^*) = 0 \text{ for } k > P(F)\} \\ {}^P D^{\geq 0} &= \{\mathcal{S}^* \in D^+(X) \mid \mathcal{H}^k(i^*\mathcal{S}^*) = 0 \text{ for } k < P(U) \text{ and } \mathcal{H}^k(j^!\mathcal{S}^*) = 0 \text{ for } k < P(F)\}. \end{aligned}$$

If $\mathcal{S}^*$ is in the heart of this T-structure, we say it is P-perverse.

[11] The reason we use P here for a perversity, departing from both our own notation, above, and from the notation in most sources on perverse sheaves (in particular [4]) is that when we use perverse sheaf theory, below, to recover intersection homology, there will be a discrepancy between the perversity P for perverse sheaves and the perversity $\bar{p}$ for the Deligne sheaf.

More generally, if X is a space with a variety of singular strata Z and P is a perversity on the stratification of X, then it is possible to glue T-structures inductively to obtain the category of P-perverse sheaves. If $j_Z : Z \hookrightarrow X$ are the inclusions, then the P-perverse sheaves are those which satisfy $\mathcal{H}^k(j_Z^*\mathcal{S}^*) = 0$ for $k > P(Z)$ and $\mathcal{H}^k(j_Z^!\mathcal{S}^*) = 0$ for $k < P(Z)$.

These two conditions turn out to be remarkably close to the conditions for $\mathcal{S}^*$ to satisfy the Deligne sheaf axioms $AX1$. In fact, the condition $\mathcal{H}^k(j_Z^*\mathcal{S}^*) = 0$ for $k > P(Z)$ is precisely the third axiom. The condition $\mathcal{H}^k(j_Z^!\mathcal{S}^*) = 0$ for $k < P(Z)$ implies that the local attaching map is an isomorphism up to degree $P(Z)-2$; see [6, page 87]. Notice that this is a less strict requirement than that for the Deligne sheaf. Thus, Deligne sheaves are perverse sheaves, but not necessarily vice versa.

The machinery developed in [4] also contains a method for creating sheaf complexes that satisfy the intersection homology axioms $AX1$, though again it is more of an axiomatic construction than the concrete construction provided in Section 8.1. Let $U \subset X$ be an open subset of X that is a union of strata, let $i : U \hookrightarrow X$ be the inclusion, and let $\mathcal{S}^*$ be a P-perverse sheaf on U. Then there is defined in [4] the "intermediate extension functor" $i_{!*}$ such that $i_{!*}\mathcal{S}^*$ is the unique extension in the category of P-perverse sheaves of $\mathcal{S}^*$ to X (meaning that the restriction of $i_{!*}\mathcal{S}^*$ to U is quasi-isomorphic to $\mathcal{S}^*$) such that for each stratum $Z \subset X - U$ and inclusion $j : Z \hookrightarrow X$,we have $\mathcal{H}^k(j^*i_{!*}\mathcal{S}^*) = 0$ for $k \geq P(Z)$ and $\mathcal{H}^k(j^!i_{!*}\mathcal{S}^*) = 0$ for $k \leq P(Z)$. We refer the reader to [4, Section 1.4] or [20, Section 5.2] for the precise definition of the functor $i_{!*}$.

In particular, suppose we let $U = X - X^{n-1}$, that $\mathcal{S}^*$ is just the local system $\mathcal{G}$, and that $\bar{p}$ is a general perversity on X. The sheaf $\mathcal{G}$ is certainly P-perverse on U with respect to the perversity $P(U) = 0$. Now let $P(Z) = \bar{p}(Z) + 1$. It follows that for each singular stratum inclusion $j : Z \hookrightarrow X$, we have $\mathcal{H}^k(j^*i_{!*}\mathcal{G}) = 0$ for $k > \bar{p}(Z)$ and $\mathcal{H}^k(j^!i_{!*}\mathcal{G}) = 0$ for $k \leq \bar{p}(Z) + 1$. In the presence of the first condition, the second condition is equivalent to the attaching map being an isomorphism up through degree $\bar{p}(Z)$; see [6, page 87]. But, according to the axioms $AX1$, these conditions are satisfied by the perversity $\bar{p}$ Deligne sheaf, which is also easily seen to be P-perverse. Thus, since $i_{!*}\mathcal{G}$ is the unique extension of $\mathcal{G}$ with these properties, $i_{!*}\mathcal{G}$ is none other than the Deligne sheaf (up to quasi-isomorphism)! Thus we can think of the Deligne process provided in Section 8.1 as a means to provide a concrete realization of $i_{!*}\mathcal{G}$.

9 Recent and future applications of general perversities

Beyond extending the results of intersection homology with GM perversities, working with general perversities makes possible new results that do not exist in "classical" intersection homology theory. For example, we saw in Sections 7 and 8 that general perversities permit the definition of PL or sheaf-theoretic intersection pairings with no restrictions on the perversities of the intersection homology classes being intersected. In this section, we review some other recent and forthcoming results made possible by intersection homology with general perversities.

Künneth theorems and cup products. In [28], general perversities were used to provide a very general Künneth theorem for intersection homology. Some special cases had been known previously. King [41] showed that for any loose perversity $I^{\bar p}H^c_*(M \times X) \cong H_*(C^c_*(M) \otimes I^{\bar p}C^c_*(X))$ when X is a pseudomanifold, M is an unstratified manifold, and $(M \times X)^i = M \times X^i$. Special cases of this result were proven earlier by Cheeger [16], Goresky and MacPherson [32, 33], and Siegel [55]. In [18], Cohen, Goresky, and Ji provided counterexamples to the existence of a general Künneth theorem for a single perversity and showed that $I^{\bar p}H^c_*(X \times Y; R) \cong H_*(I^{\bar p}C^c_*(X; R) \otimes I^{\bar p}C^c_*(Y; R))$ for pseudomanifolds X and Y and a principal ideal domain R provided either that

1. $\bar p(a) + \bar p(b) \le \bar p(a + b) \le \bar p(a) + \bar p(b) + 1$ for all a and b, *or* that
2. $\bar p(a) + \bar p(b) \le \bar p(a + b) \le \bar p(a) + \bar p(b) + 2$ for all a and b and either X or Y is locally $(\bar p, R)$-torsion free.

The idea of [28] was to ask a broader question: for what perversities on $X \times Y$ is the intersection chain complex quasi-isomorphic to $I^{\bar p}C^c_*(X; R_0) \otimes I^{\bar q}C^c_*(X; R_0)$? This question encompasses the Cohen-Goresky-Ji Künneth theorem and the possibility of both GM and non-GM perversities $\bar p, \bar q$. However, in order to avoid the fairly complicated conditions on a single perversity found by Cohen, Goresky, and Ji, it is reasonable to consider general perversities on $X \times Y$ that assign to a singular stratum $Z_1 \times Z_2$ a value depending on $\bar p(Z_1)$ and $\bar q(Z_2)$. Somewhat surprisingly, there turn out to be many perversities on $X \times Y$ that provide the desired quasi-isomorphism. The main result of [28] is the following theorem. The statement is reworded here to account for the most general case (see [28, Theorem 3.2, Remark 3.4, Theorem 5.2]), while the statement in [28] is worded to avoid overburdening the reader too much with details of stratified coefficients, which play a minimal role that paper.

Theorem 9.1. *If R is a principal ideal domain and $\bar p$ and $\bar q$ are general perversities, then $I^QH^c_*(X \times Y; R_0) \cong H_*(I^{\bar p}C^c_*(X; R_0) \otimes I^{\bar q}C^c_*(Y; R_0))$ if the following conditions hold:*

1. *$Q(Z_1 \times Z_2) = \bar p(Z_1)$ if Z_2 is a regular stratum of Y and $Q(Z_1 \times Z_2) = \bar q(Z_2)$ if Z_1 is a regular stratum of X,*
2. *For each pair $Z_1 \times Z_2$ such that Z_1 and Z_2 are each singular strata, either*
    - *(a)* $Q(Z_1 \times Z_2) = \bar p(Z_1) + \bar q(Z_2)$, or
    - *(b)* $Q(Z_1 \times Z_2) = \bar p(Z_1) + \bar q(Z_2) + 1$, or
    - *(c)* $Q(Z_1 \times Z_2) = \bar p(Z_1) + \bar q(Z_2) + 2$ and *the torsion product $I^{\bar p}H_{codim(Z_1)-2-\bar p(Z_1)}(L_1; R_0) * I^{\bar q}H_{codim(Z_2)-2-\bar q(Z_2)}(L_2; R_0) = 0$, where L_1, L_2 are the links of Z_1, Z_2 in X, Y, respectively, and codim refers to codimension in X or Y, as appropriate.*

Furthermore, if these conditions are not satisfied, then $I^QH^c_*(X \times Y; R_0)$ will not equal $H_*(I^{\bar p}C^c_*(X; R_0) \otimes I^{\bar q}C^c_*(Y; R_0))$ in general.

Of course the torsion condition in (2c) will be satisfied automatically if R is a field or if X or Y is locally $(\bar{p}, R)$- or $(\bar{q}, R)$-torsion free. Note also that it is not required that a consistent choice among the above options be made across all products of singular strata - for each such $Z_1 \times Z_2$ one can choose independently which perversity to use from among options (2a), (2b), or, assuming the hypothesis, (2c). The theorem can also be generalized further to include stratified local coefficient systems on X or Y; we leave the details to the reader.

This Künneth theorem has opened the way toward other results in intersection homology, including the formulation by the author and Jim McClure of an intersection cohomology cup product over field coefficients that they expect to be dual to the Goresky-MacPherson intersection pairing. There does not seem to have been much past research done on or with intersection cohomology in the sense of the homology groups of cochains $I_{\bar{p}}C^*(X; R_0) = \mathrm{Hom}(I^{\bar{p}}C^c_*(X; R_0); R)$. One important reason would seem to be the prior lack of availability of a geometric cup product. A cup product using the Alexander-Whitney map is unavailable in intersection homology since it does not preserve the admissibility conditions for intersection chains - in other words, breaking chains into "front p-faces and back q-faces" (see [49, Section 48]) might destroy allowability of simplices. However, there is another classical approach to the cup product that can be adapted to intersection cohomology, provided one has an appropriate Künneth theorem. For *ordinary homology*, this alternative approach is to define a diagonal map (with field coefficients) as the composite

$$H^c_*(X) \to H^c_*(X \times X) \overset{\cong}{\leftarrow} H^c_*(X) \otimes H^c_*(X),$$

where the first map is induced by the geometric diagonal inclusion map and the second is the Eilenberg-Zilber shuffle product, which is an isomorphism by the ordinary Künneth theorem with field coefficients (note that the shuffle product should have better geometric properties than the Alexander-Whitney map because it is really just Cartesian product). The appropriate Hom dual of this composition yields the cup product. This process suggests doing something similar in intersection homology with field coefficients, and indeed the Künneth theorem of [28] provides the necessary righthand quasi-isomorphism in a diagram of the form

$$I^{\bar{s}}H^c_*(X; F_0) \to I^Q H^c_*(X \times X; F_0) \overset{\cong}{\leftarrow} I^{\bar{p}}H^c_*(X; F_0) \otimes I^{\bar{q}}H^c_*(X; F_0).$$

There results a cup product

$$I_{\bar{p}}H^*(X; F_0) \otimes I_{\bar{q}}H^*(X; F_0) \to I_{\bar{s}}H^*(X; F_0)$$

when $\bar{p} + \bar{q} \geq \bar{t} + \bar{s}$.

The intersection Künneth theorem also allows for a cap product of the form

$$I_{\bar{p}}H^i(X; F_0) \otimes I^{\bar{s}}H^c_j(X; F_0) \to I^{\bar{q}}H^c_{j-i}(X; F_0)$$

for any field F and any perversities satisfying $\bar{p} + \bar{q} \geq \bar{t} + \bar{s}$. This makes possible a Poincaré duality theorem for intersection (co)homology given by cap products with a fundamental class in $I^{\bar{0}}H_n(X; F_0)$. For further details and applications, the reader is urged to consult [31].

Perverse signatures. Right from its beginnings, there has been much interest and activity in using intersection homology to define signature (index) invariants and bordism theories under which these signatures are preserved. Signatures first appeared in intersection homology in [32] associated to the symmetric intersection pairings on $I^{\bar m}H_{2n}(X^{4n};\mathbb{Q})$ for spaces X with only strata of even codimension, such as complex algebraic varieties. The condition on strata of even codimension ensures that $I^{\bar m}H_{2n}(X^{4n};\mathbb{Q}) \cong I^{\bar n}H_{2n}(X^{4n};\mathbb{Q})$ so that this group is self-dual under the intersection pairing. These ideas were extended by Siegel [55] to the broader class of Witt spaces, which also satisfy $I^{\bar m}H_{2n}(X^{4n};\mathbb{Q}) \cong I^{\bar n}H_{2n}(X^{4n};\mathbb{Q})$. In addition, Siegel developed a bordism theory of Witt spaces, which he used to construct a geometric model for *ko*-homology at odd primes. Further far reaching generalizations of these signatures have been studied by, among others, Banagl, Cappell, Libgober, Maxim, Shaneson, and Weinberger, in various combinations [1, 3, 10, 11, 9].

Signatures on singular spaces have also been studied analytically via L^2-cohomology and L^2 Hodge theory, which are closely related to intersection homology. Such signatures may relate to duality in string theory, such as through Sen's conjecture on the dimension of spaces of self-dual harmonic forms on monopole moduli spaces. Results in these areas and closely related topics include those of Müller [48]; Dai [19]; Cheeger and Dai [17]; Hausel, Hunsicker, and Mazzeo [37, 39, 38]; Saper [51, 50]; Saper and Stern [52]; and Carron [13, 15, 14]; and work on analytic symmetric signatures is currently being pursued by Albin, Leichtmann, Mazzeo and Piazza. Much more on analytic approaches to invariants of singular spaces can be found in the other papers in the present volume [30].

A different kind of signature invariant that can be defined using non-GM perversities appears in this analytic setting in the works of Hausel, Hunsicker, and Mazzeo [37, 39, 38], in which they demonstrate that groups of L^2 harmonic forms on a manifold with fibered boundary can be identified with cohomology spaces associated to the intersection cohomology groups of varying perversities for a canonical compactification X of the manifold. These *perverse signatures* are the signatures of the nondegenerate intersection pairings on $\mathrm{im}(I^{\bar p}H_{2n}(X^{4n}) \to I^{\bar q}H_{2n}(X^{4n},\partial X^{4n}))$, when $\bar p \leq \bar q$. The signature for Witt spaces mentioned above is a special case in which $\bar p = \bar q = \bar m = \bar n$ and $\partial X = \emptyset$. If X is the compactification of the interior of a compact manifold with boundary $(M,\partial M)$ and $\bar p(Z) < 0$ and $\bar q(Z) \geq \mathrm{codim}(Z) - 1$ for all singular Z, then $I^{\bar p}H_*(X) \cong H_*(M)$, $I^{\bar q}H_*(X) \cong H_*(M,\partial M)$, and in this case the perverse signature is the classical signature associated to a manifold with boundary.

Using the Lefschetz duality results of general perversity intersection homology described above, Hunsicker and the author are currently undertaking a topological study of the perverse signatures, including research on how Novikov additivity and Wall non-additivity extend to these settings.

10 Saralegi's relative intersection chains

Independently of the author's introduction of stratified coefficients, Saralegi [54] discovered another way, in the case of a constant coefficient system, to obtain an intersection chain complex that satisfies the cone formula (1) for general perversities. In [54], he used this chain complex to prove a general perversity version of the de Rham theorem on unfoldable pseudomanifolds. These spaces are a particular type of pseudomanifold on which it is possible to define a differential form version of intersection cohomology over the real numbers. This de Rham intersection cohomology appeared in a paper by Brylinski [8], though he credits Goresky and MacPherson with the idea. Brylinski showed that for GM perversities and on a Thom-Mather stratified space, de Rham intersection cohomology is Hom dual to intersection homology with real coefficients. Working on more general "unfoldable spaces," Brasselet, Hector, and Saralegi later proved a de Rham theorem in [7], showing that this result can be obtained by integration of forms on intersection chains, and this was extended to more general perversities by Saralegi in [53]. However, [53] contains an error in the case of perversities $\bar{p}$ satisfying $\bar{p}(Z) > \operatorname{codim}(Z) - 2$ or $\bar{p}(Z) < 0$ for some singular stratum Z. This error can be traced directly to the failure of the cone formula for non-GM perversities. Saralegi introduced his *relative intersection chains*[12] in [54] specifically to correct this error.

The rough idea of Saralegi's relative chains is precisely the same as the author's motivation for introducing stratified coefficients: when a perversity on a stratum Z is too high (greater than $\operatorname{codim}(Z) - 2$), it is necessary to kill chains living in that stratum in order to preserve the cone formula. The idea of stratified coefficients is to redefine the coefficient system so that such chains are killed by virtue of their coefficients being trivial. The idea of relative chains is instead to form a quotient group so that the chains living in such strata are killed in the quotient.

More precisely, let $A^{\bar{p}}C_i(X)$ be the group generated by the $\bar{p}$-allowable i-simplices of X (notice that there is no requirement that the boundary of an element of $A^{\bar{p}}C_i(X)$ be allowable), and let $X_{\bar{t}-\bar{p}}$ be the closure of the union of the singular strata Z of X such that $\bar{p}(Z) > \operatorname{codim}(Z) - 2$. Let $A^{\bar{p}}C_i(X_{\bar{t}-\bar{p}})$ be the group generated by the $\bar{p}$ allowable i-simplices with support in $X_{\bar{t}-\bar{p}}$. Then Saralegi's relative intersection chain complex is defined to be

$$S^{\bar{p}}C^c_*(X, X_{\bar{t}-\bar{p}}) = \frac{(A^{\bar{p}}C_*(X) + A^{\bar{p}+1}C_*(X_{\bar{t}-\bar{p}})) \cap \partial^{-1}\left(A^{\bar{p}}C_{*-1}(X) + A^{\bar{p}+1}C_{*-1}(X_{\bar{t}-\bar{p}})\right)}{A^{\bar{p}+1}C_*(X_{\bar{t}-\bar{p}}) \cap \partial^{-1}A^{\bar{p}+1}C_{*-1}(X_{\bar{t}-\bar{p}})}.$$

Roughly speaking, this complex consists of $\bar{p}$-allowable chains in X and slightly more allowable chains ($(\bar{p}+1)$-allowable) in $X_{\bar{t}-\bar{p}}$ whose boundaries are also either $\bar{p}$-allowable in X or $\bar{p}+1$ allowable in $X_{\bar{t}-\bar{p}}$, but then we quotient out by those chains supported in $X_{\bar{t}-\bar{p}}$. This quotient step is akin to the stratified coefficient idea of setting simplices supported in X^{n-1} to 0. In fact, there is no harm in extending Saralegi's definition by replacing $X_{\bar{t}-\bar{p}}$ by all of X^{n-1}, since the perversity conditions already guarantee that no simplex of $A^{\bar{p}}C_i(X)$ nor the boundary of any such simplex can have support in those singular strata not in $X_{\bar{t}-\bar{p}}$. In addition, there is also nothing special about the choice $\bar{p}+1$ for allowability of chains in $X_{\bar{t}-\bar{p}}$: the idea is to throw in enough singular chains supported in the singular strata so that the boundaries of any chains in $A^{\bar{p}}C_i(X)$ will also be in "the numerator" (for example, the inallowable 0-simplex in $\partial(\bar{c}x)$ that lives at the cone vertex in our example in Section 5),

[12] These should not be confused with relative intersection chains in the sense $I^{\bar{p}}C_*(X, A) \cong I^{\bar{p}}C_*(X)/I^{\bar{p}}C_*(A)$.

but then to kill any such extra chains by taking the quotient. In other words, it would be equivalent to define Saralegi's relative intersection chain complex as

$$\frac{(A^{\bar{p}}C_*(X) + S_*(X^{n-1})) \cap \partial^{-1}\left(A^{\bar{p}}C_{*-1}(X) + S_{*-1}(X^{n-1})\right)}{S_*(X^{n-1})},$$

where $S_*(X)$ is the ordinary singular chain complex.

We refer the reader to [28, Appendix A] for a proof[13] that $S^{\bar{p}}C_*(X, X_{\bar{t}-\bar{p}}; G)$ and $I^{\bar{p}}S_*(X; G_0)$ are chain isomorphic, and so, in particular, they yield the same intersection homology groups. It is not clear that there is a well-defined version of $S^{\bar{p}}C_*(X, X_{\bar{t}-\bar{p}})$ with coefficients in a local system $\mathcal{G}$ defined only on $X - X^{n-1}$, and so stratified coefficients may be a slightly broader concept. There may also be some technical advantages in sheaf theory to avoiding quotient groups.

11 Habegger and Saper's codimension $\geq$ c intersection homology theory

Finally, we discuss briefly the work of Habegger and Saper [35], in which they introduce what they call *codimension $\geq c$ intersection homology*. This is the sheafification of King's loose perversity intersection homology. In a sense, this is the opposite approach to that of stratified coefficients: stratified coefficients were introduced to provide a chain theory that agrees with the Deligne sheaf construction for superperversities, while codimension $\geq c$ intersection homology provides a Deligne-type sheaf construction whose hypercohomology yields King's intersection homology groups. Habegger and Saper work with perversities $\bar{p} : \mathbb{Z}^{\geq 2} \to \mathbb{Z}$ such that $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$ and[14] $\bar{p}(2) \geq 0$, and they work on cs-sets, which generalize pseudomanifolds (see [41, 35]). In fact, King showed in [41] that intersection homology is independent of the stratification in this setting.

The paper [35] involves many technicalities in order to obtain the most general possible results. We will attempt to simplify the discussion greatly in order to convey what seems to be the primary stream of ideas. However, we urge the reader to consult [35] for the correct details.

Given a perversity $\bar{p}$, the "codimension $\geq c$" in the name of the theory comes from considering

$$c_{\bar{p}} = \min(\{k \in \mathbb{Z}^+ | \bar{p}(k) \leq k - 2\} \cup \{\infty\}).$$

In other words, $c_{\bar{p}}$ (or simply c when the perversity is understood) is the first codimension for which $\bar{p}$ takes the values of a GM perversity. Since the condition $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$ ensures that $\bar{p}$ will be in the Goresky-MacPherson range of values for all $k \geq c$, the number c serves as somewhat of a phase transition. At points in strata of codimension $\geq c$, the cone

[13] The proof in [28] uses a slightly different definition of intersection chains with stratified coefficients than the one given here. However, for any general perversity, the intersection chains with stratified coefficients there are quasi-isomorphic to the ones discussed here, and they are isomorphic for any efficient perversity. See [28, Appendix A] for more details.

[14] Technically, they allow $\bar{p}(2) < 0$, but in this case their theory is trivial; see [35, Corollary 4.8].

formula (1) holds locally for King's singular intersection chains (i.e. we can use the cone formula to compute the local intersection homology groups in a distinguished neighborhood). For strata of codimension $< c$, the perversity $\bar{p}$ is in the "super" range, and the cone formula fails, as observed in Section 5. So, the idea of Habegger and Saper, building on the Goresky-MacPherson-Deligne axiomatic approach to intersection homology (see Section 4, above) was to find a way to axiomatize a sheaf construction that upholds the cone formula as the Deligne sheaf does for GM perversities, but only on strata of codimension $\geq c$. This idea is successful, though somewhat complicated because the coefficients now must live on $X - X^c$ and must include the sheafification on this subspace of $U \to I^{\bar{p}}S_*(U; \mathcal{G})$.

In slightly more detail (though still leaving out many technicalities), for a fixed $\bar{p}$, let $U_c = X - X^{c_{\bar{p}}}$. Then a codimension c coefficient system $\mathcal{E}^*$ is basically a sheaf on U_c that satisfies the axiomatic properties of the sheafification of $U \to I^{\bar{p}}S_*(U; \mathcal{G})$ there with respect to some stratification of U_c. These axiomatic conditions are a modification of the axioms $AX2$ (see [6, Section V.4]), which, for a GM perversity, are equivalent to the axioms $AX1$ discussed above in Section 4. We will not pursue the axioms $AX2$ in detail here, but we note that the Habegger-Saper modification occurs by requiring certain vanishing conditions to hold only in certain degrees depending on c. This takes into account the failure of the cone formula to vanish in the expected degrees (see Section 5). Then Habegger and Saper define a sheaf complex $\mathcal{P}^*_{\bar{p},\mathcal{E}^*}$ by extending $\mathcal{E}^*$ from U_c to the rest of X by the Deligne process from this point.

Among other results in their paper, Habegger and Saper show that the hypercohomology of their sheaf complex agrees (up to reindexing and with an appropriate choice of coefficients) with the intersection homology of King on PL pseudomanifolds, that this version of intersection homology is a topological invariant, and that there is a duality theorem. To state their duality theorem, let $\bar{q}(k) = k - 2 - \bar{p}(k)$, and let $\bar{q}'(k) = \max(\bar{q}(k), 0) + c_{\bar{p}} - 2$. Then, with coefficients in a field, the Verdier dual $\mathcal{D}_X\mathcal{P}^*_{\bar{p},\mathcal{E}^*}$ is quasi-isomorphic to $\mathcal{P}^*_{\bar{q}',\mathcal{D}_{U_c}(E^*)}[c_p - 2 + n]$. Roughly speaking, and ignoring the shifting of perversities and indices, which is done for technical reasons, this says that if $\bar{p} + \bar{q} = \bar{t}$ and we dualize the sheaf of intersection chains "by hand" on U_c from $\mathcal{E}^*$ to $\mathcal{D}_{U_c}\mathcal{E}^*$, then further extensions by the Deligne process, using perversity $\bar{p}$ for $\mathcal{E}^*$ and perversity $\bar{q}$ for $\mathcal{D}_{U_c}\mathcal{E}^*$, will maintain that duality. If $\bar{p}$ is a GM perversity and X is a pseudomanifold with no codimension one strata, this recovers the duality results of Goresky and MacPherson. Unfortunately, for more general perversities, there does not seem to be an obvious way to translate this duality back into the language of chain complexes, due to the complexity of the dual coefficient system $\mathcal{D}_{U_c}\mathcal{E}^*$ that appears on U_c.

One additional note should be made concerning the duality results in [35]. As mentioned above, Habegger and Saper work on cs-sets. These are more general than pseudomanifolds, primarily in that $X - X^{n-1}$ need not be dense and there is no inductive assumption that the links be pseudomanifolds. These are the spaces on which King demonstrated his stratification independence results in [41]. Thus these results are more general than those we have been discussing on pseudomanifolds, at least as far as the space X is concerned. However, as far as the author can tell, in one sense these duality results are not quite as much more general as they at first appear, as least when considering strata of X that are not in the closure of $X - X^{n-1}$. In particular, if Z is such a stratum and it lies in U_c, then the duality results on it are tautological - induced by the "by hand" dualization of the codimension c coefficient system. But if Z is not in U_c, then the pushforwards of the Deligne process cannot reach it, and $\mathcal{P}^*|_Z = 0$. So at the sheaf level the truly interesting piece of the duality still occurs in the closure of $X - X^{n-1}$. It would be interesting to understand how the choice of coefficient system and "by hand" duality on these "extraneous" strata in U_c (the strata not in the closure of $X - X^{n-1}$) influence the hypercohomology groups and the duality there. We also note that the closure of the union of the regular strata of a cs-set may still not be a pseudomanifold, due to the lack of condition on the links. It would be interesting to explore just how much more general such spaces are and the extent to which the other results we have discussed extend to them.

We refer the reader again to [35] for the further results that can be found there, including results on the intersection pairing and Zeeman's filtration.

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