Stratified fibrations and the intersection homology of the regular neighborhoods of bottom strata

Greg Friedman
Yale University

June 24, 2002

Contents

1 Introduction

The purpose of this paper is to prove the existence of a Leray-Serre-type spectral sequence for the intersection homology (with local coefficients) of the regular neighborhood (and deleted regular neighborhood) of the bottom stratum of a stratified PL-pseudomanifold. Although it is well known that the regular neighborhood of a space X is homotopy equivalent to X, this homotopy equivalence will not necessarily preserve the stratification of a filtered space and so it is not possible to determine the intersection homology of the neighborhood from that of X alone as would be possible for ordinary homology. However, the neighborhood does retain stratified versions of certain other essential properties of regular neighborhoods. For instance, it is homotopy equivalent to the mapping cylinder of a fibration *in a manner that preserves the essential properties of the stratification.* The particular fibration involved can be provided by a special case of the stratified fibrations of Hughes [10], whose properties we must first develop. In fact, we will show that any such regular neighborhood is stratum-preserving homotopy equivalent to the mapping cylinder of such a stratified fibration and then employ the stratum-preserving homotopy invariance of intersection homology. This last seems to be a well-known fact, but its proof appears to be unavailable in the literature and is thus presented below in the full generality including local coefficient systems.

Since some of the spaces involved in carrying out our program do not appear to be even topological pseudomanifolds (in the sense of [7]), it is not clear that there exists a proof (at least along these lines) that utilizes the sheaf-theoretic machinery of intersection homology, so instead we use singular intersection homology as developed in [11]; note that on a stratified pseufomanifold, the two theories are isomorphic if we assume sheaf homology with compact supports or work on a compact space. Once we have established the proper stratified analogues of fibration theory and homotopy invariance of intersection homology, many of the subsequent details directly mirror the standard proof of the existence of such spectral sequences for the homology of ordinary fibrations, though there are still several minor modification necessary in order to handle the nuances of intersection homology and local coefficients.

Our main theorem then is the following; see below for further explanations of the notations:

Theorem 1.1 (Theorem 6.1). *Let X be a finite-dimensional stratified pseudomanifold with locally finite triangulation and filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X^n = X$ such that $X_i = \emptyset$ for $i < k$. Let $N = N(X_k)$ be an open regular neighborhood of X_k, and let L be the link of the stratum X_k (if X_k is not connected, then we can treat each component separately and each component will have its own link). Then, for any fixed perversity $\bar{p}$ and local coefficient system $\mathcal{G}$ defined on $X - X_{n-2}$, there are homological-type spectral sequences $\bar{E}^r_{p,q}$ and $E^r_{p,q}$ that abut (up to isomorphism) to $IH^{\bar{p}}_i(N - X_k; \mathcal{G})$ and $IH^{\bar{p}}_i(N; \mathcal{G})$ with respective E^2 terms*

$$\bar{E}^2_{p,q} = H_p(X_k; \mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L)) \qquad\qquad E^2_{p,q} = H_p(X_k; \mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL))$$

(cL = the open cone on L), where $\mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L)$ and $\mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL)$ are local coefficient systems with respective stalks $IH^{\bar{p}}_q(L; \mathcal{G}|L)$ and $IH^{\bar{p}}_q(cL; \mathcal{G}|cL)$ (see below). Furthermore, the map $i_* : IH^{\bar{p}}_i(N - X_k; \mathcal{G}) \to IH^{\bar{p}}_i(N; \mathcal{G})$ induced by inclusion induces a map of spectral sequences which on the the E^2 terms is determined by the coefficient homomorphism $\mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L) \to \mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL)$ given by the map on the stalk intersection homology modules induced by the inclusion $L \hookrightarrow cL$.

The steps to the proof of this theorem proceed as follows: In Section 2, we recall the definition of intersection homology on a filtered space and show that intersection homology is a stratum-preserving-homotopy invariant. In Section 3, we introduce a special case of the stratified fibrations of Hughes [10] and examine some of its properties. In particular, we show that these satisfy various stratum-preserving analogues of the standard properties of fibrations. We also show, in Section 4, how a stratified fibration can determine a bundle of coefficients on its base space with coefficient stalks given by the intersection homology of the stratified fiber. In Section 5, we present some notations and facts from the works of Quinn [16], [15] and Hughes [10] on manifold weakly stratified spaces and homotopy links. Finally, in Section 6, we put this all together to prove the theorem.

We have also provided an appendix which contains a corrected version of the proof of the theorem (see [2] and [16]) that certain deformation retract neighborhoods are homotopy equivalent to the mapping cylinders of certain path spaces.

Parts of this work originally appeared as part of the author's dissertation [5], in which the results are applied to study the properties of intersection Alexander polynomials of PL-knots which are not locally-flat (see also [4]). I thank my advisor, Sylvain Cappell, for all of his generous and invaluable guidance.

For references to intersection homology theory, the reader is advised to consult [6], [7], [11], and [3].

2 Stratum-preserving homotopy invariance of intersection homology

In this section, we review some of the relevant definitions of intersection homology on a filtered space (note that we do not yet assume the spaces to be stratified in the sense of pseudomanifolds with normally locally trivial strata (e.g. [7])). Then we fill in some of the details, which are left to the reader in [15], on the invariance of intersection homology under stratum-preserving homotopy equivalence. We also show how this works for intersection homology with local coefficients.

We first recall some standard definitions. As defined in [6], a (traditional) *perversity* $\bar{p}$ is a sequence of integers $\{\bar{p}_0, \bar{p}_1, \bar{p}_2, \dots\}$ such that $\bar{p}_i \leq \bar{p}_{i+1} \leq \bar{p}_i + 1$ and such that $\bar{p}_0 = \bar{p}_1 = \bar{p}_2 = 0$ (much of the following would hold using more general perversities (see [11], [1], [5]), but for the other common case, in which $\bar{p}_2 = 1$ (superperversities), it is not clear that intersection homology with local coefficients can be well defined in a geometric manner, i.e. without sheaves; hence we omit discussion).

A *filtered space* is a space, X, together with a collection of closed subspaces

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X.$$

If we want to emphasize both the space and the filtration, we will refer to the filtered space $(X, \{X_i\})$. Note that $X_i = X_{i+1}$ is possible. We will refer to n as the (stratified) *dimension* of X and X_{n-k} as the $n-k$ *skeleton* or the *codimension* k *skeleton*. The sets $X_i - X_{i-1}$ are the *strata* of X. We call a space either *unfiltered* or *unstratified* if we do not wish to consider any filtration on it.

We can now define the singular intersection homology of X for a perversity $\bar{p}$, $IH^{\bar{p}}_*(X)$, in the usual manner: as the homology of the chain complex $IC^{\bar{p}}_*(X)$, which is the submodule generated by the $\bar{p}$-allowable chains of the singular chain complex, $C_*(X)$. A singular i-simplex $\sigma : \Delta^i \to X$ is $\bar{p}$-*allowable* if $\sigma^{-1}(X_{n-k} - X_{n-k-1})$ is contained in the $i-k+\bar{p}(k)$ skeleton of the (polyhedral) simplex Δ^i (with the usual filtration by polyhedral skeletons), and an i-chain is $\bar{p}$-*allowabe* if it and its boundary (in the chain complex sense) are linear combinations of $\bar{p}$-allowable i-simplices and $i-1$ simplices, respectively. Similarly, we can define the intersection homology with coefficients in a group or with local coefficients (see below for more details). If X is a stratified PL-pseudomanifold with filtration determined by the stratification, then this definition of intersection homology is the standard one (with compact supports, in the sheaf language) and is a topological invariant [11]. Note, however, that for a general filtered space, X, the intersection homology may not be a topological invariant, e.g. it may depend on our choice of filtration.

Since the intersection homology definitions depend only on the codimensions of the strata, we see that we can reindex a filtration by addition of a fixed integer, accompanied by the same change to the stratified dimension, without affecting the intersection homology of the space. In what follows, we shall often defer from the norm by *not* reindexing in certain situations where it is standard to do so when working with stratified pseudomanifolds. In particular, for simplicity we will usually give filtered subspaces, $Y \subset X$, the same dimension as X and filtration indexing $Y_i = Y \cap X_i$, even though the common practice for stratified pseudomanifolds is to reindex to the simplicial dimensions of Y and Y_i by subtracting the codimension of Y in X from each index.

If X and Y are two filtered spaces, we call a map $f: X \to Y$ *stratum-preserving* if the image of each component of a stratum of X lies in a stratum of Y (compare [16]). In general, it is not required that strata of X map to strata of Y of the same (co)dimension, but unless otherwise noted we will assume that X and Y have the same stratified dimension and that $f(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$. We call f a *stratum-preserving homotopy equivalence* if there is a stratum-preserving map $g: Y \to X$ such that fg and gf are stratum-preserving homotopic to the identity (where the filtration of $X \times I$ is given by the collection $\{X_i \times I\}$). We will sometimes denote the stratum-preserving homotopy equivalence of X and Y by $X \sim_{sphe} Y$ and say that X and Y are *stratum-preserving homotopy equivalent* or *s.p.h.e.*

A slightly more general notion which we will need below is the following [15]: Consider two filtered spaces X and Y with a common subset U. A map $f: X \to Y$ of filtered spaces is a *stratum-preserving homotopy equivalence near U* provided:

1. f is stratum-preserving,
2. $f|U$ is the identity,
3. There exists a neighborhood $V \subset Y$ of U and a stratum-preserving map $g: V \to X$ such that fg and gf, where defined, are stratum-preserving homotopic rel U to the inclusions (i.e. fg is stratum-preserving homotopic rel U to the inclusion $V \hookrightarrow Y$ and gf is stratum-preserving homotopic rel U to the inclusion of $f^{-1}(V)$ into X).

We now study in some detail the effect of a stratum-preserving homotopy equivalence on intersection homology. If $X \sim_{sphe} Y$, then it follows that there must be a bijective correspondence between non-empty strata of X and non-empty strata of Y determined by the mapping. We prove the following proposition which is implied in [15]:

Proposition 2.1. *If $f: X \to Y$ is a stratum-preserving homotopy equivalence with inverse $g: Y \to X$, X and Y have the same dimension n (as filtered spaces), and $f(X_i - X_{i-1}) \subset$*

$Y_i - Y_{i-1}$ and $g(Y_i - Y_{i-1}) \subset X_i - X_{i-1}$ for each i, then f and g induce isomorphisms of intersection homology $f_* : IH_i^{\bar{p}}(X) \to IH_i^{\bar{p}}(Y)$ and $g_* : IH_i^{\bar{p}}(Y) \to IH_i^{\bar{p}}(X)$.

Proof. Note that, since X and Y are the disjoint unions of their strata, the assumptions that $f(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$ and $g(Y_i - Y_{i-1}) \subset X_i - X_{i-1}$ for each i imply that $f^{-1}(Y_i - Y_{i-1}) = X_i - X_{i-1}$ for each non-empty stratum and similarly for g.

We observe that f_* and g_* are well-defined maps on intersection homology: If $\sigma : \Delta^i \to X$ is a $\bar{p}$-allowable singular i-simplex of X, then $f_*\sigma = f\sigma : \Delta^i \to Y$ is a singular i-simplex of Y. To see that it is allowable, we note that if $Y_{n-k} - Y_{n-k-1}$ is a non-empty stratum of Y, then $(f\sigma)^{-1}(Y_{n-k} - Y_{n-k-1}) = \sigma^{-1}f^{-1}(Y_{n-k} - Y_{n-k-1})$. But $f^{-1}(Y_{n-k} - Y_{n-k-1}) = X_{n-k} - X_{n-k-1}$, so $(f\sigma)^{-1}(Y_{n-k} - Y_{n-k-1}) = \sigma^{-1}(X_{n-k} - X_{n-k-1})$, which lies in the $i - k + \bar{p}(k)$ skeleton of Δ^i since σ is $\bar{p}$-allowable in X. Hence f induces a map $f_* : IC_i^{\bar{p}}(X) \to IC_i^{\bar{p}}(Y)$ which clearly also commutes with boundary maps so that it takes cycles to cycles and boundaries to boundaries, inducing a well-defined map on intersection homology.

We next show that $(fg)_*$ and $(gf)_*$ are isomorphisms, which will suffice to complete the proof. By assumption, there exists a stratum-preserving homotopy $H : X \times I \to X$ such that, for $x \in X$, $H(x, 0)$ is the identity and $H(x, 1) = gf(x)$. Now consider a $\bar{p}$-allowable i-cycle $\sigma = \sum_j a_j\sigma_j$, where the a_j are coefficients and the σ_j are i-simplices. For the (polyhedral) i-simplex Δ^i, let $[v_0, \dots, v_i]$ and $[w_0, \dots, w_i]$ be the simplices $\Delta^i \times 0$ and $\Delta^i \times 1$ in $\Delta^i \times I$. Then $\Delta^i \times I$ can be triangulated by the $i+1$ simplices $\Delta^{i+1,l} = [v_0, \cdots, v_l, w_l, \cdots, w_i]$. Now consider the singular $i+1$ chain in X given by

$$F = \sum_j a_j \sum_l (-1)^l H \circ (\sigma_j \times \mathrm{id}) \circ \sigma_{jl}, \tag{1}$$

where $(\sigma_j \times \mathrm{id}) : \Delta^i \times I \to X \times I$ is given by σ_j on the first coordinate and the identity on the second and σ_{jl} is the singular $i+1$ simplex given by the inclusion $\Delta^{i+1,l} \hookrightarrow \Delta^i \times I$. Then, by a simple computation (see [8, §2.1] for the details), $\partial F = (gf)\sigma - \sigma$. Thus $(gf)_*\sigma = \sigma$ in ordinary homology. To show that this holds in intersection homology, it remains only to check that F is a $\bar{p}$-allowable $i+1$ chain. Since σ is $\bar{p}$-allowable by assumption and $gf\sigma$ is allowable as the image of σ under a stratum-preserving map, ∂F is $\bar{p}$-allowable, and it suffices to check that each $H \circ (\sigma_j \times \mathrm{id}) \circ \sigma_{jl} : \Delta^{i+1,l} \to X$ is a $\bar{p}$-allowable singular simplex, i.e. that $\sigma_{jl}^{-1}(\sigma_j \times \mathrm{id})^{-1}H^{-1}(X_{n-k} - X_{n-k+1})$ lies in the $i + 1 - k + \bar{p}(k)$ skeleton of $\Delta^{i+1,l}$.

We must consider the inverse image of each stratum $X_{n-k} - X_{n-k-1}$. Since we filter $X \times I$ by $\{X_k \times I\}$ and H is a stratum-preserving homotopy to the identity, $H^{-1}(X_{n-k} - X_{n-k-1}) = (X \times I)_{n-k} - (X \times I)_{n-k-1} = (X_{n-k} - X_{n-k-1}) \times I$. Then $(\sigma_j \times \mathrm{id})^{-1}[(X_{n-k} - X_{n-k-1}) \times I] = [\sigma_j^{-1}(X_{n-k} - X_{n-k-1})] \times I$. Since σ_j is $\bar{p}$-allowable, this is contained in the product of I with the $i - k + \bar{p}(k) =: r$ skeleton of Δ^i (denoted Δ^i_r). Finally, we show that $\sigma_{jl}^{-1}(\Delta^i_r \times I)$ lies in the $r + 1$ skeleton of the $i + 1$-simplex $\Delta^{i+1,l}$ by construction. In fact, $\sigma_{jl}^{-1}(\Delta^i_r \times I) = (\Delta^i_r \times I) \cap [v_0, \dots, v_l, w_l, \dots, w_i]$. But $\Delta^i_r \times I = \cup[v_{\alpha_0}, \dots, v_{\alpha_r}] \times I$, where the union is taken over all sets of ordered r-tuples of integers with $0 \le \alpha_0 < \dots < \alpha_r \le i$. Furthermore, each $[v_{\alpha_0}, \dots, v_{\alpha_r}] \times I = \cup_\beta[v_{\alpha_0}, \dots, v_{\alpha_\beta}, w_{\alpha_\beta}, \dots, w_{\alpha_r}]$. That each $[v_{\alpha_0}, \dots, v_{\alpha_r}] \times I$ is a union of such simplices follows as for our original decomposition of $\Delta^i \times I$, and each such simplex lies in the given triangulation of $\Delta^i \times I$ since each is clearly a face of one of the $\Delta^{i+1,l}$. Therefore, any simplex in the intersection of $\Delta^{i+1,l}$ and $\Delta^i_r \times I$ is spanned by a set of vertices which is a subset of both $\{v_0, \dots, v_l, w_l, \dots, w_i\}$ and one of the collections $\{v_{\alpha_0}, \dots, v_{\alpha_\beta}, w_{\alpha_\beta}, \dots, w_{\alpha_r}\}$. Thus, each such simplex must have dimension less than or equal to $r+1$, and $(\Delta^i_r \times I) \cap \Delta^{i+1,l}$ must lie in the $r+1$ skeleton of $\Delta^{i+1,l}$. Hence $H \circ (\sigma_j \times \mathrm{id}) \circ \sigma_{jl}$ is a $\bar{p}$-allowable $i+1$-simplex.

This shows that $(gf)_* = g_* f_*$ is an isomorphism of intersection homology (in fact the identity). Similarly, $(fg)_* = \mathrm{id}_*$, and so f_* and g_* must each be isomorphisms.

$\square$

This proof clearly extends to hold for any constant coefficient module. To show that this also works for local systems of coefficients, it is perhaps simplest to work with an anologue of the following version of ordinary homology with local coefficients as presented in [8, §3.I]: Given a space X with local coefficient system $\mathcal{G}$ with stalk G and given action of the fundamental group, we use this data to form a bundle of modules (or "sheaf space" in the language of sheaves), which we also denote $\mathcal{G}$. Then an element of the singular chain complex $C_i(X; \mathcal{G})$ is a finite linear combination $\sum_j n_j \sigma_j$, where each σ_j is a singular simplex $\sigma_j : \Delta^i \to X$ and n_j is a lift of σ_j to $\mathcal{G}$. If n_j and m_j are both lifts of the same singular simplex, then $n_j + m_j$ is defined using the addition on the bundle, which is continuous. We also take $n_j + m_j = 0$ if it is the lift to the zero section. The coefficient of a boundary face of a simplex is given by $\partial(\sum_j n_j \sigma_j) = \sum_{j,k} (-1)^k n_j \sigma_j | [v_0, \cdots, \hat{v}_k, \cdots, v_i]$, where each n_j on the right hand side is the restricted lift $n_j | [v_0, \cdots, \hat{v}_k, \cdots, v_i]$. With these definitions, the homology of the chain complex $C_*(X; \mathcal{G})$ is the usual homology of X with local coefficients $\mathcal{G}$. Note that given a map between spaces $f : X \to Y$ covered by a bundle map of coefficient systems $\tilde{f} : \mathcal{G} \to \mathcal{H}$, then $f_*(\sum_j n_j \sigma_j)$ is defined as $\sum_j (\tilde{f} \circ n_j) \cdot (f \circ \sigma_j)$ and induces a map on homology $f_* : H_*(X; \mathcal{G}) \to H_*(Y; \mathcal{H})$.

Before getting to intersection homology, let us first show the following:

Proposition 2.2. *Suppose we are given homotopy equivalent (unfiltered) spaces X and Y, homotopy inverses $f : X \to Y$ and $g : Y \to X$, and a local coefficient system $\mathcal{G}$ over X. Then g covered by the induced map $\tilde{g} : g^*\mathcal{G} \to \mathcal{G}$ induces a homology isomorphism $g_* : H_*(Y; g^*\mathcal{G}) \to H_*(X; \mathcal{G})$.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
g^*f^*g^*\mathcal{G} & \xrightarrow{\bar{g}} & f^*g^*\mathcal{G} & \xrightarrow{\tilde{f}} & g^*\mathcal{G} & \xrightarrow{\tilde{g}} & \mathcal{G} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{g} & X & \xrightarrow{f} & Y & \xrightarrow{g} & X,
\end{array} \tag{2}$$

where $\tilde{g}$, $\tilde{f}$, and $\bar{g}$ are the maps induced by the pullback constructions. We will first show that the map gf covered by the map $\tilde{g}\tilde{f}$ induces an isomorphism $(gf)_* : H_*(X; f^*g^*\mathcal{G}) \to H_*(X; \mathcal{G})$. To do so, let H denote a homotopy $H : X \times I \to X$ from the identity to gf. Then we have an induced bundle $H^*\mathcal{G}$ over $X \times I$ and a map $\tilde{H} : H^*\mathcal{G} \to \mathcal{G}$ such that, over $X \times 0$, this is simply the identity map $\mathcal{G} \to \mathcal{G}$ and, over $X \times 1$, it is the induced map $f^*g^*\mathcal{G} \to \mathcal{G}$.

To show the surjectivity of the homology map $(gf)_* = g_* f_*$ induced by gf covered by $\tilde{g}\tilde{f}$, let $[C]$ be an element of $H_i(X; \mathcal{G})$. Then, by our definitions, $[C]$ is represented by a cycle

$C = \sum n_j\sigma_j$, where the σ_j are singular simplices, i.e. maps of Δ^i into X, and the n_j are lifts of the σ_j to $\mathcal{G}$. We can also then consider these simplices and their lifts as maps into $X \times 0$ and $H^*\mathcal{G}|X \times 0$. Next, we can extend each simplex map σ_j to the map $\sigma_j \times \mathrm{id} : \Delta^i \times I \to X \times I$. Since bundles of coefficients are covering spaces, there exist unique lifts, $\widetilde{\sigma_j \times \mathrm{id}}$, of each of these maps extending the lift over $X \times 0$. Together these provide a homotopy from C over $X \times 0$ to a new chain, say $\bar{C} = \sum \bar{n}_j\sigma_j$, where these σ_j are the same as those above identifying X and $X \times 1$, and the $\bar{n}_j$ are given by the lifts $\widetilde{\sigma_j \times \mathrm{id}}|\Delta^i \times 1$. Furthermore, $\bar{C}$ is a cycle: since C is a cycle, the sum of the lifts over each point $x \times 0 \in X \times 0$ of $\sum \partial\sigma_i$ lies in the zero section of the bundle, but, by the properties of bundles of coefficients, the liftings of the paths $x \times I \in X \times I$ collectively provide a homomorphism from $\mathcal{G}|x \times 0 \to \mathcal{G}|x \times 1$. So the sum of the lifts over $x \times 1$ at the other end of the homotopy is also 0. Now, we can compose each of the $\sigma_j \times \mathrm{id}$ covered by $\widetilde{\sigma_j \times \mathrm{id}}$ with H covered by $\tilde{H}$ to obtain a homotopy into $(X, \mathcal{G})$ from the maps representing C to the maps representing the image of $\bar{C}$ under gf covered by $\tilde{g}\tilde{f}$. By the prism construction employed above in the proof of Proposition 2.1 to break a homotopy into simplices (and the obvious extension to coefficient lifts), this gives a homology between $[C]$ and the homology class of the image $g_*f_*(\bar{C})$. Hence $(gf)_* = g_*f_*$ is surjective.

For injectivity, let $\bar{C} = \sum \bar{n}_i\sigma_i$ be a cycle representing an element of $H_i(X; f^*g^*\mathcal{G})$ which maps to 0 in $H_i(X;\mathcal{G})$ under $(gf)_*$. As in the last paragraph, we can lift $\sigma_j \times \mathrm{id} : \Delta^i \times I \to X \times I$ to an extension of $\bar{n}_i$ in $H^*\mathcal{G}$ (although this time we extend in the other direction). Again, this induces a chain C in $(X \times 0, \mathcal{G})$, and composing the homotopies and their lifts with H covered by $\tilde{H}$ induces a homotopy and, by the prism construction and its lift, a homology from the image of $\bar{C}$ to C (recall that H and $\tilde{H}$ are the identity on and over $X \times 0$, respectively). But since the image of $\bar{C}$ is 0 by assumption, C must bound another chain with local coefficients, say $D = \sum m_k\tau_k$, where the τ_k are singular $i+1$ simplices in X and the m_k are their lifts to $\mathcal{G}$. Now, we proceed as in the last paragraph: we can consider D as a chain with coefficients in $X \times 0$ covered by $\mathcal{G}$ and lift the maps $\tau_k \times \mathrm{id} : \Delta^{i+1} \times I \to X \times I$ to maps into $H^*\mathcal{G}$ extending the lifts m_k. Again this induces a chain $\bar{D} = \sum \bar{m}_i\tau_i$ in $(X \times 1, f^*g^*\mathcal{G})$, and, due to the unique path lifting property of covering spaces, it is readily verified that $\partial\bar{D} = \bar{C}$. Thus, $\bar{C}$ represents the 0 element of $H_i(X; f^*g^*\mathcal{G})$, and it follows that the homology map $(gf)_*$ is injective.

We have thus shown that the map $(gf)_* = g_*f_*$ on homology induced by gf covered by $\tilde{g}\tilde{f}$ is an isomorphism. By the same reasoning, the map fg covered by $\tilde{f}\bar{g}$ (see diagram (2)) induces a homology isomorphism $(f\bar{g})_* = f_*\bar{g}_* : H_*(Y; g^*f^*g^*\mathcal{G}) \to H_*(Y; g^*\mathcal{G})$. But together these imply that the homology map induced by f covered by $\tilde{f}$ is injective and surjective and hence an isomorphism $H_*(X; f^*g^*\mathcal{G}) \to H_*(Y; g^*\mathcal{G})$. Finally, since f covered by $\tilde{f}$ induces an isomorphism and gf covered by $\tilde{g}\tilde{f}$ induces an isomorphism, g covered by $\tilde{g}$ must induce an isomorphism. $\square$

For intersection homology with local coefficients (and traditional perversities), we can apply the same ideas. As in the stratified pseudomanifold case, we assume that the singular set has codimension at least two (i.e. $X_{n-1} = X_{n-2}$) and that a local coefficient system $\mathcal{G}$ is given on $X - X_{n-2}$. We can also think of $\mathcal{G}$ as a covering space over $X - X_{n-2}$. Now, given a $\bar{p}$-allowable i simplex, σ, it will no longer be possible, in general, to lift the entire simplex, as σ may intersect X_{n-2}. However, we can choose for coefficients lifts of $\sigma \cap (X - X_{n-2})$, which consists of σ minus, at most, pieces of its $i-2$ skeleton. So, in particular, we can lift at least the interior of the simplex and the interior of any facet ($i-1$ face). The coefficient of the boundary $\partial\sigma$ can again be defined in terms of the signed (i.e. plus or minus) restrictions of the lift of the simplex to its boundary pieces, again with the limitation that some lower-dimensional pieces (and only lower-dimensional pieces) might be missing. In this manner, the definition of the local coefficient of a simplex by its lift is well-defined for $\bar{p}$-allowable intersection simplices, and it is further well-defined (modulo lower dimensional skeleta) on all boundary pieces by taking restrictions multiplied by ± 1 since the allowability conditions ensure that the interior of each facet of an allowable simplex lies in the top stratum. Defining the chain complex $IC_*^{\bar{p}}(X;\mathcal{G})$ as the $\bar{p}$-allowable chains formed by linear combinations of singular simplices with such lifts as coefficients yields a chain complex whose homology is the intersection homology of X with coefficients in $\mathcal{G}$. This approach is clearly equivalent to the more common definition of intersection homology with local-coefficient which involves the lifts of given points of the simplices; it has the disadvantage of the lift not being defined for all points (though it is for enough!) and the advantage of not having to nitpick about which points and how to change points under boundary maps.

We should, however, note the following concerning the definitions of the last paragraph: When considering $\partial\sigma$ of a $\bar{p}$-allowable i-chain $\sigma = \sum n_j \sigma_j$ with local coefficients as defined above, a given boundary piece of a simplex σ_j may not itself be allowable, just as is the case with constant coefficients. However, by restriction and multiplication by ± 1, it will have a coefficient lifting defined over its intersection with the top stratum of X, which includes at least its interior. But, again as in the constant coefficient case, all of the non-$\bar{p}$-allowable pieces must cancel in $\partial\sigma$ since each non-zero simplex of $\partial\sigma$ must be $\bar{p}$-allowable since σ was $\bar{p}$-allowable. In other words, the coefficients, where defined, over each point of each of the non-allowable simplices in $\partial\sigma$ must add up to zero. Since the remaining boundary pieces are $\bar{p}$-allowable, their coefficient lifts are again defined, at the least, over their interiors and the interiors of their facets. Therefore, $\partial\partial\sigma$ is well-defined. It is then routine to check that $\partial\partial\sigma = 0$ just as for ordinary homology with or without coefficients, $\partial\partial\sigma$ being equal to the boundary of what remains of $\partial\sigma$ after the pieces which cancel have been removed (i.e. set to zero). Alternatively, we could note that $\partial\partial n_i\sigma_i$ is zero where the lift is defined and apply linearity (where it is not well-defined, it must be zero in $\partial\sigma$ anyway, by its non-allowability), or we could appeal directly to the equivalence of this definition of intersection chains with coefficients and the more standard definition involving lifts over well-chosen points.

Suppose now that the filtered space X is given a local coefficient system $\mathcal{G}$, as above, and that $f : X \to Y$ is a stratum-preserving homotopy equivalence with inverse g. We also again assume that X and Y have the same stratified dimension and that f and g take strata to strata of the same (co)dimension. Since f and g induce homotopy equivalences of $X - X_{n-2}$ and $Y - Y_{n-2}$ (again assuming $X_{n-1} = X_{n-2}$ and $Y_{n-1} = Y_{n-2}$), they also induce isomorphisms of the fundamental groups and determine local coefficients on Y via the pullback g^*. Furthermore, it is clear that $f^*g^*\mathcal{G}$ is bundle isomorphic to G.

We want to show that g covered by $\tilde{g} : g^*\mathcal{G} \to \mathcal{G}$ over the top stratum induces an isomorphism of intersection homology with local coefficients. But this now follows from a combination of the techniques from the proofs of Propositions 2.1 and 2.2. In particular, we simply apply the proof of Proposition 2.2, but all bundles and lifts are restricted to lie over the top stratum. Once again, we will not necessarily be able to lift entire simplices, but the homology theory works out as described in the last few paragraphs. Furthermore, since all of the homotopies in that proof can be taken to be stratum-preserving in this context, there is no trouble with extending lifts: any point that is mapped to a top stratum remains in the top stratum under all of the necessary homotopies and so its lift can be uniquely extended. Also, there is no difficulty showing that the homotopies induce homologies by breaking the homotopies of simplices up into triangulated prisms. The allowability issues are taken care of just as in the proof of Proposition 2.1, and therefore each prism piece is allowable and its coefficient is determined over, at the least, its interior and the interiors of its facets as the restriction of the lift over the entire homotopy on the top stratum. Therefore, we have shown the following:

Corollary 2.3. *Suppose that $g: Y \to X$ is a stratum-preserving homotopy equivalence, $\mathcal{G}$ is a local coefficient system on $X - X_{n-2}$ (which is the top stratum), X and Y have the same dimension n (as filtered spaces), and $f(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$ and $g(Y_i - Y_{i-1}) \subset X_i - X_{i-1}$ for each i. Then the intersection homology map $g_*: IH_i^{\bar{p}}(Y; g^*\mathcal{G}) \to IH_i^{\bar{p}}(X; \mathcal{G})$, induced by g covered over $X - X_{n-2}$ by the natural map $g^*\mathcal{G} \to \mathcal{G}$, is an isomorphism.*

Although we have have focused on stratum-preserving homotopy equivalences, which will be our primary use of the above theory, these same arguments can be easily generalized to prove the following corollary.

Corollary 2.4. *If $f_0, f_1 : X \to Y$ are stratum-preserving homotopic by a homotopy H, $\mathcal{G}$ is a local coefficient system over $X - X_{n-2}$, $\mathcal{G}'$ is a local coefficient system over $Y - Y_{n-2}$, X and Y have the same stratified dimension, $f_0(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$ (or, equivalently, $f_1(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$) for each i, and f_0, f_1 are covered over $X - X_{n-2}$ by maps $\tilde{f}_0, \tilde{f}_1 : \mathcal{G} \to \mathcal{G}'$ which are homotopic over $H|_{(X-X_{n-2})\times I}$ by a homotopy $\tilde{H}$, then $f_{0*}, f_{1*} : IH_i^{\bar{p}}(X; \mathcal{G}) \to IH_i^{\bar{p}}(Y; \mathcal{G}')$ are identical maps on intersection homology.*

Sketch of proof. For simplicity, let us omit the direct mention of the fact that the covers only lie over the top stratum; it will be assumed. Once again, we can employ a prism so that if $\sigma : \Delta^i \to X$ is a singular cycle with coefficient lift $\tilde{\sigma}$, then $H \circ (\sigma \times \mathrm{id})$ covered by $\tilde{H} \circ (\tilde{\sigma} \times \mathrm{id})$ provides a homotopy between $f_0\sigma$ with lift $\tilde{f}_0\tilde{\sigma}$ and $f_1\sigma$ with lift $\tilde{f}_1\tilde{\sigma}$. Hence, we can again employ a prism as in the proof of Proposition 2.1 to show that f_0 and f_1 induce identical maps on homology with local coefficients. □

3 Stratified fibrations

In this section, we study a special case of the notion of a stratified fibration as introduced by Hughes [10] (our definition will be less general than that of Hughes, which assumes that both of the spaces in a stratified fibration are filtered, but it will suit our purposes).

Definition 3.1. (Hughes, [10, p. 355]) If Y is a filtered space, Z and A unfiltered spaces, a map $f : Z \times A \to Y$ is *stratum-preserving along* A if, for each $z \in Z$, $f(z \times A)$ lies in a single stratum of Y.

Definition 3.2. Suppose Y is a filtered space and B is an unfiltered space (equivalently a space with one stratum). A map $p : Y \to B$ is a *stratified fibration* if, given any space Z and the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\times 0"'] & Y \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & B, \end{tikzcd} \tag{3}$$

there exists a *stratified solution* which is stratum-preserving along I, i.e. a map $\tilde{F} : Z\times I \to Y$ such that $p\tilde{F} = F$ and, for each $z \in Z$, $\tilde{F}(z, 0) = f(z)$ and $\tilde{F}(z \times I)$ lies in a single stratum of Y.

We want to show that stratified fibrations satisfy stratified analogues of several of the standard properties of fibrations, such as the existence of pullbacks and triviality over a contractible base. We obtain these results largely by reproducing the proofs for standard fibrations found in Whitehead [18], adding the relevant details where the stratifications come into play.

First, we present the following two lemmas which are special cases of [10, Lemmas 5.2 and 5.3] of Hughes and follow immediately from them:

Lemma 3.3. *Given two solutions, $\tilde{F}$ and $\tilde{G}$, of a stratified lifting problem* (3), *then there exists a homotopy $H : Z \times I \times I \to Y$ from $\tilde{F}$ to $\tilde{G}$ rel $Z \times \{0\}$ such that $pH(z, t, s) = F(z, t)$ and H is stratum-preserving along $I \times I$.*

Lemma 3.4 (Stratified relative lifting). *Suppose $p : Y \to B$ is a stratified fibration, B unstratified, (Z, A) an unstratified NDR-pair, and we are given the commutative diagram:*

$$\begin{tikzcd} Z \times \{0\} \cup A \times I \arrow[r, "f"] \arrow[d] & Y \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & B. \end{tikzcd}$$

If there is a deformation retraction $H_t : Z \times I \times I \to Z \times I$ of $Z \times I$ to $Z \times \{0\} \cup A \times I$ such that $fH_1 : Z\times I \to Y$ is stratum-preserving along I, then there exists a stratified solution $\tilde{F}$: $Z\times I \to Y$ which is stratum-preserving along I, satisfies $p\tilde{F} = F$, and $\tilde{F}|Z\times\{0\}\cup A\times I = f$. (Note: Hughes assumes in his lemma that Z is a metric space, but I don't believe this is ever used in the proof, at least supposing the definition of NDR as given in [18].)

In fact, Hughes' proof of Lemma 3.4 applies to the following slightly more general situation:

Lemma 3.5. *Suppose $p : Y \to B$ is a stratified fibration, $(Z \times I, A)$ is a DR-pair, $i : A \to Z \times I$ is the inclusion, and we are given the commutative diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ Z \times I \arrow[r, "F"] & B. \end{tikzcd}$$

If there is a deformation retraction $H_t : Z \times I \times I \to Z$ of $Z \times I$ to A such that $fH_1 : Z \times I \to Y$ is stratum-preserving along I, then there exists a stratified solution $\tilde{F} : Z \times I \to Y$ which is stratum-preserving along I, satisfies $p\tilde{F} = F$, and $\tilde{F}|A = f$.

We now define the induced stratified fibration in analogy with induced fibrations.

Definition 3.6. Suppose $p : Y \to B$ is a stratified fibration and $f : B' \to B$ is a map of spaces. We denote by Y' (or f^*Y if we want to emphasize the map f) the set $\{(b', y) \in B' \times Y | f(b') = p(y)\}$ and call it the *induced stratified fibration* or *pullback stratified fibration.* We filter Y' as a subset of $B' \times Y$, i.e. $Y'_i = \{(b', y) \in Y' | y \in Y_i\}$.

We obtain maps $f' : Y' \to Y$ and $p' : Y' \to B'$ induced by the projections. Note that f' is obviously stratum-preserving and takes the fiber over $b' \in B'$ to the fiber over $f(b')$. If f is an inclusion, then so is f'. Assuming this notation, we prove the following basic facts:

Lemma 3.7 (Universality). *If $g : Z \to Y$ is a stratum-preserving map and $q : Z \to B'$ is a map such that $pg = fq$, then there is a unique stratum preserving map $h : Z \to Y'$ such that $f'h = g$ and $p'h = q$.*

Proof.

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"] \arrow[drr, "g"] \arrow[ddr, "q"'] & & \\ & Y' \arrow[r, "f'"'] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ & B' \arrow[r, "f"] & B \end{tikzcd}$$

Based on the hypotheses, q and g define a map $(q, g) : Z \to B' \times Y$ which is stratum-preserving with respect to the product filtration on $B \times Y$. Clearly, $\mathrm{Im}(q, y) \subset Y'$, and thus (q, g) induces a stratum-preserving map to Y'. The commutativity relations are obvious. Uniqueness follows obviously from Y' being a subset of $Y \times B'$: the two projections f' and p' define the image point uniquely in Y', but these are fixed by the hypotheses. $\square$

Lemma 3.8. *The induced map $p' : Y' \to B'$ is a stratified fibration.*

Proof.

$$\begin{tikzcd}
Z\times 0 \arrow[r, "g"] \arrow[d, hook] & Y' \arrow[r, "f'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\
Z\times I \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] \arrow[urr, dotted, "\bar{G}"] & B' \arrow[r, "f"'] & B
\end{tikzcd}$$

If $g : Z \to Y'$ and $G : Z \times I \to B'$ are the data for a stratified lifting problem for the map p', then the maps fG and $f'g$ are the data for a stratified lifting problem for p, which has a stratified solution $\bar{G}$ by assumption. Now, stratify $Z \times I$ by $(Z \times I)_i = \bar{G}^{-1}(Y_i)$. Note that if $(z,t) \in (Z \times I)_i$ for some $z \in Z$ and $t \in I$, then $z \times I \in (Z \times I)_i$ by the definition of a stratified lifting. With this stratification, $\bar{G}$ is clearly a stratum-preserving map. But then $\bar{G} : Z \times I \to Y$ and $G : Z \times I \to B'$ satisfy $fG = p\bar{G}$, the hypothesis of the Universality Lemma. Hence there is a unique stratum-preserving map $\tilde{G} : Z \times I \to Y'$ such that $p'\tilde{G} = G$ and $f'\tilde{G} = \bar{G}$, and, since $\bar{G}|Z \times \{0\} = f'g$ and $G|Z \times 0 = p'g$, $\tilde{G}|Z \times \{0\} \to Y'$ is given by $\tilde{G}(z \times \{0\}) = (p'\tilde{G}(z \times 0), f'\tilde{G}(z \times 0)) = (p'g(z \times \{0\}), f'g(z \times \{0\})) \in Y' \subset B' \times Y$. But since a point in $B' \times Y$ is determined uniquely by its projections, this is also $g(z \times \{0\})$, so $\tilde{G}$ extends g. Finally, since $\tilde{G}$ is stratum-preserving on the strata we have imposed on $Z \times I$, each $z \times I$, lying in a single stratum of $Z \times I$, must map into a single stratum of Y' under $\tilde{G}$. Hence, $\tilde{G}$ is a stratified solution to the lifting problem and p' is a stratified fibration. $\square$

Corollary 3.9. *The restriction of a stratified fibration is a stratified fibration.*

Proof. Pull back over the inclusion which induces the restriction. $\square$

Corollary 3.10. *Suppose that $p : Y \to B$ is a stratified fibration, $f : B' \to B$ and $g : B'' \to B'$ are maps, $p' : Y' \to B'$ is the stratified fibration induced by p and f, and $p'' : Y'' \to B''$ is the stratified fibration induced by p' and g. Then up to natural stratified isomorphism, p'' is also the stratified fibration induced from p by the map fg.*

Proof. As for ordinary fibrations, $\phi : (fg)^*Y \to g^*f^*Y$ given by $\phi(b_2, y) = (b_2, (g(b_2), y))$ provides the isomorphism. It is clearly stratum-preserving. $\square$

Definition 3.11. Suppose that $p_i : Y_i \to B$, $i = 1, 2$, are stratified fibrations (note that these subscripts do not indicate strata). A map $H : Y_1 \times I \to Y_2$ is a *stratum-preserving fiber homotopy* (*s.p.-homotopy*) if it is stratum-preserving and $p_2H(y,t) = p_1(y)$ for all $t \in I$. $f : Y_1 \to Y_2$ is a *stratum-preserving fiber homotopy equivalence* if f is a stratum-preserving map such that $p_2 f(y_1) = p_1(y_1)$ for all $y_1 \in Y_1$ and there is a stratum-preserving map $g : Y_2 \to Y_1$ such that $p_1 g(y_2) = p_2(y_2)$ for all $y_2 \in Y_2$ and fg and gf are stratum-preserving fiber homotopic to the respective identity maps on Y_2 and Y_1.

Lemma 3.12. *Suppose that $f_0, f_1 : B' \to B$ are homotopic maps, $p : Y \to B$ is a stratified fibration, and let $p'_t : Y'_t \to B'$, $t = 0, 1$ be the induced pullback stratified fibrations. Then Y'_0*

and Y_1' are stratum-preserving fiber homotopy equivalent, i.e. there exist stratum-preserving fiber homotopy equivalences $g : Y_0' \to Y_1'$ and $h : Y_1' \to Y_0'$ and stratum-preserving fiber homotopies between gh and hg and the respective identity maps. In this situation, we will say that p_0' and p_1' have the same stratified fiber homotopy type *or are* stratum-preserving fiber homotopy equivalent (s.p.f.h.e.).

Proof. The proof follows [18, I.7.25] closely. Let $i_t : B' \to B' \times I$, $t = 0, 1$, be the inclusions of the ends, and let $f : B' \times I \to B$ be a homotopy from f_0 to f_1. Then $fi_t = f_t$. Using Corollary 3.10, it then suffices to prove that for any stratified fibration $p : Y \to B \times I$, if $p_t : Y_t \to B$ are the stratified fibrations induced by $i_t : B \to B \times I$ and Y_t has the filtration induced as a subset of Y, then p_0 and p_1 have the same stratified fiber homotopy type. So suppose that this is the set-up.

Let i_t' be the inclusion maps $Y_t \to Y$ over the inclusions i_t. The maps $\mathrm{id} \times p_0 : I \times Y_0 \to I \times B$ and the inclusions $f_0 : Y_0 \to Y$ form the data for a stratified lifting problem, and, by assumption, there exists a stratified solution $H_0 : I \times Y_0 \to Y$ such that $H_0(0, y_0) = i_0'(y_0)$ and $pH_0(s, y_0) = (s, p_0(y_0))$. Since i_0' is the inclusion, H_0 takes each $I \times y_0$ into the stratum of Y in which y_0 lies. Similarly, there is an H_1 such that $H_1(1, y_1) = i_1'(y_1)$, $pH_1(s, y_1) = (s, p_1(y_1))$, and $I \times y_1$ maps into the stratum containing y_1. Define $\phi : Y_0 \to Y_1$ and $\psi : Y_1 \to Y_0$ by $i_1'\phi(y_0) = H_0(1, y_0)$ and $i_0'\psi(y_1) = H_1(0, y_1)$. From the properties of H_t, ϕ and ψ are stratum-preserving, $p_1\phi = p_0$, and $p_0\psi = p_1$.

To see that $\psi\phi$ is stratum-preserving fiber homotopic to the identity, define $g : 1 \times I \times Y_0 \cup I \times \dot{I} \times Y_0 \to Y$ and $K : I \times I \times Y_0 \to I \times B$ by $K(s, t, y_0) = (s, p_0(y_0))$ and

$$g(s,t,x) = \begin{cases} H_0(s, y_0), & t = 0 \\ H_1(s, \phi(y_0)), & t = 1 \\ i_1'\phi(y_0), & s = 1. \end{cases}$$

g is well defined and stratum preserving on the induced strata. These maps form the data for a stratified lifting extension problem. By Lemma 3.4, it has a solution if there is a deformation retraction $R_t : I \times I \times Y_0 \times I \to I \times I \times Y_0$ to $1 \times I \times Y_0 \cup I \times \dot{I} \times Y_0$ such that $gR_1 : I \times I \times Y_0 \times I \to Y$ is stratum-preserving along I. But we can take as R the map $r \times \mathrm{id}_{Y_0}$ where r is any standard deformation retraction of $I \times I$ into $1 \times I \cup I \times \dot{I}$. Then each $I \times I \times y_0 \times I$ clearly gets mapped into a single stratum of $I \times I \times Y_0$ under R and hence under R_1, and g is stratum-preserving from its definition. Hence the conditions are met for there to be a stratified extension of g, namely $G : I \times I \times Y_0 \to Y$, such that $pG = K$ and G is stratum-preserving along the first I factor. But it is also stratum-preserving along the second I factor since clearly G is stratum-preserving along it for $s = 1$. Then $G|0 \times I \times Y_0$ is a stratum-preserving fiber homotopy between $\psi\phi$ and the identity. The other case, $\phi\psi$, is handled similarly. $\square$

Corollary 3.13. *If $p : Y \to B$ is a stratified fibration and B is contractible then p is stratified fiber homotopically trivial, i.e. there is a stratum-preserving fiber homotopy equivalence between $p : Y \to B$ and $p' : F \times B \to B$ for some stratified fiber F.*

Proof. Clearly the pullback of a map from B to a point in B gives the product stratified fibration. $\square$

We will call a stratum-preserving fiber homotopy equivalence given by the map $\phi : F \times B \to Y$ a *stratified trivialization.* We call it a *strong stratified trivialization* if, for some $b_0 \in B$ with $F = p^{-1}(b_0)$, we have $\phi(x, b_0) = x$ for all $x \in F$.

Corollary 3.14. *If (B, b_0) is a DR-pair and $p : Y \to B$ is a stratified fibration, then there exists a strong stratified trivialization $\phi : p^{-1}(b_0) \times B \to Y$. Any two such strong stratified trivializations are stratum-preserving fiber-homotopic rel $p^{-1}(b_0) \times b_0$.*

Proof. The proofs are analogous to those given for ordinary fibrations in [18, I.7.28-29]; the necessary modifications to the stratified case are either obvious or employ the same techniques used in the preceding proofs. $\square$

Finally, we will need some results on the mapping cylinders of stratified fibrations. These are the subject of the next few lemmas.

We will always use mapping cylinders (including cones) with the *teardrop topology* (see [9, §2] or [19, §3]). The *teardrop topology* on the mapping cylinder I_p of the map $p : Y \to B$ is the topology generated by the sub-basis consisting of open subsets of $Y \times (0, 1]$ and sets of the form $[p^{-1}(U) \times (0, \epsilon)] \cup U$, where U is an open set in B. This is, in fact, a basis since if U and V are two open sets of B, then within the intersection $(p^{-1}(U) \times (0, \epsilon_1) \cup U) \cap (p^{-1}(V) \times (0, \epsilon_2) \cup V)$, we have the basis element $p^{-1}(U \cap V) \times (0, \min(\epsilon_1, \epsilon_2)) \cup (U \cap V)$. The intersection of any basis elements of the first and second types or of two elements of the first type is a basis element of the first type.

If $p : Y \to B$ is a stratified fibration, Y filtered by the sets $\{Y_i\}$, then the mapping cylinder I_p is naturally filtered by B, as the bottom stratum, and the mapping cylinders of p restricted to each Y_i. There remains, however, the question of how to label the dimensions. Even if we create a consistent plan for how to handle the dimension of each $I_{p|Y_i}$, it is not clear what the dimension of B should be. Fortunately, in the applications below, it will actually be the mapping cylinder which comes equipped which a natural filtration of this form, and the filtration of Y can be considered to be induced by the intersection of that filtration with $Y \times t \subset I_p$, $t \in (0, 1]$. So, although this approach appears somewhat backwards, we will assume in what follows that the filtrations arise in this manner.

We will also have need to consider a local coefficient system on the top stratum $I_p - I_{p,n-2} = (Y - Y_{n-2}) \times (0, 1]$. By standard bundle theory, any such system, $\mathcal{G}$, will be isomorphic to $(\mathcal{G}|Y \times t) \times (0, 1]$, for any choice of $t \in (0, 1]$. Therefore, we will always assume that $\mathcal{G}$ has the form of such a product. Conversely, if we start with a coefficient system, $\mathcal{G}$, on $Y - Y_{n-2}$, then $\mathcal{G} \times (0, 1]$ gives a coefficient system on I_p, unique up to isomorphism. We will denote this coefficient system by $I_{\mathcal{G}}$. If I_p is a closed cone, $\bar{c}Y$, we will sometimes denote this induced coefficient system by $\bar{c}\mathcal{G}$ (note that we reserve the symbol cX for the open cone on the space X).

Lemma 3.15. *Suppose $f : X \to Y$ is a continuous map with mapping cylinder I_f and that Z is another space. Then $I_f \times Z = I_{f \times id_Z}$, the mapping cylinder of $f \times id_Z : X \times Z \to Y \times Z$.*

Proof. Both spaces clearly have the same underlying sets, so we focus on the topologies. $I_f \times Z$ has a basis of the form $V \times W$, where V is a basis element of I_f and W is a basis element of Z. In particular, these all have the form either $V_1 \times W$ or $V_2 \times W$, where V_1 is open in $X \times (0,1]$ and V_2 has the form $[f^{-1}(U) \times (0,\epsilon)] \cup U$ for an open subset $U \subset Y$. On the other hand, $I_{f\times \mathrm{id}_Z}$ has basis elements of the form $A = \{\text{an open subset of } (X \times Z) \times (0,1]\}$ or $[B = (f \times \mathrm{id}_Z)^{-1}(C)] \times (0,\epsilon) \cup C$ for some open $C \subset Y \times Z$. So we need to show that A and B are open in $I_f \times Z$ and $V_1 \times W$ and $V_2 \times W$ are open in $I_{f\times \mathrm{id}_Z}$.

Since the subset topologies on $X \times (0,1] \times Z \subset I_f \times Z$ and $X \times Z \times (0,1] \subset I_{f\times \mathrm{id}_Z}$ all include the standard basis elements for products, these topologies are finer than (in fact equal to) the obvious product topologies, and so the sets of type A and $V_1 \times W$ are open in the desired spaces.

The set $V_2 \times W = [f^{-1}(U) \times (0,\epsilon) \cup U] \times W$ can be rewritten as $[(f \times \mathrm{id}_Z)^{-1}(U \times W) \times (0,\epsilon)] \cup (U \times W)$. This is an element of type B, so the inclusion of $V_2 \times W$ into $I_{f\times \mathrm{id}_Z}$ is open.

Finally, the sets of the form $U \times W$ for U open in Y and W open in Z form a basis for $Y \times Z$, so we can restrict to looking at sets of type B in which C has the form $U \times W$. But then $B = [(f \times \mathrm{id}_Z)^{-1}(U \times W) \times (0,\epsilon)] \cup U \times W = [(f^{-1}(U) \times W \times (0,\epsilon)] \cup U \times W = (f^{-1}(U) \times (0,\epsilon) \cup U) \times W$, which is of the form $V_2 \times W$. Hence this inclusion is also open. □

Lemma 3.16. *Suppose $f : Y_1 \to Y_2$ is a stratum- and fiber-preserving map of stratified fibrations, $p_i : Y_i \to B$, $i = 1,2$. Let $\bar{f}$ be the map from I_{p_1} to I_{p_2} induced on the mapping cylinders by the identity on B and $f \times id_{(0,1]} : Y_1 \times (0,1] \to Y_2 \times (0,1]$. Then $\bar{f}$ is a continuous stratum-preserving map which, furthermore, takes each cone $I_{p_1|p_1^{-1}(b)}$, for $b \in B$, into the cone $I_{p_2|p_2^{-1}(b)}$.*

Proof. The only inobvious part of the statement of the lemma is that $\bar{f}$ should be continuous at points in B. If $b \in B$, then the basis elements of the topology of I_{p_2} containing $f(b) = b$ have the form $V = [p_2^{-1}(U) \times (0,\epsilon)] \cup U$, where U is a neighborhood of b in B. Then from the definitions, $\bar{f}^{-1}(V) = [f^{-1}p_2^{-1}(U) \times (0,\epsilon)] \cup \bar{f}^{-1}(U) = [p_1^{-1}(U) \times (0,\epsilon)] \cup U$. Here we have used that $\bar{f}|B$ is the identity on B and that $p_2 f = p_1$. So $\bar{f}^{-1}(V)$ is an open neighborhood of b that maps into V, and $\bar{f}$ is continuous. □

Corollary 3.17. *The map $\bar{f}$, as defined in the preceding lemma, induces a map on intersection homology $\bar{f}_* : IH_*^{\bar{p}}(I_{p_1}) \to IH_*^{\bar{p}}(I_{p_2})$. If f is covered by a bundle map on the top stratum $\tilde{f} : \mathcal{G} \to \mathcal{H}$, then $\tilde{f} \times id_{(0,1]}$ gives a map $I_{\mathcal{G}} \to I_{\mathcal{H}}$ covering $\bar{f}$ on the top stratum, and together these induce a map $f_* : IH_*^{\bar{p}}(I_{p_1}; I_{\mathcal{G}}) \to IH_*^{\bar{p}}(I_{p_2}; I_{\mathcal{H}})$.*

Lemma 3.18. *Suppose that $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ are stratum-preserving fiber homotopy inverses for the stratified fibrations $p_i : Y_i \to B$, $i = 1,2$. Then $\bar{f}$ and $\bar{g}$ as defined in the previous lemma are stratum- and cone-preserving homotopy inverses between the I_{p_i}, meaning that the homotopies from $\bar{f}\bar{g}$ and $\bar{g}\bar{f}$ to the identities are stratum-preserving and take $I_{p_i|p_i^{-1}(b)} \times I$ to $I_{p_i|p_i^{-1}(b)}$.*

Proof. Let $H : Y_1 \times I \to Y_1$ be the stratum-preserving fiber homotopy from gf to the identity. By Lemma 3.15, $I_{p_1} \times I$ is equal to the mapping cylinder of $p_1 \times \mathrm{id}_I : Y_1 \times I \to B \times I$. Define

$\bar{H} : I_{p_1} \times I \to I_{p_1}$ so that for each $t \in I$, $\bar{H}|I_{p_1} \times t$ is the map determined as in the previous lemma by extending the map $H|Y_1 \times t$ to the cylinder.

Again, continuity is clear except for points $(b,t) \in B \times I \subset I_{p_1} \times I$. Consider $\bar{H}(b,t) = b$. Again, the basis neighborhoods of b in B have the form $V = [p_1^{-1}(U) \times (0,\epsilon)] \cup U$. Since by Lemma 3.15, we can also consider $I_{p_1} \times I$ as the mapping cylinder of $p_1 \times \mathrm{id}_I$, (b,t) has a neighborhood of the form $W = [p_1^{-1}(U \times I) \times (0,\epsilon)] \cup (U \times I)$. Since H is a fiber homotopy, $\bar{H}$ takes $U \times I$ to U by projection, and since $\bar{H}$ is cone-preserving and takes each point in the cylinder to a point of the same "height", it maps points in $p_1^{-1}(U \times I) \times (0,\epsilon)$ to $p_1^{-1}(U) \times (0,\epsilon)$. Hence the neighborhood W maps into the neighborhood V, and $\bar{H}$ is continuous.

Furthermore, over $B \times 0$, since the restriction of H is the identity map of Y_1, the restriction of $\bar{H}$ is the identity map of the mapping cylinder I_{p_1}, while the restriction over $B \times 1$ of H is gf so that the restriction of $\bar{H}$ is $\overline{gf}$, which is clearly $\bar{g}\bar{f}$ from the definition of these maps. Therefore, $\bar{H}$ is a homotopy from $\overline{gf}$ to the identity. That it is cone-preserving follows immediately from the definitions and the fact that H was fiber preserving. Similarly, the fact that H was stratum-preserving and the definition of $\bar{H}$ shows that $\bar{H}$ is stratum-preserving. The proof that $\bar{f}\bar{g}$ is stratum- and cone-preserving homotopic to the identity on I_{p_2} is the same.

□

Corollary 3.19. *If a stratified fibration $p : Y \to B$ possesses a (strong) stratified trivialization $f : F \times B \to Y$, then there is also a (strong) stratum- and cone-preserving homotopy equivalence $\bar{c}F \times B \to I_p$.*

Proof. By the preceding lemma, there exists a stratum- and cone-preserving homotopy equivalence $\bar{f}$ from the mapping cylinder of the projection $F \times B \to B$ to I_p. Using Lemma 3.15, this mapping cylinder is exactly $\bar{c}F \times B$. If the trivialization is strong so that f restricted to the fiber over some b_0 is the identity map $F \times b_0 \to F_{b_0}$, then it is clear from the construction in the lemma that $\bar{f}|\bar{c}F_{b_0}$ will be the identity map $\bar{c}F \times b_0 \to \bar{c}F_{b_0}$. Furthermore, if the homotopies determining the equivalence of Y and $F \times B$ are stationary over b_0, then so will be the homotopies which determine the equivalence of I_p and $\bar{c}F \times B$ as constructed in the proof of the lemma. □

Corollary 3.20. *If $H : Y_1 \times I \to Y_2$ is a stratum-preserving fiber homotopy from f to g, $f, g : Y_1 \to Y_2$ stratum- and fiber-preserving maps, then $\bar{H} : I_{p_1} \times I \to Y_2$, defined so that $\bar{H}|(I_{p_1} \times t) = \overline{H|I_{p_1} \times t}$ for each $t \in I$, is a stratum-preserving homotopy $\bar{H} : I_{p_1} \times I \to I_{p_2}$ which maps $\bar{c}p_1^{-1}(b) \times t$ to $\bar{c}p_2^{-1}(b)$ for each $b \in B$ and $t \in I$.*

Proof. The method of proof is the same as that employed in the proof of the lemma to construct the homotopy from $\bar{g}\bar{f}$ to the identity. □

Corollary 3.21. *If f and g are stratum-preserving fiber homotopic maps of stratified fibrations, then $\bar{f}$ and $\bar{g}$ induce the same map on intersection homology $f_* = g_* : IH_*^{\bar{p}}(I_{p_1}) \to IH_*^{\bar{p}}(I_{p_2})$. If the homotopy, H, from f to g is covered on the top stratum by a homotopy of local coefficients $\tilde{H} : \mathcal{G} \times I \to \mathcal{H}$ from $\tilde{f}$ to $\tilde{g}$, then $\bar{H}$ can be covered on the top stratum* *by a homotopy $\tilde{H} \times id_{(0,1]} : I_{\mathcal{G}} \times I \to I_{\mathcal{H}}$ from $\tilde{f} \times id_{(0,1]}$ to $\tilde{g} \times id_{(0,1]}$. Then $\bar{f}$ covered by $\tilde{f} \times id_{(0,1]}$ and $\bar{g}$ covered by $\tilde{g} \times id_{(0,1]}$ induce the same map on intersection homology $f_* = g_* : IH^{\bar{p}}_*(I_{p_1}; I_{\mathcal{G}}) \to IH^{\bar{p}}_*(I_{p_2}; I_{\mathcal{H}})$.*

Proof. This follows from Corollaries 2.4 and 3.20. ☐

4 Bundles of coefficients induced by a stratified fibration

To construct bundles of modules from a stratified fibration, we will again need to generalize some of the standard arguments from the case of ordinary fibrations. We again modify some of the relevant results from [18]. Given a stratified fibration $p : Y \to B$, with B unfiltered, and a path $u : I \to B$ from b_0 to b_1, let $F_t = p^{-1}(b_t)$ and $i_t : F_t \hookrightarrow Y$ (obviously these are stratum-preserving maps if we define the stratifications on the F_t by their intersections with the strata of Y). We call a stratum preserving map $h : F_1 \to F_0$ u-s.p. admissible if there exists a stratum-preserving homotopy $H : I \times F_1 \to Y$ from h to i_1 such that $pH(t, y) = u(t)$ for all $(t, y) \in I \times F_1$. Just as for ordinary fibrations, the following are clear:

1. If u is the constant path and h is the identity map $F_0 \to F_0$, then h is u-s.p. admissible.

2. If $v : [0, 1] \to B$ and $u : [1, 2] \to B$ are paths $I \to B$ with $u(1) = v(1)$, $h : F_1 \to F_0$ is v-s.p. admissible, and $k : F_2 \to F_1$ is u-s.p admissible, then $h \circ k : F_2 \to F_0$ is $v * u$-s.p. admissible. The proof, as usual, consists of adjoining two homotopies: If H and K are the homotopies which provide the admissibility of h and k, then the admissibility of hk is given by the homotopy $L : [0, 2] \times F_2 \to Y$ defined by

$$L(t, x) = \begin{cases} H(t, k(x)), & t \in [0, 1] \\ K(t, x), & t \in [1, 2]. \end{cases}$$

   Since H, K, and k are stratum-preserving, so is L.

Given $u \in \pi_1(B; b_0, b_1)$, there always exists a u-sp admissible map: Let $f : I \times p^{-1}(b_1) \to B$ be given by $f(t, y) = u(t)$, and let $g : 1 \times p^{-1}(b_1)$ be given by $g(1, y) = y$. Then this defines a stratified homotopy lifting problem, which has a solution $G : I \times p^{-1}(b_1) \to Y$ by assumption. Then define $g' : p^{-1}(b_1) \to p^{-1}(b_0)$ by $g'(y) = G(0, y)$. This is a stratum-preserving map.

We next show that if u_0 and u_1 are homotopic rel $\dot{I}$ and h_0 and h_1 are respectively u_0- and u_1-s.p. admissible, then h_0 and h_1 are stratum-preserving homotopic: Let $F : I \times I \to B$ be a homotopy from u_0 to u_1 rel $\dot{I}$, and, for $s = 0, 1$, let H_s be the homotopy from h_s to i_1 lying over u_s. Define a map $G : I \times I \times F_1 \to B$ by $G(s, t, y) = F(s, t)$ and a map $H : (\dot{I} \times I \cup I \times 1) \times F_1 \to Y$ by

$$H(s, t, y) = \begin{cases} H_s(t, y), & s = 0, 1 \\ y, & t = 1. \end{cases}$$

These provide the data for a stratified lifting extension problem. Since p is a stratified fibration, there is a stratified solution, $\bar{H}$, by Lemma 3.4: We can deformation retract $I \times I \times F_1$ to $(\dot{I} \times I \cup I \times 1) \times F_1$ as the product of F_1 with any deformation retraction $I \times I \to \dot{I} \times I \cup I \times 1$, and clearly this retraction composed with H is stratum-preserving. $H|I \times 0 \times F_1$ then gives a stratum-preserving homotopy from h_0 to h_1.

Note, incidentally, that together the above properties imply that the fibers of Y are stratum-preserving homotopy equivalent.

The upshot of this discussion is that given a homotopy class of paths rel endpoints in B, we have defined a unique stratum-preserving homotopy class of maps from the fiber over the terminal point to the fiber over the initial point. Furthermore, this is functorial with respect to composition of classes of paths. Hence, we have established a functor from the fundamental groupoid of B to the category of filtered spaces and stratum-preserving homotopy classes of maps. This, in turn, induces a functor $\mathfrak{F}$ to abelian groups or modules by taking, for $b \in B$, $\mathfrak{F}(b) = IH_i^{\bar{p}}(p^{-1}(b))$ and as map $\mathfrak{F}(u) : IH_*^{\bar{p}}(p^{-1}(b_1)) \to IH_*^{\bar{p}}(p^{-1}(b_2))$ the map on intersection homology induced by the stratum-preserving homotopy class of maps determined by u (see Corollary 2.4). As defined in [18, Chapter VI], such a functor determines a bundle of groups (modules) over B with fiber isomorphic to $IH_i^{\bar{p}}(p^{-1}(b))$ for any b in B.

Now suppose that $Y = (Y, \{Y_n\})$, itself, is given a local coefficient system, $\mathcal{G}$, over $Y_n - Y_{n-2}$. It will again be convenient to think of $\mathcal{G}$ as a space (in fact a covering space of $Y - Y_{n-2}$ with projection $\pi : \mathcal{G} \to Y$). In the following, $\mathcal{G}$ will always lie over the top stratum, but for convenience we suppress this from the notation. Let b_0, b_1 continue to denote points of B and $F_s = p^{-1}(b_s)$, and consider the restrictions $\mathcal{G}|F_s$. As above, given a path u in B from b_0 to b_1, we can construct a stratum-preserving map $g' : F_1 \to F_0$ as the end of a stratum-preserving homotopy $G : F_1 \times I \to Y$ from the inclusion of F_1. The homotopy $G \circ (\pi \times \mathrm{id}) : \mathcal{G}|_{F_1} \times I \to Y$, together with the inclusion map $\mathcal{G}|F_1 \hookrightarrow \mathcal{G}$, gives the data for a lifting extension problem which has a solution since $\mathcal{G}$ is a covering space. Hence, there is a map $\tilde{G} : (\mathcal{G}|F_1) \times I$ which covers G over the top stratum and which is the inclusion on $(\mathcal{G}|F_1) \times 1$. Let $\tilde{g} : \mathcal{G}|F_1 \to \mathcal{G}|F_0$ be given by $\tilde{g}(z) = \tilde{G}(z, 0)$. Note that, by the properties of bundles of coefficients, $\tilde{g}$ is a bundle map and, in fact, provides an isomorphism from the stalk over each $x \in F_1$ to the stalk over $g'(x)$. Together, g' and $\tilde{g}$ induce a map $g_* : IH_*^{\bar{p}}(F_1; \mathcal{G}|F_1) \to IH_*^{\bar{p}}(F_0; \mathcal{G}|F_0)$ (recall that we filter each F by its intersections with the filtration of Y). Furthermore, g_* is independent of the choice of G and even the choice of u in its relative path class. We have seen that g' is unique up to stratum-preserving homotopy by producing a homotopy $H' : I \times I \times F_1 \to Y$ from G to the homotopy which gives any other admissible map. Hence, given any such stratum-preserving homotopy, H', from G, we can lift it again to a homotopy $I \times I \times \mathcal{G}|F_1 \to \mathcal{G}$ which covers H' and is the inclusion on each $I \times 1 \times \mathcal{G}|F_1$ by the unique path lifting property of covering spaces. On $I \times 0 \times \mathcal{G}|F_1$, this gives a homotopy from $\tilde{g}$ covering the stratum-preserving homotopy from g' to the other candidate admissible map. By Corollary 2.4, g' covered by $\tilde{g}$ and the maps at the other end of the homotopy induce the same map on intersection homology with local coefficients. Therefore, in particular, each path class $\xi \in \pi_1(B; b_0, b_1)$ induces a unique map $IH_*^{\bar{P}}(F_1; \mathcal{G}|F_1) \to IH_*^{\bar{P}}(F_0; \mathcal{G}|F_0)$.

We show functoriality of these intersection homology maps as induced by path classes in B. It is clear, by these constructions, that the map over the constant path is the identity. If $v : [0,1] \to B$ and $u : [1,2] :\to B$ are paths such that $v(1) = u(1)$, we prove that the composition of the induced maps $IH_*^{\bar{p}}(F_2; \mathcal{G}|F_2) \to IH_*^{\bar{p}}(F_1; \mathcal{G}|F_1)$ and $IH_*^{\bar{p}}(F_1; \mathcal{G}|F_1) \to IH_*^{\bar{p}}(F_0; \mathcal{G}|F_0)$ is the same as the map induced by the path $v * u$: Let $g_2' : F_2 \to F_1$, $g_1' : F_1 \to F_0$, and $g_3' : F_2 \to F_0$ be the maps induced as above by v, u, and $v * u$, respectively, as the ends of respective stratum-preserving homotopies $G_2 : F_2 \times [1,2] \to Y$, $G_1 : F_1 \times [0,1] \to Y$, and $G_3 : F_2 \times [0,2] \to Y$. We already know that g_3' and $g_1'g_2'$ are stratum-preserving homotopic since they are each $v * u$ admissible maps. In fact, we know that g_3' is given as the end of a homotopy $G_3 : F_2 \times I \to Y$ over $v * u$, while $g_1'g_2'$ can be defined as the end of the stratum-preserving homotopy, say $G_{1,2}$, which on $[1,2]$ is given by G_2 over u and on $[0,1]$ is the composition of G_1 over v with the map $g_2' \times \mathrm{id} : F_2 \times [1,2] \to F_1 \times [1,2]$. Since these each determine $v*u$-admissible maps, we have seen above how to construct a homotopy $\bar{G} : I \times [0,2] \times F_2 \to Y$ from G_3 to $G_{1,2}$ which is the inclusion on each $I \times 2 \times F_2$ and which gives a stratum-preserving homotopy between g_3' and $g_1'g_2'$ when restricted to $I \times 0 \times F_2$. But by covering space theory, we can cover $\bar{G}$ with a homotopy $I \times [0,2] \times \mathcal{G}|F_2 \to \mathcal{G}$ which is the inclusion on $I \times 2 \times \mathcal{G}|F_2$, and since the path lifting is unique, the map $I \times 0 \times \mathcal{G}|F_2$ provides a homotopy between the lifts induced over g_3 by the lift over G_3 and the lifts induced over $g_1'g_2'$ by lifting $G_{1,2}$ (which is clearly the same as the composition of the lifts over g_1' and g_2' given by the lifts of G_1 and G_2). Hence g_3' and $g_1'g_2'$, together with their lifts as constructed above, give stratum-preserving homotopic maps $F_2 \to F_0$ covered by homotopic coefficient maps, and therefore induce the same map on intersection homology with local coefficients by Corollary 2.4. Thus, the composition of the maps on intersection homology induced by u and v is the same as the map induced by $v * u$. This shows that these maps define a functor from the fundamental groupoid to the category of modules and, therefore, a local system of coefficients on B. This generalizes the above construction to include the case where Y starts with its own local coefficient system (over $Y - Y_{n-2}$). Note that these properties imply that the induced maps $IH_i^{\bar{p}}(F_1; \mathcal{G}|F_1) \to IH_i^{\bar{p}}(F_0; \mathcal{G}|F_0)$ are isomorphisms. We will call these coefficient systems $\mathcal{IH}_*^{\bar{p}}(F; \mathcal{G})$.

If I_p is the mapping cylinder of a stratified fibration $Y \to B$, we can extend these constructions to obtain bundles of coefficients $\mathcal{IH}_*^{\bar{p}}(\bar{c}F; \mathcal{G})$ over B corresponding to the projection $\Pi : I_p \to B$ as follows: For each of the stratum-preserving maps $h : F_1 \to F_0$ determined by a path in B, let $\bar{c}h : \bar{c}F_1 \to \bar{c}F_0$ be the map determined by the coning. This will also be stratum-preserving assuming the usual induced stratification on a cone. Furthermore, any stratum-preserving homotopy of h induces a stratum-preserving homotopy of $\bar{c}h$ by Corollary 3.20 (noting that a cone is trivially a mapping cylinder of a stratified fibration). Since the maps h were uniquely determined up to stratum-preserving homotopy, we see that the maps $\bar{c}h$ are also so well-defined. Furthermore, the maps and homotopies which cover these in $\mathcal{G}$ similarly extend to maps and homotopies which cover these in $\bar{c}\mathcal{G}$ (see page 14) in the obvious manner. Hence they induce unique maps $IH_*^{\bar{p}}(\bar{c}F_1; \bar{c}\mathcal{G}) \to IH_*^{\bar{p}}(\bar{c}F_0; \bar{c}\mathcal{G})$ by Corollary 3.21. Altogether, this once again forms a bundle of coefficients over B, which we denote $\mathcal{IH}_*^{\bar{p}}(\bar{c}F; \bar{c}\mathcal{G})$. Furthermore, there is clearly a bundle map $\mathcal{IH}_*^{\bar{p}}(F; \mathcal{G}) \to \mathcal{IH}_*^{\bar{p}}(\bar{c}F; \bar{c}\mathcal{G})$ induced by the inclusion $Y \hookrightarrow I_p$.

5 Weakly stratified spaces

We next introduce some definitions due to Quinn [16], [15].

A map of pairs of spaces $f : (A, B) \to (X, Y)$ is said to be *strict* if $f(B) \subset Y$ and $f : (A - B) \subset X - Y$. The space of such strict maps with the compact-open topology is denoted $\text{map}_s(A, B; X, Y)$. We can then define the *homotopy link* $\text{holink}(X, Y) = \text{map}_s(I, \{0\}; X, Y)$, i.e. the space of maps of the unit interval into X which take the point 0 into Y and all other points into the complement of Y. If X is a filtered space, we will also be interested in the *stratified homotopy link* $\text{holink}_s(X, Y)$, which is the subspace of $\text{holink}(X, Y)$ consisting of maps $I \to X$ which take 0 into Y and $(0, 1]$ into a single stratum of X. $\text{holink}_s(X, Y)$ is itself filtered by the subsets which take $(0, 1]$ into a particular skeleton of X. Note that these are subspaces of X^I and so are metrizable if X is. There are natural projections $\pi : \text{holink}(X, Y) \to Y$ and $\pi_s : \text{holink}_s(X, Y) \to Y$ defined by evaluation at 0. Where there is no room for ambiguity, we will also denote π_s by π.

A subspace $Y \subset X$ is *tame* if Y has a neighborhood $N \subset X$ such that there exists a strict map $(N \times I, Y \times I \cup N \times 0) \to (X, Y)$ which is the identity on Y and the inclusion on $N \times 1$, i.e. if there is a deformation retraction $N \to Y$ such that all points in $N - Y$ remain in $N - Y$ until time 0 (such a deformation retraction is called *nearly strict*). If X is a filtered space, then it is *weakly stratified* if for each $k > i$, $(X_{i+1} - X_i) \subset (X_{k+1} - X_k) \cup (X_{i+1} - X_i)$ is tame and the projection $\pi : \text{holink}((X_{k+1} - X_k) \cup (X_{i+1} - X_i), X_{i+1} - X_i) \to X_{i+1} - X_i$ is a fibration. The filtration of a space X is said to be a *manifold filtration* if each stratum is a manifold (we will always assume without boundary), and X is called a *manifold weakly stratified space* if it is weakly stratified by a manifold filtration. As observed in [15, p. 235], any *locally conelike* space is manifold weakly stratified and hence, in particular, stratified PL-pseudomanifolds are manifold weakly stratified (see below).

Now according to Hughes [10, Cor. 6.2], for a weakly stratified metric space X with a finite number of strata, $\pi : \text{holink}_s(X, X_k) \to X_k$ is a stratified fibration. Therefore, if $\mathcal{G}$ is a bundle of coefficients on $\text{holink}_s(X, X_k) - \text{holink}_s(X, X_k)_{n-2}$, n is the stratified dimension of $\text{holink}_s(X, X_k)$ (which is the same as the stratified dimension of X), and we let $\mathcal{L}$ denote a stratified fiber, then we can apply the results of the preceding sections to obtain bundles of coefficients $\mathcal{IH}^{\bar{p}}_*(\mathcal{L}; \mathcal{G})$ and $\mathcal{IH}^{\bar{p}}_*(\bar{c}\mathcal{L}; \bar{c}\mathcal{G})$ over X_k. These are the bundles of coefficients of the statement of Theorem 6.1, below.

6 The intersection homology of the regular neighborhood of the bottom stratum of a stratified PL-pseudomanifold

Le us recall some more definitions. Let $c(Z)$ denote the open cone on the space Z, and let $c(\emptyset)$ be a point. Generally, a *stratified paracompact Hausdorff space* Y (see [7] or [1]) is defined by a filtration

$$Y = Y_n \supset Y_{n-1} \supset Y_{n-2} \supset \cdots \supset Y_0 \supset Y_{-1} = \emptyset$$

such that for each point $y \in Y_i - Y_{i-1}$, there exists a distinguished neighborhood N of y such there is a compact Hausdorff space G (called the *link* of the component of the stratum), a filtration

$$G = G_{n-i-1} \supset \cdots \supset G_0 \supset G_{-1} = \emptyset,$$

and a homeomorphism

$$\phi : \mathbb{R}^i \times c(G) \to N$$

that takes $\mathbb{R}^i \times c(G_{j-1})$ onto Y_{i+j}.

A *PL-pseudomanifold* of dimension n is a PL space X (equipped with a class of locally finite triangulations) containing a closed PL subspace Σ of codimension at least 2 such that $X - \Sigma$ is a PL manifold of dimension n dense in X. A *stratified PL-pseudomanifold* of dimension n is a PL pseudomanifold equipped with a filtration such that $\Sigma = X_{n-2}$ and the local normal triviality conditions of a stratified space hold with the trivializing homeomorphisms ϕ being PL homeomorphisms. In fact, for any PL-pseudomanifold X, such a stratification always exists such that the filtration refines the standard filtration of X by k-skeletons with respect to some triangulation [3, Chapter I]. Furthermore, intersection homology is known to be a topological invariant of such spaces; in particular, it is invariant under choice of triangulation or stratification (see [7], [3], [11]).

Now, suppose that the filtered space $(X, \{X_i\})$ is a stratified PL-pseudomanifold with each X_i a subpolyhedron, so that, in particular, there exists a triangulation of X in which each X_i is triangulated as a subcomplex. Note that X is also a filtered polyhedron in the sense considered by Stone in [17]. Given such a filtered polyhedron and another subpolyhedron Y, then the filtered subpolyhedron $(V, \{V_i\})$ is defined (as in Stone [17, p. 4]) to be a *regular neighborhood* with *frontier* $\mathrm{fr}(V)$ if there are a triangulation B of X in which Y and the X_i are triangulated by full subcomplexes A and B_i and a first derived subdivision B' in which

1. $V = |\bar{N}(Y, B')|$,
2. $V_i = |\bar{N}(Y, B'_i)|$,
3. $\mathrm{fr}(V) = |\mathrm{lk}(Y, B')|$,

where, as usual, $|Z|$ represents the topological space underlying the simplicial complex Z and $\bar{N}(Y, B')$ is the (closed) first derived neighborhood of Y. (Stone in fact defines a more general relative regular neighborhood, but we will not need the added generality.) It is further shown in [17] that such neighborhoods satisfy the standard properties for regular neighborhoods, such as uniqueness via PL-isotopies and the generalized annulus property. We can also define an open regular neighborhood by taking the interior of V (meaning here $V - \mathrm{fr}(V)$) and its intersection with each of the V_i. In what follows, "open regular neighborhood" will always mean an open regular neighborhood in this sense.

Finally, let X be a finite-dimensional stratified PL-pseudomanifold with locally finite triangulation and such that $X_i = \emptyset$ for $i < k$ (in particular $X \subset \mathbb{R}^N$, for some N, and hence X is metrizable). Then X_k is both the lowest dimensional non-empty skeleton and stratum, and furthermore, by the definition of a stratified PL-pseudomanifold, it must be a manifold without boundary. (X_k must be an open manifold by definition, but it must also be a closed subset of X.) We will assume that X_k is connected for notational convenience, but what follows could be applied to each component separately. As a stratified PL-pseudomanifold, the stratified dimension of X is its dimension as a simplicial complex, and similarly for the dimensions of the filtering subsets, X_i. Let $N = N(X_k) = (N, \{N_i\})$ be an open regular neighborhood of X_k (in the sense of the last paragraph). Such a neighborhood exists since a closed regular neighborhood $(\bar{N}, \{\bar{N}_i\})$ exists by an application of [17, Prop. 1.11] (in fact, in Stone's system, we would relabel our X_k as X_0, and then the desired neighborhood would be the "normal $(n,0)$-regular neighborhood system" of this proposition). We also let L denote the link of the stratum X_k so that every point in X_k has an open neighborhood PL-homeomorphic to $D^k \times cL$, where D^k is the open k-dimensional ball and cL is the open cone on L. We wish to calculate the intersection homology of N and of $N - X_k$. If N has the topological structure of a fiber bundle over X_k with fiber cL then this can be computed by sheaf theoretic methods using spectral sequences (these calculations appear in Sections 5.7 and 5.8 of the author's disseration [5]). We now show that such spectral sequences exist without the need for such an assumption.

Theorem 6.1. *With N, X, and L as above and for any fixed perversity $\bar{p}$, which we sometimes omit from the notation, and local coefficient system $\mathcal{G}$ defined on $X - X_{n-2}$, there are homological-type spectral sequences $\bar{E}^r_{p,q}$ and $E^r_{p,q}$ which abut (up to isomorphism) to $IH^{\bar{p}}_i(N - X_k; \mathcal{G})$ and $IH^{\bar{p}}_i(N; \mathcal{G})$ with respective E^2 terms*

$$\bar{E}^2_{p,q} = H_p(X_k; \mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L)) \qquad\qquad E^2_{p,q} = H_p(X_k; \mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL)),$$

where $\mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L)$ and $\mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL)$ are local coefficient systems on X_k with respective stalks $IH^{\bar{p}}_q(L; \mathcal{G}|L)$ and $IH^{\bar{p}}_q(cL; \mathcal{G}|cL)$. Furthermore, the map $i_* : IH^{\bar{p}}_i(N - X_k; \mathcal{G}) \to IH^{\bar{p}}_i(N; \mathcal{G})$ induced by inclusion induces a map of spectral sequences which on the the E^2 terms is determined by the coefficient homomorphism $\mathcal{IH}^{\bar{p}}_q(L; \mathcal{G}|L) \to \mathcal{IH}^{\bar{p}}_q(cL; \mathcal{G}|cL)$ given by the map on the stalk intersection homology modules induced by the inclusion $L \hookrightarrow cL$.

Proof. Due to the invariance of intersection homology under stratum-preserving homotopy equivalences, we can replace N by any stratum-preserving homotopy equivalent space equipped with the coefficients pulled back by the equivalence map. For simplicity of notation below, we will also refer to this bundle as $\mathcal{G}$ where it is unnecessary to make a distinction. Which bundle is meant should be clear, however, from the space under consideration. To obtain a space stratum-preserving homotopy equivalent (s.p.h.e.) to N, we employ some of the constructions of Quinn. In particular, by [16, 3.2], since N is weakly stratified (in fact, locally conelike), there is a neighborhood V of X_k in N and a nearly stratum-preserving deformation retraction of V to X_k. This is a deformation retraction given by a homotopy $H : V \times I \to X$ such that $H|X_k \times I$ is the projection to X_k, $H|V \times 1$ is the inclusion $V \hookrightarrow X$, $H|V \times 0$ maps $V \times 0$ into X_k, and $H|V \times (0,1]$ is stratum preserving along $(0,1]$. In other words, the retraction is stratum preserving until the last moment at time 0 when everything must collapse into X_k. We will use this to prove the following lemma:

Lemma 6.2. *There is a nearly stratum-preserving deformation retraction R of N to X_k in N, i.e. $R : N \times I \to N$.*

Assuming the lemma for now, we can obtain a stratum-preserving homotopy equivalence between N and the mapping cylinder, I_π, of the stratified fibration $\pi : \text{holink}_s(N, X_k) \to X_k$. The idea of this equivalence is found in work by Quinn [16] and Chapman [2]. We present in the appendix to this paper a proof suggested by Bruce Hughes that fixes certain technical problems in this earlier work, specifically the lack of continuity of the holink evaluation $I_\pi \times I \to N$, $(w, s) \to w(s)$. Let us here at least describe stratum-preserving homotopy inverses $f : N \to I_\pi$ and $g : I_\pi \to N$. Given a point $y \in N$, let R_y be the retraction path of y under the nearly stratum-preserving deformation retraction R. For $x \in N$, let $d(x, X_k)$ be the distance from x to X_k; we are free to choose our regular neighborhood (or metric) so that this is always less than 1. Define $f(y)$ by $f(y) = y$ if $y \in X_k$ and $f(y) = (R_y, d(y, X_k))$ if $y \in N - X_k$ (recall that we use the teardrop topology on all mapping cylinders). In order to define g, we use an intermediate map $S : \text{holink}_s(N, X_k) \times (0, 1] \to \text{holink}_s(N, X_k)$. This homotopy should be thought of as shrinking the paths in the holink so that at time 1 we have the identity and as we approach time 0 the paths shrink to nothing. The shrinking is done in a certain way as to make the map g continuous; see the appendix for details. Define g by $g(y) = y$ for $y \in X_k$ and for $(w, s) \in \text{holink}_s(N_k, X) \times (0, 1]$ by $g(w, s) = S(w, s)(s)$; this is the evaluation of the path $S(w, s)$ at the time s.

Let the stratified dimension of I_π be the dimension of X, and filter I_π by X_k and the mapping cylinders of $\pi| \text{holink}_s(N, X_k)_i$ (note each such i is greater than k). With these definitions, $f(N_i - N_{i-1}) \subset I_{\pi,i} - I_{\pi,i-1}$, and similarly for g. Then, by the proof in the appendix, $g : I_\pi \to N$ and $f : N \to I_\pi$ are stratum preserving homotopy inverses rel X_k. In particular then, by Corollary 2.3, g must induce an isomorphism of intersection homology modules with the local coefficient system on I_π given by $g^*\mathcal{G}$. As noted above, for simplicity of notation we will also denote $g^*\mathcal{G}$ by $\mathcal{G}$ depending on the space under discussion to indicate which is meant. Note also that $g^*\mathcal{G} \cong (g^*\mathcal{G}| \text{holink}_s(N, X_k)) \times (0, 1]$ by a unique isomorphism, so we will assume that it has this form. Since these homotopy equivalences are stratum-preserving, their restrictions induce a stratum preserving homotopy-equivalence between $N - X_k$ and $I_\pi - X_k$, which is also clearly stratum-preserving homotopy equivalent to $\text{holink}_s(N, X_k)$. Hence, again by Corollary 2.3, there are isomorphisms $IH^{\bar{p}}_*(\text{holink}_s(N, X_k); \mathcal{G}) \cong IH^{\bar{p}}_*(I_\pi - X_k; \mathcal{G}) \cong IH^{\bar{p}}_*(N - X_k; \mathcal{G})$, the former induced by inclusion, the latter by g.

Letting Π stand for the obvious projection $I_\pi \to X_k$, we now filter I_π by open sets. It is unfortunate that we deal with two distinct concepts commonly referred to as filtrations. As our purpose is to construct spectral sequences, we will briefly refer to this filtration as an *SS-filtration*. We will also use the term *simplicial skeleton* to refer to the skeleton of a polyhedron in the sense of skeleta of simplicial complexes. Consider the triangulation of X_k, and let B^s be the regular neighborhood of the simplicial s-skeleton of X_k given by its star on the second derived subdivision of the triangulation. These provide an SS-filtration of X_k

and induces an SS-filtration on I_π by $J^s = \Pi^{-1}(B^s)$:

$$\begin{tikzcd}
I_\pi = J \supset \cdots \supset \arrow[d] & J^s \arrow[d] & \supset & J^{s-1} \supset \cdots \supset \arrow[d] & J^0 \arrow[d] & \supset & \emptyset \arrow[d] \\
X_k = B \supset \cdots \supset & B^s & \supset & B^{s-1} \supset \cdots \supset & B^0 & \supset & \emptyset.
\end{tikzcd}$$

These are both SS-filtrations by open sets, and thus we can define the singular intersection chain complexes $IC_i(J^s; \mathcal{G})$ and filter $IC_i(I_\pi; \mathcal{G})$ by $F_sIC_*(I_\pi; \mathcal{G}) = \operatorname{im}(IC_*(J^s; \mathcal{G}) \to IC_*(I_\pi; \mathcal{G}))$. This is obviously an increasing and exhaustive SS-filtration. The general theory of spectral sequences of SS-filtrations (see [12]) then provides a first quadrant spectral sequence with E^1 term

$$\begin{aligned} E^1_{p,q} &= H_{p+q}(F_pIC_*(I_\pi; \mathcal{G})/F_{p-1}IC_*(I_\pi; \mathcal{G})) \\ &= H_{p+q}(IC_*(J^p, J^{p-1}); \mathcal{G}) = IH_{p+q}(J^p, J^{p-1}; \mathcal{G}) \end{aligned}$$

and boundary map, d^1, given by the boundary map of the long exact intersection homology sequence of the triple (J^s, J^{s-1}, J^{s-2}). (Note that the long exact sequence of the triple for intersection homology can be defined just as in the usual case by using quotients of the intersection chain complexes.) The spectral sequence abuts to $H_*(IC_*(I_\pi; \mathcal{G})) = IH_*(I_\pi; \mathcal{G})$. Similarly, we can filter $\operatorname{holink}_s(N, X_k)$ by $\bar{J}^s = \pi^{-1}B^s$ and in the same manner obtain a spectral sequence abutting to

$$H_*(IC_*(\operatorname{holink}_s(N, X_k); \mathcal{G})) = IH_*(\operatorname{holink}_s(N, X_k); \mathcal{G}) \cong IH_*(I_\pi - X_k; \mathcal{G})$$

and with $\bar{E}^1_{p,q} = IH_{p+q}(\bar{J}^p, \bar{J}^{p-1}; \mathcal{G})$.

We can view each $\bar{J}^s$ as the subset $\bar{J}^s \times 1$ of the mapping cylinder $I_{\pi|\pi^{-1}(B^s)}$, and its filtration is the same as the induced filtration as a subset. The same is true for any pair $\pi^{-1}(U)$ and $\Pi^{-1}(U)$, for U an open subset of B. Hence these inclusions induce well-defined maps on intersection homology and, in particular, a map of spectral sequences which abuts to the map $IH_*(\operatorname{holink}_s(N, X_k); \mathcal{G}) \to IH_*(I_\pi; \mathcal{G})$ which, employing the homotopy equivalences and their induced isomorphisms, is isomorphic to the map $IH_*(N - X_k; \mathcal{G}) \to IH_*(N; \mathcal{G})$ induced by inclusion. On the E^1 terms, this map is that induced by the inclusions $IH_{p+q}(\bar{J}^p, \bar{J}^{p-1}; \mathcal{G}) \to IH_{p+q}(J^p, J^{p-1}; \mathcal{G})$.

It remains to identify the E^2 and $\bar{E}^2$ terms of these sequences and the induced map $\bar{E}^2 \to E^2$. In particular, we need to show that $E^1_{*,q}$ and $\bar{E}^1_{*,q}$ are isomorphic to the chain complexes with local coefficients $\operatorname{Cell}_*(X_k; \mathcal{IH}_q(cL; c\mathcal{G}))$ and $\operatorname{Cell}_*(X_k; \mathcal{IH}_q(L; \mathcal{G}))$ with map from the latter to the former induced by the coefficient homomorphism $\mathcal{IH}_q(L; \mathcal{G}) \to \mathcal{IH}_q(cL; c\mathcal{G})$.

For I_π, we define the collections $\{\sigma^i_\alpha\}$, $\{\hat{\sigma}^i_\alpha\}$, $\{\tilde{\sigma}^i_\alpha\}$, $\{\partial\hat{\sigma}^i_\alpha\}$, and $\{\partial\tilde{\sigma}^i_\alpha\}$ as follows: Let $\{\sigma^i_\alpha\}$ denote the collection of i-simplices of B, let $\hat{\sigma}^i_\alpha$ denote the open neighborhood of σ^i_α in the second derived subdivision, and let $\tilde{\sigma}^i_\alpha = \Pi^{-1}(\hat{\sigma}^i_\alpha)$. For each σ^i_α, let $\partial\hat{\sigma}^i_\alpha$ denote $\hat{\sigma}^i_\alpha \cap B^{i-1}$, i.e. the neighborhood of the boundary of $\hat{\sigma}^i_\alpha$, and let $\partial\tilde{\sigma}^i_\alpha$ denote $\tilde{\sigma}^i_\alpha \cap J^{i-1} = \Pi^{-1}(\partial\hat{\sigma}^i_\alpha)$. We claim that $H_i(IC_*(J^p, J^{p-1}); \mathcal{G}) \cong \oplus_\alpha IH_i(\tilde{\sigma}^p_\alpha, \partial\tilde{\sigma}^p_\alpha; \mathcal{G})$ for all i. The claim is obvious for $p = 0$. For

$p > 0$, we proceed by induction on the simplices. In particular, order the simplices of each dimension by placing an ordering on the indices α, and let A^p_j equal the union of the first j of the $\tilde\sigma^p_\alpha$. We will show by induction that $IH_i(A^p_j, A^p_j \cap J^{p-1}; \mathcal{G}) = \oplus_{\alpha\le j} IH_i(\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha; \mathcal{G})$ for all j, which will imply the claim since $A^p_j \subset A^p_{j+1}$ for all j, $\cup A^p_j = J^p$, and $J^p \cap J^{p-1} = J^{p-1}$. The formula obviously holds for $j = 1$. For $j > 1$, consider the Mayer-Vietoris sequence

$$\begin{aligned} \to IH_i(A^p_j \cap \tilde\sigma^p_{j+1}, A^p_j \cap J^{p-1} \cap \tilde\sigma^p_{j+1}; \mathcal{G}) \\ \to IH_i(A^p_j, A^p_j \cap J^{p-1}; \mathcal{G}) \oplus IH_i(\tilde\sigma^p_{j+1}, \partial\tilde\sigma^p_{j+1}; \mathcal{G}) \\ \to IH_i(A^p_{j+1}, A^p_{j+1} \cap J^{p-1}; \mathcal{G}) \to . \end{aligned}$$

Now, we have $A^p_j \cap \tilde\sigma^p_{j+1} \subset J^{p-1}$: In the base B, the intersection of $\cup^j_{i=0}\sigma^p_i$ and σ^p_{j+1} must lie in a lower dimensional skeleton and the intersection of their neighborhoods will lie in the neighborhood of this lower dimensional skeleton hence in B^{p-1}, and we can then consider the inverse images of these sets under the projection. Therefore, the intersection terms of the sequence are $IH_i(A^p_j \cap \tilde\sigma^p_{j+1}, A^p_j \cap J^{p-1} \cap \tilde\sigma^p_{j+1}; \mathcal{G}) = 0$. This suffices to prove the claim by a possibly infinite induction. Note, by the way, that it is in order to employ such Mayer-Vietoris sequences that we must use the open sets $\hat\sigma$ and their inverse images instead of simply the closed simplices, because Mayer-Vietoris sequences for intersection homology require open sets.

The corresponding results hold for $I_\pi - X_k \sim_{sphe} \text{holink}_s(N, X_k)$ by considering the analogously defined $\bar\sigma^p_\alpha = \pi^{-1}(\hat\sigma^p_\alpha)$, etc., and in this case we obtain $H_i(IC_*(\bar J^p, \bar J^{p-1}); \mathcal{G}) \cong \oplus_\alpha IH_i(\bar\sigma^p_\alpha, \partial\bar\sigma^p_\alpha; \mathcal{G})$. Now we must examine the $IH_i(\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha; \mathcal{G})$ and $IH_i(\bar\sigma^p_\alpha, \partial\bar\sigma^p_\alpha; \mathcal{G})$.

By definition, $\bar\sigma^p_\alpha = \pi^{-1}(\hat\sigma^p_\alpha)$ and $\tilde\sigma^p_\alpha = \Pi^{-1}(\hat\sigma^p_\alpha)$, which is the mapping cylinder of the restriction of π to $\bar\sigma^p_\alpha$. According to the proof of [15, Theorem 2], this restriction is precisely the projection $\text{holink}_s((N - B) \cup \hat\sigma^p_\alpha, \hat\sigma^p_\alpha) \to \hat\sigma^p_\alpha$, which is a stratified fibration by [10, Cor. 6.2]. By Corollary 3.14, above, there is a strong stratified trivialization $\mathfrak{h}_\alpha : \hat\sigma^p_\alpha \times \mathcal{L} \to \text{holink}_s(N, \hat\sigma^p_\alpha) = \bar\sigma^p_\alpha$, where $\mathcal{L}$ is the stratified fiber over some point of $\hat\sigma^p_\alpha$. By Corollary 3.19, we obtain a strong stratum-preserving fiber homotopy equivalence $h_\alpha : \hat\sigma^p_\alpha \times \bar c\mathcal{L} \to \tilde\sigma^p_\alpha$. By restricting the equivalences to the inverse images $\pi^{-1}(\partial\hat\sigma^p_\alpha)$, we see that the pairs $(\bar\sigma^p_\alpha, \partial\bar\sigma^p_\alpha)$ and $(\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha)$ are stratum-preserving fiber homotopy equivalent to the pairs $(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \mathcal{L}$ and $(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \bar c\mathcal{L}$. Also, if e_α is the basepoint of $\hat\sigma^p_\alpha$ over which the the trivializations are identities and $\mathcal{L}_\alpha$ is the corresponding fiber, then, by bundle theory, $\mathfrak{h}^*_\alpha\mathcal{G} \cong \hat\sigma^p_\alpha \times (\mathcal{G}|\mathcal{L}_\alpha)$ and $h^*_\alpha\mathcal{G} \cong \hat\sigma^p_\alpha \times (\mathcal{G}|\bar c\mathcal{L}_\alpha)$. Hence by the Künneth theorem for the intersection homology of a product one of whose factors is a manifold [11] (and its obvious generalizations to manifold pairs and a local coefficient system on the filtered space) and by the homotopy equivalence $(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \sim_{h.e.} (\sigma^p_\alpha, \partial\sigma^p_\alpha)$, we obtain the formulas

$$\begin{aligned} IH_i(\bar\sigma^p_\alpha, \partial\bar\sigma^p_\alpha; \mathcal{G}) &\cong IH_i((\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \mathcal{L}; h^*_\alpha\mathcal{G}) \cong IH_i((\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \mathcal{L}; \hat\sigma^p_\alpha \times (\mathcal{G}|\mathcal{L}_\alpha)) \\ &\cong H_p(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \otimes IH_{i-p}(\mathcal{L}; \mathcal{G}) \cong H_p(\sigma^p_\alpha, \partial\sigma^p_\alpha) \otimes IH_{i-p}(\mathcal{L}; \mathcal{G}) \\ IH_i(\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha; \mathcal{G}) &\cong IH_i((\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \bar c\mathcal{L}; \mathfrak{h}^*_\alpha\mathcal{G}) \cong IH_i((\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \times \bar c\mathcal{L}; \hat\sigma^p_\alpha \times (\mathcal{G}|\bar c\mathcal{L}_\alpha)) \\ &\cong H_p(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha) \otimes IH_{i-p}(\bar c\mathcal{L}; \mathcal{G}) \cong H_p(\sigma^p_\alpha, \partial\sigma^p_\alpha) \otimes IH_{i-p}(\bar c\mathcal{L}; \mathcal{G}). \end{aligned}$$

(If $\mathcal{G}$ is a bundle of R modules for some commutative ring R, then technically terms $H_p(\hat{\sigma}^p_\alpha, \partial\hat{\sigma}^p_\alpha)$ should be take with R coefficients, but we will suppress this from the notation for simplicity.) These are cellular simplices with coefficients in $IH_{i-p}(\mathcal{L};\mathcal{G})$ and $IH_{i-p}(\bar{c}\mathcal{L};\mathcal{G})$ (recall that on these spaces we are using $\mathcal{G}$ to stand for $g^*\mathcal{G}$). We want to see that $IH_*(c\mathcal{L};g^*\mathcal{G}) \cong IH_*(cL;\mathcal{G})$ and $IH_*(\mathcal{L};g^*\mathcal{G}) \cong IH_*(L;\mathcal{G})$.

We first show that $\bar{c}\mathcal{L} \sim_{s.p.h.e.} cL$ (recall that L is our geometric link). This will also imply $\mathcal{L} \sim_{s.p.h.e.} L$ since the stratum preserving condition implies that $c\mathcal{L} - * \sim_{s.p.h.e.} cL - *$, where $*$ represents the cone point, and clearly $\bar{c}\mathcal{L} - * \sim_{s.p.h.e.} c\mathcal{L} - * \sim_{s.p.h.e.} \mathcal{L}$ and $cL - * \sim_{s.p.h.e.} L$. Once again we have $\tilde{\sigma}^p_\alpha = \Pi^{-1}(\hat{\sigma}^p_\alpha)$, which is the mapping cylinder of the restriction of π to $\bar{\sigma}^p_\alpha$, and this restriction is precisely $\text{holink}_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$. Let us assume, as we are free to have imposed earlier, that we have chosen a fine enough triangulation so that each $\hat{\sigma}^i_\alpha$ has a neighborhood in N that is PL-homeomorphic to $\hat{\sigma}^i_\alpha \times cL$; this is possible since each point in X_k has a neighborhood of the form $D^k \times cL$, and we can use generalized subdivisions (see e.g. [14, §16]) to find a fine enough triangulation so that each $\hat{\sigma}^i_\alpha$ lies in some such D^k. Then the restriction of $D^k \times cL$ to the product over $\hat{\sigma}^i_\alpha$ provides such a neighborhood, which we will denote N^α.

Now, define a continuous function $\delta : \hat{\sigma}^p_\alpha \to (0,\infty)$ such that for each $x \in \hat{\sigma}^p_\alpha$, $B(x,\delta(x)) \subset N^\alpha$, where $B(x,\delta(x))$ is the closed ball in N of radius $\delta(x)$ centered at x. For example, the function given by one half the distance from the boundary of the closure of N^α in X will suffice. Then, following [16, p. 453], we can define $\text{holink}^\delta((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ to be the subset of $j \in \text{holink}((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ such that $\text{Im}(j : I \to N)$ lies within $\delta(j(0))$ of $j(0)$. Similarly, we can define $\text{holink}^\delta_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$. According to [16, Lemma 2.4], the inclusion $\text{holink}^\delta((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha) \hookrightarrow \text{holink}((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ is a fiber homotopy equivalence. Since the proof of this lemma consists of appropriately "shrinking" each path of the holink along itself, restriction provides a stratum-preserving homotopy equivalence of $\text{holink}^\delta_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ and $\text{holink}_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$. On the other hand, by our choice of δ, clearly $\text{holink}^\delta_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha) = \text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha)$, and by the same arguments, the inclusion $\text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha) \hookrightarrow \text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha)$ must be a stratum-preserving fiber homotopy equivalence. Hence, we have shown that $\text{holink}_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ is stratum-preserving fiber homotopy equivalent to $\text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha)$, and this induces a stratum-preserving fiber homotopy equivalence of the mapping cylinders of each of the projections to $\hat{\sigma}^p_\alpha$ by Corollary 3.18. By definition, we already know that the mapping cylinder of the projection of $\text{holink}_s((N-X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ is $\tilde{\sigma}_\alpha$, and by the same methods as our original proof that $I_\pi \sim_{s.p.h.e.} N$, we can show that N^α is stratum-preserving homotopy equivalent (by a map we will denote ϕ) to the mapping cylinder of the projection $\pi_{N^\alpha} : \text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha) \to \hat{\sigma}^p_\alpha$ (as our nearly stratum-preserving deformation retraction of $N^\alpha \cong \hat{\sigma}^p_\alpha \times cL$ to $\hat{\sigma}^p_\alpha$, we can take the product with the identity on $\hat{\sigma}^p_\alpha$ of the obvious nearly stratum-preserving retraction of cL to it cone point). Hence we have $N^\alpha \sim_{s.p.h.e} \tilde{\sigma}^p_\alpha$. Finally, we know that $\tilde{\sigma}^p_\alpha \sim_{s.p.h.e.} \hat{\sigma}^p_\alpha \times \bar{c}\mathcal{L}$, while $N^\alpha \cong_{P.L} \hat{\sigma}^p_\alpha \times cL$ by a stratum-preserving homeomorphism. Since $\hat{\sigma}^p_\alpha$ is contractible and unfiltered, we at last conclude:

$$\bar{c}\mathcal{L} \sim_{s.p.h.e.} \hat{\sigma}^p_\alpha \times \bar{c}\mathcal{L} \sim_{s.p.h.e.} \tilde{\sigma}^p_\alpha \sim_{s.p.h.e.} N^\alpha \cong_{P.L.} \hat{\sigma}^p_\alpha \times cL \sim_{s.p.h.e.} cL.$$

In order to show that $IH_*(c\mathcal{L}; g^*\mathcal{G}) \cong IH_*(cL; \mathcal{G})$, we must recall that the bundle over $\mathcal{L}$ is actually that determined by the pullback of $\mathcal{G}$ under the stratum-preserving homotopy equivalence $g : I_\pi \to N$ and show that the homotopy equivalences of the preceding paragraph are covered by the proper coefficient maps. In particular, for any α, let us identify cL with one of its isomorphic images in N^α under the inclusion $* \times cL \hookrightarrow \hat{\sigma}_\alpha \times cL$, for some $*$, followed by the isomorphism $\hat{\sigma}_\alpha \times cL \to N^\alpha$. Let $h : N^\alpha \to \tilde{\sigma}_\alpha$ denote the stratum-preserving homotopy equivalence determined as in the preceding paragraph. Then $h^*g^*\mathcal{G} = (gh)^*\mathcal{G}$ is a coefficient system on N^α. We want to see that it is equivalent to $\mathcal{G}|N^\alpha$. Let $\pi_{N^\alpha} : \text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha) \to \hat{\sigma}^p_\alpha$, $\pi^\delta_{N^\alpha} : \text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha) \to \hat{\sigma}^p_\alpha$, and $\pi^\delta : \text{holink}^\delta_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha) \to \hat{\sigma}^p_\alpha$ be the projections. Let $\psi : \text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha) \to \text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha)$ be the stratum-preserving fiber homotopy inverse to the inclusion. By the proof of [16, Lemma 2.4], if $\gamma \in \text{holink}_s(N^\alpha, \hat{\sigma}^p_\alpha)$, then $\psi\gamma : [0, 1] \to N^\alpha$ is the path γ composed with multiplication by a number $r \in (0, 1)$, which may depend on γ. Then ψ induces a map $\bar{\psi} : I_{\pi_{N^\alpha}} \to I_{\pi^\delta_{N^\alpha}}$. Also, the inclusion $\text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha) \hookrightarrow \text{holink}_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ induces an inclusion $I_{\pi^\delta} \to I_\pi$, and the equality $\text{holink}^\delta_s(N^\alpha, \hat{\sigma}^p_\alpha) = \text{holink}^\delta_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)$ induces an equality $I_{\pi^\delta_{N^\alpha}} = I_{\pi^\delta}$. With this notation, gh is the composition of maps

$$N^\alpha \xrightarrow{\phi} I_{\pi_{N^\alpha}} \xrightarrow{\bar{\psi}} I_{\pi^\delta_{N^\alpha}} = I_{\pi^\delta} \hookrightarrow \tilde{\sigma}^p_\alpha \hookrightarrow I_\pi \xrightarrow{g} N$$

(recall that $\tilde{\sigma}^p_\alpha$ is the mapping cylinder of the projection $\pi|_{\pi^{-1}(\hat{\sigma}^p_\alpha)} = \pi|_{\text{holink}_s((N - X_k) \cup \hat{\sigma}^p_\alpha, \hat{\sigma}^p_\alpha)}$). If $d(\cdot, \cdot)$ represents distance in X and $(s, t, x) \in \hat{\sigma}^p_\alpha \times (0, 1) \times L \subset \hat{\sigma}^p_\alpha \times cL - \hat{\sigma}^p_\alpha$ and $\gamma_{(s,t,x)}$ is the path of the retraction of (s, t, x) to its cone point, $s \in \hat{\sigma}^p_\alpha$, under the nearly stratum-preserving deformation retraction of N^α to $\hat{\sigma}^p_\alpha$, then we can calculate $gh(s, t, x)$ as follows: $\phi(s, t, x) = (d((s, t, x), X_k), \gamma_{(s,t,x)}) \in I_{\pi_{N^\alpha}}$; $\bar{\psi}\phi(s, t, x) = (d((s, t, x), X_k), \psi\gamma_{(s,t,x)})$; the next three maps are inclusions; and finally $gh(s, t, x) = S(\psi\gamma_{(s,t,x)}, d((s, t, x), X_k))(d((s, t, x), X_k))$, which lies on the line from (s, t, x) to the cone point $s \in \hat{\sigma}^p_\alpha$ (recall that we have assumed that all point in N are distance < 1 from X_k and S retracts all paths along themselves). Therefore, if, for a given $t \in (0, 1)$, $H : \hat{\sigma}^p_\alpha \times (0, 1) \times L \times I \to \hat{\sigma}^p_\alpha \times (0, 1) \times L$ is any of the stratum-preserving deformation retractions to $\hat{\sigma}^p_\alpha \times t \times L$ which takes each $s \times (0, 1) \times x$ to (s, t, x), then $H \circ (gh|_{\hat{\sigma}^p_\alpha \times t \times L} \times \text{id}_I)$ is a homotopy from $gh|_{\hat{\sigma}^p_\alpha \times t \times L}$ to the inclusion $i : \hat{\sigma}^p_\alpha \times t \times L \hookrightarrow N^\alpha$. Therefore, $(gh|_{\hat{\sigma}^p_\alpha \times t \times L})^*\mathcal{G} \cong \mathcal{G}|_{\hat{\sigma}^p_\alpha \times t \times L}$. But then, employing some bundle theory and recalling that $\mathcal{G}$ is only defined over the top stratum, we have

$$\mathcal{G} \cong \mathcal{G}|_{\hat{\sigma}^p_\alpha \times t \times L} \times (0, 1) \cong (gh|_{\hat{\sigma}^p_\alpha \times t \times L})^*\mathcal{G} \times (0, 1) \cong (gh)^*\mathcal{G}|_{\hat{\sigma}^p_\alpha \times t \times L} \times (0, 1) \cong (gh)^*\mathcal{G}.$$

Therefore, $IH^{\bar{p}}_*(N^\alpha; h^*g^*\mathcal{G}) \cong IH^{\bar{p}}_*(N^\alpha; \mathcal{G}|N^\alpha)$, and, by Corollary 2.3, we have isomorphisms $IH^{\bar{p}}_*(\tilde{\sigma}_\alpha; g^*\mathcal{G}) \cong IH^{\bar{p}}_*(N^\alpha; h^*g^*\mathcal{G})$. Furthermore, there are isomorphisms $IH^{\bar{p}}_*(N^\alpha; \mathcal{G}|N^\alpha) \cong IH^{\bar{p}}_*(cL; \mathcal{G}|cL)$ since the stratum-preserving homotopy equivalence of cL and N^α is induced by inclusion. Similarly, we obtain $IH^{\bar{p}}_*(\tilde{\sigma}_\alpha; g^*\mathcal{G}) \cong IH^{\bar{p}}_*(\bar{c}\mathcal{L}; g^*\mathcal{G}|\bar{c}\mathcal{L})$ due to the strong stratum-preserving trivialization $\hat{\sigma}_\alpha \times c\mathcal{L} \to \tilde{\sigma}_\alpha$, which is the identity on $* \times \mathcal{L}$, and the stratum-preserving homotopy equivalence given by the inclusion $\bar{c}\mathcal{L} \to \hat{\sigma}_\alpha \times \bar{c}\mathcal{L}$. Lastly, since $IH_*(c\mathcal{L}; g^*\mathcal{G}) \cong IH_*(\bar{c}\mathcal{L}; g^*\mathcal{G})$, induced by inclusion, we conclude $IH_*(c\mathcal{L}; g^*\mathcal{G}) \cong IH_*(cL; \mathcal{G})$.

In the same manner, we can use these homotopy equivalences to show that $IH_*(\mathcal{L}; g^*\mathcal{G}) \cong IH_*(L; \mathcal{G})$: the restriction $h|N^\alpha - \hat{\sigma}_\alpha$ is a stratum-preserving homotopy equivalence to $\tilde{\sigma}_\alpha - \hat{\sigma}_\alpha$, and $gh|N^\alpha - \hat\sigma_\alpha$ is stratum-preserving homotopic to the inclusion. As above, this gives an isomorphism $IH^{\bar p}_*(cL-*;\mathcal{G}|cL-*) \cong IH^{\bar p}_*(\bar c\mathcal{L}-*;g^*\mathcal{G}|\bar c\mathcal{L}-*)$. But $IH^{\bar p}_*(cL-*;\mathcal{G}|cL-*) \cong IH^{\bar p}_*(L;\mathcal{G}|L)$ and $IH^{\bar p}_*(\bar c\mathcal{L}-*;g^*\mathcal{G}|\bar c\mathcal{L}-*) \cong IH^{\bar p}_*(\mathcal{L};g^*\mathcal{G}|\mathcal{L})$, as induced by the inclusions (for the former, the inclusion can be considered as $L \hookrightarrow L\times t \subset cL$ for any $t\in(0,1)$; also note that the $\mathcal{G}|(L\times t)$ are all isomorphic, as $\mathcal{G}|cL \cong (\mathcal{G}|L\times s)\times(0,1)$ for any $s\in(0,1)$). We should also observe that, as seen above in our study of local coefficient systems on stratified fibrations (and their mapping cylinders), these intersection homology groups are independent, up to isomorphism, of our particular choice of fiber $\mathcal{L}$.

At this point, by putting together the results of the preceding paragraphs, we have shown that E^1 and $\bar E^1$ are composed of the correct cellular chain modules. It remains to verify that the boundary maps are those compatible with the cellular chain complex of X_k with the desired local coefficient systems and that the map between the two systems is the desired one. This will be done over the following pages. $\square$

Let us first dispense with the lemma whose proof was deferred.

Proof of lemma 6.2. According to [16, Prop. 3.2] (also see [15, p. 241]) and the remarks preceding Definition A1 of [10]), there exists a neighborhood V of X_k in N and a nearly stratum-preserving deformation retraction $r: V\times I\to X$ of V to X_k. Note that r may take points in V outside of V during the retraction process, which is the reason this lemma is necessary. Let N' be another open regular neighborhood of X_k such that $\bar N'\subset \text{int}(V)$. Such an N' exists because we can find a closed regular neighborhood of X_k in $\text{int}(V)$ (as by the construction of [17, Prop. 1.11]), but this will clearly also be a regular neighborhood of X_k in X. Furthermore, we have $\bar N'\subset N$. By [17, Prop. 1.5], $\text{cl}(\bar N-\bar N')$ is PL-isomorphic to the product $\text{fr}(\bar N')\times I$, and by an isomorphism which takes each $\text{cl}(X_i\cap(\bar N-\bar N'))$ to $(X_i\cap \text{fr}(\bar N'))\times I$. Hence, there is a homotopy which gives a deformation retraction of N to $\bar N'$ which is stratum preserving and maps N into N, namely the homotopy which retracts $\text{cl}(\bar N-\bar N')\cong \text{fr}(\bar N')\times I$ to $\text{fr}(\bar N')\times\{0\}$ while leaving the rest of N fixed. So, if we consider N as $N'\cup_{\text{fr}(\bar N')\times\{0\}}\text{fr}(\bar N')\times I$ and represent each point in $\text{fr}(\bar N')\times I$ by (y,s), we define

$$H(t,x)=\begin{cases} x, & x\in N-\text{cl}(\bar N-\bar N')\\ (y,ts), & x\in \text{cl}(\bar N-\bar N'), x=(y,s).\end{cases}$$

This is clearly continuous. Now we can define $R: N\times I\to N$ as

$$R(t,x)=\begin{cases} H(3t-2,x), & t\in[2/3,1]\\ H(0,x)=r(1,H(0,x)), & t\in[1/3,2/3]\\ r(3t,H(0,x)), & t\in[0,1/3].\end{cases}$$

This is the homotopy which shrinks N to $N'\subset V$ by deformation retracting $\text{fr}(\bar N')\times I$ as time decreases from 1 to 2/3, then pauses in the time interval $[1/3,2/3]$, and then finally, for time in $[0,1/3]$, deformation retracts $H(0,N')$ to X_k utilizing the retraction r on V. Furthermore, R is nearly stratum-preserving since H is stratum-preserving and r is nearly stratum-preserving, and continuity follows from the continuity of H and r. Hence R is our desired deformation retraction. $\square$

We will now complete the proof of Proposition 6.1 by showing that the E^1 terms constructed above together with the boundary maps given by the spectral sequences are chain isomorphic to cellular chain complexes on B with local coefficients determined as above from the stratified fibration. The proof is essentially that given in the usual computation of the E^2 term of the spectral sequence of a fibration (see, e.g., Whitehead [18, Chapter XIII]). We run through the details in this situation mainly to double check that our departures are consistent with the usual the proof. We make the following initial remarks with the hope of clarifying the specific points of departure:

1. We are using intersection homology as opposed to the more traditional homology theories of the standard presentations. This technicality is dealt with by the use of stratified fibrations. We have already employed the stratified fibration $\text{holink}_s(X, X_k) \to X_k$ to construct both bundles of coefficients. However, in the following, as in the standard proof, it is the local triviality that comes to the fore.

2. We need to allow a bundle of local coefficients on the total space N (or, more specifically, on the complement of $N \cap X_{n-2}$). For ordinary homology this complication is tacitly present in Whitehead's treatment for extraordinary homology theories. Similarly, local coefficients fit more-or-less seemlessly into the following, only warranting an occasional remark to keep us honest.

3. Due to our need for all the subsets to be open in order to employ the Mayer-Vietoris arguments for intersection homology used in the first part of the proof, we have been saddled with using the groups $H_*(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha)$ as the generators of our cellular chain complex on X_k. However, this presents no difficulty as the homotopy equivalences $(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha) \sim_{h.e.} (\sigma_\alpha, \partial\sigma_\alpha)$ induce canonical chain isomorphisms between the cellular chain complex so-defined and that of the standard cellular chain complex. After a brief initial statement, below, we will usually suppress the distinction.

4. In contrast to the above generality, the following discussion is simplified by the fact that we use a fixed triangulation of X_k to impart its structure as a CW-complex. Hence, we can treat all characteristic maps as inclusions (although, once again, we take the unusual approach of including the open simplices $(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha)$).

Completion of the proof of Theorem 6.1. We will employ the following conventions: For the homology of $X_k = B$ with local coefficients, we will employ the paradigm of computing homology with local coefficients by using coefficients in the module over a basepoint of each relative singular simplex generating each cellular simplex (as opposed to defining the coefficient as a lift as we did above). In this manner, we will define cellular homology of B with local coefficients as in [18]. We use the given triangulation of B to define it as a (regular) cell complex. Hence, with local coefficient system $\mathcal{M}$ with fiber M, we define $H_*(B; \mathcal{M})$ as the homology of the chain complex of B with local coefficients

$$\begin{aligned}\Gamma_*(B; \mathcal{M}) = \oplus_\alpha H_*(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha; \mathcal{M}) &\cong H_*(B^*, B^{*-1}; \mathcal{M})\\ &\cong \oplus_\alpha H_*(\sigma_\alpha, \partial\sigma_\alpha; \mathcal{M}) \cong \oplus_\alpha M,\end{aligned}$$

with the boundary maps given by those of the the long exact sequence of the triple (B^*, B^{*-1}, B^{*-2}) with coefficient maps handled as usual for homology with local coefficients (see [18]). Again, it is the direct sum $\oplus_\alpha H_*(\sigma_\alpha, \partial\sigma_\alpha; \mathcal{M})$ which is generally used to define the cellular homology chain complexes, but the above string of isomorphisms and those induced in the obvious places by the homotopy equivalences of the σ_α and the $\hat{\sigma}_\alpha$ allow us to redefine the cellular homology in this manner. Note that each $H_*(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha; \mathcal{M}) \cong H_*(\sigma_\alpha, \partial\sigma_\alpha; \mathcal{M})$ can be considered as generated by the singular simplex given by the characteristic inclusion $i_\alpha : \Delta^i \to \sigma_\alpha$ with coefficient in the module over the leading vertex $e_\alpha := i_\alpha(e_0)$.

We have already seen that there is an isomorphism

$$\Gamma_p(B; \mathcal{IH}^{\bar{p}}_q(\bar{c}\mathcal{L}; \mathcal{G}|\bar{c}\mathcal{L})) \xrightarrow{\cong} IH^{\bar{p}}_{p+q}(J^p, J^{p-1}; \mathcal{G}|J^p),$$

in which $\bar{c}\mathcal{L}$ is the closed cone on the fiber of the stratified fibration $\pi : \mathrm{holink}_s(N, X_k) \to B = X_k$ up to stratum-preserving homotopy equivalence. Let us specify this isomorphism, which we will denote λ, by determining its action on the cellular simplex with coefficient $x[i_\alpha]$, where $[i_\alpha]$ denotes the homology class of the generating singular chain i_α in $H_p(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha)$ and $x \in IH^{\bar{p}}_q(\bar{c}\mathcal{L}_\alpha; \mathcal{G})$, $\bar{c}\mathcal{L}_\alpha$ the cone on the fiber over $e_\alpha = i_\alpha(e_0)$. Let h_α be the strong stratified trivialization $h_\alpha : (\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha) \times \bar{c}\mathcal{L}_\alpha \to (\tilde{\sigma}_\alpha, \partial\tilde{\sigma}_\alpha)$ obtained in the first part of the proof from a strong stratified trivialization from $\hat{\sigma}_\alpha \times \mathcal{L}_\alpha$ to $\mathrm{holink}_s(N, \hat{\sigma}_\alpha)$ (and covered by the induced map $h^*_\alpha(\mathcal{G}|\hat{\sigma}_\alpha) \to (\mathcal{G}|\tilde{\sigma}_\alpha)$); and let $[i_\alpha]\times : IH^{\bar{p}}_q(\bar{c}\mathcal{L}_\alpha; \mathcal{G}) \to IH^{\bar{p}}_{p+q}(\hat{\sigma}_\alpha \times \bar{c}\mathcal{L}, \partial\hat{\sigma}_\alpha \times \bar{c}\mathcal{L}; h^*_\alpha\mathcal{G})$ be the map given by the cross product of the Künneth theorem and using the unique isomorphism $\hat{\sigma}_\alpha \times (\mathcal{G}|\bar{c}\mathcal{L}_\alpha) \xrightarrow{\cong} h^*_\alpha\mathcal{G}$ which is the identity over e_α. Let $j_\alpha : (\tilde{\sigma}_\alpha, \partial\tilde{\sigma}_\alpha) \to (J^p, J^{p-1})$ be the inclusion. Then, by reversing the order of the steps by which we previously demonstrated this isomorphism, we see that λ can be given by

$$\lambda(x[i_\alpha]) = j_{\alpha*}h_{\alpha*}([i_\alpha] \times x).$$

To show that, for each fixed q, $E^1_{*,q}$ with boundary d^1 is chain isomorphic to $\Gamma_p(B; \mathcal{IH}^{\bar{p}}_q(\bar{c}\mathcal{L}; \mathcal{G}|\bar{c}\mathcal{L}))$, it remains to prove that the following diagram commutes for each p:

$$\begin{tikzcd}
\Gamma_p(B; \mathcal{IH}^{\bar{p}}_q(\bar{c}\mathcal{L}; \mathcal{G}|\bar{c}\mathcal{L})) \arrow[r, "\partial"] \arrow[d, "\lambda"'] & \Gamma_{p-1}(B; \mathcal{IH}^{\bar{p}}_q(\bar{c}\mathcal{L}; \mathcal{G}|\bar{c}\mathcal{L})) \arrow[d, "\lambda"'] \\
IH^{\bar{p}}_{p+q}(J^p, J^{p-1}; \mathcal{G}|J^p) \arrow[r, "\partial"] & IH^{\bar{p}}_{p+q-1}(J^{p-1}, J^{p-2}; \mathcal{G}|J^p).
\end{tikzcd}$$

(Note: we use the symbol "∂" to refer to either "boundary" map.) Let $\Gamma_*(\hat{\sigma}^p_\alpha)$ be the cellular chain complex generate by $H_*(\hat{\sigma}^p_\alpha, \partial\hat{\sigma}^p_\alpha) \cong H_p(\hat{\sigma}^p_\alpha, \partial\hat{\sigma}^p_\alpha)$. Let $\Gamma_*(\partial\hat{\sigma}^p_\alpha)$ be the cellular chain complex induced on $\partial\hat{\sigma}^p_\alpha$; in other words $\oplus_{\beta<\alpha}H_{p-1}(\hat{\sigma}^{p-1}_\beta, \partial\hat{\sigma}^{p-1}_\beta)$, the sum over the facets of σ_α; similarly for local coefficients. Let $i_\alpha : \hat{\sigma}_\alpha \hookrightarrow B$ and $j_\alpha : (\tilde{\sigma}^p_\alpha, \partial\tilde{\sigma}^p_\alpha) \hookrightarrow (J^p, J^{p-1})$ be the inclusions. Let $\mathcal{L}_\beta$ and $\mathcal{L}_\alpha$ stand for the fibers over the basepoints of the given generators of $H_{p-1}(\hat{\sigma}_\beta, \partial\hat{\sigma}_\beta)$ and $H_p(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha)$. We consider the following cube diagram whose front face is diagram (6):

$$
\begin{tikzcd}
H_p(\hat\sigma^p_\alpha, \partial\hat\sigma^p_\alpha; IH^{\bar p}_q(\bar c\mathcal{L}_\alpha;\mathcal{G}|\bar c\mathcal{L}_\alpha)) \arrow[rr, "\partial"] \arrow[dr, "i_{\alpha *}"] \arrow[dd, "\lambda"'] & & \oplus_{\beta<\alpha} H_{p-1}(\hat\sigma^{p-1}_\beta, \partial\hat\sigma^{p-1}_\beta; IH^{\bar p}_q(\bar c\mathcal{L}_\alpha;\mathcal{G}|\bar c\mathcal{L}_\alpha)) \arrow[dr, "i_{\alpha *}"] \arrow[dd, "\lambda'"' near start] & \\
& \Gamma_p(B;\mathcal{IH}^{\bar p}_q(\bar c\mathcal{L};\mathcal{G}|\bar c\mathcal{L})) \arrow[rr, "\partial"] \arrow[dd, "\lambda"' near start] & & \Gamma_{p-1}(B;\mathcal{IH}^{\bar p}_q(\bar c\mathcal{L};\mathcal{G}|\bar c\mathcal{L})) \arrow[dd, "\lambda"] \\
IH^{\bar p}_{p+q}(\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha;\mathcal{G}) \arrow[rr, "\partial" near end] \arrow[dr, "j_{\alpha *}"] & & IH^{\bar p}_{p+q-1}(\partial\tilde\sigma^p_\alpha, \partial\tilde\sigma^p_\alpha \cap J^{p-2};\mathcal{G}) \arrow[dr, "j_{\alpha *}"] & \\
& IH^{\bar p}_{p+q}(J^p, J^{p-1};\mathcal{G}|J^p) \arrow[rr, "\partial"] & & IH^{\bar p}_{p+q-1}(J^{p-1}, J^{p-2};\mathcal{G}|J^p)
\end{tikzcd}
\tag{4}
$$

The top back map of the cube (4) is the boundary map with constant coefficients. The map λ' is determined by the restriction of the strong trivialization over $\hat{\sigma}^p_\alpha$, not by the trivialization given over each $\hat{\sigma}_\beta$, i.e. $\lambda'(x[i_\beta]) = \dot{h}_{\alpha*}([i_\beta]\times x)$, where $\dot{h}_\alpha$ is the restriction of h_α to $\partial\hat{\sigma}_\alpha \times \bar{c}\mathcal{L}_\alpha$ and $[i_\beta]\times$ is the cross product of the Künneth theorem together with the bundle isomorphism $[\hat{\sigma}_\alpha \times (\mathcal{G}|\bar{c}\mathcal{L}_\alpha)]|(\hat{\sigma}_\beta \times \mathcal{L}_\alpha) \overset{\cong}{\to} h^*_\alpha\mathcal{G}|(\hat{\sigma}_\beta \times \mathcal{L}_\alpha)$. Note also that the $i_{\alpha*}$ on the top right edge is induced by the inclusion on the spaces $\hat{\sigma}_\beta$ and the map on coefficients determined by the map of the coefficient system over the path class in $\hat{\sigma}_\alpha$ between the basepoints of the generators of $H_p(\hat{\sigma}_\alpha, \partial\hat{\sigma}_\alpha)$ and $H_{p-1}(\hat{\sigma}_\beta, \partial\hat{\sigma}_\beta)$.

The top and bottom of this cube are clearly commutative. On the left side, it is easy to see then that the difference between the maps $j_{\alpha*}\lambda$ and $\lambda i_{\alpha*}$ is the difference between first including into a direct sum and then mapping the direct sum componentwise versus first mapping components and then including into the direct sum. These are equivalent and yield commutativity of the left side of the cube. The commutativity of the right side of the cube is harder to prove and will be deferred until we complete the rest of the proof of Proposition 6.1.

Finally, we show that the back face of the cube commutes: If i_β are the facets of i_α, suitably ordered, then by standard cellular theory, $\partial[i_\alpha] = \sum_{\beta=0}^p (-1)^\beta [i_\beta]$. Within $\tilde{\sigma}_\alpha$, we have $\lambda(x[i_\alpha]) = h_{\alpha*}([i_\alpha]\times x)$. Now we compute:

$$\begin{aligned}
\lambda'\partial(x[i_\alpha]) &= \lambda' \sum_{\beta=0}^{p} (-1)^\beta x[i_\beta] \\
&= \sum_{\beta=0}^{p} (-1)^\beta \dot{h}_{\alpha*}([i_\beta]\times x) \\
&= \dot{h}_{\alpha*}\left(\sum_{\beta=0}^{p} (-1)^\beta [i_\beta]\times x\right) \\
&= \dot{h}_{\alpha*}\partial([i_\alpha]\times x) \\
&= \partial h_{\alpha*}([i_\alpha]\times x) \\
&= \partial\lambda(x[i_\alpha]).
\end{aligned}$$

The fourth equality is due to the naturality of the Künneth theorem and, more particularly, the Eilenberg-Zilber map for intersection homology ([11]).

We have shown the commutativity of all faces of the cube but for the front face (and the deferred righ face, which we now assume to commute). Thus we have

$$\begin{aligned}
\lambda\partial i_{\alpha*} &= \lambda(i_\alpha|\partial\hat{\sigma}_\alpha)_*\partial \\
&= (j_\alpha|\partial\tilde{\sigma}_\alpha)_*\lambda'\partial \\
&= (j_\alpha|\partial\tilde{\sigma}_\alpha)_*\partial\lambda \\
&= \partial j_{\alpha*}\lambda \\
&= \partial\lambda i_{\alpha*}.
\end{aligned}$$

Hence, we have established commutativity of diagram (6) on every term of the direct sum decomposition of $\Gamma_p(B; \mathcal{IH}^{\bar{p}}_q(\bar{c}\mathcal{L}; \mathcal{G}|\bar{c}\mathcal{L}))$ and therefore commutativity in general. So we have shown that $E^1_{*,q}$ and $\mathrm{Cell}_*(X_k;\mathcal{IH}^{\bar p}_q(cL;\mathcal{G}))$ are chain equivalent as desired (recalling once again that $IH^{\bar p}_*(L;\mathcal{G})\cong IH^{\bar p}_*(\mathcal{L};\mathcal{G})$). The proof for $\bar E^1_{*,q}$ and $\mathrm{Cell}_*(X_k;\mathcal{IH}^{\bar p}_q(L;\mathcal{G}))$ is entirely analogous using the $\bar\sigma_\alpha$ instead of the $\tilde\sigma_\alpha$.

Finally, we need to see that the map $i_*:\bar E^1_{*,*}\to E^1_{*,*}$ which is induced by the inclusion $i:\mathrm{holink}_s(N,X_k)\hookrightarrow I_\pi$ is given by the cellular chain map $j:\mathrm{Cell}(X_k,\mathcal{IH}^{\bar p}_q(\mathcal{L};\mathcal{G}))\to \mathrm{Cell}(X_k,\mathcal{IH}^{\bar p}_q(\bar c\mathcal{L};\mathcal{G}))$ induced by the map on the bundle of coefficients that is induced on the fiber intersection homology by the inclusion $\mathcal{L}\hookrightarrow\bar c\mathcal{L}$. In other words, we must show that the following diagram commutes:

$$\begin{array}{ccc}
\Gamma_*(B;\mathcal{IH}^{\bar p}_q(\mathcal{L};\mathcal{G}|\mathcal{L})) & \xrightarrow{j_*} & \Gamma_*(B;\mathcal{IH}^{\bar p}_q(\bar c\mathcal{L};\mathcal{G}|\bar c\mathcal{L}))\\
{\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\lambda}\big\downarrow\\
IH^{\bar p}_{*+q}(\bar J^*,\bar J^{*-1};\mathcal{G}|\bar J^*) & \xrightarrow{i_*} & IH^{\bar p}_{*+q}(J^*,J^{*-1};\mathcal{G}|J^*).
\end{array}$$

But this commutes for each $*$ as follows directly by considering how the maps λ act on each summand of $\Gamma_*(B;\mathcal{IH}^{\bar p}_q(\mathcal{L};\mathcal{G}|\mathcal{L}))$ or $\Gamma_*(B;\mathcal{IH}^{\bar p}_q(\bar c\mathcal{L};\mathcal{G}|\bar c\mathcal{L}))$ determined by the cell $\hat\sigma_\alpha$. It is only necessary to employ the naturality of the Künneth theorem and the fact that each trivializing stratum-preserving fiber homotopy equivalence $h_\alpha:(\hat\sigma_\alpha,\partial\hat\sigma_\alpha)\times\bar c\mathcal{L}\to(\tilde\sigma_\alpha,\partial\tilde\sigma_\alpha)$ as defined retricts to a stratum-preserving fiber homotopy equivalence $\mathfrak{h}_\alpha:(\hat\sigma_\alpha,\partial\hat\sigma_\alpha)\times\mathcal{L}\to(\bar\sigma_\alpha,\partial\bar\sigma_\alpha)$ (in fact h_α was defined by extending $\mathfrak{h}_\alpha$).

We have already seen that λ is a chain map. That i_* and j_* are, in fact, maps of chain complexes follows from the naturality of homology and intersection homology with respect to maps between local coefficient systems over a single base space (this follows for intersection homology with local coefficients just as for ordinary homology with local coefficients).

By the naturality of homology, this chain map induces the desired map of E^2 terms of the spectral sequences. Note that, by the formula for the intersection homology of a cone (see [11]), this map is the identity for $*<n-k-1-\bar p(n-k)$ and the the 0 map for $*\geq n-k-1-\bar p(n-k)$, since this is true for each map $IH^{\bar p}_*(L;\mathcal{G})\to IH^{\bar p}_*(cL;\mathcal{G})$ induced by $L\hookrightarrow cL$. $\square$

Proof that the right face of the cube commutes. We conclude by showing that the right side of the cube (4) commutes. Actually, we begin with the corresponding cube for the $\bar E^1$ terms of the spectral sequence for the intersection homology of the stratified fibration $\mathrm{holink}_s(N,X_k)\to X_k$. The proof is a modified version of that of [18, XIII.4.7], which is again largely simplified by the fact that the maps are mostly inclusions.

Replacing each term with its direct sum decomposition, we must prove that the following diagram commutes:

$$\begin{array}{ccc}
\oplus_{\beta<\alpha}H_{p-1}(\hat\sigma^{p-1}_\beta,\partial\hat\sigma^{p-1}_\beta;IH^{\bar p}_q(\mathcal{L}_\alpha;\mathcal{G}|\mathcal{L}_\alpha)) & \xrightarrow{i_{\alpha*}} & \oplus_\beta H_{p-1}(\hat\sigma^{p-1}_\beta,\partial\hat\sigma^{p-1}_\beta;IH^{\bar p}_q(\mathcal{L}_\beta;\mathcal{G}|\mathcal{L}_\beta))\\
{\scriptstyle\lambda'}\big\downarrow & & {\scriptstyle\lambda}\big\downarrow\\
\oplus_{\beta<\alpha}IH^{\bar p}_{p+q-1}(\bar\sigma^{p-1}_\beta,\partial\bar\sigma^{p-1}_\beta;\mathcal{G}|\bar\sigma^{p-1}_\beta) & \xrightarrow{j_{\alpha*}} & \oplus_\beta IH^{\bar p}_{p+q-1}(\bar\sigma^{p-1}_\beta,\partial\bar\sigma^{p-1}_\beta;\mathcal{G}|\bar\sigma^{p-1}_\beta).
\end{array}$$

The top and bottom maps are induced by inclusions and, for $i_{\alpha*}$, the map of the local coefficient system determined by the path class in $\hat\sigma_\alpha$ between the basepoints of the generators of $H_p(\hat\sigma_\alpha,\partial\hat\sigma_\alpha)$ and $H_{p-1}(\hat\sigma_\beta,\partial\hat\sigma_\beta)$. The issue is that λ is defined using the strong stratified trivialization over each $\hat\sigma_\beta^{p-1}$, while λ' is defined using the restriction of the strong stratified trivialization over $\hat\sigma_\alpha^p$. It suffices to prove that this diagram commutes on each summand. It is clear that if $\beta \neq \beta'$, then the maps $H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\alpha; \mathcal{G}|\mathcal{L}_\alpha)) \to IH_{p+q-1}^{\bar p}(\bar\sigma_{\beta'}^{p-1}, \partial\bar\sigma_{\beta'}^{p-1}; \mathcal{G}|\sigma_\beta^{p-1})$ are each the zero map, so it suffices to show that

$$\begin{CD}
H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\alpha; \mathcal{G}|\mathcal{L}_\alpha)) @>{i_{\alpha*}}>> H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\beta; \mathcal{G}|\mathcal{L}_\beta))\\
@V{\lambda'}VV @V{\lambda}VV\\
IH_{p+q-1}^{\bar p}(\bar\sigma_\beta^{p-1}, \partial\bar\sigma_\beta^{p-1}; \mathcal{G}|\bar\sigma_\beta^{p-1}) @>{j_{\alpha*}}>> IH_{p+q-1}^{\bar p}(\bar\sigma_\beta^{p-1}, \partial\bar\sigma_\beta^{p-1}; \mathcal{G}|\bar\sigma_\beta^{p-1})
\end{CD}$$

commutes.

The bottom map is a canonical isomorphism, while, again, the top is induced by the identity of the space $\hat\sigma_\beta$ and the map on coefficients, u, determined over the path class in α between the basepoints of the generators of $H_p(\hat\sigma_\alpha, \partial\hat\sigma_\alpha)$ and $H_{p-1}(\hat\sigma_\beta, \partial\hat\sigma_\beta)$.

So, applying the universal coefficient theorem and the Künneth theorem for intersection homology, we identify $H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\beta; \mathcal{G}|\mathcal{L}_\beta))$ with

$$IH_{p-1}^{\bar p}(\hat\sigma_\beta^{p-1} \times \mathcal{L}_\beta, \partial\hat\sigma_\beta^{p-1} \times \mathcal{L}_\beta; \hat\sigma_\beta^{p-1} \times (\mathcal{G}|\mathcal{L}_\beta)),$$

and similarly for α. From the definitions of λ and λ', we need to show the commutativity of

$$\begin{CD}
H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\alpha; \mathcal{G}|\mathcal{L}_\alpha)) @>{i_{\alpha*}}>> H_{p-1}(\hat\sigma_\beta^{p-1}, \partial\hat\sigma_\beta^{p-1}; IH_q^{\bar p}(\mathcal{L}_\beta; \mathcal{G}|\mathcal{L}_\beta))\\
@VVV @VVV\\
IH_{p-1}^{\bar p}(\hat\sigma_\beta^{p-1} \times \mathcal{L}_\alpha, \partial\hat\sigma_\beta^{p-1} \times \mathcal{L}_\alpha; \hat\sigma_\beta^{p-1} \times (\mathcal{G}|\mathcal{L}_\alpha)) @>{(\mathrm{id}\times u)_*}>> IH_{p-1}^{\bar p}(\hat\sigma_\beta^{p-1} \times \mathcal{L}_\beta, \partial\hat\sigma_\beta^{p-1} \times \mathcal{L}_\beta; \hat\sigma_\beta^{p-1} \times (\mathcal{G}|\mathcal{L}_\beta))\\
@V{\mathfrak{h}_{\alpha*}}VV @VV{\mathfrak{h}_{\beta*}}V\\
IH_{p+q-1}^{\bar p}(\bar\sigma_\beta^{p-1}, \partial\bar\sigma_\beta^{p-1}; \mathcal{G}|\bar\sigma_\beta^{p-1}) @>{j_{\alpha*}}>> IH_{p+q-1}^{\bar p}(\bar\sigma_\beta^{p-1}, \partial\bar\sigma_\beta^{p-1}; \mathcal{G}|\bar\sigma_\beta^{p-1}).
\end{CD} \tag{5}$$

The bottom vertical maps $\mathfrak{h}_{\alpha*}$ and $\mathfrak{h}_{\beta*}$ are induced by $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta$ and covered by the respective coefficient maps $[\hat\sigma_\alpha \times (\mathcal{G}|\mathcal{L}_\alpha)]|(\hat\sigma_\beta \times \mathcal{L}_\alpha) \to (\mathfrak{h}_\alpha^*\mathcal{G})|(\hat\sigma_\beta \times \mathcal{L}_\alpha) \to \mathcal{G}$ and $\hat\sigma_\beta \times (\mathcal{G}|\mathcal{L}_\beta) \to \mathfrak{h}_\beta^*\mathcal{G} \to \mathcal{G}$. Note that the first map of each of these compositions is given by a unique isomorphism compatible with the identity over the base point. The map u will be the map determined by extending the inclusion of $\mathcal{L}_\alpha$ over a path.

The top half of this diagram commutes by the naturality of the universal coefficient and Künneth theorems. To show that the bottom commutes, we find a stratum-preserving homotopy between $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta \circ (\mathrm{id} \times u)$, covered by a coefficient homotopy $\hat\sigma_\beta^{p-1} \times (\mathcal{G}|\mathcal{L}_\alpha) \times I \to \mathcal{G}|\bar\sigma_\beta^{p-1}$. For this, we will use a stratified lifting extension problem.

Let e_α and e_β be the base points for the singular simplices which generate $H_p(\hat\sigma_\alpha, \partial\hat\sigma_\alpha)$ and $H_{p-1}(\hat\sigma_\beta, \partial\hat\sigma_\beta)$. Given a path $v : I \to \hat\sigma_\alpha$ from e_β to e_α and the projection $p : \mathcal{L}_\alpha \times I \to I$, let $\tilde v$ denote a stratum preserving extension over $vp : \mathcal{L}_\alpha \times I \to B$ of the inclusion map $\mathcal{L}_\alpha \hookrightarrow \operatorname{holink}_s(N; X_k)$, and let u_t be the induced stratum-preserving map from $\mathcal{L}_\alpha$ to $\pi^{-1}(v(t))$ (so that $u = u_0$ is the map induced between the fibers over the endpoints).

We first construct a stratum-preserving homotopy from $\mathfrak{h}_\alpha i_{e_\beta}$ to u, where $i_x : \mathcal{L}_\alpha \to \hat{\sigma}_\alpha \times \mathcal{L}_\alpha$ is the inclusion over $x \in \hat{\sigma}_\alpha$. Consider the diagram

$$\begin{array}{ccc} \dot{I} \times I \times \mathcal{L}_\alpha \cup I \times 1 \times \mathcal{L}_\alpha & \xrightarrow{f} & \bar{\sigma}_\alpha \\ \downarrow & & \pi\downarrow \\ I \times I \times \mathcal{L}_\alpha & \xrightarrow{g} & \hat{\sigma}_\alpha, \end{array}$$

where $g(t,s,l) = v(s)$ and

$$f(t,s,l) = \begin{cases} u_s(l), & t=0 \\ \mathfrak{h}_\alpha i_{v(s)}, & t=1 \\ \mathfrak{h}_\alpha(l), & s=1. \end{cases}$$

Since $\mathfrak{h}_\alpha$ is a strong stratified trivialization, f is well-defined, and it is stratum-preserving by the properties of the component maps. If K is the product of the identity map on $\mathcal{L}_\alpha$ with a deformation retraction of the DR-pair $(I \times I, \dot{I} \times I \cup I \times 1)$, then fK_1 is stratum-preserving along I and, by Lemma 3.5, there exists a stratified solution, G, to the lifting extension problem. It is easy to check that G is stratum-preserving along $I \times I$ (it is stratum-preserving along one I by the lemma and along the other because it is along $s=1$). In particular, the restriction of $G|I \times 0 \times \mathcal{L}_\alpha$ over $v(0) = e_\beta$, being stratum-preserving along I, gives a stratified homotopy between $\mathfrak{h}_\alpha i_{e_\beta}$ and u. Let us denote this homotopy by $\tilde{u} : I \times \mathcal{L}_\alpha \to \mathcal{L}_\beta$.

Now consider the diagram

$$\begin{array}{ccc} \dot{I} \times \hat{\sigma}_\beta \times \mathcal{L}_\alpha \cup I \times \{e_\beta\} \times \mathcal{L}_\alpha & \xrightarrow{\eta} & \bar{\sigma}_\beta \\ i\downarrow & & \pi\downarrow \\ I \times \hat{\sigma}_\beta \times \mathcal{L}_\alpha & \xrightarrow{\tau} & \hat{\sigma}_\beta, \end{array}$$

where $\tau(t,s,l) = s$ and

$$\eta(t,s,l) = \begin{cases} \mathfrak{h}_\alpha(s,l), & t=1 \\ \mathfrak{h}_\beta(s,u(l)), & t=0 \\ \tilde{u}(t,l), & s=e_\beta. \end{cases}$$

As $\mathfrak{h}_\beta$ is a strong stratified trivialization and $\tilde{u}$ is a stratified homotopy from u to $h_\alpha i_{e_\beta}$, η is well defined. Furthermore, by the stratum-preserving properties of the components, η is stratum-preserving with respect to the product filtration of $I \times \hat{\sigma}_\beta \times \mathcal{L}_\alpha$. Therefore, if K' is the product of the identity map on $\mathcal{L}_\alpha$ with a deformation retraction of the DR-pair $(I \times \sigma_\beta, \dot{I} \times \sigma_\beta \cup e_\beta \times I)$ (see [18, I.5.2]), then $\eta K'$ is stratum-preserving along I and, by Lemma 3.5, there exists a stratified solution, $\tilde{\eta}$ to the lifting extension problem. As $\tilde{\eta}$ is stratum-preserving along I, $\tilde{\eta}$ is a stratum-perserving map, and, in particular, a stratum-preserving homotopy between $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta \circ (\mathrm{id} \times u)$. In addition, these constructions can all be lifted in the exact same manner to give covering homotopies of all of the local coefficient maps (see also the section on the construction of the bundle of coefficients of a stratified fibration, above). Hence $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta \circ (\mathrm{id} \times u)$, covered by their coefficient maps, induce the same maps on intersection homology and diagram (5) commutes.

This completes the proof of the commutativity of the right side of the cube for the $\bar{E}^1$ terms. For the E^1 terms, the commutativity follows from that of the $\bar{E}^1$ terms by our standard methods for turning maps and homotopies of stratified fibrations into maps and homotopies of their mapping cylinders. In slightly greater detail: we could build another cube with the $\bar{E}^1$ terms on the right, the E^1 terms on the left, and with maps between them induced by the obvious inclusions. The front, back, top, and bottom commute by the naturality of inclusions, the naturality inherent in our definitions of λ and λ', the nature of the construction of the h_γ from the $\mathfrak{h}_\gamma$, and the same for all of the coefficient covers. We have just proven that the left side commutes, hence so does the right. Also, we could note that the right face of the cube for the $E^1_{*,q}$ terms is isomorphic to that for the $\bar{E}^1_{*,q}$ terms if $* < n-k-1-\bar{p}(n-k)$ and identically zero for $* \geq n-k-1-\bar{}(n-k)$ (again, byt the formula for the intersection homology of a cone in [11]).

$\square$

This, at last, completes the proof of Theorem 6.1.

In this appendix, we provide a proof of (a slight generalization of) a theorem found in works of Quinn [16] and Chapman [2]. It is necessary to provide some details since the proofs in those works make use of a map that turns out not to be continuous.

A The stratum-preserving homotopy equivalence of a deformation retract neighborhood of a pure subset with the mapping cylinder of its holink

In this appendix, we provide a proof of a theorem found in works of Chapman [2] and Quinn [16]. It is necessary to provide some details since the proofs in those works make use of a map that turns out not to be continuous. The proof below was suggested by Bruce Hughes.

Suppose that Y is a manifold weakly stratified space and that X is a *pure* subset, i.e. X is a closed subspace that is the union of connected strata $Y_k - Y_{k-1}$. The subsets of interest in this paper are the connected components of bottom strata of Y. Suppose also that X has a neighborhood N in Y and that there exists a nearly stratum-preserving strong deformation retraction $R : N \times I \to N$ that takes N into X (recall that *nearly-stratum preserving* means that R respects the stratification except at time 0 when all of N is mapped into X). The theorem asserts that under these assumptions, N is stratum-preserving homotopy equivalent to the mapping cylinder of the holink evaluation map $\pi : \mathrm{holink}_s(N, X) \to X$ given by evaluation of paths at time 0.:

Proposition A.1. *Given a nearly stratum-preserving deformation retraction $R : N \times I \to N$ of N to X, there exists a stratum-preserving homotopy equivalence between N and the mapping cylinder I_π of the holink evaluation $\pi : \mathrm{holink}_s(N, X) \to X$.*

Proof. Recall that we assume the *teardrop topology* on the mapping cylinder I_π of the holink evaluation $\pi : \mathrm{holink}_s(N, X) \to X$, i.e. the topology generated by the basis of open subsets of $\text{holink}_s(N,X)\times(0,1]$ with the product topology and sets of the form $[\pi^{-1}(U)\times(0,\epsilon)]\cup U$, where U is an open set in X.

Also recall ([16, Lemma 2.4] and [10]) that given a map $\delta : X \to (0,\infty)$, the inclusion $\text{holink}_s^{\delta}(N,X) \hookrightarrow \text{holink}_s(N,X)$ is a stratum and fiber preserving homotopy equivalence (where $\text{holink}_s^{\delta}(N,X)$ is the subset of paths w in $\text{holink}_s(N,X)$ that are contained entirely within distance $\delta(w(0))$ of their endpoints $w(0)$). Furthermore, the proof of this fact comes from constructing a deformation retraction of $\text{holink}_s(N,X)$ into $\text{holink}_s^{\delta}(N,X)$ by shrinking paths along themselves. Similarly, we can note that if $\delta'(x) < \delta(x)$ then $\text{holink}_s^{\delta'}(N,X) \hookrightarrow \text{holink}_s^{\delta}(N,X)$ is also a stratum and fiber preserving homotopy equivalence also given by a deformation retraction that shrinks paths and hence remains entirely in $\text{holink}_s^{\delta}(N,X)$. In fact, just restrict the retraction of the whole space $\text{holink}_s(N,X)$ to $\text{holink}_s^{\delta}(N,X)$. We now define a shrinking map $S : \text{holink}_s(N,X)\times(0,1] \to \text{holink}_s(N,X)$ as follows: on $\text{holink}_s(N,X)\times[1/2,1]$, define S as a deformation retraction that shrinks $\text{holink}_s(N,X)$ into $\text{holink}_s^{1/2}(N,X)$ (scaled to occur on $[1/2,1]$ instead of $[0,1]$); on $[1/3,1/2]$, define S as the composition of $S(\cdot,1/2)\times\text{id}_{[1/3,1/2]}$ followed by a deformation retraction from $\text{holink}_s^{1/2}(N,X)$ into $\text{holink}_s^{1/3}(N,X)$ (again scaling I to the interval $[1/3,1/2]$); and so on. So S is stratum and fiber preserving continuous homotopy that shrinks all of the paths in the holink. We can extend S to a map $\bar{S} : I_\pi \to I_\pi$ as follows: Let $\bar{S}$ be the the identity on the base X, and on $\text{holink}_s(N,X)\times(0,1]$ define $\bar{S}$ by $\bar{S}(w,s) = (S(w,s),s)$. The map $\bar{S}$ is continuous: this is clear on $I_\pi - X$; for a point $x \in X$, notice that any neighborhood of the form $[\pi^{-1}(U)\times(0,\epsilon)]\cup U$, U an open subset of X with $x\in U$, gets mapped into itself under $\bar{S}$.

We can now define the stratum-preserving homotopy inverses $f : N \to I_\pi$ and $g : I_\pi \to N$. Given a point $y \in N$, let R_y be the retraction path of y under the nearly stratum-preserving deformation retraction R. Define f by $f(y) = y$ if $y \in x$ and $f(y) = (R_y, d(y,X))$ if $y \in N - X$. Here $d(y,X)$ is the distance from y to X. Define g by $g(y) = y$ for $y \in X$ and for $(w,s) \in \text{holink}_s(N,X)\times(0,1]$ by $g(w,s) = S(w,s)(s)$; this is the evaluation of the path $S(w,s)$ at the time s. We first show that f and g are continuous, then that they are in fact stratum-preserving homotopy inverses.

For f, continuity is clear except at points in X. So choose $x \in X$, and suppose that we are given an open set $V \subset I_\pi$ of the form $[\pi^{-1}(U)\times(0,\epsilon)]\cup U$ such that U is an open set in X that contains $f(x) = x$. We need to find a neighborhood of x in N that maps into V. Let $\delta = \min(\epsilon, d(x, X-U))$, where the distance d is that of N restricted to X. Let $B(x,\delta)$ be the open neighborhood of x in N of radius δ. Consider the set $R^{-1}(B) \subset N\times I$. Note that this open set includes $B\cap X$, and so by standard general topology (see, e.g., [13]), it includes a set of the form $W\times I$ where W is an open neighborhood of $B\cap X$ in N. Hence, in particular, W is an open neighborhood of x. So let us consider $f(w)$ for $w \in W$. If $w \in W\cap X$, then $w \in B\cap X \subset U$ so that $f(w) \in V$. If $w \in W - X\cap W$, then $f(w) = (R_w, d(w,X))$. The path R_w is the image of $R(w,\cdot)$, but by construction, this path lies entirely within B. Hence the 0 endpoint must lie in $B\cap X \subset U$, and $R_w \subset \pi^{-1}(U)$. Also, since $w = R(w,1) \in B$, $d(w,X) < \epsilon$. This shows that $f(w) \subset V$. We conclude that $f(W) \subset V$, and since our choice of V was arbitrary, f is continuous.

Next, we show that g is continuous. Again, continuity is clear at points not in $X \subset I_\pi$. So let $x \in X$, and consider a neighborhood of $x \in N$. Since N has the metric topology, we can assume that this neighborhood has the form $B = B(x, \epsilon)$, the ball of radius ϵ about x in N. So let U be the open ball about x in X of radius $\epsilon/2$, and choose n so that $1/n < \epsilon/2$. Consider the subset of I_π given by $V = [\pi^{-1}(U) \cup (0, 1/n)] \cup U$. We will see that $g(V) \subset B$. If $x \subset U$, then $g(x) = x \in U \subset B \cap X$. For points of the form $(w, s) \in \text{holink}_s(N, X) \times (0, 1]$, $g(w, s) = S(w, s)(s)$. Since $s < 1/n < \epsilon/2$ by assumption, it follows from the construction of S that $S(w, s)$ is a path whose image lies within $\epsilon/2$ of its endpoint in U. So $d(g(w, s), x) \leq d(g(w, s), w(0)) + d(w(0), x) < \epsilon/2 + \epsilon/2 = \epsilon$, the latter $\epsilon/2$ occuring because we have assume $w(0) \subset U$ (since $w \in \pi^{-1}(U)$) and U has radius $\epsilon/2$ from x. Thus g is continuous.

Now let us show that gf is stratum-preserving homotopy-equivalent to the identity. Again, if $y \in X$, $gf(y) = y$, but if $y \in N - X$, $gf(y) = g(R_y, d(y, X)) = S(R_y, d(y, X))(d(y, X))$. Since S retracts paths along themselves, $gf(y) = R_y(s') = R(y, s')$, where s' is a time that depends continuously on y. The homotopy from gf to the identity is given by $(y, t) \to R(y, t + (1 - t)s')$: at $t = 0$, $y \to gf(y)$ and at $t = 1$, $y \to R(y, 1) = y$. This homotopy is stratum-preserving because R is. Again, there are no problems seeing the continuity at points in $N \times I - X \times I$. For points $(x, t) \in X \times I$, we can emulate the arguments whereby we showed that f was continuous by looking only at points in a small neighborhood of x that also have retraction paths contained entirely in a small neighborhood of x.

Finally, we must show that fg is stratum-preserving homotopy-equivalent to the identity. Again, let us first calculate gf. For $x \in X \subset I_\pi$, $fg(x) = x$, while for $(w, s) \in \text{holink}_s(N, X) \times (0, 1] \subset I_\pi$, $fg(w, s) = f(S(w, s)(s)) = (R_{S(w,s)(s)}, d(S(w, s)(s), X))$. Again, let us choose s' so that $S(w, s)(s) = w(s')$ and note that s' depends continuously on w and s. We now define a homotopy $h : I_\pi \times I \to I_\pi$: let $h|X \times I$ be the projection onto X, and let the first coordinate of the image in $\text{holink}_s(N, X) \times (0, 1]$ be determined by the paths given as follows (where p_i is the projection of $\text{holink}_s(N, X) \times (0, 1]$ onto its ith factor):

$$p_1 h((w, s), u)(t) = \begin{cases} R(w(t), 1), & u = 1 \\ R(w(ut + (1 - u)s'), \frac{t}{t-u+1}), & 0 \leq t \leq s'u, 0 \leq u < 1 \\ R(w(ut + (1 - u)s'), \frac{s'u}{s'u-u+1}(1 - \frac{t-s'u}{s'-s'u}) + (u + (1 - u)s')(\frac{t-s'u}{s'-s'u}), & \\ & s'u \leq t \leq s', 0 \leq u < 1 \\ R(w(tu + (1 - u)s'), u + (1 - u)t), & s' \leq t \leq 1, 0 \leq u < 1 \end{cases} \tag{6}$$

For the second coordinate, we have $p_2 h((w, s), u) = us + (1 - u)d(S(w, s)(s), X)$.

The reason these formulas need be so complicated is that we must homotop the paths in such a manner that all paths are in X at time $t = 0$ (i.e. to keep all paths in the holink). For each path w, the first coordinate of this homotopy takes it to the path of the retraction of the point $w(s')$, sweeping through the retractions of the points $w(t)$, $0 \leq t \leq s'$ along the way.

Note that when $u = 1$, we have $p_1 h((w, s), 1)(t) = R(w(t), 1) = w(t)$, and for $u = 0$, we have $p_1 h((w, s), 0)(t) = R(w(s'), t) = R(S(w, s)(s), t) = R_{S(w,s)(s)}(t)$. Also, $p_1 h((w, s), u)(0) = R(w((1 - u)s'), 0) \in X$, and so long as $s > 0$, the second input into R in the above formulae will be > 0 so long as $t > 0$. So, for all u the first coordinate gives us a legitimate element of holink(N,X). That our homotopy is stratum-preserving follows form the fact that R is stratum-preserving and that paths in $\mathrm{holink}_s(N,X)$ are nearly stratum-preserving.

To show the continuity of h, we divide into two cases, for points in $X \times I$ and for points in $\mathrm{holink}_s(N,X) \times (0,1] \times I$. The latter points map into $\mathrm{holink}_s(N,X) \times (0,1]$ and so to show continuity here we need only check continuity of the component functions (or equivalently, of $p_1 h$ and $p_2 h$). The continuity of $p_2 h$ on $\mathrm{holink}_s(N,X) \times (0,1] \times I$ is clear from the definition. Let us show the continuity of $p_1 h$ on $\mathrm{holink}_s(N,X) \times (0,1] \times I$. By general topology, it suffices to show that $E(p_1 h \times \mathrm{id}_I) : \mathrm{holink}_s(N,X) \times (0,1] \times I \times I \to N$ is continuous, where E is the holink evaluation $(w,t) \to w(t)$. Observe that for the chosen domain $s > 0$ and so $s' > 0$. Looking at equation (6), we see that this map is clearly continuous except possibly for points where $u = 1$ and t is equal to s' or 0. So let us directly check continuity at such points.

The potential difficulty at points $(w, s, s', 1)$ is, in fact, illusory. The above equations have been written to make clear where linear interpolations are occuring, and a rewriting of the third line of the equation by simplifying the algebra will illustrate that the denominator occurs in a neighborhood of $(s')^2$ when u is near 1. It is then easy to check that $p_1 h$ is continuous in the neighborhoods of such points.

Consider then a point $(w_0, s_0, 1, 0)$. Near this point, the governing equations will be $R(w(t), 1)$ for $u = 1$ and $R(w(ut+(1-u)s'), \frac{t}{t-u+1})$ for $u < 1$. By definition, $h((w_0, s_0), 1)(0) = w_0(0) \in X \subset N$. Let B be an ϵ neighborhood of $w_0(0)$ in N. We find an open neighborhood of $(w_0, s_0, 1, 0)$ that maps into B. Without constructing such a neighborhood explicity we indicate that one is easy to obtain using the product topology on $\mathrm{holink}_s(N,X) \times (0,1] \times I \times I$. Let us briefly keep fixed w_0 and s_0 (and hence $s'_0 = s'(w_0, s_0)$). Now, as observed in the proof of continuity of f, given a δ (say $\epsilon/2$), x has a neighborhood U in N such that any retraction path under R that starts in this neighborhood remains δ close to x. If we take t sufficiently small, then $w_0(t)$ will be in this neighborhood and so will $R(w_0(ut+(1-u)s'_0), \frac{t}{t-u+1})$ because these points will lie in the retractions paths of the points on w_0 in the neighborhood. Thus we obtain continuity on the restriction to fixed w_0 and s_0. But now observe that since we use the compact-open topology on $\mathrm{holink}_s(N,X)$, we can choose a neighborhood of w_0 in $\mathrm{holink}_s(N,X)$ so that all paths in this neighborhood are very close to the path w_0. We can also choose a small interval about s_0 and since s' depends continuously on w and s, we can choose these neighborhoods so that s' also varies only slightly. Hence it is clear that for sufficiently refined choices of neighborhoods of w_0 and s_0 and for the same choices of sufficiently small neighborhoods of $t = 0$ and $u = 1$, the image points under $R(w(ut + (1-u)s'), \frac{t}{t-u+1})$ or $R(w(t), 1)$ will remain close to $w(0)$. In particular, with u sufficiently close to 1 and t sufficiently close to 0, all the $w(ut + (1-u)s')$ will remain within our neighborhood U of $w_0(0)$ and so the retraction paths utilized will all remain within $\epsilon/2$ of $w_0(0)$.

To prove that h is a continuous homotopy on all of $I_\pi \times I$, it remains only to check continuity at points of the form $(x,t) \in X \times I \subset I_\pi \times I$. Let V be a basis element of I_π that contains $h(x,t) = x$. Then V is of the form $[\pi^{-1}(U) \times (0,\epsilon)] \cup U$, and we need to find an open neighborhood W of (x,t) in $I_\pi \times I$ that maps into V under h. Recall (Lemma 3.15) that the topology of $I_\pi \times I$ is the same as that given if we consider this set as the mapping cylinder of $\pi \times \mathrm{id}_I$. So we will try to find a W of the form $[(\pi \times \mathrm{id}_I)^{-1}(Z) \times (0,\delta)] \cup Z$, where

Z is an open neighborhood of (x,t) in $X \times I$.

Let D be the open ball about x in N of radius $\rho' < \min(\epsilon/2, d(x, X-U))$, where the distance d is that obtained from N by restricting to X. Consider $R^{-1}(D) \subset N \times I$, and let $C \times I$ be an open neighborhood of $(D \cap X) \times I$ within $R^{-1}(D)$ (again, such a neighborhood exists by standard topology). Now let B be a neighborhood of x in X of radius $\rho < \rho'/2$, and choose δ' so that $0 < \delta' < d(B, N-C)$ (note, of course, that it is possible to choose such a δ' by construction).

Let us choose n such that $1/n < \delta'$ and $1/n < \epsilon/2$, and choose $\delta < \min(\rho, 1/n)$. Let B be as above, the open ball about x in X of radius ρ, and let $Z = B \times I$. Then $Z \subset U \times I$, and with this choice of Z and δ, let W be defined as above. We will show that $h(W) \subset V$. If $(x,u) \in Z \times I \subset X \times I$, then $h(x,t) = x \subset B \subset U \subset V$. So let us consider a point in W of the form $((w,s),u)$. We need to show that $h((w,s),u) \in \pi^{-1}(U) \times (0,\epsilon)$. Since $p_2h((w,s),u) = us + (1-u)d(S(w,s)(s), X)$, p_2h lies in $(0,1]$ between s and $d(S(w,s)(s), X)$. By assumption, $s < \delta < \epsilon/2$, and $d(S(w,s)(s), X) < \epsilon/2$ since $S(w,s)$ is a path contained within $\epsilon/2$ of $w(0) \in X$ by the construction of the map S. Thus $p_2h((w,s),u) < \epsilon$. So finally, we need only check that $p_1h((w,s),u)(0) \subset U$ to conclude that $h(W) \subset V$. As we observed above, $p_1h((w,s),u)(0) = R(w((1-u)s'), 0)$, and as u varies these are simply the end points of the retraction paths of the points on w between 0 and s'. But then the images of w in this range must also be images of $S(w,s)$ and each such point has a distance $< \delta'$ from $x \subset B$ and so lies in C. But then, again by construction, each of these points has a retraction path which lies entirely within D and, in particular then, has its endpoint in $D \cap X \subset U$. At last, we conclude that $h(W) \subset V$ and so h is continuous.

$\square$

References

[1] Sylvain E. Cappell and Julius L. Shaneson, *Singular spaces, characteristic classes, and intersection homology*, Annals of Mathematics **134** (1991), 325–374.

[2] T. A. Chapman, *Concordances of Hilbert cube manifolds and tubular neighborhoods of finite-dimensional manifolds*, Geometric Topology (James C. Cantrell, ed.), New York: Academic Press, 1970, pp. 581–596.

[3] A. Borel et. al., *Intersection cohomology*, Progress in Mathematics, vol. 50, Birkhauser, Boston, 1984.

[4] Greg Friedman, *Intersection Alexander polynomials*, preprint.

[5] ______, *Polynomial invariants of non-locally-flat knots*, Ph.D. thesis, New York University, New York, NY, 2001.

[6] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[7] ______, *Intersection homology II*, Invent. Math. **72** (1983), 77–129.

[8] Allen Hatcher, *Algebraic topology I*, see http://math.cornell.edu/~hatcher.

[9] Bruce Hughes, *Stratifications of teardrops*, Fundamenta Mathematicae **161** (1999), 305–324.

[10] ______, *Stratified path spaces and fibrations*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 351–384.

[11] Henry C. King, *Topological invariance of intersection homology without sheaves*, Topology Appl. **20** (1985), 149–160.

[12] John McCleary, *User's guid to spectral sequences*, Publish or Perish, Inc., Wilmington, Delaware, 1985.

[13] James R. Munkres, *Topology: A first course*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[14] ______, *Elements of algebraic topology*, Addison-Wesley, Reading, MA, 1984.

[15] Frank Quinn, *Intrinsic skeleta and intersection homology of weakly stratified sets*, Geometry and topology (Athens, GA, 1985), Lecture Notes in Pure and Appl. Math., vol. 105, Dekker, New York, 1987, pp. 225–241.

[16] ______, *Homotopically stratified sets*, J. Amer. Math. Soc. **1** (1988), 441–499.

[17] David A. Stone, *Stratified polyhedra*, Lecture Note in Mathematics, vol. 252, Spring-Verlag, Berlin-Heidelberg-New York, 1972.

[18] George W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

[19] Bruce Hughes, Laurence R. Taylor, Shmuel Weinberger, Bruce Williams, *Neighborhoods in stratified spaces with two strata*, Topology **39** (2000), 873–919.

Several diagrams in this paper were typeset using the TEX commutative diagrams package by Paul Taylor.