Generalizations of intersection homology with duality over the integers

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Contents

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Abstract

We provide a generalization of the Deligne sheaf construction of intersection homology theory and a corresponding generalization of Poincaré duality on pseudomanifolds such that the Goresky-MacPherson, Goresky-Siegel, and Cappell-Shaneson duality theorems all arise as special cases. Unlike classical intersection homology theory, our duality theorem holds with ground coefficients in an arbitrary PID and with no "locally torsion free" conditions on the underlying space.

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1 Introduction

Our goal in this paper is to construct generalized versions of the Deligne sheaf and a generalized Poincaré duality theorem on stratified pseudomanifolds such that the Goresky-MacPherson, Goresky-Siegel, and Cappell-Shaneson duality theorems for intersection homology all occur as special cases. In particular, our duality theorem will hold with ground coefficients in an arbitrary PID and with no "locally torsion free" conditions on the underlying space. In order to explain this result and its context, we begin by recalling some historical background.

Background and results. In [14], Goresky and MacPherson introduced intersection homology for a closed oriented PL stratified pseudomanifold X and showed that if $\bar{p}$, $\bar{q}$ are complementary perversity parameters[1] (i.e. $\bar{p}(Z) + \bar{q}(Z) = \text{codim}(Z) - 2$ for all singular strata Z of X), then the intersection pairing

$$I^{\bar{p}}H_i(X) \otimes I^{\bar{q}}H_{\dim(X)-i}(X) \to \mathbb{Z}$$

becomes nonsingular after tensoring with $\mathbb{Q}$. This provides an important generalization of Poincaré duality to non-manifold spaces. In [15], and in the broader context of *topological* stratified pseudomanifolds, Goresky and MacPherson further refined this intersection homology version of Poincaré duality into the statement that there is a quasi-isomorphism of sheaf complexes over X:

$$\mathcal{P}^*_{\bar{p}} \sim_{qi} (\mathcal{D}\mathcal{P}^*_{\bar{q}})[-\dim(X)]. \tag{1}$$

Here $\mathcal{P}^*_{\bar{r}}$ denotes the "Deligne sheaf" with perversity $\bar{r}$ (this is an iteratively-constructed sheaf complex characterized by nice axioms whose hypercohomology groups give intersection homology via $\mathbb{H}^i_c(X; \mathcal{P}^*_{\bar{r}}) \cong I^{\bar{r}}H_{n-i}(X)$ — see [15]), the symbol $\mathcal{D}$ denotes the Verdier dualizing functor, the sheaf complex $\mathcal{S}^*[n]$ is the shifted sheaf complex of $\mathcal{S}^*$ with $(\mathcal{S}^*[n])^i \cong \mathcal{S}^{i+n}$, and $\sim_{qi}$ denotes quasi-isomorphism. The stratified pseudomanifold X is also no longer required to be compact, but the ground ring of coefficients is required in [15] to be a field.

In [16], Goresky and Siegel explored the duality properties of Deligne sheaves with coefficients in a principal ideal domain, demonstrating that, in general, one cannot hope for a version of (1) in this generality. The obstruction occurs in the form of torsion in local intersection homology groups at the singular points of X. This led to the definition of a locally $\bar{p}$-torsion-free space. More precisely, a stratified pseudomanifold is locally $\bar{p}$-torsion-free (with respect to the PID R) if for each x in each singular stratum Z of codimension k, the torsion subgroup of $I^{\bar{p}}H_{k-2-\bar{p}(Z)}(L; R)$ vanishes, where L is the link of x. If X is such a space, then (1) holds with coefficients in R, leading to certain other nice "integral" properties of duality and homology such as nonsingular linking pairings and a universal coefficient theorem.

[1] In early work on intersection homology, e.g. [14, 15, 3, 16], perversities were only considered that took the same value on all strata of the same codimension. We employ a slightly revisionist history in this introduction by stating the theorems in a form more consonant with more general notions of perversity; see [9, 10].

In [4], Cappell and Shaneson proved a "superduality" theorem, which holds in a situation that can be considered somewhat the opposite of that of Goresky and Siegel. Cappell and Shaneson showed that if the stratified pseudomanifold X possesses the property that all local intersection homology groups are torsion, then (1) holds provided $\bar{p}$ and $\bar{q}$ are "superdual", meaning $\bar{p}(Z) + \bar{q}(Z) = \operatorname{codim}(Z) - 1$ for all singular strata Z of X. While this statement seems more drastic than that of Goresky-Siegel in terms of the number of dimensions for which there is a local intersection homology condition, it follows from the proof that one could impose this "torsion only" condition in just one dimension per link[2].

In Complement 3.3 of [2], Beilinson, Bernstein, and Deligne construct a t-structure on the derived category $D(X, \mathbb{Z})$ of sheaves of abelian groups (or R-modules over a Dedekind ring) on a space X that takes into account torsion by setting

$$
\begin{aligned}
{}^{n+}D^{\leq 0} &= \{K \in D(X, \mathbb{Z}) \mid H^i(K) = 0 \text{ for } i > 1 \text{ and } H^i(K) \otimes \mathbb{Q} = 0\} \\
{}^{n+}D^{\geq 0} &= \{K \in D(X, \mathbb{Z}) \mid H^i(K) = 0 \text{ for } i < 0 \text{ and } H^0(K) \text{ is torsion free}\}.
\end{aligned}
$$

If X is stratified and equipped with a perversity $\bar{p}$, they observe that one can glue such t-structures over strata to obtain a t-structure $({}^{\bar{p}^+}D^{\leq 0}(X, \mathbb{Z}), {}^{\bar{p}^+}D^{\geq 0}(X, \mathbb{Z}))$. It is then noted that Verdier duality interchanges this t-structure with the standard t-structure of the form $({}^{D\bar{p}}D^{\leq 0}(X, \mathbb{Z}), {}^{D\bar{p}}D^{\geq 0}(X, \mathbb{Z}))$, where we use $D\bar{p}$ to denote the dual perversity to $\bar{p}$.

In this paper, we work through the details of a generalization of this Beilinson-Bernstein-Deligne construction from the point of view of intersection homology and Deligne sheaves, meaning, among other things, that we shall focus on the explicit construction of sheaf complexes as opposed to the more abstract construction of t-structures. We will provide a detailed modification of the Deligne sheaf construction such that a version of (1) holds over a PID for any topological stratified pseudomanifold. Furthermore, rather than asking our Deligne sheaves to be either "all torsion" or "no torsion" at the truncation dimensions as in [2], we allow mixed situations, and, in fact, we will take as part of our perversity information a set of primes on each stratum and allow torsion at the truncation dimension only with respect to those primes. Verdier duality then interchanges the set of primes with its complement. As we shall see in Section 5, this leads to some interesting duality results, even for quite simple spaces. We will also demonstrate how the duality theorems of Goresky-MacPherson, Goresky-Siegel, and Cappell-Shaneson all occur as special cases.

More specifically, in order to implement our construction, we generalize the notion of perversity from that of a function

$$\bar{p} : \{\text{singular strata of X}\} \to \mathbb{Z}$$

to that of a function

$$\vec{p} = (\vec{p}_1, \vec{p}_2) : \{\text{singular strata of X}\} \to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R)),$$

[2]It is also worth noting that Cappell and Shaneson use local coefficient systems on the complement of the singular locus throughout [4] so that their local intersection homology groups are akin to Alexander modules of knots. This explains how it is possible for each local intersection homology group to be torsion, even in degree zero.

where $\mathfrak{P}(R)$ is the set of primes of the PID R and $\mathbb{P}(\mathfrak{P}(R))$ is its power set (the set of all subsets). We refer to such functions as "torsion-sensitive perversities" or "ts-perversities", and we denote our associated Deligne sheaf complex as $\mathscr{P}^*_{\vec{p}}$. In the case that $\vec{p}_2(Z) = \emptyset$ for all singular strata Z, $\mathscr{P}^*_{\vec{p}}$ is isomorphic to the classical Deligne sheaf $\mathcal{P}^*_{\vec{p}_1}$. The complementary ts-perversity $\vec{q}$ to a ts-perversity $\vec{p}$ is defined by $\vec{q}_1(Z) = \operatorname{codim}(Z) - 2 - \vec{p}_1(Z)$ and $\vec{q}_2(Z) = D\vec{q}_1(Z)$, where $D\vec{q}_1(Z)$ is the complement of $\vec{q}_1(Z)$ in $\mathfrak{P}(R)$. Then, for complementary $\vec{p}, \vec{q}$, our generalized duality statement has the form

$$\mathscr{P}^*_{\vec{p}} \sim_{qi} (\mathcal{D}\mathscr{P}^*_{\vec{q}})[-\dim(X)]. \tag{2}$$

To derive from this statement the pre-existing duality statements mentioned above, the following facts will be shown below:

1. With coefficients over a field, $\mathscr{P}^*_{\vec{p}} \cong \mathcal{P}^*_{\vec{p}_1}$. Thus (2) reduces to the Goresky-MacPherson version of (1).

2. When X is locally $\bar{p}$-torsion-free over the ground ring R, again $\mathscr{P}^*_{\vec{p}} \cong \mathcal{P}^*_{\vec{p}_1}$ and (2) reduces to the Goresky-Siegel version of (1).

3. If the local intersection homology of X is all torsion and $\vec{p}_2(Z) = \mathfrak{P}(R)$ for all singular strata Z, then $\mathscr{P}^*_{\vec{p}} \cong \mathcal{P}^*_{\vec{p}_1+1}$, where $\vec{p}_1 + 1$ is the perversity defined by $(\vec{p}_1 + 1)(Z) = \vec{p}_1(Z) + 1$. Also, since this forces $\vec{q}_2(Z) = \emptyset$ for all Z, $\mathscr{P}^*_{\vec{q}} \cong \mathcal{P}^*_{\vec{q}_1}$, and (2) reduces to the Cappell-Shaneson version of (1).

It would be interesting to have a geometric formulation of the hypercohomology groups $\mathbb{H}^*(X; \mathscr{P}^*_{\vec{p}})$ in terms of simplicial or singular chains with certain restrictions, as is the case for intersection homology theory and the Deligne sheaf $\mathcal{P}^*_{\bar{r}}$.

Motivation. Let us attempt to provide some brief motivation for why the Goresky-Siegel or Cappell-Shaneson conditions are necessary for duality over a PID in the classical formulation of intersection homology and why one might expect to find something like the Beilinson-Bernstein-Deligne t-structure or $\mathscr{P}^*_{\vec{p}}$. To simplify this discussion, we work over $\mathbb{Z}$ and suppose perversity values depend only on codimension as in [14]. We also will not attempt to get too deeply into technical details here; we will limit ourselves to presenting the basic idea.

Recall the Deligne sheaf is defined by a process of consecutive pushforwards and truncations. In the original Goresky-MacPherson formulation, if $X = X^n \supset X^{n-2} \supset \cdots$ is a stratified pseudomanifold and $\mathcal{P}^*_{\bar{p}}(k)$ is the Deligne sheaf defined over $X - X^{n-k}$ (or, if $k = 2$, $\mathcal{P}^*_{\bar{p}}(2)$ is a locally constant sheaf of coefficients over $\mathbb{Z}$), then one extends $\mathcal{P}^*_{\bar{p}}(k)$ to $\mathcal{P}^*_{\bar{p}}(k+1)$ on $X - X^{n-k-1}$ as

$$\mathcal{P}^*_{\bar{p}}(k+1) = \tau_{\leq \bar{p}(k)} Ri_{k*}\mathcal{P}^*_{\bar{p}}(k),$$

where i_k is the inclusion $X - X^{n-k} \hookrightarrow X - X^{n-k-1}$, τ is the sheaf complex truncation functor, and $\bar{p}(k)$ is the common value of $\bar{p}$ on all strata of codimension k. In particular, it follows that at a point $x \in X^{n-k}$, $k \geq 2$, with link L, we have $H^i((\mathcal{P}^*_{\bar p})_x) = 0$ for $i > \bar p(k)$, while for $i \leq \bar p(k)$, we have $H^i((\mathcal{P}^*_{\bar p})_x) = \mathbb{H}^i(L; \mathcal{P}^*_{\bar p})$.

On the other hand, using the properties of Verdier duality (see [3]), one obtains a universal coefficient-flavored calculation that looks like this[3]:

$$H^i((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x) \cong \mathrm{Hom}(H^{n-i}(f^!_x \mathcal{P}^*_{\bar q}), R) \oplus \mathrm{Ext}(H^{n-i+1}(f^!_x \mathcal{P}^*_{\bar q}), R),$$

where $f_x : x \to X$ is the inclusion. If we were working instead with coefficients in a field F, the Ext term would vanish, and so $H^i((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x) \cong \mathrm{Hom}(H^{n-i}(f^!_x \mathcal{P}^*_{\bar q}), F)$. One of the steps in proving the Goresky-MacPherson duality isomorphism (1) then involves showing[4]that $H^{n-i}(f^!_x \mathcal{P}^*_{\bar q}) = 0$ for $i > \bar p(k)$, which is compatible with our computation for $H^i((\mathcal{P}^*_{\bar p})_x)$. With a bit more work, one then shows the sheaf complexes $\mathcal{D}\mathcal{P}^*_{\bar q}[-n]$ and $\mathcal{P}^*_{\bar p}$ are in fact quasi-isomorphic.

However, with coefficients over $\mathbb{Z}$, we have the following problem: From the truncations in the definition of $\mathcal{P}^*_{\bar p}$, we must have $H^{\bar p(k)+1}((\mathcal{P}^*_{\bar p})_x) = 0$. Meanwhile, from the duality computation, we have

$$H^{\bar p(k)+1}((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x) \cong \mathrm{Hom}(H^{n-(\bar p(k)+1)}(f^!_x \mathcal{P}^*_{\bar q}), \mathbb{Z}) \oplus \mathrm{Ext}(H^{n-(\bar p(k)+1)+1}(f^!_x \mathcal{P}^*_{\bar q}), \mathbb{Z}).$$

The observation of the last paragraph that $H^{n-i}(f^!_x \mathcal{P}^*_{\bar q}) = 0$ for $i > \bar p(k)$ holds for any PID coefficients and implies that $H^{n-(\bar p(k)+1)}(f^!_x \mathcal{P}^*_{\bar q}) = 0$. However, it will not generally be true that $H^{n-\bar p(k)}(f^!_x \mathcal{P}^*_{\bar q}) = 0$, and so

$$H^{\bar p(k)+1}((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x) \cong \mathrm{Ext}(H^{n-\bar p(k)}(f^!_x \mathcal{P}^*_{\bar q}), \mathbb{Z})$$

might not be zero, in which case we could not have $H^{\bar p(k)+1}((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x) \cong H^{\bar p(k)+1}((\mathcal{P}^*_{\bar p})_x)$. However, $H^{n-\bar p(k)}(f^!_x \mathcal{P}^*_{\bar q})$ will be finitely generated and if it were also torsion-free, then $H^{\bar p(k)+1}((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x)$ would indeed vanish! It turns out one could then continue on to complete the argument that $\mathcal{P}^*_{\bar p}$ and $\mathcal{D}\mathcal{P}^*_{\bar q}[-n]$ are quasi-isomorphic. This is the source of the Goresky-Siegel condition which, with a bit more computation, implies that $H^{n-\bar p(k)}(f^!_x \mathcal{P}^*_{\bar q})$ is torsion-free. See [16] for details[5].

The Cappell-Shaneson computation is remarkably similar "from the other side". If we extend our perversity from $\bar p$ to $\bar p + 1$ (but keep $\bar q$ the same), then it is acceptable to have $H^{\bar p(k)+1}((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x)$ not vanish, but as we have seen, it must be isomorphic to the torsion group $\mathrm{Ext}(H^{n-\bar p(k)}(f^!_x \mathcal{P}^*_{\bar q}), \mathbb{Z})$, as $\mathrm{Hom}(H^{n-(\bar p(k)+1)}(f^!_x \mathcal{P}^*_{\bar q}), \mathbb{Z})$ still vanishes. This is problematic if $H^i((\mathcal{D}\mathcal{P}^*_{\bar q}[-n])_x)$ is not all torsion, but if we assume it is torsion, then again this turns out to be enough to dodge catastrophe and allow the original Goresky-MacPherson quasi-isomorphism argument to go through.

The preceding arguments should lead one to the thought that it might be possible to make the duality quasi-isomorphism arguments "come out alright", provided one is able to exercise sufficient control on when (and what kind of) torsion is allowed to crop up in local

[3]See Section 4.3 for more details.

[4]For the technicalities, see [3], in particular Step (b) of the proof of Theorem V.9.8 and the $(2b)$ implies $(1'b)$ part of the proof of Proposition V.4.9.

[5]And be mindful of the different indexing convention!

intersection homology groups and when it is not. Indeed, such control at the level of spaces is precisely the idea behind the Goresky-Siegel and Cappell-Shaneson conditions. We will pursue an alternative route by allowing the space to be arbitrary while instead building such torsion control into the definition of the Deligne sheaf complex. This is precisely what the second component $\vec{p}_2$ of our torsion sensitive perversities will do: it is a switch indicating what kind of torsion the strata are permitted to have in their local intersection homology groups at the cut-off dimension. This information is assimilated into the ts-Deligne sheaf via a modified "torsion-tipped" version of the truncation functor that, rather than simply cutting off all stalk cohomology of a sheaf complex at a given dimension, permits a certain torsion subgroup of the stalk cohomology to continue to exist for one dimension above the cutoff, analogously to the Beilinson-Bernstein-Deligne construction. The ts-Deligne sheaf then incorporates this torsion-tipped truncation according to the instructions given by $\vec{p}_2$.

This is the main idea of the paper. The rest is details!

Outline of the paper. Section 2 contains some algebraic preliminaries. In Section 3, we introduce the torsion-tipped truncation functor. Then in Section 4, we construct the torsion-sensitive Deligne sheaf, demonstrate that it satisfies a set of characterizing axioms generalizing the Deligne sheaf axioms of Goresky and MacPherson [15], and prove our duality theorem, Theorem 4.15. Finally, in Section 5, we conclude with an example, computing the hypercohomology groups of ts-Deligne sheaves for pseudomanifolds with isolated singularities and showing how their duality relates to classical Poincaré-Lefschetz duality for manifolds with boundary. This leads us to formulate and prove some results concerning manifold theory that are not so obvious from more direct approaches.

Prerequisites and assumptions. We assume the read has some background in intersection homology theory along the lines of Goresky-MacPherson [15], Borel [3], or Banagl [1]. We will also freely utilize the author's generalizations to arbitrary perversity functions, for which background can be found in [6, 9, 10]. Accordingly, we may also allow stratified pseudomanifolds to possess codimension one strata.

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2 Some algebraic preliminaries

Throughout the paper, R will be a principal ideal domain (PID). We let $\mathfrak{P}(R)$ be the set of equivalence classes of primes of R, where two primes $p, q \in R$ are equivalent if $p = uq$ for some unit $u \in R$. In practice, we fix a representative of each equivalence class and identify the class with its representative prime, i.e. we think of $\mathfrak{P}(R)$ as a set of specific primes, one from each equivalence class.

Let $\wp \subset \mathfrak{P}(R)$ be a set of primes of R. Define the *span of* $\wp$, $S(\wp)$, to be the set

$$S(\wp) = \left\{ n \in R \mid n = \prod_{i=1}^{s} p_i^{m_i}, \text{ where } p_i \in \wp \text{ and } m_i, s \in \mathbb{Z}_{\geq 0} \right\}.$$

In other words, $S(\wp)$ is the set of products of powers of primes in $\wp$. We allow $s = 0$ (which is necessary when $\wp = \emptyset$), and in this case we interpret the product to be 1. In particular, $S(\emptyset) = \{1\}$, and $\{1\} = S(\emptyset) \subset S(\wp)$ for all $\wp$, so, in particular, $S(\wp)$ is never empty. Also, notice that $0 \notin S(\wp)$ for any $\wp$.

If M is an R-module, we define $T^{\wp}M$ to be the submodule of elements of M annihilated by elements of $S(\wp)$, i.e.

$$T^{\wp}M = \{s \in M \mid \exists n \in S(\wp) \text{ such that } ns = 0\}.$$

This is a well-defined submodule: if $s, t \in T^{\wp}M$ with $ns = mt = 0$ for $n, m \in S(\wp)$ and $r \in R$, then $nm \in S(\wp)$ and $n(rs) = r(ns) = 0$, $(mn)(s+t) = m(ns) + n(mt) = 0$, $n0 = 0$, and $n(-s) = -ns = 0$. We will refer to $T^{\wp}M$ as the submodule of elements of M possessing $S(\wp)$-torsion. Note that if $\wp = \emptyset$, then $T^{\wp}M = 0$ for any M.

Recall (see [18, Section III.7]) that any finitely-generated R-module M can be written as a direct sum $M \cong R^{r_M} \oplus \bigoplus_p M(p)$, where r_M is the rank of M and p ranges over $\mathfrak{P}(R)$ (though there will only be a finite number of non-trivial summands) and each $M(p) \cong R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_{s_p}})$. This is the PID generalization of the fundamental theorem of finitely generated abelian groups, and the submodule $M(p) \subset M$ consists of precisely those elements of M that are annihilated by some non-negative power of p. In particular, $T^{\wp}M \cong \bigoplus_{p \in \wp} M(p)$.

Clearly, if we identify M with $R^{r_M} \oplus \bigoplus_p M(p)$ and $T^{\wp} \cong \bigoplus_{p \in \wp} M(p)$ with the obvious submodule, then $M/T^{\wp}M \cong R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p)$. In fact, this is the only way to obtain this quotient:

Lemma 2.1. *If $M \cong R^{r_M} \oplus \bigoplus_p M(p)$ is a finitely-generated R-module and $A \subset M$ is a submodule such that $M/A \cong R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p)$, then $A = T^{\wp}M$.*

Proof. Without loss of generality, we use the given isomorphism to identify M with $R^{r_M} \oplus \bigoplus_p M(p)$ and $T^{\wp}M$ with the summand $\bigoplus_{p \in \wp} M(p)$.

By hypothesis, M/A contains no $S(\wp)$-torsion. This implies that $T^{\wp}M \subset A$. For otherwise, suppose that $x \in T^{\wp}M$, $x \notin A$. The quotient class $[x]$ of x in M/A is non-zero, but it is $S(\wp)$-torsion (if $rx = 0$ in M, then $r[x] = 0$ in M/A). Since $R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p)$ has no $S(\wp)$-torsion, this would be a contradiction.

On the other hand, suppose $x \in A$ but $x \notin T^{\wp}M$. Given the direct sum structure of M, we can write $x = x_{\wp} + x_c$, where $x_{\wp} \in T^{\wp}M$ and $x_c \neq 0$ is in the complementary summand $R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p)$. Since we have already established $T^{\wp}M \subset A$, it follows that $x_c \in A$. Let $\langle x_c \rangle$ be the submodule of M generated by x_c. Then M/A is a quotient of $M/(T^{\wp}M \oplus \langle x_c \rangle) \cong (R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p))/\langle x_c \rangle$. But no such quotient can be homeomorphic to $R^{r_M} \oplus \bigoplus_{p \notin \wp} M(p) \cong M/A$, another contradiction.

So $A = T^{\wp}M$. $\square$

We also recall that, in analogy with abelian groups, if $M \cong R^{r_M} \oplus \bigoplus_p M(p)$ is finitely-generated, then[6]

$$\begin{aligned}
\operatorname{Hom}(M,R) &\cong \operatorname{Hom}(R^{r_M},R) \oplus \operatorname{Hom}\left(\bigoplus_p M(p), R\right)\\
&\cong \operatorname{Hom}(R^{r_M},R)\\
&\cong (\operatorname{Hom}(R,R))^{r_M}\\
&\cong R^{r_M},
\end{aligned}$$

while

$$\begin{aligned}
\operatorname{Ext}(M,R) &\cong \operatorname{Ext}(R^{r_M},R) \oplus \operatorname{Ext}\left(\bigoplus_p M(p), R\right)\\
&\cong \operatorname{Ext}\left(\bigoplus_p M(p), R\right)\\
&\cong \bigoplus_p \operatorname{Ext}(M(p),R)\\
&\cong \bigoplus_p \operatorname{Ext}(R/(p^{\nu_1}) \oplus \cdots \oplus R/(p^{\nu_{s_p}}), R)\\
&\cong \bigoplus_p \bigoplus_{i=1}^{s_p} \operatorname{Ext}(R/(p^{\nu_i}), R)\\
&\cong \bigoplus_p \bigoplus_{i=1}^{s_p} R/(p^{\nu_i}).
\end{aligned}$$

In particular, $\operatorname{Hom}(M,R)$ is free with the rank of M, and $\operatorname{Ext}(M,R)$ is isomorphic to the torsion submodule of M.

3 Torsion-tipped truncation

We wish to define an endofunctor $\mathfrak{t}^{\wp}_{\leq k}$, which we will call the *$\wp$-torsion-tipped truncation* functor, in the category of cohomologically indexed complexes of sheaves of R-modules on a space X. We will first define $\mathfrak{t}^{\wp}_{\leq k}$ as an endofunctor of presheaves.

Let A^* be a presheaf complex on X with boundary map d. Let $W^{\wp}A^j$ be the presheaf of weak $\wp$-boundaries in degree j, which we define to be

$$W^{\wp}A^j(U) = \{s \in A^j(U) \mid \exists n \in S(\wp) \text{ such that } ns \in \operatorname{im}(d : A^{j-1}(U) \to A^j(U))\}.$$

Notice that if $\wp = \emptyset$, then $W^{\emptyset}A^j(U) = \operatorname{im}(d : A^{j-1}(U) \to A^j(U))$; in particular $\operatorname{im}(d : A^{j-1}(U) \to A^j(U))\} = W^{\emptyset}A^j(U) \subset W^{\wp}A^j(U)$ for all $\wp$.

[6]Since all modules will be R modules, we write Hom and Ext rather than Hom_R and Ext_R.

Then $W^\wp A^j$ is a presheaf. First of all, for each U, $W^\wp A^j(U)$ is a submodule of $A^j(U)$: Suppose $s, t \in A^j(U)$ are such that $ms, nt \in \operatorname{im}(d : A^{j-1}(U) \to A^j(U))$ for $m, n \in S(\wp)$. Then then for any $r \in R$, $m(rs) = r(ms) \in \operatorname{im}(d : A^{j-1}(U) \to A^j(U))$ (since d is a module homomorphism), so $rs \in W^\wp A^j$. And then also $mn(s+t) = n(ms) + m(nt) \in \operatorname{im}(d : A^{j-1}(U) \to A^j(U))$, but if $m, n \in S(\wp)$ then also $mn \in S(\wp)$, so $s + t \in W^\wp A^j$. Clearly also 0 and $-s$ are in $W^\wp A^j$. So $W^\wp A^j(U)$ is an R-module. Also, $W^\wp A^j$ is a preseheaf, using the fact that restriction commutes with boundaries: if $u \in A^{j-1}(U)$ and $du = ns$ for some $u \in A^j(U)$, $n \in S(\wp)$, and if $V \subset U$, we have $du|_V = ns|_V$. Thus $s|_V \in W^\wp A^j(V)$.

Now define

$$(\mathfrak{t}^\wp_{\le k} A^*)^i = \begin{cases} 0, & i > k+1, \\ W^\wp A^{k+1}, & i = k+1, \\ A^i, & i \le k. \end{cases}$$

This is a presheaf complex: we have seen that we have legitimate presheaves at all levels and furthermore, as already observed, $\operatorname{im}(d : A^k \to A^{k+1}) = W^\emptyset A^{k+1} \subset W^\wp A^{k+1}$.

Suppose $f : A^* \to B^*$ is a chain map of presheaf complexes. Then if $du = ns$ for some $u \in A^k$, $n \in S(\wp)$, we see that $df(u) = f(du) = f(ns) = nf(s)$, so $s \in W^\wp A^{k+1}$ implies $f(s) \in W^\wp B^{k+1}$. So f induces in the obvious way a map $\mathfrak{t}^\wp_{\le k}(f)$, and $\mathfrak{t}^\wp_{\le k}$ is a functor. Additionally, it is clear that we always have a monomorphism $\mathfrak{t}^\wp_{\le k} A^* \hookrightarrow A^*$.

Lemma 3.1.

$$H^i(\mathfrak{t}^\wp_{\le k} A^*(U)) = \begin{cases} 0, & i > k+1, \\ T^\wp H^{k+1}(A^*(U)), & i = k+1, \\ H^i(A^*(U)), & i \le k. \end{cases}$$

Furthermore, the homology isomorphisms or torsion submodule isomorphisms, in the respective degrees, are induced by the inclusion $\mathfrak{t}^\wp_{\le k} A^* \hookrightarrow A^*$.

Proof. This is trivial in all degrees save $i = k+1$. Notice that the chain inclusion $\mathfrak{t}^\wp_{\le k} A^*(U) \to A^*(U)$, induces a map $f : H^{k+1}(\mathfrak{t}^\wp_{\le k} A^*(U)) \to H^{k+1}(A^*(U))$. If $s \in W^\wp A^{k+1}(U)$, then for some $n \in S(\wp), u \in A^k$, we have $ns = du$, so the image of f must lie in $T^\wp H^{k+1}(A^*(U))$. Conversely, given a cycle s representing an element of $T^\wp H^{k+1}(A^*(U))$, by the definition of $TH^{k+1}(A^*(U))$, there must be some $n \in S(\wp)$ and $u \in A^k(U)$ such that $ns = du$. Thus f is surjective. Now suppose $s \in W^\wp A^{k+1}(U)$ and $f(s) = 0$. Then there is a $u \in A^k(U)$ such that $du = s$. But then this relation also holds in $\mathfrak{t}^\wp_{\le k} A^*(U)$ and s represents 0 in $H^{k+1}(\mathfrak{t}^\wp_{\le k} A^*(U))$. Thus f is an isomorphism $H^{k+1}(\mathfrak{t}^\wp_{\le k} A^*(U)) \to T^\wp H^{k+1}(A^*(U))$. □

Remark 3.2. If $\wp = \emptyset$, then the Lemma demonstrates that $\mathfrak{t}^\wp_{\le k} A^*(U)$ has the cohomology we obtain from the standard truncation functor, $\tau_{\le k} A^*(U)$. In fact, it is not difficult to see that this cohomology isomorphism is induced by an inclusion $\tau_{\le k} A^*(U) \hookrightarrow \mathfrak{t}^\wp_{\le k} A^*(U)$.

We can now extend $\mathfrak{t}^\wp_{\le k}$ to a functor of sheaves by sheafification, i.e. if $\mathscr{S}^*$ is a sheaf complex, define $\mathfrak{t}^\wp_{\le k} \mathscr{S}^*$ as the sheafification of the presheaf $U \to \mathfrak{t}^\wp_{\le k}(\mathscr{S}^*(U))$. Furthermore, there is a canonical injection of sheaves $\mathfrak{t}^\wp_{\le k} \mathscr{S}^* \hookrightarrow \mathscr{S}^*$.

Lemma 3.3. *Suppose $\mathscr{S}^*$ is a sheaf complex on X and $x \in X$. Then,*

$$H^i((\mathfrak{t}^{\wp}_{\leq k}\mathscr{S}^*)_x) = \begin{cases} 0, & i > k+1, \\ T^{\wp}H^{k+1}(\mathscr{S}^*_x), & i = k+1, \\ H^i(\mathscr{S}^*_x), & i \leq k. \end{cases}$$

Furthermore, the homology isomorphisms or torsion submodule isomorphisms, in the respective degrees, are induced by the inclusion $\mathfrak{t}^{\wp}_{\leq k}\mathscr{S}^* \hookrightarrow \mathscr{S}^*$.

Proof. By basic sheaf theory and the definitions above, $H^i((\mathfrak{t}^{\wp}_{\leq k}\mathscr{S}^*)_x) \cong H^i(\varinjlim_{x\in U} \mathfrak{t}^{\wp}_{\leq k}(\mathscr{S}^*(U)))$, which, by the properties of direct limits, is isomorphic to $\varinjlim_{x\in U} H^i(\mathfrak{t}^{\wp}_{\leq k}(\mathscr{S}^*(U)))$. Applying Lemma 3.1 and basic sheaf theory proves the lemma for $i \neq k+1$.

For $i = k+1$, note that there are natural maps

$$\varinjlim_{x\in U} H^{k+1}(\mathfrak{t}^{\wp}_{\leq k}(\mathscr{S}^*(U))) \cong \varinjlim_{x\in U} T^{\wp}H^{k+1}(\mathscr{S}^*(U)) \hookrightarrow \varinjlim_{x\in U} H^{k+1}(\mathscr{S}^*(U)) \to H^{k+1}(\mathscr{S}^*_x)$$

whose composite image must lie in $T^{\wp}H^{k+1}(\mathscr{S}^*_x)$ because each element of each $T^{\wp}H^{k+1}(\mathscr{S}^*(U))$ is $S(\wp)$-torsion. We claim that this produces an isomorphism $\varinjlim_{x\in U} T^{\wp}H^{k+1}(\mathscr{S}^*(U)) \to T^{\wp}H^{k+1}(\mathscr{S}^*_x)$. To see that this is onto, recall that any element $s_x \in T^{\wp}H^{k+1}(\mathscr{S}^*_x)$ must be represented by a section $s \in \mathscr{S}^{k+1}(V)$ for some neighborhood V of x, and furthermore, since s is $S(\wp)$-torsion in the stalk homology, there must be a germ $t_x \in \mathscr{S}^k_x$ such that $dt_x = ns_x$ for some $n \in S(\wp)$. Let t be an element of $\mathscr{S}^k(V')$, for some open V' such that $t|_x = t_x$. Since $dt_x = ns_x$, we must have $dt = ns$ on some open $V'' \ni x$, $V'' \subset V \cap V'$. But therefore s represents an element of $T^{\wp}H^{k+1}(\mathscr{S}^*(V''))$ whose image under $T^{\wp}H^{k+1}(\mathscr{S}^*(V'')) \to \varinjlim_{x\in U} T^{\wp}H^{k+1}(\mathscr{S}^*(U)) \to T^{\wp}H^{k+1}(\mathscr{S}^*_x)$ is s_x. This establishes surjectivity. Injectivity is established similarly: if $\tilde{s} \in \varinjlim_{x\in U} T^{\wp}H^{k+1}(\mathscr{S}^*(U))$ is represented by $s \in TH^{k+1}(\mathscr{S}^*(V))$ and $s|_x = 0$ in $H^{k+1}(\mathscr{S}^*_x)$, then there is a $t \in \mathscr{S}^*(V')$, $V' \subset V$ such that $dt = s$, whence $\tilde{s} = 0$. $\square$

Remark 3.4. Following on Remark 3.2, if $\wp = \emptyset$, then the inclusion $\tau_{\leq k}\mathscr{S}^* \hookrightarrow \mathfrak{t}^{\emptyset}_{\leq k}\mathscr{S}^*$ induces a quasi-isomorphism.

4 The torsion-sensitive Deligne sheaves

In this section, we define our $\wp$-torsion-sensitive Deligne sheaf, first in Section 4.1 for perversities that depend only on codimension and then more generally in Section 4.1.1. In Section 4.2, we investigate the axiomatic and constructibility properties of the ts-Deligne sheaf. The duality theorem then follows in Section 4.3.

4.1 Definitions

Let $\mathfrak{P}(R)$ be the set of primes of R (up to unit), and let $\mathbb{P}(\mathfrak{P}(R))$ be its power set (so elements of $\mathbb{P}(\mathfrak{P}(R))$ are sets of primes of R). Let a *skeletal torsion-sensitive perversity* (or simply *skeletal ts-perversity*) be a function $\vec{p}: \mathbb{Z}_{\geq 1} \to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R))$. We denote the components of $\vec{p}(k)$ by $(\vec{p}_1(k), \vec{p}_2(k))$. Notice that the function $\vec{p}_1$ is a (loose) perversity in the usual sense [17].

Now let $X \supset X^{n-1} \supset \cdots$ be a topological stratified n-pseudomanifold. Let $U_k = X - X^{n-k}$, let $X_{n-k} = X^{n-k} - X^{n-k} = U_{k+1} - U_k$, and let $i_k : U_k \hookrightarrow U_{k+1}$ be the inclusion. Let $\mathcal{E}$ be a locally constant sheaf of finitely generated R-modules on U_1 for a principal ideal domain R. We define a torsion sensitive Deligne sheaf[7] (or ts-Deligne sheaf) $\mathscr{P}^*$ (or, if we must, $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$) inductively. Let $\mathscr{P}^*_1 = \mathcal{E}$ on U_1, and suppose $\mathscr{P}^*_k$ is defined on U_k. Then we inductively define $\mathscr{P}^*_{k+1} = \mathfrak{t}^{\vec{p}_2(k)}_{\leq \vec{p}_1(k)} Ri_{k_*}\mathscr{P}^*_k$, where $\mathfrak{t}^{\vec{p}_2(k)}_{\leq \vec{p}_1(k)}$ is the $\vec{p}_2(k)$-torsion-tipped truncation functor. Let $\mathscr{P}^* = \mathscr{P}^*_{n+1}$.

Perhaps a slicker way to write the definition is as follows:

Definition 4.1. Given a skeletal ts-perversity $\vec{p}$, define

$$\mathfrak{t}_{\leq \vec{p}(k)} = \mathfrak{t}^{\vec{p}_2(k)}_{\leq \vec{p}_1(k)}.$$

Then we define the *torsion-sensitive Deligne sheaf* (or *ts-Deligne sheaf*) by

$$\mathscr{P}^*_{X,\vec{p},\mathcal{E}} = \mathfrak{t}_{\leq \vec{p}(n)} Ri_{n*} \cdots \mathfrak{t}_{\leq \vec{p}(1)} Ri_{1*}\mathcal{E}.$$

So $\mathscr{P}^*$ is defined just like the Deligne sheaf, but using torsion-tipped truncation with respect to a set of primes specified by the ts-perversity.

Example 4.2. If $\vec{p}_2(k) = \emptyset$ for all k, then Remark 3.4 implies that $\mathscr{P}^*$ is quasi-isomorphic to the standard Deligne sheaf $\mathcal{P}^*$ as defined in [15].

More generally, $\mathscr{P}^*$ is the standard Deligne sheaf if, for each k, we have $T^{\vec{p}_2(k)}H^{\vec{p}_1(k)+1}((Ri_{k*}\mathscr{P}^*_k)_x) = 0$ for each $x \in X_{n-k}$. If X is locally $\bar{p}$-torsion-free in the sense of Goresky and Siegel [16], this will be the case for any $\vec{p}$ such that $\vec{p}_1 = \bar{p}$.

Example 4.3. Suppose $H^{\vec{p}_1(k)+1}((Ri_{k*}\mathscr{P}^*_k)_x)$ is always a torsion R-module, and that $\vec{p}$ is a ts-perversity with $\vec{p}_2(k) = \mathfrak{P}(R)$, the set of all primes in R, for all k. Then the complex $\mathscr{P}^*_{\vec{p}}$ is the same as the Deligne sheaf $\mathcal{P}^*_{\vec{p}_1+1}$, where $\vec{p}_1 + 1$ is the perversity whose value on k is $\vec{p}_1(k) + 1$. Such Deligne sheaves arise in the Cappell-Shaneson superduality theorem [4].

4.1.1 A variation for general ts-perversities

While the original perversity functions of Goresky and MacPherson were functions $\bar{p} : \mathbb{Z}_{\geq 2} \to \mathbb{Z}$ (with additional restrictions), in recent years it has become useful to define more general perversities $\bar{p}$: {singular strata of X} $\to \mathbb{Z}$, where by *singular strata* we mean connected components of $X^k - X^{k-1}$, $k < n$. Deligne sheaves that accommodate such perversities were built in [9] via a modification of the the standard sheaf complex truncation functor $\tau_{\leq m}$.

[7] We apologize to the reader for the plethora of "P"s. Unfortunately, it is standard to refer to primes, power sets, and Deligne sheaves all with variants of the letter "P". The other likely candidate letter for Deligne sheaves, "D", is unfortunately reserved for the Verdier dualizing sheaf and the Verdier dualizing functor. Additionally, we will use the same R both for our ground ring and for labeling right derived functors, though context should prevent any confusion.

In this section, we will describe how to likewise generalize the torsion-tipped truncation functor. It will thus be possible to define a modified Deligne sheaf $\mathscr{P}^*_{\vec{p}}$ for *torsion-sensitive perversities* (abbreviated *ts-perversities*) $\vec{p}$: {singular strata of X} $\to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R)$.

Remark 4.4. The reader who is interested primarily in intersection homology with Goresky-MacPherson perversities, or who might just wish to keep things more simple through a first reading, might safely skip this section. Proofs in later sections are presented using the general ts-Deligne sheaf for general ts-perversities given here, but it will not be difficult for the reader to "scale back" these proofs to conform to the context of the ts-Deligne sheaf of Definition 4.1.

Definition 4.5. Let $\mathcal{A}^*$ be a sheaf complex on X, and let $\mathfrak{F}$ be a locally-finite collection of subsets of X. Let $|\mathfrak{F}| = \cup_{V\in\mathfrak{F}} V$. Let $\vec{P}$ be a function $\mathfrak{F} \to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R))$. We intentionally do not assume that $\mathfrak{F}$ is necessarily the collection of all singular strata of X, and so $\vec{P}$ is not technically a perversity; below, $\mathfrak{F}$ will generally be the set of strata of X of a given dimension.

If $F \subset \mathfrak{F}$ is a subset, define $\inf_F \vec{P}$ to be the ts-perversity $(\inf\{\vec{P}_1(V) \mid V \in F\}, \cap_{V\in F}\vec{P}_2(V))$. Note that, conceivably, $\inf\{\vec{P}_1(V) \mid V \in F\}$ could be $-\infty$, which we will allow - in our applications, any truncation in dimension < 0 will be the trivial complex of 0 sheaves, so all truncations in negative dimensions are equivalent.

We now wish to define a sheaf $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*$. It will be the sheafification of the presheaf $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*$. Let

$$\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*(U) = \begin{cases} \Gamma(U;\mathcal{A}^*), & U\cap|\mathfrak{F}| = \emptyset, \\ \Gamma(U;\mathfrak{t}_{\le\inf_{\{V\in\mathfrak{F}\mid U\cap V\neq\emptyset\}}\vec{P}}\mathcal{A}^*), & U\cap|\mathfrak{F}|\neq\emptyset. \end{cases}$$

If $\inf_{\{V\in\mathfrak{F}\mid U\cap V\neq\emptyset\}}\vec{P} = -\infty$, then we let $\mathfrak{t}_{\le\inf_{\{V\in\mathfrak{F}\mid U\cap V\neq\emptyset\}}\vec{P}}\mathcal{A}^* = 0$.

We observe that restriction is well-defined. In particular, if $W \subset U$ and we let $F(W) = \{V \in \mathfrak{F} \mid W\cap V \neq \emptyset\}$ and $F(U) = \{V \in \mathfrak{F} \mid U \cap V \neq \emptyset\}$, then $F(W) \subset F(U)$. So in the definition of $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*(W)$, the infs and intersections are being taken over subset of what they are for $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*(U)$. So if $\inf_{F(W)}\vec{P} = (a(V),\wp(V))$ and $\inf_{F(U)}\vec{P} = (a(U),\wp(U))$, then $a(U) \le a(W)$ and $\wp(U) \subset \wp(W)$. But in general if $a \le a'$ are integers and $\wp \subseteq \wp'$ are sets of primes of R, there are natural inclusions $\mathfrak{t}^{\wp}_{\le a}\mathcal{A}^* \hookrightarrow \mathfrak{t}^{\wp'}_{\le a'}$. So such an inclusion induces a well-defined restriction homomorphism $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*(U) \to \mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*(W)$.

We now let the *generalized ts-truncation sheaf* $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*$ be the sheafification of $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^*$.

For maps $f : \mathcal{A}^* \to \mathcal{B}^*$ of sheaf complexes over X, we can define $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}f$ in the obvious way. In fact, $\mathfrak{T}^{\mathfrak{F}}_{\le\vec{P}}f$ is well-defined by applying the truncation functors on the appropriate subsets, and we obtain $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}f$ again by passing to limits in the sheafification process.

The following lemma contains the key facts we will need about the generalized ts-truncation; they all follow immediately from the definition and the properties of $\mathfrak{t}^{\wp}_{\le m}$.

Lemma 4.6. *1. $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}$ is an endofunctor of sheaf complexes on X.*

2. There is an inclusion of sheaf complexes $\mathfrak{t}^{\mathfrak{F}}_{\le\vec{P}}\mathcal{A}^* \hookrightarrow \mathcal{A}^*$.

3. $(\mathfrak{t}^{\mathfrak{F}}_{\leq \vec{P}}\mathcal{A}^*)|_{X-|\mathfrak{F}|} = \mathcal{A}^*|_{X-|\mathfrak{F}|}$

4. *Suppose $\mathfrak{F}$ has the property that for each $V \in \mathfrak{F}$ and each $x \in V$, there is a neighborhood U of x such that $U \cap V' = \emptyset$ for each $V' \in \mathfrak{F}$ such that $V' \neq V$. Then for each $V \in F$, $(\mathfrak{t}^{\mathfrak{F}}_{\leq \vec{P}}\mathcal{A}^*)|_V = (\mathfrak{t}_{\leq \vec{P}(V)}\mathcal{A}^*)|_V$.*

Remark 4.7. It follows from the last statement of the lemma that if $\mathfrak{F} = \{X\}$, then $\mathfrak{t}^{\mathfrak{F}}_{\leq \vec{P}}\mathcal{A}^* = \mathfrak{t}_{\leq \vec{P}(X)}\mathcal{A}^*$, which is a $\wp$-torsion-tipped truncation in the sense of our original definition.

Remark 4.8. $\mathfrak{T}^{\mathfrak{F}}_{\leq \vec{P}}\mathcal{A}^*$ will not necessarily be a sheaf, so the sheafification in the definition is necessary. This is true even when all $\vec{P}_2(V) = \emptyset$ so that all truncation functors are the classical ones; see [9, Remark 3.5] for an example.

Now, suppose that $\vec{p} : \{\text{singular strata of } X\} \to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R))$ is a ts-perversity. We will abuse our earlier notation and write $\mathfrak{t}^{X_k}_{\leq \vec{p}}$, allowing X_k to stand for the set of connected components of $X_k = X^k - X^{k-1}$ and letting $\vec{p}$ also refer to its restriction to the components of X_k.

Definition 4.9. Given a ts-perversity $\vec{p} : \{\text{singular strata of } X\} \to \mathbb{Z} \times \mathbb{P}(\mathfrak{P}(R))$ and a locally constant sheaf of finitely generate R-modules $\mathcal{E}$ on $X - X^{n-1}$, let the *torsion-sensitive Deligne sheaf* (or *ts-Deligne sheaf*) be defined by

$$\mathscr{P}^*_{X,\vec{p},\mathcal{E}} = \mathfrak{t}^{X_0}_{\leq \vec{p}} Ri_{n*} \dots \mathfrak{t}^{X_{n-1}}_{\leq \vec{p}} Ri_{1*}\mathcal{E}.$$

This generalizes both our construction in the preceding section and the construction of the Deligne sheaf for general perversities in [9].

4.2 Axiomatics and constructibility

We define a set of axioms analogous to the Goresky-MacPherson axioms Ax1 and show that they characterize $\mathscr{P}^*$. The treatment parallels the work of [15] and the exposition of [3, Section V.2].

Let X be a stratified n-pseudomanifold and let $\mathcal{E}$ be a locally constant sheaf of finitely generated R-modules on U_1 over a principal ideal domain R. For a sheaf complex $\mathscr{S}^*$ on X, let $\mathscr{S}^*_k = \mathscr{S}^*|_{U_k}$.

Recall the notation from Section 2. We say $\mathscr{S}^*$ satisfies the axioms TAx1$(X, \vec{p}, \mathcal{E})$ (or simply TAx1) if

1. $\mathscr{S}^*$ is quasi-isomorphic to a complex that is bounded and that is 0 for $* < 0$;
2. $\mathscr{S}^*|_{U_1}$ is quasi-isomorphic to $\mathcal{E}$;
3. if $x \in Z \subset X_{n-k}$, where Z is a singular stratum, then $H^i(\mathscr{S}_x) = 0$ for $i > \vec{p}_1(Z) + 1$ and $H^{\vec{p}_1(Z)+1}(\mathscr{S}_x)$ is $S(\vec{p}_2(Z))$-torsion;
4. if $x \in Z \subset X_{n-k}$, where Z is a singular stratum, then the attachment map $\alpha_k : \mathscr{S}^*_{k+1} \to Ri_{k*}\mathscr{S}^*_k$ induces stalkwise cohomology isomorphisms at x in degrees $\leq \vec{p}_1(Z)$ and it induces stalkwise cohomology isomorphisms $H^{\vec{p}_1(Z)+1}(\mathscr{S}^*_{k+1,x}) \to T^{\vec{p}_2(Z)}H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{S}^*_k)_x)$.

Theorem 4.10. *The sheaf complex $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ satisfies the axioms TAx1$(X,\vec{p},\mathcal{E})$, and any sheaf complex satisfying TAx1$(X,\vec{p},\mathcal{E})$ is quasi-isomorphic to $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$.*

The theorem relies on the following lemma.

Lemma 4.11. *Suppose $\mathscr{S}^*$ satisfies the axioms TAx1$(X,\vec{p},\mathcal{E})$. Then, for $k>0$, the sheaf complex $\mathscr{S}^*_{k+1}$ is quasi-isomorphic to $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}} Ri_{k*}\mathscr{S}^*_k$.*

Proof. By the functoriality of the truncation functors and their inclusion properties, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{S}^*_{k+1} \arrow[r, "\alpha_k"] & Ri_{k*}\mathscr{S}^*_k \\
\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\mathscr{S}^*_{k+1} \arrow[u, "\beta"] \arrow[r, "\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k"] & \mathfrak{t}^{X_{n-k}}_{\leq\vec{p}} Ri_{k*}\mathscr{S}^*_k. \arrow[u, "\gamma"']
\end{tikzcd}$$

The map β is a quasi-isomorphism by axiom (3), the properties of $\mathfrak{t}$, and Lemma 3.3.

At $x\in Z\subset X_{n-k}$, the map $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k$ is evidently an isomorphism in degrees $i>\vec{p}_1(Z)+1$. In degrees $i\leq\vec{p}_1(Z)$, α_k is a quasi-isomorphism by axiom (4) and γ is a quasi-isomorphism by the properties of $\mathfrak{t}$ and Lemma 3.3; thus $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k$ is a quasi-isomorphism in this range, as well. Finally, consider the diagram

$$\begin{tikzcd}
H^{\vec{p}_1(Z)+1}(\mathscr{S}^*_{k+1,x}) \arrow[r] & T^{\vec{p}_2(Z)}H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{S}^*_k)_x) \\
H^{\vec{p}_1(Z)+1}((\mathfrak{t}_{\leq\vec{p}}\mathscr{S}^*_{k+1})_x) \arrow[u, "\beta"] \arrow[r, "\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k"] & H^{\vec{p}_1(Z)+1}((\mathfrak{t}_{\leq\vec{p}}Ri_{k*}\mathscr{S}^*_k)_x). \arrow[u]
\end{tikzcd}$$

By Lemma 3.3, the righthand map is an isomorphism induced by the inclusion. The top map is induced by α and is an isomorphism by axiom (4). We have already seen that β induces an isomorphism. Thus the bottom map must be an isomorphism, and $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k$ is a quasi-isomorphism of sheaves.

Together, β and $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}\alpha_k$ provide the desired quasi-isomorphism of the lemma. □

Proof of Theorem 4.10. It is evident from the definition of $\mathscr{P}^*$ that it satisfies the axioms. Conversely, suppose $\mathscr{S}^*$ satisfies the axioms and that $\mathscr{S}^*_k$ is quasi-isomorphic to $\mathscr{P}^*_k$ for some k. This is true for $\mathscr{S}^*_1$ by axiom (2). By the preceding lemma, $\mathscr{S}^*_{k+1}=\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}Ri_{k*}\mathscr{S}^*_k$. But by the induction hypothesis, this is quasi-isomorphic to $\mathfrak{t}^{X_{n-k}}_{\leq\vec{p}}Ri_{k*}\mathscr{P}^*_k$, which is $\mathscr{P}^*_{k+1}$. □

Theorem 4.12. *Let $\mathfrak{X}$ denote the stratification of the stratified pseudomanifold X. The sheaf complex $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ is $\mathfrak{X}$-cohomologically locally constant ($\mathfrak{X}$-clc), $\mathfrak{X}$-cohomologically constructible ($\mathfrak{X}$-cc), and cohomologically constructible (cc), using the definitions of [3, Section V.3.3].*

Proof. The proof follows from the machinery developed in Section V.3 of Borel [3]. This theorem is completely analogous to Borel's Proposition V.3.12. The only additional observation needed is that $\mathfrak{t}_{\leq \vec{p}}^{X_{n-k}}$ preserves the properties of being $\mathfrak{X}$-cc. □

As in [15, 3], it will be useful to have some slight reformulations of the axioms. First we have the following lemma, which shows that axiom (3) can be replaced by an equivalent condition if we assume *a priori* that $\mathcal{S}^*$ is $\mathfrak{X}$-cc.

Lemma 4.13. *Suppose $\mathcal{S}^*$ is $\mathfrak{X}-$cc and satisfies axiom TAx1(3). Then TAx1(4) is equivalent to the following condition: Suppose $x \in Z \subset X_{n-k}$, $k > 0$, and let $j : Z \hookrightarrow X$ be the inclusion; then*

1. $H^i((j^!\mathcal{S}^*)_x) = 0$ *for* $i \leq \vec{p}_1(Z) + 1$,
2. $H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x) \cong H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)/T^{\vec{p}_2(Z)}H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$.

Proof. For $x \in Z$, there is a long exact sequence (see [3, V.1.8(7)])

$$\longrightarrow H^i((j^!\mathcal{S}^*)_x) \longrightarrow H^i(\mathcal{S}_x^*) \xrightarrow{\alpha} H^i((Ri_{k*}\mathcal{S}_{k*}^*)_x) \longrightarrow .$$

Suppose $\mathcal{S}^*$ satisfies TAx1(4). Then we have $H^i((j^!\mathcal{S}^*)_x) = 0$ for $i \leq p_1(Z) + 1$, noting that α remains injective in degree $p_1(Z) + 1$. Around degree $p_1(Z) + 2$ and using TAx(3), the sequence specializes to

$$0 \longrightarrow H^{\vec{p}_1(Z)+1}(\mathcal{S}_x^*) \xrightarrow{\alpha} H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x) \longrightarrow H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x) \longrightarrow 0,$$

and since α is an isomorphism onto $T^{\vec{p}_2(Z)}H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$, it follows that $H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x) \cong H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)/T^{\vec{p}_2(Z)}H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$.

Conversely, if $j^!\mathcal{S}^*$ satisfies the conditions stated in the lemma, then certainly α is an isomorphism on cohomology for $i \leq \vec{p}_1(Z)$. Around $H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x)$, we have the same specialized sequence as above, so $H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x) \cong H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)/\mathrm{im}(\alpha)$. But also by assumption, $H^{\vec{p}_1(Z)+2}((j^!\mathcal{S}^*)_x) \cong H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)/T^{\vec{p}_2(Z)}H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$. Now since $\mathcal{S}^*$ is $\mathfrak{X}$-cc, so is $Ri_{k*}\mathcal{S}_k^*$ by [3, Corollary V.3.11], and thus $H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_{k*}^*)_x)$ is finitely-generated. By Lemma 2.1, $T^{\wp}M$ is the unique subgroup $\bigoplus_{p\in\wp} M(p)$, such that $M/T^{\wp}M \cong R^{f_M} \oplus \bigoplus_{p\notin\wp} M(p)$. Thus α must take $H^{\vec{p}_1(Z)+1}(\mathcal{S}_x^*)$ onto $T^{\vec{p}_2(Z)}H^{p_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$. □

Next, we say $\mathscr{S}^*$ satisfies the axioms TAx1'$(X, \vec{p}, \mathcal{E})$ (or simply TAx1') if it is $\mathfrak{X}$-cc and

1. $\mathscr{S}^*$ is quasi-isomorphic to a complex that is bounded and that is 0 for $* < 0$;
2. $\mathscr{S}^*|_{U_1}$ is quasi-isomorphic to $\mathcal{E}$;
3. if $x \in Z \subset X_{n-k}$, where Z is a singular stratum, then $H^i(\mathscr{S}_x) = 0$ for $i > \vec{p}_1(Z) + 1$ and $H^{\vec{p}_1(Z)+1}(\mathscr{S}_x)$ is $S(\vec{p}_2(Z))$-torsion ;
4. if $x \in Z \subset X_{n-k}$, where Z is a singular stratum, and $f_x : x \hookrightarrow X$ is the inclusion, then

(a) $H^i(f_x^! \mathcal{S}^*) = 0$ for $i \le \vec{p}_1(Z) + n - k + 1$

(b) $H^{p_1(Z)+n-k+2}(f_x^! \mathcal{S}^*) \cong H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)/T^{\vec{p}_2(Z)} H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{S}_k^*)_x)$.

If $x \in Z \subset X_{n-k}$ and $\ell_x : x \hookrightarrow Z$, $j : Z \hookrightarrow U_k \cup Z$, and $f_x : x \hookrightarrow U_k \cup Z$ are inclusions, then$f_x = j \circ \ell_x$, so $f_x^! = \ell_x^! j^!$. So $H^i(f_x^! \mathcal{S}^*) = H^i(\ell_x^! j^! \mathcal{S}^*)$, which, since Z is an $n-k$ dimensional manifold, is isomorphic to $H^{i-n+k}((j^! \mathcal{S}^*)_x)$, by [3, Proposition V.3.7.b]. We use here that $j^! \mathcal{S}^*$ is $\mathfrak{X}$-clc, which follows from $\mathcal{S}^*$ being $\mathfrak{X}$-clc by [3, Proposition V.3.10]. Thus, since any sheaf complex satisfying TAx1 is $\mathfrak{X}$-cc by Theorem 4.12, and in light of Lemma 4.13, we have the following theorem:

Theorem 4.14. *TAx1' is equivalent to TAx1.*

4.3 Duality

Let $\mathcal{D}_X$ be the Verdier dualizing functor on the space X. Given a ts-perversity $\vec{p}$, let $D\vec{p} = (D\vec{p}_1, D\vec{p}_2)$ be the ts-perversity with $D\vec{p}_1(Z) = \text{codim}(Z) - 2 - p_1(Z)$ and $D\vec{p}_2(Z) = \mathfrak{P}(R) - \vec{p}_2(Z)$, the complement of $\vec{p}_2(Z)$ in the set of primes (up to unit) of R. Notice the $D\vec{p}_1$ is the perversity that is complementary to the perversity $\vec{p}_1$. Let $\mathcal{E}$ be a locally constant sheaf of finitely generated free R-modules on U_1 for a principal ideal domain R.

Theorem 4.15. *$(\mathcal{D}_X \mathscr{P}^*_{X,\vec{p},\mathcal{E}})[-n]$ is quasi-isomorphic to $\mathscr{P}^*_{X,D\vec{p},(\mathcal{D}_{U_1}\mathcal{E})[-n]}$ by a quasi-isomorphism that extends the identity morphism of $(\mathcal{D}_{U_1}\mathcal{E})[-n]$ on U_1.*

Before providing the proof, we make some observations and present some corollaries.

Remark 4.16. If our base ring is in fact a field, then each $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ is in fact equal to the Deligne sheaf $\mathcal{P}^*_{X,\vec{p}_1,\mathcal{E}}$, where $\vec{p}_1$ is the first component of $\vec{p}$. In this case, Theorem 4.15 reduces to the duality theorem of Goresky and MacPherson [15] if $\vec{p}_1$ is a Goresky-MacPherson perversity. If $\vec{p}_1$ is a general perversity, Theorem 4.15 with field coefficients reduces to the duality theorem proven in [9].

Suppose R is a PID, $\bar{p}$ is a general perversity, and X is locally $(\bar{p}, \mathcal{E})$-torsion-free in the sense of [16] (see also [9]), i.e. for each singular stratum Z and each $x \in Z$, the R-module $I^{\bar{p}}H_{\text{codim}(Z)-2-\bar{p}(Z)}(L_x; \mathcal{E})$ is R-torsion-free, where L_x is the link of x in X. In this case[8], again $\mathscr{P}^*_{X,\vec{p},\mathcal{E}} = \mathcal{P}^*_{X,\bar{p},\mathcal{E}}$ for any $\vec{p}$ such that $\vec{p}_1(Z) = \bar{p}(Z)$, and Theorem 4.15 reduces to the duality theorem of Goresky and Siegel [16] if $\bar{p}$ is a Goresky-MacPherson perversity or the duality theorem proven in [9] for more general perversities.

Finally, suppose that $\bar{p}$ is a Goresky-MacPherson perversity and that for each singular stratum Z and each $x \in Z$, $I^{\bar{p}}H_*(L_x; \mathcal{E})$ is R-torsion. Suppose further that $\vec{p}$ is a ts-perversity with $\vec{p}_2(Z) = \mathfrak{P}(R)$ for all singular strata Z. Then $\mathscr{P}^*_{X,\vec{p},\mathcal{E}} = \mathcal{P}^*_{X,\vec{p}_1+1,\mathcal{E}}$, where

[8] This can be shown by induction. By definition, $\mathscr{P}^*_1 \cong \mathcal{P}^*_1 \cong \mathcal{E}$. Now assuming $\mathcal{P}^*_k \cong \mathscr{P}^*_k$, we have for $x \in Z \subset X_{n-k}$, $H^i((Ri_{k*}\mathcal{P}^*_k)_x) \cong \varinjlim_{x\in U} \mathbb{H}^i(U - U \cap Z; \mathcal{P}^*_k) \cong \varinjlim_{x \in U} I^{\bar{p}}H^{\infty}_{n-i}(U - Z; \mathcal{E})$. But we may assume U is chosen from the cofinal system of distinguished neighborhoods of x, and thus $U - Z \cong \mathbb{R}^{n-k+1} \times L$. So then each $I^{\bar{p}}H^{\infty}_{n-i}(U - Z; \mathcal{E})$ is isomorphic to $I^{\bar{p}}H_{k-i-1}(L; \mathcal{E})$. Thus $H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{P}^*_k)_x) \cong I^{\bar{p}}H_{k-2-\vec{p}_1(Z)}(L; \mathcal{E})$, which is torsion free by assumption. And this implies that $\mathscr{P}^*_{k+1} \cong \mathfrak{t}^{X_{n-k}}_{\le \vec{p}} Ri_{k*}\mathscr{P}^*_k \cong \mathfrak{t}^{X_{n-k}}_{\le \vec{p}} Ri_{k*}\mathcal{P}^*_k \cong \tau^{X_{n-k}}_{\le \bar{p}} Ri_{k*}\mathcal{P}^*_k \cong \mathcal{P}^*_{k+1}$, using the notation of [9].

$\vec{p}_1 + 1$ is the $\mathbb{Z}$-valued perversity such that $(\vec{p}_1 + 1)(Z) = \vec{p}_1(Z) + 1$ for all singular Z. Then also $\mathscr{P}^*_{X,D\vec{p},(\mathcal{D}_{U_1}\mathcal{E})[-n]} = \mathcal{P}^*_{X,\bar{q},(\mathcal{D}_{U_1}\mathcal{E})[-n]}$, where $\bar{q}$ is the $\mathbb{Z}$-valued perversity such that $(\vec{p}_1+1)(Z) + \bar{q}(Z) = \vec{p}_1(Z) + 1 + \bar{q}(Z) = \operatorname{codim}(Z) - 1$. With these assumptions Theorem 4.15 reduces to the Superduality Theorem of Cappell and Shaneson [4]. Note that in order to have $\mathscr{P}^*_{X,\vec{p},\mathcal{E}} = \mathcal{P}^*_{X,\vec{p}_1+1,\mathcal{E}}$ it is in fact sufficient to require only $I^{\bar{p}}H_{k-2-\bar{p}(Z)}(L_x;\mathcal{E})$ to be torsion.

Corollary 4.17. *Let X be an n-dimensional stratified pseudomanifold, and let $\mathcal{E}$ be a locally constant sheaf of finitely generated free R-modules on U_1 for a principal ideal domain R. Let $T\mathbb{H}^*$ and $F\mathbb{H}^*$ denote, respectively, the R-torsion submodule and R-torsion-free quotient module of $\mathbb{H}^*$, and let $Q(R)$ denote the field of fractions of R.*

Suppose $\operatorname{Ext}(\mathbb{H}^{n-i+1}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}}), R)$ *is a torsion R-module (for example, if $\mathbb{H}^{n-i+1}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}})$ is finitely generated). Then*

$$F\mathbb{H}^i(X;\mathscr{P}^*_{X,D\vec{p},\mathcal{D}_{U_1}\mathcal{E}}) \cong \operatorname{Hom}(\mathbb{H}^{n-i}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}}), R) \cong \operatorname{Hom}(F\mathbb{H}^{n-i}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}}), R)$$

and

$$T\mathbb{H}^i(X;\mathscr{P}^*_{X,D\vec{p},\mathcal{D}_{U_1}\mathcal{E}}) \cong \operatorname{Ext}(\mathbb{H}^{n-i+1}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}}), R) \cong \operatorname{Hom}(T\mathbb{H}^{n-i+1}_c(X;\mathscr{P}^*_{X,\vec{p},\mathcal{E}}), Q(R)/R).$$

In particular, if X is compact and orientable,

$$F\mathbb{H}^i(X;\mathscr{P}^*_{X,D\vec{p},R}) \cong \operatorname{Hom}(F\mathbb{H}^{n-i}_c(X;\mathscr{P}^*_{X,\vec{p},R}), R)$$

and

$$T\mathbb{H}^i(X;\mathscr{P}^*_{X,D\vec{p},R}) \cong \operatorname{Hom}(T\mathbb{H}^{n-i+1}_c(X;\mathscr{P}^*_{X,\vec{p},R}), Q(R)/R).$$

Proof. These statements follow directly from the theorem, using the universal coefficient short exact sequence for Verdier duality and basic homological algebra. See [3, 16, 7] for more details. □

Proof of Theorem 4.15. Note that since $\mathcal{E}$ is a local system of finitely-generated free R-modules, $(\mathcal{D}_{U_1}\mathcal{E})[-n]$ is also a locally constant sheaf of finitely-generated free R-modules by [3, Section V.7.10]. Thus, as in [15, 3], it suffices to verify that $(\mathcal{D}_X\mathscr{P}^*_{X,\vec{p},\mathcal{E}})[-n]$ satisfies the axioms for $\mathscr{P}^*_{X,D\vec{p},(\mathcal{D}_{U_1}\mathcal{E})[-n]}$. However, we do not have available the reformulation into a version of the Goresky-MacPherson axioms Ax2, so our proof will have to proceed a bit differently from those in [15, 3]; instead we emulate the proof of [4, Theorem 3.2] and utilize the axioms TAx1'.

Constructibility. By [3, Corollary V.8.7], $\mathcal{D}_X\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ is $\mathfrak{X}$-clc and $\mathfrak{X}$-cc because $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ is by Theorem 4.12.

Axiom TAx1'(2). Let $j : U_1 \hookrightarrow X$ be the inclusion. Since U_1 is open in X, $j^! = j^*$, and thus if $\mathbb{D}^*_X$ is the Verdier dualizing sheaf on X, $j^*\mathbb{D}^*_X = j^!\mathbb{D}^*_X = \mathbb{D}^*_{U_1}$. Now for any sheaf complex, $\mathcal{D}_X\mathcal{S}^* \cong RHom(\mathcal{S}^*, \mathbb{D}^*_X) \cong Hom(\mathcal{S}^*, \mathbb{D}^*_X)$, since $\mathbb{D}^*$ is injective in the construction of [3]. Furthermore, it is clear from the construction of the sheaf functor Hom that $Hom(\mathcal{S}^*, \mathbb{D}^*_X)|_{U_1} \cong Hom(\mathcal{S}^*|_{U_1}, \mathbb{D}^*_{U_1}) \cong \mathcal{D}_{U_1}(\mathcal{S}^*|_{U_1})$. Thus since $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}|_{U_1} \cong \mathcal{E}$, it follows that $(\mathcal{D}_X \mathscr{P}^*_{X,\vec{p},\mathcal{E}})[-n]$ is quasi-isomorphic to $(\mathcal{D}_{U_1}\mathcal{E})[-n]$ on U_1. This demonstrates axiom TAx1'(2).

Axiom TAx1'(3). Next, let $x \in Z \subset X_{n-k}$, $k > 0$. Let $j : Z \to U_k \cup Z$, $\ell_x : x \hookrightarrow Z$, and $f_x : x \hookrightarrow X$ all be the inclusion maps, and let us abbreviate $\mathscr{P}^*_{X,\vec{p},\mathcal{E}}$ as simply $\mathscr{P}^*$.

Then

$$\begin{aligned} H^i((\mathcal{D}_X \mathscr{P}^*[-n])_x) &\cong H^{i-n}(f_x^* D_X \mathscr{P}^*) \\ &\cong H^{i-n}(\mathcal{D}_x(f_x^! \mathscr{P}^*)) && \text{see footnote}^9 \\ &\cong \mathrm{Hom}(H^{n-i}(f_x^! \mathscr{P}^*), R) \oplus \mathrm{Ext}(H^{n-i+1}(f_x^! \mathscr{P}^*), R) && \text{by [3, Section V.7.7].} \end{aligned}$$

Since $\mathscr{P}^*$ satisfies TAx1'$(X, \vec{p}, \mathcal{E})$, we know $H^i(f_x^! \mathscr{P}^*) = 0$ for $i \le p_1(Z) + n - k + 1$ and $H^{p_1(Z)+n-k+2}(f_x^! \mathscr{P}^*) \cong H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)/T^{\vec{p}_2(Z)} H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)$. Thus $H^i((\mathcal{D}_X \mathscr{P}^*[-n])_x) = 0$ for $n - i + 1 \le p_1(Z) + n - k + 1$, i.e. for $i \ge k - p_1(Z) = D\vec{p}_1(Z) + 2$. Furthermore,

$$\begin{aligned} H^{D\vec{p}_1(Z)+1}((\mathcal{D}_X \mathscr{P}^*[-n])_x) &\cong \mathrm{Hom}(H^{n-D\vec{p}_1(Z)-1}(f_x^! \mathscr{P}^*), R) \oplus \mathrm{Ext}(H^{n-\vec{D}p_1(Z)}(f_x^! \mathscr{P}^*), R) \\ &= \mathrm{Hom}(H^{\vec{p}_1(Z)+n-k+1}(f_x^! \mathscr{P}^*), R) \oplus \mathrm{Ext}(H^{\vec{p}_1(Z)+n-k+2}(f_x^! \mathscr{P}^*), R) \\ &= \mathrm{Ext}(H^{\vec{p}_1(Z)+n-k+2}(f_x^! \mathscr{P}^*), R) \\ &= \mathrm{Ext}(H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)/T^{\vec{p}_2(Z)} H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x), R) \end{aligned}$$

Since $H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)$ is finitely generated by the constructibility assumptions and since $H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)/T^{\vec{p}_2(Z)} H^{\vec{p}_1(Z)+1}((Ri_{k*}\mathscr{P}^*_k)_x)$ has no $S(\vec{p}_2(Z))$-torsion, $H^{D\vec{p}_1(Z)+1}((\mathcal{D}_X \mathscr{P}^*[-n])_x)$ must then consist entirely of $S(D\vec{p}_2(Z))$-torsion.

This demonstrates TAx1'(3).

Axiom TAx1'(4). Next, consider

$$\begin{aligned} H^i(f_x^! \mathcal{D}_X \mathscr{P}^*[-n]) &\cong H^{i-n}(f_x^! \mathcal{D}_X \mathscr{P}^*) \\ &\cong H^{i-n}(\mathcal{D}_x \mathscr{P}^*_x) && \text{by [3, Proposition V.8.2]} \\ &\cong \mathrm{Hom}(H^{n-i}(\mathscr{P}^*_x), R) \oplus \mathrm{Ext}(H^{n-i+1}(\mathscr{P}^*_x), R) && \text{by [3, Section V.7.7].} \end{aligned}$$

Since $\mathscr{P}^*$ satisfies TAx1'$(X, \vec{p}, \mathcal{E})$, we know that $H^i(\mathscr{P}^*_x) = 0$ for $i > \vec{p}_1(Z) + 1$ and $H^{\vec{p}_1(Z)+1}(\mathscr{P}_x)$ is $S(\vec{p}_2(Z))$-torsion. This immediately implies $H^i(f_x^! \mathcal{D}_X \mathscr{P}^*[-n]) = 0$ if $n - i > p_1(Z) + 1$, i.e. if $i \le n - \vec{p}_1(Z) - 2 = D\vec{p}_1(Z) + n - k$. Furthermore, if $i = D\vec{p}_1(Z) + n - k + 1$, then $n - i = \vec{p}_1(Z) + 1$, and we still have $n - i + 1 > p_1(Z) + 1$, so $H^{D\vec{p}_1(Z)+n-k+1}(f_x^! \mathcal{D}_X \mathscr{P}^*[-n]) \cong \mathrm{Hom}(H^{\vec{p}_1(Z)+1}(\mathscr{P}^*_x), R)$. But $H^{\vec{p}_1(Z)+1}(\mathscr{P}^*_x)$ is torsion by the axioms for $\mathscr{P}$, so also $H^{D\vec{p}_1(Z)+n-k+1}(f_x^! \mathcal{D}_X \mathscr{P}^*[-n])$ vanishes.

It remains to show that

$$\begin{aligned} &H^{D\vec{p}_1(Z)+n-k+2}(f_x^! \mathcal{D}_X \mathscr{P}^*[-n]) \\ &\qquad \cong H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X \mathscr{P}^*[-n])_x)/T^{D\vec{p}_2(Z)} H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X \mathscr{P}^*[-n])_x). \end{aligned}$$

[9]This is well-known. For an official proof, we can use that $j^! \mathcal{D}\mathcal{S}^* \cong \mathcal{D} j^* \mathcal{S}^*$ for a closed inclusion j by [3, Proposition V.8.2] and that $\mathcal{D}\mathcal{D}\mathcal{S}^* = \mathcal{S}^*$ by [3, Theorem V.8.10] to conclude $\mathcal{D}_x(f_x^! \mathscr{P}^*) \cong \mathcal{D}_x(f_x^! \mathcal{D}_X \mathcal{D}_X \mathscr{P}^*) \cong \mathcal{D}_x \mathcal{D}_x f_x^* \mathcal{D}_X \mathscr{P}^* \cong f_x^* \mathcal{D}_X \mathscr{P}^*$ in the derived category $D^b(X)$.

As all modules are finitely-generated by the constructibility assumptions, by Lemma 2.1 it suffices to show that $H^{D\vec{p}_1(Z)+n-k+2}(f_x^!\mathcal{D}_X\mathscr{P}^*[-n])$ has the same rank as $H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X\mathscr{P}^*[-n])_x)$ and that the torsion subgroup of $H^{D\vec{p}_1(Z)+n-k+2}(f_x^!\mathcal{D}_X\mathscr{P}^*[-n])$ is equal to $T^{\vec{p}_2(Z)}H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X\mathscr{P}^*[-n])_x)$.

From our formula above,

$$H^{D\vec{p}_1(Z)+n-k+2}(f_x^!\mathcal{D}_X\mathscr{P}^*[-n]) \cong \operatorname{Hom}(H^{-D\vec{p}_1(Z)+k-2}(\mathscr{P}_x^*), R) \oplus \operatorname{Ext}(H^{-D\vec{p}_1(Z)+k-1}(\mathscr{P}_x^*), R).$$

Clearly the first summand determines the rank, while the second summand determines the torsion. Additionally, since $H^{-D\vec{p}_1(Z)+k-1}(\mathscr{P}_x^*) = H^{\vec{p}_1(Z)+1}(\mathscr{P}_x^*)$, which is $S(\vec{p}_2(Z))$-torsion by the axioms for $\mathscr{P}_x^*$, so all torsion of $H^{D\vec{p}_1(Z)+n-k+2}(f_x^!\mathcal{D}_X\mathscr{P}^*[-n])$ is automatically $S(\vec{p}_2(Z))$-torsion.

Let us first consider the rank. By axiom TAx1'$(X, \vec{p}, \mathcal{E})$,

$$\begin{aligned} H^{-D\vec{p}_1(Z)+k-2}(\mathscr{P}_x^*) &= H^{\vec{p}_1(Z)}(\mathscr{P}_x^*) \\ &\cong H^{\vec{p}_1(Z)}((Ri_{k*}\mathscr{P}_k^*)_x) \\ &\cong \varinjlim_{x\in U} \mathbb{H}^{\vec{p}_1(Z)}(U; Ri_{k*}\mathscr{P}_k^*) \\ &\cong \varinjlim_{x\in U} \mathbb{H}^{\vec{p}_1(Z)}(U-Z; \mathscr{P}_k^*). \end{aligned} \tag{3}$$

We may assume that the U are all distinguished neighborhoods of x, in which case $U - X \cong \mathbb{R}^{n-k+1} \times L$, where L is a compact link, and, by [3, Lemma V.3.8.b],

$$\mathbb{H}^{\vec{p}_1(Z)}(U-Z; \mathscr{P}^*) \cong \mathbb{H}^{\vec{p}_1(Z)}(L; \mathscr{P}^*) \cong \mathbb{H}_c^{p_1(Z)-k+1+n}(U-Z; \mathscr{P}^*).$$

Furthermore, the direct systems are constant over distinguished neighborhoods of x, and $\mathbb{H}^{\vec{p}_1(Z)}(L; \mathscr{P}^*)$ is finitely generated as L is compact and $\mathscr{P}^*$ is $\mathfrak{X}$-cc by Theorem 4.12, so therefore $\mathscr{P}^*|_L$ is clearly $\mathfrak{X}$-cc and thus cc by [3, Proposition 3.10.e].

On the other hand,

$$\begin{aligned} H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X\mathscr{P}_k^*[-n])_x) &\cong \varinjlim_{x\in U} \mathbb{H}^{D\vec{p}_1(Z)+1}(U; Ri_{k*}\mathcal{D}_X\mathscr{P}_k^*[-n]) \\ &\cong \varinjlim_{x\in U} \mathbb{H}^{D\vec{p}_1(Z)+1-n}(U-Z; \mathcal{D}\mathscr{P}^*) \\ &\cong \varinjlim_{x\in U} (\operatorname{Hom}(\mathbb{H}_c^{-D\vec{p}_1(Z)-1+n}(U-Z; \mathscr{P}^*), R) \\ &\qquad \oplus \operatorname{Ext}(\mathbb{H}_c^{-D\vec{p}_1(Z)+n}(U-Z; \mathscr{P}^*), R)) \\ &\cong \varinjlim_{x\in U} (\operatorname{Hom}(\mathbb{H}_c^{p_1(Z)-k+1+n}(U-Z; \mathscr{P}^*), R) \\ &\qquad \oplus \operatorname{Ext}(\mathbb{H}_c^{p_1(Z)-k+2+n}(U-Z; \mathscr{P}^*), R)), \end{aligned} \tag{4}$$

where again the direct sequence is constant.

So the rank of $H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X\mathscr{P}_k^*[-n])_x)$ is the rank of $\operatorname{Hom}(\mathbb{H}_c^{p_1(Z)-k+1+n}(U-Z; \mathscr{P}^*), R)$, which is the rank of $\mathbb{H}_c^{p_1(Z)-k+1+n}(U-Z; \mathscr{P}^*), R)$, which is the rank of $\mathbb{H}^{\vec{p}_1(Z)}(U-Z; \mathscr{P}^*)$, which is the rank of $H^{-D\vec{p}_1(Z)+k-2}(\mathscr{P}_x^*)$.

Similarly, the torsion submodule of $H^{D\vec{p}_1(Z)+n-k+2}(f_x^! \mathcal{D}_X \mathscr{P}^*[-n])$ is isomorphic to $\mathrm{Ext}(H^{-D\vec{p}_1(Z)+k-1}(\mathscr{P}^*_x), R)$ (see Section 2), which is isomorphic to the torsion submodule of $H^{-D\vec{p}_1(Z)+k-1}(\mathscr{P}^*_x)$. By a computation equivalent to (3), this is isomorphic to the torsion of $\varinjlim_{x\in U} \mathbb{H}^{\vec{p}_1(Z)+1}(U-Z;\mathscr{P}^*_k)$, which, since the directed set is constant over distinguished neighborhoods, is isomorphic to the torsion of $\mathbb{H}^{\vec{p}_1(Z)+1}(U-Z;\mathscr{P}^*_k)$, which is isomorphic to the torsion of $\mathbb{H}_c^{p_1(Z)-k+2+n}(U-Z;\mathscr{P}^*)$. But now using (4) again, this is precisely the torsion of $H^{D\vec{p}_1(Z)+1}((Ri_{k*}\mathcal{D}_X\mathscr{P}^*_k[-n])_x)$. As already noted, the torsion of $H^{D\vec{p}_1(Z)+n-k+2}(f_x^! \mathcal{D}_X \mathscr{P}^*[-n])$ must be all $S(\vec{p}_2(Z))$-torsion, so this suffices.

This verifies Axiom TAx1'(4).

Axiom TAx1'(1). Lastly, we need to demonstrate that $H^i((\mathcal{D}_X\mathscr{P}^*[-n])_x) = 0$ for $i<0$, and hence complete axiom TAx1'(1). This requires a bit of an induction argument, so we prove it next as a separate lemma. $\square$

Lemma 4.18. *For any $x \in X$, $H^i((\mathcal{D}_X\mathscr{P}^*[-n])_x) = 0$ for $i<0$.*

Proof. We have

$$\begin{aligned} H^i((\mathcal{D}_X\mathscr{P}^*[-n])_x) &\cong \varinjlim_{x\in U} \mathbb{H}^{i-n}(U;\mathcal{D}_X\mathscr{P}^*) \\ &\cong \varinjlim_{x\in U} \mathrm{Hom}(\mathbb{H}_c^{n-i}(U;\mathscr{P}^*),R)\oplus \mathrm{Ext}(\mathbb{H}_c^{n-i+1}(U;\mathscr{P}^*),R), \end{aligned}$$

and the systems are essentially constant over distinguished neighborhoods of x by [3, Proposition V.3.10] since $\mathscr{P}^*$ is $\mathfrak{X}$-clc. So it suffices to show $\mathbb{H}^j_c(U;\mathscr{P}^*)=0$ for $j>n$ and for any distinguished neighborhood U of x.

We will perform an induction argument over the depth of X, utilizing Lemma V.9.5 of [3], according to which if Y is a stratified space and there exists an $\ell\in\mathbb{Z}$ such that $H^j(\mathcal{S}^*_y)=0$ for $j>\ell-n+k$ whenever $y\in X_{n-k}$, then $\mathbb{H}^j_c(Y;\mathcal{S}^*)=0$ for $j>\ell$. Thus, taking $\ell=n$, we must show $H^j(\mathcal{S}^*_y)=0$ for $j>k$ for each $y\in X_{n-k}$.

First, we note that since $\mathscr{P}^*|_{U_1}\cong\mathcal{E}$, we have both $H^i(\mathscr{P}^*_x)=0$ for $i>0$ and $x\in U_1$ and, for any Euclidean neighborhood U of x, $\mathbb{H}^i_c(U;\mathcal{E})=0$ for $i>n$ by classical manifold theory.

Now, we assume as induction hypothesis that for all $K\in\mathbb{Z}$, $0\le K<k$, $H^j(\mathscr{P}^*_y)=0$ for $j>K$ whenever $y\in X_{n-K}$ and that $\mathbb{H}^i_c(U;\mathscr{P})=0$ for $i>n$ if U a distinguished neighborhood of such a y. We will show the corresponding facts for points in X_{n-k}.

Let $x\in X_{n-k}$. Then $H^i(\mathscr{P}^*_y)\cong\varinjlim_{x\in U}\mathbb{H}^i(U;\mathscr{P}^*)$, and from the definition of $\mathscr{P}^*$, each $\mathbb{H}^i(U;\mathscr{P}^*)$ is a subgroup of $\mathbb{H}^i(U-Z;\mathscr{P}^*)$. Since $U-Z\cong\mathbb{R}^{n-k-1}\times L$, where L is the $k-1$ dimensional link of Z, $\mathbb{H}^i(U-Z;\mathscr{P}^*)\cong\mathbb{H}^i(L;\mathscr{P}^*|_L)$ by [3, Lemma V.3.8.b]. Recall that if $y\in X_{n-K}\cap L$, then $y\in L_{k-1-K}$, and since L is compact, $\mathbb{H}^j(L;\mathscr{P}^*|_L)=\mathbb{H}^j_c(L;\mathscr{P}^*|_L)$. If $y\in L_{k-1-K}$, then since $y\in X_{n-K}$, we have by induction hypothesis that $H^j(\mathscr{P}^*_y)=0$ for $j>K=(k-1)-(k-1)+K$, and we know this for all $y\in Y$. Therefore, by [3, Lemma V.9.5], $\mathbb{H}^j(L;\mathscr{P}^*|_L)=\mathbb{H}^j_c(L;\mathscr{P}^*|_L)=0$ for $j>k-1$. It follows that $H^i(\mathscr{P}^*_y)=0$ for $i>k-1$, and so also for $i>k$. But then again by [3, Lemma V.9.5] $\mathbb{H}^i_c(U;\mathscr{P}^*)=0$ for $i>n$.

This completes the induction.

$\square$

5 Torsion-tipped truncation and manifold duality

In this section, we provide an interesting example by computing $\mathbb{H}^*(X;\mathscr{P}^*)$, where X is a PL pseudomanifold with just one singular point v and $R=\mathbb{Z}$, and relating these groups to the homology groups of the ∂-manifold obtained by removing a distinguished neighborhood of v. We then use manifold techniques to verify (abstractly) the isomorphisms guaranteed by Corollary 4.17. It would be interesting to have a proof that the isomorphisms of the Corollary are induced by geometric intersection and linking pairings as is the case for intersection homology with field coefficients (see [14, 8]). We leave this question for future research.

Let X be a compact $\mathbb{Z}$-oriented n-dimensional PL stratified pseudomanifold with stratification $X=X^n\supset X^0=\{v\}$, where v is a single point. Then X has the form $X\cong M\cup_{\partial M}\bar{c}(\partial M)$, where M^n is a compact $\mathbb{Z}$-oriented PL manifold with boundary. Let $U=X-v$. Let $\mathcal{O}$ be the constant orientation sheaf with $\mathbb{Z}$ coefficients on U, let $i:U\hookrightarrow X$ be the inclusion, and let $k\in\mathbb{Z}$. Let $\mathscr{P}^*=\mathscr{P}^*_{X,\vec{p},\mathcal{O}}$ be the ts-Deligne sheaf with $\vec{p}_1(\{v\})=k$ and $\vec{p}_2(\{v\})=\wp$ for some $\wp\in\mathbb{P}(\mathfrak{P}(\mathbb{Z}))$. If $k<0$, then $\mathscr{P}^*$ is the extension by 0 of (an injective resolution of) $\mathcal{O}$. If $k\geq 0$, then $\mathscr{P}^*=\mathfrak{t}^{\wp}_{\leq k}Ri_*\mathcal{O}$. If $\wp=\emptyset$, then $\mathscr{P}^*$ would be the classical Deligne sheaf for the perversity $\bar{p}$ with $\bar{p}(\{v\})=k$, and its hypercohomology would be the classical perversity $\bar{p}$ intersection homology.

To simplify notation, we let $\mathbb{H}^i(X;\mathscr{P}^*_{X,\vec{p},\mathcal{O}})$ be denoted by $N^{\vec{p}}H_{n-i}(X)$. We also let $\vec{q}=D\vec{p}$, so that $\vec{q}_1(\{v\})=n-k-2$ and $\vec{q}_2(\{v\})$ is the complement of $\wp$ in $\mathfrak{P}(\mathbb{Z})$. Further, note that $\mathcal{D}\mathcal{O}\cong\mathcal{O}$, so $(\mathcal{D}_X\mathscr{P}^*_{X,\vec{p},\mathcal{D}\mathcal{O}})[-n]=\mathscr{P}^*_{X,\vec{q},\mathcal{O}}$, with hypercohomology groups $N^{\vec{q}}H_{n-i}(X)$.

We begin by computing $N^{\vec{p}}H_{n-i}(X)$ as best as possible in terms of the homology groups of M. For comparison, it is worth recalling that the standard computation involving the cone formula and the Mayer-Vietoris sequence gives[10]

$$I^{\bar{p}}H_{n-i}(X)\cong\begin{cases}H_{n-i}(M), & i>k+1,\\ \operatorname{im}(H_{n-i}(M)\to H_{n-i}(M,\partial M)), & i=k+1,\\ H_{n-i}(M,\partial M), & i<k+1.\end{cases}$$

As noted above, this will then also be the computation for $N^{\vec{p}}H_{n-i}(X)$ when $\vec{p}_2(\{v\})=\emptyset$.

Recall that X is compact by assumption, so $N^{\vec{p}}H_{n-i}(X)=\mathbb{H}^i(X;\mathscr{P}^*)=\mathbb{H}^i_c(X;\mathscr{P}^*)$. Therefore, to study $N^{\vec{p}}H_{n-i}(X)$, we can use that the adjunction triangle yields a long exact sequence [5, Remark 2.4.5.ii]

$$\to\mathbb{H}^i_c(U;\mathscr{P}^*)\to\mathbb{H}^i_c(X;\mathscr{P}^*)\to\mathbb{H}^i_c(v;\mathscr{P}^*)\to.$$

We know the restriction of $\mathscr{P}^*$ to U is quasi-isomorphic to $\mathcal{O}$, so

$$\mathbb{H}^i_c(U;\mathscr{P}^*)\cong\mathbb{H}^i_c(U;\mathcal{O})\cong H^c_{n-i}(U)\cong H^c_{n-i}(M).$$

[10]For $k<0$ or $k>\dim(M)-2$, this computation continues to hold if we interpret $I^{\bar{p}}H_i(X)$ in the sense of intersection homology with general perversities treated in [9, 10]. In those papers, the notation $I^{\bar{p}}H_*(X;\mathbb{Z}_0)$ was used, but here we conform to the notation of [13].

Furthermore, applying Lemma 3.3,

$$\begin{aligned}\mathbb{H}^i_c(v;\mathscr{P}^*) &\cong \mathbb{H}^i(v;\mathscr{P}^*)\\ &\cong \begin{cases} 0, & i>k+1,\\ T^{\wp}H^{k+1}((Ri_*\mathcal{O})_v), & i=k+1,\\ H^i((Ri_*\mathcal{O})_v), & i\le k.\end{cases}\end{aligned}$$

But

$$\begin{aligned}H^i((Ri_*\mathcal{O})_v) &\cong \varinjlim_{v\in U}\mathbb{H}^i(U;Ri_*\mathcal{O})\\ &\cong \varinjlim_{v\in U}\mathbb{H}^i(U-v;\mathcal{O})\\ &\cong \varinjlim_{v\in U}H^{\infty}_{n-i}(U-v;\mathcal{O}).\end{aligned}$$

Restricting to a cofinal sequence of conical neighborhoods, this becomes simply $H^{\infty}_{n-i}(\partial M\times(0,1);\mathcal{O})\cong H_{n-i-1}(\partial M)$. Similarly, $T^{\wp}H^{k+1}((Ri_*\mathcal{O})_v)\cong T^{\wp}H_{n-k-2}(\partial M)$.

So if we denote $\mathbb{H}^i_c(X;\mathscr{P}^*)$ by $N^{\vec{p}}H_{n-i}(X)$, our exact sequences look like

$$\to H_{n-i}(M)\to N^{\vec{p}}H_{n-i}(X)\to H_{n-i-1}(\partial M)\to$$

for $i\le k$, like

$$\to H_{n-i}(M)\to N^{\vec{p}}H_{n-i}(X)\to 0\to$$

for $i>k+1$, and at the transition, we have

$$\begin{aligned}&\to H_{n-k-1}(\partial M)\to H_{n-k-1}(M)\to N^{\vec{p}}H_{n-k-1}(X)\\ &\qquad\to T^{\wp}H_{n-k-2}(\partial M)\to H_{n-k-2}(M)\to N^{\vec{p}}H_{n-k-2}(X)\to 0.\end{aligned}\tag{5}$$

It is therefore immediate that $N^{\vec{p}}H_i(X)\cong H_i(M)$ for $i\le n-k-3$. Furthermore, the inclusion $\mathscr{P}^*\hookrightarrow Ri_*\mathcal{O}$ induces a map between the corresponding long exact adjunction sequences. The sequence for $Ri_*\mathcal{O}$ is simply the sheaf-theoretic long exact (compactly supported) cohomology sequence of the pair $(M,\partial M)$, and so it follows from the five lemma that $N^{\vec{p}}H_i(X)\cong H_i(M,\partial M)$ for $i\ge n-k$. It also follows from this that all maps in the sequence for $\mathscr{P}^*$ are the evident ones. For $i=n-k-2,n-k-1$, we see that $N^{\vec{p}}H_{n-k-2}(X)\cong \operatorname{cok}(T^{\wp}H_{n-k-2}(\partial M)\to H_{n-k-2}(M))$. The module $N^{\vec{p}}H_{n-k-1}(X)$ is a bit more complicated, but we can nonetheless compute it using the following lemma.

Lemma 5.1. *Let $\partial_*: H_{n-k-1}(M,\partial M)\to H_{n-k-2}(\partial M)$ be the boundary map of the exact sequence, and let $\mathfrak{q}^{\wp}$ be the quotient $\mathfrak{q}^{\wp}: H_{n-k-2}(\partial M)\to H_{n-k-2}(\partial M)/T^{\wp}H_{n-k-2}(\partial M)$. Then $N^{\vec{p}}H_{n-k-1}(X)\cong\ker(\mathfrak{q}^{\wp}\partial_*)$.*

Proof. Consider the following diagram of exact sequences induced by the inclusion $\mathscr{P}^*\hookrightarrow Ri_*\mathcal{O}$:

$$
\begin{array}{ccccccccc}
H_{n-k-1}(\partial M) & \longrightarrow & H_{n-k-1}(M) & \xrightarrow{\tilde{j}} & N^{\vec{p}}H_{n-k-1}(X) & \xrightarrow{f} & T^{\wp}H_{n-k-2}(\partial M) & \longrightarrow & H_{n-k-2}(M) \\
= \big\downarrow & & = \big\downarrow & & g \big\downarrow & & h \big\downarrow & & = \big\downarrow \\
H_{n-k-1}(\partial M) & \xrightarrow{i} & H_{n-k-1}(M) & \xrightarrow{j} & H_{n-k-1}(M,\partial M) & \xrightarrow{\partial_*} & H_{n-k-2}(\partial M) & \longrightarrow & H_{n-k-2}(M)
\end{array}
\tag{6}
$$

The map h here is the inclusion of the $S(\wp)$-torsion subgroup of $H_{n-k-2}(\partial M)$.

From the diagram, if $x \in N^{\vec{p}}H_{n-k-1}(X)$, then $g(x) \in H_{n-k-1}(M, \partial M)$ maps to a $S(\wp)$-torsion element in $H_{n-k-2}(\partial M)$ under the boundary map. Thus $N^{\vec{p}}H_{n-k-1}(X)$ must map into $\ker(\mathfrak{q}^{\wp}\partial_*)$.

We now proceed with diagram chases akin to those in the proof of the five lemma.

To see that g maps onto $\ker(\mathfrak{q}^{\wp}\partial_*)$, suppose $u \in \ker(\mathfrak{q}^{\wp}\partial_*)$. Then $\partial_*(u) \in T^{\wp}H_{n-k-2}(\partial M)$, so $\partial_*(u)$ is in the image of h. Since the image of $\partial_*(u)$ in $H_{n-k-2}(M)$ must be 0 (from the long exact sequence on the bottom), it follows that $\partial_*(u)$, as an element of $T^{\wp}H_{n-k-2}(\partial M)$ must be in the image of f. Let $x \in N^{\vec{p}}H_{n-k-1}(X)$ be such that $hf(x) = \partial_* u \in H_{n-k-2}(\partial M)$. Then $\partial_*(g(x)) = \partial_*(u)$ from the diagram, i.e. $\partial_*(g(x) - u) = 0$, so there is a $z \in H_{n-k-1}(M)$ such that $j(z) = g(x) - u$. But $j(z) = g\tilde{j}(z)$, so $g\tilde{j}(z) = g(x) - u$, whence $u = g(x) - g\tilde{j}(z) = g(x - \tilde{j}(z))$. Therefore u is in the image of g and so g maps onto $\ker(\mathfrak{q}^{\wp}\partial_*)$.

For injectivity, suppose $x \in N^{\vec{p}}H_{n-k-1}(X)$ and $g(x) = 0$. Then $\partial_* g(x) = hf(x) = 0$, but h is injective, so $f(x) = 0$ and $x = \tilde{j}(y)$ for some $y \in H_{n-k-1}(M)$. This implies that $j(y) = g\tilde{j}(y) = g(x) = 0$, so $y = i(z)$ for some $z \in H_{n-k-1}(\partial M)$. But then $x = \tilde{j}(y) = \tilde{j}i(z) = 0$, from the short exact sequence. $\square$

So, altogether, we see that

$$
N^{\vec{p}}H_i(X) \cong \begin{cases} H_i(M, \partial M), & i \geq n-k, \\ \ker(H_i(M, \partial M) \xrightarrow{\mathfrak{q}^{\wp}\partial_*} H_{i-1}(\partial M)/T^{\wp}H_{i-1}(\partial M)), & i = n-k-1, \\ \operatorname{cok}(T^{\wp}H_i(\partial M) \to H_i(M)), & i = n-k-2, \\ H_i(M), & i \leq n-k-3. \end{cases} \tag{7}
$$

In particular, $N^{\vec{p}}H_i(X) \cong I^{\vec{p}_1}H_i(X)$ for $i \neq n-k-2, n-k-1$.

For reference, if we replace $\vec{p}$ with its dual $\vec{q}$, we see that similarly

$$
N^{\vec{q}}H_j(X) \cong \begin{cases} H_j(M, \partial M), & j \geq k+2, \\ \ker(H_{k+1}(M, \partial M) \xrightarrow{\mathfrak{q}^{D\wp}\partial_*} H_{k+1}(\partial M)/T^{D\wp}H_{k+1}(\partial M)), & j = k+1, \\ \operatorname{cok}(T^{D\wp}H_j(\partial M) \to H_j(M)), & j = k, \\ H_j(M), & j \leq k-1 \end{cases}
$$

in which case $N^{\vec{q}}H_i(X) \cong I^{\vec{q}_1}H_i(X)$ for $i \neq k, k+1$.

Corollary 4.17 implies that there must be isomorphisms

$$FN^{\vec{p}}H_i(X) \cong \mathrm{Hom}(FI^{\bar{q}}H_{n-i}(X), \mathbb{Z}) \qquad \text{and} \qquad TN^{\vec{p}}H_i(X) \cong \mathrm{Hom}(TI^{\bar{q}}H_{n-i-1}(X), \mathbb{Q}/\mathbb{Z}), \tag{8}$$

where, as before, given an abelian group G, we let TG denote the torsion subgroup of G and $FG = G/TG$.

Next, we would like to see how these isomorphisms (8) relate to the known isomorphisms from Lefschetz duality. For such a simple pseudomanifold, many of the isomorphisms of (8) correspond precisely to the known duality isomorphisms of intersection homology, which themselves can be described in terms of the intersection and torsion linking pairing on the manifold M. However, we will not provide here the technical proof that the pairings induced by the sheaf isomorphism of Theorem 4.15 correspond to the classical intersection and linking pairings; this turns out to be a difficult result. We refer the reader to [12] for a proof that sheaf-theoretic duality is compatible with geometric intersection pairings. The author hopes to provide a similar result concerning linking pairings in the future.

What we will first look at here is the extent to which the isomorphisms of (8) are abstractly reflected by isomorphisms in Lefschetz duality, meaning that we will see that Lefschetz duality guarantees isomorphisms between the same groups though without showing that we obtain the *same* isomorphisms. Then we will observe that some of the dualities of (8) are not so obvious from classical manifold theory. Finally, we will show that some of the less expected abstract isomorphisms really do reflect intersection and linking pairings on manifolds. Altogether then, our results about intersection homology will have led us to formulate and prove some unexpected results concerning manifold theory.

We begin with the following easy observations:

1. We have seen that $N^{\vec{p}}H_i(X) \cong H_i(M)$ for $i \leq n-k-3$, while $N^{\vec{q}}H_j(X) \cong H_i(M, \partial M)$ for $j > k+1$. So for $i \leq n-k-3$, there exist isomorphisms of the form (8) by classical Lefschetz duality.

2. Similarly, we have $N^{\vec{p}}H_i(X) \cong H_i(M, \partial M)$ for $i \geq n-k$, while $N^{\vec{q}}H_i(X) \cong H_i(M)$ for $i \leq k-1$. So, again, there exist isomorphisms of the form (8) by classical Lefschetz duality when $i \geq n-k+1$ and also for the classical Lefschetz torsion pairing when $i = n-k$.

3. When $i = n-k$, the torsion-free part of (8) also follows abstractly from Lefschetz duality, since

$$\begin{aligned} FN^{\vec{q}}H_j(X) &\cong F(\mathrm{cok}(T^{D\wp}H_j(\partial M) \to H_j(M))) \\ &\cong FH_j(M). \end{aligned}$$

4. We have seen that $N^{\vec{p}}H_{n-k-2}(X) \cong \mathrm{cok}(T^{\wp}H_{n-k-2}(\partial M) \to H_{n-k-2}(M))$, and so $FN^{\vec{p}}H_{n-k-2}(X) \cong FH_{n-k-2}(M)$. Once again, $I^{\bar{q}}H_{k+2}(X) \cong H_{k+2}(M, \partial M)$, so there is an isomorphism as in (8) by Lefschetz duality.

By contrast, the remaining isomorphisms

$$\begin{aligned} FN^{\vec{p}}H_{n-k-1}(X) &\cong \mathrm{Hom}(FN^{\vec{q}}H_{k+1}(X), \mathbb{Z}) \\ TN^{\vec{p}}H_{n-k-1}(X) &\cong \mathrm{Hom}(TN^{\vec{q}}H_{k}(X), \mathbb{Q}/\mathbb{Z}) \\ TN^{\vec{p}}H_{n-k-2}(X) &\cong \mathrm{Hom}(TN^{\vec{q}}H_{k+1}(X), \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

are more complex and less evident from the classical manifold point of view, though, by Corollary 4.17, such isomorphisms must exist. We will provide explicit such isomorphisms via intersection and linking forms on M in the special case where $\wp = \mathfrak{P}(\mathbb{Z})$, the set of all primes, and $D\wp = \emptyset$. The same arguments would also handle the case with $\wp$ and $D\wp$ reversed. The more general situation seems to be quite a bit more delicate, and we will not take it up here. We will also shift to working with piecewise-linear (PL) manifolds so that we may make geometric arguments, though the results hold on topological manifolds with a little more work and interpreting the intersection pairing as the Lefschetz dual of the cup product pairing.

Lemma 5.2. *If M is a PL manifold with non-empty boundary and $\vec{p}_2(\{v\}) = \mathfrak{P}(\mathbb{Z})$, the intersection pairing on M induces a nonsingular pairing between $FN^{\vec{p}}H_{n-k-1}(X) \subset FH_{n-k-1}(M, \partial M)$ and $FN^{\vec{q}}H_{k+1}(X) \subset FH_{k+1}(M, \partial M)$.*

Proof. As indicated in the statement of the lemma, we identify $FN^{\vec{p}}H_{n-k-1}(X)$ with $F\ker(\mathfrak{q}^{\wp}\partial_*) \subset H_{n-k-1}(M, \partial M)$ and $FN^{\vec{q}}H_{k+1}(X)$ with $F\ker(\mathfrak{q}^{D\wp}\partial_*) \subset H_{k+1}(M, \partial M)$. As $D\wp = \emptyset$, the latter group is really just $F\ker(\partial_*) \subset H_{k+1}(M, \partial M)$.

We define $f : FN^{\vec{q}}H_{k+1}(X) \to \mathrm{Hom}(FN^{\vec{p}}H_{n-k-1}(X), \mathbb{Z})$ via intersection pairings. Suppose $\xi \in FN^{\vec{q}}H_{k+1}(X)$. Then $\partial_*\xi = 0$, and $\xi = j(x)$ for some $x \in H_{k+1}(M)$ by the long exact sequence in (6). Define the homomorphism $f(\xi)$ so that if $y \in FN^{\vec{p}}H_{n-k-1}(X)$ then $(f(\xi))(y) = x \pitchfork y$, where $\pitchfork$ denotes the Lefschetz duality intersection pairing on M. We first check this is well-defined.

The intersection pairing is trivial on torsion elements, so f is well defined on the torsion free quotients. Next, we show that f is independent of the choice of x. For this, suppose $z \in \ker(H_{k+1}(M) \to H_{k+1}(M, \partial M))$. We will shows that $z \pitchfork y = 0$. So if x' is another preimage of ξ in $H_{k+1}(M)$, then $x - x' \in \ker(H_{k+1}(M) \to H_{k+1}(M, \partial M))$, so $(x - x') \pitchfork y = 0$ and $x \pitchfork y = x' \pitchfork y$. It will follow that f is independent of the choice of x. So let $z \in \ker(H_{k+1}(M) \to H_{k+1}(M, \partial M))$. Then z is represented by a chain in ∂M. Now if $y \in \ker(\mathfrak{q}^{\wp}\partial_*)$, then for some $m \in S(\wp)$, we have $m\partial_* y = 0 \in H_{n-k-2}(\partial M)$, and this implies $m\partial_* y$, which is represented by $m\partial y$, is itself a boundary in ∂M, say[11] $m\partial y = \partial Y$ for some $Y \in C_{n-k-1}(\partial M)$. So $my - Y$ is a cycle in M that also represents my in $H_{k+1}(M, \partial M)$. But then $my - Y$ is homologous to a cycle u in the interior of M. In particular, u and z can be represented by disjoint cycles in M. So, in M, the intersection number of z and u is 0. But the intersection number between z and u represents $z \pitchfork my$ as $my = u \in H_{n-k-1}(M, \partial M)$. So $z \pitchfork my = m(z \pitchfork y) = 0$, and $z \pitchfork y$ must be 0. Thus f is independent of the choice of x.

[11] We will have occasion to abuse notation by sometimes letting the same symbol refer to both a chain and the homology class it represents.

We also observe that $f(x)(y_1+y_2) = f(x)(y_1)+f(x)(y_2)$ by the basic properties of intersection products. To show that f is a homomorphism, we note that if $\xi_1, \xi_2 \in FN^{\vec{q}}H_{k+1}(X)$ and $j(x_1) = \xi_1$, $j(x_2) = \xi_2$, then $j(x_1+x_2) = \xi_1+\xi_2$, and so

$$\begin{aligned} f(\xi_1+\xi_2)(y) &= (\xi_1+\xi_2) \pitchfork y \\ &= \xi_1 \pitchfork y + \xi_2 \pitchfork y \\ &= f(\xi_1)(y) + f(\xi_2)(y). \end{aligned}$$

Altogether, we have now shown that f is a well-defined homomorphism.

Next we show that f is injective. Recall that, by Lefschetz duality, $FH_{k+1}(M) \cong \operatorname{Hom}(FH_{n-k-1}(M,\partial M),\mathbb{Z})$ and $FH_{k+1}(M,\partial M) \cong \operatorname{Hom}(FH_{n-k-1}(M),\mathbb{Z})$ via the intersection pairing. Let $\xi \in FN^{\vec{q}}H_{k+1}(X) \cong F\ker(\partial_*)$ with $\xi \neq 0$. We will show that $f(\xi) \neq 0$, which implies injectivity. The class ξ is represented by a cycle x in M, which also represents an element of $FH_{k+1}(M)$. As $0 \neq \xi \in FH_{k+1}(M,\partial M)$, by Lefschetz duality, there must be a $y \in FH_{n-k-1}(M)$ such that $x \pitchfork y \neq 0$. Furthermore, the intersection number continues to be the same if we think of a chain representing y as instead representing an element of $FH_{n-k-1}(M,\partial M)$, while x can be represented by an element of $H_{k+1}(M)$. Therefore, the class of the chain representing y must be non-zero in $FH_{n-k-1}(M,\partial M)$, and, since it's in the image of $FH_{n-k-1}(M)$, it must be in $\ker(\partial_*)$ and hence in $F\ker(\mathfrak{q}^{\wp}\partial_*)$. Therefore, given a non-zero $\xi \in FN^{\vec{q}}H_{k+1}(X)$, with x a preimage of ξ in $H_{k+1}(M)$, we have found a $y \in FN^{\vec{p}}H_{n-k-1}(X)$ such that $x \pitchfork y \neq 0$. It follows that $f(\xi) \neq 0$, and thus f is injective.

For surjectivity, we note that $\mathfrak{q}^{\wp}\partial_*$ has free image (as $\wp = \mathfrak{P}(\mathbb{Z})$), so the group $\ker(\mathfrak{q}^{\wp}\partial_*) = FN^{\vec{p}}H_{n-k-1}(X)$ is a direct summand of $FH_{n-k-1}(M,\partial M)$. Let y be a generator of $\ker(\mathfrak{q}^{\wp}\partial_*)$, and let $\{y'_j\}$ be a collection of elements of $FH_{n-k-1}(M,\partial M)$ that together with y form a basis. Let $\{y''_\ell\}$ be a collection of elements of $\ker(\mathfrak{q}^{\wp}\partial_*)$ that together with y form a basis. As $\ker(\mathfrak{q}^{\wp}\partial_*) \subset FH_{n-k-1}(M,\partial M)$, ever y''_ℓ must be a linear combination of the $\{y'_j\}$. Now, let $x \in FH_{k+1}(M)$ be the Lefschetz dual of y in the pairing between $FH_{n-k-1}(M,\partial M)$ and $FH_{k+1}(M)$. In other words, let x be the unique element with $x \pitchfork y = 1$, while $x \pitchfork y'_j = 0$ for each of the y'_j. Let ξ be the image of x in $FH_{k+1}(M,\partial M)$; then $\xi \in \ker(\partial_*) = FN^{\vec{q}}H_{k+1}(X)$. We must have $f(\xi)(y) = 1$, while all $f(\xi)(y''_\ell) = 0$. So ξ is a dual to y in the pairing between $FN^{\vec{p}}H_{n-k-1}(X)$ and $FN^{\vec{q}}H_{k+1}(X)$. Since y was an arbitrary generator of $F\ker(\mathfrak{q}^{\wp}\partial_*)$, we see that we can construct a dual basis in $FN^{\vec{q}}H_{k+1}(X)$ to our basis of $FN^{\vec{p}}H_{n-k-1}(X)$, and it follows that f is surjective. $\square$

Lemma 5.3. *If M is a PL manifold with non-empty boundary and $\vec{p}_2(\{v\}) = \mathfrak{P}(\mathbb{Z})$, the linking pairing on M induces a nonsingular pairing between $TN^{\vec{p}}H_{n-k-1}(X)$ and $TN^{\vec{q}}H_k(X)$ and a nonsingular pairing between $TN^{\vec{p}}H_{n-k-2}(X)$ and $TN^{\vec{q}}H_{k+1}(X)$.*

Proof. Given that $\vec{p}_2(\{v\}) = \mathfrak{P}(\mathbb{Z})$, the pairing involving $TN^{\vec{p}}H_{n-k-1}(X)$ actually reduces to the standard torsion Lefschetz pairing. To see this, we first notice that $TN^{\vec{p}}H_{n-k-1}(X)$ must be the torsion elements of $\ker(H_{n-k-1}(M,\partial M) \xrightarrow{\mathfrak{q}^{\wp}\partial_*} H_{n-k-2}(\partial M)/T^{\wp}H_{n-k-2}(\partial M))$. But this is precisely[12] $TH_{n-k-1}(M,\partial M)$, itself, as any torsion element of $H_{n-k-1}(M,\partial M)$

[12] We can see here one reason that a general choice of $\wp$ would make things much more complicated: $TN^{\vec{p}}H_{n-k-1}(X)$ would have to contain of all the $S(\wp)$-torsion of $TH_{n-k-1}(M,\partial M)$, but also any other torsion elements that happen to be in $\ker \partial_*$.

that is not in $\ker \partial_*$ has its image in $TH_{n-k-2}(M)$, and so dies under $\mathfrak{q}^{\wp} = \mathfrak{q}^{\mathfrak{P}(\mathbb{Z})}$. Thus $TN^{\vec{p}}H_{n-k-1}(X) \cong TH_{n-k-1}(M, \partial M) \cong \mathrm{Hom}(TH_k(M), \mathbb{Q}/\mathbb{Z})$, via the linking pairing. But now we notice that $I^{\vec{q}}H_k(M) \cong \mathrm{cok}(T^{D\wp}H_k(\partial M) \to H_k(M)) = H_k(M)$, as $D\wp$ is empty and thus $T^{D\wp}H_k(\partial M) = 0$. So

$$TN^{\vec{p}}H_{n-k-1}(X) \cong \mathrm{Hom}(TI^{\vec{q}}H_k(M), \mathbb{Q}/\mathbb{Z}),$$

via the classical linking duality.

Now, we consider $TN^{\vec{p}}H_{n-k-2}(X)$. By (7),

$$N^{\vec{p}}H_{n-k-2}(X) \cong \mathrm{cok}(T^{\wp}H_{n-k-2}(\partial M) \to H_{n-k-2}(M)) = \mathrm{cok}(TH_{n-k-2}(\partial M) \to H_{n-k-2}(M)).$$

So if we let $U = \mathrm{im}(TH_{n-k-2}(\partial M) \to TH_{n-k-2}(M))$; then $TN^{\vec{p}}H_{n-k-2}(X) \cong TH_{n-k-2}(M)/U$. Meanwhile

$$\begin{aligned} N^{\vec{q}}H_{k+1}(X) &\cong \ker(H_{k+1}(M, \partial M) \xrightarrow{\mathfrak{q}^{D\wp}\partial_*} H_{k+1}(\partial M)/T^{D\wp}H_{k+1}(\partial M)) \\ &\cong \ker(H_{k+1}(M, \partial M) \xrightarrow{\partial_*} H_{k+1}(\partial M)) \\ &\cong \mathrm{im}(H_{k+1}(M) \to H_{k+1}(M, \partial M)), \end{aligned}$$

since $D\wp = \emptyset$. For brevity, let $W = \mathrm{im}(H_{k+1}(M) \to H_{k+1}(M, \partial M)) \cong N^{\vec{q}}H_{k+1}(X)$, and let $\odot : TH_{n-k-2}(M) \otimes TH_{k+1}(M, \partial M) \to \mathbb{Q}/\mathbb{Z}$ denote the linking pairing operation[13]. Define $f : TN^{\vec{p}}H_{n-k-2}(X) \to \mathrm{Hom}(TW, \mathbb{Q}/\mathbb{Z})$ by $f(x)(y) = x \odot y$. We must first show that this is well-define by showing that $x \odot y = 0$ if $x \in U$. But in this case x is represented by a cycle in ∂M and if $mx = 0 \in TH_{n-k-2}(\partial M)$, $m \neq 0$, then $mx = \partial z$ for some chain z in ∂M. But, by definition, y is represented by a cycle in M, which we can assume is supported in the interior of M. Thus $z \pitchfork y = 0$, so $x \odot y = 0$.

Consider the inclusion $TW \hookrightarrow TH_{k+1}(M, \partial M)$. By classical manifold linking duality, the linking pairing induces an isomorphism $TH_{n-k-2}(M) \to \mathrm{Hom}(TH_{k+1}(M, \partial M), \mathbb{Q}/\mathbb{Z})$. Since TW is a subgroup of $TH_{k+1}(M, \partial M)$ and $\mathbb{Q}/\mathbb{Z}$ is an injective group, we have a surjection $\mathrm{Hom}(TH_{k+1}(M, \partial M), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(TW, \mathbb{Q}/\mathbb{Z})$ induced by restriction. The composition $g : TH_{n-k-2}(M) \to \mathrm{Hom}(TW, \mathbb{Q}/\mathbb{Z})$ induces f, which we therefore see is onto.

Next, since we already know $U \subset \ker g$, to show that f is injective, it now suffices to show $\ker g \subset U$. By counting,

$$|TH_{n-k-2}(M)| = |\ker g| \cdot |\mathrm{im} g| = |\ker g| \cdot |\mathrm{Hom}(TW, \mathbb{Q}/\mathbb{Z})| = |\ker g| \cdot |TW|.$$

Consider the linking duality isomorphism $TH_{k+1}(M, \partial M) \to \mathrm{Hom}(TH_{n-k-2}(M), \mathbb{Q}/\mathbb{Z})$. Since $U \subset TH_{n-k-2}(M)$ and $\mathbb{Q}/\mathbb{Z}$ is an injective group, the map $\mathrm{Hom}(TH_{n-k-2}(M), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(U, \mathbb{Q}/\mathbb{Z})$ is surjective, and thus we have a composite surjection $h : TH_{k+1}(M, \partial M) \to \mathrm{Hom}(U, \mathbb{Q}/\mathbb{Z})$. So

$$|TH_{k+1}(M, \partial M)| = |\ker h| \cdot |\mathrm{im} h| = |\ker h| \cdot |\mathrm{Hom}(U, \mathbb{Q}/\mathbb{Z})| = |\ker h| \cdot |U|.$$

[13] Recall that the linking number can be described geometrically as follows: if x, y are cycles in general position with $mx = \partial z$ and $ny = \partial u$, $m, n \neq 0$, then $x \odot y = \frac{z \pitchfork y}{m} = \frac{x \pitchfork u}{n} \in \mathbb{Q}/\mathbb{Z}$, where now $\pitchfork$ denotes the intersection number on chains in general position. A derivation of this formula in the dual cohomological setting can be found in [6].

We have already seen that U and TW are orthogonal under the linking pairing, thus h induces a surjective homomorphism $TH_{k+1}(M,\partial M)/TW \twoheadrightarrow \mathrm{Hom}(U,\mathbb{Q}/\mathbb{Z})$. In particular, $TW \subset \ker h$. We will see that in fact $\ker(h) \subset TW$, so $\ker(h) = TW$. Therefore,

$$\begin{aligned}\ker(g) &= |TH_{n-k-2}(M)| \div |TW| \\ &= |TH_{k+1}(M,\partial M)| \div |TW| \\ &= |\ker(h)| \cdot |U| \div |TW| \\ &= |TW| \cdot |U| \div |TW| \\ &= |U|,\end{aligned}$$

which implies $\ker g = U$.

To prove the claim that $\ker h \subset TW$, suppose $x \in TH_{k+1}(M,\partial M)$ and $x \notin W$. Then $\partial x \neq 0 \in TH_k(\partial M)$. However, since x is a torsion element, there exists a $z \in C_{k+2}(M)$ such that $\partial z = mx + z'$, where $m \neq 0$ and z' is a chain in ∂M. Then $m\partial x = -\partial z' \in C_k(\partial M)$. Now since $TH_k(\partial M) \cong \mathrm{Hom}(TH_{n-k-2}(\partial M),\mathbb{Q}/\mathbb{Z})$ by the linking pairing $\odot_{\partial M}$ in ∂M, there is a $y \in TH_{n-k-2}(\partial M)$ such that $\partial x \odot_{\partial M} y = \frac{-1}{m} z' \pitchfork_{\partial M} y \neq 0$ (see e.g. [11, Appendix]). But $z' \pitchfork_{\partial M} y = \pm z \pitchfork_M y$, where the subscript indicates the space in which we are computing the intersection number, after moving chains into general position (which does not alter homology classes). Therefore $\partial x \odot_{\partial M} y = \pm\frac{1}{m} z \pitchfork_M y$. But now thinking of y as representing an element of U and of z as a chain rel ∂M, in which case $\partial z = mx$, we have $\frac{1}{m} z \pitchfork_M y = x \odot_M y$. As this linking number is not 0, we have shown that if $x \notin TW$, then $h(x) \neq 0$. Thus $\ker h \subset TW$.

□

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