ON THE RELATIONSHIP BETWEEN DIFFERENTIABLE MANIFOLDS AND COMBINATORIAL MANIFOLDS.[1]

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1. Introduction. J. H. C. Whitehead has shown in [11] that every compact differentiable manifold can be assigned a combinatorial structure which is essentially unique. (Precise statements of Whitehead's results are given in §3.) We will see, on the other hand, that a given combinatorial manifold may either (a) have no differentiable structure at all which is compatible with its given combinatorial structure; or (b) have several, essentially distinct, differentiable structures which are compatible with it. (Theorems 2 and 1 of §4. For further discussion see §5.) Assertion (a) is due to R. Thom [10] and is based on his construction of Pontrjagin classes for triangulated manifolds.

An appendix to this paper attempts to set a foundation for the study of differentiable manifolds with boundary. A second appendix is concerned with Whitehead's C^r-triangulations.

2. Definitions. We remark first that the word "differentiable" will be used to mean "differentiable of class C^∞"; and that the word "manifold" will mean "manifold with boundary". Our manifolds are required to be separable (i.e. have a countable basis); but are not necessarily connected.

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1. This paper is a continuation of the author's paper [6]. By a combinatorial manifold we mean a manifold in the sense of Newman [7] and Alexander [1]. Definitions will be given in §2.

The underlying space of a simplicial complex K is denoted by $|K|$. A triangulation (K,f) of a space X consists of a simplicial complex K, together with a homeomorphism f of $|K|$ onto X. A second triangulation (K_1,f_1) of X is called a subdivision of (K,f) if the composition $f^{-1}f_1: |K_1| \longrightarrow |K|$ maps simplexes linearly into simplexes. We will also say that the triangulated space (K_1,f_1,X) is a subdivision of the triangulated space (K,f,X).

Two triangulated spaces (K_1,f_1,X_1) and (K_2,f_2,X_2) are isomorphic if K_1 is isomorphic to K_2. They are combinatorially equivalent if they have isomorphic subdivisions. In particular we will say that a triangulated space is a combinatorial n-cell (or (n-1)-sphere) if it is combinatorially equivalent to the n-simplex (or its boundary).

A triangulated space is a combinatorial n-manifold if the star neighborhood of every vertex is a combinatorial n-cell. (An equivalent condition would be that the link of every j-simplex, $0 \leq j < n$, should be either a combinatorial (n-j-1)-sphere or a combinatorial (n-j-1)-cell.)

The concept of differentiable manifold (that is C^∞-manifold with boundary) will be defined in Appendix I. A triangulation (K,f) of a differentiable n-manifold is a C^∞-triangulation if, for each n-simplex σ of $|K|$, the map $f|\sigma$ is differentiable (see Appendix I) and has Jacobian of rank n at all points. (This definition is equivalent to that given by Whitehead, although not identical to it. See Appendix II.) Alternatively we will say that the differentiable structure for M^n is compatible with the triangulation (K,f), if (K,f) is a C^∞-triangulation.

3. The Theorems of Whitehead. (These theorems have been restated in the author's terminology.)

W1. Every compact[2] differentiable manifold has a C^{∞}-triangulation. In fact every C^{∞}-triangulation of the boundary can be extended to a C^{∞}-triangulation of the whole manifold.

W2. If $(K.f)$ is a C^{∞}-triangulation of M^n, then the triangulated space (K,f,M^n) is a combinatorial manifold.

W3. If (K_1,f_1) and (K_2,f_2) are C^{∞}-triangulations of the same compact[2] differentiable manifold M^n, then the triangulated spaces (K_1,f_1,M^n), (K_2,f_2,M^n) are combinatorially equivalent.

Without the hypothesis of differentiability, these assertions would reduce to extremely difficult unsolved problems. (Namely: the triangulability of manifolds; the question of whether every triangulated manifold is a combinatorial manifold; and the Hauptvermutung.)

For the proof of W1 see [11] theorems 3, 7, 10, together with the remarks at the bottom of page 822. For W3 see theorem 8, together with these same remarks. In the case of an unbounded manifold, W2 follows from theorem 5. The extension to a manifold with boundary is straightforward.

These theorems suggest the following definition. Two compact differentiable manifolds M_1^n, M_2^n are combinatorially equivalent if, for some (and therefore for all) C^{∞}-triangulations (K_1,f_1) and (K_2,f_2) of M_1^n and M_2^n respectively the triangulated spaces (K_1,f_1,M_1^n) and

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2. Whitehead remarks (p. 823) that the hypothesis of compactness is not necessary.

(K_2, f_2, M_2^n) are combinatorially equivalent.

4. The Main Theorems.

Theorem 1. *There exists a differentiable 7-manifold which is combinatorially equivalent to the sphere* S^7, *but not diffeomorphic to* S^7.

Theorem 2. (R. Thom) *There exists a closed combinatorial manifold* (K, f, M^8) *such that no differentiable structure for* M^8 *is compatible with the given triangulation* (K, f), *or with any subdivision of* (K, f).

In order to prove theorem 1 it will be sufficient (comparing [6]) to prove the following.

Lemma 1. *Let* M^n *be a closed differentiable manifold. If there exists a differentiable real valued function* g *on* M^n *with only two critical points, both being non-degenerate, then* M^n *is combinatorially equivalent to* S^n.

Proof. Suppose that g has maximum $+1$ and minimum -1. Consider the submanifolds M_+^n, M_-^n defined by the inequalities $g \geq 0$, $g \leq 0$. The argument given in [6] shows that each of these manifolds is diffeomorphic to the solid n-ball. By W1 we can choose a C^∞-triangulation (K_+, f_+) of M_+^n and extend it to a C^∞-triangulation (K, f) of M^n. Let (K_-, f_-) denote the resulting triangulation of M_-^n. By W3 the triangulated spaces $(K_\pm, f_\pm, M_\pm^n)$ are combinatorial n-cells. But a triangulated space obtained by matching the boundaries of two combinatorial n-cells is clearly a combinatorial n-sphere. This proves lemma 1 and theorem 1.

Proof of theorem 2 (following [10]). Consider the manifold B_k^8

with boundary M_k^7 as defined in [6]. Form a space X_k^8 from B_k^8 by adjoining a cone over M_k^7. Choose a C^∞-triangulation of B_k^8, and extend it to a triangulation (K,f) of X_k^8 by using the standard triangulation for a cone. Lemma 1, together with W2, implies that (K,f,X_k^8) is a combinatorial manifold.

Let $p_i \in H^{4i}(X_k^8, Q)$ denote the Pontrjagin classes as defined by Thom [10]; and let $\nu \in H_8(X_k^8, Z)$ be a suitably chosen generator. As in [6] we obtain $\langle p_1^2, \nu \rangle = 4k^2$; and therefore

$$\langle p_2, \nu \rangle = (45 + 4k^2) / 7 .$$

But for $k \not\equiv \pm 1 \pmod 7$, this expression is not an integer; which implies that p_2 is not the Pontrjagin class corresponding to any differentiable structure for X_k^8. This proves theorem 2.

5. Miscellaneous remarks. It is interesting to ask whether the phenomena described in theorems 1 and 2 can occur in lower dimensions. Cairns has proved in [5] that for $n \leq 4$ every unbounded combinatorial n-manifold possesses a differentiable structure. Cairns' proof actually shows that every unbounded combinatorial n-manifold, $n \leq 4$, has a subdivision which possesses a compatible differentiable structure. The corresponding question for $n = 5,6,7$ remains open.

It is definitely necessary to allow subdivision, since Cairns has given in [4] an example of a combinatorial 4-manifold such that the star neighborhood of a certain vertex can not be imbedded in euclidean 4-space by any linear homeomorphism. Lemma 7 of Appendix II implies that this combinatorial manifold cannot possess any compatible differentiable structure.

J. R. Munkres has proved (unpublished) that, if two closed differentiable 2-manifolds are homeomorphic, then they are necessarily diffeomorphic. The corresponding question for $3 \leq n \leq 6$ remain open.

Up to this point we have restricted attention to C^∞-manifolds and C^∞-maps. The discussion could have been carried out using C^r-manifolds, $1 \leq r \leq \infty$. For example Whitehead's theorem W1, W2, W3 hold[3] with "C^∞" replaced by "C^r". Our excuse for sticking to the C^∞-case is provided by the following theorems of Whitney [13]: Every unbounded C^r-manifold can be given a compatible C^∞-structure. Every C^r-isomorphism between two unbounded C^∞-manifolds can be approximated by a C^∞-isomorphism (or diffeomorphism). Thus, for the problems which we are considering, it is sufficient to consider the C^∞-case.

The situation is less clear for real analytic manifolds. However suppose we call a real analytic manifold imbeddable if it possesses an analytic imbedding in some euclidean space. (Compare Bochner [3]. The existence of non-imbeddable analytic manifolds is an unsolved problem.) Then (see Whitney [13]) every unbounded C^∞-manifold can be given a compatible real analytic structure which is imbeddable; and any C^∞-isomorphism between imbeddable analytic manifolds can be approximated by an analytic-isomorphism. So again it is sufficient to consider the C^∞ case.

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3. Whitehead assumes (p. 822) that $r \geq 3$, but this assumption could be removed, making use of our lemma 3.

Appendix I. Differentiable manifolds.

Let H^n be a closed half-space in euclidean space R^n. By an n-dimensional coordinate system on a space X we will mean a collection $\{(U_\alpha, f_\alpha)\}$ of pairs, where the U_α are open sets covering X, and where each f_α is a homeomorphism of U_α onto either H^n or R^n. An n-manifold is a Hausdorff space which possesses such a coordinate system. We assume that all manifolds considered are separable and hence paracompact.

The set of points in X which correspond to boundary points of H^n forms the boundary $\dot{X}$. A manifold is unbounded if its boundary is vacuous. (Note that the boundary of any n-manifold is an unbounded (n-1)-manifold.) A manifold is closed if it is compact and unbounded.

Let A be any subset of euclidean space R^m. A map $f\colon A \longrightarrow R^n$ is differentiable of class C^r (or briefly, is a C^r-map), $1 \leq r \leq \infty$, if for every point $a \in A$ there exists a neighborhood U of a in R^m, and a map $g\colon U \longrightarrow R^n$, such that g has continuous j-th partial derivatives for all $j \leq r$, and such that $f(x) = g(x)$ for $x \in A \cap U$. (Compare Whitney [12].)

A coordinate system $\{(U_\alpha, f_\alpha)\}$ for X is a C^r-coordinate system if, for each α_1, α_2 the map

$$f_{\alpha_2} f_{\alpha_1}^{-1}\colon\ f_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2}) \longrightarrow R^n$$

is a C^r-map. Two C^r-coordinate systems $\{(U_\alpha, f_\alpha)\}$, $\{(V_\beta, g_\beta)\}$ are C^r-equivalent if the composite system $\{(U_\alpha, f_\alpha), (V_\beta, g_\beta)\}$ is also a C^r-coordinate system.

By a <u>differentiable manifold of class</u> C^r (or C^r-manifold) we mean[4] a manifold together with a C^r-equivalence class of C^r-coordinate systems. (This equivalence class is called a C^r-structure.) Note that the boundary of any C^r-manifold can itself be considered as a C^r-manifold. Also any open subset of a C^r-manifold is a C^r-manifold.

Let M^m, N^n be C^r-manifolds with representative coordinate systems $\{(U_\alpha, f_\alpha)\}$, $\{(V_\beta, g_\beta)\}$; and let A be any subset of M^m. A map $h: A \longrightarrow N^n$ is a C^r-<u>map</u> if for each α, β the map

$$g_\beta h f_\alpha^{-1}: f_\alpha(A \cap U_\alpha \cap h^{-1}(U_\beta)) \longrightarrow R^n$$

is differentiable of class C^r.

A homeomorphism h of M^m into N^n (or onto N^n) is called a C^r-<u>imbedding</u> (or C^r-<u>isomorphism</u>) if h is a C^r-map and has Jacobian of rank m at all points. A C^∞-isomorphism is also called a <u>diffeomorphism</u>.

Some basic properties of C^r-manifolds follow.

Lemma 2. <u>Let</u> A <u>be a subset of the</u> C^r-<u>manifold</u> M^m <u>and let</u> N^n <u>be an unbounded</u> C^r-<u>manifold</u>. <u>Then every</u> C^r-<u>map</u> $h: A \longrightarrow N^n$ <u>can be extended to a</u> C^r-<u>map of a neighborhood of</u> A <u>into</u> N^n.

(Compare Whitney [12].)

Lemma 3. <u>If</u> M^n <u>is a</u> C^r-<u>manifold with boundary</u> $\dot{M}^n$, <u>then there exists a neighborhood</u> U <u>of</u> $\dot{M}^n$ <u>and a</u> C^r-<u>isomorphism</u>

$$g: \dot{M}^n \times [0,1) \longrightarrow U$$

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4. This definition differs from the definition of "variété à bord" used by Thom [9].

satisfying $g(y,0) = y$.

(The corresponding statement for topological manifolds seems very difficult.)

Lemma 4. Let M^n be the unbounded manifold obtained by piecing together two bounded C^r-manifolds M_1^n, M_2^n by means of a C^r-isomorphism $h: \dot{M}_1^n \longrightarrow \dot{M}_2^n$. Then M^n can be given a C^r-structure compatible with that of M_1^n and M_2^n.

(By "compatible" we mean that the natural homeomorphisms $j_i: M_i^n \longrightarrow M^n$, $i = 1,2$, are C^r-imbeddings. This lemma was used without proof in [6] and [9].)

Lemma 5. Any C^r-manifold M_1^n can be imbedded in an unbounded C^r-manifold M^n of the same dimension. If M_1^n is compact then M^n can be chosen as a closed manifold.

The basic tool used in proving these lemmas is the partition of unity. By a C^r-partition of unity on a C^r-manifold M^n is meant a collection $\{p_\beta\}$ of C^r-maps $p_\beta : M^n \longrightarrow R$ satisfying

$$(1) \quad p_\beta(x) \geq 0 ; \qquad\qquad (2) \quad \sum_\beta p_\beta(x) = 1;$$

and (3) the collection $\{\text{Carrier } p_\beta\}$ is locally finite. Here "Carrier p_β" denotes the closure of the set of x in M^n with $p_\beta(x) \neq 0$. We will say that $\{p_\beta\}$ is associated with an open covering $\{V_\beta\}$ of M^n if Carrier $p_\beta \subset V$ for each β.

Lemma 6. Given any open covering $\{V_\beta\}$ of a C^r-manifold M^n, there exists an associated C^r-partition of unity $\{p_\beta\}$.

Proof. It is well known that there exists a C^r-partition of unity $\{q_\gamma\}$ such that each set (Carrier q_γ) is contained in some set V_β of the given covering. (For the case of an unbounded manifold, see deRham [8], p. 4. The extension to manifolds with boundary is straightforward.) Now for each index γ choose $\beta(\gamma)$ so that

$$\text{Carrier } q_\gamma \subset V_{\beta(\gamma)};$$

and define

$$p_{\beta_0}(x) = \sum_{\beta(\gamma)=\beta_0} q_\gamma(x).$$

Then $\{p_\beta\}$ is the required partition of unity.

Proof of lemma 2. First suppose that the image space is the euclidean space R^n. From the definition of a C^r-map it follows that for each point $a \in A$ we can choose an open neighborhood V_a and a C^r-map $h_a: V_a \longrightarrow R^n$ which agrees with h on $V_a \cap A$. Let V be the union of these sets V_a, and let $\{p_a\}$ be C^r-partition of unity in V associated with the covering $\{V_a\}$. Then the required extension $H: V \longrightarrow R^n$ is given by $H(x) = \sum p_a(x)h_a(x)$. (It is to be understood that $p_a(x)\, h_a(x) = 0$ for $x \notin V_a$.)

Now let N^n be any unbounded C^r-manifold. According to Whitney [13], lemma 19, there exists a C^r-imbedding $i: N^n \longrightarrow R^{2n+1}$. By lemma 23 of [13], there exists a neighborhood U of $i(N^n)$ in R^{2n+1}, and a C^r-retraction $\rho: U \longrightarrow i(N^n)$. By the previous remarks, we can extend $ih: A \longrightarrow R^{2n+1}$ to a C^r-map $H: V \longrightarrow R^{2n+1}$. Now $i^{-1}\rho H: H^{-1}(U) \longrightarrow N^n$ is the required extension of h.

Proof of lemma 3. From lemma 2 we see that, for some neighborhood V_1 of $\dot{M}^n$, there exists a C^r-retraction $\rho : V_1 \longrightarrow \dot{M}^n$. (This follows by substituting $M^n, \dot{M}^n, \dot{M}^n$, and the identity map, respectively, for M^m, N^n, A and h in lemma 2.)

Let $\lambda : M^n \longrightarrow R$ be a C^r-map which vanishes on $\dot{M}^n$ but is positive on $M^n - \dot{M}^n$ and has non-vanishing gradient on $\dot{M}^n$. (It is clearly possible to choose a C^r-map $\lambda_\alpha : U_\alpha \longrightarrow R$ satisfying this condition for each coordinate neighborhood U_α. If $\{p_\alpha\}$ is a C^r-partition of unity associated with $\{U_\alpha\}$, then

$$\lambda(x) = \sum p_\alpha(x)\lambda_\alpha(x)$$

will be the required function on M^n.)

Define the C^r-map $\mu : V_1 \longrightarrow \dot{M}^n \times R$ by $\mu(x) = (\rho(x), \lambda(x))$. Let V_2 denote the set of points x in V_1 such that the Jacobian of μ at x has the maximal rank n. It is clear that V_2 is a neighborhood of $\dot{M}^n$. If μ is not one-one on V_2 then we will construct a smaller neighborhood V_3 such that $\mu | V_3$ is one-one.

Let D be the subset of $V_2 \times V_2$ consisting of all pairs (y_1, y_2) with $\mu(y_1) = \mu(y_2)$ but $y_1 \neq y_2$. Since $\mu | V_2$ is a local homeomorphism, it follows that D is a closed set. Choose any metric $d(x,y)$ for the space V_2, and define the distance between points (x_1, x_2) and (y_1, y_2) of $V_2 \times V_2$ as $\mathrm{Max}(d(x_1,y_1), d(x_2,y_2))$. Define $\delta(x) > 0$ as the distance of the point (x,x) from the set D. Let V_3 be the set of all points x of V_2 with $d(x, \rho(x)) < \delta(\rho(x))$.

Clearly V_3 is a neighborhood of $\dot{M}^n$. If $\mu(x_1) = \mu(x_2)$ with $x_1, x_2 \in V_3$, then $\rho(x_1) = \rho(x_2)$. Since x_1 and x_2 have distance less than $\delta(\rho(x))$ from $\rho(x_1)$, it follows that $(x_1, x_2) \notin D$; and therefore $x_1 = x_2$. Thus $\mu | V_3$ is one-one.

Choose a C^r-map $\varepsilon : \dot{M}^n \longrightarrow R$ so that $\varepsilon(y) > 0$, and so that the region $W = \{(y,t) \mid 0 \leq t < \varepsilon(y)\}$ is contained in $\mu(V_3)$. (This can be constructed, using a C^r-partition of unity on $\dot{M}^n$) Let $V_4 = \mu^{-1}(W) \cap V_3$ and define

$$g: \dot{M}^n \times [0,1) \longrightarrow V_4$$

by

$$g(y,t) = \mu^{-1}(y_1, \varepsilon(y)t).$$ Then g is the required C^r-isomorphism. This completes the proof.

<u>Proof of lemma</u> 4. Choose C^r-isomorphisms $g_i: \dot{M}_i^n \times [0,1) \longrightarrow U_i$ as in lemma 3 $(i = 1, 2)$. Let U be the open set $j_1(U_1) \cup j_2(U_2)$ of M^n. A homeomorphism $g: \dot{M}_1^n \times (-1,1) \longrightarrow U$ is defined by

$$g(y,t) = \begin{cases} j_1\, g_1(y, t) & t \geq 0 \\ j_2\, g_2(h(y), -t) & t \leq 0 . \end{cases}$$

In order to define a C^r-structure on a manifold, it is sufficient to define compatible C^r-structures on open sets covering the manifold. But M^n is covered by the sets $j_1(M_1^n - \dot{M}_1^n)$, $j_2(M_2^n - \dot{M}_2^n)$, and U. Since the differentiable structures on these sets which are induced by j_1, j_2, and g respectively, are compatible, this completes the proof.

Proof of lemma 5. This follows from lemma 4 by taking $M_2^n = M_1^n$; with h the identity map of $\dot{M}_1^n$. The resulting manifold M^n is called the "double" of M_1^n.

Appendix II. C^r-triangulations.

A triangulation (K,f) of a C^r-manifold M^n will be called a C^r-triangulation if, (1), for each closed n-simplex σ of $|K|$ the map $f|\sigma$ is a C^r-map[5]; and (2), this map $f|\sigma$ has Jacobian of rank n at all points.

For every point x of $|K|$ let N_x denote the open star neighborhood of x in $|K|$, and let $T_{f(x)}$ denote the tangent space of M^n at $f(x)$. Following Whitehead, we define a linear map $F_x: N_x \longrightarrow T_{f(x)}$ as follows. For each $y \in N_x$ there is a unique linear map $g_y: [0,1] \longrightarrow N_x$ satisfying $g_y(0) = x$, $g_y(1) = y$. Define $F_x(y)$ as the velocity vector of the curve $fg_y: [0,1] \longrightarrow M^n$ at 0.

Lemma 7. *If (K,f) is a C^r-triangulation of M^n, then for each x in $|K|$ the map F_x is a homeomorphism of N_x into the euclidean space $T_{f(x)}$.*

The relationship of our definition of C^r-triangulation to that of Whitehead can now be described as follows. Whitehead considers only a bounded manifold M^n which is imbedded in an unbounded manifold M_*^n. By lemma 5, this is no restriction on M^n. He calls a triangulation

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5. See Appendix I. The simplex σ is to be considered as a rectilinear subset of a euclidean n-space.

(K,f) of M^n a C^r-triangulation if it satisfies the following two conditions:

(1') For each j-simplex τ of $|K|$, $1 \leq j \leq n$, the map $f|\tau$ can be extended to a C^r-map of a neighborhood of τ in euclidean j-space into M^n_*.

(2') For each point x of $|K|$, the map F_x is a homeomorphism of N_x into $T_{f(x)}$.

It follows easily from lemmas 2 and 7 that this definition is equivalent to our earlier definition.

Proof of lemma 7. Let σ be any n-simplex of $|K|$ incident to x. Since $f|\sigma$ has Jacobian of rank n at x, it follows that

(a) $F_x|(\sigma \cap N_x)$ is a homeomorphism.

This implies:

(b) Let y be any point of N_x. Then $F_x(z) \neq F_x(y)$ for all $z \neq y$ in a sufficiently small neighborhood (namely $N_y \cap N_x$ of y.

If y is not a boundary point of $|K|$ this implies that the map F_x has a local degree d_y at y, which is well defined up to sign (See [2] XII, §2. By the boundary $|\dot{K}|$ of $|K|$ we mean $f^{-1}(\dot{M}^n)$.

Let w be an interior point of the n-simplex σ. The following statement is not hard to verify.

(c) If $c: [0,1] \longrightarrow M^n$ is a differentiable curve which is tangent to $fg_w: [0,1] \longrightarrow M^n$ at 0, then $c(t) \in f(\sigma)$ for small values of t.

This implies:

(d) Let w be an interior point of σ, and let z be any point of N_x other than w. Then $F_x(z) \neq F_x(w)$. (In fact (d) follows from (a) if $z \in \sigma$, and from (c) if $z \notin \sigma$.

Now let y be any point of N_x which is not a boundary point of $|K|$. Choose an n-simplex σ incident to y, According to (d) the points $F_x(\sigma \cap N_x)$ are covered only once by F_x. From this it follows that the local degree d_y must be ± 1. Therefore F_x maps any neighborhood of y onto a neighborhood of $F_x(y)$; or in other words:

(e) F_x restricted to $N_x - |\dot{K}|$ is an open mapping.

We are now ready to prove by induction on n that the linear map F_x is one-one. Suppose $F_x(y) = F_x(z)$ with $y \neq z$. If y and z both belong to the boundary $|\dot{K}|$ then this contradicts the induction hypothesis. If one of the points, say y, is in $N_x - |\dot{K}|$, then F_x maps $N_y \cap N_x$ onto a neighborhood of $F_x(z)$ which must intersect the interior of $F_x(\sigma)$ for any n-simplex σ containing z. By (d) this implies that the interior of σ intersects $N_y \cap N_x$, and therefore that σ contains y. But this contradicts (a); which completes the proof.

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