Some free actions of cyclic groups on spheres

J. Milnor, December 1963

Let \( p \geq 5 \) be prime and let \( n \geq 5 \) be odd. This note will show that the cyclic group \( \mathbb{Z}/n \) of order \( n \) can act differentiably on the \( n \)-sphere, without fixed points, in infinitely many different ways. These actions are "different" in the sense that the corresponding quotient manifolds \( M = S^n/\mathbb{Z}/n \) can be distinguished by their Reidemeister-Franz-de Rham torsion invariants. Hence two such "different" manifolds \( M, M' \) cannot have the same simple homotopy type, cannot be piecewise-linearly homeomorphic, and cannot be diffeomorphic. (It is not known whether or not \( M \) and \( M' \) can be homeomorphic.)

First let me review the basic properties of the torsion invariant, following [3], [4]. Let \( K \) be a finite, connected CW-complex and let \( \Pi \) denote the fundamental group of \( K \). Let

\[
f: \mathbb{Z}[\Pi] \longrightarrow \mathbb{C}
\]

be a ring homomorphism from the integral group ring to the complex numbers. If the homology groups \( H_1(K; \mathbb{C}_f) \) are all zero (homology with local coefficients twisted by \( f \)) then the torsion invariant

\[\Delta_f K \in \mathbb{C}_0/\mathbb{C}_0 \, f\Pi \]

is defined. (Here \( \tilde{K} \) denotes the universal covering complex, \( \mathbb{C}_0 \) the multiplicative group of non-zero complex numbers, and \( f\Pi \) the subgroup generated by \( f(\Pi) \) and \( \pm 1 \).) To simplify the notation we will henceforth leave off the tilde, and write simply \( \Delta_f K \).
Similarly, given a pair $K, L$ with $H_\pi(K, L; C_\pi) = 0$ the torsion $\Delta_\pi(K, L)$ is defined. This satisfies the identity

\[ \Delta_\pi(K, L) = \Delta_\pi K \Delta_\pi L, \]

providing that the three terms are defined. (If two out of three are defined, then the third is automatically defined.)

If $W$ is a triangulated manifold of dimension $n$ with boundary $\partial W$, then the following duality theorem holds. We must assume that $|f(t)| = 1$ for $t \in \tau = \tau_1(W)$. Then

\[ \Delta_\pi(\partial W) = (\Delta_\pi W)(\Delta_\pi W)^\varepsilon(n) \]

where $\overline{\Delta}$ denotes the complex conjugate and $\varepsilon(n) = (-1)^n$. We will also need the following variant form. If $M$ is a triangulated manifold without boundary of dimension $n - 1$ then

\[ \Delta_\pi M = (\Delta_\pi M)^\varepsilon(n). \]

Now consider an h-cobordism $(W; M, M')$. That is, assume that $W$ is a smooth manifold with boundary $M + M'$, and that both $M$ and $M'$ are deformation retracts of $W$. Choosing a $C^1$-triangulation of $(W; M, M')$ we will assume that the torsion

\[ \Delta_\pi M \in C_\pi/\pm f \tau \]

is defined.
Lemma 1. Then $\Delta M'$ is defined, and equal to

$$(\Delta W)(\Delta (W, M)(\Delta (W, M))^\epsilon(n)).$$

**Proof.** Since $M$ is a deformation retract of $W$, it is clear that $\Delta (W, M)$ is defined. Thus $\Delta W$ is defined, and similarly $\Delta M'$ is defined. Consider the duality statement

$$\Delta (bW) = (\Delta W)(\Delta (W))^\epsilon(n).$$

Since $\Delta (bW) = (\Delta M)(\Delta M')$ and since $\Delta W = (\Delta M)(\Delta (W, M))$, this can be rewritten as

$$(\Delta M)(\Delta M') = (\Delta M)(\Delta (W, M)(\Delta (W, M))^\epsilon(n)(\Delta (W, M))^\epsilon(n).$$

Now dividing through by

$$\Delta M = (\Delta M)^\epsilon(n)$$

we obtain the required formula

$$\Delta M' = (\Delta M)(\Delta (W, M)(\Delta (W, M))^\epsilon(n).$$

Henceforth we will assume that the dimension $n$ of $W$ is even. Thus Lemma 1 can be rewritten in the form

$$(4) \quad \Delta M' = (\Delta M)|\Delta (W, M)|^2.$$ 

Suppose that we are given the manifold $M$ with fundamental group $\Pi$, and wish to construct the h-cobordism $(W; M, M')$. 
Lemma 2 (Stallings). If \( \dim(M) \geq 5 \) then the h-cobordism \((W; M, M')\) can be constructed so that \( \Delta_{f}(W, M) \) is equal to the image in \( C_{0}^{+} \) of any unit of the ring \( \mathbb{Z}[\Pi] \).

Proof. Stallings actually observes that the h-cobordism can be constructed so that the Whitehead torsion invariant \( \tau(W, M) \) is any desired element of the Whitehead group
\[
\text{Wh}(\Pi) = \text{GL}(\infty, \mathbb{Z}[\Pi])/(\text{Commutators}, \pm \Pi).
\]
(See Stallings [6, §2].) In particular if \( u \) is a unit of \( \mathbb{Z}[\Pi] \) then \( W \) can be chosen so that \( \tau(W, M) \) is the element of \( \text{Wh}(\Pi) \) corresponding to the matrix
\[
\begin{pmatrix}
  u & 1 \\
  1 & \ddots \\
  \vdots & \ddots & \ddots
\end{pmatrix}
\in \text{GL}(\infty, \mathbb{Z}[\Pi]).
\]
It is then clear that \( \Delta_{f}(W, M) \) is equal to the image of \( u \) in \( C_{0}^{+} \). (Compare Cockcroft [1], or [3, pg. 589].) This completes the proof.

Thus in order to construct examples of h-cobordisms, we need only look for units in \( \mathbb{Z}[\Pi] \). To be more specific, let us now assume that \( \Pi \) is cyclic of order \( p \) with generator \( t \). Define \( f: \mathbb{Z}[\Pi] \to \mathbb{C} \) by \( f(t) = \exp(2\pi i/p) \).

Lemma 3 (Higman). If \( p \geq 5 \) is an integer of the form \( 6k + 1 \) then \( \mathbb{Z}[\Pi] \) contains a unit \( u \) with \( |f(u)| \neq 1 \).
5.

**Proof.** This follows easily from Higman [2]. Alternatively, here is a direct proof. Let

\[ u = t + t^{-1} - 1 \]

so that \( f(u) = 2 \cos(2\pi/p) - 1 \neq -1 \). To see that \( u \) is a unit it is only necessary for the reader to verify the identity

\[ u(l + t - t^3 - t^4 + t^6 + t^7 - - - - + t^{p-1}) = 1 \]

for \( p \equiv 1 \pmod{6} \); or

\[ u(- l + t^2 + t^3 - t^5 - t^6 + - - - - + t^{p-3} + t^{p-2}) = 1 \]

for \( p \equiv -1 \pmod{6} \). This completes the proof.

Now combining the three lemmas we have the following.

**Theorem.** Let \( M \) be a smooth manifold of odd dimension \( \geq 5 \) whose fundamental group is cyclic of order \( p = 6k + 1, \ p \geq 5 \). Then there exist infinitely many manifolds \( M_1, M_2, M_3, \ldots \) which are h-cobordant to \( M \), but such that no two have the same simple homotopy type.

**Proof.** For each integer \( m \) we can choose the h-cobordism \( (W_m; M, M_m) \) so that

\[ \Delta_f(W_m, M) = |f(u^m)|. \]
Then

$$\Delta^M_\pi = (\Delta^M_\pi)^{|f(u)|^{2^m}}.$$ 

Since $|f(u)| \neq 0,1$ the real numbers $\Delta^M_\pi$ are all distinct.

This does not yet prove that the $M_\pi^m$ all have distinct simple homotopy types, since the invariant $|\Delta^M_\pi|$ depends on the choice of $f$. But there are only finitely many homomorphisms from $Z[\pi]$ to $\mathbb{C}$, so out of the infinite sequence $M_1, M_2, \ldots$ one can certainly extract an infinite subsequence consisting of pairwise distinct manifolds. This completes the proof.

In particular let us apply this theorem to a lens space

$$L = S^{2k-1}/\pi.$$ 

The resulting h-cobordant manifolds $L_1, L_2, \ldots$ will all have universal covering spaces diffeomorphic to the sphere. (See Smale [5].) Thus we have infinitely many distinct free actions of the cyclic group $\pi$ on $S^{2k-1}$. But there are only finitely many orthogonal actions of $\pi$ on $S^{2k-1}$. Thus we have:

**Corollary.** For $2k - 1 > 5$ and $p$ prime $> 5$ there exist infinitely many smooth fixed point free actions of the cyclic group of order $p$ on $S^{2k-1}$ which are not smoothly equivalent to orthogonal actions.

It would be interesting to know whether any corresponding phenomenon occurs in dimension 3.
References

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