Lectures on
Homology Operations
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Introduction and historical comments

These notes are chapter 2 of the lecture notes of a course I gave at Brown University in the spring of 1976. This chapter is the heart of the course—the geometry of maps of polyhedral cycles to Euclidean spaces. Chapter 1 will discuss maps of smooth surfaces to the plane and to 3-space, centering on the work of Whitney. Chapter 3 will show how the constructions of chapter 2 can be used to give geometric descriptions of the dual Steenrod operations in mod 2 homology.

My discussion of double points of cycles generalizes that of Hudson for manifolds [H]. The geometry of Steenrod homology operations was announced in [M1] and developed in [M2] and [HM].

I begin with a summary of definitions of polyhedral topology. The main idea is that of "localizing" a polyhedral map \( f \) at a point \( x \) to obtain a link map \( Lf_x \). The global topology of \( f \) will be described using the topology of its links \( Lf_x \). This is an old idea which seems very natural from Rourke and Sanderson's definition of a polyhedron as a locally conical space [RS].

The second section is about spreading out, usually called "general position". This is also an old technique, given its modern form by Zeeman [Z].
In the next section I give a geometric definition of singular homology. This idea is due to Sullivan. Similar interpretations of his idea occur in [A], [RS], [BRS], and [M2].

The fourth section contains a generalization of the classical simplicial definition of the degree of a map. The concept of an irreducible cycle is borrowed from algebraic geometry.

Next a definition of linking numbers is given using the degree (cf. [AH], [Mi]), and intersection numbers are defined using local linking numbers. This is a relatively undeveloped idea of Lefschetz [L].

After these preliminaries come the double point and branch point cycles. (The name "branch point" for a nonimmersion point was coined by Zeeman [Z].) Let \( f : X^k \to \mathbb{R}^n \) be a spread-out map of a \( k \)-cycle to Euclidean \( n \)-space with \( k \leq n \). I show that the set of double points of \( f \) in \( X \) is a cycle, provided that double points are counted with multiplicities defined using local linking numbers. Any two such maps have homologous double point cycles, so the homology class of double point cycles of spread-out maps \( X^k \to \mathbb{R}^n \) is an intrinsic invariant of \( X \). Next the analogous results are proved for branch points. Finally I show that the double point class and the branch point class are equal, using an idea of Tom Banchoff [B].
At the end of each section are some illustrations and exercises. Statements in these notes which are not proved are intended to be exercises for the reader.

In lieu of chapter 1, I recommend chapter 1 of Seifert and Threlfall [ST] and the papers of Whitney [W] and Banchoff [B] on characteristic cycles, as a geometric introduction to chapter 2.

In chapter 3 I will give an axiomatic characterization of the double point class [HM], give a combinatorial formula for it [BM], and discuss its relation with Smith operations [Wu].

Many thanks go to Dave Damiano and James Stormes, who did most of the work of writing up these notes.

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§1. The polyhedral category

This section is a brief introduction to the category of polyhedra and piecewise-linear maps. It contains all the background material necessary for understanding this chapter. For a more detailed and more general discussion of the PL category, see [RS].

By $\mathbb{R}^n$ we will mean Euclidean $n$-space. $\mathbb{R}^n$ embeds in $\mathbb{R}^{n+1}$ in the first $n$ coordinates $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. A $k$-plane in $\mathbb{R}^n$, $k \leq n$, is a translated $k$-dimensional vector subspace of $\mathbb{R}^n$. The span of $\{x_0, \ldots, x_k\} \subset \mathbb{R}^n$ is the smallest plane containing $\{x_0, \ldots, x_k\}$. In terms of coordinates, $\text{span}\{x_0, \ldots, x_k\} = \{x | x = t_0x_0 + \ldots + t_kx_k, t_i \in \mathbb{R}, \sum t_i = 1\}$. We say that the points $x_0, \ldots, x_k$ are independent if their span has dimension $k$ or, equivalently, if $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent vectors.

A $k$-simplex $\sigma$ is the convex hull of $k+1$ independent points $x_0, \ldots, x_k$ in $\mathbb{R}^n$. That is, $\sigma = \{x | x = \sum t_i x_i, \sum t_i = 1, t_i \geq 0\}$. The points $x_0, \ldots, x_k$ are the vertices of $\sigma$. They are uniquely determined by $\sigma$. To denote that $\sigma$ is a $k$-simplex we will write $\sigma^k$. The simplex $\tau$ is a face of $\sigma$, written $\tau \subset \sigma$, if all the vertices of $\tau$ are vertices of $\sigma$. (Notice that $\sigma \subset \sigma$.) The interior of $\sigma$, $\sigma^\circ$, consists of all the points of $\sigma$ which are not in any proper face of $\sigma$. (Or, if the dimension of $\sigma$ is greater than zero, $\sigma^\circ$ is the topological interior of $\sigma$ in the plane spanned by the vertices of $\sigma$.)
A simplicial complex $K$ is a collection of simplices in $\mathbb{R}^n$ which satisfies three conditions. First, if $\sigma, \tau \in K$ and $\sigma \neq \tau$, then $\sigma \cap \tau = \emptyset$. Second, if $\sigma \in K$ and $\tau < \sigma$ then $\tau \in K$. Third, if $\tau \in K$, there is a neighborhood $U$ of $\tau$ in $\mathbb{R}^n$ such that $\{ \sigma \in K | \tau < \sigma \text{ and } \sigma \cap U \neq \emptyset \}$ is finite. The realization of the simplicial complex $K$ is the topological space $|K| = \bigcup_{\sigma \in K} \sigma$.

A polyhedron $X$ is a topological subspace of $\mathbb{R}^n$ such that $X = |K|$ for some simplicial complex $K$. A subpolyhedron of $X$ is a subspace of $X$ which is a polyhedron. The complex $K$ is called a triangulation of $X$. Notice that $|K|$ is compact if and only if $K$ is finite. $X$ has dimension $\leq k$ if $X$ is the realization of a complex all of whose simplices have dimension $\leq k$.

The complex $J$ in $\mathbb{R}^n$ is a subdivision of $K$ if $|J| = |K|$ and each simplex of $J$ is contained in a simplex of $K$. For example, given $x \in |K|$, there is a subdivision $J$ of $K$ which has $x$ as a vertex.

**Proposition 1.1** Let $K$ and $L$ be simplicial complexes in $\mathbb{R}^n$. If $|K| = |L|$, then $K$ and $L$ have a common subdivision. \[\square\]

If $A$ and $B$ are subsets of $\mathbb{R}^n$, their join $AB$ is the set
\[
\{ \lambda a + \mu b | a \in A, b \in B ; \lambda, \mu \in \mathbb{R} ; \lambda, \mu \geq 0 ; \lambda + \mu = 1 \}
\]. $AB$ consists of all points on straight line segments with endpoints in both $A$ and $B$. If $A = \emptyset$ then we define $AB = B$. The sets $A$ and $B$ are called joinable if for each $x$ in $AB$ there is only one representation of $x$ as $x = \lambda a + \mu b$ with $a, b, \lambda$ and $\mu$ as above.
We write \( aB \) for \( \{a\}B \). If \( \{a\} \) and \( B \) are joinable, \( aB \) is called the cone with apex \( a \) and base \( B \). For example, if \( B \subset \mathbb{R}^{n-1} \), then using the embedding \( \mathbb{R}^{n-1} \to \mathbb{R}^n \) we have the cones \( C_+(B) = (0, \ldots, 0, 1)B \) and \( C_-(B) = (0, \ldots, 0, -1)B \).

A suspension of a subset \( B \) of \( \mathbb{R}^n \) is the union of two cones \( aB \) and \( a'B \) whose intersection is \( B \). For example, if \( B \subset \mathbb{R}^{n-1} \), we have the suspension \( S(B) = C_+(B) \cup C_-(B) \) (Figure 1.1). If \( X \) is a compact polyhedron, and \( \{a\} \) and \( X \) are joinable in \( \mathbb{R}^n \), then the cone \( aX \) is a polyhedron. Thus if \( X \) is a compact polyhedron, so are \( C_+(X) \), \( C_-(X) \), and \( S(X) \).

If \( X \) is a polyhedron and \( x \in X \), then a link of \( x \) in \( X \), denoted by \( LX_x \), is any compact subpolyhedron of \( X \) such that \( \{x\} \) and \( LX_x \) are joinable and the cone \( xLX_x \) is a neighborhood of \( x \) in \( X \). More generally, if \( \sigma \in K \), \( |K| = X \), a link of \( \sigma \) in \( X \) is any compact subpolyhedron \( LX_\sigma \) of \( X \) such that \( \sigma \) and \( LX_\sigma \) are joinable and \( \sigma LX_\sigma \) is a neighborhood of \( \sigma^\circ \) in \( X \). It follows from proposition 1.2 below that any two links of \( X \) in \( x \) are (piecewise linearly) homeomorphic.

The simplicial link of \( \sigma \in K \) is defined to be the subcomplex \( LK_\sigma = \{ \tau \in K \mid \sigma \cap \tau = \emptyset \} \) and there exists \( \omega \in K \) with \( \sigma < \omega \) and \( \tau < \omega \).
The realization of $LK_\sigma$ is a suitable choice for $LX_\sigma$ (Figure 1.2).

If $X$ is a polyhedron in $\mathbb{R}^m$ and $Y$ is a polyhedron in $\mathbb{R}^n$, the map $f : X \to Y$ is polyhedral (or piecewise-linear, or PL) if the graph of $f$ is a polyhedron in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. If $X$ is compact, then $f : X \to Y$ is polyhedral if and only if there exist triangulations $K$ of $X$ and $L$ of $Y$ such that for each $\sigma \in K$, $f(\sigma)$ is a simplex of $L$ and $f|\sigma : \sigma \to f(\sigma)$ is the restriction of a linear map from the span of $\sigma$ to the span of $f(\sigma)$. Equivalently, if $x_0, \ldots, x_k$ are the vertices of $\sigma \in K$, then $f(x_0), \ldots, f(x_k)$ are the vertices (possibly with repetitions) of a simplex of $L$, and $f(\sum_i t_i x_i) = \sum_i t_i f(x_i)$ for $\sum_i t_i = 1$ and $t_i \geq 0$. In this situation, we will say that $f$ is simplicial with respect to the triangulations $K$ and $L$.

Let $A$ and $B$ be subsets of $\mathbb{R}^{n-1}$. If $f : A + B$, define $C_+(f) : C_+(A) \to C_+(B)$ by

$$C_+(f)((0, \ldots, 0, \lambda) + uA) = (0, \ldots, 0, \lambda) + uf(a).$$

$C_-(f) : C_-(A) \to C_-(B)$ is defined similarly, and $S(f) : S(A) \to S(B)$ is the union of $C_+(f)$ and $C_-(f)$. If $A$ and $B$ are compact polyhedra, and $f : A + B$ is PL, then $C_+(f)$, $C_-(f)$, and $S(f)$ are PL.

Now let $f : X \to Y$ be a PL map, and $x \in X$. Suppose that there is a neighborhood $U$ of $x$ in $X$ such that $U \cap f^{-1}(f(x)) = \{x\}$. (If $X$ is compact, this is true for all $x \in X$ if and only if $f$ is finite; i.e., $f^{-1}(y)$ is a finite set.
for all \( y \in Y \). Then we can choose a link \( LX_{x} \) and a link \( LY_{f(x)} \) so that \( f(LX_{x}) \subset LY_{f(x)} \). A link of \( f \) at \( x \) is then the restriction of \( f \) to \( LX_{x} \),

\[
Lf_{x} : LX_{x} \rightarrow LY_{f(x)}
\]

(Figure 1.3).

**Proposition 1.2** Let \( f : X \rightarrow Y \) be a PL map. If \( l : L_{1} \rightarrow L_{2} \) and \( l' : L_{1}' \rightarrow L_{2}' \) are links of \( f \) at \( x \in X \), there are PL homeomorphisms \( h_{1} : L_{1} \rightarrow L_{1}' \) and \( h_{2} : L_{2} \rightarrow L_{2}' \) such that \( l' \circ h_{1} = h_{2} \circ l \).

Thus any two choices for \( Lf_{x} \) are polyhedrally equivalent.

**Exercises**

1. If \( X \) and \( Y \) are compact polyhedra, then so are \( X \cup Y \) and \( X \cap Y \).

2. The composition of polyhedral maps is polyhedral.

3. The subspace \( X \) of \( \mathbb{R}^{n} \) is **locally conical** if each point \( x \in X \) has a cone neighborhood \( xL \) in \( X \), where \( L \) is compact. Show that \( X \) is locally conical if and only if \( X \) is a polyhedron (cf. [RS]).

4. Define a locally conical map of locally conical spaces. Show that \( f : X \rightarrow Y \) is locally conical if and only if \( f \) is polyhedral.
5. Describe the possible links of a finite polyhedral map from the plane to itself. Do the same for maps of $\mathbb{R}^2$ to $\mathbb{R}^3$.
Figure 1.1. \( S(B) = C_+(B) \cup C_-(B) \)

\[ x = |K| \]

Figure 1.2. \( IX_\sigma = |LK_\sigma| \)

Figure 1.3. \( f(LX_x) = LYf(x) \)
§2. The spreading out theorem

In this section we recall the double point set and the branch point set of a map, and prove the spreading out theorem, which plays an important role in the following theory.

Let \( f : A \to B \) be a continuous map of topological spaces with \( A \) compact. Let \( D^0(f) = \{ x \in A | \text{there exists } y \neq x \text{ with } f(y) = f(x) \} \). The closure of \( D^0(f) \), denoted \( D(f) \), is the double point set of \( f \). The closed set \( B(f) = \{ x \in A | \text{for every neighborhood } U \text{ of } x, \text{there exist } y, w \in U, y \neq w, f(y) = f(w) \} \) is the branch point set. Clearly \( D(f) = D^0(f) \cup B(f) \), but this is not necessarily a disjoint union. Examples of branch point sets are the fold set of a stable map of a surface to \( \mathbb{R}^2 \) and the pinch point set of a stable map of a surface to \( \mathbb{R}^3 \) (chapter 1).

Proposition 2.1 If \( X \) and \( Y \) are polyhedra, \( X \) compact, and \( f : X \to Y \) is a PL map, then \( D(f) \) and \( B(f) \) are polyhedra.

Proof. Let \( K \) and \( L \) be triangulations of \( X \) and \( Y \) with respect to which \( f \) is simplicial. Then \( D^0(f) \) is the union of the interiors of all the simplices \( \sigma \) of \( K \) such that there exists \( \tau \in K, \tau \neq \sigma \), with \( f(\tau) = f(\sigma) \). Thus \( D(f) \) is a union of simplices of \( K \). Similarly, \( B(f) \) is the union of all the simplices \( \sigma \in K \) such that there exist \( \tau, \omega \in K, \sigma < \tau \), \( \sigma < \omega \), \( \tau \neq \omega \), \( f(\tau) = f(\omega) \).
Suppose that $f : X \to Y$ is a PL map from the compact polyhedron $X$ to the polyhedron $Y$, where $\dim X \leq \dim Y$. The map $f$ is spread-out if

1. $f$ is finite (i.e. for all $y \in Y$, $f^{-1}(y)$ is finite)

2. $\dim D(f) \leq 2\dim X - \dim Y$. In other words, the codimension of $D(f)$ ($\dim X - \dim D(f)$) is greater than or equal to the codimension of $f$ ($\dim Y - \dim X$).

If $\dim X = \dim Y$, condition (2) is vacuous. If $\dim Y > 2\dim X$, then $f$ is spread-out means $f$ is an embedding. If $\dim Y = 2\dim X$, then $f$ is spread-out implies $f$ is an immersion. (Figure 2.1)

There are three versions of the spreading out theorem. We prove the first using the simplicial approximation theorem and a general position argument. The proofs of the second and third versions are refinements of the proof of the first version. (The second will be used to prove 7.3 and 8.3, and the third to prove 9.2.)

Let $R^n_+ = \{x_1, \ldots, x_n\} \in R^n \mid x_n > 0\}$. 

**Theorem 2.2** (Spreading-out theorem) Let $X$ be a compact polyhedron of dimension $k$, and let $g : X \to R^n$ be a continuous map, with $n \geq k$. Then

(a) Given $\epsilon > 0$ there is a spread-out map $f : X \to R^n$ with $||f(x) - g(x)|| < \epsilon$ for every $x \in X$.

(b) If $A \subset X$ is a compact subpolyhedron and $g|A$ is finite,
there exists a PL map \( f : X \to \mathbb{R}^n \) such that \( \| f(x) - g(x) \| < \epsilon \)
for all \( x \in X \), \( f|A = g|A \), and \( \dim(D(f) \setminus A) \leq 2\dim(X \setminus A) - n \).
If \( g(X) \subseteq \mathbb{R}_+^n \) and \( g(A) \subseteq \mathbb{R}_+^{n-1} \), then \( f \) can be chosen so that \( f(X) \subseteq \mathbb{R}_+^n \) and \( f^{-1}(\mathbb{R}_+^{n-1}) = A \).

(c) Let \( p : \mathbb{R}_+^n \to \mathbb{R}_+^{n-1} \) by \( p(x_1, \ldots, x_n) = (x_2, \ldots, x_n) \). Again let \( A \) be a compact subpolyhedron of \( X \). Suppose that \( g(X) \subseteq \mathbb{R}_+^n \), \( g(A) \subseteq \mathbb{R}_+^{n-1} \), \( g|A \) is finite, and \( p \circ g|A \) is finite. Then there exists a PL map \( f : X \to \mathbb{R}^n \) such that \( \| f(x) - g(x) \| < \epsilon \) for all \( x \in X \), \( f(X) \subseteq \mathbb{R}_+^n \), \( f^{-1}(\mathbb{R}_+^{n-1}) = A \), \( f|A = g|A \), \( \dim(D(f) \setminus A) \leq 2\dim(X \setminus A) - n \), and \( \dim(D(p \circ f) \setminus A) \leq 2\dim(X \setminus A) - n + 1 \).

The following is an immediate corollary.

**Corollary 2.3** Any compact \( k \)-dimensional polyhedron embeds in \( \mathbb{R}^{2k+1} \) and immerses in \( \mathbb{R}^{2k} \).

**Remark.** The theorem is true with \( \mathbb{R}^n \) replaced by an arbitrary PL \( n \)-manifold, i.e. a polyhedron locally PL homeomorphic with \( \mathbb{R}^n \).

**proof of (a).** By the simplicial approximation theorem there is a PL map \( g' : X \to \mathbb{R}^n \) such that \( \| g'(x) - g(x) \| < \epsilon/2 \) for all \( x \in X \). Let \( K \) be a triangulation of \( X \) with respect to which \( g' \) is simplicial.

Let \( v_0, \ldots, v_m \) be an ordering of the vertices of \( K \). The new map \( f \) will be linear on the simplices of \( K \), so \( f \) will be determined by its values on the vertices of \( K \). Let
\[ f(v_0) = g'(v_0) \] . Now suppose that \( f(v_0), \ldots, f(v_i) \) have already been defined.

Let \( L_i \) be the union of all the proper planes in \( \mathbb{R}^n \) spanned by the subsets of \( \{ f(v_0), \ldots, f(v_i) \} \). Since \( \dim L_i < n \), it follows that \( \mathbb{R}^n \setminus L_i \) is dense in \( \mathbb{R}^n \). Let \( f(v_{i+1}) \) be any point in \( \mathbb{R}^n \setminus L_i \) such that \( \| f(v_{i+1}) - g'(v_{i+1}) \| < \varepsilon/2 \). Proceeding in this manner we define \( f \) on all the vertices of \( K \).

Now we verify that \( f \) has the desired properties. Suppose \( x \in \sigma^o \). If \( v_{i0}, \ldots, v_{is} \) are the vertices of \( \sigma \), then
\[
x = \sum_j t_j v_{ij} , \quad \sum_j t_j = 1 , \quad t_j > 0 .
\]
Thus
\[
\| f(x) - g'(x) \| = \| \sum_j t_j f(v_{ij}) - \sum_j t_j g'(v_{ij}) \|
\]
\[
= \| \sum_j t_j (f(v_{ij}) - g'(v_{ij})) \|
\]
\[
< \sum_j t_j \varepsilon/2 = \varepsilon/2 .
\]
Thus \( \| f(x) - g(x) \| < \varepsilon \) for all \( x \in X \).

In order to show \( f \) is finite we must show that \( f|\sigma \) is injective for each \( \sigma \in K \). But if \( f|\sigma \) were not injective, some vertex of \( \sigma \) would have been mapped to a point in the span of the preceding vertices of \( \sigma \).

We prove that \( \dim D(f) \leq 2 \dim X - n \) by showing that if \( \sigma \) is a \( k \)-simplex and \( \tau \) is an \( \lambda \)-simplex of \( K \), then
\[
\dim(f(\sigma^o) \cap f(\tau^o)) \leq k + \lambda - n .
\]
This is sufficient since
\[ f(D(f)) \subseteq \bigcap_{\sigma, \tau} (f(\sigma^\circ) \cap f(\tau^\circ)) \], and \( f \) has already been proven finite, so \( f \) preserves dimension. Let \( v_{i0}, \ldots, v_{is} \) be the vertices of \( \sigma \) and \( \tau \) with no repetitions. Then since the dimension of the intersection of two linear spaces is the sum of the dimensions of the spaces minus the dimension of their span,

\[
\dim(f(\sigma^\circ) \cap f(\tau^\circ)) \leq k + \ell - \dim(\text{span}\{f(v_{i0}), \ldots, f(v_{is})\})
\]

\[ = k + \ell - \min\{s, n\} \]

by the construction of \( f \). If \( s \leq n \) then the images of the open simplices are disjoint (the closed simplices intersect on a \((k+\ell-n)\)-dimensional face) so in general we can say the dimension of intersection is \( \leq k + \ell - n \).

**Exercises**

1. Extend the proof of 2.2(a) to the case when the target is an arbitrary polyhedral manifold.

2. For each \( n > 0 \) find an example of an \( n \)-dimensional polyhedron which does not embed in \( \mathbb{R}^{2n} \).
Figure 2.1a. \( f: \mathbb{R}^1 \rightarrow \mathbb{R}^2 \) Spread-out

Figure 2.1b. \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) Spread-out

Figure 2.1c. \( f: \mathbb{R}^1 \rightarrow \mathbb{R}^2 \) not Spread-out
§3. Geometric homology

In this section we show how to define the homology of a topological space using maps of polyhedra into the space.

The simplicial complex $K$ is purely $k$-dimensional if each simplex of $K$ is a face of a $k$-simplex. If $K$ is purely $k$-dimensional, the boundary (mod 2) of $K$ is the subcomplex $\partial K = \{ \tau | \tau < \sigma^{k-1} \text{, } \sigma^{k-1} \text{ is a face of an odd number of } k\text{-simplices of } K\}$.

**Proposition 3.1** If $|K| = |L|$ and $K$ is purely $k$-dimensional, then $L$ is purely $k$-dimensional, and $|\partial K| = |\partial L|$.

Therefore we say that a polyhedron $X$ is purely $k$-dimensional if $X = |K|$, where $K$ is purely $k$-dimensional. A polyhedral $k$-chain (mod 2) is a compact, purely $k$-dimensional polyhedron. If $X = |K|$ is purely $k$-dimensional, its boundary (mod 2) is $\partial X = |\partial K|$. A polyhedral $k$-cycle (mod 2) is a $k$-chain with empty boundary.

If $A$ is a topological space, we are going to define the $k^{th}$ mod 2 homology of $A$ to be the set of maps $f : X \to A$, where $X$ is a $k$-cycle, modulo those maps $f$ such that $f = F|X$, where $F : W \to A$, and $W$ is a $(k+1)$-chain with boundary $X$.

Suppose that the $k$-chain $X$ is contained in the $k$-chain $Y$. Recall that the frontier of $X$ in $Y$, or $\text{Fr}(Y,X)$, is the set
of points \( y \in Y \) such that every neighborhood of \( y \) in \( Y \) contains points of both \( X \) and \( Y \setminus X \). Clearly \( \text{Fr}(Y,X) \subset \partial X \) if and only if \( \partial X = \text{Fr}(Y,X) \cup (X \cap \partial Y) \) (Figure 3.1).

Now let \((A,B)\) be a pair of spaces. We define \( Z_k(A,B) \) to be the set of all continuous maps \( f : (X,\partial X) + (A,B) \), where \( X \) is a \( k \)-chain, and \( B_k(A,B) \) to be the subset of \( Z_k(A,B) \) consisting of those maps which are boundaries in the following sense. The map \( f : (X,\partial X) + (A,B) \) is in \( B_k(A,B) \) if there is a \((k+1)\)-chain \( W \) together with

1. a polyhedral embedding \( e : X + \partial W \) such that \( \text{Fr}(\partial W,e(X)) \subset e(\partial X) \), and
2. a map \( F : (W,C\partial(\partial W\setminus e(X))) + (A,B) \) such that \( F\circ e = f \) .
(Figure 3.2)

\( Z_k(A,B) \) is an abelian semigroup under disjoint union, and \( B_k(A,B) \) is a subsemigroup.

**Definition.** \( H_k(A,B) = Z_k(A,B)/B_k(A,B) \), the \( k \)-th mod 2 geometric homology group of \((A,B)\). The group \( H_k(A) \) equals \( H_k(A,\emptyset) \).

Note that \( H_k(A,B) \) is indeed a group, since if \( f : (X,\partial X) + (A,B) \) is in \( Z_k(A,B) \), then \( f + f \) is in \( B_k(A,B) \), as can be seen by taking \( W = X \times I \). If \( f \in Z_k(A,B) \), the element of \( H_k(A,B) \) determined by \( f \) is called the homology class of \( f \). It is denoted by \([f]\).
The boundary morphism $\partial : H_k(A,B) \to H_{k-1}(B)$ is defined by
$\partial[f] = [f|\partial X]$. 

If $g : (A,B) \to (C,D)$ is a map, then $g_*[f] = H_k(g)[f] = [g \circ f]$. If $X$ is a $k$-chain, the fundamental class $[X] \in H_k(X, \partial X)$ is the homology class of the identity map $(X, \partial X) \to (X, \partial X)$.

Theorem 3.2 The pair $(H, \partial)$ satisfies the axioms of Eilenberg and Steenrod [ES].

Therefore, geometric mod 2 homology is naturally isomorphic with ordinary mod 2 homology on the category of pairs of triangulable spaces. If $X = |K|$ is a $k$-chain, let $<X>$ be the fundamental class of $X$ in the $k$th simplicial homology group of $(X, \partial X)$. The class $<X>$ is represented by the sum of all the $k$-simplices of $K$. The transformation from geometric homology to simplicial homology is defined by sending the class of $f : (X, \partial X) \to (A,B)$ to $f_\#<X>$, where $f_\#$ is the induced map in simplicial homology.

In fact, $H$ is naturally isomorphic with mod 2 singular homology for all pairs of spaces (D. Sullivan). Integral geometric homology can be defined similarly; it is isomorphic with integral singular homology.

Exercises

1. If $X$ is a $k$-chain, then $\partial(\partial X) = \emptyset$. 

2. How is the definition of \( H_k(A,B) \) affected by the removal of condition (1) in the definition of \( B_k(A,B) \) ?

3. Define the suspension homomorphism \( \sigma_k : H_k(A,B) \to H_{k+1}(SA,SB) \). Prove that it is an isomorphism. (This can be done directly or by using theorem 3.2.)

4. Show that if \( X \) is a polyhedron then every class in \( H_k(X) \) is represented by an embedding of a \( k \)-cycle in \( X \).

5. Give a similar definition of integral homology.
Figure 3.1. \( \text{Fr}(Y,X) \neq \partial X \) and \( \text{Fr}(Y,X) \subset \partial X \)

Figure 3.2. A map \( f: (X, \partial X) \rightarrow (A,B) \) in \( B_1(A,B) \)
§4. Degree

The theory of linking and intersection will allow us to distinguish between qualitatively different types of double points. In particular, we will find that we do not want to count certain double points, because of their local linking behavior. If $X$ is a k-cycle, and $f : X \to \mathbb{R}^n$ is spread out, $n > k$, then the double points of $f$ which we do count will form a cycle in $X$ (Figure 4.1).

We will carry out part of the program of Lefschetz [L, ch. IV, §6] to define intersection in the homology of a manifold in terms of linking numbers. We adopt the convention that a "number" means an integer mod 2. Much of this section and the next could be extended to the integers by taking orientations into account.

Let $Y^k$ be a k-cycle. $Y$ is irreducible if $Y$ is not the union of two k-cycles properly contained in $Y$.

Let $f : X^k \to Y^k$ be a PL map of k-cycles, with $Y^k$ irreducible. The degree of $f$ may be defined as follows. Choose triangulations of $X$ and $Y$ with respect to which $f$ is simplicial. Let $\sigma$ be a k-simplex of $Y$ and $y \in \sigma^o$. The degree of $f$ is then the number (mod 2) of points of $f^{-1}(y)$. This number is well defined, for if we let
\[ Y_0 = \mathcal{C} \{ y \in Y \mid f^{-1}(y) \text{ has an even number of points} \}, \]
\[ Y_1 = \mathcal{C} \{ y \in Y \mid f^{-1}(y) \text{ has an odd number of points} \}, \]

then \( Y_0 \) and \( Y_1 \) are k-cycles and \( Y = Y_0 \cup Y_1 \), so either \( Y = Y_0 \) or \( Y = Y_1 \).

**Proposition 4.1** Let \( f : X^k \to Y^k \) be a PL map of k-cycles, with \( Y \) irreducible. If \( f \in B_k(Y) \), then the degree of \( f \) is zero.

**Proof.** Let \( W \) be a \((k+1)\)-chain with \( \partial W = X \), and \( F : W \to Y \) a PL map with \( F|X = f \). Fix triangulations of \( W \) and \( Y \) for which \( F \) is simplicial. If \( y \) is a point in the interior of a k-simplex of \( Y \), then \( F^{-1}(y) \) is a \( 1 \)-chain with \( \partial F^{-1}(y) = f^{-1}(y) \). As the boundary of a \( 1 \)-chain must have an even number of points, the degree of \( f \) is zero (Figure 4.2). \( \square \)

Now suppose that \( g : X^k \to Y^k \) is any continuous map of k-cycles, with \( Y \) irreducible. Define the degree of \( g \) to be the degree of a PL map homotopic to \( g \). The degree of \( g \) is well defined, since if \( f_0 \) and \( f_1 \) are homotopic PL maps, the relative simplicial approximation theorem implies that \( f_0 \) and \( f_1 \) are PL homotopic. In other words, there is a PL map \( H : X \times I \to Y \) such that \( H(x,0) = f_0(x) \) and \( H(x,1) = f_1(x) \) for all \( x \in X \). Thus proposition 4.1 implies that \( \deg(f_0) = \deg(f_1) \).

The degree of a continuous map has several basic properties.
Proposition 4.2

(a) If \( f \in B_k(Y^k) \), \( \deg(f) = 0 \).

(b) If \( f \) is homotopic to \( g \), \( \deg(f) = \deg(g) \).

(c) \( \deg(f + g) = \deg(f) + \deg(g) \).

(d) \( \deg(g \circ f) = \deg(g)\deg(f) \).

(e) \( \deg(S(f)) = \deg(f) \).

Proof. Property (a) follows from proposition 4.1 and the relative simplicial approximation theorem. Properties (c) through (e) are immediate consequences of the definition of degree. (Note that for (d), the target of \( g \) = the source of \( f \) must be irreducible. Implicit in (e) is the assertion that if \( Y \) is irreducible, then \( SY \) is irreducible.) Property (b) follows from (a) and (c).

Exercises

1. The \( k \)-cycle \( Y \) is irreducible if and only if the \( \mathbb{Z}/2 \) vector space \( H_k(Y) \) has dimension one. If \( f : X \rightarrow Y \) is a continuous map of \( k \)-cycles with \( Y \) irreducible, then \( f_*[X] = \deg(f)[Y] \).

2. Reprove proposition 4.2 using exercise 1.

3. Prove that an \( n \)-ball does not retract to its boundary. Deduce Brouwer's theorem that any map from an \( n \)-ball to itself must have a fixed point.

4. Define the degree in integral homology. Let \( X \) and \( Y \) be closed orientable polyhedral surfaces. Show that there exists a continuous map \( f : X \rightarrow Y \) with nonzero integral degree if and only if genus \( (X) \geq \text{genus} \ (Y) \).
Figure 4.1. The double points of $f$ and $g$ are qualitatively different. This difference is detected by linking theory.

Figure 4.2. $\partial F^{-1}(y) = f^{-1}(y)$.
§5. Linking

Let \( x^k \) and \( y^\ell \) be cycles, and \( f : x^k \to \mathbb{R}^n \), \( g : y^\ell \to \mathbb{R}^n \) be continuous maps, where \( n = k + \ell + 1 > 1 \). Assuming \( f(x) \cap g(y) = \emptyset \), we may define

\[
D(f,g) : X \times Y \to S^{n-1}
\]

\[
D(f,g)(x,y) = \frac{f(x) - g(y)}{||f(x) - g(y)||}
\]

where \( ||(x_1, \ldots, x_n)|| = \sum_{i=1}^{n} |x_i| \), and \( S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid ||(x_1, \ldots, x_n)|| = 1\} \), the standard polyhedral \((n-1)\)-sphere.

(Even if \( f \) and \( g \) are PL, \( D(f,g) \) will not in general be PL.) The linking number of \( f \) and \( g \) is defined by

\[
L(f,g) = \text{deg}(D(f,g))
\]

(Figure 5.1).

We can also define the linking number of two maps to a sphere. Let \( x^k \) and \( y^\ell \) be cycles, \( k > 0 \), \( \ell > 0 \), and let \( f : x^k \to S^n \), \( g : y^\ell \to S^n \), where \( n = k + \ell + 1 \), with \( f(x) \cap g(y) = \emptyset \). Choose an embedding \( e \) of \( B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid ||(x_1, \ldots, x_n)|| \leq 1\} \) in \( S^n \) such that \( f(x) \cup g(y) \subset e(B^n) \). Let \( h : e(B^n) \to \mathbb{R}^n \) be defined by \( h(e(b)) = b \), and define

\[
L(f,g) = L(h \circ f, h \circ g)
\]
Of course it must be checked that this definition is independent of the choice of embedding e.

Our definitions have omitted certain special cases. The linking number in Euclidean space has been defined only for \( k + l > 0 \). If \( k = l = 0 \), then \( \mathcal{L}(f,g) \) can be defined only if either \( X^0 \) or \( Y^0 \) is an even 0-cycle (that is, consists of an even number of points). We then define \( \mathcal{L}(f,g) \) to be the "degree" of \( D(f,g) : X \times Y \to S^0 \); that is, the number (mod 2) of points of \( D(f,g)^{-1}(z) \), where \( z \) is either of the two points of \( S^0 \).

The linking number in a sphere has been defined only for \( k,l > 0 \). To extend the definition, say for \( k = 0 \), we must require that \( X \) be an even 0-cycle. We then choose \( x_0 \in X \) and \( e : B^n \to S^n \) such that \( f(X \setminus \{x_0\}) \cup g(Y) \subset e(B^n) \) and \( f(x_0) \in S^n \setminus e(B^n) \). Defining \( \mathcal{L}(f,g) \) as before, one easily verifies that \( \mathcal{L}(f,g) \) is independent of these choices. Notice that if \( k = l = 0 \), then the linking number in \( S^1 \) is defined only if both \( X \) and \( Y \) are even 0-cycles.

Linking numbers have several basic properties. Let \( f \) and \( g \) be maps of cycles to \( R^n \) or \( S^n \) such that \( \mathcal{L}(f,g) \) is defined. Let \( \text{Im}(g) \) denote the image of the map \( g \).

**Proposition 5.1**

(a) If \( f \in B_k(R^n \setminus \text{Im}(g)) \) [or \( B_k(S^n \setminus \text{Im}(g)) \)], then \( \mathcal{L}(f,g) = 0 \).
(b) If $f_1$ is homotopic to $f_2$ by a homotopy which is disjoint from $\text{Im}(g)$, then $\mathcal{L}(f_1, g) = \mathcal{L}(f_2, g)$.

(c) $\mathcal{L}(f, g) = \mathcal{L}(g, f)$.

(d) $\mathcal{L}(f_1 + f_2, g) = \mathcal{L}(f_1, g) + \mathcal{L}(f_2, g)$.

(e) $\mathcal{L}(f \circ h, g) = (\deg h)\mathcal{L}(f, g)$.

(f) (for linking in a sphere only) $\mathcal{L}(Sf, i \circ g) = \mathcal{L}(f, g)$, where $i : S^n \to S^{n+1}$ is the usual inclusion (Figure 5.2).

**proof.** Properties (a)-(e) are easy consequences of properties (a)-(e) of degree. To prove (f), let $m : (SX) \times Y \to S(X \times Y)$ be the obvious map. $S(X \times Y)$ will not in general be irreducible, but if $z \in \sigma^o$, where $\sigma$ is a $(k+l+1)$-simplex of a triangulation of $S(X \times Y)$ with respect to which $m$ is simplicial, then there will be precisely one point in $m^{-1}(z)$. As in property (e), we may conclude that

$$\deg(SD(f, g) \circ m) = \deg(SD(f, g)).$$

Furthermore, $SD(f, g) \circ m$ is homotopic to $D(Sf, i \circ g)$. It follows that

$$\mathcal{L}(Sf, i \circ g) = \deg(D(Sf, i \circ g))$$

$$= \deg(SD(f, g) \circ m)$$

$$= \deg(SD(f, g))$$

$$= \deg(D(f, g))$$

$$= \mathcal{L}(f, g).$$
Exercises

1. Give a definition of integral linking numbers (cf. [AH]).

2. Give an example of a pair of disjoint simple closed curves in $\mathbb{R}^3$ such that their integral linking number is zero, but neither curve is contractible in the complement of the other.
Figure 5.1. $\mathcal{L}(f,g) = \deg D(f,g)$. 
Figure 5.2. $\mathcal{L}(S_f, i \circ g) = \mathcal{L}(f, g)$. 
§6. Intersection numbers

Let $f : x^k \to \mathbb{R}^n$ and $g : y^\ell \to \mathbb{R}^n$ be PL maps, where $x^k$ is a $k$-chain, $y^\ell$ is an $\ell$-chain, $k + \ell = n$, $k > 0$, and $\ell > 0$. Suppose that $f^{-1}(g(y))$ and $g^{-1}(f(x))$ are finite sets, and that $f^{-1}(g(\partial Y))$ and $g^{-1}(f(\partial X))$ are empty. If $x \in X \setminus \partial X$, then $LX_x$ is a $(k-1)$-cycle. If $y \in Y \setminus \partial Y$, $LY_y$ is an $(\ell-1)$-cycle. If $z \in \mathbb{R}^n$, $LR_z^n$ is a polyhedral $(n-1)$-sphere. The intersection number of $f$ and $g$ is defined to be

$$\mathcal{J}(f,g) = \sum_{x,y} \mathcal{L}(Lf_x, Lg_y), \quad f(x) = g(y).$$

(This means $\sum_{I} \mathcal{L}(Lf_x, Lg_y)$, $I = \{(x,y)|f(x) = g(y)\}$). We will always use such an abbreviation in writing sums.) Notice that the dimensions are correct:

$$\dim(LX_x) + \dim(LY_y) + 1 = \dim LR_z^n, \quad z = f(x) = g(y).$$

If $k = 0$ and $\ell > 0$ then we define

$$\mathcal{J}(f,g) = \sum_{x,y} \deg(Lg_y), \quad f(x) = g(y).$$

If $k = \ell = 0$ we define $\mathcal{J}(f,g)$ to be the number (mod 2) of ordered pairs $(x,y)$ such that $f(x) = g(y)$. (Figure 6.1)

This definition easily extends to intersection numbers in any PL manifold of dimension $n$. At this point we depart from the
program of Lefschetz. We will return to it in chapter 3.

**Theorem 6.1  (The linking theorem)** Let \( f : X^k \to R^n \) and \( g : Y^\ell \to R^n \) be PL maps of chains for which \( \mathcal{L}(f,g) \) is defined. If \( \exists Y = \emptyset \), then

\[
\mathcal{L}(f,g) = \mathcal{L}(f|_{\exists X}, g) .
\]

(Figure 6.2)

**Corollary 6.2** Let \( f : X^k \to R^n \) and \( g : Y^\ell \to R^n \) be PL maps of chains for which \( \mathcal{L}(f,g) \) is defined. If \( \exists X = \emptyset \) and \( \exists Y = \emptyset \), then \( \mathcal{L}(f,g) = 0 \).

If \( f : (X,\exists X) \to (Z,\exists Z) \) is a map of pairs, let \( \exists f : \exists X \to \exists Z \) be the restriction of \( f \).

**Corollary 6.3** Let \( f : (X^k, \exists X) \to (B^n, S^{n-1}) \), and \( g : (Y^\ell, \exists Y) \to (B^n, S^{n-1}) \) be PL maps, where \( X^k \) is a \( k \)-chain, \( Y^\ell \) is an \( \ell \)-chain, \( k + \ell = n \), \( f^{-1}(g(Y)) \) and \( g^{-1}(f(X)) \) are finite, \( f^{-1}(g(\exists Y)) = \emptyset \), \( g^{-1}(f(\exists X)) = \emptyset \), \( f^{-1}(S^{n-1}) = \exists X \), and \( g^{-1}(S^{n-1}) = \exists Y \). Then

\[
\mathcal{L}(f,g) = \mathcal{L}(\exists f, \exists g) .
\]

(Figure 6.3)

Corollary 6.2 is an immediate consequence of the theorem.
Corollary 6.3 is proved as follows. Let \( \tilde{X} = X \cup C_+ (\partial X) \), 
\( \tilde{Y} = Y \cup C_+ (\partial Y) \), and \( \mathbb{B}^n = B^n \cup C_+ (S^{n-1}) \). Let \( h : \tilde{B}^n \rightarrow S^n \) be 
the obvious PL homeomorphism. Define \( \tilde{f} : \tilde{X} \rightarrow S^n \) and 
\( \tilde{g} : \tilde{Y} \rightarrow S^n \) by \( \tilde{f} = h \circ (f \cup C_+ (\partial f)) \), 
\( \tilde{g} = h \circ (g \cup C_+ (\partial g)) \). Since \( \tilde{X} \) and \( \tilde{Y} \) are cycles, 
corollary 6.2 implies that \( \mathcal{L}(\tilde{f}, \tilde{g}) = 0 \).

The observation that

\[
\mathcal{L}(\tilde{f}, \tilde{g}) = \mathcal{L}(f, g) + \mathcal{L}(\partial f, \partial g)
\]

proves corollary 6.3. \( \blacksquare \)

proof of theorem 6.1 (for \( k, \lambda > 0 \)). Let \( P = \{ p_1, \ldots, p_s \} = 
\{ f(X) \cap g(Y) \}. \) Let \( f_i = \sum_{f(x) = p_i} f_x \) and \( g_i = \sum_{g(x) = p_i} g_x \).

The theorem is a consequence of the following three statements:

(a) \( \mathcal{L}(f, g) = \sum_i \mathcal{L}(f_i, g_i) \)

(b) \( \mathcal{L}(f_i, g_i) = \mathcal{L}(j_i \circ f_i, g) \), where \( j_i : LR^n_{p_i} \rightarrow R^n \)
is the inclusion.

(c) \( \sum_i \mathcal{L}(j_i \circ f_i, g) = \mathcal{L}(\partial f, g) \).

Statement (a) follows directly from the definition of \( \mathcal{L} \).

Statement (c) is a consequence of the additivity of \( \mathcal{L} \) (proposition 5.1(d)), which implies that

\[
\sum_i \mathcal{L}(j_i \circ f_i, g) + \mathcal{L}(\partial f, g) = \mathcal{L}(\sum_i j_i \circ f_i + \partial f, g).
\]
Let \( W = \text{C} \cup_{f(x) \in \text{P}} x \text{L}X_x \). Since \( \partial W = \partial X \cup \bigcup_{f(x) \in \text{P}} \text{L}X_x \), and \( f(W) \subset R^n \setminus g(Y) \), proposition 5.1(a) implies that \( 0 = \mathcal{L}(f|\partial W, g) = \mathcal{L}(\partial f + \sum_i i \circ f_i, g) \).

Finally, statement (b) will be proved by constructing a homology between \( g \) and the "suspension" of \( g_i \), in the complement of the image of \( f_i \).

More precisely, let \( B \) be a compact polyhedron in \( R^n \) containing \( f(X) \cup g(Y) \) and let \( e : B \to S^n \) be a PL embedding with the following properties:

\begin{enumerate}
  \item \( e(p_i) = (0, \ldots, 0, -1) \)
  \item \( e(LR^n) = S^{n-1} \)
  \item if \( g(y) = p_i \) and \( \lambda \in \text{L}X_y \), then \( e(g(\lambda y + \mu \lambda)) = (0, \ldots, 0, -\lambda) + \mu e(g(\lambda)) \), where \( \lambda, \mu > 0 \), \( \lambda + \mu = 1 \).
\end{enumerate}

Now let \( f' = e \circ f \), \( g' = e \circ g \), \( f'_i = e \circ f_i \), \( g'_i = e \circ g_i \), and \( L = \text{L}X \). Thus \( \text{Im}(f'_i) \subset S^{n-1} \) and \( g'_i : L \to S^{n-1} \) with \( g(y) = p_i \).

\( \text{Im}(g'_i) \cap \text{Im}(f'_i) = \emptyset \). We shall construct an \((l+1)\)-chain \( W \) with \( \partial W = Y \cup S(L) \) and a map \( H : W \to S^n \setminus \text{Im}(f'_i) \) such that \( H|Y = g' \) and \( H|S(L) = S(g'_i) \) (Figure 6.4). It then follows that, if \( j : S^{n-1} \to S^n \) is the inclusion,
\[ \mathcal{L}(j_\circ f_i, g) = \mathcal{L}(j \circ f_i', g') \quad \text{by 5.1(e)} \]
\[ = \mathcal{L}(j \circ f_i', S(g'_1)) \quad \text{by 5.1(a)} \]
\[ = \mathcal{L}(f_i', g'_1) \quad \text{by 5.1(f)} \]
\[ = \mathcal{L}(f_i', g'_1) \quad \text{by 5.1(e)}. \]

**Construction of \( W \) and \( H \):** Let \( Y' = Cl(Y \setminus \cup yLY_y) \), and \( g(y) = p_i \).

Let \( A_+ = Y' \cup (L \times I) \cup C_+(L) \), with the identifications of \( L \times \{0\} \) with \( L = 3Y' \) and of \( L \times \{1\} \) with \( L = 3C_+(L) \). Thus \( A_+ \) is an \((\&+1)\)-cycle. Let \( B_+ = C_+(S^{n-1}) \), and let \( \alpha_+ : A_+ \rightarrow B_+ \) be defined by

\[ \alpha_+|_{Y'} = g'|_{Y'} \]
\[ \alpha_+|_{C_+(L)} = S(g'_1)|_{C_+(L)} \]
\[ \alpha_+((\ell, t)) = g'(\ell) \quad \text{for} \quad (\ell, t) \in L \times I. \]

Thus \( \text{Im}(\alpha_+) \cap \text{Im}(f'_1) = \emptyset \). Let \( V_+ \) be a cone with base \( A_+ \).

Since \( B_+ \) is contractible, there is a map \( \beta_+ : V_+ \rightarrow B_+ \) with \( \beta_+|_{A_+} = \alpha_+ \). And \( \beta_+ \) can be chosen so that \( \beta_+(1)(S^{n-1}) = \alpha_+(1)(S^{n-1}) \), so \( \text{Im}(\beta_+) \cap \text{Im}(f'_1) = \emptyset \).

Now let \( A_- = \cup yLY_y \cup (L \times I) \cup C_-(L) \), with identifications along \( L \times \{0\} \) and \( L \times \{1\} \) as above. Let \( B_- = C_-(S^{n-1}) \). Define \( \alpha_- : A_- \rightarrow B_- \) by

\[ \alpha_-|_{g(y) = p_i yLY_y} = g'|_{g(y) = p_i yLY_y} \]
\[ \alpha_\cdot |C_\cdot (L) = S(g_\cdot |f_\cdot )|C_\cdot (L) \] ,
\[ \alpha_\cdot (\ell, t) = g(\ell) \text{ for } (\ell, t) \in L \times I . \]

Let \( V_\cdot \) be a cone with base \( A_\cdot \), and define \( \beta_\cdot : V_\cdot \to B_\cdot \) by
\[ \beta_\cdot (\lambda c + \mu a) = (0, \ldots, 0, -\lambda) + \mu \alpha_\cdot (a) , \]
where \( a \in A_\cdot \) and \( c \) is the apex of \( V_\cdot \).

Let \( W \) be the union of \( V_+ \) and \( V_- \) identified along \( L \times I \) (Figure 6.5). Define \( H : W \to S^n \) by \( H|V_+ = \beta_+ \) and \( H|V_- = \beta_- \).
Then \( W \) and \( H \) have the desired properties. \( \blacksquare \)

**Exercises**

1. What does theorem 6.1 say about winding numbers (\( k = 1 \), \( \ell = 0 \), and \( n = 2 \))?

2. Prove that a nonorientable polyhedral surface can't be embedded in \( \mathbb{R}^3 \) (cf. chapter 1). (Hint: If the surface were embedded in \( \mathbb{R}^3 \), there would exist a curve in \( \mathbb{R}^3 \) crossing the surface once.)
Figure 6.1. $f: x^1 \rightarrow \mathbb{R}^2$ and $g: y^1 \rightarrow \mathbb{R}^2$ with $\mathcal{L}(f,g) = 0$.

a. $X = \mathbb{D}^2$, $Y = S^1$

b. $X = [0,1]$, $Y = S^2$

Figure 6.2. $\mathcal{L}(f,g) = \mathcal{L}(f|\partial X, g)$

Figure 6.3. $\mathcal{L}(f,g) = \mathcal{L}(\partial f, \partial g)$
Figure 6.4. A homology between $g$ and the "suspension" of $g_i$.

Figure 6.5. $W =$ the union of $V_+$ and $V_-$ along $L \times I$. 
§7. The double point cycle

Let \( X^k \) be a \( k \)-cycle, and let \( f : X^k \to \mathbb{R}^n, \quad n > k \), be a spread-out map. Define a \((2k-n)\)-chain \( \mathcal{D}(f) \subset D(f) \) as follows. Let \( K \) be a triangulation of \( X \) with respect to which \( f \) is simplicial, and let \( \sigma \) be a \((2k-n)\)-simplex in \( D(f) \). Since \((f, \partial f)\) is spread-out, \( \sigma \not\in \partial X \). Define

\[
d_f(\sigma) = \bigoplus_{\sigma'} \mathcal{L}(L_f\sigma', L_f\sigma) \quad \text{dim}(\sigma') = 2k-n, \quad \sigma' \neq \sigma, \quad f(\sigma') = f(\sigma),
\]

and let \( \mathcal{D}(f) \) be the union of all \((2k-n)\)-simplexes \( \sigma \) of \( K \) such that \( d_f(\sigma) \neq 0 \) (Figure 7.1). Note that \( \dim(LX_\sigma) = k - (2k-n) - 1 = n - k - 1 \), \( LR^n_f(\sigma) \) is a sphere of dimension \( n - (2k-n) - 1 = 2n - 2k - 1 \), and \( \Im(L_f\sigma') \cap \Im(L_f\sigma) = \emptyset \), since \( \dim(D(f)) < 2k - n \). Thus \( \mathcal{L}(L_f\sigma', L_f\sigma) \) is defined.

We will prove that \( \mathcal{D}(f) \) is a cycle. This can be seen intuitively as follows. If \( \tau \) is a \((2k-n-1)\)-simplex of \( \mathcal{D}(f) \), then the number of \((2k-n)\)-simplices \( \sigma \) of \( \mathcal{D}(f) \) such that \( \tau < \sigma \) is equal to the sum of the intersection numbers \( \mathcal{L}(L_f\tau', L_f\tau) \), where \( \tau' \neq \tau \) and \( f(\tau') = f(\tau) \), plus the number of points of \( D(L_f\tau) \). Each intersection number \( \mathcal{L}(L_f\tau', L_f\tau) \) is zero because \( LX_\tau \) and \( LX_\tau \) are cycles (corollary 6.2). The link \( LX_\tau \) is an \((n-k)\)-cycle, and \( L_f\tau : LX_\tau \to S^{2n-2k} \) is spread-out. Therefore \( L_f\tau \) has an even number of double points \( \mathcal{D}(L_f\tau) \), since double points - in the source! - occur in pairs (Figure 7.2).

More generally, suppose that \( X \) is a \( k \)-chain, and that
(f, ∂f) : (X, ∂X) → (R^n_+, R^{n-1}) is spread-out, n > k. This means that f and ∂f are spread-out and f^{-1}(R^{n-1}) = ∂X. (Recall that R^n_+ = \{(x_1, ..., x_n) ∈ R^n | x_n ≥ 0\}.) Defining D(f) just as above, we have the following basic result.

**Theorem 7.1** Let X^k be a k-chain. If (f, ∂f) : (X^k, ∂X) → (R^n_+, R^{n-1}), n > k, is spread-out, then

\[ ∂D(f) = D(∂f) \]

(Figure 7.3)

**Corollary 7.2** If X^k is a k-cycle and f : X^k + R^n, n > k, is spread-out, then D(f) is a (2k-n)-cycle.

**Corollary 7.3** Let X^k be a k-chain and let (f, ∂f) and (g, ∂g) be spread-out maps from (X, ∂X) to (R^n_+, R^{n-1}), n > k.

Let i : D(f) + D(g) → (X, ∂X) and j : D(g) + D(g) → (X, ∂X)

be the inclusions. Then i + j ∈ B_{2n-k}(X, ∂X).

**Definition.** Let X be a k-chain. For each integer i > 0, the double point class D^i(X) ∈ H_{k-i}(X, ∂X) is the homology class of the inclusion map D(f) + D(f) → (X, ∂X), where (f, ∂f) : (X, ∂X) → (R^n_+, R^{n-1}) is a spread-out map. (D^i(X) is well-defined by corollary 7.3.) D^0(X) = [X] ∈ H_k(X, ∂X), the fundamental class of X.

**Corollary 7.2** follows immediately from the theorem.
proof of 7.3. Let \( h : \partial X \times I \to \mathbb{R}^{n-1} \) be a homotopy from \( \partial f \) to \( \partial g \) such that \( h(x, t) = \partial f(x) \) for all \( t \in [0, \varepsilon] \) and \( h(x, t) = \partial g(x) \) for all \( t \in [1 - \varepsilon, 1] \) for some \( \varepsilon > 0 \). Let \( (e, \partial e) : (\mathbb{R}_+^n \times I, \partial (\mathbb{R}_+^n \times I)) \to (\mathbb{R}_+^{n+1}, \mathbb{R}^n) \) be a PL homeomorphism. Define \( h' : \partial (X \times I) \to \mathbb{R}^n \) by

\[
h'(x, t) = \begin{cases} 
\partial e(h(x, t), t) & \text{if } x \in \partial X \\
\partial e(f(x), t) & \text{if } t \in [0, \varepsilon] \\
\partial e(g(x), t) & \text{if } t \in [1 - \varepsilon, 1]
\end{cases}
\]

By theorem 2.2(b), there is a spread out map \( \tilde{h}' : \partial (X \times I) \to \mathbb{R}^n \) such that \( h'(x, t) = \tilde{h}'(x, t) \) if \( t \in [0, \varepsilon] \cup [1 - \varepsilon, 1] \).

Let \( H : (X \times I, \partial (X \times I)) \to (\mathbb{R}_+^{n+1}, \mathbb{R}^n) \) be an extension of \( \tilde{h}' \), such that \( H(x, t) = e(f(x), t) \) for all \( t \in [0, \varepsilon] \) and \( H(x, t) = e(g(x), t) \) for all \( t \in [1 - \varepsilon, 1] \). By theorem 2.2(b), there is a spread out map \( \tilde{H} : X \times I \to \mathbb{R}_+^{n+1} \) such that \( \tilde{H}(x, t) = H(x, t) \) if \( t \in [0, \varepsilon] \cup [1 - \varepsilon, 1] \) or \( x \in \partial X \), and \( \tilde{H}^{-1}(\mathbb{R}^n) = \partial (X \times I) \).

Now let \( W = D(\tilde{H}) \), and let \( F : W \to X \) be the restriction of the projection \( X \times I \to X \). Theorem 7.1 implies that \( W \) is a (2k-n+1)-chain with \( \partial W = \partial D(\tilde{H}) = D(\partial \tilde{H}) \). Thus \( D(f) \) and \( D(g) \) are contained in \( \partial W \). Furthermore, \( Fr(\partial W, D(f)) \subset \partial D(f) \) and \( Fr(\partial W, D(g)) \subset \partial D(g) \), since \( \tilde{H} \) agrees with \( H \) near \( X \times \{0\} \).
and $X \times \{1\}$. Finally, $F(\partial W \setminus (\partial f) \cup \partial g)) \subset \partial X$ since $H(\partial X \times I) \subset \mathbb{R}^n$. Thus $F : W \to X$ is a homology between $\partial f \subset X$ and $\partial g \subset X$ (Figure 7.4).

**Proof of 7.1** Let $K$ be a triangulation of $X$ with respect to which $f$ is simplicial, and let $\tau$ be a $(2k-n-1)$-simplex of $K$. Let $\partial f(\tau)$ be the number (mod 2) of $(2k-n)$-simplices $\sigma$ such that $d_f(\tau) \neq 0$ and $\tau < \sigma$. We must show that $\partial f(\tau) = 0$ if $\tau \not\in \partial X$ and $\partial f(\tau) = d_{\partial f}(\tau)$ if $\tau \in \partial X$.

We claim the following:

(a) $\partial d_f(\tau) = \sum_{\tau'} \partial(\partial f_{\tau'}) \cup \partial f_{\tau} \quad \tau' \neq \tau, \quad f(\tau') = f(\tau)$.

(b) $\partial(\partial f_{\tau}) \cup \partial f_{\tau} = \partial(\partial\partial f_{\tau}) \cup \partial\partial f_{\tau}$.

(c) If $\tau \not\in \partial X$, then $\partial(\partial X_{\tau}) = \emptyset$. If $\tau \in \partial X$, then $\partial(\partial X_{\tau}) = \partial(\partial X)_{\tau}$. 


From (a) and (b) we conclude that

\[ \partial_d f(\tau) = \sum_{\tau'} L(\partial f_{\tau'}, \partial f_{\tau}) , \quad \tau' \neq \tau , \quad f(\tau') = f(\tau) . \]

Thus (c) implies that if \( \tau \notin \partial X \), then \( \partial_d f(\tau) = 0 \), and that if \( \tau \in \partial X \), then

\[ \partial_d f(\tau) = \sum_{\tau'} L(\partial f_{\tau'}, \partial f_{\tau}) , \quad \tau' \neq \tau , \quad \partial f(\tau') = \partial f(\tau) , \]

which equals \( d_{\partial f}(\tau) \), as desired.

Proof of (c): If \( \tau \) is an i-simplex of \( K \), \( \dim(LK_{\tau}) = k - i - 1 \). The \( l \)-simplices of \( \tau LK_{\tau} \) are all the joins \( \tau a \), where \( a \) is an \((l\text{-}i\text{-}l)\)-simplex of \( LK_{\tau} \). If \( a, b \in LK_{\tau} \), then \( a < b \) if and only if \( \tau a < \tau b \). Thus \( a \in \partial LK_{\tau} \) if and only if \( \tau a \in \partial X \) if and only if \( a \in L(\partial X)_{\tau} \) (Figure 7.5).

Proof of (a): If \( \tau \in K \) and \( a \) is a simplex of \( LK_{\tau} \), then \( L(LK_{\tau})_{\alpha} = LK_{\tau a} \) (Figure 7.6). If \( \tau \) is an \((i\text{-}l)\)-face of an \( i \)-simplex \( \sigma \in K \), and \( \sigma' \) is an \( i \)-simplex of \( K \) such that \( f(\sigma') = f(\sigma) \), then there exists a unique \((i\text{-}l)\)-face \( \tau' \) of \( \sigma' \) such that \( f(\tau') = f(\tau) \).

Now let \( \tau \) be a \((2k-n-l)\)-simplex of \( K \). Then

\[ \partial_d f(\tau) = \sum_{\sigma} d_{f}(\sigma) , \quad \dim(\sigma) = 2k - n , \quad \tau < \sigma \]
= \sum_{\sigma', \sigma} \mathcal{L}(\text{L}(\text{f}_{\sigma'^{\prime}}), \text{L}(\text{f}_{\sigma})), \dim(\sigma') = \dim(\sigma) = 2k - n, \sigma' \neq \sigma, \text{f}(\sigma') = \text{f}(\sigma), \tau < \sigma

= \sum_{\alpha, \beta} \mathcal{L}(\text{L}(\text{L}(\text{f}_{\alpha}^{\prime}), \text{L}(\text{f}_{\beta}))), \dim(\tau') = \dim(\tau) = 2k - n - 1, \dim(\alpha) = \dim(\beta) = 0, \tau' \neq \tau, \alpha \neq \beta, \text{f}(\tau') = \text{f}(\tau), \text{f}(\alpha) = \text{f}(\beta)

= \sum_{\alpha, \beta} \mathcal{L}(\text{L}(\text{L}(\text{f}_{\alpha}^{\prime}), \text{L}(\text{f}_{\beta}))), \alpha \neq \beta, \text{f}(\alpha) = \text{f}(\beta)

+ \sum_{\alpha, \beta} \mathcal{L}(\text{L}(\text{L}(\text{f}_{\alpha}^{\prime}), \text{L}(\text{f}_{\beta}))), \tau' \neq \tau, \text{f}(\tau') = \text{f}(\tau), \text{f}(\alpha) = \text{f}(\beta).

In this last expression, every term in the first summand appears twice, so it equals zero (mod 2). The second summand equals

\[ \sum_{\tau'} \mathcal{L}(\text{L}(\text{L}_{\tau}^{\prime}), \text{L}(\text{L}_{\tau})), \tau' \neq \tau, \text{f}(\tau') = \text{f}(\tau). \]

Proof of (b): If \( \text{f}(\tau') = \text{f}(\tau) \notin \mathbb{R}^{n-1} \), then \( \tau, \tau' \notin \exists X \), \( L_X_{\tau} \), and \( L_X_{\tau} \) are \((n-k)\)-cycles, and \( \text{L}(\mathbb{R}^n_+) \text{f}(\tau) \) is a \((2n-2k)\)-sphere. Thus corollary 6.2 implies that \( \mathcal{L}(\text{L}(\text{f}_{\tau}^{\prime}), \text{L}(\text{f}_{\tau})) = 0 \). On the other hand, if \( \text{f}(\tau') = \text{f}(\tau) \in \mathbb{R}^{n-1} \), then \( \tau, \tau' \in \exists X \), \( L_X_{\tau} \), and \( L_X_{\tau} \) are \((n-k)\)-chains, and \( \text{L}(\mathbb{R}^n_+) \text{f}(\tau) \) is a \((2n-2k)\)-ball. Thus corollary 6.3 implies that \( \mathcal{L}(\text{L}(\text{f}_{\tau}^{\prime}), \text{L}(\text{f}_{\tau})) = \mathcal{L}(\exists \text{L}_X_{\tau}^{\prime}, \exists \text{L}_X_{\tau}) \).

Let \( Y^k \subseteq X^k \) be \( k \)-chains such that \( \text{Fr}(X, Y) \subseteq \exists Y \) or equivalently, \( \exists Y = \text{Fr}(X, Y) \cup (Y \cap \exists X) \). The restriction morphism

\[ \tau(X, Y) : H_1(X, \exists X) \rightarrow H_1(Y, \exists Y) \]
is defined by the diagram

\[
\begin{array}{c}
H_i(X, \partial X) \\
\downarrow \\
H_i(X, \partial X \cup \text{Cl}(X \setminus Y)) \cong H_i(Y, \text{Fr}(X, Y) \cup (Y \cap \partial X)) \\
\end{array}
\xrightarrow{r(X, Y)} H_i(Y, \partial Y)
\]

Proposition 7.4 If \( Y^k \subseteq X^k \) are \( k \)-chains with \( \text{Fr}(X, Y) \subseteq \partial Y \), then \( r(X, Y)D^i(X) = D^i(Y) \) for all \( i \geq 0 \).

Proof. If \( k \) or \( i \) equals zero, this is clear. If \( k \) and \( i \) are positive, let \( n = k + i \), and let

\[
Q_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0 \text{ and } x_{n-1} \geq 0 \right\},
\]

\[
\partial Q = \left\{ (x_1, \ldots, x_n) \in Q_n \mid x_n = 0 \text{ or } x_{n-1} = 0 \right\}.
\]

There exists a spread-out map \( f : (X, \partial X) \to (\mathbb{R}_+^n, \mathbb{R}^{n-1}) \) such that \( f^{-1}(Q) = Y \) and \( f^{-1}(\partial Q) = \partial Y \). Let \( h : (Q, \partial Q) \to (\mathbb{R}_+^n, \mathbb{R}^{n-1}) \) be a PL homeomorphism. Then

\[
D^i(Y) = [D(h \circ f | Y)] = r(X, Y) [D(f)] = r(X, Y) D^i(X).
\]

Exercises

1. If \( X \) is an orientable \( k \)-cycle, and \( f : X \to \mathbb{R}^n \), \( n > k \), is spread out, is \( D(f) \) orientable?

2. Let \( X \) be the Möbius strip. Construct an embedding of \( X \) in
$R^3$ so that $\exists x \in R^2$. Prove that there exists no embedding of $X$ in $R_+^3$ with $\exists x \in R^2$.

3. If $f : X^k \to M^n$ is a spread-out map of the $k$-cycle $X$ to the polyhedral $n$-manifold $M$, $k < n$, then $D(f)$ is a cycle whose homology class in $X$ depends only on the homotopy class of $f$.

4. $D^i(SX) = \sigma D^i(X)$ (cf. exercise 3 of §3).

5. If $X$ and $Y$ are $k$-cycles and $f : X \to Y$ is a map, then $f_\ast D^i(X) = \deg(f) D^i(Y)$.

6. Illustrate proposition 7.4 with $X$ the Klein bottle and $Y$ the Moebius strip.
Figure 7.1. $\mathbb{D}(f)$. 
Figure 7.2. $\mathbb{D}(L_\mathbf{f}_\tau)$ must consist of an even number of points.

Figure 7.3. $\mathbb{D}(\mathbf{f}) = \mathbb{D}(\delta\mathbf{f})$. 
Figure 7.4. \(\mathcal{D}(f)\) is homologous to \(\mathcal{D}(g)\).
Figure 7.5. $\partial LX_\tau = L(\partial X)_\tau$

Figure 7.6. $L(LX_\tau)_\alpha = LX_{\tau\alpha}$
§8. The branch point cycle

Let \( X^k \) be a \( k \)-chain, and let \( (f, \delta f) : (X, \partial X) \to (R^n_+, R^{n-1}) \) be spread-out, \( n > k \). Define a \((2k-n-1)\)-chain \( B(f) \subseteq B(f) \) as follows. Let \( K \) be a triangulation of \( X \) with respect to which \( f \) is simplicial, and let \( \tau \) be a \((2k-n-1)\)-simplex in \( B(f) \). Since \((f, \delta f)\) is spread-out, \( \tau \not\in \partial X \). Define

\[
 b_f(\tau) = \sum_{\{\sigma, \sigma'\} \in \mathcal{L}(L_f(\sigma), L_f(\sigma'))} \mathcal{L}_f(\sigma', \sigma) \text{, } \dim(\sigma) = \dim(\sigma') = 2k - n \text{, } \sigma \neq \sigma' \text{, } f(\sigma) = f(\sigma') \text{, } \tau < \sigma \text{, } \tau < \sigma' .
\]

This sum is taken over all unordered pairs \( \{\sigma, \sigma'\} \). Let \( B(f) \) be the union of all \((2k-n-1)\)-simplices \( \tau \) of \( K \) such that \( b_f(\tau) \neq 0 \). (Figure 8.1).

If \( n = k \), \( B(f) \subseteq B(f) \) is defined as follows. Let \( K \) and \( L \) be triangulations of \( X^k \) and \( R^n_+ \) with respect to which \( f \) is simplicial. Let \( \tau \in K \) be a \((k-1)\)-simplex of \( B(f) \), and let \( t = f(\tau) \in L \). Since \((f, \delta f)\) is spread-out, \( t \not\in R^{n-1} \). Let \( s \) and \( s' \) be the two \( k \)-simplices of \( L \) which have \( t \) as a face. Let \( a \) be the (integral) number of \( k \)-simplices \( \sigma \in K \) such that \( f(\sigma) = s \) and \( \tau < \sigma \), and let \( b \) be the number of \( k \)-simplices \( \sigma' \in K \) such that \( f(\sigma') = s' \) and \( \tau < \sigma' \). Since \( \tau \not\in \partial X \), \( a + b \) is even. Let \( b_f(\tau) = \frac{1}{2}(a - b) \) (mod 2), and let \( B(f) \) be the union of all \((k-1)\)-simplices \( \tau \) of \( K \) such that \( b_f(\tau) \neq 0 \).

We will prove that if \( X \) is a cycle, then \( B(f) \) is a cycle. This can be seen intuitively (for \( n > k \)) as follows. If \( \omega \) is a \((2k-n-2)\)-simplex of \( B(f) \), the number of \((2k-n-1)\)-simplices \( \tau \)
of \( B(f) \) such that \( \omega < \tau \) equals the number of points in \( B(Lf_\omega) \). The link \( L_\omega X \) is an \((n-k+1)\)-cycle, and \( Lf_\omega : L_\omega X \to S^{2n-2k+1} \) is a spread-out map. Such a map always has an even number of branch points, since the image of the branch points in \( S^{2n-2k+1} \) is the boundary of the image of the double points (Figure 8.2).

**Theorem 8.1** Let \( x^k \) be a \( k \)-chain. If \( (f, \partial f) : (x^k, \partial x) \Rightarrow (R^n_+, R^{n-1}) \), \( n \geq k \), is a spread-out map, then

\[
\partial B(f) = B(\partial f)
\]

**Corollary 8.2** If \( x^k \) is a \( k \)-cycle and \( f : x^k \to \mathbb{R}^n \), \( n \geq k \), spread-out, then \( B(f) \) is a \((2k-n-1)\)-cycle.

**Corollary 8.3** Let \( x^k \) be a \( k \)-chain, and let \( (f, \partial f) \) and \( (g, \partial g) \) be spread-out maps from \((x, \partial x)\) to \((R^n_+, R^{n-1})\), \( n \geq k \). Let \( i : (B(f), \partial B(f)) \to (x, \partial x) \) and \( j : (B(g), \partial B(g)) \to (x, \partial x) \) be the inclusions. Then \( i + j \in B_{2n-k}(x, \partial x) \).

Corollary 8.2 follows immediately from the theorem, and the proof of 8.3 is just like the proof of 7.3.

**Definition.** Let \( x \) be a \( k \)-chain. For each integer \( i > 0 \), the branch point class \( B_i(x) = H_{k-i}(x, \partial x) \) is the homology class of the inclusion map \( (B(f), \partial B(f)) \to (x, \partial x) \), where \( (f, \partial f) : (x, \partial x) \Rightarrow (R^n_+, R^{n-1}) \) is a spread-out map. \( B_i(x) \) is well-defined by corollary 8.3.) \( B_0(x) = [x] \in H_k(x, \partial x) \).
Proof of 8.1 \( (n > k) \). Let \( K \) be a triangulation of \( X \) with respect to which \( f \) is simplicial, and let \( \omega \) be a \((2k-n-2)\)-simplex of \( X \). Let \( \partial_b f(\omega) \) be the number \( \text{mod } 2 \) of \((2k-n-1)\)-simplices \( \tau \) such that \( b_f(\tau) \neq 0 \) and \( \omega < \tau \). We must show that \( \partial_b f(\omega) = 0 \) if \( \omega \notin \partial X \) and \( \partial_b f(\omega) = b_{\partial f}(\omega) \) if \( \omega \in \partial X \).

If \( v \in LX_\omega \) and \( \tau = \omega v \in \omega LX_\omega \), then \( v \in B(Lf_\omega) \) if and only if \( \tau \in B(f) \), so \( \partial_b f(\omega) \) is the number of points in \( B(Lf_\omega) \). On the other hand, if \( \omega \in \partial X \), then

\[
b_{\partial f}(\omega) = \sum_{\{\tau, \tau'\}} L(L\partial f_\tau, L\partial f_{\tau'}) \quad \text{dim}(\tau) = \text{dim}(\tau') = 2k - n - 1, \\
\tau \neq \tau', \quad f(\tau) = f(\tau'), \\
\omega < \tau, \quad \omega < \tau'
\]

\[
= \sum_{\{v, v'\}} L(L\partial f_\omega)_v, L(L\partial f_\omega)_{v'} \quad \text{dim}(v) = \text{dim}(v') = 0, \\
v \neq v', \quad f(v) = f(v').
\]

Therefore, theorem 8.1 is a consequence of the following lemma applied to \( (g, \partial g) = (Lf_\omega, L\partial f_\omega) \).

Lemma 8.4 If \( Y^\ell \) is an \( \ell \)-chain and \( (g, \partial g) : (Y, \partial Y) \to (B^{2\ell-1}, S^{2\ell-2}) \) is a spread-out map, the number of points in \( B(g) \) equals

\[
\sum_{\{v, v'\}} L(L\partial g_v, L\partial g_{v'}) \quad \text{dim}(v) = \text{dim}(v') = 0, \\
v \neq v', \quad g(v) = g(v').
\]

(Figure 8.3).
proof. Let $K$ and $L$ be triangulations of $Y^l$ and $B^{2l-1}$ with respect to which $g$ is simplicial. For each 1-simplex $\sigma$ of $L$, let

$$i(\sigma) = \sum_{\{\tau, \tau'\}} \mathcal{L}(Lg_{\tau}, Lg_{\tau'}) \quad \text{dim}(\tau) = \text{dim}(\tau') = 1 \quad \tau \neq \tau' \quad g(\tau) = g(\tau') = \sigma.$$ 

Let $u$ be a vertex of $B^{2l-1}$. If $u \notin S^{2l-2}$, then

$$\sum_{u < \sigma} i(\sigma) = \sum_{u = g(v)} b^g(v) = \text{number of points in } B(g) \text{ with } g(v) = u.$$ 

If $u \in S^{2l-2}$, then

$$\sum_{u < \sigma} i(\sigma) = \sum_{\sigma} \sum_{\{\tau, \tau'\}} \mathcal{L}(Lg_{\tau}, Lg_{\tau'}) \quad u < \sigma \quad \tau \neq \tau' \quad g(\tau) = g(\tau') = \sigma$$

$$= \sum_{\{v, v'\}} \sum_{\{w, w'\}} \mathcal{L}(Lg_v, Lg_{v'}), \quad \mathcal{L}(Lg_w, Lg_{w'}), \quad v \neq v' \quad g(v) = g(v') = u \quad g(w) = g(w')$$

$$= \sum_{\{v, v'\}} \mathcal{L}(Lg_v, Lg_{v'}) \quad v \neq v' \quad g(v) = g(v') = u$$

$$= \sum_{\{v, v'\}} \mathcal{L}(Lg_v, Lg_{v'}) \quad v \neq v' \quad g(v) = g(v') = u.$$ 

(The last step is justified by corollary 6.3.)

Therefore, $\sum_{u} \sum_{u < \sigma} i(\sigma)$ equals this last expression plus the number of points in $B(g)$. But $\sum_{u} \sum_{u < \sigma} i(\sigma)$ is zero.
(mod 2), since each l-simplex \( \sigma \) appears twice in the sum, once for each endpoint. 

**Proposition 8.5** If \( Y^k \subseteq X^k \) are \( k \)-chains with \( \text{Fr}(X,Y) \subseteq \partial Y \), then \( r(X,Y) \mathbb{B}^i(X) = \mathbb{B}^i(Y) \) for all \( i \geq 0 \).

This can be proved just as proposition 7.4, or it can be derived from the following result.

If \( X^k \) is a \( k \)-chain and \( f : X^k \to \mathbb{R}^n \) is a spread-out map, \( n \geq k \), the number \( b_f(\tau) \) can be defined as above for any \((2k-n-1)\)-simplex \( \tau \notin \partial X \). Let \( \mathbb{B}(f) \) be the union of all such simplices \( \tau \) for which \( b_f(\tau) \neq 0 \).

**Proposition 8.6** If \( X^k \) is a \( k \)-chain, and \( f : X^k \to \mathbb{R}^n \), \( n \geq k \), is spread-out, then \( \mathbb{B}(f) \subseteq \partial X \), and the homology class of the inclusion \( (\mathbb{B}(f), \mathbb{B}(f)) \to (X, \partial X) \) is \( \mathbb{B}^{n-k+1}(X) \).

**Remark.** The corresponding statement about \( \mathbb{D} \) is false (Figure 8.4).

**proof.** The statement that \( \mathbb{B}(f) \subseteq \partial X \) follows from the proof of theorem 8.1. That \( (\mathbb{B}(f), \mathbb{B}(f)) \to (X, \partial X) \) represents \( \mathbb{B}^{n-k+1}(X) \) follows from the fact that \( f \) is homotopic to a map \( g : X \to \mathbb{R}^n \) with \( g(X) \subseteq \mathbb{R}^n_+ \) and \( g(\partial X) \subseteq \mathbb{R}^{n-1}_+ \), and an argument similar to the proof of corollary 7.3. 

\( \Box \)
Exercises

1. Prove theorem 8.1 for $n = k$.

2. Let $Y$ be an $k$-cycle and let $g : Y \to \mathbb{R}^{2k}$ be a spread-out map. The crossing number of $g$ is

$$c(g) = \sum_{\{p,q\}} L(g_p, g_q), \quad g(p) = g(q),$$

summed over all unordered pairs $\{p,q\}$. Show that if $X$ is a $k$-cycle and $f : X \to \mathbb{R}^n$ is spread-out, $n \geq k$, then $\mathbb{B}(f)$ is the union of all $(2n-k-1)$-simplices $\tau$ of a triangulation of $f$ with $c(Lf_\tau) \neq 0$.

3. Illustrate theorem 8.1 with $X$ the Moebius strip and $n = 2$.

4. Illustrate lemma 8.4 with $Y$ the Moebius strip.
Figure 8.1. \((f, \partial f): (\mathbb{R}^2, \partial x) \rightarrow (\mathbb{R}_+, \mathbb{R}^2)\)

Figure 8.2. \(f: S^2 \rightarrow \mathbb{R}^3\) spread-out. The image of the branch points is the boundary of the image of the double points.
Figure 8.3. \((g, \delta g): (Y^2, \delta Y) \rightarrow (B^3, S^2)\) spreadout. Note: To avoid confusion not all the simplices have been illustrated.

Figure 8.4. \(Y^1 \subset X^1\) such that \(\mathcal{D}(f|Y) \neq \mathcal{D}(f) \cap Y\)
§9. The double-branch homology

The final major result of this chapter will be that $\mathbb{E}^i(X) = \mathbb{E}^i(X)$ for all $i \geq 0$. In the proof of this theorem we will need the following lemma.

**Lemma 9.1** Let $x^k$ and $y^l$ be cycles, and $f : x^k \rightarrow R^n$, $g : y^l \rightarrow R^n$, $n = k + l + 1$, be PL maps such that $f(x) \cap g(y) = \emptyset$. Let $p : R^n \rightarrow R^{n-1}$ be defined by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$, and let $\pi : R^n \rightarrow R$ be defined by $\pi(x_1, \ldots, x_n) = x_n$. Suppose that $\{(x, y) \in X \times Y \mid pf(x) = pg(y)\}$ is finite. Then

$$\mathcal{L}(f, g) = \bigcup_{x, y} \mathcal{L}(Lp_f x, Lp_g y), \quad pf(x) = pg(y), \quad \pi f(x) < \pi g(y).$$

(Figure 9.1).

**Proof.** Choose $T \in R$ such that $\pi g(y) - \pi f(x) < T$ and $\pi g(y) < T$ for all $x \in X$ and $y \in Y$. Let $W = (X \times [0, T]) \cup C_+(X)$, with the identification of $X \times \{T\}$ with $X \in C_+(X)$. Define $F : W \rightarrow R^n$ as follows:

$$F(x, t) = f(x) + (0, \ldots, 0, t), \quad (x, t) \in X \times [0, T],$$
$$F(\lambda c + \mu x) = \mu f(x) + (0, \ldots, 0, T), \quad \lambda c + \mu x \in C_+(X).$$

By theorem 6.1, $\mathcal{L}(f, g) = \mathcal{L}(F, g)$. To complete the proof, we will show that if $F(x, t) = g(y)$, then

$$\mathcal{L}(Lp_f x, Lp_g y) = \mathcal{L}(LF(x, t), Lg_y).$$
Let $z = pf(x)$. Since $F(x,t) = g(y)$, $z = pg(y)$ as well.

Let $h : R^n \to R^n$ be the map $h(s_1, ..., s_n) = (s_1, ..., s_{n-1}, s_n + pg(y))$, and let $i : R^{n-1} \to R^n$ be the inclusion $i(s_1, ..., s_{n-1}) = (s_1, ..., s_{n-1}, 0)$. Let $h' = Lh_i(z)$ and $i' = Li_z$. By proposition 5.1(f),

$$\mathcal{L}(Lp_f, Lp_g) = \mathcal{L}(SLp_f, i' \circ Lp_g)$$

$$= \mathcal{L}(h' \circ SLp_f, h' \circ i' \circ Lp_g) .$$

Now $h' \circ SLp_f$ is homotopic to $LF(x,t)$, and $h' \circ i' \circ Lp_g$ is homotopic to $Lg$, by homotopies with disjoint images.

So proposition 5.1(b) implies that

$$\mathcal{L}(h' \circ SLp_f, h' \circ i' \circ Lp_g) = \mathcal{L}(LF(x,t), Lg) .$$

(Figure 9.2)

**Theorem 9.2** If $X^k$ is a $k$-chain, then $D^i(X) = B^i(X) \in H_{k-i}(X, \partial X)$ for $i > 0$.

(Figure 9.3)

**Proof.** This is true by definition for $i = 0$. We prove the theorem for $i > 1$. The case $i = 1$ requires a separate argument.

Let $n = k + i$, and let $p : R^n_+ \to R^{n-1}_+$, $p(s_1, ..., s_n) = (s_2, ..., s_n)$, and $\pi : R^n_+ \to R$, $\pi(s_1, ..., s_n) = s_1$. By theorem 2.2(c), there exist spread-out maps $(f, \partial f) : (X, \partial X) \to (R^n_+, R^{n-1}_+)$ and $(g, \partial g) : (X, \partial X) \to (R^{n-1}_+, R^{n-2})$ such that $g = p \circ f$.
and \( g^{-1}(R^{n-2}) = f^{-1}(R^{n-1}) = \partial X \). Let \( K \) be a triangulation of \( X \) with respect to which \( f \) and \( g \) are simplicial.

Let \( \sigma, \sigma' \) be \((k-i+1)\)-simplices of \( K \) such that \( g(\sigma') = g(\sigma) \).

Let \( f(\sigma') \neq f(\sigma) \), it follows that either \( \pi f(s) < \pi f(s') \) for all \( s \in \sigma^o \) and \( s' \in (\sigma')^o \) with \( g(s) = g(s') \), or \( \pi f(s) > \pi f(s') \) for all \( s \in \sigma^o \) and \( s' \in (\sigma')^o \) with \( g(s) = g(s') \). We will abbreviate these two conditions to \( \pi f(\sigma) < \pi f(\sigma') \) or \( \pi f(\sigma) > \pi f(\sigma') \).

If \( \sigma \) is a \((k-i+1)\)-simplex of \( K \), we define

\[
\partial_g^{-}(\sigma) = \sum_{\sigma' \neq \sigma, g(\sigma') = g(\sigma), \pi f(\sigma') > \pi f(\sigma)} L(\sigma', \sigma).
\]

Let \( D^-(g) \subseteq D(g) \) be the union of all the \((k-i+1)\)-simplices \( \sigma \) of \( K \) such that \( \partial_g^{-}(\sigma) \neq 0 \).

If \( \tau \) is a \((k-i)\)-simplex of \( K \) with \( \tau \notin \partial X \), let

\[
\partial_d^{-}(\tau) = \sum_{\tau \subset \sigma} \partial_g^{-}(\sigma).
\]

We must show that

\[
\partial_d^{-}(\tau) = d_f(\tau) + b_g(\tau).
\]

By definition,

\[
\partial_d^{-}(\tau) = \sum_{\sigma', \sigma, g(\sigma') = g(\sigma), \partial \sigma' \cap \partial \tau} L(\sigma', \sigma), \quad \tau \subset \sigma, \quad \sigma' \neq \sigma,
\]

\[
\pi f(\sigma') > \pi f(\sigma).
\]
If $g(\sigma') = g(\sigma)$ and $f(\sigma') \neq f(\sigma)$, then $f(\sigma') \cap f(\sigma)$ is either empty, or a common face of $f(\sigma')$ and $f(\sigma)$. Thus this last expression equals

(i) \[ \sum_{\sigma', \sigma} \mathcal{L}(Lg_{\sigma'}, Lg_{\sigma}) , \quad \tau < \sigma , \quad \tau < \sigma' , \quad \sigma' \neq \sigma , \quad g(\sigma') = g(\sigma) , \quad \pi f(\sigma') > \pi f(\sigma) \]

(ii) \[ + \sum_{\nu', \nu} \mathcal{L}(Lg_{\tau', \nu'}, Lg_{\tau \nu}) , \quad \tau' \neq \tau , \quad f(\tau') = f(\tau) , \]
\[ g(\nu') = g(\nu) , \quad \pi f(\nu') > \pi f(\nu) \]

(iii) \[ + \sum_{\nu', \nu} \mathcal{L}(Lg_{\tau', \nu'}, Lg_{\tau \nu}) , \quad \tau' \neq \tau , \quad f(\tau') \neq f(\tau) , \]
\[ g(\nu') = g(\nu) , \quad \pi f(\tau') > \pi f(\tau) , \quad g(\tau') = g(\tau) . \]

The summand (i) equals $b_g(\tau)$. Lemma 9.1 applied to $Lf_{\tau'}$ and $Lf_{\tau}$ shows that the summand (ii) equals

\[ \sum_{\tau'} \mathcal{L}(Lf_{\tau'}, Lf_{\tau}) , \quad \tau' \neq \tau , \quad f(\tau') = f(\tau) , \]

which is $d_f(\tau)$. The summand (iii) equals

\[ \sum_{\tau'} \mathcal{L}(Lg_{\tau'}, Lg_{\tau}) , \quad \tau' \neq \tau , \quad f(\tau') \neq f(\tau) , \]
\[ \pi f(\tau') > \pi f(\tau) , \quad g(\tau') = g(\tau) . \]

Since $\tau \neq \partial X$, $L X_{\tau}$ and $L X_{\tau'}$ are cycles. By corollary 6.2, $\mathcal{L}(Lg_{\tau'}, Lg_{\tau}) = 0$, so (iii) is zero. \[ \square \]
Exercises

1. Prove $\mathbb{D}^1 = \mathbb{B}^1$. Illustrate this for the Klein bottle.

2. Discuss the integral version of lemma 9.1.
Figure 9.1. \( \mathcal{L}(f,g) = \sum_{x,y} \mathcal{L}(pf_x, pg_y) \) 

\[ pf(x) = pg(y), \]

\[ \pi f(x) < \pi g(y) \]

---

\[ \text{SLpf}_x (\text{SLX}_x) = LF(x,t) (L(X \times I)(x,t)) \]

---

Figure 9.2. \( \mathcal{L}(LF(x,t), Lg_y) = \mathcal{L}(Lpf_x, Lpg_y) \)
Figure 9.3. $\mathbb{H}(f)$ is homologous to $\mathbb{B}(g)$. 
References


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Lectures on Homology Operations
Chapter 3

by

Clint McCrory and Kent Johnson

§1. Homology operations

§2. The geometric operations $\phi^i$

§3. The intersection axiom

§4. Steenrod operations

§5. A combinatorial formula for $\phi^i$

§6. Smith operations
§1. Homology operations

Let \((H, \partial)\) be a homology theory satisfying the first six axioms of Eilenberg and Steenrod [ES, p. 10-11], for example mod 2 geometric homology (section 3 of chapter 2).

A **homology operation** \(\theta\) is a natural transformation from the functor \(H\) to itself. That is, for each pair of spaces \((A, B)\) there is a function \(\theta(A, B) : H(A, B) \to H(A, B)\) such that for every continuous map \(f : (A, B) \to (C, D)\) the diagram

\[
\begin{array}{ccc}
H(A, B) & \xrightarrow{H(f)} & H(C, D) \\
\downarrow{\theta(A, B)} & & \downarrow{\theta(C, D)} \\
H(A, B) & \xrightarrow{H(f)} & H(C, D)
\end{array}
\]

commutes. The operation \(\theta\) is **stable** if it commutes with the boundary transformation; that is, if the diagram

\[
\begin{array}{ccc}
H(A, B) & \xrightarrow{\partial} & H(B) \\
\downarrow{\theta(A, B)} & & \downarrow{\theta(A)} \\
H(A, B) & \xrightarrow{\partial} & H(B)
\end{array}
\]

commutes for all \((A, B)\). \(\theta\) has **degree** \(i\) if it lowers degrees of all homology classes by \(i\), that is, if
\[ \theta(A,B)(H_k(A,B)) \subset H_{k-1}(A,B) \]

for all \( k \).

**Example.** Let \((H, \partial)\) be mod 2 simplicial homology theory. The **Bockstein operation** \( \beta \) is a stable homology operation of degree one,

\[ \beta = \beta(K,L) : H_k(K,L) \to H_{k-1}(K,L) . \]

It can be defined as follows (we take \( L = \emptyset \) for simplicity). If \( c \) is any simplicial mod 2 cycle in \( K \), choose an integral chain \( \tilde{c} \) such that \( \tilde{c}_2 = c \), where \( \tilde{c}_2 \) is the mod 2 reduction of \( \tilde{c} \). Then \( (\partial \tilde{c})_2 = \partial(\tilde{c}_2) = \partial c = 0 \), so all the coefficients in \( \partial \tilde{c} \) are even, and therefore \( \frac{1}{2} \partial \tilde{c} \) is still an integral chain. Furthermore \( \partial(\frac{1}{2} \partial \tilde{c}) = \frac{1}{2} \partial^2 \tilde{c} = 0 \), i.e. \( \frac{1}{2} \partial \tilde{c} \) is an integral cycle. The Bockstein operations sends the class of \( c \) to the class of the mod 2 reduction of \( \frac{1}{2} \partial \tilde{c} \):

\[ \beta[c] = [\frac{1}{2} \partial \tilde{c}] . \]

It is very easy to check that \( \beta[c] \) is independent of the choice of \( \tilde{c} \).

**Remark.** The operation \( \beta \) can be defined similarly in mod 2 singu-
lar homology. In the next section we shall see that $\beta$ has a geometric interpretation in terms of double points or branch points.

For example if $K$ is a triangulation of the Klein bottle and $\langle K \rangle \in H_2(K)$ is represented by the sum of all the 2-simplices in $K$, then $\beta\langle K \rangle \neq 0$. On the other hand if $T$ is a triangulation of the torus then $\beta\langle T \rangle = 0$. (Figure 1.1)

To illustrate how homology operations can be used, we are now able to prove that there is no continuous map from the torus to the Klein bottle with nonzero mod 2 degree. For if there were such a map $f$, then $f_*\langle T \rangle = \langle K \rangle$, so we would have $\beta f_*\langle T \rangle = \beta\langle K \rangle \neq 0$. But $\beta f_*\langle T \rangle = f_* \beta \langle T \rangle$ by naturality, and $f_* \beta \langle T \rangle = 0$ since $\beta\langle T \rangle = 0$, giving a contradiction.

**Exercises**

1. Show that a homology operation is stable if and only if it commutes with the suspension homomorphism.

2. Show that every homology operation $\theta$ is additive, i.e. $\theta(A,B)$ is a homomorphism for all $(A,B)$. (This observation is due to Steenrod.)

3. Check that the simplicial Bockstein homomorphism $\beta$ is well-defined. Show that $\beta \circ \beta = 0$, and that $\beta$ is stable.

4. Let $X = |K|$ be a polyhedral $k$-cycle, and let $\langle X \rangle \in H_k(K)$ be
the simplicial fundamental class of $X$. Show that $B(f)$ represents $\beta\langle X \rangle$, where $f : X \to R^k$ is a spread-out map which is simplicial with respect to $K$. 
Figure 1.1. The Bockstein operation.
§2. The geometric operations $\phi^i$

If $X$ is a geometric $k$-chain then the homology class $\mathbb{M}^i(X) = B^i(X) \in H_{k-i}(X, \partial X)$ of chapter 2 can be interpreted as an operation on mod 2 geometric homology as follows. We define

$$\phi^i = \phi^i(A, B) : H_k(A, B) \rightarrow H_{k-i}(A, B)$$

for all $k$ by sending the homology class of the element $f : (X, \partial X) \rightarrow (A, B)$ of $Z_k(A, B)$ to the class of $f_*\mathbb{M}^i(X) :$

$$\phi^i[f] = f_*\mathbb{M}^i(X).$$

To show that this is a good definition, we have to check that $\phi^i$ does not depend on the choice of $f$ in its homology class and that $\phi^i$ is a natural transformation.

**Proposition 2.1** If $f : (X, \partial X) \rightarrow (A, B)$ is in $B_k(A, B)$ then $f_*\mathbb{M}^i(X) = 0$ for all $i$.

**proof:** By definition of $B_k(A, B)$ there exists a $(k+1)$-chain $W$ and a commutative diagram

$$
\begin{array}{ccc}
(e(X), e(\partial X)) & \xrightarrow{\iota} & (\partial W, C_\partial W \setminus e(X)) \\
\downarrow \approx & & \downarrow \approx \\
(X, \partial X) & \xrightarrow{f} & (A, B)
\end{array}
$$
where $e$ is a polyhedral embedding and $i$ and $j$ are inclusions. Let $r : H_{k-i}(\partial W) \rightarrow H_{k-i}(e(X), e(\partial X))$ be the restriction homomorphism (see proposition 7.4 of chapter 2). Then we have

$$f_* D^i(X) = F_* i_* e_* D^i(X)$$
$$= F_* i_* D^i(e(X))$$
$$= F_* i_* r_* D^i(\partial W)$$
(by proposition 7.4 of chapter 2)
$$= F_* i_* r_* D^i(W)$$
(by theorem 7.1 of chapter 2)
$$= 0$$

since $j_*$ and $(i_* r_*)$ are consecutive homomorphisms in the long exact sequence of the triple $(W, \partial W, (\partial W \setminus e(X)))$.

**Proposition 2.2** $\phi^i$ is a natural transformation.

**proof:** Let $g : (A, B) \rightarrow (C, D)$ be a continuous map, and let $f : (X, \partial X) \rightarrow (A, B)$ be in $Z_k(A, B)$. Then $g_* \phi^i[f] = g_* f_* D^i(X) = (g \circ f)_* D^i(X) = \phi^i[g \circ f] = \phi^i g_*[f]$.

**Proposition 2.3** The operation $\phi^i$ has the following properties:

1. $\phi^i$ is stable.
2. $\phi^0$ is the identity transformation.
3. $\phi^1$ is the Bockstein transformation.
4. If $x \in H_k(A, B)$ then $\phi^i(x) = 0$ for all $i \geq k$.

**proof:** (1) follows from theorem 7.1 of chapter 2. (2) follows from
the definition of $D^i(X)$, as does (4) for $i > k$. For $i = k$, (4) is true since double points occur in pairs in the source (cf. figure 7.2 of chapter 2). And (3) comes from exercise 4 of section 1 and the definition of $IB^1$.

Exercises

1. Prove (4) of proposition 2.3 for $i = k$ by using (1) and the cone on a polyhedral chain.

2. Prove (3) of proposition 2.3 by using the double point cycle.

3. Prove that $\phi^i$ commutes with suspension by using the double point cycle.

4. Give a geometric proof of proposition 2.1.
§3. The intersection axiom

The homology operation $\phi^i$ is characterized by its relation to the self-intersection of a homology class in a manifold.

Let $M^n$ be a PL n-manifold (without boundary), and let $X^k \subset M^n$ be a k-chain and $Y^\ell \subset M^n$ an $\ell$-chain, with $k + \ell \geq n$, $k > 0$, and $\ell > 0$. Suppose that $X$ and $Y$ are in general position, i.e. $\dim(X \cap Y) \leq k + \ell - n$, $\dim(X \cap \emptyset Y) \leq k + \ell - n - 1$, and $\dim(\partial X \cap Y) \leq k + \ell - n - 1$. Then an intersection chain $X.Y \subset X \cap Y$ can be defined as follows.

Triangulate $M$ so that $X$ and $Y$ are subcomplexes. For each $(k+\ell-n)$-simplex $\sigma$ of $X \cap Y$, define

$$i(\sigma) = \mathcal{L}(LX_\sigma, LY_\sigma),$$

the linking number of the $(n-k-1)$-cycle $LX_\sigma$ and the $(n-\ell-1)$-cycle $LY_\sigma$ in the $(2n-k-\ell-1)$-sphere $LM_\sigma$. Let $X.Y$ be the union of all the $(k+\ell-n)$-simplices $\sigma$ of $X \cap Y$ such that $i(\sigma) \neq 0$.

Proposition 3.1 If $X$ and $Y$ are chains in general position in the manifold $M$, then

$$\emptyset(X,Y) = (\emptyset X).Y + X.(\emptyset Y).$$
This is an easy consequence of the linking theorem (6.1 of chapter 2). (Cf. figure 3.1)

**Lemma 3.2** If $\alpha \in H_k(M)$ and $\beta \in H_\ell(M)$, where $M$ is a manifold, there is a cycle $X \subset M$ representing $\alpha$ and a cycle $Y \subset M$ representing $\beta$ such that $X$ and $Y$ are in general position.

This follows from the spreading out technique of chapter 2, §2.

Now if $\alpha \in H_k(M^n)$ and $\beta \in H_\ell(M^n)$, we define $\alpha \beta \in H_{k+\ell-n}(M^n)$ to be the homology class of $X \cdot Y$, for $X$ and $Y$ as in lemma 3.2. It is not hard to show that this intersection product is a well-defined associative, commutative bilinear pairing

$$H_k(M^n) \times H_\ell(M^n) \to H_{k+\ell-n}(M^n).$$

This definition is due to Lefschetz [L, ch. IV §6].

(Alexandroff and Hopf probably would have developed this definition in volume two of their book [AH], had it been written. They developed linking in Euclidean space in volume one.) The definition can be extended to non-embedded geometric chains (cf. §6 of chapter 2), and to a manifold with boundary, modulo a (PL) subset:

$$H_k(M^n, A) \times H_\ell(M^n, A) \to H_{k+\ell-n}(M^n, A).$$
Theorem 3.3

1) Let $M$ be a compact $n$-dimensional PL submanifold (with boundary) of $\mathbb{R}^n$, and let $A$ be a subpolyhedron of $M$. If $\alpha \in H_{n-1}(M,A)$ then $\psi_i^*(\alpha) = \alpha \cdot \alpha \in H_{n-2i}(M,A)$.

2) If $\psi_i$ is any mod 2 homology operation of degree $i$ such that $\psi_i^*(\alpha) = \alpha \cdot \alpha$ for all $\alpha$ as in (1), then $\psi_i = \phi_i$.

proof of (1). Replacing A by a regular neighborhood, we can assume that $A = N$, a compact $n$-manifold with boundary. Let $(X, \partial X) \subset (\text{int } M, \text{int } N)$ be an embedded chain representing $\alpha$.

For each vector $v \in \mathbb{R}^n$, define $g_v : X \to \mathbb{R}^n$ by $g_v(x) = x + v$. Let $W = (X \times 0) \cup (\partial X \times I) \cup (X \times 1)$, and define $h_v : W \to \mathbb{R}^n$ by

$$h_v(x,0) = x,$$
$$h_v(x,t) = x + tv,$$  $x \in \partial X$,  $0 \leq t \leq 1$,
$$h_v(x,1) = g_v(x).$$

There exists a vector $v \in \mathbb{R}^n$ such that

a) $g_v(tX, \partial X) \subset (\text{int } M, \text{int } N)$ for all $t \in [0,1]$,

b) $X$ and $g_v(X)$ are in general position,

c) $h_v$ is spread-out.
Choose such a vector $v$, and let $Y = q_v(X)$, $h = h_v$. It follows from the definition of the double point cycle $\Delta$ that

$$[X,Y] = i_*r(W,X \times 0)\Delta(h) \in H_{n-2i}(M,N),$$

where $i : (X \times 0, \partial X \times 0) \to (M,N)$ is the inclusion and $r(W,X \times 0)$ is the restriction homomorphism. By proposition 7.4 of chapter 2, $r(W,X \times 0)\Delta(h)$ represents $\Delta(X \times 0)$, so $i_*r(W,X \times 0)\Delta(h)$ represents $\phi^i(\alpha)$, and (1) is proved.

**Proof of (2).** It suffices to show that $\psi^i[X] = \phi^i[X]$ for any $k$-chain $X$. For suppose $\alpha \in H_k(A,B)$ is represented by $f : (X, \partial X) \to (A,B)$, so that $\alpha = f_*[X]$. Then $\psi^i(\alpha) = \psi^i_f_*[X] = f_*\psi^i[X] = f_*\phi^i[X] = \phi^i_f_*[X] = \phi^i(\alpha)$. In fact it suffices to show that $\psi^i[Z] = \phi^i[Z]$ for any $k$-cycle $Z$. For if $X$ is a $k$-chain, then the space $dX$, obtained by gluing together two copies of $X$ along $\partial X$, is a cycle.

Letting $r : H_*(dX) \to H_*(X, \partial X)$ be the restriction homomorphism, we then have $\psi^i[X] = \psi^i_r[dX] = r\psi^i[dX] = r\phi^i[dX] = \phi^i_r[dX] = \phi^i[X]$.

If $Y$ is a $k$-chain which embeds piecewise-linearly in $R^{k+i}$, then $\psi^i[Y] = \phi^i[Y]$. For let $(M,N)$ be a regular neighborhood of $(Y, \partial Y)$ in $R^{k+i}$, and let $i : (Y, \partial Y) \to (M,N)$ be the inclusion. Then $i_*\psi^i[Y] = \psi^i_i_*[Y] = (i_*[Y]).(i_*[Y]) = \phi^i_i_*[Y] = i_*\phi^i[Y]$, so $\psi^i[Y] = \phi^i[Y]$ since $i_*$ is an isomorphism.
Now let $Z$ be a $k$-cycle. We shall construct a $k$-chain $Y \subset Z$ such that

a) $Fr(Z,Y) = \emptyset Y$, 

b) the restriction homomorphism $r : H_{k-i}(Z) \rightarrow H_{k-i}(Y, \emptyset Y)$ is injective, and 

c) $Y$ embeds in $R^{k+i}$.

This will prove (2), because $r\psi^i[Z] = \psi^i r[Z] = \psi^i[Y] = \phi^i[Y] = \phi^i r[Z] = r\phi^i[Z]$, so $\psi^i[Z] = \phi^i[Z]$.

So it remains to construct the $k$-chain $Y \subset Z$. Let $K$ be a triangulation of $Z$, and let $Y$ be the closure of the complement of a regular neighborhood of the $(k-i-1)$-skeleton $K_{k-i-1}$ of $K$. Then (a) is clear. To see (b) consider the following commutative diagram:

$$
\begin{array}{ccc}
H_{k-i}(Z) & \xrightarrow{r} & H_{k-i}(Y, \emptyset Y) \\
\downarrow u & & \downarrow v \\
H_{k-i}(Z, |K_{k-i-1}|) & \xrightarrow{w} & H_{k-i}(Z, Cl(Z\setminus Y))
\end{array}
$$

Here $u, v, w$ are induced by inclusions. The map $u$ is injective since $H_{k-i}(|K_{k-i-1}|) = 0$. The map $v$ is an excision isomorphism, and $w$ is an isomorphism because $|K_{k-i-1}|$ is a strong deformation retract of $Cl(Z\setminus Y)$. Therefore $r$ is injective.
Finally we show that (c) holds, for an appropriate choice of triangulation $K$. Choose $K$ so that there is a spread-out map $f : Z \to R^{k+1}$ which is simplicial on $K$. Then $D(f) \subset |K_{k-1}|$. Now modify the second barycentric subdivision $K''$ as follows. Move the barycenters of the $(k-1)$-simplices $\sigma$ of $K$ to points $b(\sigma)$ such that $f(b(\sigma_1)) = f(b(\sigma_2))$ only if $\sigma_1 = \sigma_2$. Call the resulting subdivision $\tilde{K}'$. For each simplex $\omega'$ of $\tilde{K}'$ there is a unique simplex $\sigma$ of $K$ such that the interior of $\omega'$ is contained in the interior of $\sigma$. For each such $\omega'$, move the barycenter of $\omega'$ to a point $b(\omega')$ very close to $b(\sigma)$. Call the resulting shifted second barycentric subdivision $\tilde{K}''$. Choose the points $b(\omega')$ so close to the corresponding points $b(\sigma)$ that $f(\text{Star}(b(\sigma_1),\tilde{K}'')) \cap f(\text{Star}(b(\sigma_2),\tilde{K}'')) \neq \emptyset$ only if $\sigma_1 = \sigma_2$.

Now let $Y$ (respectively $\tilde{Y}$) be the union of all the simplices of $K''$ (respectively $\tilde{K}''$) which have no vertices in the subdivision of $K_{k-1}$. Then $Y$ is PL homeomorphic to $\tilde{Y}$, and $f|\tilde{Y}$ is an embedding, by construction of $Y$.

Exercises

1. Prove proposition 3.1.

2. What are the signs in the integral version of proposition 3.1?

3. Carry out the proof that the intersection product is well-defined in homology. (You will need a stronger version of lemma 3.2.)
4. Go through the proof of 3.3(2) for $Z$ the Klein bottle. In other words, show that if $\psi$ has the intersection property then $\psi[Z] = \phi[Z]$. 

5. Let $L$ be a subcomplex of the simplicial complex $K$. Construct a strong deformation retraction of $N$ to $|L|$, where $N$ is the stellar neighborhood of $L''$ in $K''$ (second barycentric subdivisions).
Figure 3.1. $\delta(X,Y) = (\delta X) \cdot Y + X \cdot (\delta Y)$.
§4. Steenrod operations

The set $O$ of all operations on mod 2 homology is a ring, with addition $(\theta + \psi)(\alpha) = \theta(\alpha) + \psi(\alpha)$, multiplication $(\theta \psi)(\alpha) = \theta(\psi(\alpha))$ and multiplicative identity 1 equal to the identity transformation. Furthermore $O$ is graded by degree. Any $\theta \in O$ can be written uniquely as a sum $\theta = \theta^0 + \theta^1 + \theta^2 + \ldots$, where $\theta^i$ has degree $i$. (If $\alpha$ is a homology class of dimension $k$ then $\theta^i(\alpha) = 0$ for $i > k$, so $\theta(\alpha) = \theta^0(\alpha) + \theta^1(\alpha) + \ldots + \theta^k(\alpha)$, a finite sum.)

Define $\phi \in O$ by $\phi = \phi^0 + \phi^1 + \phi^2 + \ldots$. Since $\phi^0 = 1$ there exists a unique operation $\bar{\phi} \in O$ such that $\phi \bar{\phi} = 1 = \bar{\phi} \phi$. The components $\bar{\phi}^i$ of $\bar{\phi}$ can be calculated inductively from those of $\phi$ from the identity $(1 + \phi^1 + \phi^2 + \ldots)(\bar{\phi}^0 + \bar{\phi}^1 + \bar{\phi}^2 + \ldots) = 1$.

Since mod 2 cohomology $H^k(A, B)$ can be identified with $\text{Hom}(H_k(A, B), \mathbb{Z}/2)$, the dual of the $\mathbb{Z}/2$ vector space $H_k(A, B)$, we can define a pairing $<,>$ between cohomology and homology as follows. If $\alpha \in H^k(A, B)$ and $\beta \in H_k(A, B)$ then

$$<\alpha, \beta> = \begin{cases} 
\alpha(\beta) & \text{if } k = l \\
0 & \text{if } k \neq l
\end{cases}.$$
Furthermore, we can define an action of $\phi^i$ on cohomology by the identity

$$<\phi^i(\alpha), \beta> = <\alpha, \phi^i(\beta)>.$$ 

Let $Sq^i$ be the $i$th Steenrod operation on mod 2 cohomology [SE].

**Theorem 4.1** $\phi^i = Sq^i$.

**Proof.** We can just as well make $Sq^i$ act on homology, by requiring that

$$<\alpha, Sq^i(\beta)> = <\overline{Sq}^i(\alpha), \beta>$$

(cf. [MS, p.136]), and the theorem is equivalent to the statement that $\phi^i = Sq^i$ on homology. To show this we need only check that the action of $Sq^i$ on homology is natural and satisfies the intersection axiom, by theorem 3.1.

a) $Sq^i$ is natural, i.e. $Sq^i f_* (\beta) = f_* Sq^i (\beta)$ for all homology classes $\beta$ and maps $f$. This follows from the naturality of $\overline{Sq}^i$ on cohomology: $<\alpha, Sq^i f_* \beta> = <\overline{Sq}^i \alpha, f_* \beta> = <f_* \overline{Sq}^i \alpha, \beta> = <\overline{Sq}^i f_* \alpha, \beta> = <f_* \alpha, Sq^i \beta> = <\alpha, f_* Sq^i \beta>.$

b) $Sq^i$ satisfies the intersection property of 3.3(1) To prove this we need a lemma.
Let \( Sq = Sq^0 + Sq^1 + Sq^2 + \ldots \), and let \( \sim \) be the cap product pairing.

**Lemma 4.2**

1. \( Sq(\alpha \sim \beta) = Sq\alpha \sim Sq\beta \).

2. If \( M \) is a compact \( n \)-dimensional submanifold of \( \mathbb{R}^n \), then \( Sq[M] = [M] \).

The proof of this lemma is left as an exercise. (Hints are given below.)

Now let \( \alpha \in H^i(M \setminus N, (M \setminus N) \cap \partial M) \) be the Poincaré dual of \( \beta \), i.e. \( \alpha \sim [M] = \beta \). Then

\[
Sq^i \beta = Sq^i (\alpha \sim [M]) \\
= \sum_j Sq^j \alpha \sim Sq^{i-j} [M] \quad \text{by (1)} \\
= Sq^i \alpha \sim [M] \quad \text{by (2)} \\
= (\alpha \sim \alpha) \sim [M] \quad \text{by [SE, p.1, (3)]} \\
= \beta \cdot \beta .
\]

This completes the proof of theorem 4.1.

The proof of the following theorem illustrates how the geometry of \( \phi \) and the algebra of \( Sq \) can be used together.

**Theorem 4.3** Let \( n \) be a positive integer. The real projective space \( \mathbb{P}^{2n} \) cannot be polyhedrally embedded in \( \mathbb{R}^{2n+1-1} \) or polyhedrally immersed in \( \mathbb{R}^{2n+1-2} \).
proof. Let $\alpha$ be a generator of $H^1(\mathbb{P}^n)$ . Then $<\alpha,\text{Sq}^{2n-1}[\mathbb{P}^n]> = \text{Sq}^{2n-1}\alpha, [\mathbb{P}^n]$ $= <\alpha^{2n}, [\mathbb{P}^n]> \neq 0$ . That $\text{Sq}^{2n-1}\alpha = \alpha^{2n}$ is proved by a simple induction (cf. [SE, p.34]). Therefore $\text{Sq}^{2n-1}[\mathbb{P}^n] \neq 0$ . But $\text{Sq}^{2n-1}[\mathbb{P}^n] = \phi^{2n-1}[\mathbb{P}^n] = \text{ID}^{2n-1}[\mathbb{P}^n]$ , the class of the double point cycle of a spread-out map

$$
\mathbb{P}^n + R^{2n+1-1}
$$

so $\mathbb{P}^n$ can't be embedded in $R^{2n+1-1}$ . Since $\text{Sq}^{2n-1}[\mathbb{P}^n] = \text{ID}^{2n-1}[\mathbb{P}^n]$ , the class of the branch point cycle of a spread-out map

$$
\mathbb{P}^n + R^{2n+1-2}
$$

neither can $\mathbb{P}^n$ be immersed in $R^{2n+1-2}$ . (Clearly any immersion can be approximated by a spread-out immersion.)

Remark. This theorem can be strengthened to the nonexistence of topological embeddings and immersions (cf. [M2,§7], [HM,§9]).

A challenging open problem is to give geometric proofs of the Cartan formula [SE, p. 1] and the Adem relations [SE, p. 2] for the Steenrod operations using double points and branch points.

Exercises

1. Let $x_1, x_2, \ldots$ be noncommuting symbols. Define $\overline{x}_1, \overline{x}_2, \ldots$
by the identity \((1 + x_1 + x_2 + \ldots)(1 + \overline{x}_1 + \overline{x}_2 + \ldots) = 1\).

Show that \((1 + \overline{x}_1 + \overline{x}_2 + \ldots)(1 + x_1 + x_2 + \ldots) = 1\).

2. Assuming the Steenrod axioms for \(\text{Sq}^i\) [SE, p. 1], show that \(\text{Sq}(\alpha \wedge \beta) = \text{Sq}\alpha \wedge \text{Sq}\beta\) [MS, p. 136, problem 11-F].

3. Using Thom's formula for the Stiefel-Whitney classes of a bundle [MS, p. 91] and the preceding exercise, show that \(\text{Sq}^i[M]\) is Poincaré dual to \(\overline{w}^i(M)\), the \(i\)-th Stiefel-Whitney class of the stable normal bundle of \(M\).

4. Using exercise 3, prove part (2) of lemma 4.2.

5. Prove geometrically that \(\text{Sq}^1\text{Sq}^{2i+1} = 0\).

6. Prove that if the polyhedron \(X\) embeds piecewise-linearly in \(\mathbb{R}^n\), or immerses piecewise-linearly in \(\mathbb{R}^{n-1}\), and \(\alpha \in H_k(X)\), then \(\text{Sq}^i(\alpha) = 0\) for \(i \geq n - k\).

7. Do exercise 6 with "piecewise-linearly" replaced by "topologically".
§ 5. A combinatorial formula for $\phi^i$ (cf. [BM])

Let $K$ be a triangulation of the $k$-cycle $X$, and choose an ordering of the vertices of $K$. For each ordered pair $(\sigma, \tau)$ of top dimensional simplices of $K$, define $\mu(\sigma, \tau) \in \mathbb{Z}/2\mathbb{Z}$ as follows:

Let $v_1, \ldots, v_s$ be the vertices of $\sigma$ and $\tau$ in the given order. (If a vertex belongs to both $\sigma$ and $\tau$, it should be listed only once.) Let $\mu(\sigma, \tau) = 1$ if $v_1, v_3, v_5, \ldots$ are in $\sigma$ and $v_2, v_4, v_6, \ldots$ are in $\tau$, and $\mu(\sigma, \tau) = 0$ otherwise.

Let $c_{k-i}(K)$ be the simplicial mod 2 $(k-i)$-chain

$$c_{k-i}(K) = \sum_{(\sigma, \tau)} \mu(\sigma, \tau) \sigma \cap \tau,$$

summed over all ordered pairs of simplices $(\sigma, \tau)$ of $K$ such that $\dim(\sigma \cap \tau) = k - i$. Let $[X] \in H_k(X)$ be the fundamental class of $X$.

Theorem 5.1 The chain $c_{k-i}(K)$ is a mod 2 cycle, and its homology class is $\phi^i[X]$.

For example, let $K$ be the octahedron, with vertices ordered as in figure 5.1. With this ordering,

$$c_1(K) = \langle 1,2 \rangle + \langle 2,3 \rangle + \langle 3,4 \rangle + \langle 4,5 \rangle + \langle 5,6 \rangle + \langle 6,1 \rangle.$$
To see that the coefficient of \( <1,2> \) is 1, we calculate \( \mu(\sigma,\tau) \) for all ordered pairs \((\sigma,\tau)\) of 2-simplices such that \( \sigma \cap \tau = <1,2> \). The only 2-simplices with \( <1,2> \) as a face are \( \sigma = <1,2,3> \) and \( \tau = <1,2,6> \). The list of vertices occurring in \( \sigma \) and \( \tau \) is \( 1,2,3,6 \). Since 1 and 3 are in \( \sigma \), while 2 and 6 are in \( \tau \), we have \( \mu(\sigma,\tau) = 1 \); on the other hand, \( \mu(\tau,\sigma) = 0 \). The other coefficients of \( c_1(K) \) are determined similarly.

If we choose a different ordering for the vertices of \( K \), we may get a different chain \( c_{k-1} \). For example, if we use the ordering of figure 5.2 for the octahedron, then

\[
c_1(K) = <1,2> + <2,6> + <6,5> + <5,1> + <2,4> + <4,5> + <5,3> + <3,2>.
\]

The reader can also check that \( c_0(K) = 0 \) using ordering #1, while \( c_0(K) = \sum_{i=1}^{6} <i> \) using ordering #2.

To prove the theorem, let \( f : X + \mathbb{R}^n \), \( n = k + i - 1 \), be a map which is linear on each simplex of \( K \) and which sends the \( i \)th vertex of \( K \) to \( (t_i, t_i^2, \ldots, t_i^n) \), where \( t_i < t_j \) if \( i < j \). In other words, \( f \) maps the vertices of \( K \), in order, to points of the "moment curve" \( (t, t^2, \ldots, t^n) \). [N.B. \( f \) is not necessarily simplicial with respect to \( K \), since the images of simplices may cross each other.] The homology class of \( B(f) \) is \( \phi^i[X] \), provided \( f \) is spread-out. Thus it suffices to prove (1)
f is spread-out, and (2) \( \mathcal{B}(f) \) is the union of the \((k-i)\)-simplices with nonzero coefficient in \( c_{k-i}(K) \). In other words, there is a sub-division \( K' \) of \( K \) such that \( f \) is simplicial with respect to \( K' \), and if the \((k-i)\)-simplex \( \omega' \) of \( K' \) is contained in the \((k-i)\)-simplex \( \omega \) of \( K \), then

\[
\mathcal{B}_f(\omega') = \sum_{\sigma \cap \tau = \omega} \mu(\sigma, \tau).
\]

Before discussing (1) and (2), we show some examples of such maps \( f \). If \( K \) is the octahedron with the ordering of figure 5.1 then the image of \( f : K + R^2 \) is illustrated by figure 5.3. The fold set of \( f \) is the perimeter of the image, so \( \mathcal{B}(f) \) is the support of the chain \( c_1(K) \) associated with the ordering of figure 5.1.

As another example, if \( K \) is the octahedron with the ordering of figure 5.2 then the vertex \( \langle l \rangle \) is in \( \mathcal{B}(f) \) for \( f : K + R^3 \). The image of the top of \( K \) in \( R^3 \) is illustrated by figure 5.4. So \( \langle l \rangle \) is a "pinch point" of the image, and hence \( \langle l \rangle \) is in \( \mathcal{B}(f) \). (In fact the images of all the vertices under \( f \) for this ordering are pinch points.)

We now proceed to prove (1) and to discuss the proof of (2).

(1) \( f \) is spread-out. In fact, if \( v_0, \ldots, v_s \) are distinct vertices of \( K \), \( s \leq n \), then the span of \( f(v_0), \ldots, f(v_s) \) in
$R^n$ has dimension $s$. (This implies that $F$ is spread-out, by
the proof of the spreading out theorem.) For suppose $f(v_i) =
(t_{i_1}, t_{i_2}, \ldots , t_{i_n})$, where $t_i \leq t_j$ if and only if $i \leq j$. The
matrix

$$
\begin{bmatrix}
1 & t_0 & t_0^2 & \cdots & t_0^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & t_s & t_s^2 & \cdots & t_s^n \\
\end{bmatrix}
$$

has rank $s$, since

$$
\det
\begin{bmatrix}
1 & t_0 & t_0^2 & \cdots & t_0^s \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & t_s & t_s^2 & \cdots & t_s^s \\
\end{bmatrix}
= \prod_{i>j} (t_i - t_j) \neq 0
$$

So the dimension of $\text{Span}(f(v_0), \ldots , f(v_s))$ is $s$.

(2) $\mathbb{B}(f)$ equals the support of $c_{k-i}(K)$. The key to proving
this is the following lemma of Arnold Shapiro:

Lemma 5.2 Let $A$ and $B$ be $q$-simplices in $R^m$, $m = 2q$ (or
let $A$ be a $q$-simplex, $B$ a $(q+1)$-simplex, and $m = 2q + 1$ ),
with distinct vertices lying on the moment curve $C$. Then
$A \cap B \neq \emptyset$ if and only if the vertices of $A$ and $B$ alternate
along C.

Now the proof of (2) goes roughly as follows. First recall that the coefficient \( b_f(\omega') \) of \( \omega' \) in \( B(f) \) is \( b_f(\omega') = \sum L(\alpha', \beta') \), where the sum is taken over all pairs \( \{\alpha', \beta'\} \) of \( (k-i+1) \)-simplices in \( K' \), such that \( \omega' < \alpha' \), \( \omega' < \beta' \), \( \alpha' \neq \beta' \), and \( f(\alpha') = f(\beta') \).

If \( \sigma \) and \( \tau \) are \( k \)-simplices of \( K \) with \( \omega < \sigma \), \( \omega < \tau \), and \( \mu(\sigma, \tau) = 1 \), then the definition of \( \mu(\sigma, \tau) \), together with lemma 5.2, implies that there are simplices \( s = <v_1, v_3, v_5, ... > < \sigma \) and \( t = <v_2, v_4, v_6, ... > < \tau \) (where \( v_1, v_2, v_3, ... \) are the vertices of \( \sigma \) and \( \tau \) in order) whose images under \( f \) intersect, say in a point \( p \). Let \( a \) and \( b \) be those points in \( s \) and \( t \) respectively such that \( f(a) = p = f(b) \). Then \( x = aw \) and \( y = bw \) are \( (k-i+1) \)-simplices with \( \omega < x \), \( \omega < y \), \( x \neq y \), and \( f(x) = f(y) \). Furthermore, \( L(\alpha', \beta') = 1 \). (This follows from corollary 6.3 of chapter 2, since if \( D \) is a little \( (2i-2) \)-ball transverse to \( f(x) = f(y) \), with boundary \( LR_{f(a)}^{N} \), then \( f(LX_\alpha) \) is the boundary of \( f(\sigma) \cap D \), \( f(LX_\beta) \) is the boundary of \( f(\tau) \cap D \), and \( f(\sigma) \cap D \), \( f(\tau) \cap D \) intersect simply.) Finally, given a \( (k-i) \)-simplex \( \omega' \) of \( K' \), with \( \omega' \) contained in \( \omega \), each such pair \( \{x, \beta\} \) determines a pair \( \{x', \beta'\} \) in \( K' \) with \( \omega' < x' \), \( \omega' < \beta' \), \( x' \neq \beta' \), \( f(x') = f(\beta') \), and \( L(\alpha'_x, \beta'_x) = 1 \). For further details of the proof of theorem 5.1, see [BM].
Remarks.

(1) As a corollary of theorems 4.1 and 5.1, we get a combinatorial formula for the action of the Steenrod operations $Sq^i$ on homology. This formula is very similar to Steenrod's original (1947) formula for $Sq^i$ in cohomology, using "cup-i products".

(2) By exercise 3 of section 4, theorem 4.1 also gives a combinatorial formula for the normal Stiefel-Whitney classes of a manifold. There is a similar formula for the tangential Stiefel-Whitney classes, due to Goldstein and Turner [GT].

Exercises

1. Prove Shapiro's lemma. Can it be generalized to the case $\dim A + \dim B = m$?

2. Find a combinatorial proof that $c_{k-1}(K)$ is a cycle. (This problem was suggested by Lee Rudolph.)

3. If $K$ is the octahedron, which mod 2 $l$-cycles of $K$ can occur as $c_l(K)$ for some ordering of the vertices of $K$?
Figure 5.4.
§ 6. Smith operations

Let $T$ be a piecewise-linear involution of the polyhedron $X$, that is, a map from $X$ to itself which is PL and such that $T(T(x)) = x$ for all $x \in X$. Let $K$ be a triangulation of $X$ for which $T$ is simplicial (that is, $\sigma \in K$ implies that $\sigma T \in K$), and such that the fixed point set $X_T$ of $T$, $X_T = \{x \in X | T(x) = x\}$, supports a subcomplex of $K$. (Such a triangulation exists for any PL involution.)

Let $C_i$ be the group of mod 2 simplicial $i$-chains of $K$, let $C_* = \bigoplus_i C_i$, and let $T_* : C_* \to C_*$ be the morphism induced by $T$. Then $T_*$ is a morphism of chain complexes, i.e. $T_* \partial c = \partial T_* c$ for all $c \in C_*$. 

Let $C_i^T = \{c \in C_i | T_* c = c\}$ be the group of equivariant $i$-chains of $K$. (N.B. This is not the same as the group generated by the equivariant $i$-simplices.) Then $C_*^T = \bigoplus_i C_i^T$ is a subcomplex of $C_*$, for if $c \in C_i$ and $T_* c = c$, then $T_* \partial c = \partial T_* c = \partial c$, so that $\partial (C_i^T) \subseteq C_{i-1}^T$.

Let $H_*^T(X)$, the $T$-equivariant homology of $X$, be the homology of $C_*^T$. In other words, $H_*^T(X)$ is the group of equivariant $i$-cycles modulo boundaries of equivariant $(i+1)$-chains. $H_*^T(X)$ is independent of the triangulation $K$.

In the late 1930's, P. A. Smith defined homomorphisms...
A : $H_i^T(X) \rightarrow H_i(X^T)$

B : $H_i^T(X) \rightarrow H_{i-1}^T(X)$

(where all coefficients are $\mathbb{Z}/2\mathbb{Z}$) as follows.

If $c = \sum j \sigma_j$ is a chain in $C_i^T$, then $Ac = \sum_{T#\sigma_j = \sigma_j} \sigma_j$, the sum of all the simplices $\sigma_j$ of $c$ such that $T#\sigma_j = \sigma_j$.

Choose a chain $c'$ in $C_i$ so that

$$c = Ac + c' + T#c'$$

($c'$ is not unique), and let $Bc = \partial c'$ . ($Bc$ is a boundary, but not necessarily the boundary of an equivariant chain, so $Bc$ need not represent $0 \in H_{i-1}^T(X)$.)

To see that $A$ and $B$ make sense and that they are well-defined in homology, we need these facts:

1) $\partial Ac = A\partial c$

2) If $\partial c = 0$, then $T#Bc = Bc$

3) $\partial Bc = 0$

4) If $c = Ac + c'' + T#c''$ then $\partial c'' = \partial c' + \partial d$ for some $d \in C_i^T$. 
5) If \( c = \partial e, \ e \in C_{i+1}^T \), and \( e = Ae + e' + T_\# e' \), then we can let \( c' = \partial e' \).

We apply these statements as follows. (1) says that \( A \) is a chain map, so \( A \) induces a map in homology. (2) and (3) say that if \( c \) is an equivariant cycle, so is \( Bc \). (4) says that the class of \( Bc \) in \( H_{i-1}^T \) doesn't depend on the choice of \( c' \), and (4) together with (5) imply that if \( c \) is the boundary of an equivariant chain, then so is \( Bc \). (For if \( c' = \partial e' \) then \( Bc = \partial c' = \partial \partial e' = 0 \).)

The proofs of (1)-(5) are easy:

1) \( \partial c = \partial (Ac + c' + T_\# c') = \partial Ac + \partial c' + T_\# \partial c' \), so \( A\partial c = A\partial Ac + A(\partial c' + T_\# \partial c') = \partial Ac \), because \( A(b + T_\# b) = 0 \) for any chain \( b \), and \( T_\# \sigma = \sigma \) for every \( \sigma \) in \( \partial Ac \).

2) If \( \partial c = 0 \), then \( \partial Ac = A\partial c = A0 = 0 \), so the equation \( \partial c = \partial Ac + \partial c' + T_\# \partial c' \) becomes \( T_\# \partial c' = \partial c' \), or \( T_\# Bc = Bc \).

3) \( \partial Bc = \partial \partial c' \) because \( Bc = \partial c' \), and \( \partial \partial c' = 0 \).

4) We have \( Ac + c' + T_\# c' = c = Ac + c'' + T_\# c'' \), so \( c' + c'' = T_\#(c' + c'') \), i.e. \( c' + c'' \in C_i^T \). But \( \partial c'' = \partial c' + \partial (c' + c'') \), so we can let \( d = c' + c'' \).

5) We have \( c = \partial e = \partial Ae + \partial e' + T_\# \partial e' \), and \( \partial Ae = A\partial e = Ac \), so \( c = Ac + \partial e' + T_\# \partial e' \).
Having defined $A$ and $B$, now define

$$F^n : H_i^T(X) \to H_{i-n}(X^n)$$

by $F^n = AB \cdots B$.

In the early 1950's, Wu Wen-tsün showed that mod 2 homology operations could be defined by using these Smith homomorphisms for the canonical involution on the square of a space.

$$T : X \times X \to X \times X , \ T(x,y) = (y,x) .$$

Wu observed that if $\alpha \in H_k(X)$, then $\alpha \times \alpha$ defines an element of $H_{2k}^T(X \times X)$. He defined a Smith operation $S_m : H_k(X) \to H_{k-m}(X)$ by

$$S_m(\alpha) = F^{k+m}(\alpha \times \alpha) \in H_{k-m}(\Delta) = H_{k-m}(X) ,$$

where $\Delta \subset X \times X$ is the diagonal, which is the fixed point set of $T$.

It is easy to show that if $c$ is a cycle in $X$, then $c \times c$ is an equivariant cycle in $X \times X$. However, it is not obvious (although it's true) that if $c$ is a boundary in $X$ then $c \times c$ the boundary of an equivariant chain in $X \times X$. (Wu overlooked the non-triviality of this in his book [Wu]!)
For example, if $d$ is a 1-simplex and $c = 3d$, then $c \times c$ is not the boundary of $d \times d$, because $d \times d$ has dimension two! But $c \times c$ is the boundary of an equivariant chain, namely the sum of the two diagonals in $d \times d$.

To avoid this problem, we work with a geometric $k$-cycle $X$, and just define $\text{Sm}^i[X]$, where $[X] \in H_k(X)$ is the fundamental class. If $\mathcal{K} \in H_{2k}^T(X \times X)$ is the canonical class, we set

$$\text{Sm}^i[X] = F^{k+i}(\mathcal{K}) \in H_{k-i}(X).$$

**Theorem 6.1 (Wu)** $\text{Sm}^i[X] = \text{Sq}^i[X]$.

Wu's proof of this theorem is a direct combinatorial calculation, which is difficult to read.

Here is the outline of a geometric proof, using branch points and double points. If $N \geq n$ let $p_{N,n} : \mathbb{R}^N + \mathbb{R}^n$ be the orthogonal projection onto the first $n$ coordinates. Given a polyhedral $k$-cycle $X$, choose an embedding $f_N : X \to \mathbb{R}^N$ such that the maps $f_n = p_{N,n} \circ f_N : X \to \mathbb{R}^n$ are spread-out if $n \geq k$, and have $\dim f_n^{-1}(x) \leq k - n$ for all $x \in \mathbb{R}^n$ if $n \leq k$.

Choose a triangulation $K$ of $X$ so that all the maps $f_n$ are simplicial with respect to $K$. Let $L$ be the first barycentric subdivision of the cell complex $K \times K$. The diagonal of $X \times X$ is a subcomplex of $L$. 
Now let $C$ be the sum of all the $2k$-simplices of $L$. Then $F^{k+i}C = \text{AB}^{k+i}C$ represents $\text{Sm}^i[X]$, by definition. I claim that the choices involved in defining $\text{B}^{k+i}$ can be made so that $\text{AB}^{k+i}C = \mathbb{B}(f_{k+i-1})$. This implies theorem 6.1, since $\mathbb{B}(f_{k+i-1})$ represents $\text{Sq}^i[X]$, by theorem 4.1.

For each $j$ define $\mathcal{D}(f_j)$ and $\mathcal{D}^+(f_j)$ to be mod 2 $j$-chains whose supports are contained respectively in the sets

$$\mathcal{C} \{ (x,y) \in X \times X \mid x \neq y, f_j(x) = f_j(y) \},$$

$$\mathcal{C} \{ (x,y) \in X \times X \mid x \neq y, f_j(x) = f_j(y), \pi_{j+1} f_{j+1}(x) > \pi_{j+1} f_{j+1}(y) \},$$

where $\pi_{j+1} : R^{j+1} \to R^1$ is projection onto the last coordinate.

The actual multiplicities of $\mathcal{D}(f_j)$ and $\mathcal{D}^+(f_j)$ are defined using local linking numbers—this is left to the reader. These chains have the following properties:

1) $\mathcal{D}(f_j) = \mathcal{D}^+(f_j) + T_+ \mathcal{D}^+(f_j)$

2) $\exists \mathcal{D}^+(f_j) = \begin{cases} \mathcal{D}(f_{j+1}) & \text{if } j < k - 1 \\ \mathbb{B}(f_j) + \mathcal{D}(f_{j+1}) & \text{if } j \geq k - 1 \end{cases},$

where $\mathbb{B}(f_j)$ is identified with a cycle in the diagonal, and $\mathbb{B}(f_{k-1})$ is just $[X]$.

Using (1) and (2), we shall show

3) $B_j^iC = \begin{cases} \mathcal{D}(f_j) & \text{if } j < k \\ \mathbb{B}(f_{j-1}) + \mathcal{D}(f_j) & \text{if } j \geq k \end{cases}$. 

This is trivial if \( j = 0 \). If (3) holds for \( B^{j-1}C \), then the terms in the equation
\[
B^{j-1}C = AB^{j-1}C + (B^{j-1}C)' + T_#(B^{j-1}C)'
\]
can be chosen using (1):
\[
AB^{j-1}C = \begin{cases} 
0 & \text{if } j \leq k \\
\mathbb{B}(f_{j-2}) & \text{if } j > k
\end{cases}
\]
and we can take \((B^{j-1}C)' = \partial^+(f_{j-1})\), so \( B^jC = B(B^{j-1}C) = \partial(B^{j-1}C) = \partial \partial^+(f_{j-1}) \), and so (2) implies (3).

It follows from (3) that \( AB^jC = \mathbb{B}(f_{j-1}) \) if \( j \geq k \), which is what we wanted.

For example, let \( X \) be an octagon, with maps \( f_2 : X \to R^2 \), \( f_1 : X \to R^1 \), \( f_0 : X \to R^0 \) as illustrated in figure 6.1. The chains \( \partial^+(f_1) \) and \( \mathbb{B}(f_1) \) are illustrated in figure 6.2, which shows that if \( C \) is the sum of all the 2-simplexes of the triangulation \( L \) of \( X \times X \) defined above, then
\[
C = \partial^+(f_0) + T_# \partial^+(f_0),
\]
\[
\partial \partial^+(f_0) = \mathbb{B}(f_0) + \partial^+(f_1) + T_# \partial^+(f_1),
\]
\[
\partial \partial^+(f_1) = \mathbb{B}(f_1) + \partial^+(f_2) + T_# \partial^+(f_2).
\]
Exercises

1. Show that if $T$ is a PL involution of $X$, there is a triangulation $K$ of $X$ for which $T$ is simplicial, and such that $X^T$ is a subcomplex of $K$.

2. Show that $H^T_k(X)$ is independent of the choice of triangulation $K$.

3. Illustrate the proof of theorem 6.1 using a PL embedding of the projective plane in $R^4$. 
Figure 6.1.

Figure 6.2.
References*


*Additional references can be found at the end of chapter 2.