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The Surgery Obstruction Groups of C. T. C. Wall

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1. Introduction

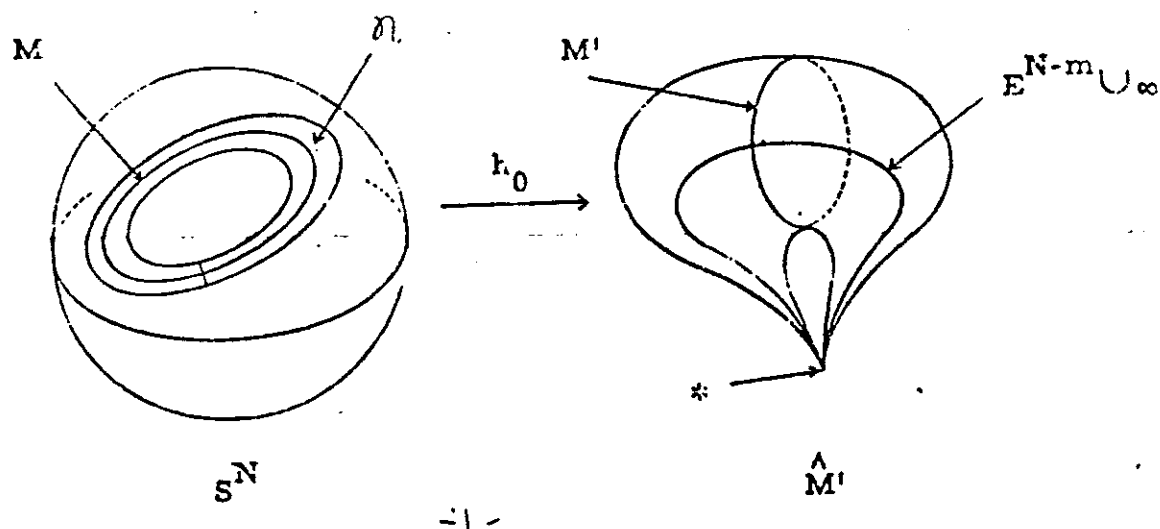
In these notes, we give an expository account of the construction of Wall's Surgery Obstruction Groups [13]. Our account touches on selected topics in the theory of piecewise linear and differentiable manifolds; we refer the reader to [4], [6].

We may take as our starting point the following problem: When is a simple homotopy equivalence of closed smooth manifolds $f: M \rightarrow M'$ homotopic to a diffeomorphism? The following condition is necessary: the two tangent bundles TM and TM' should be indistinguishable; that is, $f^* TM'$ and TM should be equivalent bundles.

Now suppose we are given a simple homotopy equivalence f such that $f^* TM' = TM$; in particular, there is an equivalence of the stable normal bundles $\nu(M \subset S^N)$ and $\nu(M' \subset S^N)$ which covers f . For simplicity, we will suppose $\nu(M' \subset S^N)$ is trivial.

The equivalence of stable normal bundles then defines a map of a tubular neighborhood \mathcal{U} of M in S^N to $M' \times E^{N-m}$ which sends fibres diffeomorphically to fibres, where $m = \dim M$. It follows that this equivalence induces a map $h_0: S^N \rightarrow \hat{M}' := M' \times S^{N-m} / M' \times \infty$ with $h_0(S^N - \mathcal{U}) = *$, the basepoint of \hat{M}' (Figure 1).

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Similarly, the identity on M' induces a map $h_1 : S^N \rightarrow \hat{M}'$.

We now make one further assumption; we suppose h_0 and h_1 are homotopic, say by $h : S^N \times [0, 1] \rightarrow \hat{M}' \times [0, 1]$. It is not hard to see that h_0 and h_1 are ^{each} transverse [7] along M' , and so we may apply the Relative Transversality Theorem to obtain $\hat{h} : S^N \times [0, 1] \rightarrow \hat{M}' \times [0, 1]$ transverse along $M' \times [0, 1]$. Thus, $W^* = \hat{h}^{-1}(M' \times [0, 1])$ is a cobordism between M and M' . Note that the map $\hat{h}|_W$ is covered by a bundle map: $\nu(W \subset S^N \times [0, 1]) \rightarrow \nu(M' \times [0, 1] \subset \hat{M}' \times [0, 1]) = \nu(M' \times [0, 1] \subset S^N \times [0, 1])$.

We thus obtain a cobordism W , a proper map $\varphi = \hat{h}|_W : W \rightarrow M' \times [0, 1]$ and an equivalence $F : \nu(W \subset E^N \times [0, 1]) \rightarrow \varphi^* \nu(M' \times [0, 1] \subset E^N \times [0, 1])$ of stable normal bundles.

Remarks:

- (a) $\varphi|_{\partial W}$ is a simple homotopy equivalence.
- (b) W is compact.
- (c) φ has degree 1, as in §3.

We now attempt to simplify (W, φ, F) by a series of steps (surgeries) so as to obtain (W', φ', F') say, with $\varphi' : W' \rightarrow M \times [0, 1]$ a simple homotopy equivalence, $\varphi' = \varphi$ on $\partial W' = \partial W$. W' would then be an

s-cobordism, and M and M' diffeomorphic.

We will see that there is a geometric obstruction to making this reduction; a surgery problem (W, φ, F) determines an element of an abelian group $L_n(\pi_1 M', \omega M')$ called the Wall Surgery Obstruction Group. If the element so determined is zero, we may reduce W to an s-cobordism. The group $L_n(\pi_1 M', \omega M')$ depends only on the fundamental group of M' and the orientation homomorphism $\omega : \pi_1 M' \rightarrow \mathbb{Z}_2$.

2. The Surgery Problem

In what follows, we will consider the surgery problem in the piecewise linear category; we are given (W, φ, F) , W^n a compact piecewise linear cobordism between two (possibly empty) piecewise linear manifolds, φ a degree 1 proper map to another such cobordism X^n with $\varphi|_{\partial W}$ a simple homotopy equivalence (see §4) $\partial W \rightarrow \partial X$, and F an equivalence of the stable normal bundles νW and $\varphi^* \nu X$.

Let us describe this last condition more fully. A piecewise linear bundle over W is thought of as an R^N bundle in the sense of Steenrod [12], whose fibres are glued together by piecewise linear homeomorphisms $R^N \rightarrow R^N$ preserving the origin (There are in fact some technical problems here, which we suppress). Equivalence of such bundles is the usual equivalence in the sense of Steenrod.

In the smooth category, we single out the tangent bundle and stable normal bundle of a manifold W . We may represent the tangent bundle TW (W smooth, closed) by a tubular neighborhood of the diagonal $\Delta W \subset W \times W$ in a natural way; introduce a Riemannian metric on TW and send $(x, v) \in TW$ to $(x, \exp_x v)$ for sufficiently small v . The stable normal bundle νW is represented similarly, by a tubular neighborhood of an embedding $M \subset E^N$.

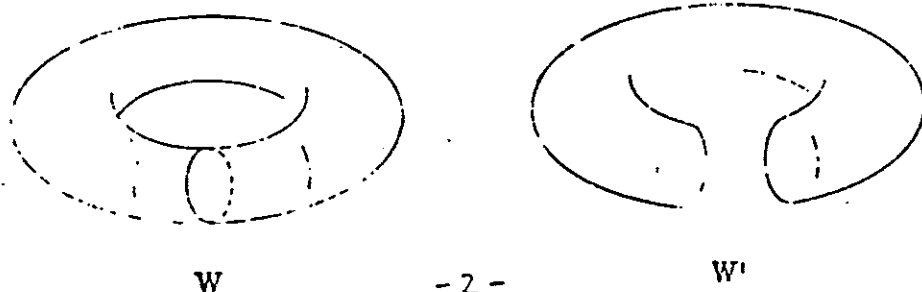
We now construct the tangent bundle of a piecewise linear manifold by taking a neighborhood U of the diagonal $\Delta W \subset W \times W$, together with the 0-section $\Delta : W \rightarrow U$ and bundle projection $\pi_2 : U \rightarrow W$. It is a

theorem of Kuiper and Lashof [5] that U contains in a natural way a piecewise linear R^n bundle called the tangent bundle of W . We construct the stable normal bundle of W by considering the interior of a regular neighborhood of M in R^N ; this can be shown to contain a piecewise linear R^{N-n} bundle [2].

Remark: In what follows, we could replace $(X, \partial X)$ by a Poincaré pair, that is, a pair of finite complexes satisfying Poincaré Duality in the dimension of W , see §3. The stable normal bundle νX is obtained in this case from a construction of Spivak [11].

Let us now turn to the problem of simplifying (W, φ, F) as hinted at in §1, by the method of surgery.

The effect of surgery on W is to remove an embedded $S^k \times D^{n-k}$ from the interior of W and glue in its place $D^{k+1} \times S^{n-k-1} = \partial(D^k \times D^{n-k}) - S^k \times \text{Int } D^{n-k}$ along $S^k \times S^{n-k-1} = (S^k \times D^{n-k}) \cap (D^{k+1} \times S^{n-k-1})$, thus obtaining W' , see figure 2.



The selection of the embedding $S^k \times D^{n-k} \subset W$ and the effect of surgery on the map φ are best described in terms of the homotopy groups of φ which we now define.

An element of $\pi_{k+1}(\varphi)$ is represented by a pair (f, g) as follows:

$$\begin{array}{ccc} S^k & \subset & D^{k+1} \\ f \downarrow & & \downarrow g \\ W & \xrightarrow{\varphi} & X \end{array} \quad (\text{the diagram commutes}).$$

Two such define the same element of $\pi_{k+1}(\varphi)$ if they are homotopic by homotopies which preserve the commutativity of the diagram. We will speak of doing surgery on a class in $\pi_{k+1}(\varphi)$ as in the following theorem.

Theorem 1: Let $\alpha \in \pi_{k+1}(\varphi)$ with $2k \leq n$. α determines a regular homotopy class of immersions of $S^k \times D^{n-k}$ in $\text{Int } W^n$. We can do surgery on α so as to obtain (W', φ', F') if this class contains an embedding.

Remark. $\bar{f}: S^k \times D^{n-k} \rightarrow W^n$ is an immersion if it is locally an embedding.

Two immersions lie in the same regular homotopy class if they are homotopic through immersions.

Proof of Theorem 1: Let (f, g) represent α , thus g is a nullhomotopy of φf ; $f: S^k \rightarrow W$. We may extend (f, g) to (\hat{f}, \hat{g}) defined on $S^k \times D^{n-k}$, $D^{k+1} \times D^{n-k}$ respectively, using the natural projections. By hypothesis we have a bundle equivalence $F: \nu W \rightarrow \varphi^* \nu X$. Then

$$(1) \hat{f}^* \nu W \rightarrow \hat{f}^* \varphi^* \nu X = \hat{g}^* \nu X \rightarrow \varepsilon^N \rightarrow \nu(S^k \times D^{n-k} \subset E^{n+N})$$

is a bundle equivalence; the nullhomotopy \hat{g} determines the equivalence with the trivial bundle ε^N .

Combining this with the equivalence

$$(2) T(S^k \times D^{n-k}) \oplus \hat{f}^* TW = T(S^k \times D^{n-k}) \oplus \hat{f}^* TW$$

and noting that for any manifold Q , $TQ \oplus \nu(Q \subset E^N) = TE^N|_Q = \varepsilon^N$, we obtain the equivalence represented by the horizontal arrow in

$$(3) \begin{array}{ccc} T(S^k \times D^{n-k}) \oplus \varepsilon^M & \longrightarrow & \hat{f}^* TW \oplus \varepsilon^M \\ & \searrow & \downarrow \\ & & TW \oplus \varepsilon^M \end{array}$$

The vertical arrow in (3) is a bundle monomorphism (fibres are mapped homeomorphically onto fibres) and is obtained from the construction

of the induced bundle [12].

Thus the diagonal arrow in (3) is a bundle monomorphism. It is a result of bundle theory [2] that this monomorphism is homotopic through bundle monomorphisms to the sum of a monomorphism $T(S^k \times D^{n-k}) \rightarrow TW$ and the identity $\epsilon^M \rightarrow \epsilon^M$.

We now appeal to the Immersion Classification Theorem of Haefliger and Poenaru [1]. This theorem states that a bundle monomorphism $\Phi: T(S^k \times D^{n-k}) \rightarrow TW$ determines a regular homotopy class of immersions of $S^k \times D^{n-k}$ in W . Any representative immersion \bar{f} determines a monomorphism $df: T(S^k \times D^{n-k}) \rightarrow TW$ which is homotopic through monomorphisms to Φ . df can be represented as the map $\bar{f} \times \bar{f}$ restricted to a neighborhood of the diagonal in $(S^k \times D^{n-k}) \times (S^k \times D^{n-k})$.

Thus (f, g) determines a regular homotopy class of immersions of $S^k \times D^{n-k}$ in $\text{Int } W$; it can be shown that this class is independent of the representative (f, g) of $\alpha \in \pi_{k+1}(\varphi)$. If this class contains an embedding \bar{f} , we can do surgery as described on page 4 so as to obtain $W' = W - \bar{f}(S^k \times D^{n-k}) \cup_{\bar{f}} D^{k+1} \times S^{n-k-1}$.

Since $\varphi\bar{f}$ is nullhomotopic, $\varphi|_{W - \bar{f}(S^k \times D^{n-k})}$ extends to $\varphi': W' \rightarrow X$. Finally, one can check that the bundle equivalence $F|_{\text{Im } \bar{f}}$ corresponds under df to the standard equivalence $\nu(S^k \times D^{n-k} \subset E^n \subset E^{N+n}) \rightarrow \epsilon^N$. It is not hard to see that this last equivalence, when restricted to $S^k \times S^{n-k-1}$ extends to an equivalence $\nu(D^{k+1} \times S^{n-k-1} \subset E^n \subset E^{N+n}) \rightarrow \epsilon^N$ (Both equivalences are restrictions of $\nu(\partial(D^{k+1} \times D^{n-k}) \subset E^{n+1} \subset E^{N+n+1}) \rightarrow \epsilon^N$).

We now define $F': \nu W' \rightarrow \varphi'^* \nu X$ by F on $\nu W' |_{W' - \bar{f}(D^{k+1} \times S^{n-k-1})}$ and by $\nu(D^{k+1} \times S^{n-k-1}) \rightarrow \epsilon^N \rightarrow (\varphi' |_{D^{k+1} \times S^{n-k-1}})^* \nu X$; the first arrow is the equivalence described above, the second is obtained from the nullhomotopy of $\varphi' |_{D^{k+1} \times S^{n-k-1}}$ as constructed. One can check that F' is a well defined bundle equivalence; this completes the proof of Theorem 1.

Corollary: If $2k < n$, we can do surgery on $\alpha \in \pi_{k+1}(W)$.

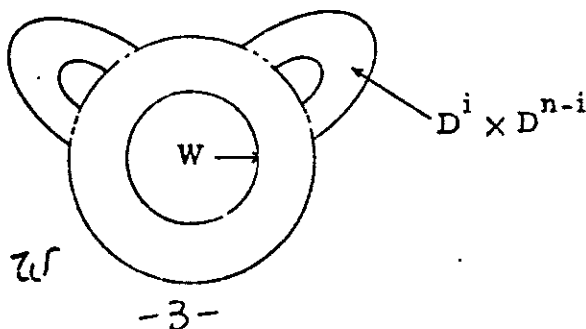
We appeal to the "General Position Theorem" [4]; we can move the

vertices of $\bar{f}(S^k \times 0)$ by a small ambient isotopy of W^n so that the singular set of $\bar{f}|S^k \times 0$ has dimension equal to the dimension of k -planes in n -space in general position, namely $2k-n < 0$. Thus \bar{f} can be chosen in the given regular homotopy class so that \bar{f} is an embedding on $S^k \times 0$. \bar{f} must then embed some neighborhood [8], and we obtain an embedding of $S^k \times D^{n-k}$ in the regular homotopy class of \bar{f} by shrinking the factor D^{n-k} .

Theorem 2: If $2k \leq n$ we can make φ k -connected by a finite number of surgeries on homotopy classes α in dimension $\leq k$.

Proof of Theorem 2: We replace X by the mapping cylinder of φ and φ by an inclusion. Attach the simplexes of $X - W$ to W one at a time so as to obtain a sequence of subcomplexes of X . Let $X_0 \subset X_1$ be a typical pair of successive subcomplexes, which differ by an r -simplex, $r \leq k$.

Now suppose (W', φ', F') is obtained from (W, φ, F) by a finite number of surgeries. It follows from the definition of surgery that W' is cobordant to W (we construct a cobordism \mathcal{W} by adding handles $D^{i+1} \times D^{n-i}$ to $W \times I \subset W \times I$ along $S^i \times D^{n-i}$, see figure 3).



We will suppose inductively that we have obtained \mathcal{W} by adding a finite number of handles of index $\leq k$, together with a homotopy equivalence $\Phi: \mathcal{W} \rightarrow X_0$. Then \mathcal{W} is obtained from W' by adding handles of index $\geq n+1-k \geq k+1 \geq r+1$.

Hence, $W' \subset \mathcal{W}$ is r -connected. Now let $\varphi' = \Phi|W' : W' \rightarrow X$, so

$$(4) \quad \pi_i(\varphi') = \pi_i(W' \times I \cup_{\varphi'} X, W') = \pi_i(W' \times I \cup_{\Phi} X, W') = \pi_i(X, X_0), \quad i \leq r.$$

Mapping Cylinder

Choose $\alpha \in \pi_r(\varphi')$ corresponding to the element of $\pi_r(X, X_0)$ determined by the r -simplex of $X_1 - X_0$. Since $2(r-1) < n$, we can do surgery on α so as to obtain (W'', φ'', F'') together with a cobordism $\mathcal{W} \cup D^r \times D^{n-r}$ of W with W'' , and a homotopy equivalence $\mathcal{W} \cup D^r \times D^{n-r} \rightarrow X_1$ extending Φ .

This completes the inductive step; to complete the proof of the theorem we take $X_0 = W \cup (k\text{-skeleton of } X)$ so that $X_0 \subset X$ is k -connected. Using (4) with $r = k$, we see that φ' is k -connected.

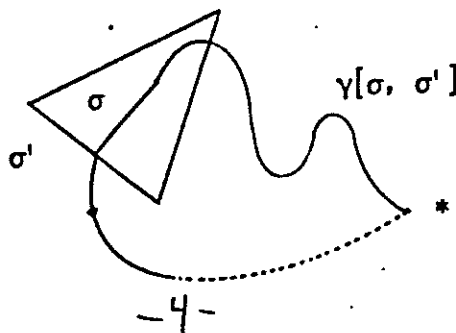
3. Poincaré Duality

We define duality with respect to a fundamental class $[X]$ in homology with twisted integer coefficients $H_n(X, \partial X; \mathbb{Z}^t)$, X a manifold. We first define the orientation homomorphism $\omega : \pi_1 X \rightarrow \{\pm 1\}$ as follows: An element of $\pi_1 X$ maps to $+1$ if transport along a representative loop preserves orientation, -1 otherwise.

$H_n(X, \partial X; \mathbb{Z}^t)$ is defined as the n th homology of the chain complex of ordinary simplicial chains on X with special boundary operator $\bar{\partial}$:

$$(\bar{\partial}f)\sigma' = \sum_{\sigma} [\sigma' : \sigma] \omega(\gamma[\sigma, \sigma']) f_{\sigma}.$$

Here, f is the usual functional notation for chains; f_{σ} is an integral multiple of σ . $[\sigma' : \sigma]$ is the incidence number, and the loop $\gamma[\sigma, \sigma']$ is defined as in figure 4; we suppose that each simplex σ of X is joined to the basepoint $*$ by a prescribed path from the barycentre $\hat{\sigma}$ of σ .



Now choose an orientation of a neighborhood of $*$. Note that when σ is an n -simplex, we may transport the orientation at $*$ along the prescribed

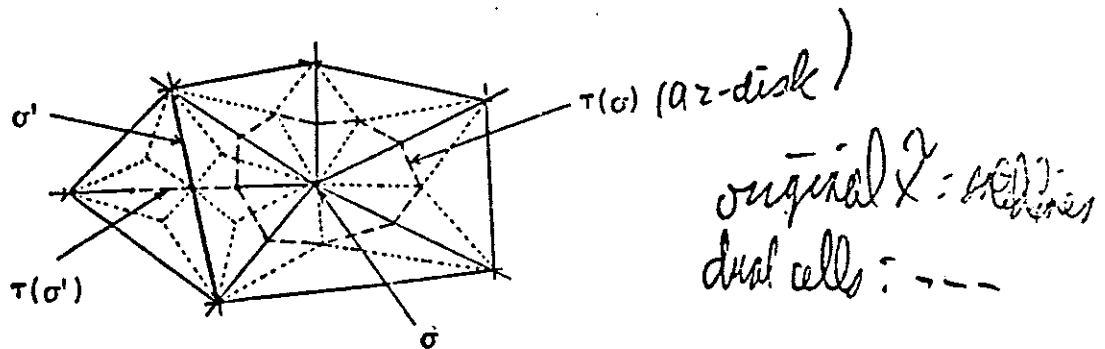
path so as to orient σ . Using this orientation we define a relative n -chain $\eta :=$ the sum of all the n -simplexes of X so oriented.

It is not hard to check that η represents a relative cycle $[X] \in H_n(X, \partial X; \mathbb{Z}^t)$; in fact $[X]$ generates $H_n(X, \partial X; \mathbb{Z}^t) = \mathbb{Z}$. We now show that η induces, by cap product, an isomorphism $C^*(X) \rightarrow C_*(X; \partial X)$ of the cochains on X with the relative chains on $(X, \partial X)$.

must be in $\pi_1 X$ We interpret $C^*(X)$ as ordinary simplicial cochains on the natural triangulation of the universal cover \tilde{X} of X . $\pi_1 X$ acts on \tilde{X} as a transformation group, hence the group ring $\Lambda := \mathbb{Z}\pi_1 X$ acts on $C^*(X)$. Thus $C^*(X)$ is a finitely generated free Λ -module with prescribed basis; the basis elements are determined by using the prescribed paths above to select lifts of the simplexes of X to \tilde{X} .

$\rightarrow \sim$ We interpret $C_*(X, \partial X)$ as the chain complex on the relative chains on X modulo $\pi^{-1}\partial X$ ($\pi: \tilde{X} \rightarrow X$ is the natural projection), considered as a Λ -module as above, but now the simplicial chains are taken with respect to the dual cell decomposition of \tilde{X} .

To define the dual cell decomposition, we pass to the first derived, \tilde{X}' . A typical simplex of \tilde{X}' can be written $\langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_k \rangle :=$ the simplex spanned by the barycentres $\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_k$ of simplexes σ_i of \tilde{X} with $\sigma_0 < \sigma_1 < \dots < \sigma_k$. Note that this expression defines a natural orientation for this simplex. Now let σ be a simplex of \tilde{X} . The dual cell $\tau(\sigma)$ corresponding to σ is defined as the subcomplex of \tilde{X}' generated by all simplexes of the form $\langle \hat{\sigma}, \hat{\sigma}_1, \dots, \hat{\sigma}_n \rangle$, $\sigma < \sigma_1 < \dots < \sigma_n$, see figure 5.



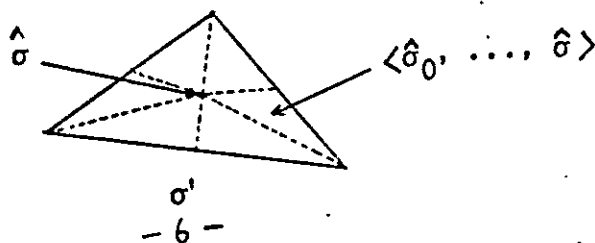
It is clear that $\dim \sigma + \dim \tau(\sigma) = n$. It can be shown [4] that $\tau(\sigma)$ triangulates a ball and $\bigcup_{\sigma} \tau(\sigma)$ gives a cell decomposition of \tilde{X} .

We now define the cap product $\eta \cap : C^*(X) \rightarrow C_*(X, \partial X)$. For consistency of notation, we will represent the elements of $C^*(X)$, $C_*(X, \partial X)$ by simplexes in the first derived and η by η' . Let $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$ be a typical simplex of η' , $\bar{\sigma}$ an elementary p -cochain corresponding to the p -simplex σ of \tilde{X} . We define:

$$(1) \langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle \cap \bar{\sigma} = (\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle, \bar{\sigma}') \langle \hat{\sigma}_p, \dots, \hat{\sigma}_n \rangle.$$

Here, $(\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle, \bar{\sigma}')$ denotes the value of $\bar{\sigma}'$ on $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$.

Observe that this expression is non zero precisely when $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$ is a simplex of σ' , that is, when $\hat{\sigma}_p = \hat{\sigma}$ (figure 6).

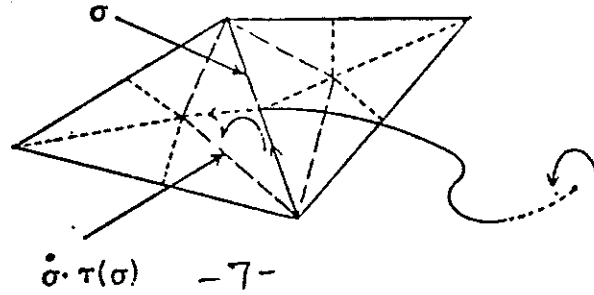


But then $\langle \hat{\sigma}_p, \dots, \hat{\sigma}_n \rangle$ is a principal simplex of $\tau(\sigma)$. Thus the $(n-p)$ -simplexes in $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle \cap \bar{\sigma}$ are precisely those which make up $\tau(\sigma)$.

On the other hand, it follows from general principles [10] that the dual cells of \tilde{X} can be oriented so that the homology $H_*(X, \partial X)$ of $(X, \pi^{-1}\partial X)$ as a Λ -module can be calculated from the homology of the complex $C_*(X, \partial X)$. We can obtain this orientation geometrically as follows, for the basis elements of $C_*(X, \partial X)$ as a Λ -module given by the cells $\tau(\sigma)$, $\bar{\sigma}$ a basis element of $C^*(X)$ chosen previously:

Observe that the join $\hat{\sigma} \cdot \tau(\sigma)$ is an n -dimensional ball in \tilde{X} . If we choose an orientation at the base point of \tilde{X} (a lift of the orientation at $*$) we obtain an orientation of $\hat{\sigma} \cdot \tau(\sigma)$ by transport along a lift of the prescribed path from $*$ to $\pi\hat{\sigma}$. The orientation of $\tau(\sigma)$ is determined by the condition that the orientation of σ followed by the orientation of

each principal simplex of $\tau(\sigma)$ gives the orientation of $\dot{\sigma} \cdot \tau(\sigma)$ (figure 7).



We now look at the effect of $\eta \cap$ on the oriented elementary cochain $\bar{\sigma}$. First note that we may orient the simplex $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$ of η' concordant with the n -simplex of η in which it lies, say by $\alpha \langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$, $\alpha = \pm 1$. Similarly, we orient each simplex $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$ of σ' concordant with σ , say by $\beta \langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$. The formula (1) above gives

$$\alpha \langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle \cap \bar{\sigma}' = \alpha \beta \langle \hat{\sigma}_p, \dots, \hat{\sigma}_n \rangle$$

and $\beta \langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$ followed by $\alpha \beta \langle \hat{\sigma}_p, \dots, \hat{\sigma}_n \rangle$ gives $\alpha \langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$, so that $\eta' \cap \bar{\sigma}'$ is the sum of the principal simplexes of $\tau(\sigma)$ oriented as above, that is, the oriented cell $\tau(\sigma)$.

Thus, $\eta \cap : C^*(X) \longrightarrow C_*(X, \partial X)$ is a correspondence between oriented basis elements, and extends Λ -linearly to an isomorphism. A calculation [10] shows that

$$\partial(\eta' \cap \bar{\sigma}') = (-1)^{n-p+1} (\eta' \cap \bar{\partial \sigma}')$$

and so $\eta \cap$ preserves cycles and boundaries; similarly for $(\eta \cap)^{-1}$. Hence, $\eta \cap$ induces Λ -isomorphisms $[X] \cap : H^p(X) \longrightarrow H_{n-p}(X, \partial X)$; the Poincaré Duality Isomorphisms.

Remark: The bases of $C^*(X)$ and $C_*(X, \partial X)$ set into correspondence by $\eta \cap$ may also be obtained as follows: Consider the triangulation of X , together with the dual cell decomposition of X . A choice of prescribed paths from the basepoint $*$ of X to the barycentres $\hat{\sigma}$ of simplexes of X

gives a prescribed lift of each simplex σ and each dual cell $\tau(\sigma)$ to \tilde{X} ; note that $\hat{\sigma} \in \text{Int } \tau(\sigma)$. These lifts determine the bases described above.

A map $\varphi : (W, \partial W) \rightarrow (X, \partial X)$ has degree 1 if it preserves the twisted integer coefficients and fundamental classes defined above. We will now give the form in which the Poincaré Duality Isomorphisms are used.

Theorem 3: Let $\varphi : (W, \partial W) \rightarrow (X, \partial X)$ have degree 1. Then the exact homology and cohomology sequences for $(W, \partial W)$ split as direct sums of the corresponding sequences for $(X, \partial X)$ and sequences $K_* = \text{Ker } \varphi_*$ (φ_* onto) and $K^* = \text{Coker } \varphi^*$ (φ^* 1-1) respectively. Further, the duality map $[W] \cap$ induces isomorphisms

$$K^*(W) \rightarrow K_*(W, \partial W) \text{ and } K^*(W, \partial W) \rightarrow K_*(W).$$

Proof of Theorem 3: Write $H^k(H^k)$ for the cohomology of W or $(W, \partial W)$ (X or $(X, \partial X)$) and $H_{n-k}(H^1_{n-k})$ for the homology of W or $(W, \partial W)$ (X or $(X, \partial X)$) respectively. The following diagram commutes by naturality of cap products [10]:

$$\begin{array}{ccccccc} 0 & \longleftarrow & K^k & \longleftarrow & H^k & \xleftarrow{\varphi^*} & H^k & \longleftarrow & 0 \\ & & & & \downarrow & \swarrow \psi_* & \searrow \psi_* & \downarrow & \\ & & [W] \cap & & & & & & [X] \cap \\ & & & & \downarrow & \swarrow \psi_* & \searrow \psi_* & \downarrow & \\ 0 & \longrightarrow & K_{n-k} & \longrightarrow & H_{n-k} & \xrightarrow{\varphi_*} & H^1_{n-k} & \longrightarrow & 0 \end{array}$$

Now $\varphi_* \psi_* = 1$ and $\psi^* \varphi^* = 1$ so we obtain the desired splitting in each dimension; $H^k = K^k \oplus \varphi^* H^k$, $H_{n-k} = K_{n-k} \oplus \psi_* H^1_{n-k}$. It is not hard to show that $[W] \cap$ preserves the splitting. Finally, note that φ^* and φ_* are homomorphisms of exact sequences, that is, the sequence operations preserve K^* and K_* .

4. Some Algebraic Results

A Λ -module P is called projective if the diagram

$$\begin{array}{ccc} & P & \\ \swarrow & \downarrow & \\ A & \longrightarrow B & \longrightarrow 0 \end{array}$$

can be completed, that is, if for any Λ -modules A and B as in the diagram, the map denoted by the diagonal arrow exists so as to make the diagram commute.

Lemma: P is projective if and only if P is a direct summand of a free module.

Proof of the Lemma: Suppose P projective, let F be a free module mapping onto P . Then the sequence

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \swarrow & \parallel & \\ 0 & \longrightarrow & \text{Ker } f & \longrightarrow & F & \xrightarrow{f} & P \longrightarrow 0 \end{array}$$

splits, and so $P \oplus \text{Ker } f = F$.

Conversely, let $P \oplus P' = F$. Then

$$\begin{array}{ccc} P & & P \oplus P' = F \\ \downarrow & & \downarrow \\ A \longrightarrow B \longrightarrow 0 \end{array} \text{ extends to } \begin{array}{ccc} & P \oplus P' = F & \\ \swarrow & \downarrow & \\ A & \longrightarrow B & \longrightarrow 0 \end{array}$$

and we may restrict to P the map given by the diagonal arrow.

Now let A_* , B_* be chain complexes of projective modules and $f: A_* \rightarrow B_*$ a chain map which induces homology isomorphisms. We will show that f is a chain homotopy equivalence, hence f also induces isomorphisms in cohomology. To see this, write C_* for the chain complex of the mapping cylinder of f ; $C_n = A_{n-1} \oplus B_n$ with boundary operator $\partial(C): C_{n+1} \rightarrow C_n$; $\partial(C)(\alpha, \beta) = \partial(A)\alpha + (-1)^n f\alpha + \partial(B)\beta$. C_* is then an

acyclic complex of projective modules. We will exhibit a chain homotopy equivalence of C_* with the trivial chain complex 0_* ; it then follows that f is a chain homotopy equivalence.

Since $H_*(C_*)$ is trivial, we have short exact sequences

$$(1) \quad 0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow Z_{i-1} \longrightarrow 0$$

for all i , $Z_0 = C_0$. By induction, the definition, and the Lemma above, Z_i is projective for all i , and the sequences split, giving for C_* :

$$(2) \quad \longrightarrow Z_k \oplus Z_{k-1} \longrightarrow Z_{k-1} \oplus Z_{k-2} \longrightarrow \dots \longrightarrow Z_0 \longrightarrow 0$$

where the maps are the obvious compositions of projections and inclusions.

Now let F be the map which sends C_* to 0_* , G the map which sends 0_* to C_* . Clearly, $FG = 0$, the identity on 0_* . Also, $GF = 0$. We now define a boundary operator ∂ such that $\partial(C)\partial + \partial\partial(C) = 1 - GF = 1$. Such an operator ∂ is given by composition of projections and inclusions:

$$Z_k \oplus Z_{k-1} \longrightarrow Z_{k+1} \oplus Z_k$$

We will call the Λ -module S stably free if it is the direct summand of a free module, with free complement. If S is finitely generated, we will require that the free module and the complement also be finitely generated.

Theorem 4: Let $\varphi : (W, \partial W) \longrightarrow (X, \partial X)$. Suppose, with $\Lambda := \mathbb{Z}\pi_1 X$ as coefficients, $H_i(\varphi) = 0$ for $i \neq k$; and $H^{k+1}(\varphi; L) = 0$ for any Λ -module L . Then $H_k(\varphi)$ is a finitely generated stably free Λ -module.

Proof of Theorem 4: We replace $(X, \partial X)$ by the mapping cylinder of φ , and φ by the inclusion. We then calculate the homology of φ from the chain complex $C_* := C_*(X, W \cup \partial X)$.

For $i \leq k$ we have the short exact sequences (1), with $Z_0 = C_0$. Thus, by induction, Z_i is projective and these sequences split for $i \leq k$. It follows that $H_k(\varphi)$ is finitely generated; it is the image

*ck is 1/2
why is 7/11 f.g. because it is surjective*

$C_k \rightarrow Z_k \rightarrow H_k(\varphi)$ of a finitely generated Λ -module.

Now observe that the inclusion $\iota : C'_* \subset C_*$

$$C'_* : \dots \rightarrow C_{k+2} \rightarrow C_{k+1} \rightarrow Z_k \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

induces isomorphisms in homology. By the remarks above, ι induces isomorphisms in cohomology; in particular, for any Λ -module L , $H^{k+1}(C'_*; L) = 0$.

Let us regard $\partial : C_{k+1} \rightarrow B_k$ as a cochain on C'_* . Since $(c, \delta\partial) = (\partial c, \partial) = \partial\partial c = 0$, it is a cocycle. But now, with L above set equal to B_k , $H^{k+1}(C'_*, B_k) = 0$ and so the cocycle is a coboundary, say

$$\begin{array}{ccccccc} \rightarrow & C_{k+2} & \rightarrow & C_{k+1} & \xrightarrow{\partial} & Z_k & \rightarrow 0 \rightarrow \\ & & & \searrow \partial & & \downarrow & \\ & & & & & B_k & \end{array}$$

thus $0 \rightarrow B_k \rightarrow Z_k \rightarrow H_k(\varphi) \rightarrow 0$ splits, $Z_k = B_k \oplus H_k(\varphi)$ and $H_k(\varphi)$ is projective. We also have

$\Rightarrow B_k$ is proj.

$$(3) \quad B_k \oplus H_k(\varphi) \oplus Z_{k-1} = Z_k \oplus Z_{k-1} = C_k.$$

Now consider the acyclic complexes C''_* , C'''_* :

$$C''_* : \dots \rightarrow C_{k+2} \rightarrow C_{k+1} \rightarrow B_k \rightarrow 0 \rightarrow \dots$$

$$C'''_* : \dots \rightarrow 0 \rightarrow Z_{k-1} \rightarrow C_{k-1} \rightarrow C_{k-2} \rightarrow \dots$$

$C''_* \oplus C'''_*$ is of course also an acyclic complex of projective modules, hence by (2) the sum of the odd terms equals the sum of the even terms. Thus, *(odd terms = even terms)*

$$(4) \quad B_k \oplus Z_{k-1} \oplus C_{k+2} \oplus C_{k-2} \oplus \dots = C_{k+1} \oplus C_{k-1} \oplus \dots$$

and adding $H_k(\varphi)$ to both sides of (4) we have, using (3),

$$C_k \oplus C_{k+2} \oplus C_{k-2} \oplus \dots = H_k(\varphi) \oplus C_{k+1} \oplus C_{k-1} \oplus \dots$$

It follows that $H_k(\varphi)$ is stably free.

A stable basis of a stably free Λ -module S with free complement F , is a basis for the free module $F' = S \oplus F$.

Theorem 4, continued: $H_k(\varphi)$ has a preferred equivalence class of stable bases.

In order to select this equivalence class, we sketch the definition of the torsion of the chain complex C_* ; for precise information, see [4]. Note that C_* has a preferred base as in §3. Now choose some stable basis for $H_k(\varphi)$, as well as bases for Z_i, B_i .

For $i < k$, the short exact sequences in (1) split and we can compare the bases of C_i with the bases of $Z_i \oplus Z_{i-1}$, and so obtain matrices a_i . For $i = k$, we obtain a matrix a_k from (3) which expresses the basis for C_k in terms of the basis for $B_k \oplus H_k(\varphi) \oplus Z_{k-1}$. Also by (3), B_k is projective, thus $C_{k+1} = Z_{k+1} \oplus B_k$, and we get a corresponding matrix a_{k+1} . We obtain matrices $a_i, i > k+1$ as for $i < k$.

The matrices a_i are of different sizes, but they all are nonsingular and so determine elements $[a_i]$ in $GL(\Lambda) = \varinjlim_r GL_r(\Lambda) (\dots \subset GL_r(\Lambda) \subset GL_{r+1}(\Lambda)$. We now consider the subgroup of $GL(\Lambda)$ generated by all the elementary matrices which could reasonably be considered in this context, bearing in mind that Λ is a group ring. The factor group of $GL(\Lambda)$ by this subgroup is called the Whitehead Group $Wh(\pi_1 X)$. The class of

$$\dots [a_{i+1}]^{-1} [a_i] [a_{i-1}]^{-1} [a_{i-2}] \dots [a_0] \in GL(\Lambda)$$

in $Wh(\pi_1 X)$ is called the torsion $\tau(C_*)$ of C_* . $\tau(C_*)$ is independent of the choice of bases for Z_i, B_i ; but it does depend on the choice of bases for $C_*, H_k(\varphi)$. In fact, if we choose another stable base for $H_k(\varphi)$ whose matrix with respect to the first base represents τ , an arbitrary element of $Wh(\pi_1 X)$, the new and old torsions are related by the formula:

$$\tau_{\text{new}}(C_*) = \tau_{\text{old}}(C_*) + (-1)^k \tau.$$

We may thus choose a stable basis for $H_k(\varphi)$ so that $\tau(C_*) = 0$. This defines an equivalence class of preferred bases for $H_k(\varphi)$; two representative stable bases differ by the elementary transformations referred to above.

Remarks:

(a) If C_* is the chain complex of the mapping cylinder of a homotopy equivalence φ , and $\tau(C_*) = 0$, we say that φ is a simple homotopy equivalence.

(b) An isomorphism $L \rightarrow L'$ of stably free Λ -modules with prescribed bases is called simple if the image of the prescribed base for L differs from the prescribed base for L' by the above elementary transformations, that is, if the difference is expressed by a matrix which represents $0 \in \text{Wh}(\pi_1 X)$.

5. The Even-Dimensional Wall Groups

In this section, we will consider the surgery problem (W, φ, F) , with W a manifold of dimension $n = 2k$. By Theorem 2, we may suppose that φ is k -connected. Thus, φ is homologically and cohomologically k -connected.

On the other hand, we have an exact sequence

$$\rightarrow H_{i+1}(W, \partial W) \xrightarrow{\varphi_*} H_{i+1}(X, \partial X) \rightarrow H_{i+1}(\varphi) \rightarrow H_i(W, \partial W) \xrightarrow{\varphi_*} H_i(X, \partial X) \rightarrow$$

in which each φ_* is onto, by Theorem 3. Hence, $H_{i+1}(\varphi) = K_i(W, \partial W)$ for all i , and so $K_i(W, \partial W) = 0$ for $i < k$. Again by Theorem 3, $K^i(W)$ and hence $K^i(W, \partial W) = 0$ for $i > n-k$ ($\varphi|_{\partial W}$ is a homotopy equivalence). Using a similar sequence involving φ^* , we have that $K^i(W, \partial W) = 0$ for $i < k$, $K_i(W, \partial W) = 0$ for $i > n-k$.

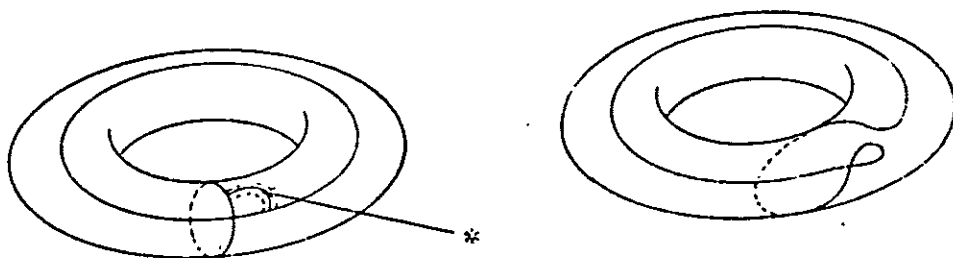
Thus, $K_i(W, \partial W) = H_{i+1}(\varphi) = 0$ for $i \neq k$, $K_k(W, \partial W) = H_{k+1}(\varphi) = \pi_{k+1}(\varphi)$ by the Hurewicz Theorem [3]. A similar argument shows that $H^{k+2}(\varphi; L) = 0$ for any Λ -module L . Hence, by Theorem 4, $G : \pi_{k+1}(\varphi) = H_{k+1}(\varphi) = K_k(W, \partial W)$ is a finitely generated stably free Λ -module with a preferred equivalence class of bases.

We now describe the elements of G as representative immersions \bar{f}

and $\pi_{k+1}(\varphi)$ is the image of \bar{f} in G (see Th. 4)

in the regular homotopy classes given by Theorem 1, $\bar{f}: S^k \times D^{n-k} \rightarrow W$. As we now regard these classes as elements of a Λ -module, we prescribe also a lift $S^k \times D^{n-k} \rightarrow \tilde{W}$ for each such representative; alternatively, a path leading from the basepoint of $\bar{f}(S^k \times D^{n-k})$ to the basepoint $*$ of W .

We add two immersed spheres by taking their connected sum along an embedded arc tracing out the prescribed paths in a natural way (figure 8).



-8-

Let \bar{f}_0, \bar{f}_1 denote two such representative immersions of $S^k \times D^{n-k}$ in W , $f_0 = \bar{f}_0|_{S^k \times 0}$, $f_1 = \bar{f}_1|_{S^k \times 0}$. By "general position" we may suppose that $S_0 := \text{Im } f_0$ and $S_1 := \text{Im } f_1$ intersect like hyperplanes of dimension k in general position in E^{2k} . That is, we suppose that each point of $S_0 \cap S_1$ has a neighborhood U piecewise linearly equivalent to $\Delta_0^k \cdot \Delta_1^k \subset E^{2k}$ so that $S_0 \cap U$ maps to $\Delta_0^k \subset E^k \times 0$, $S_1 \cap U$ to $\Delta_1^k \subset 0 \times E^k$ under this equivalence. We say that f_0 and f_1 intersect transversely.

The intersection number $\lambda(S_0, S_1)$ is now defined as an element of Λ ; at each point P of intersection we assign an element $g_P \in \pi_1 X$, $\epsilon_P = \pm 1$. g_P is the class of the loop leading from $*$ to the basepoint of S_1 by the prescribed path, along a path in S_1 to P , along S_0 to its basepoint and back along the prescribed path to $*$.

To define ϵ_P we orient W at $*$ and transport this orientation along the prescribed path to the basepoint of S_0 and along S_0 to P . ϵ_P is 1 if the orientation of S_0 followed by that of S_1 agrees with the orientation at P , -1 otherwise. We define $\lambda(S_0, S_1)$ as the sum of $\epsilon_P g_P$ over all points $P \in S_0 \cap S_1$.

Similarly, we define the self-intersection number $\mu(S_0)$ of an immersed sphere S_0 with transverse self-intersections. $\mu(S_0)$ takes values in the factor group Λ/I which we define as follows: Let $\alpha \rightarrow \bar{\alpha}$ be the involution on Λ given by: $\alpha = \sum g \rightarrow \bar{\alpha} = \sum g \omega(g)g^{-1}$. The subgroup I is defined as $\{\nu - (-1)^{k-} \bar{\nu}\} \subset \Lambda$.

Now if S_0 is immersed with transverse self-intersections, then two branches cross at each point P of self-intersection. If we order these branches arbitrarily, we can compute ϵ_P and g_P as above. On interchanging the order, we obtain $(-1)^k \omega(g_P) \epsilon_P$ and g_P^{-1} respectively. Thus the difference $\epsilon_P g_P - (-1)^k \omega(g_P) \epsilon_P g_P^{-1} = \epsilon_P g_P - (-1)^k \bar{\epsilon}_P g_P$ is an element of I , and so the class $\mu(S_0)$ of $\sum \epsilon_P g_P$ in Λ/I does not depend on the ordering of the branches.

Theorem 5: The above intersection numbers define a map

$\lambda : G \times G \rightarrow \Lambda$ such that

$$(1) \lambda(x, y\beta + y'\beta') = \lambda(x, y)\beta + \lambda(x, y')\beta'; \beta, \beta' \in \Lambda$$

$$(2) \lambda(y, x) = (-1)^{k-} \lambda(x, y).$$

Self-intersections define a map $\mu : G \rightarrow \Lambda/I$ such that

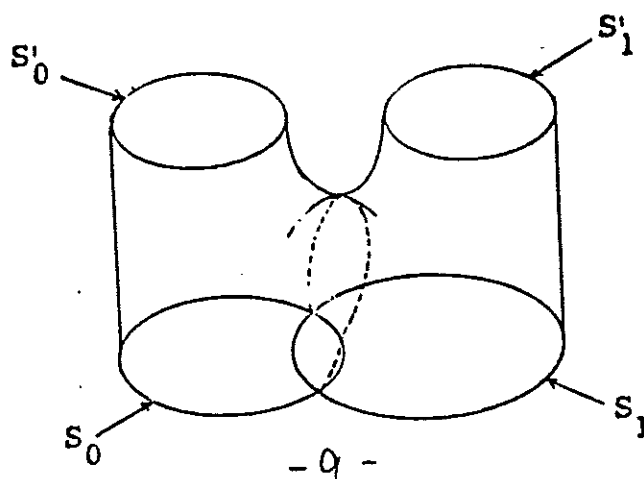
$$(3) \lambda(x, x)I = \mu(x) + (-1)^{k-} \mu(x)$$

$$(4) \lambda(x, y)I = \mu(x+y) - \mu(x) - \mu(y)$$

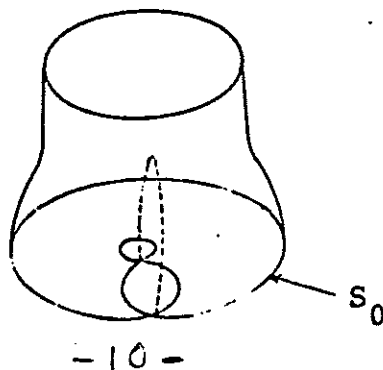
$$(5) \mu(x\alpha) = \bar{\alpha}\mu(x)\alpha; \alpha \in \Lambda$$

Proof of Theorem 5: We must first show that λ defined above is an invariant of the regular homotopy classes. To this end, let S_0, S_1 be transversely intersecting representatives, regularly homotopic to S'_0, S'_1 respectively. We now place the regular homotopies, regarded as level preserving maps of $S^k \times I$ in $W^{2k} \times I$, in general position.

The set of points of intersection then has dimension $2(k+1) - (2k+1) = 1$. It can be shown that the free faces of this set which do not meet $W \times \partial I$ can be removed by further perturbations. It follows that we can choose the regular homotopies so that they intersect only in circles and intervals (figure 9).



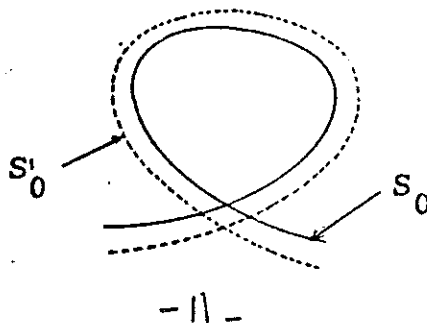
Thus the intersection number λ as a function of I is constant except at a finite number of critical points where paired intersections appear or disappear together. Thus λ is constant; $\lambda(S_0, S_1) = \lambda(S'_0, S'_1)$. Similarly, $\mu(S_0)$ is a regular homotopy invariant (figure 10).



We now prove (1). Note first of all that the form λ is bilinear over the integers; we simply choose the connecting tubes (figure 8) so as to contain no points of intersection. In replacing S_1 by $S_1 g$, we replace g_P by $g_P g$ at each intersection point. Thus, $\lambda(S_0, S_1 g) = \lambda(S_0, S_1) g$ and (1) follows.

To prove (2), we argue as in the definition of μ given above; on interchanging S_0 and S_1 , $\epsilon_P g_P$ is replaced by $(-1)^{k-\overline{\epsilon_P g_P}}$.

To establish (3), we compute $\lambda(x, x)$ by taking two representatives for x , say $S_0 = f_0(S^k) = \bar{f}_0(S^k \times 0)$ and $S'_0 = \bar{f}_0(S^k \times 1)$ (figure 11).



Each self-intersection of S_0 gives rise to two intersections of S_0 and S'_0 , in which each ordering of the branches occurs once. Thus, with $\mu(x) = \sum \epsilon_P g_P$, $\lambda(x, x) = \sum [\epsilon_P g_P + (-1)^k \overline{\epsilon_P g_P}] = \mu(x) + (-1)^k \mu(x) \pmod{1}$.

As to (4), the self-intersections of $S_0 \# S_1$ are those of S_0 , those of S_1 , and the mutual intersections of S_0 and S_1 .

We show (5) for $\alpha = g \in \pi_1 X$. As in the proof of (1), $\epsilon_P g_P$ becomes $\omega(g) \epsilon_P g^{-1} g_P g = \overline{g_P} \epsilon_P g_P g$. The case in which α is an arbitrary element of Λ is proved by a lengthy computation, which we omit.

Theorem 5, continued: The adjoint map $A\lambda : G \rightarrow \text{Hom}_\Lambda(G, \Lambda)$ defined by sending x to the map $y \rightarrow \lambda(x, y)$ is a simple isomorphism of G with its preferred stable basis and $\text{Hom}_\Lambda(G, \Lambda)$ with a preferred stable basis which we describe below.

We give $\text{Hom}_\Lambda(G, \Lambda)$ a right Λ -module structure by defining a left Λ -module structure on G as follows: $\lambda x = x \overline{\lambda}$. With respect to this structure, $A\lambda$ as defined above is a Λ -homomorphism by (1) and (2). To show $A\lambda$ is a simple isomorphism, we consider the following diagram:

$$\begin{array}{ccccc}
 K_k(W, \partial W) & \xleftarrow{i_*} & K_k(W) & \xleftarrow{[W] \cap} & K^k(W, \partial W) \\
 \downarrow A\lambda & & & & \downarrow j^* \\
 \text{Hom}_\Lambda(K_k(W, \partial W), \Lambda) & \xleftarrow{j^{**}} & \text{Hom}_\Lambda(H_{k+1}(\varphi), \Lambda) & \xleftarrow{\mathcal{U}} & H^{k+1}(\varphi)
 \end{array}
 \quad (6)$$

Here, i_* is an isomorphism, for the extreme terms in the exact sequence

$$\longrightarrow K_k(\partial W) \longrightarrow K_k(W) \xrightarrow{i^*} K_k(W, \partial W) \longrightarrow K_{k-1}(\partial W) \longrightarrow$$

vanish. j^* is an isomorphism by an argument similar to that on page 17, and j^{**} is the isomorphism algebraically induced by j^* . \mathcal{U} is the isomorphism given by the Universal Coefficient Theorem [10]. Since $[W] \cap$ is an isomorphism, it follows that $A\lambda$ will be an isomorphism if the diagram (6) commutes.

We now prove commutativity; first proceed clockwise around the diagram starting at $x \in K_k(W, \partial W)$. We obtain

$$\begin{aligned} x &\longrightarrow x \longrightarrow ([W] \cap)^{-1} x \longrightarrow [h \longrightarrow (h, \delta([W] \cap)^{-1} x)] \longrightarrow \\ &\longrightarrow [\partial h \longrightarrow (h, \delta([W] \cap)^{-1} x) = (\partial h, ([W] \cap)^{-1} x)] = [y \longrightarrow (y, ([W] \cap)^{-1} x)] . \end{aligned}$$

Represent x, y by immersed spheres S_0, S_1 respectively, which intersect transversely. In order to calculate $(S'_0, (\eta' \cap)^{-1} S'_1)$, where we regard S'_0 and S'_1 as k -chains in the first derived, we lift S_0 and S_1 to \tilde{W} using the prescribed paths from $*$. Note that S_0, S_1 are not necessarily simply connected.

Let σ be a typical simplex of S_0 , $g \in \pi_1 W$. Then $(\eta' \cap)^{-1} S'_1$ evaluated on $\sigma'g$ is zero, unless a principal simplex of the dual cell $\tau(\sigma g)$ lies in S'_1 , that is, when $\hat{\sigma}g \in S_1$, see page 10.

In this case, with Δ_0^k, Δ_1^k as in the definition of transversality, $(\sigma'g, (\eta' \cap)^{-1} S'_1)$ is equal to $(\Delta_0^k, (\Delta_0^k \cdot \Delta_1^k \cap)^{-1} \Delta_1^k)g$ where the orientations of $\Delta_0^k, \Delta_1^k \subset \Delta_0^k \cdot \Delta_1^k \subset E^{2k}$ are obtained from those of $S_0, S_1 \subset W$ at $\hat{\sigma}$.

It is not hard to see that the dual cell of Δ_0^k is the second derived neighborhood δ of $0 = \Delta_0^k \cap \Delta_1^k$ in Δ_1^k , where $\Delta_0^k, \Delta_1^k \subset \Delta_0^k \cdot \Delta_1^k$ have the usual orientations. Thus, with the usual orientations, $(\Delta_0^k, (\Delta_0^k \cdot \Delta_1^k \cap)^{-1} \Delta_1^k) = (\Delta_0^k, (\Delta_0^k \cdot \Delta_1^k \cap)^{-1} \delta) = 1$. It follows that $(\sigma'g, (\eta' \cap)^{-1} S'_1) = \pm 1$ according as the orientation of σ at $\hat{\sigma}$ obtained from S_0 followed by that of S_1 at $\hat{\sigma}$ agrees or not with the orientation of W at $\hat{\sigma}$.

Summing over all the points of intersection of S_1 and translates of S_0 by elements of $\pi_1 W$, we obtain

$$(7) \quad \lambda(x, y) = (y, ([W] \cap)^{-1} x),$$

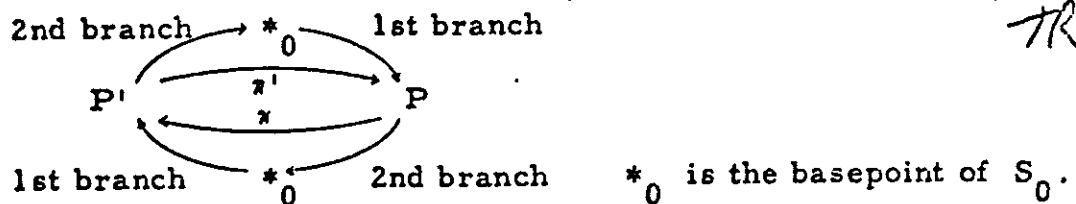
thus the diagram (6) is commutative, and $A\lambda$ is an isomorphism.

To define the preferred stable basis for $\text{Hom}_\Lambda(G, \Lambda)$ we first observe that the arguments of §4 imply that $H^{k+1}(\varphi)$ is stably free with a preferred stable basis which makes the torsion of $C^*(\varphi)$ vanish. $j^{**}\mathcal{U}$ sends this basis to the desired stable basis for $\text{Hom}_\Lambda(G, \Lambda)$. It can be shown that this is exactly the image under $A\lambda$ of the preferred stable basis of $K_k(W, \partial W)$. Thus, with this convention, $A\lambda$ is a simple isomorphism, and Theorem 5 is proved.

Theorem 6: Let $x \in G$. If $k \geq 3$, x is represented by an embedding if and only if $\mu(x) = 0$.

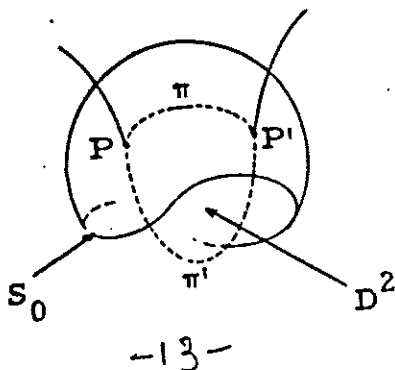
Proof of Theorem 6: An embedded sphere has self-intersection number zero. Conversely, by hypothesis, for some ordering of the branches at each self-intersection point of a representative immersed sphere S_0 , $\sum \epsilon_P g_P \in \mathbb{Z}$. Let ϵg be a typical term in this sum; then the term $-(-1)^k \omega(g) \epsilon g^{-1}$ must also appear. If ϵg is associated with the self-intersection P , $-(-1)^k \omega(g) \epsilon g^{-1}$ with P' , then by reordering the branches at P' , we obtain ϵg and $-\epsilon g$ respectively.

With this ordering, the loop in S_0 defined by paths π, π' chosen as in figure 12 is the difference gg^{-1} and so nullhomotopic.



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By "general position" we may represent this loop by an embedded circle S^1 bounding an embedded disc $D^2 \subset W$ such that $D^2 \cap S_0 = S^1$ (figure 13); this requires $\dim W > 4$.



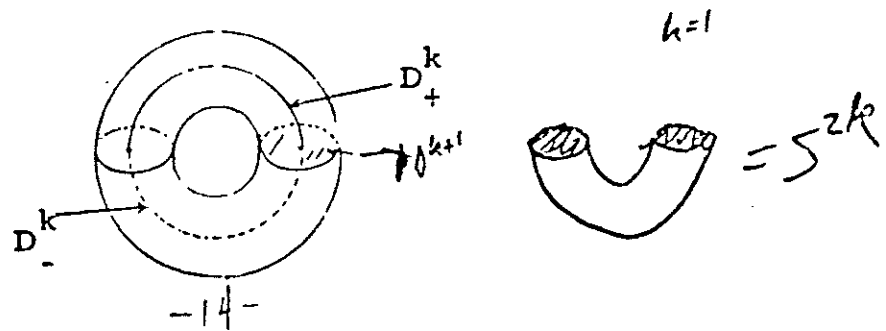
We may now pass a neighborhood of π' in S_0 across D^2 by an isotopy fixed outside a neighborhood of D^2 so that the new π' misses π . The effect of this procedure is to remove the self-intersections P and P' . For details, see [4] or [6]. Thus, we may remove the self-intersections inductively in pairs so as to obtain an embedding representing x .

Remark: Similarly, if $\lambda(x, y) = 0$ we can remove the mutual intersections of x and y , if $k \geq 3$.

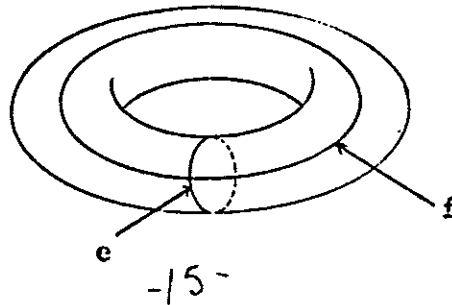
We now show how the algebraic structure given in Theorem 5 is connected with the surgery problem. We call (G, λ, μ) as in Theorem 5 a Special Hermitian Form. In what follows, we will need to assume that G is free with an actual preferred class of bases. We can make G free by adding a finite number of free summands; but we must at the same time extend λ and μ .

This can all be done geometrically in a natural way by doing surgery on the trivial element of $\pi_k(\varphi)$. We represent this element by a nullhomotopic immersed $S^{k-1} \times D^{k+1}$. By "general position" applied to the nullhomotopy, we may assume $S^{k-1} \times 0$ is embedded, and is isotopic to a $(k-1)$ -sphere inside an n -disc $D^{2k} \subset W$. By Zeeman's Unknotting Theorem [4] we may suppose that this $(k-1)$ -sphere is the standard $S^{k-1} \subset D^{2k}$.

Now observe that surgery on the standard $S^{k-1} \subset S^{2k}$ gives $S^k \times S^k$, for $S^k \times S^k = S^k \times \partial D^{k+1} = (D_+^k \cup D_-^k) \times \partial D^{k+1} = [\partial(D_-^k \times D^{k+1}) - \partial D_-^k \times D^{k+1}] \cup (D_+^k \times S^k) \subset [S^{2k} - S^{k-1} \times D^{k+1}] \cup D^k \times S^k$ (figure 14).



It follows that surgery on $0 \in \pi_k(\varphi)$ replaces W by the connected sum $W \# S^k \times S^k$, G by $G \oplus \Lambda e \oplus \Lambda f$ where e, f are the classes of $S^k \times 1$ and $1 \times S^k$ respectively. Further, $\lambda(e, e) = \lambda(f, f) = 0$, $\lambda(e, f) = 1$, $\lambda(f, e) = (-1)^k$, $\mu(e) = \mu(f) = 0$ with the usual orientations on $S^k \times 1$, $1 \times S^k \subset S^k \times S^k$ (figure 15).



The Special Hermitian Form given by $(\Lambda e \oplus \Lambda f, \lambda, \mu)$ where λ is described by the matrix

$$\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

with respect to the basis e, f ; $\mu(e) = \mu(f) = 0$, is called a standard plane, and the orthogonal direct sum of standard planes is called a kernel.

We now show that the orthogonal direct sum $(G, \lambda, \mu) \oplus (G, -\lambda, -\mu)$ is a kernel. First, we prove the following lemma.

Lemma 7: Let $(\bar{G}, \bar{\lambda}, \bar{\mu})$ be a Special Hermitian Form in which \bar{G} has a free submodule \bar{H} with $\lambda(\bar{H} \times \bar{H}) = 0$, $\mu(\bar{H}) = 0$. Also suppose the map $\bar{G}/\bar{H} \rightarrow \text{Hom}_{\Lambda}(\bar{H}, \Lambda)$ induced by $\bar{\lambda}$ is a simple isomorphism with respect to a preferred basis for \bar{G}/\bar{H} and the basis dual to a preferred basis for \bar{H} .

Then $(\overline{G}, \overline{\lambda}, \overline{\mu})$ is a kernel.

Proof of Lemma 7: Let $\{\overline{e}_i\}$ be the preferred basis for \overline{H} . It follows that the dual basis $\{\overline{e}_i^*\}$ induces, by $\overline{\lambda}$, a preferred base of $\overline{G}/\overline{H}$, say $\{\overline{f}_i\}$; $\lambda(\overline{e}_i, \overline{f}_j) = \delta_{ij}$. If $\overline{\mu}_i \in \overline{\mu}(\overline{f}_i)$ and

$$\overline{f}_j = \overline{f}_j + (-1)^{k-1} [\overline{\mu}_j \overline{e}_j + \sum_{i < j} \lambda(\overline{f}_i, \overline{f}_j) \overline{e}_i],$$

one can check that $\overline{\mu}(\overline{f}_i) = 0$, $\overline{\lambda}(\overline{e}_i, \overline{f}_j) = \delta_{ij}$, $\lambda(\overline{f}_i, \overline{f}_j) = 0$ and so the basis $\{\overline{e}_i, \overline{f}_j\}$ gives $(\overline{G}, \overline{\lambda}, \overline{\mu})$ the structure of a kernel. This proves Lemma 7.

Now write $(\overline{G}, \overline{\lambda}, \overline{\mu}) = (G, \lambda, \mu) \oplus (G, -\lambda, -\mu)$, \overline{H} for the submodule of \overline{G} generated by $\{\overline{e}_i\} = \{(e_i, e_i)\}$. It is not hard to see that $\overline{\lambda}(\overline{H} \times \overline{H}) = 0$, $\overline{\mu}(\overline{H}) = 0$. Also, the map $\overline{G}/\overline{H} \rightarrow \text{Hom}_{\Lambda}(\overline{H}, \Lambda)$ induced by $\overline{\lambda}$ is a simple isomorphism, for with respect to the basis $\{(0, e_i)\}$ of $\overline{G}/\overline{H}$ and the dual basis $\{\overline{e}_i^*\}$ of $\text{Hom}_{\Lambda}(G, \Lambda)$, the matrix of this transformation has (i, j) th place $\overline{\lambda}((0, e_i), (e_j, e_j)) = -\lambda(e_i, e_j)$. That is, this matrix is the negative of the matrix of the isomorphism $\Lambda\lambda$ with respect to the preferred bases. Hence, by the Lemma, $(G, \lambda, \mu) \oplus (G, -\lambda, -\mu)$ is a kernel.

We introduce an equivalence relation on Special Hermitian Forms by identifying two forms (G, λ, μ) and (G', λ', μ') if there are kernels K, K' so that

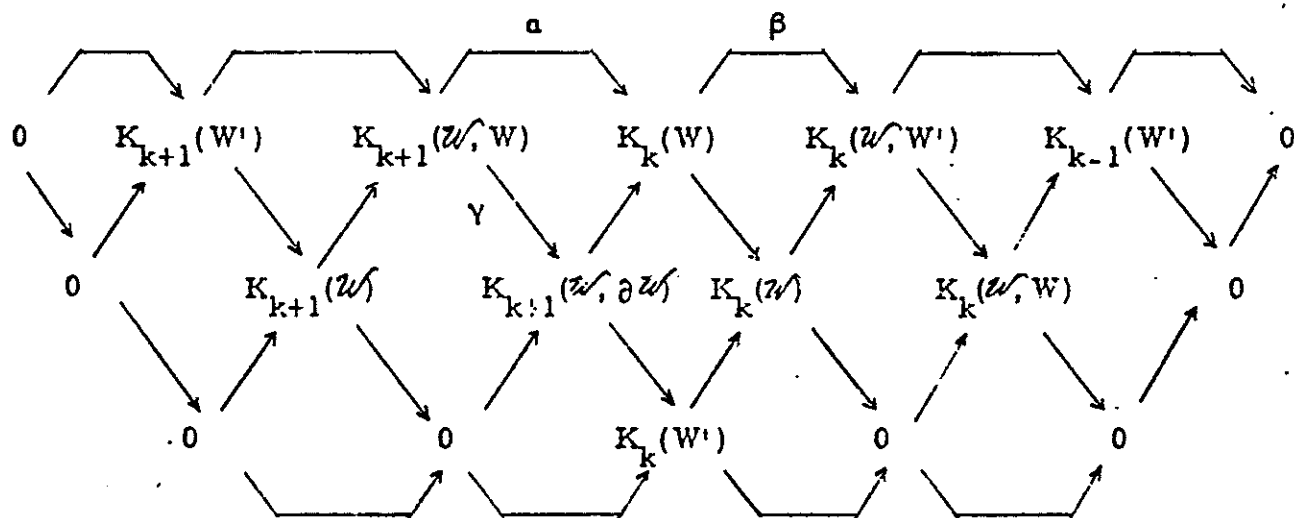
$$(G, \lambda, \mu) \oplus K = (G', \lambda', \mu') \oplus K'.$$

Since the sum of kernels is a kernel, this equivalence relation respects addition and so defines an associative, commutative addition on the equivalence classes of forms. Note that a kernel represents an identity element, and that we have demonstrated above the existence of inverses. Thus, the equivalence classes of forms form an abelian group, the Wall Surgery Obstruction Group $L_{2k}(\pi_1 X, \omega X)$.

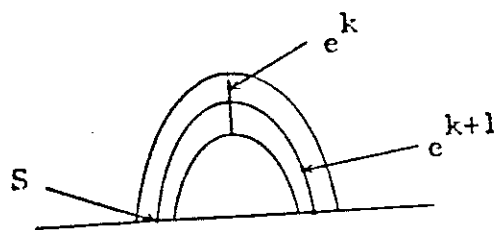
Theorem 8: Let (W^{2k}, φ, F) be a surgery problem, θ the class of (G, λ, μ) given by Theorem 5 in $L_{2k}(\pi_1 X, \omega X)$. If $\theta = 0$ we can do surgery

to make φ a simple homotopy equivalence, provided $k \geq 3$.

Proof of Theorem 8: After a finite number of surgeries on the trivial element of $\pi_k(\varphi)$, we may suppose $K_k(W) = K_k(W, \partial W)$ is a kernel, with standard basis $e_1 \dots e_r, f_1 \dots f_r$, $\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$, $\lambda(e_i, f_j) = \delta_{ij}$, $\mu(e_i) = \mu(f_j) = 0$. Since $\mu(e_r) = 0$, the class $e_r \in K_k(W) = \pi_{k+1}(\varphi)$ is represented by an embedded sphere S (Theorem 6) on which we can do surgery (Theorem 1), obtaining, say, (W', φ', F') . Let \mathcal{W} denote the union of $W \times I$ with a $(k+1)$ -handle whose attaching sphere is S , so that $\partial \mathcal{W} = W \cup W' \cup \partial W \times I$. Then, in the following commutative diagram of exact sequences, the groups not listed vanish since φ is k -connected:



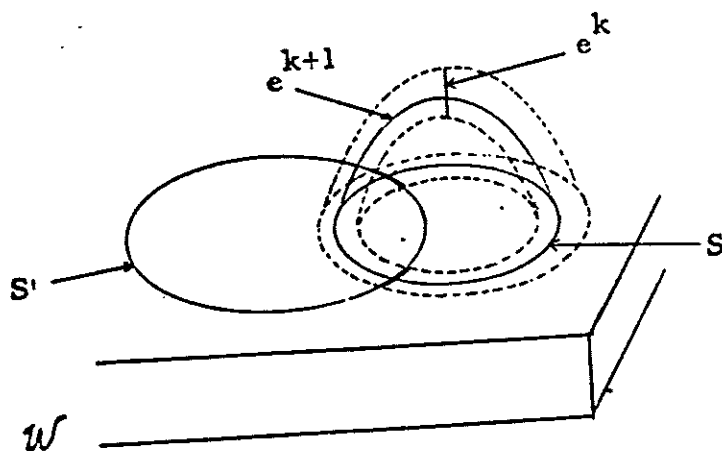
We now show that α is 1-1 and β is onto. In fact, the homomorphism $\alpha : K_{k+1}(\mathcal{W}, W) \rightarrow K_k(W)$ takes e^{k+1} (the core of the attached handle) to its attaching sphere $\partial e^{k+1} = S$. Note that $K_{k+1}(\mathcal{W}, W)$ is generated by e^{k+1} , since $\mathcal{W} \simeq W \cup e^{k+1}$. Since S represents e_r , a generator, α is 1-1 (figure 16).



W

We will show that $\beta : K_k(W) \rightarrow H_k(W) \rightarrow H_k(W, W')$ takes x to $\lambda(e_r, x)[e^k, \partial e^k]$, where e^k is the k -cell dual to e^{k+1} in the attached handle. Note that $W \simeq W' \cup e^k$.

Let x , a generator of $K_k(W)$, be represented by an immersed sphere S' which meets S transversely. Write $h = e^k \times e^{k+1}$, $\dot{h} = e^k \times S$. Under excision, $S' \subset W \times 1$ represents $(S' \cap h, S' \cap \dot{h})$ in the homology of (h, \dot{h}) , which is generated by $(e^k, \partial e^k)$. Since S and S' intersect transversely, each component of $(S' \cap h, S' \cap \dot{h})$ is homologous to $\pm(e^k, \partial e^k)$ according as the orientation sign ε_P at the corresponding point $P \in S \cap S'$ is ± 1 (figure 17).



Summing over the points of $S \cap S'$, we have the desired result for generators, hence generally.

Since $\lambda(e_r, f_r) = 1$, β is onto. Now from the commutative exact diagram, $K_{k-1}(W')$ and $K_{k+1}(W') = 0$, so φ' is k -connected. Also,

$K_{k+1}(\mathcal{W}, \partial\mathcal{W}) = \text{Ker } \beta =$ the submodule generated by $e_1 \dots e_r, f_1 \dots f_{r-1}$. Further, $K_k(W') = K_{k+1}(\mathcal{W}, \partial\mathcal{W})/\text{Im } \gamma$, and $\text{Im } \gamma = \Lambda e_r$. So $K_k(W')$ is generated by $e_1 \dots e_{r-1}, f_1 \dots f_{r-1}$, and moreover these elements can be represented by spheres disjoint from S in $(W \times 1) \cap W'$.

Also, the preferred base of $K_k(W)$ is chosen so that the torsion of $C_*(\varphi)$ vanishes. Inductively, the same holds for $K_k(W')$. Hence, finally, we obtain (W', φ', F') with $K_k(W') = 0$ and the torsion of $C_*(\varphi') = 0$. Hence, φ' is a simple homotopy equivalence, and Theorem 8 is proved.

Let us make some calculations, for $\pi_1 X = 1$. In this case, $\omega = 1$, $\Lambda = \mathbb{Z}$, the involution on Λ is the identity, and $\text{Wh}(\pi) = 0$. In particular, every isomorphism of a \mathbb{Z} module is simple. Now consider Special Hermitian Forms (G, λ, μ) as given by Theorem 5. There are two cases:

Case 1: k even: $= 2s$. Here, $(-1)^k = 1$, and $I = 0$. G is a finitely generated free abelian group, λ is a nondegenerate symmetric bilinear form over the integers, with associated quadratic form μ . λ is even, that is, $\lambda(x, x)$ is even for all x .

We define a homomorphism $L_{4s}(1, 1) \rightarrow \mathbb{Z}$ by $\sigma/8 : (G, \lambda, \mu) \rightarrow \sigma(\lambda)/8$; $\sigma(\lambda)$ is the signature of λ , that is, the number of positive terms less the number of negative terms after λ is diagonalized over \mathbb{Q} .

Note that $\sigma/8$ is additive. Also,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $\sigma/8$ vanishes on kernels. It is known [9] that the signature of a nondegenerate even symmetric bilinear form over the integers is divisible by 8, thus $\sigma/8$ is a well defined homomorphism.

We now show that $\sigma/8$ is 1-1. If $\sigma(\lambda) = 0$, it is known [9] that λ has a nontrivial zero, say $\lambda(\beta_1, \beta_1) = 0$ with $\beta_1 \neq 0$; we may as well assume that β_1 is indivisible, that is, β_1 is not a proper integral multiple

of some other element of G . Since λ is nondegenerate, $A\lambda$ is an isomorphism and there is an α with $\lambda(\beta_1, \alpha) \neq 0$. Choose α so that $\lambda(\beta_1, \alpha) > 0$ is as small as possible, then, by the division algorithm, for any γ , $\lambda(\beta_1, \gamma)$ is a multiple of $\lambda(\beta_1, \alpha)$. Again since $A\lambda$ is an isomorphism, β_1 is divisible by $\lambda(\beta_1, \alpha)$, so that $\lambda(\beta_1, \alpha) = 1$.

Now let $\beta_2 = \alpha - (\lambda(\alpha, \alpha)/2)\beta_1$; note that $\lambda(\alpha, \alpha)$ is divisible by 2. It follows that $\lambda(\beta_1, \beta_2) = 1$, $\lambda(\beta_2, \beta_2) = 0$ and $\mu(\beta_1) = \mu(\beta_2) = 0$. Thus $\mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2$ is a kernel. Let H denote the set of $\alpha \in G$ with $\lambda(\beta_1, \alpha) = \lambda(\beta_2, \alpha)$. Then $\alpha - \lambda(\alpha, \beta_2)\beta_1 - \lambda(\alpha, \beta_1)\beta_2 \in H$ and so $G = \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2 + H$. If $\gamma \in \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2 \cap H$, then $A\lambda(\gamma)$ vanishes on H and on $\mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2$. Thus the last sum is direct, and by induction G is a kernel.

To see that $\sigma/8$ is onto, consider the 8×8 symmetric matrix (a_{ij}) whose diagonal elements are all equal to 2, with $a_{i,i+1} = a_{i+1,i} = 1$ for $i \leq 6$, $a_{5,8} = a_{8,5} = 1$ and all other entries zero. (a_{ij}) represents a nondegenerate even symmetric bilinear form $\lambda : \mathbb{Z}^8 \times \mathbb{Z}^8 \rightarrow \mathbb{Z}$ with signature 8.

Case 2: k odd: $= 2s+1$. In this case, G is again a finitely generated free abelian group, λ is a nondegenerate antisymmetric bilinear form over the integers, and μ has values in $\mathbb{Z}/2\mathbb{Z}$.

Note that $\lambda(x, x) = 0$ for all x . Proceeding as in Case 1 we can find a symplectic basis $e_1 \dots e_r, f_1 \dots f_r$ for G , that is, a basis such that $\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$, $\lambda(e_i, f_j) = \delta_{ij}$. We define a homomorphism $c : L_{4s+2}(1, 1) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by $c(\mu) = \sum \mu(e_i)\mu(f_i)$, the Arf Invariant of μ . $c(\mu)$ is independent of the choice of symplectic base.

Now c is 1-1. For, if $c(\mu) = 0$ it follows that the number of values of i for which $\mu(e_i) = \mu(f_i) = 1$ is even. Group these generators in consecutive pairs. Then the transformation

$$\begin{aligned} e'_1 &= e_1 + e_2 & f'_1 &= f_1 \\ e'_2 &= e_2 & f'_2 &= f_1 + f_2 \end{aligned}$$

gives a new decomposition of $\Lambda e_1 \oplus \Lambda f_1 \oplus \Lambda e_2 \oplus \Lambda f_2$ into two planes with

$$\lambda(e_i', e_j') = \lambda(f_i', f_j') = 0, \lambda(e_i', f_j') = \delta_{ij}, \mu(e_1') = \mu(f_2') = 0.$$

By induction, we obtain a set of generators $e_1' \dots e_r', f_1' \dots f_r'$ and a submodule H generated by $e_1', f_2', \dots, e_{r-1}', f_r'$ such that $\lambda(H \times H) = 0$, $\mu(H) = 0$. It is not hard to see that the map $G/H \rightarrow \text{Hom}(H, \mathbb{Z})$ is a simple isomorphism, and now G is a kernel by Lemma 7.

To see that c is onto, take $G = \mathbb{Z}e \oplus \mathbb{Z}f$ with $\mu(e) = \mu(f) = 1$, $\lambda(e, e) = \lambda(f, f) = 0$, $\lambda(e, f) = 1$.

We now show that each element of $L_{2k}(\pi, \omega)$, π finitely presented, $\omega : \pi \rightarrow \mathbb{Z}_2$, is a surgery obstruction. First observe that, given π, ω we can find a closed manifold X of dimension $2k-1 \geq 5$ such that $\pi_1 X = \pi$, and ω is the orientation homomorphism for X .

To see this, consider a wedge of circles, one for each generator of π , with 2-discs attached by the prescribed relations. The resulting 2-complex K has $\pi_1 K = \pi$. We now realize $\omega : \pi \rightarrow \mathbb{Z}_2$ by a map $\kappa : K \rightarrow \mathbb{RP}(\infty) = K(\mathbb{Z}_2, 1)$. By "general position" we may as well suppose that κ is an embedding of K in $\mathbb{RP}(2k)$.

Let X be the boundary of a regular neighborhood of κK in $\mathbb{RP}(2k)$. It is not hard to see, using "general position" that $\kappa_* : \pi_1 K \rightarrow \pi_1 X$ is an isomorphism. Hence $\pi_1 X = \pi$. Also, since $\nu(X \subset \mathbb{RP}(2k))$ is trivial, it is not hard to show that the first Stiefel-Whitney Class of X is induced by ω . Hence ω is the orientation homomorphism for X .

Remarks:

- (a) If $\omega = 1$, we could simply take κ an embedding of K in E^{2k} .
- (b) We obtain even dimensional examples from $X \times S^3$.

Theorem 9: Let X^{2k-1} be a compact manifold with fundamental group π and orientation homomorphism ω , $2k-1 \geq 5$. Then we can find a cobordism W and a map $\varphi : (W, \partial_- W, \partial_+ W) \rightarrow (X \times I, X \times 0 \cup \partial X \times I, X \times 1)$ of degree one, together with an equivalence $F : \nu W \rightarrow \varphi^* \nu(X \times I)$ such that

(8) $\varphi|_{\partial_- W}$ is the identity

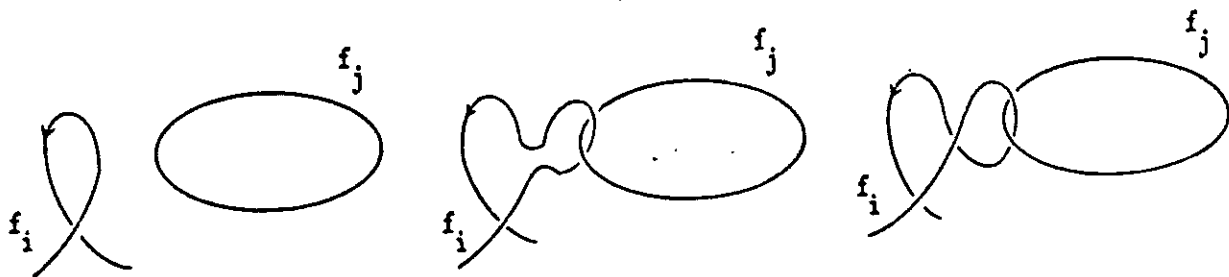
(9) $\varphi|_{\partial_+ W}$ is a simple homotopy equivalence

(10) the surgery obstruction θ for (W, φ, F) is a prescribed element of $L_{2k}(\pi, \omega)$.

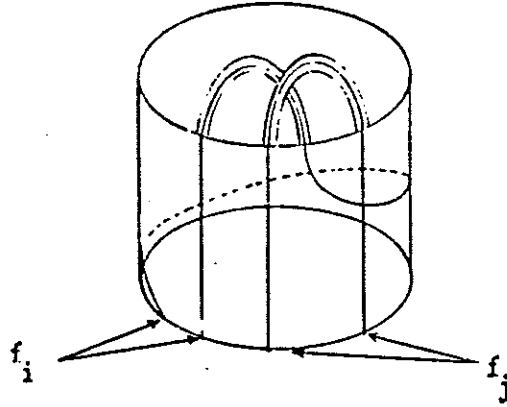
Proof of Theorem 9: Represent the prescribed element of $L_{2k}(\pi, \omega)$ by a Special Hermitian Form (G, λ, μ) with preferred base e_1, \dots, e_r of G . Choose r disjoint $(2k-1)$ -discs D_i in $\text{Int } X$, and let f_1, \dots, f_r be r standard embeddings, $f_i : S^{k-1} \times D^k \rightarrow D_i$.

Now extend f_i to a regular homotopy \overline{F}_i , regarded as an immersion of $S^{k-1} \times D^k \times I \rightarrow X \times I$ so that $f'_i = \overline{F}_i|_{S^{k-1} \times D^k \times 1}$ is also an embedding. As in Theorem 5, we can compute the self-intersections and mutual intersections of $F_i = \overline{F}_i(S^{k-1} \times 0 \times I)$. We can in fact choose $\mu(F_i) = \mu(e_i)$ and $\lambda(F_i, F_j) = \lambda(e_i, e_j)$.

To see this, it is sufficient to introduce a single intersection or self-intersection with intersection number $\pm g$, $g \in \pi$, since intersection and self-intersection numbers are additive under composition of regular homotopies. We join $P \in f_i(S^{k-1} \times 0)$ to $P' \in f'_i(S^{k-1} \times 0)$ by a path obtained by composing a representative for g on the left and right with the prescribed paths for $f_i(S^{k-1} \times 0)$ and $f'_i(S^{k-1} \times 0)$, and deform a neighborhood of P along this path and across a disc transverse to $f'_i(S^{k-1} \times 0)$ centered at P' . We may change the sign ± 1 as necessary by reversing the orientation along this path (figure 18).



Now use the attaching maps f'_i to attach k -handles to $X \times I$ along $X \times 1$; let W be the resulting manifold (figure 19).



The nullhomotopy of f_i in D_i can be used to define an extension $\varphi : W \rightarrow X \times I$ of the identity on $X \times I$, so that $\varphi^* \nu(X \times I)$ is equivalent to νW .

It is clear that (8) holds. To prove (10), note that φ is k -connected and that $K_k(W) = K_k(W, \partial_- W)$ has a preferred class of bases represented by spheres, each obtained by gluing together the core of an attached handle with an F_i , and spanning $f_i(S^{k-1} \times 0)$ by a k -disc in D_i . It follows that the (G, λ, μ) of Theorem 5 are as prescribed.

To show (9), refer to the diagram (6) on page 21. Here, we assume $A\lambda$ is a simple isomorphism, hence so is i_* , and from the exact sequence, $K_i(\partial W) = K_i(\partial_+ W) = 0$ for all i . Note that W is obtained from $\partial_+ W$ by adding handles of index greater than 2, so $\pi_1 \partial_+ W = \pi$. Thus, $\varphi|_{\partial_+ W}$ is a simple homotopy equivalence, and Theorem 9 is proved.

6. The Odd-Dimensional Wall Groups

We now consider the surgery problem (W, φ, F) with W a manifold of dimension $n = 2k+1$. We may suppose that surgery has already been done to make φ k -connected, by Theorem 2. From the exact sequence

$$\longrightarrow H_{i+1}(W, \partial W) \xrightarrow{\varphi_*} H_{i+1}(X, \partial X) \longrightarrow H_{i+1}(\varphi) \longrightarrow H_i(W, \partial W) \xrightarrow{\varphi_*} H_i(X, \partial X) \longrightarrow$$

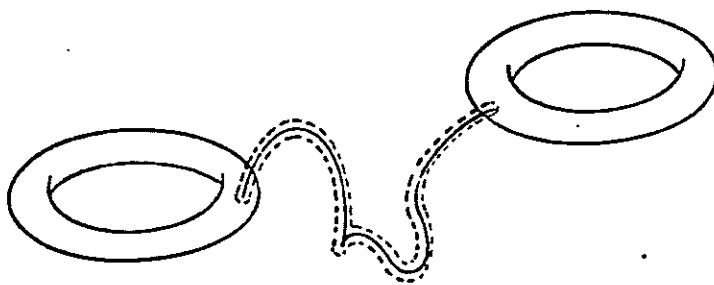
and Theorem 3, it follows, as on page 17, that $K_i(W, \partial W) = 0$ for $i \neq k, k+1$.

Further, $K_k(W, \partial W) = K_k(W) = \pi_{k+1}(\varphi)$.

By "general position," we may describe the generators of $K_k(W)$ by disjoint

embeddings $g_i: S^k \times D^{k+1} \rightarrow W$, $i = 1, \dots, r$ in the regular homotopy classes of immersions given by Theorem 1. As in the even-dimensional case, we regard these classes as elements of a Λ -module, thus we prescribe paths leading from the basepoints of $g_i(S^k \times D^{k+1})$ to the basepoint $*$ of W . We will suppose that each such path is an embedding $I \rightarrow W$, and that different paths meet only in $*$.

In what follows, $g_i(S^k \times D^{k+1})$ will denote the g_i -image of $S^k \times D^{k+1}$ together with a regular neighborhood of the prescribed path (figure 20).

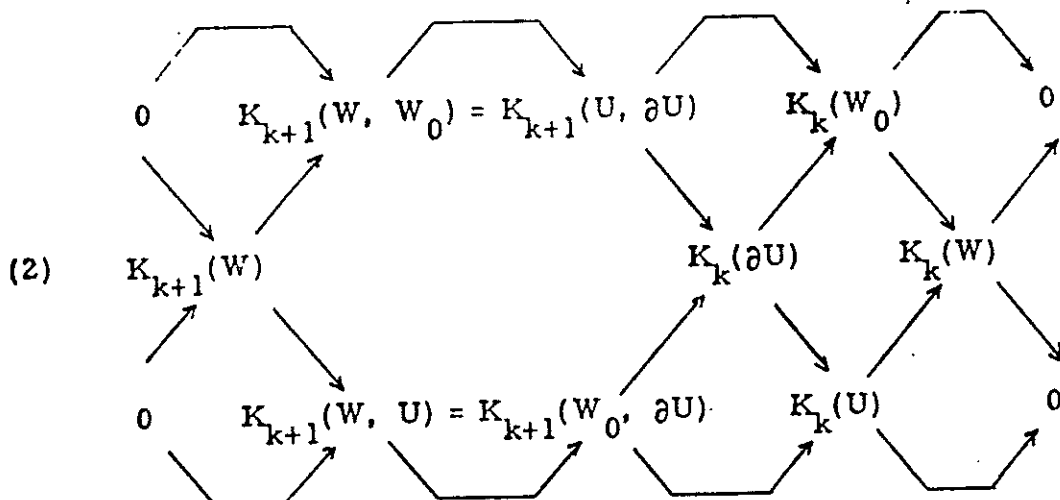


Let $U = \bigcup_{i=1}^r g_i(S^k \times D^{k+1})$. It will be convenient to replace φ by a homotopic map so that

$$(1) \varphi U = D, \varphi \partial U = \partial D$$

for a $2k+1$ disc $D \subset X$. We obtain this map by first noting that we can replace φ by a homotopic map so that $\varphi U \subset D$, and approximating this map by one which is an embedding near an interior point of U ; then deforming the image of U along radii in D .

Now write $W_0 = W - \text{Int } U$ and consider the following commutative diagram of exact sequences in which $K_k(W, U) = K_k(W_0, \partial U) = 0$ by construction:



Note that $K_{k+1}(W, U)$ is the single nonvanishing group in $H_*(C(\varphi), W \cup (U \times I))$. It is not hard to show that $H^{k+3}(C(\varphi), W \cup (U \times I)) = 0$, so that $K_{k+1}(W, U)$ is a finitely generated stably free Λ -module with a preferred class of stable bases, by Theorem 4.

We may in fact suppose $K_{k+1}(W, U)$ is free and based. To see this, represent $0 \in K_k(W)$ by a standard embedding of $S^k \times D^{k+1}$ in a $2k+1$ disc in W whose image, say U_0 , satisfies condition (1) above. We also suppose U_0 contains a regular neighborhood of a prescribed path to $*$. We write W as $W \# S^{2k+1}$, with $U_0 \subset S^{2k+1}$.

Now $H_{i+1}(S^{2k+1}, U_0) = H_i(S^k) = 0$ in dimensions $1 \leq i < 2k$, $i \neq k$, and $H_{k+1}(S^{2k+1}, U_0) = \Lambda$. Apply excision to the mapping cylinder; it follows that

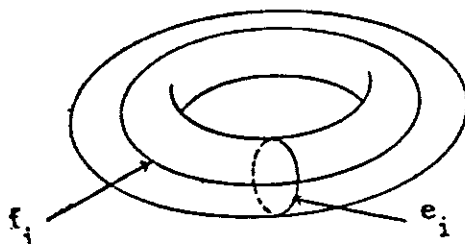
$$K_{k+1}(S^{2k+1}, U_0) = H_{k+1}(S^{2k+1}, U_0), \text{ and so } K_{k+1}(W, U \cup U_0) =$$

$$K_{k+1}(W, U) \oplus K_{k+1}(S^{2k+1}, U_0) = K_{k+1}(W, U) \oplus \Lambda. \text{ Thus we may take}$$

$K_{k+1}(W, U)$ free and based by adding a finite number of copies of U_0 to U .

Again by excision, we have that $K_{k+1}(U, \partial U) = H_{k+1}(U, \partial U)$, $K_k(\partial U) = H_k(\partial U)$ and $K_k(U) = H_k(U)$. Each of these modules has a preferred class of bases, which we now describe. Let e_1, \dots, e_r represent the classes $(g_i(1 \times D^{k+1}), g_i(1 \times S^k))$ in $(U, \partial U)$; we will also write e_1, \dots, e_r for the corresponding classes of $g_i(1 \times S^k)$ in ∂U . Let f_1, \dots, f_r represent the classes $g_i(S^k \times 1)$ in ∂U ; we also write f_1, \dots, f_r for the corresponding

classes $g_i(S^k \times 0)$ in U (figure 21).



Thus, the sequence from upper left to lower right in (2) becomes

$$(3) \quad 0 \longrightarrow K_{k+1}(U, \partial U) \longrightarrow K_k(\partial U) \longrightarrow K_k(U) \longrightarrow 0$$

$$e_1 \dots e_r \quad e_1 \dots e_r f_1 \dots f_r \quad f_1 \dots f_r$$

where we suppose the orientations have been chosen so that $\lambda(e_i, f_j) = \delta_{ij}$. Of course, $\mu(e_i) = \mu(f_i) = 0$, so this choice of basis gives $K_k(\partial U)$ the structure of a kernel.

Now consider the module $K_k(W_0)$. Replacing W by W_0 in (6) of §5, and proceeding clockwise from $K_k(W_0)$, we have that $K_k(W_0) = \text{Hom}_\Lambda(K_{k+1}(W_0, \partial W_0), \Lambda)$. Thus, $K_k(W_0)$ is a finitely generated free Λ -module, with preferred basis dual to the preferred basis for $K_{k+1}(W_0, \partial W_0) = K_{k+1}(W_0, \partial U)$ ($\partial W_0 = \partial U \cup \partial W$, $\varphi|_{\partial W}$ is a homotopy equivalence).

It follows that the sequence from lower left to upper right in (2) splits, and the preferred basis $\{e_i^*\}$ for $K_{k+1}(W_0, \partial U)$ together with the preferred basis $\{e_i^*\}$ for $K_k(W_0)$ forms a basis for $K_k(\partial U)$:

$$(4) \quad 0 \longrightarrow K_{k+1}(W_0, \partial U) \longrightarrow K_k(\partial U) \longrightarrow K_k(W_0) \longrightarrow 0$$

$$e_1^* \dots e_r^* \quad e_1^* \dots e_r^* e_1^* \dots e_r^* \quad e_1^* \dots e_r^*$$

Further, the sequence (4) is the exact homology sequence associated with the short exact sequence of chain complexes of the mapping cylinders corresponding to $\partial U \longrightarrow W_0 \longrightarrow (W_0, \partial U)$, and the bases $\{e_i^*\}$, $\{e_i^*\}$ have

been chosen to make the torsions of the two associated chain complexes vanish. Now with the basis $e_1 \dots e_r, f_1 \dots f_r$ of $K_k(\partial U)$ from (3), the torsion of the associated chain complex corresponding to ∂U also vanishes.

A standard formula for Whitehead Torsion [4] now shows that the torsion of

$$(5) \quad 0 \longrightarrow K_{k+1}(W_0, \partial U) \longrightarrow K_k(\partial U) \longrightarrow K_k(W_0) \longrightarrow 0$$

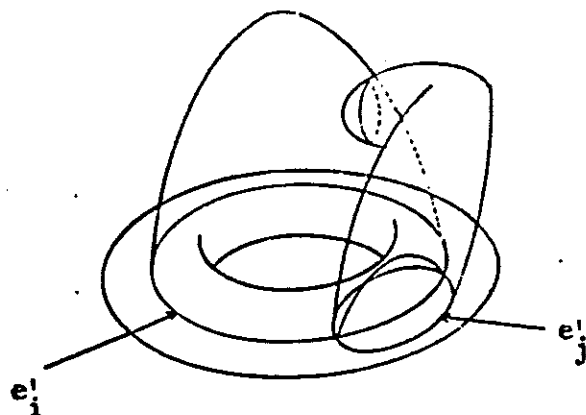
$$e'_1 \dots e'_r \quad e_1 \dots e_r f_1 \dots f_r \quad e_1^* \dots e_r^*$$

vanishes; it is a linear combination of the torsions of the associated chain complexes. Thus the transformation defined by $e_i \longrightarrow e'_i, f_i \longrightarrow e_i^*$ is a simple isomorphism of $K_k(\partial U)$.

We will now show that the submodule H of $K_k(\partial U)$ generated by $e'_1 \dots e'_r$ satisfies the hypotheses of Lemma 7. It then follows that we can obtain basis elements f'_1, \dots, f'_r from e_1^*, \dots, e_r^* by an elementary transformation so that $\{e'_i, f'_j\}$ also gives $K_k(\partial U)$ the structure of a kernel.

We need only show that $\lambda(H \times H) = 0, \mu(H) = 0$. Now a generator e'_i of H , as a linear combination of $e_1, \dots, e_r, f_1, \dots, f_r$, is represented by a sphere. Since e'_i lies in the image of $K_{k+1}(W_0, \partial U)$, it bounds in W_0 ; in fact, by the Relative Hurewicz Theorem [3], the representative sphere is nullhomotopic in W_0 . We therefore obtain maps $d_i : (D^{k+1}, \partial D^{k+1}) \longrightarrow (W_0, \partial U)$ so that the restriction $\partial D^{k+1} \longrightarrow \partial U$ of d_i represents e'_i .

We may suppose $d_i D^{k+1}$ and $d_j D^{k+1}$ meet transversely in a finite set of circles and arcs with both ends representing intersections of $d_i \partial D^{k+1}$ and $d_j \partial D^{k+1}$ with opposite sign (figure 22). It follows that $\lambda(e'_i, e'_j) = 0$.



Similarly, we may suppose $d_i D^{k+1}$ has transverse self-intersections; the self-intersections of $d_i \partial D^{k+1}$ then occur in pairs as endpoints of arcs of self-intersections of $d_i D^{k+1}$. It follows as above that $\mu(e'_j) = 0$. Thus, $\lambda(H \times H) = 0$, $\mu(H) = 0$ and we obtain a basis $\{e'_i, f'_i\}$ from Lemma 7 which gives $K_k(\partial U)$ the structure of a kernel:

$$(6) \quad 0 \longrightarrow K_{k+1}(W_0, \partial U) \longrightarrow K_k(\partial U) \longrightarrow K_k(W_0) \longrightarrow 0$$

$$e'_1 \dots e'_r \quad e'_1 \dots e'_r f'_1 \dots f'_r \quad f'_1 \dots f'_r$$

Now the transformation of $K_k(\partial U)$ defined by $e_i \longrightarrow e'_i, f_i \longrightarrow f'_i$ is an element α of $SU_r(\Lambda)$, the group of automorphisms of the standard kernel, that is, the group of simple isomorphisms of the standard kernel which preserve λ and μ .

α depends on the choice of the f_1, \dots, f_r given by Lemma 7. If β is an automorphism corresponding to another choice of f_1, \dots, f_r , $\beta\alpha^{-1}$ lies in $TU_r(\Lambda)$, the subgroup of automorphisms of $SU_r(\Lambda)$ which leave H invariant and induce a simple isomorphism of H .

α also depends on the choice of embeddings g_i representing regular homotopy classes corresponding to generators of $K_k(W)$. We will show that making a new choice of embeddings g_i in the given regular homotopy classes has the effect of replacing α by $\alpha\gamma$, where γ lies in a subgroup of $TU_r(\Lambda)$. In particular, the double coset $TU_r(\Lambda)\alpha TU_r(\Lambda)$ does not

depend on the choices in Lemma 7, or on the choice of representative embeddings in the regular homotopy classes.

To see this, choose embeddings \hat{g}_i of $S^k \times D^{k+1}$ in W , regularly homotopic to g_i ; let G_i be the restriction to $S^k \times I$ of a regular homotopy $S^k \times D^{k+1} \times I \rightarrow W \times I$ between g_i and \hat{g}_i . Let $\hat{U} = \bigcup_{i=1}^r \hat{g}_i(S^k \times D^{k+1})$, $\hat{W}_0 = W - \hat{U}$. Then the sequence (3) is isomorphic, by $\hat{g}_i g_i^{-1}$ to the sequence

$$(3) \quad 0 \longrightarrow K_{k+1}(\hat{U}, \partial\hat{U}) \longrightarrow K_k(\partial\hat{U}) \longrightarrow K_k(\hat{U}) \longrightarrow 0$$

$$\hat{e}_1 \dots \hat{e}_r \quad \hat{e}_1 \dots \hat{e}_r \hat{f}_1 \dots \hat{f}_r \quad \hat{f}_1 \dots \hat{f}_r$$

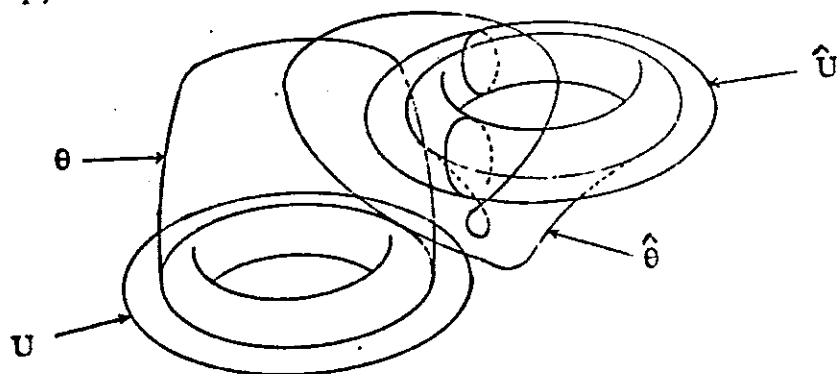
Also, there is a natural isomorphism \hat{k} of $K_{k+1}(W_0, \partial U) \oplus K_k(W_0)$ with $K_{k+1}(\hat{W}_0, \partial\hat{U}) \oplus K_k(\hat{W}_0)$, which sends the preferred basis $e_1, \dots, e_r, f_1, \dots, f_r$ to a preferred basis $\hat{e}_1, \dots, \hat{e}_r, \hat{f}_1, \dots, \hat{f}_r$. We will show that \hat{k} sends the basis $e_1, \dots, e_r, f_1, \dots, f_r$ to the basis $\hat{e}_1, \dots, \hat{e}_r, \hat{f}_1 - \sum \lambda(F_j, F_1)\hat{e}_j, \dots, \hat{f}_r - \sum \lambda(F_j, F_r)\hat{e}_j; \hat{e}_j, \hat{f}_j$ as in (3). It follows that the automorphism \hat{a} corresponding to \hat{g}_i , given by $\hat{e}_i \rightarrow \hat{e}_i, \hat{f}_i \rightarrow \hat{f}_i$ is the composition $\alpha\gamma$ of α with a transformation $\gamma \in TU_r(\Lambda)$.

In order to prove the above formula for the effect of \hat{k} on $e_1, \dots, e_r, f_1, \dots, f_r$, we give an explicit description of the isomorphism $K_{k+1}(W_0, \partial U) \rightarrow K_{k+1}(\hat{W}_0, \partial\hat{U})$ induced by restricting \hat{k} . Let θ be a typical relative cycle representing an element of $K_{k+1}(W_0, \partial U)$; we have seen that we may represent θ by an immersion $\theta: (D^{k+1}, \partial D^{k+1}) \rightarrow (W_0, \partial U)$. Thus, $\partial\theta$ is a linear combination of the standard generators of $K_k(\partial U)$,

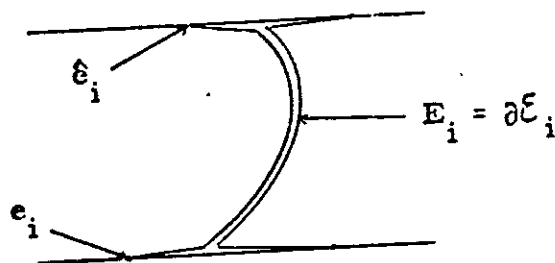
$$\partial\theta = \sum \epsilon_i e_i + \sum \phi_i f_i.$$

Now G_1, \dots, G_r induce regular homotopies E_i of $g_i(1 \times S^k)$ representing e_i , and F_i of $g_i(S^k \times 1)$ representing f_i in $K_k(\partial U)$. By the covering homotopy property for immersions [1], we can find a regular

homotopy $\Theta : D^{k+1} \times I \rightarrow W \times I$ from θ to $\hat{\theta}$, say, which covers the regular homotopy of $\partial\theta$ induced by $E_1, \dots, E_r, F_1, \dots, F_r$ (figure 23).



We may suppose E_i is an isotopy, whose image lies in a neighborhood of $g_i(1 \times D^{k+1}) \cup \hat{g}_i(1 \times D^{k+1}) \cup \pi_i$, where π_i is a path from $g_i(1 \times 0)$ to $\hat{g}_i(1 \times 0)$, π_i disjoint. Thus, E_i bounds, say $E_i = \partial \mathcal{E}_i$ (figure 24).



We now calculate $\partial\hat{\theta}$. If η represents the fundamental class in $W \times I$, $\eta|_{W \times \{0, 1\}} = \eta_1 - \eta_0$ represents the difference of the fundamental class of $W \times 1$ and that of $W \times 0$. Thus,

$$\begin{aligned} \pm \delta(\eta \cap)^{-1} \Theta &= (\eta \cap)^{-1} \partial\Theta = (\eta \cap)^{-1} (\hat{\theta} - \theta + \sum \epsilon_i E_i + \sum \varphi_i F_i) = \\ &= (\eta_1 \cap)^{-1} \hat{\theta} + (\eta_0 \cap)^{-1} \theta + \delta \sum \epsilon_i (\eta \cap)^{-1} \mathcal{E}_i + \sum \varphi_i (\eta \cap)^{-1} F_i. \end{aligned}$$

It follows, using (7) of §5, that

$$\begin{aligned} \lambda(\sum F_j, \sum \varphi_i F_i) &= \langle \sum F_j, \sum \varphi_i (\eta \cap)^{-1} F_i \rangle = - \langle \sum F_j, (\eta_1 \cap)^{-1} \hat{\theta} + (\eta_0 \cap)^{-1} \theta \rangle = \\ &= - \langle \sum \partial F_j, (\eta_1 \cap)^{-1} \hat{\theta} + (\eta_0 \cap)^{-1} \theta \rangle = \sum - \langle \hat{g}_j(S^k \times 0), (\eta_1 \cap)^{-1} \hat{\theta} \rangle + \\ &\quad \langle g_j(S^k \times 0), (\eta_0 \cap)^{-1} \theta \rangle = - \sum \lambda(\hat{g}_j(S^k \times 0), \hat{\theta}). \end{aligned}$$

We may assume that F_i and F_j meet transversely, thus $\text{Int } \hat{\theta} \cap \partial \hat{U}$ consists of spheres of the form $1 \times S^k$ (figure 23), one for each intersection of $\hat{g}_j(S^k \times 0)$ with $\text{Int } \hat{\theta}$. Hence,

$$\begin{aligned} \partial \hat{\theta} &= \sum \epsilon_i \hat{e}_i + \sum \varphi_i \hat{f}_i + \sum \lambda(\hat{g}_j(S^k \times 0), \hat{\theta}) \hat{e}_j \\ &= \sum \epsilon_i \hat{e}_i + \sum \varphi_i (\hat{f}_i - \sum \lambda(F_j, F_i) \hat{e}_j) \end{aligned}$$

as was to be proved.

Now let $\sigma \in \text{SU}_2(\Lambda)$ denote the automorphism of the standard plane whose matrix with respect to the standard basis is given by:

$$(7) \quad \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

Let $\text{SU}(\Lambda)$ be the direct limit of $\dots \text{SU}_r(\Lambda) \subset \text{SU}_{r+1}(\Lambda) \subset \dots$ and $\text{TU}(\Lambda)$ the limit of $\dots \text{TU}_r(\Lambda) \subset \text{TU}_{r+1}(\Lambda) \subset \dots$. Define $\text{RU}(\Lambda)$ as the subgroup of $\text{SU}(\Lambda)$ generated by σ and $\text{TU}(\Lambda)$.

It can be shown algebraically that $\text{TU}(\Lambda) \subset \text{SU}'(\Lambda) \subset \text{RU}(\Lambda)$, where $\text{SU}'(\Lambda)$ is the commutator subgroup of $\text{SU}(\Lambda)$ [13].* Hence, $\text{RU}(\Lambda)$ is normal, and the quotient $\text{SU}(\Lambda)/\text{RU}(\Lambda)$ is an abelian group, the Wall Surgery Obstruction Group $L_{2k+1}(\pi_1 X, \omega X)$.

Note that the double coset $\text{TU}_r(\Lambda) \alpha \text{TU}_r(\Lambda)$ determined above by the surgery problem (W, φ, F) defines a left coset $\alpha \text{RU}(\Lambda)$ of the normal subgroup $\text{RU}(\Lambda)$ and hence an element of $L_{2k+1}(\pi_1 X, \omega X)$.

Theorem 10: Let (W^{2k+1}, φ, F) be a surgery problem, θ the element of $L_{2k+1}(\pi_1 X, \omega X)$ determined by $\alpha \text{RU}(\Lambda)$. If $\theta = 0$, we can do surgery so as to make φ a simple homotopy equivalence, provided $k \geq 2$.

Proof of Theorem 10: $\alpha \text{RU}(\Lambda)$ represents 0 when $\alpha = \dots \sigma \dots \tau \dots \sigma \dots \tau^{-1} \dots \in \text{RU}(\Lambda)$. Thus, $\dots (\tau^{-1})^{-1} \dots \sigma^{-1} \dots \tau^{-1} \dots \sigma^{-1} \dots$

* It can also be shown that $\text{RU}(\Lambda)$ is generated by $\text{SU}'(\Lambda)$ and σ .

We first show how to realize multiplication by elements τ^{-1} , σ^{-1} geometrically.

Now multiplication by $\sigma = (-1)^k \sigma^{-1}$ corresponds to a surgery on an embedding $g'_1 : S^k \times D^{k+1} \rightarrow W$ chosen so that $g'_1(1 \times S^k)$ represents e'_1 , $g'_1(S^k \times 1)$ represents f'_1 , in $K_k(\partial U)$. We obtain g'_1 by choosing an embedding $S^k \times D^{k+1} \rightarrow W$ whose core represents $f'_1 \in K_k(U)$, expressed as a linear combination of the standard basis elements f_1, \dots, f_r . The formula for the intersection numbers $\lambda(e_i, f_j)$, $\lambda(e'_i, f'_j)$ shows that $g'_1(1 \times S^k)$ represents e'_1 . Since $\partial(D^{k+1} \times D^{k+1}) = S^k \times D^{k+1} \cup D^{k+1} \times S^k$, the effect of this surgery is to replace e'_1 by $(-1)^k f'_1$, f'_1 by e'_1 , that is, to multiply by σ .

On the other hand, $\tau \in TU(\Lambda)$ is the product of elementary transformation of the type represented by

(8) a permutation matrix

(9) a matrix which sends e'_1 to $\pm e'_1 g$, f'_1 to $\pm \omega(g) f'_1 g$

(10) a matrix which sends e'_1 to $e'_1 + e'_2$, e'_2 to e'_2 , f'_1 to f'_1 , f'_2 to $f'_2 - f'_1$

The transformation (8) may be obtained by interchanging the order of the g'_i . (9) corresponds to choosing a new prescribed path for g'_1 and a new orientation along that path. To obtain (10), we replace g'_1, g'_2 by $g'_1 \# g'_2, g'_2$.

It follows that we can reduce a to 1 by a finite number of surgeries. By further surgeries, we can replace 1 by $\sigma \oplus \dots \oplus \sigma$ (r copies). The effect of the corresponding automorphism of $K_k(\partial U)$ is as follows:

$e_i \rightarrow (-1)^k f_i$, $f_i \rightarrow e_i$. This means that (6) has the form

$$(6') \quad 0 \rightarrow K_{k+1}(W_0, \partial U) \rightarrow K_k(\partial U) \rightarrow K_k(W_0) \rightarrow 0$$

$$e'_1 \dots e'_r \quad (-1)^k f'_1 \dots (-1)^k f'_r \quad e_1 \dots e_r \quad f'_1 \dots f'_r$$

Now by the commutativity of the diagram (2), the map $K_{k+1}(W, U) \rightarrow K_k(U)$ is a simple isomorphism, and so $K_{k+1}(W) = K_k(W) = 0$. Hence, φ is a homotopy equivalence. Since the stable bases at each stage were chosen to make the torsion of $C_*(\varphi)$ vanish, φ is a simple homotopy equivalence. This proves Theorem 10.

We now show that each element of $L_{2k+1}(\pi, \omega)$, π finitely presented, $\omega : \pi \rightarrow \mathbb{Z}_2$, is a surgery obstruction. As in §5, we can find a closed manifold X^{2k} such that $\pi_1 X = \pi$ and ω is the orientation homomorphism for X .

Theorem 11: Let X^{2k} be a compact manifold with fundamental group π and orientation homomorphism ω , $2k \geq 6$. Then we can find a cobordism W and a map $\varphi : (W, \partial_- W, \partial_+ W) \rightarrow (X \times I, X \times 0 \cup \partial X \times I, X \times 1)$ of degree one, together with an equivalence $F : \nu W \rightarrow \varphi^* \nu(X \times I)$ such that

(11) $\varphi|_{\partial_- W}$ is the identity

(12) $\varphi|_{\partial_+ W}$ is a simple homotopy equivalence

(13) the surgery obstruction θ for (W, φ, F) is a prescribed element of $L_{2k+1}(\pi, \omega)$.

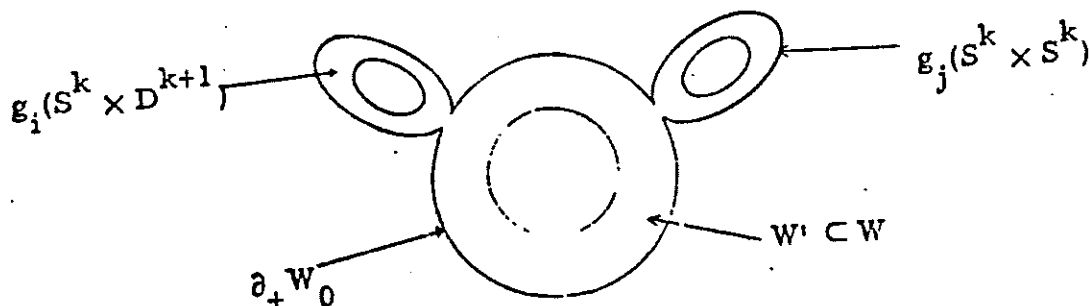
Proof of Theorem 11: Represent the prescribed element of $L_{2k+1}(\pi, \omega)$ by $a \in SU_r(\Lambda)$. Now do surgery on r copies of the trivial element of $\pi_k(1_X)$; by the remarks preceding figures 3 and 14 we obtain a cobordism W' by attaching r handles of index k to $X \times I$ along $X \times 1$ so that $\partial_+ W'$ is the connected sum of X with r copies of $S^k \times S^k$. We also obtain $\varphi' : W' \rightarrow X \times I$ of degree one, $F' : \nu W' \rightarrow \varphi'^* \nu(X \times I)$ an equivalence, with $\varphi'|_{\partial_- W'} = 1_X$.

Write g_1, \dots, g_r for the embeddings of $S^k \times S^k$ in $\partial_+ W'$ obtained above. Let $\{e_i, f_i\}$ be the generators of $K_k(\partial_+ W')$ represented by $g_i(1 \times S^k)$, $g_i(S^k \times 1)$ respectively. Then $\{e_i, f_i\}$ gives $\partial_+ W'$ the structure of a kernel. Let $e'_i = ae_i$, $f'_i = af_i$; then $\{e'_i, f'_i\}$ also gives $\partial_+ W'$ the structure of a kernel.

It follows by Theorem 8 that we can do surgery on $(\partial_+ W', \varphi'|_{\partial_+ W'}, F'|_{\nu \partial_+ W'})$ corresponding to the class of (G, λ, μ) determined by $\{e'_i, f'_i\}$. We thus obtain (W'', φ'', F'') with $\partial_- W'' = \partial_- W'$, $\varphi'' : W'' \rightarrow X \times I$ of degree one, and $F'' : \nu W'' \rightarrow \varphi''^* \nu(X \times I)$ an equivalence, such that $\varphi''|_{\partial_- W''} = \varphi'|_{\partial_- W'}$ and $\varphi''|_{\partial_+ W''}$ is a simple homotopy equivalence.

We can now obtain (W, φ, F) by putting together (W', φ', F') and

(W'', φ'', F'') . It is clear that (11) and (12) hold. To prove (13), note that the surgeries of Theorem 8 correspond to handles of index $k+1$ attached along $\partial_+ W'$. Thus, the generators of $K_k(W)$ are represented by embeddings $\bar{g}_1, \dots, \bar{g}_r$ of $S^k \times D^{k+1}$ in W' extending g_1, \dots, g_r in a natural way (figure 25).



Also, these generators correspond to f_1, \dots, f_r in $K_k(U)$, where $U = \bigcup_{i=1}^r \bar{g}_i(S^k \times D^{k+1})$.

On the other hand, the preferred basis of $K_{k+1}(W, U) = K_{k+1}(W_0, \partial U)$ is given by the classes of the cores of the handles of index $k+1$ attached to $\partial_+ W'$ as in Theorem 8. These were chosen so that their boundaries represent e'_1, \dots, e'_r so the automorphism defining the surgery obstruction for (W, φ, F) sends e_i to e'_i . It is not hard to show that this automorphism also sends f_i to f'_i , hence it represents the prescribed element of $L_{2k+1}(\pi, \omega)$.

7. Some Further Properties of the Groups $L_n(\pi, \omega)$

We now mention some further results without giving proofs. Observe that, with $n = 2k$ or $2k+1$, k occurs in the definition of $L_n(\pi, \omega)$ exactly in the form of factors $(-1)^k$. Thus, on replacing k by $k+2$ we obtain the same Wall Surgery Obstruction Group. That is, $L_{n+4}(\pi, \omega) = L_n(\pi, \omega)$.

Further, this periodicity is generated by multiplication by $\mathbb{C}P(2)$. This means the following: Given the surgery problem (W, φ, F) we obtain an element $\theta \in L_n(\pi, \omega)$. On the other hand, the surgery problem $(W \times \mathbb{C}P(2), \varphi \times 1_{\mathbb{C}P(2)}, F \times 1_{\nu \mathbb{C}P(2)})$ determines an element

$\theta' \in L_{n+4}(\pi, \omega) = L_n(\pi, \omega)$. The first sentence of this paragraph states that $\theta' = \theta$.

Another useful fact is the following: If we have a commutative diagram of homomorphisms

$$\begin{array}{ccc} & \pi & \omega \\ h \downarrow & \searrow & \nearrow \\ \pi' & & \omega' \end{array} \quad \begin{array}{c} \\ \\ \mathbb{Z}_2 \end{array}$$

we get a homomorphism $l_n(h) : L_n(\pi, \omega) \rightarrow L_n(\pi', \omega')$, and it can be checked that (L_n, l_n) is a functor.

We now reformulate the results of the preceding sections. Define the bordism classes $B_n(X)$ as the set of equivalence classes of triples (W^n, φ, F) as in the surgery problem; two such, say (W, φ, F) and (W', φ', F') are equivalent if there is a triple $(\mathcal{W}, \Phi, \mathcal{F})$, $\Phi : \mathcal{W} \rightarrow X \times I$, $\mathcal{F} : \nu \mathcal{W} \rightarrow \nu(X \times I)$ an equivalence, such that $\partial \mathcal{W} = W - W'$, $\Phi|_W = \varphi$, $\Phi|_{W'} = \varphi'$, $\mathcal{F}|_{\nu W} = F$, $\mathcal{F}|_{\nu W'} = F'$. Of course, (W, φ, F) and (W', φ', F') are equivalent if and only if (W', φ', F') can be obtained from (W, φ, F) by surgery.

We can define a map $\mathcal{H} : B_n(X) \rightarrow L_n(\pi_1 X, \omega X)$ by sending a representative element (W, φ, F) to its surgery obstruction θ . It turns out that the element θ does not depend upon the choice of representative. It follows that \mathcal{H} is a homomorphism, where addition in $B_n(X)$ is defined using connected sum.

By Theorems 8 and 10, for $n \geq 6$ $\text{Ker } \mathcal{H}$ is the subset of $B_n(X)$ with representatives (W, φ, F) satisfying: φ is a simple homotopy equivalence. By Theorems 9 and 11 and the additivity of \mathcal{H} , $L_n(\pi, \omega)$ is the \mathcal{H} -image of the subset of $B_n(X)$ with representatives (W, φ, F) satisfying: $\varphi|_{\partial W}$ is a simple homotopy equivalence.

We now give a brief description of the form in which the results on surgery are usually applied. For convenience, we give this formulation in the smooth case; all the preceding theorems carry over to this case

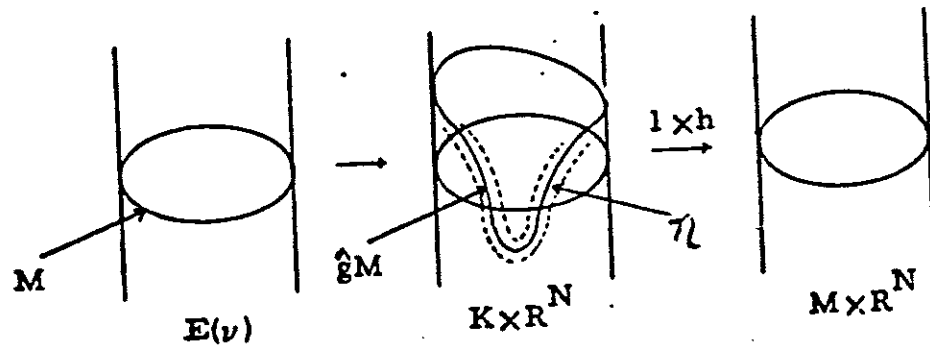
using the theory of smooth manifolds rather than the piecewise linear theory.

Let us first define F_k as the set of basepoint preserving homotopy equivalences of S^k . Note that a proper homotopy equivalence of E^k extends to the one point compactification and so defines an element of F_k . In particular, the orthogonal group O_k lies in F_k . Passing to the limit ($F_k \subset F_{k+1}$ by suspension), $O \subset F$. We may then form F/O ; this turns out to be the fibre of a fibration:

$$F/O \rightarrow BO \rightarrow BF.$$

Now a homotopy class of maps $M \rightarrow F/O$, M a closed, smooth manifold, determines a class of maps into BO , the classifying space for vector bundles over M , so that a representative classifying map becomes trivial in BF . Thus, $M \rightarrow F/O$ determines an equivalence class of fibre homotopy trivializations of vector bundles over M . Conversely, such an equivalence class determines a homotopy class $M \rightarrow F/O$.

We will now see that a simple homotopy equivalence $h: K \rightarrow M$ determines an element of $[M, F/O]$. Let g be a homotopy inverse of h . Then for N sufficiently large we can approximate $g: M \rightarrow K \subset K \times \mathbb{R}^N$ by an embedding \hat{g} , with normal bundle ν . Let $E(\nu)$ be the total space of ν ; then the total space of the unit disc bundle is diffeomorphic to the interior of a tubular neighborhood \mathcal{N} of the image of g (figure 26).



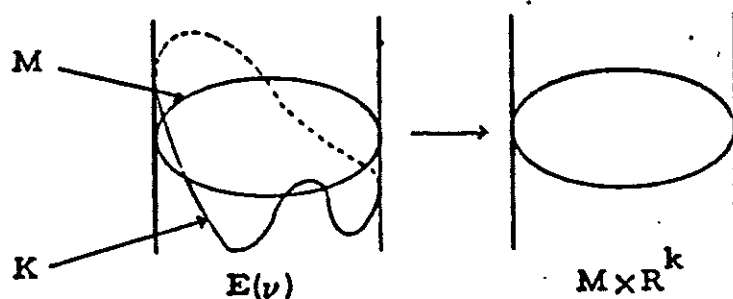
It can be shown that the region between $K \times$ (a large disc in \mathbb{R}^N) and \mathcal{N} is an h-cobordism. By the method of proof of the Weak

h-cobordism Theorem, we may conclude that $K \times \mathbb{R}^N - \mathcal{N}$ is a product $\partial \mathcal{N} \times [0, \infty)$, and so obtain a diffeomorphism $E(\nu) \rightarrow K \times \mathbb{R}^N$. Composing this with $h \times 1 : K \times \mathbb{R}^N \rightarrow M \times \mathbb{R}^N$, we obtain a fibre homotopy trivialization of $E(\nu)$, that is, an element of $[M, F/O]$.

It turns out that this correspondence induces a map $hS(M) \rightarrow [M, F/O]$, where $hS(M)$ is the set of equivalence classes of simple homotopy equivalences $h : K \rightarrow M$ defined as follows: $h : K \rightarrow M$ and $h' : K' \rightarrow M$ are identified when h is homotopic to the composition of h' with a diffeomorphism $K \rightarrow K'$:

$$\begin{array}{ccc} K & \xrightarrow{h} & M \\ \downarrow & \nearrow h' & \\ K' & & \end{array}$$

We will now define a map $[M, F/O] \rightarrow L_n(\pi_1 M, \omega M)$. Choose a representative fibre homotopy trivialization $H : E(\nu) \rightarrow M \times \mathbb{R}^k$, ν a vector bundle over M . We may approximate H by a map \hat{H} , transverse regular along $M \times 0$. Then $K = \hat{H}^{-1}(M \times 0)$ is a smooth manifold, $\varphi : K \rightarrow E(\nu) \xrightarrow{\hat{H}} M \times 0 \rightarrow M$ is a smooth map, and the framing of the normal bundle $\nu(M \times 0 \subset M \times \mathbb{R}^k)$ induces a framing of the normal bundle $\nu(K \subset E(\nu))$. This framing can be used to define an equivalence $F : \nu(K \subset \mathbb{R}^N) \rightarrow \varphi^* \nu(M \subset \mathbb{R}^N)$; then (K, φ, F) determines an element of $L_n(\pi_1 M, \omega M)$ (figure 27).



Finally, we have a map $L_{n+1}(\pi_1 M, \omega M) \rightarrow hS(M)$, as follows: Given $\theta \in L_{n+1}(\pi_1 M, \omega M)$ there is a cobordism W , a map $\varphi : (W, \partial W) \rightarrow (M \times I, M \times \partial I)$ and an equivalence $F : \nu W \rightarrow \varphi^* \nu(M \times I)$

with $\varphi|_{\partial_- W}$ the identity, $\varphi|_{\partial_+ W}$ a simple homotopy equivalence, so that the surgery problem (W, φ, F) determines θ . Then $\varphi|_{\partial_+ W} : \partial_+ W \rightarrow M$ determines an element of $hS(M)$.

It turns out that all the above maps are well defined. A theorem of Sullivan states that the sequence of pointed sets:

$$L_{n+1}(\pi_1 M, \omega M) \rightarrow hS(M) \rightarrow [M, F/O] \rightarrow L_n(\pi_1 M, \omega M) \rightarrow$$

is exact.

Here is a selected list of calculations; p an odd prime:

$L_n(\pi, 1)$	$n \backslash \pi$	1	\mathbb{Z}_2	\mathbb{Z}_p	$ \pi $ odd
1	0	0		.	$4L_1(\pi, 1) = 0$
2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}^{(p-1)/2}$	$\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$
3	0	\mathbb{Z}_2		0	0
4	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}^{(p-1)2}$		$\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$

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