Intersection Homology and Perverse Sheaves

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Introduction

This report is about intersection homology and perverse sheaves. Intersection homology is the subject of Chapter 1, which may be read independently of the rest. This introduction will concern only perverse sheaves, which are perhaps less well known at present.

The first thing to know about perverse sheaves is that they are neither sheaves nor are they perverse [1]. They have in common with sheaves the fact that you can take the cohomology of them, the fact that they form an abelian category, and the fact that to construct one, it is enough to construct it locally everywhere. The adjective "perverse" is a reference to certain vanishing conditions that their cohomology satisfies, which are unfamiliar looking to someone who is used to ordinary cohomology [2].

The second thing to know is that Perverse sheaves are one of the most natural and fundamental objects in topology. Their naturality may be seen through their beautiful formal properties. Their fundamental importance is clear from the list of problems in diverse areas of mathematics that have been reformulated in terms of perverse sheaves and solved using them.

Although perverse sheaves are geometric objects, it has been difficult for geometrically minded mathematicians to absorb the theory. There are two reasons for this. The first reason is a special case of a very general problem of mathematical exposition: Geometry tends to be explained in a way that is algebraically natural, rather than geometrically natural, since algebra is closer to language than geometry is. The second reason is of a more technical nature: Perverse sheaves are defined in terms of the derived category of the category of ordinary sheaves. However, perverse sheaves are much simpler and more natural ordinary sheaves, let alone their derived category. We know this from their formal properties. Therefore, if we believe in the essential simplicity of mathematics, the definition starting from ordinary sheaves must not be the most elegant one.

In these notes, we give a definition of perverse sheaves which relies on Morse theory as its fundamental tool, rather than on ordinary sheaves or derived categories. Since Morse theory is very geometric in nature, the hope is that a

[1] "Les faisceaux pervers n'etant ni des faisceaux, ni pervers, la terminologie requiert une explication." [BBD] p. 10.

[2] An example of such a vanishing condition is the following: Let $\mathcal{P}$ be a perverse sheaf on a complex vector space C^n. Suppose that the support of $\mathcal{P}$ is a complex subvariety V of complex dimension is k, (so its real dimension is $2k$). Then both the cohomology of $\mathcal{P}$ and the compact support cohomology of $\mathcal{P}$ are zero outside ranges of degrees of length k, just as if V had real dimension k. For ordinary constructible sheaves, this is true for the cohomology, but not for the cohomology with compact supports. See section 1.8.2

Actually, the original meaning of perverse was a non-transversality property of the chains for intersection homology [GM1], as explained in section 1.1 . This turns out to imply the vanishing conditions just referred to.

presentation of based on it will appeal to geometers.

The rest of this introduction will have the character of a New York Times article rather than a piece of mathematics. I suggest that the reader go directly to the definition of a perverse sheaf in sections 4.2 and 4.3 and see directly what kind of a beast it is.

0.1 Formal properties

The perverse sheaves on a complex manifold X form an abelian category, i.e. the notions of injections, surjections, kernels, cokernels, exact sequences, adding maps between the same pair of perverse sheaves, and direct sums all make sense and have the usual properties. The category is Artinian, i.e. every perverse sheaf has a finite composition series whose successive quotients are irreducible perverse sheaves.

The irreducible perverse sheaves are just the middle perversity intersection homology sheaves of subvarieties of X. Also, intersection homology sheaves of subvarieties with perversities "close to middle" are perverse. Therefore intersection homology theory and the theory of perverse sheaves are closely intertwined.

Perverse sheaves are closed under the vanishing cycle functor and the nearby cycle functor, for a family of varieties parameterized by a curve. Irreducible perverse sheaves have a remarkable closure property under complex analytic maps (see section 1.7). Perverse sheaves arise from differential equations: The category of holonomic $\mathcal{D}$-modules with regular singularities on X is equivalent to the category of perverse sheaves on X.

0.2 History

The whole theory came together during a few dramatic months of 1980. Intersection homology sheaves, due to Goresky and myself, had been around for several years; they were topological in nature. Holonomic $\mathcal{D}$-modules with regular singularities, due to Kashiwara and Kawai, had also been around for several years: they obviously form an abelian category which was linked to topology by the Riemann-Hilbert correspondence, due to Kashiwara and to Mebkhout. In 19980, proof of the Kazhdan-Lusztig conjectures, due to Beilinson and Bernstein and to Brylinski and Kashiwara, established that the Riemann- Hilbert correspondence embedded intersection homology into holonomic $\mathcal{D}$-modules with regular singularities. Now that intersection homology sat inside an abelian category, the identification of that category was immediately carried out by Beilinson. Bernstein, Deligne, and Gabber. That category is perverse sheaves.

This thumbnail sketch already shows that many mathematicians contributed in an essential way to the creation of perverse sheaves. The detailed story has many more nuances, and it has been written about in more detail a number of times. Surveys with some historical material can be found in [K], [BBD], [M], [Br], and [S].

0.3 Singularities

If the only spaces you are interested in are nonsingular manifolds and the only maps you are interested in are fibrations, then perverse sheaves won't matter at all to you. Perverse sheaves involve singularities in an essential way. The subspace Y of X that is the support of a perverse sheaf will usually be singular, and even when it's not, the perverse sheaf itself will usually have singularities. (The theory of the exceptional perverse sheaves on a manifold without singularities is carried in Chapter 2 as scholium. Its triviality illuminates the structure of the general theory.)

The interest in singular spaces is itself a very important phenomenon in the history of mathematics. Until approximately 1960, very few mathematicians took singular spaces seriously. There had been a magnificent outpouring of topological results about manifolds that was climaxing just about then: Lefschetz theory, deRham cohomology, Hodge theory, characteristic numbers, cobordism, Hirzebruch Riemann-Roch, multiple differentiable structures, surgery, handlebodies, Atiyah-Singer theory. It was one of the most exciting chains of development in the whole history of mathematics. The standard orthodoxy of the time was that if you had a singularity, you should resolve it and get a manifold. During this time, only a few independent pioneers who had the fortune to be ahead of their time (like Marie-Hélèn Schwartz, and Istvan Fary) were looking at singular spaces.

This is not the place for a history of singularity theory, but there has certainly been a swing of the pendulum. Now there is even a journal about singularity theory. I would like to think that this swing is another magnificent story, and that perverse sheaves are an important step of that story.

0.4 Applications of intersection homology and perverse sheaves

There has been a renaissance of applications of topology, some of which can be traced to the use intersection homology and perverse sheaves. The reason is that many mathematical objects of interest in mathematics have singular spaces that are naturally associated to them. Then it often happens that the perverse sheaves on the associated stratified space reflect the structure of the object itself in a deep way. This report is not about the applications of perverse sheaves, although an index of them some of them is included in chapter 6. However, even the most skeptical observer would agree that algebraic geometry, differential equations and analysis, and Lie group theory have all been affected by perverse sheaves in an essential way.

0.5 About this report

The goal here, as stated above, is to give an introduction to intersection homology and perverse sheaves that is purely geometric and topological. An attempt has been made to choose the definition for each object that has the most geometric appeal.

The sheaf theoretic and algebraic languages have been slighted here. This can perhaps be forgiven in light of the fact that there are rather many high quality introductions to those aspects. Unfortunately for readers who prefer the geometric approach, all of the applications of the theory have been written up in the sheaf theoretic and algebraic language. A future version is planned to contain a dictionary from the geometric version presented here to the language that dominates all of the literature.

Proofs of many statements are not given. In many case, no proof is known that proceeds entirely within the geometric framework of this report. In these cases, the truth of the theorem is known because of the dictionary mentioned above. In other cases, proofs were omitted which would not add to the conceptual framework developed here.

The format includes many exercises, interwoven into the text, of varying levels of difficulty. They are an integral part of the text. It is strongly recommended that the reader not skip them.

In those cases where proofs are given, they are given in the exercises. The main reason for this is that geometric proofs are usually much simpler than their expression in terms of language. So many readers will find it easier to read an exercise that gives the main ideas of the proof, than to read a proof with all of the details and lots of notation.

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Chapter 1

Intersection Homology

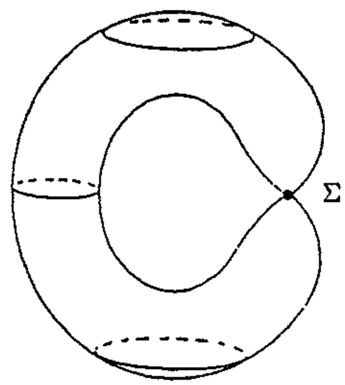
Intersection homology is a topological homology theory of a topological space Y with coefficients in a local system L. In the event that Y is a nonsingular manifold and L is a global nonsingular local system, then the intersection homology is just the ordinary homology Y with coefficients in L. So the interest of intersection homology comes when Y or L has singularities. In this case, the intersection homology continues to have desirable properties, such as Poincaré duality and Hodge theory, that the ordinary homology of a manifold has.

The most important examples of perverse sheaves on a space X are intersection homology sheaves of the closure Y of a stratum of X. The middle perversity intersection homology sheaves form the irreducible perverse sheaves. The logarithmic and sublogarithmic intersection homology sheaves are perverse sheaves that are not irreducible, but which are important in many applications.

Intersection homology is interesting in its own right, independent of its connection with perverse sheaves. Therefore, in this section we will give its definition and properties in more generality than that which we need in the rest of this report. In particular, we will use an arbitrary real stratification of X, not just a complex analytic one.

1.1 Definition of Intersection Homology.

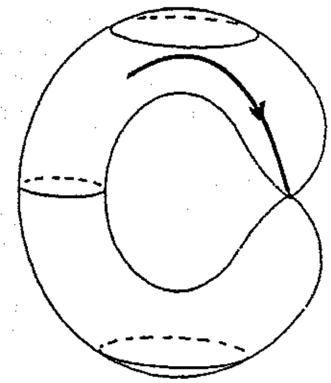
Notational conventions. We will let X be a Whitney stratified manifold, and we will focus attention on a subspace Y of X which is the closure of a single stratum of X. We denote the dense stratum in Y by Y_0 and we call it the *nonsingular part* of Y (even though Y may be nonsingular at points not in Y_0). We denote $Y - Y_0$ by Σ and we call it the *singular set* of Y. Since Y is the closure of a single stratum, it has pure dimension, and we call that dimension n. The strata of X contained in Y will be denoted Y_α. The set of strata contained in Σ will be denoted by $\hat{\Sigma}$. In other words $\hat{\Sigma}$ is the set of all strata of Y except for Y_0. It is called the *set of singular strata* of Y. The (real) dimension of Y will be denoted by n. dWe define the *codimension* of a stratum Y_α, denoted $\text{codim}Y_\alpha$ to be $n - \dim Y_\alpha$. If $Y_\alpha \in \hat{\Sigma}$, then $\text{codim}Y_\alpha > 0$.



The closure Y of a stratum $Y_0 = Y - \Sigma$

We give ourselves a local system $l : L \longrightarrow Y_0$ over the nonsingular stratum Y_0 of Y.

Further, we choose either the piecewise linear or the subanalytic good class of subsets (see 7.3.2) in X so that all of the strata of X are in this class. All geometric chains will be defined with respect to this good class of subsets. By a geometric chain ξ in Y_0, we mean a geometric chain with support that is closed in Y_0. However, just because the support is closed in Y_0 doesn't mean that it is closed in X. It may "run off the edge" of Y_0 into Σ. We denote by $|\xi|$ the closure in X of the support of ξ .



A geometric chain in Y_0 that "runs off the edge" into Σ

DEFINITION. A *perversity* is a function $p : \hat{\Sigma} \longrightarrow \mathbf{Z}$ from the set of singular strata of Y to the integers.

We note that this is a more general definition of a perversity than that found in [GM1]. Likewise, some of the other definitions have been generalized slightly. However, the concepts introduced in [GM1] are always special cases of those studied here.

DEFINITION. If i is an integer and p is a perversity, a subspace $Z \subseteq Y$ is called (p, i) *allowable* if the dimension of Z is $\leq i$ and, for each stratum $Y_\alpha \subseteq \Sigma$, we have $\dim(Z \cap Y_\alpha) \leq i - \text{codim}Y_\alpha + p(Y_\alpha)$.

In this definition, saying that a set has negative dimension should be taken as saying that the set is empty.

DEFINITION. A geometric i-chain ξ in Y_0 with coefficients in L is called *p-allowable* if the closure $|\xi|$ in X of the support of ξ is (p, i) allowable and the closure $|\partial\xi|$ in X of the support of the boundary of ξ is $(p, i-1)$ allowable. The space of p-allowable i-chains in Y_0 with coefficients in L is denoted by $I^pC_i(Y, L)$.

DEFINITION. The *Intersection Homology complex* of Y with perversity p and coefficients in L is the chain complex

$$\cdots \longrightarrow I^pC_{i+1}(Y, L) \longrightarrow I^pC_i(Y, L) \longrightarrow I^pC_{i-1}(Y, L) \longrightarrow \cdots$$

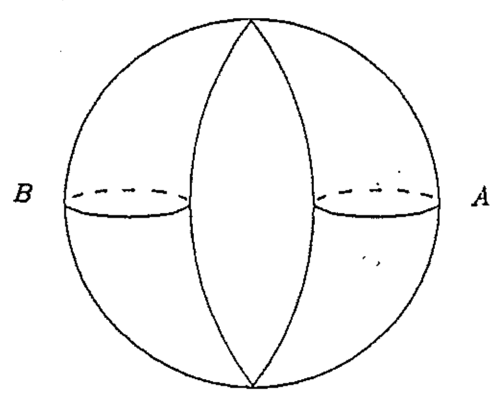
DEFINITION. The i^{th} *Intersection Homology group* of Y with perversity p and coefficients in L, denoted $I^pH_i(Y; L)$ is the i^{th} homology group of the Intersection Homology complex of Y with perversity p and coefficients in L.

So intersection homology is defined similarly to the way that ordinary homology of Y_0 is, with the exception that "allowability" conditions are placed on the dimensions in which the closures of the chains can meet the singularities of Y. If you find that the numerology of these allowability conditions is less than transparent, you are not alone. The first observation to make on them is the following: If the perversity is zero, i.e. $p(Y_\alpha) = 0$ for all strata $Y_\alpha \subseteq \Sigma$, then the condition says that the codimension of $Y_\alpha \cap |\xi|$ in $|\xi|$ is at least the codimension of Y_α in Y and the codimension of $Y_\alpha \cap |\partial\xi|$ in $|\partial\xi|$ is also at least the codimension of Y_α in Y. This looks like a transversality condition; it says that the cycles dip into the singularities as little as can be expected. As the perversity $p(Y_\alpha)$ grows, this transversality condition is relaxed, and are allowed to go deeper into the singularities. If you regard transversality as the nicest behavior that a cycle can have with respect to a singularity, then lack of transversality can be thought of as "perverse". This was the original source of the word perversity.

1.2 Examples of intersection homology groups

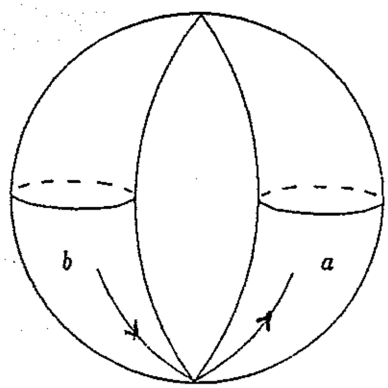
The only way to get used to the effect of the allowability restrictions on the intersection homology groups is to look at some examples. Four examples are given here.

Example 1: The suspension of two circles. We consider two circles A and B and we let Y be the suspension of the disjoint union of A and B; $Y = \mathrm{Susp}(A \cup B)$. The space Y can also be visualized as the result of taking two spheres (or two bananas) $\mathrm{Susp}\,A$ and $\mathrm{Susp}\,B$ and joining them twice at points.



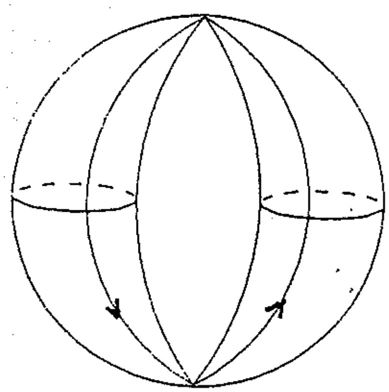
The suspension of two circles.

What is the usual homology of $\mathrm{Susp}(A \cup B)$? We will try to understand it geometrically. The zeroth homology group $H_0(Y)$ is just Q, since Y is connected. The generating cycle may be thought of as either a point a lying on A or a point b lying on B. These two points are homologous (i.e. they lie in the same homology class) because of a 1-chain which connects them. We will call this 1-chain [Cone$a$] − [Cone$b$], since the cone on any subset of $A \cup B$ is embedded in the suspension of that subset, which is further embedded in Susp$(A \cup B)$. The minus sign indicates that the chain is oriented up on one side and down on the other, so as not to have a boundary at the vertex in Σ.



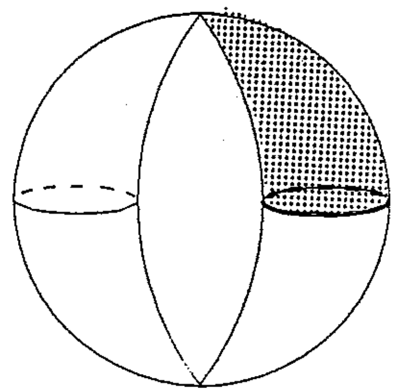
A 1-chain [Cone$a$] − [Cone$b$] whose boundary is $a - b$.

The first homology group $H_1(Y)$ is the Q, generated by a circle that goes up on one sphere and down on the other. We call this cycle [Susp$a$] − [Susp$b$]



The generator of $H_1(Y)$.

There are two other 1-cycles that one might be worried about. These are the fundamental class $[A]$ of the circle A and the fundamental class $[B]$ of the circle B. However, $[A]$ is zero since it is the boundary of $[\,\mathrm{Cone}A]$, the chain carried by the cone over A. Similarly for $[B]$.



The class $[A]$ is homologous to zero since it bounds $[\,\mathrm{Cone}A]$.

Finally, $H_2(Y)$ is $Q \oplus Q$ generated by $[\,\mathrm{Susp}A]$and $[\,\mathrm{Susp}B]$, the two lobes of Y.

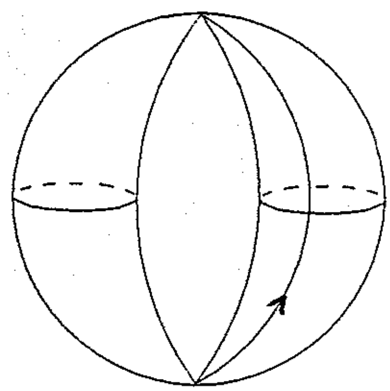
Now, in the same spirit, let's see what the intersection homology of Y is. First, we have to stratify Y. We choose the stratification where $Y_1 = \Sigma$ is the union of top point and the bottom point. So $Y_0 = Y - \Sigma$ is the whole nonsingular part of Y. Second, we have to choose a local system over Y_0. We choose the trivial local system $L = Q$. So we are calculating $I^pH_i(Y; Q)$ or $I^pH_i(Y)$ for short. Third, we have to choose a perversity. Since there is only one singular stratum, the choice of a perversity p is simply the choice of an integer $p(Y_1)$.

Lets choose the perversity $p(Y_1) = 0$ first for illustration. With this perversity, the allowability conditions come down to this: 2-chains may hit the singular set (provided that their boundary doesn't). One chains and zero chains must miss the singular set. Now, $I^0H_0(Y)$ is $Q \oplus Q$ generated by $[a]$ and $[b]$. These two 0-cycles are not homologous now because the homology between them (which was $[\,\mathrm{Cone}a] - [\,\mathrm{Cone}b]$ and was illustrated in a picture above) is not allowed. The first intersection homology group $I^0H_1(Y)$ is zero. The cycles $[\,\mathrm{Susp}a]$ and $[\,\mathrm{Susp}b]$ are not allowed, and the cycles $[A]$ and $[B]$ are boundaries of $[\,\mathrm{Cone}A]$ and $[\,\mathrm{Cone}B]$, which are allowed. The second intersection homology group $I^0H_2(Y)$ is $Q \oplus Q$ as before, generated by $[\,\mathrm{Susp}A]$ and $[\,\mathrm{Susp}B]$, both of which are allowed.

There are two other interesting perversities: $p(Y_1) = -1$ and $p(Y_1) = 1$. Rather than following out the argument in detail for each group, we make a chart of the generators.

i	$H_i(Y)$	$I^{-1}H_i(Y)$	$I^0H_i(Y)$	$I^1H_i(Y)$
2	[Susp$A$], [Susp$B$]	0	[Susp$A$], [Susp$B$]	[Susp$A$], [Susp$B$]
1	[Susp$a$] − [Susp$b$]	$[A], [B]$	0	[Susp$a$], [Susp$b$]
0	$[a] = [b]$	$[a], [b]$	$[a], [b]$	0

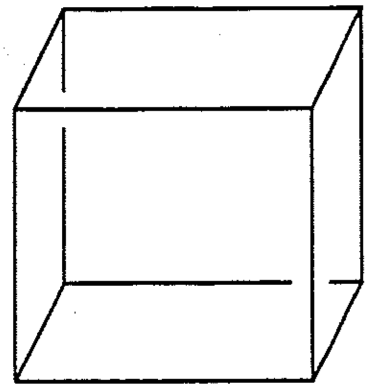
Perhaps the only thing on this chart that needs comment is the entry for $I^1H_1(Y)$. The claim is, that this is two dimensional, and that the generators are [Susp$a$] and [Susp$b$]. Let's consider [Susp$a$].



The generator [Susp$a$] of $I^1H_1(Y)$.

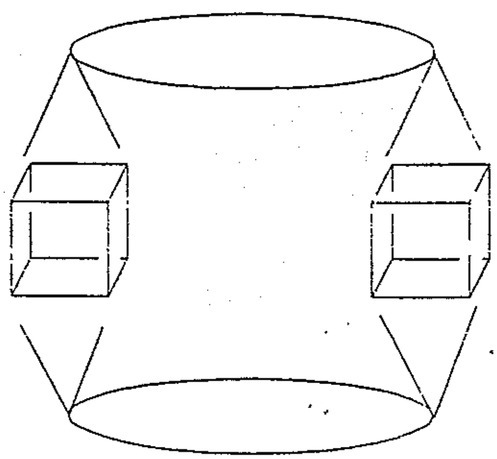
Why is this a cycle? Doesn't it have boundary at the top and the bottom? To answer, remember the important point that the cycles are in Y_0, not in Y. Only their support is in Y. So any boundary in Σ doesn't count.

Example 2. The suspension of the three torus crossed with the circle. Now, let's consider an example with a singular set of dimension greater than zero. We take (SuspT^3) $\times S^1$ where T^3 is the three torus. We may imagine the three torus as a cube with opposite faces identified.



The three torus T^3.

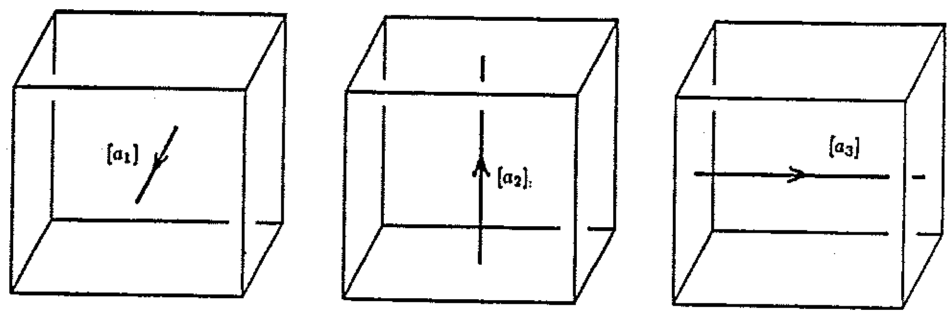
Then we suspend it and cross the result with the circle to get a five dimensional stratified space. Its singular set Σ is the top of the suspension crossed with the circle and the bottom of the suspension crossed with the circle. We make this one stratum Y_1 and we let Y_0 be the complement. This five dimensional space admits the following pictorial representation:



$(\,\mathrm{Susp} T^3) \times S^1$

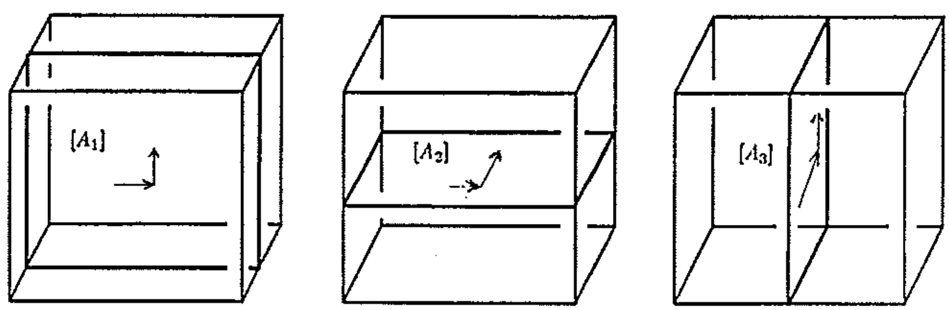
First, we need names for the generators of the homology of the three torus T^3. The zeroth homology is generated by the fundamental class $[p]$ of a point p. The first homology is generated by the fundamental classes of three circles

a_1, a_2, and a_3, which we may picture as follows:



Generators $[a_1], [a_2]$, and $[a_3]$ for $H_1(T^3)$

The second homology is generated by the fundamental classes of three 2-tori A_1, A_2, and A_3, which we may picture as follows:



Generators $[A_1], [A_2]$, and $[A_3]$ for $H_2(T^3)$

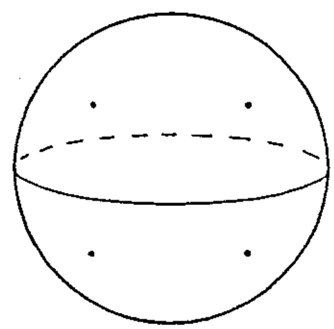
Now in terms of these generators and their suspensions, we will give generators for the intersection homology with constant coefficients Q, and with perversities $p(Y_1) = 0$, $p(Y_1) = 1$, and $p(Y_1) = 2$.

i	$I^0H_i(Y)$	$I^1H_i(Y)$	$I^2H_i(Y)$
5	$[(\operatorname{Susp} T^3) \times S^1]$	$[(\operatorname{Susp} T^3) \times S^1]$	$[(\operatorname{Susp} T^3) \times S^1]$
4	$[\operatorname{Susp} T^3]$	$[\operatorname{Susp} T^3], [(\operatorname{Susp} A_i) \times S^1]$	$[\operatorname{Susp} T^3], [(\operatorname{Susp} A_i) \times S^1]$
3	$[A_i \times S^1]$	$[\operatorname{Susp} A_i]$	$[\operatorname{Susp} A_i], [(\operatorname{Susp} a_i) \times S^1]$
2	$[A_i], [a_i \times s^1]$	$[a_i \times S^1]$	$[\operatorname{Susp} a_i]$
1	$[a_i], [p \times S^1]$	$[a_i], [p \times S^1]$	$[p \times S^1]$
0	$[p]$	$[p]$	$[p]$

In each case, if a symbol is missing, the explanation is either that the cycle is not allowable or that the cycle is killed by the chain supported on a cone over that cycle.

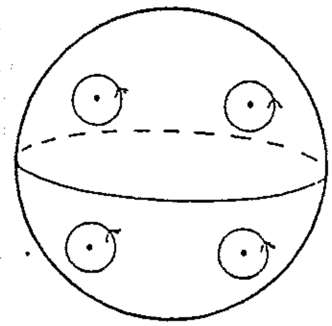
Exercise 1.1. Verify all of the entries in the table.

Example 3. A twisted local system over the sphere minus four points. The next exapmle is chosen to illustrate that an twisted local system can make intersection homology interesting even when the space Y is nonsingular. We take Y to be the 2-sphere S^2. We choose $\Sigma = Y_1$ to be the union of four points on the sphere.



The "singular" set Σ on Y.

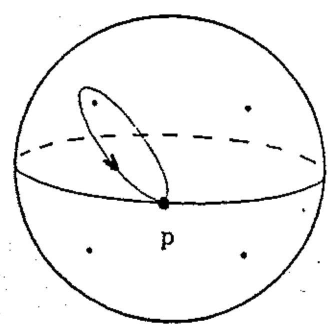
Now we take a local system L on Y_0 whose fiber is just the rational numbers Q and whose monodromy maps are multiplication by -1 on the loops that go once around any of the four points of Y_1.



The local system has monodromy -1 on each of these loops .

In this case, although Y is smooth at the four points in Σ, the local system has singularities at them. Let's compute the intersection homology for the perversity 0, i.e. $p(Y_1) = 0$.

First of all, $I^0H_0(Y;L) = 0$. An element of $I^0H_0(Y;L)$would have to be a point in the nonsingular part Y_0 together with an orientation (+ or $-$) and a lift to the local system. We will call this data $[p]$. Now, build a 1-chain ξ by taking the lift and moving it by monodromy around one of the points in Y_1 back to p. The following picture gives the support of ξ:

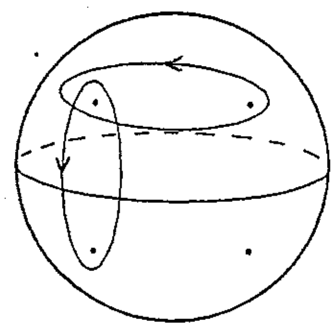


A 1-chain ξ whose boundary is $2[p]$

The boundary of ξ is $2[p]$. (If the local system did not twist, the two boundary terms of a chain of the shape of ξ would cancel and give 0. The twisting makes them add rather than subtract.)

Next $I^0H_2(Y;L) = 0$. A geometric cycle representing a class in $I^0H_2(Y;L)$ would have to be a 2–cycle (with support on all of Y) with a global lift to L. But L has no global lifts except 0.

Finally, $I^0H_1(Y;L) = Q \oplus Q$. To get a 1–cycle, we should have an oriented circle in Y_0 together with a lift to the local system. To have a lift to the local system, the circle should go around an even number of points in Σ. The two generators can be chosen to be represented by these circles:



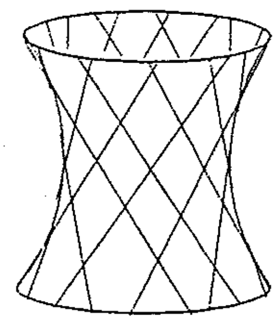
Cycles giving generators for $I^0H_1(Y;L)$

(It is not so visually apparent that these cycles generate the homology group, because our intuition for local systems is not so developed. The reader should try to find some more 1–cycles.)

Exercise 1.2. Show that the result of this calculation is the same if we choose any other perversity.

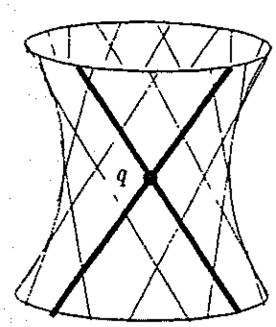
Exercise 1.3. Show that if we choose $2n$ points in Σ instead of 4 points (and let the local system have monodromy -1 around each of them), then $I^0H_1(Y;L) = Q^{2n-2}$ and $I^0H_0(Y;L) = I^0H_2(Y;L) = 0$.

Example 4. A quadric 3-fold. As a final example, we choose an example that is a complex algebraic variety, but has complex dimension more than 1. Since it's too big to draw, we will draw pictures of the same variety over the reals and just translate into the complexes in our minds. First of all, we take a nonsingular quadric surface H in complex projective 3–space, say $x^2 + y^2 = z^2 + w^2$ in homogeneous coordinates, or $x^2 + y^2 = z^2 + 1$ in Affine coordinates. As is well known, this is a doubly ruled surface: it has two families of flat projective lines on it. (The real picture is sometimes called the "hyperboloid of one sheet".)



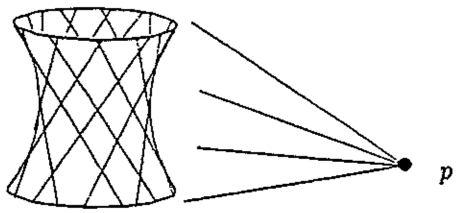
The quadric surface H

Topologically, H is just $S^2 \times S^2$, the product of two 2- spheres. The two foliations by 2-spheres coming from the two projections are the two rulings by straight projective lines. (To translate from the affine real picture to the projective complex picture, every time you see a straight line you should replace it by a 2-sphere.) So H has a CW-decomposition with one 0-cell q, two 2-cells A and B, and one 4-cell $G = H - (A \cup B)$. These may be pictured as follows:



A CW decomposition for H

Now, we embed the complex projective 3-space into X =complex projective 4-space, and we choose a point p outside of it. We let Y be the projective cone, the union of all complex projective lines joining points in h to p.



Y is the projective cone over H

This example has two other descriptions. To a topologist, it is just the Thom space of a complex line bundle over H with Chern class $(1,1)$. It is also the first singular Schubert variety: It is all planes in the Grassmannian of complex 2-dimensional subspaces of C^4 which meet a given fixed 2-plane in a subspace of dimension at least 1. (In this guise, this example has a lot of historical importance as one of the main examples that led to the Kazhdan-Lusztig conjectures.)

The ordinary homology of Y is easy to compute, since Y has a CW- decomposition $Y = p \cup Cq \cup CA \cup CB \cup CG$, where C denotes the open complex projective cone, i.e. the cone without the point p. Therefore $H_0(Y) = Q$, $H_2(Y) = Q$, $H_4(Y) = Q \oplus Q$, $H_6(Y) = Q$, and all of the others are zero. One might pause to consider why it is that $H_2(Y) = Q$ even though there are three obvious 2-cycles, $[\bar{A}]$, $[\bar{B}]$, and $[\overline{Cq}]$. This is because $[\bar{A}]$ is homologous to $[\overline{Cq}]$ by a homology that "swings" it through $\overline{CA}$; similarly for $[\bar{B}]$ and $[\overline{Cq}]$.

The variety Y has an obvious stratification with $\Sigma = Y_1 = p$ and $Y_0 = Y - p$. Let's compute the intersection homology with the perversity $p(Y_1) = 1$.

The answer is $I^1H_0(Y) = Q$ generated by $[q]$, $I^1H_2(Y) = Q \oplus Q$ generated by $[\bar{A}]$ and $[\bar{B}]$, $I^1H_4(Y) = Q \oplus Q$ generated by $[\overline{CA}]$ and $[\overline{CB}]$, $I^1H_6(Y) = Q$ generated by $[Y]$, and all of the others are zero. So the answer differs from the ordinary homology calculation only in degree 2. Why are $[\bar{A}]$ and $[\bar{B}]$ not homologous now when they were before? The answer lies in the fact that any homology between them, like the one described above, must contain the singular point p in its support, but the allowability conditions for perversity 1 won't permit that.

Exercise 1.4. Compute $I^pH_i(Y)$ for the other perversities p. HINT: if $p(y_1) = 2$, you will get the same thing as the ordinary homology of Y

1.3 Theorems holding for all perversities

We begin here a fairly systematic account of the properties of intersection homology. In addition to the goal of providing an introduction to the subject, we want to gather together in one place a number of statements that are scattered throughout the literature.

We will organize the results according to what the required perversity restrictions are. One reason for this is the following: In [GM1] and [GM2], some results were proved with unnecessary restrictions on the perversity, in order not to have to constantsly switch assumptions. However, this has led to some confusion on the matter, which we want to clear up here.

While assuming as little as possible on the perversities, we will assume as much as possible on the coefficients: we are treating only local systems of modules over Q. Modules over other rings are very interesting, and lead to deep torsion questions. There is already a full literature on this, for the interested reader.

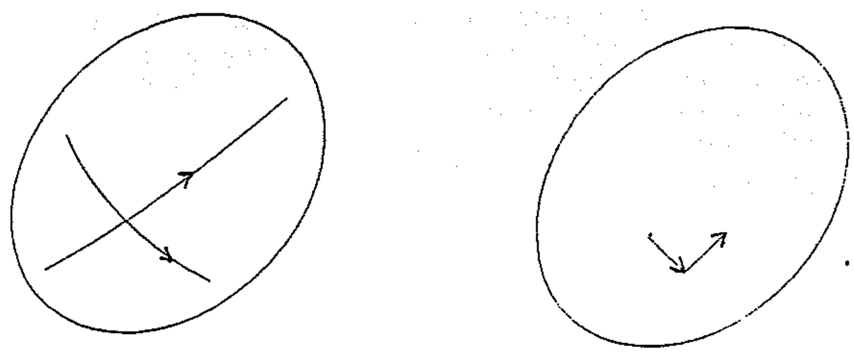
1.3.1 Poincaré duality

DEFINITION. Suppose that L and L' are two local systems on Y_0. We say that L and L' are *dual* if we are given a *duality* map of local systems $d : L \otimes L' \longrightarrow \mathcal{O}$ that, when restricted to each fiber, is a perfect (nonsingular) pairing of vector spaces. Here $\mathcal{O}$ is the orientation local system on Y_0.

DEFINITION. Suppose that ξ and ξ' are two geometric cycles in Y_0 of dimensions i and j where $i + j = k$. (We call ξ and ξ' cycles of *complimentary dimension*). Then ξ and ξ' are said to be *transverse* if

- $|\xi| \cap |\xi'|$ consists of finitely many points, all of which are in the interior of Y_0 (i.e. not in Σ)..
- At each point y of $|\xi| \cap |\xi'|$, both $|\xi|$ and $|\xi'|$ are nonsingular.
- At each point y of $|\xi| \cap |\xi'|$, $T_y|\xi| \oplus T_y|\xi'| = T_yY_0$.

Let ξ be a geometric cycle in Y_0 with coefficients in L, and let ξ' be a geometric cycle in Y_0 with coefficients in L'. Suppose ξ and ξ' are transverse geometric cycles in Y_0 of complimentary dimension, and suppose that L and L' are dual local systems. Then we can assign an *intersection number* $\mathcal{I}(\xi, \xi')$ to ξ and ξ' by the following procedure. The rational number $\mathcal{I}(\xi, \xi')$ is a sum over intersection points $y \in |\xi| \cap |\xi'|$ of local contributions $\mathcal{I}_y(\xi, \xi')$. To define $\mathcal{I}_y(\xi, \xi')$, we observe that over y we have two naturally determined elements of the stalk of $\mathcal{O}$. The first is the multiplicity of y in ξ tensored with the multiplicity of y in ξ', moved to the stalk of $\mathcal{O}$ by the pairing d. The second is the orientation of $T_y Y_0$ i.e. of $T_y|\xi| \oplus T_y|\xi'|$ constructed by taking the orientation of $T_y|\xi|$from ξ followed by the orientation of $T_y|\xi'|$ from ξ'(see the picture below). Since the stalk of $\mathcal{O}$ at y is one dimensional, the first element is some rational multiple of the second element. That rational multiple is $\mathcal{I}_y(\xi, \xi')$.



The orientations of ξ and of ξ' (left) induce an orientation of Y_0 (right).

DEFINITION. Two perversities p and q are said to be *dual* if for each stratum Y_α in Σ, we have $p(Y_\alpha) + q(Y_\alpha) = t(Y_\alpha) = \text{codim} Y_\alpha - 2$. We notate the duality of p and q by writing $p = q^*$ or $q = p^*$.

Now with these definitions, we can state the Poincaré Duality Theorem.

Theorem 1.1 ([GM1],[GM2],[GM3].) *Suppose that p and q are dual perversities, L and L' are dual local systems on Y_0, and i and j are complementary dimensions. Then there is a unique* intersection *pairing*

$$\mathcal{I} : I^p H_i(Y; L) \otimes I^q H_j(Y; L') \longrightarrow Q$$

which, if ξ and ξ' are transverse cycles, takes $[\xi] \otimes [\xi']$ to $\mathcal{I}(\xi, \xi')$. This pairing is nonsingular, i.e. the two vector spaces $I^p H_i(Y; L)$ and $I^q H_j(Y; L')$ are vector space duals of each other with respect to this pairing.

This was historically the first serious result about intersection homology.

The reader should take the time to check that this theorem holds for each of the examples in section 1.2. In the third example, the local system L is dual to itself. In the other three examples, the local system is trivial and Y_0 is orientable, so the local systems in those examples are also their own duals. In each case, explicit cycles are given representing generators of each intersection homology group, so the intersection pairing $\mathcal{I}$ can be actually computed.

Exercise 1.5. Show that in each example, the actual bases given are dual bases with respect to the pairing.

1.3.2 Relative intersection homology and the long exact sequence

DEFINITION. A subset R of X is called *smoothly enclosed* if R is a submanifold with boundary of the same dimension as X, and the boundary of R is transverse to all of the strata X_α of X. The subset R^0 is the interior of R; it is an open subset of X. A subset U of Y is called *smoothly enclosed* if it is the intersection with Y of a smoothly enclosed subset R of X. Then $U^0 = Y \cap R^0$ is the interior of U.

Suppose that U is a smoothly enclosed subset of Y. Then we define the intersection chain complex $I^pC_i(U;L)$ to be the subcomplex of the intersection chain complex $I^pC_i(X;L)$ consisting of chains whose support lies in U^0. The i^{th} intersection homology group of U with coefficients in L, denoted $I^pH_i(U;L)$ is the i^{th} homology group of the Intersection Homology complex $I^pC_i(U;L)$.

Suppose further that V is the intersection of U with another smoothly enclosed subset of Y. Then we define the intersection chain complex $I^pC_i(U,V;L)$ to be $I^pC_i(U;L)/I^pC_i(V;L)$. The i^{th} intersection homology group of the pair (U,V) with coefficients in L, denoted $I^pH_i(U,V;L)$ is the i^{th} homology group of the Intersection Homology complex $I^pC_i(U,V;L)$.

Exercise 1.6. Show that there is a long exact sequence

$$\cdots \longrightarrow I^pH_i(U;L) \longrightarrow I^pH_i(U,V;L) \longrightarrow I^pH_{i-1}(V;L) \longrightarrow I^pH_{i-1}(U;L) \longrightarrow \cdots$$

Exercise 1.7. Show that excision holds: Suppose that W is a smoothly enclosed subset of Y that is contained in V^0 (so $U - W^0$ is also a smoothly enclosed subset), then

$$I^pH_i(U - W^0, V - W^0; L) \xrightarrow{\sim} I^pH_i(U,V;L)$$

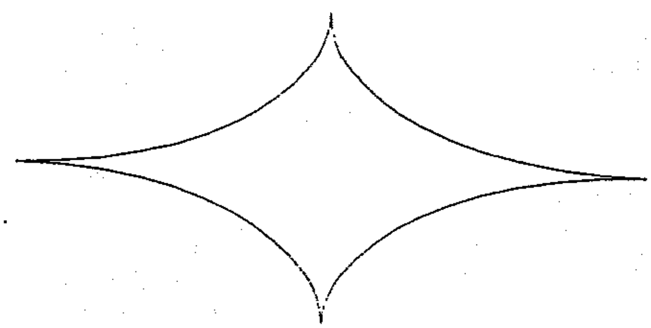
is an isomorphism.

1.3.3 Simplicial intersection homology

Ordinary homology of a simplicial can be calculated from a small chain complex, consisting of the simplicial chains. This is one of the oldest themes in topology, dating from Poincaré, and it is the basis of combinatorial topology.

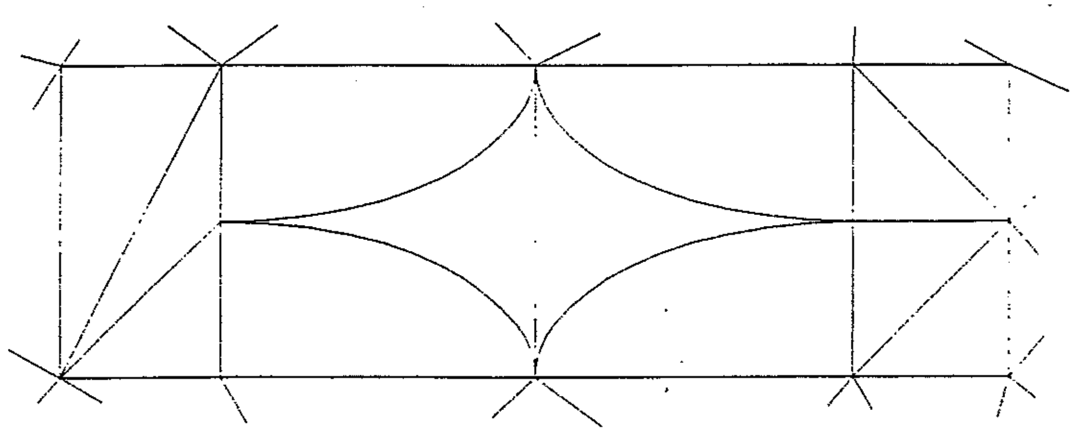
DEFINITION. A triangulation T of Y with the property that every stratum of Y is a union of interiors of simplices of T is called a triangulation *adapted to the stratification of* Y.

Triangulations adapted to the stratification of Y always exist (see [G]). Consider the following stratification of the plane of the page:



A stratification of the plane

The following is a triangulation adapted to it (The lines of the original stratification are edges of triangles):



A triangulation adapted to the stratification

DEFINITION. Suppose that T is a stratification of Y adapted to the stratification. A *closed support simplicial i-chain* of Y_0 is a simplicial chain with respect to T such that every simplex Δ with nonzero coefficient in the chain satisfies

$\star$ the interior of Δ is contained in Y_0.

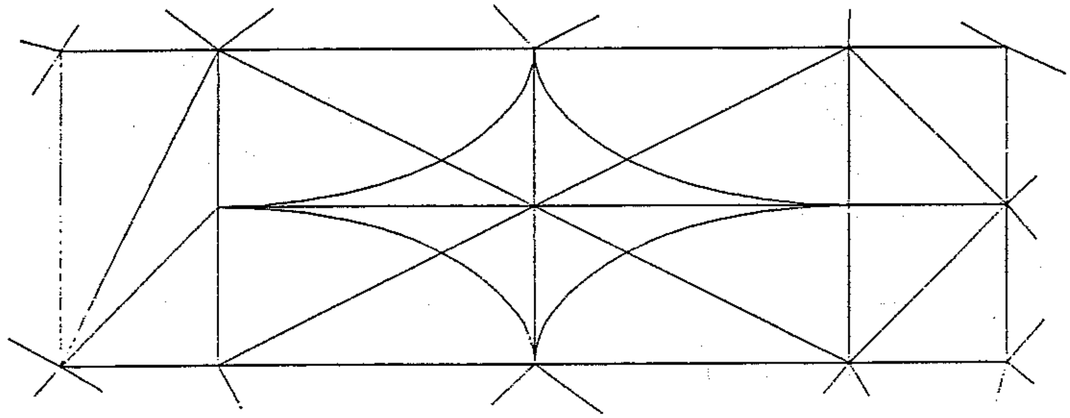
(Note that because of $\star$, it makes sense to speak of a closed support simplicial i-chain of Y_0 with coefficients in a local system L over Y_0.) The closed support simplicial chains form a chain complex, by neglecting all simplices of the boundary which do not satisfy $\star$. We denote this chain complex by $C_i^T(Y_0; L)$. The *support* of a chain in $C_i^T(Y_0; L)$ is the union of all simplices of T satisfying $\star$ which have nonzero coefficient in that chain. (The chain complex $C_i^T(Y_0; L)$ is isomorphic to the quotient chain complex $C_*^T(Y)/C_*^T(\Sigma)$, where C^T represents simplicial chains with respect to T.)

DEFINITION. A closed support simplicial i-chain ξ in Y_0 with coefficients in L is called *p-allowable* if the support $|\xi|$ of ξ is (p, i) allowable and the support $|\partial\xi|$ of the boundary of ξ is $(p, i-1)$ allowable. The space of p-allowable i-chains in Y with coefficients in L is denoted by $I^pC_i^T(Y, L)$. The simplicial Intersection Homology complex of Y with perversity p and coefficients in L is the chain complex

$$\cdots \longrightarrow I^pC_{i+1}^T(Y, L) \longrightarrow I^pC_i^T(Y, L) \longrightarrow I^pC_{i-1}^T(Y, L) \longrightarrow \cdots$$

One might expect that the intersection homology group $I^pH_i(Y; L)$ can be calculated as the homology of the complex $I^pC_i^T(Y; L)$, if the triangulation T is adapted to the stratification of Y. This is false. However, with a mild restriction on the triangulation, it becomes true.

DEFINITION. A triangulation T adapted to the stratification of Y is called *flaglike* if it satisfied the following condition: Let S_k be the closure of the union of all of the i-dimensional strata Y. Then we require that the intersection of any simplex Δ of T with S_i is a single face of Δ, for each k.



A flaglike triangulation of the same stratified space

Flaglike triangulations exist. For example, if T is any triangulation of Y which is adapted to the stratification, then the barycentric subdivision of T is flaglike.

Theorem 1.2 ([GM4]) *If the triangulation T is flaglike, then the intersection homology group $I^pH_i(Y;L)$ can be calculated as the homology of the simplicial intersection homology complex $I^pC_i^T(Y;L)$.*

Exercise 1.8. Find a space Y and a triangulation T such that the intersection homology group $I^pH_i(Y;L)$ is not homology of the simplicial intersection homology complex $I^pC_i^T(Y;L)$.

Remark. There is also a version of singular homology that computes intersection homology [Kin].

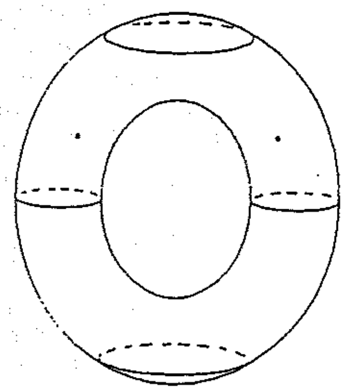
1.4 The extreme perversities.

The reader learning about intersection homology for the first time may want to skip now to section 1.6 on theorems holding for the middle perversity.

The definition of intersection homology above gives a plethora of groups, one for each perversity p. Many of these are really important. However, some of them are not really different each other or from classical (= pre- intersection homology) invariants. We explore this phenomenon in this section.

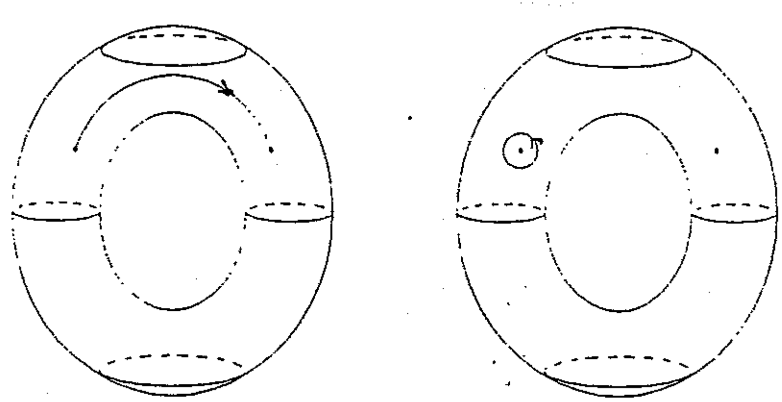
1.4.1 Beyond the extremes

Example 5. Let Y be the torus T^2 stratified with a zero dimensional stratum $Y_1 = \Sigma$ consisting of two points.



Y with its stratification.

Let -1 be the perversity p where $p(\Sigma) = -1$, and let 1 be the perversity p where $p(\Sigma) = 1$. Then $I^{-1}H_i(Y, Q) = H_i(Y, Q)$ and $I^1 H_i(Y, Q) = H_i(Y - \Sigma, Q)$. The following figure shows 1-cycles that are nonzero in $I^{-1}H_1(Y, Q)$ and $I^1 H_1(Y, Q)$.



Cycles for $I^{-1}H_1(Y, Q)$ (left) and $I^1 H_1(Y, Q)$ (right)

Exercise 1.9. Fix a singular stratum Y_α in Y. Let p' be the restriction of p to $\hat{\Sigma} - \{Y_\alpha\}$. Define the group $I^{p'}H_i(Y - Y_\alpha; L)$. (Note that $Y - Y_\alpha$ is often not a locally compact space.) Show that if $p(Y_\alpha) < 0$, then $I^p H_i(Y; L) = I^{p'} H_i(Y - Y_\alpha; L)$.

Exercise 1.10. Again, fix a singular stratum Y_α in Y, and let p' be the restriction of p to $\hat{\Sigma} - \{Y_\alpha\}$. Define the relative intersection homology group $I^{p'}H_i(Y, Y_\alpha; L)$. Show that if $p(Y_\alpha) > \text{codim}Y_\alpha - 2$, then $I^pH_i(Y; L) = I^{p'}H_i(Y, Y_\alpha; L)$.

These two exercises show that if the value of the perversity on a stratum y_α is less than zero or more than $\text{codim}Y_\alpha - 2$, then the intersection homology group $I^pH_i(Y; L)$ has an alternative interpretation as the intersection homology of the remainder of the space Y with Y_α either deleted or moded out. Now consider the case that Y_α has codimension one. Then any value of $p(S_\alpha)$ is either < 0 or $> \text{codim}S_\alpha - 2 = -1$. Therefore, the intersection homology group brings nothing new to a codimension one stratum of Y. For this reason, one sometimes wants to assume that Y has no codimension one strata.

DEFINITION. A stratification of Y is called a *pseudomanifold stratification* if it has no codimension one strata. A closure Y of a stratum of X is called a *pseudomanifold* if it admits a pseudomanifold stratification.

Note that all complex analytic stratifications of a complex analytic manifold X are pseudomanifold stratifications, since in that case every stratum has even real dimension. Therefore, every complex analytic subspace Y (of pure dimension) of a complex analytic manifold $X$is a pseudomanifold.

A similar argument would seem to exclude the consideration of all perversities such that $p(S_\alpha)$ is < 0 or $p(S_\alpha) > \text{codim}S_\alpha - 2$. However, we will sometimes find these perversities convenient for the study of perverse sheaves.

1.4.2 The zero perversity and the top perversity

What happens in the case of these two bounding perversities? This has a nice answer, at least for local systems on Y_0 which are restrictions of local systems on Y (for example, for the trivial local system $\boldsymbol{Q}$).

DEFINITION. The *zero perversity* $\bar{0}$ is defined by $\bar{0}(Y_\alpha) = 0$ for all strata $Y_\alpha \subseteq \Sigma$. The *top perversity* $\bar{t}$ is defined by $\bar{t}(Y_\alpha) = \text{codim}Y_\alpha - 2$ for all strata $Y_\alpha \subseteq \Sigma$.

DEFINITION. The space Y is *normal* if the link of every stratum Y_α in Y is connected. (If Y is also a pseudomanifold, then the property of being normal turns out to be independent of the stratification.)

Theorem 1.3 ([GM1], section 4.3) *Suppose that $\bar{L}$ is a local system on Y and that L is its restriction to Y_0. If Y is a normal pseudomanifold, then $I^tH_i(Y; L) = H_i(Y; \bar{L})$. If Y is a normal pseudomanifold and , $\mathcal{O}$ is the orientation sheaf on Y_0, which is n- dimensional, then $I^{\bar{0}}H_i(Y; L \otimes \mathcal{O}) = H_{n-i}(Y; \bar{L})$.*

This theorem was proved in ([GM1], section 4.3) for trivial coefficients and oriented Y_0, but the proof carries over to this generality.

According to this theorem, if Y_0 is oriented and L is the restriction of a local system on Y, then the intersection homology with zero or top perversity is just ordinary homology or cohomology. However, if L is not the restriction of a local system on Y, then even in these extreme perversities the intersection homology can be interesting.

1.5 The invariant range perversities

1.5.1 Topological invariance

Now we come to the question of topological invariance of intersection homology. The groups $I^pH_i(Y;L)$ are defined using a lot of non- topological baggage. First of all, the geometric cycles themselves use an analytic structure on Y. Next, we choose a stratification of Y. How much do these choices matter?

Example 5 is a case of a manifold whose intersection homology changes when it is re-stratified, for an extreme perversity. However, Theorem 1.3 shows that for the zero or top perversities and for appropriate local systems, the intersection homology is just the ordinary homology or cohomology. These, of course, are defined without reference to the choices made, so this establishes topological invariance.

DEFINITION. A *classical perversity* is a function $\bar{p}$ from the integers greater than one to the integers $\bar{p} : \mathbf{Z}_{>1} \longrightarrow \mathbf{Z}$ with the properties that $\bar{p}(2) = 0$ and $\bar{P}(i+1)$ is either $\bar{P}(i)+1$ or $\bar{P}(i)$ for $i > 2$. Given a classical perversity, the *associated perversity* p is defined by $p(Y_\alpha) = \bar{p}(\text{codim}Y_\alpha)$ for $Y_\alpha \subseteq \Sigma$, for any pseudomanifold stratification of Y.

Theorem 1.4 ([GM2], section 4) *Suppose that Y is a pseudomanifold, $\bar{L}$ is a local system on Y and that $\bar{p}$ is a classical perversity. Then for any pseudomanifold stratification of Y, we get an intersection homology group $I^pH_i(Y;L)$ where p is the associated perversity and L is the restriction of $\bar{L}$ to Y_0. This group $I^pH_i(Y;L)$ is independent of the stratification chosen, and in fact depends only on Y as a topological space and $\bar{L}$ as a topological local system over Y.*

1.5.2 Lack of homotopy invariance

Exercise 1.11. Let Y be the the space obtained from gluing a complex projective plane to a four-sphere along a two sphere. The two sphere is embedded in the complex projective plane as a complex projective line, and it is embedded in the four sphere as an "equator". Let Y' be the one point join of two four spheres. Show that Y and Y' have the same homotopy type. Show, however, that $I^m H_2(Y, Q) \neq I^m H_2(Y', Q)$, where m is the middle perversity (which is in the invariant range) defined in the next section.

Nevertheless, intersection homology is interesting from a homotopy point of view. See the notion of stratified homotopy in [W], for instance.

1.5.3 Functoriality under placid maps

DEFINITION. ([GM5].) A *placid map* $f : Y \longrightarrow Y'$ is a continuous map for which there exist stratifications of Y and Y' such that the inverse image of a stratum of Y' of codimension c in Y' is a union of strata in Y each of codimension at least c in Y.

For example, any map to a manifold is a placid map. Fibrations in topology and flat maps in algebraic geometry are placid maps.

Exercise 1.12. Show that for any invariant range perversity, $I^p H_*(\bullet, Q)$ is both a covariant and a contravariant functor for placid maps. The contravariant induced map will shift degrees. (You may want to assume that the map has desirable geometric properties to do this exercise. To do it in the generality stated, one can use the techniques of [GM2].)

1.5.4 The Kunneth theorem

Theorem 1.5 ([CGL].) *Suppose that p is a classical perversity such that $p(a) + p(b) \leq p(a+b) \leq p(a) + p(b) + 2$. Then*

$$\bigoplus_{i+j=k} I^p H_i(Y; L) \otimes I^p H_j(Y'; L') = I^p H_k(Y \times Y'; L \otimes L')$$

1.6 The middle perversity and the Kähler package.

The Poincaré Duality theorem of the last section said that a general intersection homology group is dually paired to another intersection homology group. For many applications, however, one wants a group that is dually paired to itself. When does this happen?

DEFINITION. The perversity m is a *middle perversity* for Y if the dual perversity to m is m itself, i.e. if $m(Y_\alpha) + m(Y_\alpha) = \text{codim}Y_\alpha - 2$ for all strata Y_α in Σ.

Clearly, a middle perversity can exist only if $\text{codim}Y_\alpha$ is even all strata Y_α in Σ. Then the middle perversity is unique and given by $m(Y_\alpha) = (\text{codim}Y_\alpha)/2-1$. In this case, the stratification is a pseudomanifold stratification. Note that the middle perversity is associated to a classical perversity, so the topological invariance theorem applies for the middle perversity.

Convention. The middle perversity is so important that if no perversity is mentioned, then the middle perversity is assumed by default. So we have

$$IH_i(Y;L) = I^m H_i(Y;L)$$

$$IC_i(Y;L) = I^m C_i(Y;L)$$

Corollary 1.6 (to Poincaré Duality) *Suppose that m is a middle perversity for Y, and that Y is compact. Suppose further that L is dually paired to itself (for example, suppose Y_0 is oriented and L is Q). Then if $i+j=k$,*

$$\mathcal{I} : IH_i(Y;L) \otimes IH_j(Y;L') \longrightarrow Q$$

is a nonsingular dual pairing.

Exercise 1.13. Show that for ordinary rational homology of a compact singular complex variety, the Betti numbers in complementary dimension don't even have to be equal.

This middle perversity intersection homology satisfies the same Poincaré duality that the ordinary homology of a manifold does. This historically led to the question of whether it satisfies other properties of ordinary homology. (See [CGM] for an early paper in this direction.)

1.6.1 Complex manifolds and the Hodge decomposition

From the point of view of geometry, some of the most interesting theorems satisfied by the ordinary homology of a smooth manifold are those that apply to complex projective varieties. We will refer to these theorems collectively as the *Kähler Package.* Each theorem of this section represents a monumental piece of mathematics, both as to its importance and as to its proof. We don't want to go in to complete details. This section can be taken as an overview.

Further assumptions. We assume for the rest of this section that X is a smooth complex algebraic variety and that all stratifications are complex algebraic. We denote the real dimension of In particular, Y will be complex algebraic. Now is a good time to reiterate that someone interested only in constant coefficients can skip all mention of properties of the local system, because all conditions will be satisfied for constant coefficients. (Since Y_0 will be complex analytic in what follows, it is oriented.)

Theorem 1.7 Conjectured in [CGM], proved in [S]

If L is a complex local system that is a polarized variation of Hodge Structure, then there is a Hodge decomposition

$$IH_i(Y;L) = \bigoplus_{p+q=n-i} H^{p,q}$$

where p and q are positive integers, and

$$\overline{H^{p,q}} = H^{q,p}$$

Exercise 1.14. Show that for ordinary homology of a compact singular variety, the odd Betti numbers don't have to be even (which they would be if it had a Hodge decomposition). Hint: Use Example 1 of section 1.2.

1.6.2 The Hard Lefschetz Theorem

For the next theorem, we suppose that X is a complex projective space. Then there is a large family of complex hyperplanes H in X, which are themselves complex projective spaces of one complex dimension less (therefore two real dimensions less). Given a geometric i-cycle ξ in Y_0, we can always find a hyperplane H in this family that is *transverse* to $|\xi|$, i.e. that is transverse in the usual sense to each stratum of a stratification of $|\xi|$. In that case, there is a geometric $(i-2)$-cycle called the *intersection cycle*, denoted $[H]\cap\xi$ supported on $H\cap|\xi|$. We refer to [G2] for the construction of $[H]\cap\xi$, noting only that the construction hinges on the fact that both X and H are canonically oriented, since they are complex analytic varieties. We observe the following two properties of $[H]\cap\xi$ from [G2]:

Proposition 1.8 *If ξ is allowable with perversity p, then $[H]\cap\xi$ is allowable with perversity p. The class in $I^pH_i(Y;L)$represented by $[H]\cap\xi$ is independent of the transverse hyperplane H chosen.*

Theorem 1.9 Hard Lefschetz[BBD]) *If L has geometric origin ([BBD], p. 162), then the map*

$$([H]\cap)^i : IH_{n/2+i}(Y;L) \longrightarrow IH_{n/2-i}(Y;L)$$

is an isomorphism for all i.

Exercise 1.15. Show that the corresponding statement would be false for ordinary homology. Hint: Use Example 1 of section 1.2.

1.6.3 The L^2 cohomology

Another fundamental result about the ordinary homology is the deRham theorem. This says that the homology can be calculated using differential forms. Now, if we want to generalize the deRham theorem to singular spaces, we are in trouble since it is not clear what a differential form on a singular space should be. The reader will have noticed that the whole philosophy of intersection homology is to study a singular space Y using chains on the nonsingular part Y_0, where the singularities Σ enter only by putting a restriction on what chains we should study. So, the proper analogue of a differential form on Y should be a differential form on Y_0 (which is a non-compact manifold) with some sort of restriction.

The most natural restriction on a differential form on a noncompact manifold, from the point of view of analysis, is that the differential form should be square integrable. This is an old theme; it was already well developed by the time of deRham's book.

DEFINITION. A *square integrable*, or L^2, differential form on Y_0 is a form such that

$$\int_{Y_0} \omega \wedge *\omega < \infty$$

and also (in order to make the space of L^2 differential forms into a complex) we require that the same condition should hold for $d\omega$.

$$\int_{Y_0} d\omega \wedge *(d\omega) < \infty$$

(This definition resembles the definition of the p-allowable geometric chains, in that there also the same condition is put on the chains and their boundaries.)

DEFINITION. The L^2 *cohomology* $H^i_{(2)}(Y_0)$ is the i^{th} cohomology of the complex of L^2 differential forms on Y_0.

These definitions also make sense in the presence of a local system L on Y_0, provided that L has a smoothly varying positive definite inner product on each fiber.

We could hope that under some assumptions, we have $IH_{n-i}(Y;L) = H^i_{(2)}(Y_0;L)$. (The dimension shift is because L^2 cohomology uses cohomology numbering of groups, whereas intersection homology is uses homology numbering.) There is one problem: the definition of the Hodge star operator $*$, and therefore the definition of L^2 cohomology, depends on the choice of a Riemannian metric on Y_0. Easy examples show that the L^2 cohomology changes when the metric changes, so long as Y_0 is non-compact). In fact, whether the L^2 cohomology is finite dimensional or not depends on the metric. So we are led to the question:

Question. Are there naturally occurring classes of metrics on Y_0 and on L for which we have $I^H_{k-i}(Y;L) = H^i_{(2)}(Y_0;L)$?

A geometrically minded person who wants to know what the two things could possibly have to do with each other might consult [BGM] for a simplicial version.

This question has been the subject of intensive work, an has been answered positively in a number of cases. Both because of my own ignorance and because of the complexity of the situation, and because the problem is somewhat tangential to the aim of this report, it is not possible here to do more than sketch the situation. (Also, the situation is constantly changing.)

- For a complete metric on a curve, the question was proved by Zucker [Z1]
- The existence of a (non-Kähler) metric for which the question has a positive answer was established by Cheeger [C], who had already considered the L^2 cohomology groups involved before he had ever heard of intersection homology.
- For the incomplete metric on Y_0 induced from its embedding in X, the question was conjectured by Cheeger, Goresky, and MacPherson [CGM] and was proved for two dimensional varieties by Hsiang and Pati.
- If Y is nonsingular and Σ is a divisor with normal crossings, the question was proved for a complete metric by Cattani, Kaplan, and Schmid [CKS], and by Kashiwara and Kawai [KK].
- If Y is the Baily-Borel compactification of a quotient of a Hermitian symmetric domain by an arithmetic subgroup of its automorphism group, the metric is the usual metric induced from the Hermitian symmetric domain, and L arises from a finite dimensional representation of the automorphism group, then the question was conjectured by Zucker [Z2] and was proved by Saper and Stern [SS] and by Looijenga [L], after some special cases had been established by Borel, Casselman, and Zucker.
- . If Y has only point singularities, for a nice class of complete Kähler metrics, the question was proved by Saper [Sa].

It is important to emphasize the difficulty of the problems involved in proving theorems like those summarized briefly above.

1.6.4 The Lefschetz fixed point theorem

Proposition 1.10 *Let $f : Y \longrightarrow Y$ be a placid map. Then there exist unique classes $[f] \in IH_n(Y \times Y, Q)$ and $[\Delta] \in IH_n(Y \times Y, Q)$ satisfying the properties: 1) $[f]$ and $[\Delta]$ both project to the fundamental class $[Y]$ of Y, under projection to the second factor (which is a placid map) and 2) $[f]$ (resp. $[\Delta]$) can be represented by cycles that lie in an arbitrarily small neighborhood of the graph of f (resp. the diagonal).*

DEFINITION. The *Lefschetz number* $L(f)$ of f is

$$L(f) = \sum(-1)^i trace\,(f^* IH^i(Y, Q) \longrightarrow IH^i(Y, Q))$$

Theorem 1.11 ([GM5].)

$$\mathcal{I}([f], [\Delta]) = L(f)$$

Consequently, if f has no fixed points, then $L(f)$ must be zero.

1.7 The Decomposition Theorem

Until here in our development, intersection homology has been useful only if we are interested in singular spaces Y (or spaces with a local system with singularities). However, the decomposition theorem shows that intersection homology intervenes in an essential way for a map $f : Z \longrightarrow Y$ even if both $Z$and Y are nonsingular algebraic manifolds.

Theorem 1.12 The Decomposition Theorem. (Conjectured in [GeM], Conjecture 2.10 and proved in [BBD], Theorems 6.2.5 and 6.2.10.)

Consider a proper algebraic map $f : Z \longrightarrow X$ of complex algebraic varieties. We assume that X is smooth, but Z may be singular. Then there exists:

- *A stratification $X = \bigcup_\alpha X_\alpha$ of X,*
- *A list of enriched strata $E_\beta = (X_\beta, L_\beta)$ where X_β is a stratum of X and L_β is a local system over X_β, and*
- *For each enriched stratum E_β, a polynomial in t, $\phi^\beta = \sum_j \phi_j^\beta t^j$*

such that for any pair of smoothly enclosed subsets $A \subseteq B \subseteq X$ in X, we have

$$IH_i(f^{-1}B, f^{-1}A) = \bigoplus_{\beta} \bigoplus_{j+k=i} IH_k\left((B \cap \overline{X_\beta}), (A \cap \overline{X_\beta}); L_\beta\right) \otimes V_j^\beta$$

where V_j^β is a rational vector space of dimension ϕ_j^β, i.e. $V_j^\beta = Q^{\phi_j^\beta}$.

The enriched strata may be chosen to be irreducible.

If the map f is projective and every component of Z has the same dimension, then all of the local systems L_β are self-dual, and all the polynomials $\phi^\beta = \sum_j \phi_j^\beta t^j$ satisfy the following properties (where d_β is $\dim Z - \dim X_\beta$):

- Poincaré Duality $\phi_j^\beta = \phi_{d_\beta/2-j}^\beta$
- Hard Lefschetz *If $j \geq d_\beta/2$, then $\phi_j^\beta \geq \phi_{j+2}^\beta$.*

Explanation of notations The enriched stratum $E_\beta = (X_\beta, L_\beta)$ is said to be *irreducible* if L_β is an irreducible local system, i.e. if it is zero on all but one connected component Z of X_β, and on Z it corresponds to an irreducible irreducible representation of the fundamental group of Z. The map $f : Z \longrightarrow X$ is *projective* if it admits a factorization $Z \longrightarrow X \times \mathbf{CP}^N \longrightarrow X$ where the map $X \times \mathbf{CP}^N \longrightarrow X$ is the projection onto the first factor.

Remark. Taking $B = X$ and $A =$, we get a direct sum decomposition of the intersection homology of Z. For each enriched stratum E_β, the summand of the decomposition theorem looks a lot like a Kunneth formula. So the decomposition theorem has the following fanciful interpretation:

The intersection homology of Z looks like the intersection homology of a disjoint union of varieties D_β, $IH_i(Z) = IH_i(\bigcup D_\beta, L)$ where $D_\beta = \overline{X_\beta} \times F_\beta$, and L restricted to D_β is the pull-up of L_β from X_β. Here F_β is a fictitious projective "fiber variety" such that $IH_j(F_\beta) = V_j^\beta$. The polynomials ϕ^β have the interpretation of the intersection homology Poincaré polynomials of F_β. The "fiber variety" has dimension d_β, so every variety D_β has the same dimension as Z. The "Poincaré Duality" and "Hard Lefschetz" relations on the polynomials ϕ^β are the Poincaré duality theorem and the hard Lefschetz theorem for F_β.

To emphasize that these "fiber varieties" are really fictitious, note that ϕ_0^β can be zero.

Exercise 1.16. Show that if $Z = \overline{X_\alpha} \times F$ for some compact variety F and if $f : Z \longrightarrow X$ is projection on the first factor followed by inclusion, then the decomposition theorem holds with only one enriched variety $E_\beta = (X_\alpha, Q)$ and ϕ^β given by the intersection Poincaré polynomial of F. (So in this case, the fiction of the remark is real.)

Remark. At least one summand of the decomposition theorem will always correspond to an enriched variety $(X_\alpha, L_\alpha)$where $\overline{X_\alpha}$ is the image of f. The other strata X_β occurring in summands of the decomposition theorem are, in some sense, the strata over which f is "singular". The projection of a product onto the first factor can be considered a nonsingular map this sense. See the exercises at the end of the section for examples.

1.7.1 Functoriality

The isomorphism between the two sides in the statement of the decomposition theorem is natural, in that it commutes with maps induced by inclusions of pairs of subsets of X, and with connecting homomorphisms:

Theorem 1.13 (Functoriality of the decomposition theorem.) *Suppose that we have two pairs of smoothly enclosed subsets $A \subseteq B \subseteq X$ and $A' \subseteq B' \subseteq X$ in X, such that $A \subseteq A'$ and $B \subseteq B'$. Then the following diagram commutes:*

$$\begin{array}{ccc} IH^i(f^{-1}B, f^{-1}A) & = & \bigoplus_\beta \bigoplus_{j+k=i} IH_k\left((B \cap \overline{X_\beta}), (A \cap \overline{X_\beta}); L_\beta\right) \otimes V_j^\beta \\ \downarrow & & \downarrow \\ IH^i(f^{-1}B', f^{-1}A') & = & \bigoplus_\beta \bigoplus_{j+k=i} IH_k\left((B' \cap \overline{X_\beta}), (A' \cap \overline{X_\beta}); L_\beta\right) \otimes V_j^\beta \end{array}$$

where the vertical maps on each side are given by the maps on intersection homology induced form the inclusion of pairs.

Suppose that we have a single pair of smoothly enclosed subsets $A \subseteq B \subseteq X$ in X. Then the following diagram commutes:

$$\begin{array}{ccc} IH^i(f^{-1}B, f^{-1}A) & = & \bigoplus_\beta \bigoplus_{j+k=i} IH_k\left((B \cap \overline{X_\beta}), (A \cap \overline{X_\beta}); L_\beta\right) \otimes V_j^\beta \\ \downarrow \partial_* & & \downarrow \partial_* \\ IH^i(f^{-1}A') & = & \bigoplus_\beta \bigoplus_{j+k=i} IH_k\left((A' \cap \overline{X_\beta}); L_\beta\right) \otimes V_j^\beta \end{array}$$

where the vertical maps on each side are given by the connecting homomorphism maps on intersection homology.

1.7.2 Uniqueness

The remarkable thing about the decomposition theorem is the existence. Uniqueness of the set of enriched varieties E_β and the polynomials ϕ^β is very much more elementary. The only complexity how to formulate uniqueness, given that the choice of the stratification of X is somewhat arbitrary. (If one stratification works, then any refinement of that stratification also works.)

Theorem 1.14 Uniqueness. *If the stratification $X = \bigcup_\alpha X_\alpha$ of X is fixed, then the list of irreducible enriched strata $E_\beta = (X_\beta, L_\beta)$ and the polynomials ϕ^β are* *determined uniquely by the conditions. If $X = \bigcup_\alpha X_\alpha$ and $X = \bigcup_\gamma X'_\gamma$ are two different stratifications, and the decomposition thoeorm holds for each of them. Then there will exist a bijection between the irreducible enriched varieties $E_\beta = (X_\beta, L_\beta)$ and $E'_{\beta'} = (X'_{\beta'}, L'_{\beta'})$ such that (X_β, L_β) corresponds to $(X'_{\beta'}, L'_{\beta'})$ if and only if $\overline{S(L_\beta)} = \overline{S(L'_{\beta'})}$ and $L_\beta|(S(L_\beta) \cap S(L'_{\beta'}) = L'_{\beta'}|(S(L_\beta) \cap S(L'_{\beta'})$. (Here, SL denotes the set over which L is defined and nonzero.) The polynomials ϕ for the corresponding enriched subvarieties are equal.*

Exercise 1.17. Prove this uniqueness statement.

1.7.3 Generalizations

One would like to replace the trivial local system Q in $IH(Z_\beta)$ by an arbitrary local system. While there are no counterexamples known for any local system, it is unknown at the moment just how general the theorem can be made. It is proved for local systems of "geometric origin" [BBD].

Likewise, one would like to generalize the decomposition theorem to complex analytic maps rather than complex algebraic maps. This has been carried out by Morihiko Saito, again for local systems of "geometric origin", as part of his theory of mixed Hodge modules.

1.7.4 Examples of the decomposition theorem

The force of the decomposition theorem is difficult to understand without seeing its application in a number of special cases.

In the following series of exercises, we consider resolutions of singularities. (A map $f : Z \longrightarrow Y$ is a resolution of singularities if Z is nonsingular, f is proper, and if there exists an open dense set $Y_0 \in Y$ such that the restriction $(f|f^{-1}(Y_0)) : f^{-1}(Y_0) \longrightarrow Y_0$ is a homeomorphism.)

Exercise 1.18. Show that if $f : Z \longrightarrow \overline{X_\alpha}$ is a resolution of singularities, then one enriched stratum of the decomposition theorem for f is (X_α, Q).

Exercise 1.19. Using the previous exercise, conclude that the intersection homology of any complex algebraic variety is a subspace of the ordinary homology of any of its resolutions of singularities.

Exercise 1.20. Show that if the resolution of singularities $f : Z \longrightarrow \overline{X_\alpha}$ is an isomorphism outside of a finite set of points x_γ in $\overline{X_\alpha}$, then in addition to (X_α, Q), the only other possible enriched subvarieties in the decomposition theorem are of the form $E_\beta =(x_\gamma, Q)$. In this case, show that $\phi^\beta_j = b_j(f^{-1}(x_\gamma)) - b_{d-j}(f^{-1}(x_\gamma))$ if $j \geq d/2$. Here d is the dimension of X_α and

$b_j(f^{-1}(x_\gamma))$ is the j^{th} ordinary homology Betti number of $f^{-1}(x_\gamma)$. Determine ϕ_j^β for the other values of j by Poincaré duality.

Exercise 1.21. Continue consideration of a resolution of singularities $f : Z \longrightarrow \overline{X_\alpha}$. The map f is called *small* if for any point $x \in X_\gamma \subset \overline{X_\alpha}$ such that $X_\gamma \neq X_\alpha$, the dimension of $f^{-1}(x)$ is strictly less than $(\dim X_\alpha - \dim X_\gamma)/2$. Show in this case that the only enriched stratum occurring in the decomposition theorem for f is (X_α, Q).

Exercise 1.22. Show that if an algebraic variety admits a small resolution, then the intersection homology of the variety is the ordinary homology of the resolution. Show that for such varieties, the statements of the Kähler package are clear from the classical statements.

Exercise 1.23. Let $X = \text{Hom}(C^n, C^p)$ be the manifold of $n \times p$ matrices. It is stratified by the rank: X_k is the set of matrices of rank k. We denote the closure of X_k by $\overline{X_k}$. Calculate the intersection homology group $IH_i(\overline{X_k})$ as a function of n, p, k, and i. Hint: Suppose that $n \geq p$. Then let Z be the variety a point of which consists of the data a k-plane ξ in C^p together with a linear map fromC^n to ξ. Show that Z is a small resolution of $\overline{X_k}$.

Remark. There are other examples where intersection homology can be calculated through small resolutions. See, for example, [Ze].

1.7.5 Further material on the decomposition theorem

The decomposition theorem is considered further in section 5.2 , after Fary functors have been introduced. The interested reader can jump to that section, reading only the definition of Fary functors from section 4.3.2 . There, the discussion focuses more on the role of the local systems.

1.8 Theorems that hold for close to middle perversities.

I would like to tell a piece of history here, since it serves as well for an introduction to this section as anything else I could write. In 1978 and 1979, Mark Goresky and I were immersed in the philosophy that this chapter has been building so far: There are some nice theorems about intersection homology, but the best of them hold only for the middle perversity. The philosophy was that the middle intersection homology group of a singular space was the true analogue of the ordinary homology of a compact manifold. (We were particularly happy about this, because when we first introduced intersection homology it was criticized because there were "too many groups", one for each perversity.) We were developing Stratified Morse Theory in those years, and we were applying it to prove the Lefschetz Hyperplane Theorem. What surprised us was that our proof, indeed the whole theory, applied to a more general class of perversities just as well as it did to the middle one. This was totally unexpected at the time. We were puzzled, and worse, we were disappointed. We didn't want lots of interesting groups, we wanted one very interesting group. I recall this at length to explain to myself why we passed so close to a very important discovery without seeing it. For, as we found in 1980, the more general class of perversities give Perverse Sheaves. And it is Perverse Sheaves that represent the natural generality of our proofs. That idea, to over- simplify, is the main theme of this report, and it will be developed at length later.

First, let's define a class of perversities.

DEFINITION. A perversity p is *close to middle* if

$$\frac{\text{codim} S_\alpha}{2} \geq p(S_\alpha) \geq \frac{\text{codim} S_\alpha}{2} - 2$$

for all strata $S_\alpha \subseteq \Sigma$.

DEFINITION. The *logarithmic perversity* is the perversity $p(S_\alpha) = (\text{codim} S_\alpha)/2$, and the *sublogarithmic perversity* is the perversity $p(S_\alpha) = (\text{codim} S_\alpha)/2 - 2$

Note that the middle perversity is close to middle, as is the logarithmic perversity and the sublogarithmic perversity. There are plenty of classical perversities that are close to middle, but not all perversities close to the middle are classical even if $p(Y_\alpha)$ depends only on the codimension of Y_α. For example, all three perversities considered in example 1 were close to middle, whereas only one of them was classical.

1.8.1 The Lefschetz Hyperplane Theorem

As for the Hard Lefschetz Theorem, we suppose that X is a complex projective space. We say that the hyperplane H is *transverse* to Y if it is transverse to every stratum Y_α of Y. Most hyperplanes in the family of all complex hyperplanes H in X are transverse to Y. If H is transverse to Y, the space $Y \cap H$ is stratified by strata $Y_\alpha \cap H$, and we can take as its nonsingular set $Y_0 \cap H$. Call ρ the inclusion of $Y \cap H$ into Y. Given a geometric i-chain ξ in $Y_0 \cap H$ with coefficients in $L|(Y_0 \cap H)$, we define $\rho_* \xi$ to be that same chain considered as a cycle in Y_0.

Proposition 1.15 *If ξ is allowable with perversity p, then $\rho_*\xi$ is allowable with perversity p. Therefore ρ_* induces a chain map, $\rho_* : I^p H_i(Y \cap H; L|(Y_0 \cap H)) \longrightarrow I^p H_i(Y; L)$.*

Theorem 1.16 ([GM7],[GM8]) *If p is close to middle perversity, then the map*

$$\rho_* : I^pH_i(Y \cap H; L|(Y_0 \cap H)) \longrightarrow I^pH_i(Y; L)$$

is an isomorphism for $i < n/2 - 1$ *and is a surjection for* $i = n/2 - 1$.

1.8.2 Homology of Stein spaces

A Stein space is a complex analytic space which admits a closed embedding in C^N for some N. So we can let X be C^N and we let Y be a Stein space.

Theorem 1.17 ([GM7],[GM8]) *Let Y be a Stein space of (real) dimension k, and let p be a perversity that is close to middle. Then the intersection homology group* $I^pH_i(Y; L)$ *vanishes for* $i > n/2$. *Also the intersection homology with closed support* $I^p_{\text{cl}}H_i(Y; L)$ *vanishes for* $i < n/2$.

Exercise 1.24. Show that the vanishing result for the intersection homology with closed support does not hold for the ordinary homology with closed support of a Stein space. Consider the union of two transverse complex 2- planes in C^4.

Chapter 2

Interlude: Perverse Sheaves on Manifolds.

This chapter, which is not logically necessary for the main results that we are aiming at, is a preamble for the next chapter.

This preamble may serve as a paradigm of bad mathematics. We will give a long and complicated definition, and then we will state a structure theorem saying that the thing defined is really just a simple, well-known mathematical object. The long and complicated definition is that of a perverse sheaf a manifold X, stratified with only one stratum. The simple object that it turns out to be equivalent to is a local system on X.

There are two reasons for having this preamble. The first reason is this: While the definition of a perverse sheaf on a manifold is long and complicated, it is very natural and familiar for someone for someone who knows a little homology theory and a little Morse theory. However, the definition of a perverse sheaf in general is a very straight-forward generalization of the definition of a perverse sheaf on a manifold. Therefore, the reader familiar with the contents of this preamble will find the definition of a perverse sheaf very natural. And one of the points of this book is to give a definition of perverse sheaves which makes them look natural to begin with.

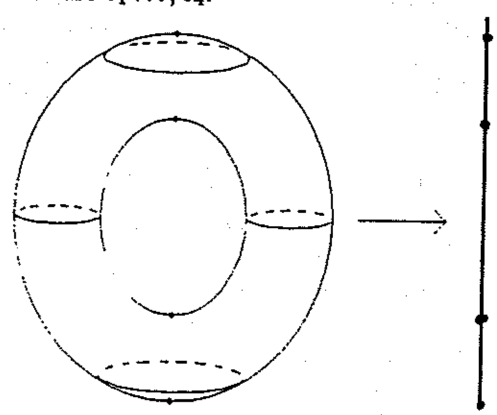
The second reason for having this preamble is that the proofs in the next chapter are structurally the same as those here. However here they appear without the necessity of a lot of techniques of stratification theory. Since stratification theory appears rather technical until you are used to it, the essential ideas are clearer in the preamble.

The reader who is interested in perverse sheaves and who wants to get at the heart of the matter quickly may skip this chapter and refer to it as needed.

Notational conventions. Throughout this chapter, X will be a (real) oriented smooth manifold (without boundary). We will denote by n the dimension of X. We will assume that n is even. This assumption is not necessary for the theory we present, however the assumption makes the notations easier. This assumption will not detract from the applications of perverse sheaves: Complex analytic varieties, which are what we ultimately want to study, have even real dimensions.

2.1 Classical Morse Theory

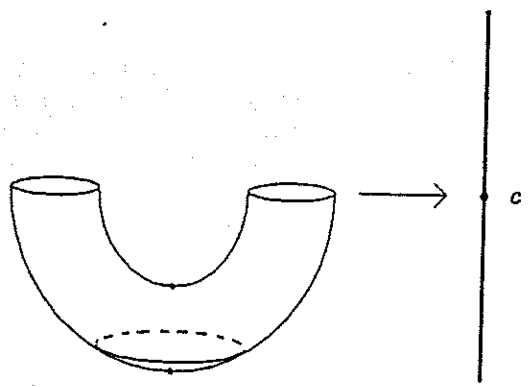
If $f : X \longrightarrow \boldsymbol{R}$ is a smooth real valued function on X, then a *critical point* of f is a point $p \in X$ such that all of the partial derivatives of f vanish at p, i.e. $df(p) = 0$. A *critical value* of f is a real number $v \in \boldsymbol{R}$ such that $v = f(p)$ for some critical point p. For example, in the following picture, X is a torus and f is the orthogonal projection to a vertical line. The critical points are $p_1, \ldots, p_4$ and the critical values are $v_1 \ldots, v_4$.



The critical points and critical values of a real valued function.

If $c \in \boldsymbol{R}$ is any real number, we define the *truncation* of X by f at a, notated $X_{f<c}$ to be the subset of X where f takes values less than c, i.e. $X_{f<c} =$

$f^{-1}((-\infty, c))$.



The truncation of X by f at c

If c is not a critical value of f, then $X_{f<c}$ is the interior of a smooth manifold with smooth boundary (by the implicit function theorem).

The object of Classical Morse Theory (CMT) is to study how the homology groups of $X_{f<c}$ changes as c varies. (Of course, modern Morse theory also studies how the diffeomorphism type of $X_{f<c}$ changes as c varies. However in this report, we are interested mainly in homological invariants. The statement as made is historically true of Morse theory as originally conceived by Morse, incidentally.) The first result is this.

Theorem 2.1 (CMT Part A). *As c varies in the open interval between two adjacent critical values, the homology groups of $X_{f<c}$ remain constant. More specifically, if $b < c$ and if there is no critical value in the closed interval $[b, c]$, then*

$$H_i(X_{f<c}, X_{f<b}) = 0$$

so, by the long exact sequence for homology, the map induced by inclusion

$$H_i(X_{f<b}) \longrightarrow H_i(X_{f<c})$$

is an isomorphism.

To go further, we need some more definitions of Morse theory. Consider a critical point p of a smooth function $f : X \longrightarrow \mathbf{R}$. The Hessian $\mathcal{H}$ of f at p is the quadratic form on T_pX which is the second order Taylor expansion of $f(x) - f(p)$. Namely,

$$\mathcal{H}(\bar{x}_1 \cdots, \bar{x}_n) = \begin{bmatrix} \bar{x}_1 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

Here $(x_1, \cdots, x_n)$ are local coordinates about p in X and $(\bar{x}_1, \cdots, \bar{x}_n)$ are the induced linear coordinates in T_pX. As a quadratic form on T_pX, the Hessian is independent of the local coordinates chosen. The function f is said to have a *Morse singularity* at p if the Hessian quadratic form is nondegenerate, i.e. if the determinant of the matrix is nonzero. In this case, define an *up space* U_p at p to be a maximal vector subspace of the tangent space T_pX on which the Hessian is positive definite. Define a *down space* D_p at p to be a maximal vector subspace on which it is negative definite. It is a result of the theory of real quadratic forms that all up spaces have the same dimension; as do all down spaces. Furthermore, the direct sum decomposition $T_pX = U_p \oplus D_p$ always holds. Sylvester's Theorem of Inertia states that the only invariant of $\mathcal{H}$ as a quadratic form is the information contained in the dimensions of U_p and D_p. Given that the dimension of X is fixed (it's n), we need only one integer to express these two dimensions. There are two standard choices for that integer:

DEFINITION. The *signature* of the Hessian $\sigma(\mathcal{H})$, also called the *signature of f at p* is the dimension of U_p minus the dimension of D_p.

DEFINITION. The *Morse Index* of the Hessian $\mu(\mathcal{H})$, also called the *Morse index of f at p* is the dimension of D_p.

The signature is more usual among algebraists and the Morse index is more usual among geometers. Note that σ and μ determine each other, since $\sigma = \dim U_p - \dim D_p = (\dim U_p + \dim D_p) - 2\dim D_p = n - 2\dim D_p = n - 2\mu$. Or $\mu = (n - \sigma)/2$. Also, note that σ is always even since n is even.

Theorem 2.2 (CMT Part B). *If $b < c$ are not critical values, if there is only one critical point p with critical value v in the closed interval $[b, c]$, and if f has a Morse singularity at p with Morse index μ, then*

$$H_i(X_{f<c}, X_{f<b}) = \begin{matrix} 0 & \text{if } i \neq \mu \\ Q & \text{if } i = \mu \end{matrix}$$

So, by the long exact sequence for homology, we have the Morse alternative: *as a varies from b to c,* Either *the μ^{th} Betti number of $X_{f<a}$ is increased by one, and all of the other Betti numbers are unchanged,* Or *the $(\mu-1)^{st}$ Betti number of $X_{f<a}$ is decreased by one, and all of the others are unchanged.*

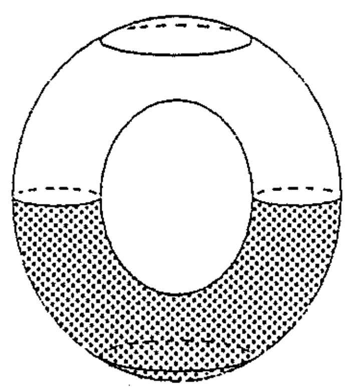
Exercise 2.1. Observe that for all four critical points in the torus example above, the first Morse alternative occurs. Construct an example in which the second Morse alternative occurs.

Remark. The reason for making a fuss about the difference between μ and σ is that for perverse sheaves we will use $\bar{\sigma} = \sigma/2$ in the place that μ appears in the theorem above. In fact, that one substitution can be seen as engendering the whole theory of perverse sheaves.

There is much more than what we have said so far to Classical Morse theory. Functions with only Morse singularities are called *Morse functions.* There is a very strong existence theorem for Morse functions: the form an open dense set in the space of all real valued functions, with the appropriate topology, so any function can be approximated by a Morse function. There is also beautiful characterization of Morse functions with distinct critical values: They are the structurally stable functions, i.e. the functions with the property that sufficiently nearby functions have the same topological type. We won't need these facts here.

2.2 Opposed Pairs of Smoothly Enclosed Subsets.

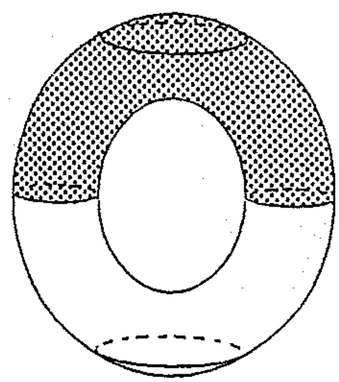
DEFINITION. A subset of X is called *smoothly enclosed* if it is of the form $X_{f\leq c}$ for some smooth function f and for some value c which is not a critical value of f. In other words, a smoothly enclosed subset of X is the closure of the truncation of X by some smooth function f at a number c which is not a critical value of f. If $G = X_{f\leq c}$ is a smoothly enclosed subset of X, then the interior of G is notated G^0. It is $X_{f<c}$. Smoothly enclosed subsets of X are just n dimensional submanifolds with boundary whose boundary is smooth.



A smoothly enclosed subset R of X

If R is a smoothly enclosed subset of X, then we will often have occasion to consider the closure of the complement of R, which we denote by $\sim R$ and we call the *complimentary* smoothly enclosed set. For example, if $R = X_{f\leq c}$, then

$\sim R$ is the subset of X where f takes values greater than or equal to c, which we denote by $X_{f \geq c}$. The set $\sim R$ is also smoothly enclosed, since $\sim R = X_{-f \leq -c}$.



The complementary smoothly enclosed subset $\sim R$.

DEFINITION. A *smoothly varying family* of smoothly enclosed subsets of X is a family $R(t)$ for $t \in \boldsymbol{R}$ of subsets such that there is a smooth function $X \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ notated $(x,t) \mapsto f_t(x)$ with no critical points having critical value a, such that $R(t) = f_t^{-1}((-\infty, a]) = X_{f_t \leq a}$.

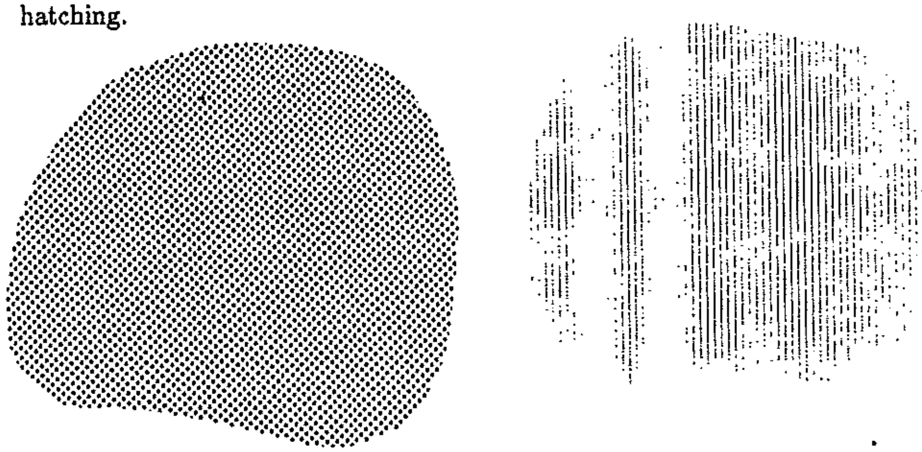
The term "smoothly varying" is justified by the following version of Ehresmann's theorem.

Proposition 2.3 *Suppose $R(t)$ is a smoothly varying family of smoothly enclosed subsets of X. Then there is a one parameter family of diffeomorphisms of X onto itself $F_t : X \longrightarrow X$ such that $F_t(R(0)) = R(t)$.*

DEFINITION. Two smoothly enclosed subsets R and G are said to be *opposed* under the following conditions:

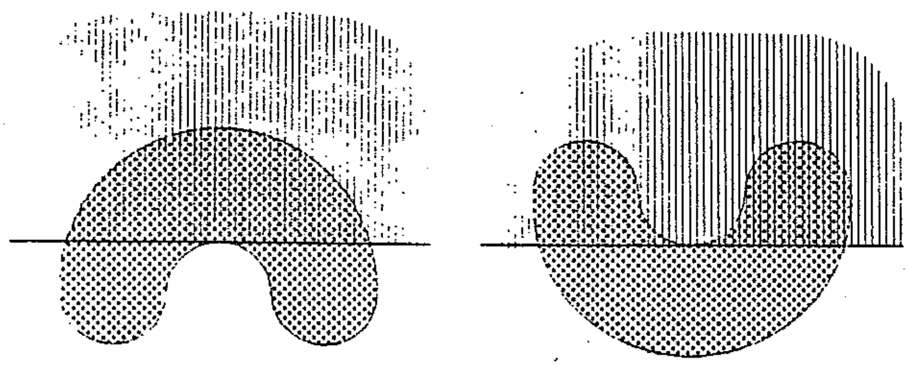
- Suppose $R = X_{f \leq a}$ and $G = X_{g \leq b}$. Then for no $p \in X$ does it happen that $f(p) = a$, $g(p) = b$, and $df(p)$ is a positive multiple of $dg(p)$.
- The subset $X - (R^0 \cup G^0)$, called the *support* of the opposed pair, is compact.

For short, we will use the expression *an opposed pair* (R, G) in X for a pair of smoothly enclosed subsets R and G of X which are opposed. We will call the first subset R of an opposed pair (R, G) the *red* subset and the second subset G the *green* subset. Lacking the possibility of reproducing colors here, we will indicate the red subset in pictures by dots and the green subset by cross hatching.



red green

The geometric meaning of the definition of an opposed pair is that at all points where the boundary of R is tangent to the boundary of G, the sets R and G must lie on opposite sides of the common tangent plane. In the following pictures, you should think of the page you are reading as lying in a coordinate chart in the 2-sphere.



Not An opposed pair ~~Not~~ An opposed pair

Note that the statement that (R, G) is an opposed pair is symmetric in R and G. If $f : X \longrightarrow \mathbf{R}$ is a proper map, then $(X_{f \geq a} X_{f \leq b})$ is an opposed pair whenever a and b are not critical values of f.

We want to put a put a partial order on the set of opposed pairs in X.

DEFINITION. Suppose that (R, G) and (R', G') are two opposed pairs in X.

We say that (R,G) *covers* (R',G') , and we write $(R,G) \sqsupseteq (R',G')$ if both $R \supseteq R'$ and $G \subseteq G'$.

Proposition 2.4 *The relation* $\sqsupseteq$ *satisfies*

- *Reflexivity:* $(R,G) \sqsupseteq (R,G)$.
- *Transitivity: if* $(R,G) \sqsupseteq (R',G')$ *and* $(R',G') \sqsupseteq (R'',G'')$, *then* $(R,G) \sqsupseteq (R'',G'')$.
- *If* $(R,G) \sqsupseteq (R',G')$, *then* $(G',R') \sqsupseteq (G,R)$.
- *If* $(R,G) \sqsupseteq (R',G')$, *then* $(\sim R',\sim G') \sqsupseteq (\sim R,\sim G)$.

DEFINITION. A *smoothly varying family* of opposed pairs in X is a family $(R(t),G(t)), t \in \boldsymbol{R}$ such that $R(t)$ and $G(t)$ separately are smoothly varying families of smoothly enclosed subsets of X and such that for all t, the pair $(R(t),G(t))$ is opposed.

DEFINITION. Suppose that (R,G) and (R',G') are two opposed pairs in X. We say that (R,G) covers (R',G') *by deformation* if there is a smoothly varying family of opposed pairs in X is a family $(R(t),G(t)), t \in \boldsymbol{R}$ such that $(R,G) = (R(0),G(0))$, $(R',G') = (R(1),G(1))$, and $(R(t),G(t)) \sqsupseteq (R(1),G(1))$ for all $t \leq 1$.

Proposition 2.5 *The four statements of proposition remain valid when the relation "covers" is replaced by the relation "covers by deformation".*

2.3 The ordinary homology perverse sheaf.

DEFINITION (provisional). The *homology perverse sheaf* $\mathcal{H}$ is the rule which assigns to any pair of opposed smoothly bounded subsets R and G of X and to any integer i the rational vector space $\mathcal{H}^i(X, \genfrac{}{}{0pt}{}{R}{G})$, according to the following rule:

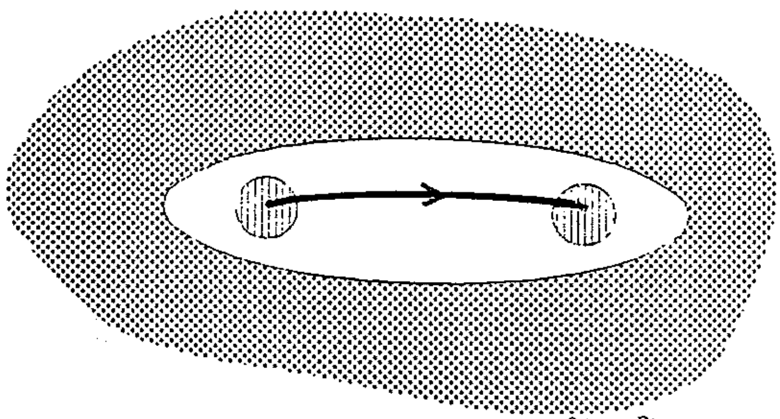
$$\mathcal{H}^i(X, \genfrac{}{}{0pt}{}{R}{G}) := H_{n/2-i}(X-R, G^0 \cap (X-R); Q) \quad (2.1)$$

$$= H_{n/2-i}((X-R) \cup G^0, G^0; Q) \quad (2.2)$$

Here, H_i is the usual homology group. The two expressions on the right are equal to each other by excision.

Remarks. By definition $\mathcal{H}^i(X, \frac{R}{G})$ is just a homology group of a subset of X relative to another subset. But there are some strange looking changes from the usual notation. The first is a shift in the numbering scheme, where the zeroth homology group is placed in degree n, the top homology group is placed in degree $-n$, and the middle homology group is placed in degree 0. This numbering scheme is the standard one in perverse sheaf theory, so getting used to it now will save effort in the long run. One merit of it is that duality has a particularly appealing expression in this numbering scheme.

The other difference from ordinary homology notation is in the labeling of the sets. The way to think of it is this: an element of $\mathcal{H}^i(X, \frac{R}{G})$ is represented by an $(n-i)$-chain ξ in X such that ξ avoids the red set R (get it?), and it is allowed to have boundary in the green set G.



A chain representing a class in $\mathcal{H}^0(X, \frac{R}{G})$.

One way of explaining the naturality of this notation is the following: Intersection Homology, which is the quintessential perverse sheaf, is neither cohomology nor homology, but rather something in between. We want to use a notation that favors neither cohomology nor homology conventions. In relative cohomology $H^i(X,R)$, the cocycles representing classes are required to be zero on the subspace R. In relative homology $H_i(S,G)$, the cycles are allowed to go to G but may have boundary there. In other words, there is an analogy "homology is to the green set as cohomology is to the red set". To emphasize this, we state the following proposition which shows that if we switch homology and cohomology and we switch the roles of R and G, we get the same thing.

Proposition 2.6

$$\mathcal{H}^i(X, \tfrac{R}{G}) := H^{n/2+i}(X-G, R\cap(X-G); Q) \qquad (2.3)$$

$$= H^{n+i}((X-G)\cup R^0, R^0; Q) \qquad (2.4)$$

PROOF. The equality with the definition of $\mathcal{H}^i(X, \frac{R}{G})$ is just the Lefschetz Duality Theorem (remember that X is oriented). The equality of the two expressions on the right is excision in cohomology.

We will call either of the two expressions on the right the *cohomology definition* of $\mathcal{H}$. Justified by this proposition, we may say that the homology perverse sheaf and the cohomology perverse sheaf are exactly the same thing.

***Remark.** There is an analogy between the roles of the red set R and the green set G on the one hand, and Diriclet and Neumann boundary conditions from P.D.E. on the other hand. (The analogy becomes exact if we represent homology classes by harmonic forms.)

The following section contains a list of the defining properties of a perverse sheaf $\mathcal{P}^i(X, \frac{R}{G})$ on a manifold X. The main example to keep in mind while reading it is $\mathcal{P}^i(X, \frac{R}{G}) = \mathcal{H}^i(X, \frac{R}{G})$.

2.3.1 Definition of a perverse sheaf on a manifold.

DEFINITION. A *Perverse sheaf* on a manifold on X is a device $\mathcal{P}$ which does the following three things:

- $\mathcal{P}$ assigns to each opposed pair (R, G) in X and to each integer i a finite dimensional vector space over the rationals $\mathcal{P}^i(X, \frac{R}{G})$.
- Whenever (R, G) and (R', G') are two opposed pairs in X such that $(R, G) \sqsupseteq (R', G')$, then $\mathcal{P}$ gives a map denoted $\mathcal{R}^*$

$$\mathcal{R}^* : \mathcal{P}^i(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^i(X, \tfrac{R'}{G'})$$

  called the *restriction* map.
- Whenever (R, C) and $(\sim C, G)$ are opposed subsets of X, $\mathcal{P}$ gives a map denoted ∂^*

$$\partial^* : \mathcal{P}^i(X, \tfrac{R}{C}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{\sim C}{G})$$

  called the *coboundary homomorphism.*

The device $\mathcal{P}$ will qualify as a perverse sheaf if it satisfies the following axioms called the *modified Eilenberg - Steenrod axioms:*

- **Functoriality.** For any opposed pair (R, G), the map

$$\mathcal{R}^* : \mathcal{P}^i(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^i(X, \tfrac{R}{G})$$

  is an isomorphism. For any triple of opposed pairs (R, G), (R', G'), and (R'', G'') such that $(R, G) \sqsupseteq (R', G') \sqsupseteq (R'', G'')$, we have a diagram of restriction maps which commutes:

$$\begin{array}{ccccc} & & \mathcal{P}^i(X, \frac{R'}{G'}) & & \\ & \nearrow & & \searrow & \\ \mathcal{P}^i(X, \frac{R}{G}) & & \longrightarrow & & \mathcal{P}^i(X, \frac{R''}{G''}) \end{array}$$

- **Naturality.** Whenever (R,G), $(\sim C,G)$, (R',G'), and $(\sim C',G')$ are all opposed pairs, and $(R,C) \sqsupseteq (R',C')$ and $(\sim C,G) \sqsupseteq (\sim C',G')$then the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}^i(X, \frac{R}{C}) & \xrightarrow{\partial^*} & \mathcal{P}^{i+1}(X, \frac{\sim C}{G}) \\ \downarrow & & \downarrow \\ \mathcal{P}^i(X, \frac{R'}{C'}) & \xrightarrow{\partial^*} & \mathcal{P}^{i+1}(X, \frac{\sim C'}{G'}) \end{array}$$

  where the vertical arrows are restriction maps.

- **Exactness.** Whenever (R,C), $(\sim C,G)$, and (R,G) are opposed subsets of X with $(R,G) \sqsupseteq (R,C)$ and $(\sim C,G) \sqsupseteq (R,G)$, then the following sequence is exact:

$$\cdots \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{C}) \xrightarrow{\partial^*} \mathcal{P}^i(X, \tfrac{\sim C}{G}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{G}) \longrightarrow \cdots$$

- **Excision.** Whenever (R,G) and (R,G') are two opposed pairs in X such that $(R,G) \sqsupseteq (R,G')$, if $G \cap \sim R = G' \cap \sim R$, then $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R}{G'})$ is a isomorphism. Dually, whenever (R,G) and (R',G) are two opposed pairs in X such that $(R,G) \sqsupseteq (R',G)$, if $R \cap \sim G = R' \cap \sim G$, then $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R'}{G})$ is a isomorphism.
- **Homotopy.** Suppose (R,G) covers (R',G') by deformation. Then the map $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R'}{G'})$ is an isomorphism.
- **Dimension.** Let $f : X \longrightarrow \boldsymbol{R}$ is a smooth proper function with only one critical point $p \in X$ with critical value in the interval $[b,a]$, and assume that f has a Morse singularity at p . Then $\mathcal{P}^i(X, \frac{X_{f \geq a}}{X_{f \leq b}}) = 0$ for unless $i = -\bar{\sigma}$ where $\bar{\sigma}$ is on half of the signature of the Hessian quadratic form of F at p (see definitions in section 1).

DEFINITION. A *Fary functor* is a device that does the same three things and satisfies all of the axioms for a perverse sheaf except for the last one, the Dimension axiom, and which satisfies instead the following additional axiom:

- There exists a number N such that $\mathcal{P}^i(X, \frac{R}{G}) = 0$ unless $-N \leq i \leq N$.

Thus a perverse sheaf is a Fary functor satisfying the dimension axiom.

DEFINITION. Given a Fary functor $\mathcal{F}$, there is another Fary functor $\mathcal{TF}$ *translated* Fary functor which defined by $\mathcal{TF}^i(X, \frac{R}{G}) = \mathcal{F}^{i+1}(X, \frac{R}{G})$.

Exercise 2.2. Show that if $\mathcal{F}$ is a Fary functor, then $\mathcal{TF}$ is a Fary functor. Show, however, that if $\mathcal{F}$ is a perverse sheaf, then $\mathcal{TF}$ is not necessarily a perverse sheaf. (As a much harder exercise, show that $\mathcal{F}$ and $\mathcal{TF}$ are both perverse sheaves only if $\mathcal{F}$ is zero on every opposed pair. Hint: read the next few sections first.)

2.4 The homology perverse sheaf satisfies the axioms.

First of all, we should complete the definition of the homology perverse sheaf. This may be done from either the homology point of view or the cohomology point of view.

The homology formulation.

DEFINITION. The homology perverse sheaf $\mathcal{H}$, is the following:

- The vector space $\mathcal{H}^i(X, {}^R_G) = H_{n-i}(X - R, G^0 \cap (X - R); Q)$.
- The restriction map

$$\mathcal{R}^* : \mathcal{H}^i(X, {}^R_G) \longrightarrow \mathcal{H}^i(X, {}^{R'}_{G'})$$

  is just the map

$$H_{n-i}(X - R, G^0 \cap (X - R)) \longrightarrow H_{n-i}(X - R', G'^0 \cap (X - R'))$$

  induced by the inclusion of pairs.
- The coboundary homomorphism

$$\partial^* : \mathcal{H}^i(X, {}^R_C) \longrightarrow \mathcal{H}^{i+1}(X, {}^{\sim C}_G)$$

  is just the homomorphism

$$H_{n-i}(X - R, C \cap (X - R)) \xrightarrow{\partial_*} H_{n-i-1}(C \cap (X - R) \longrightarrow H_{n-i-1}(C) =$$
$$= H_{n-i-1}((X- \sim C) \longrightarrow H_{n-i-1}((X- \sim C), G \cap (X- \sim C))$$

  where the unlabeled maps are induced by inclusions of pairs.

The cohomology formulation.

DEFINITION. The (co)homology perverse sheaf $\mathcal{H}$, is the following:

- The vector space $\mathcal{H}^i(X, {R \atop G}) = H^{n+i}(X - G, R \cap (X - G); Q)$.
- The restriction map

  $$\mathcal{R}^* : \mathcal{H}^i(X, {R \atop G}) \longrightarrow \mathcal{H}^i(X, {R' \atop G'})$$

  is the map

  $$H^{n+i}(X - G, R \cap (X - G)) \longrightarrow H^{n+i}(X - G', R' \cap (X - G'))$$

  induced by the inclusion of pairs.
- The coboundary homomorphism

  $$\partial^* : \mathcal{H}^i(X, {R \atop C}) \longrightarrow \mathcal{H}^{i+1}(X, {\sim C \atop G})$$

  is the homomorphism

  $$H^{n+i}(X - C, R \cap (X - C)) \longrightarrow H^{n+i}(X - C) \xrightarrow{\partial^*} H^{n+i+1}(X, X - C) =$$
  $$= H^{n+i+1}(X, \sim C) \longrightarrow Hn + i + 1(X - G, \sim C - G)$$

  where the unlabeled maps are induced by the inclusions of pairs.

2.4.1 Verification of the axioms.

In order to establish that the ordinary homology perverse sheaf as defined above is in fact a perverse sheaf, we must verify the Modified Eilenberg- Steenrod Axioms. From either the homology version or the cohomology version, all of the axioms except the last two, Dimension and Homotopy, are immediate consequences of the similarly named properties of homology (or cohomology). We leave these as an exercise. Furthermore, the Dimension axiom for ordinary homology is just a restatement of the fundamental homological result of Morse Theory, as formulated in Section 2.1. This leaves the homotopy axiom.

2.4.2 The homotopy axiom.

The homotopy axiom asks us to consider two opposed pairs $(R, G) = (R(0), G(0))$, and $(R', G') = (R(1), G(1))$ selected from a smoothly varying family of opposed pairs $(R(t), G(t))$. It states that if $(R(t), G(t)) \sqsupseteq (R(1), G(1))$ for all $t \leq 1$ then the restriction map is an isomorphism. So, in order to prove it, we should investigate to what extent the opposed pairs in the family resemble each other. Under an additional transversality condition, we will actually have that all of the opposed pairs in the family are homeomorphic.

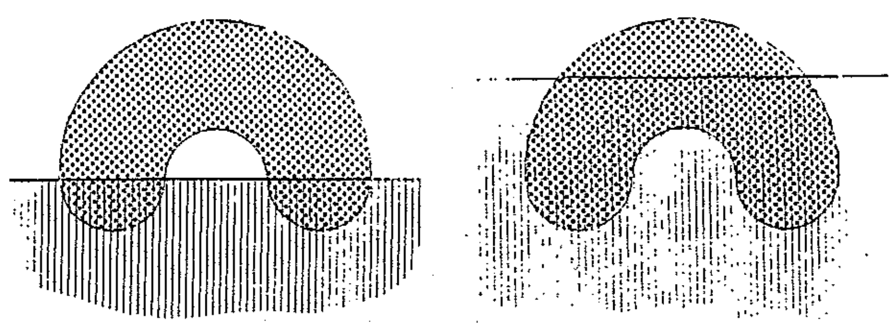
DEFINITION. Suppose that R and G, smoothly enclosed subsets of X, are defined by $R = X_{f \leq a}$ and $G = X_{g \leq b}$. Then the pair (R, G) is said to be *transverse* if for no $p \in X$ does it happen that $f(p) = a$, $g(p) = b$, and $df(p)$ is a real multiple (positive or negative) of $dg(p)$.

So transverse pairs are opposed, but not vice versa.

Proposition 2.7 *Let $R(t)$ and $G(t)$ be a smoothly varying families of smoothly enclosed subsets of X. Suppose that for each $t \in \boldsymbol{R}$, $R(t)$ is transverse to $G(t)$. Then there is a one parameter family of diffeomorphisms of X onto itself $F_t : X \longrightarrow X$ such that $F_t(R(0)) = R(t)$ and $F_t(G(0)) = G(t)$.*

This is a slightly extended version of Eheresmann's theorem and it is proved in the same way: On $X \times \boldsymbol{R}$ a vector field can be found (by partitions of unity) which projects to the unit speed vector field on $\boldsymbol{R}$ and which is tangent to the boundaries of $\cup_t R(t)$ and $\cup_t G(t)$ whenever it lies in them. Then the required diffeomorphism is obtained by integrating this vector field.

The proposition just stated would be false if we only assumed $(R(t), G(t))$ is an opposed pair, rather than a transverse pair, as the following two examples show: (In each case, $R(t)$ remains constant, while $G(t)$ is the part of the page below a horizontal line which is rising as t increases.)



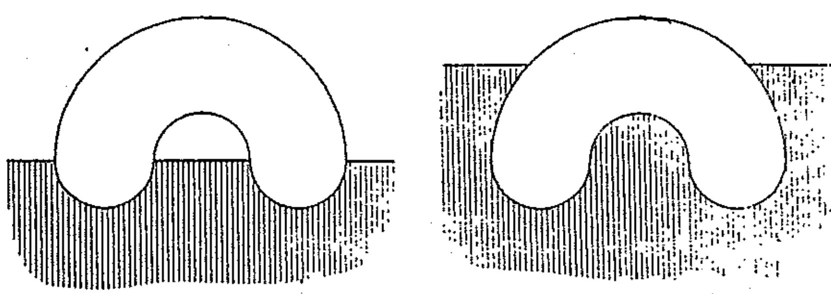
A family of opposed subsets.

In both cases, the last picture is not homeomorphic to the first one. The problem arises when the boundary of the set R becomes tangent to the boundary of the set S, i.e. when transversality fails.

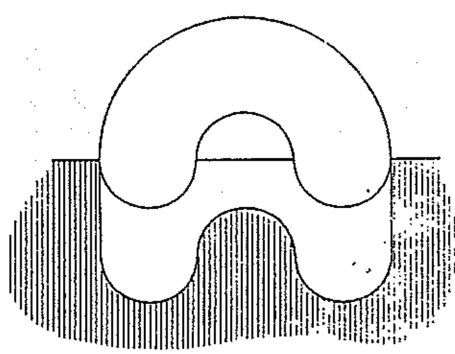
Suppose that $(R, G) = (R(0), G(0))$, and $(R', G') = (R(1), G(1))$for a smoothly varying family of pairs $(R(t), G(t))$, and suppose that if $(R(t), G(t)) \sqsupseteq (R(1), G(1))$ for all $t \leq 1$. If $(R(t), G(t))$ is a transverse pair for each t then the proposition above clearly shows that the restriction map

$$\mathcal{R}^* : \mathcal{P}^i(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^i(X, \tfrac{R'}{G'})$$

is an isomorphism. On the other hand, one may verify by hand that this is also true for the two opposed but non-transverse families illustrated above. In fact, in each of these two examples, the inclusion of the topological pair $(X - R, (X - R) \cap G)$ into $(X - R', (X - R') \cap G')$ is a homotopy equivalence of pairs. The homotopy inverses can be viewed as follows:



Inclusion of topological pairs.



The image of a homotopy inverse to the inclusion.

Exercise 2.3. Carry out the general proof of the homotopy axiom. HINT: Replace the family of opposed pairs with a new family with the same beginning and ending pair, sueh that the new family has the following property:

The interval $[0,1]$ can be broken up into subintervals on each of which either the generalized Eheresman's theorem holds or on which the deformation is a generalization of the pictures above.

Exercise 2.4. Let $\rho : Z \longrightarrow Y$ be a fibration and let m be an integer. Show that the following is a Fary functor:

$$\mathcal{F}^i(X, \substack{R \\ G}) = H_{m-i}(\rho^{-1}(X-R), \rho^{-1}(G^0 \cap (X-R)); Q)$$

Exercise 2.5. Let $\rho : Z \longrightarrow Y$ be a covering projection. Show that the following is a perverse sheaf:

$$\mathcal{F}^i(X, \substack{R \\ G}) = H_{n/2-i}(\rho^{-1}(X-R), \rho^{-1}(G^0 \cap (X-R)); Q)$$

2.5 The dimension axiom

This section can be regarded as an extended exercise in the axioms of a perverse sheaf. The idea developed here is key to the proofs of some of the main theorems.

We want to investigate what the dimension axiom is asserting about a Fary functor. So let us assume that $\mathcal{F}$ is a Fary functor. Further, we consider, as in the case of the dimension axiom, a smooth proper function $f : X \longrightarrow \boldsymbol{R}$ with only one critical point $p \in X$ with critical value in the interval $[b,a]$, and we assume that f has a Morse singularity at p with signature σ and Morse index μ. The dimension axiom is about the calculation of $\mathcal{F}^i(X, \substack{X_{f\geq a} \\ X_{f\leq b}})$.

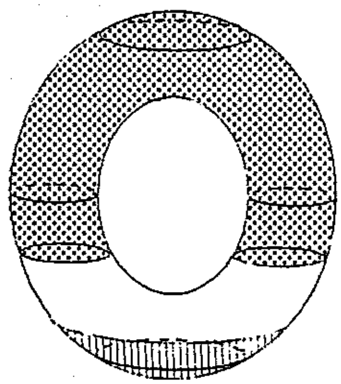
Theorem 2.8 *For any Fary functor $\mathcal{F}$,*

$$\mathcal{F}^i(X, \substack{X_{f\geq a} \\ X_{f\leq b}}) = \mathcal{F}^i(X, \substack{\sim B \\ S(\mu-1)})$$

where B is a n-ball embedded in X and $S(k)$ is a tubular neighborhood of a k-sphere embedded in B.

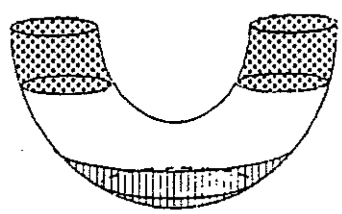
This theorem says that, instead of the millions of possible topological types of pairs $(X_{f\geq a}, X_{f\leq b})$ we have only to consider one topological type for each Morse index, namely $(\sim B, S(\mu-1))$.

Let's consider an example of this theorem. Take the first critical point on the torus with Morse index one. Then the pair $(X_{f\geq a}, X_{f\leq b})$ looks like this:



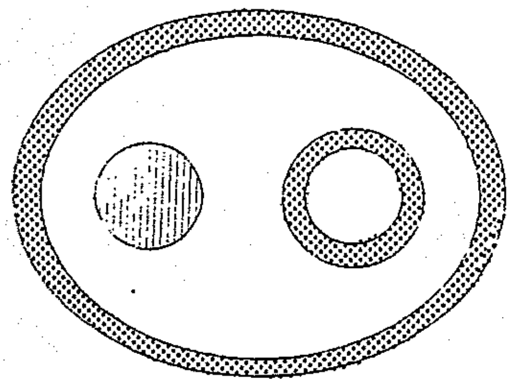
A pair $(X_{f\geq a}, X_{f\leq b})$ for the dimension axiom

All of the action is in the lower half of the torus, so we may consider it alone.



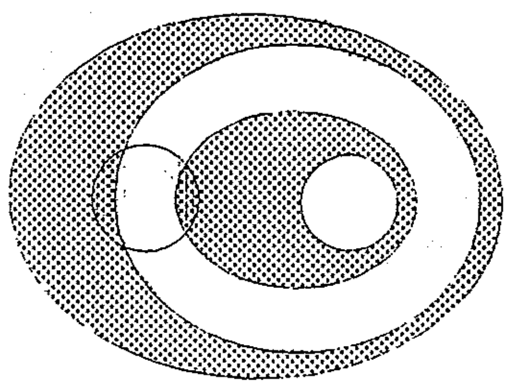
The lower half of the last picture

But this is just an annulus.

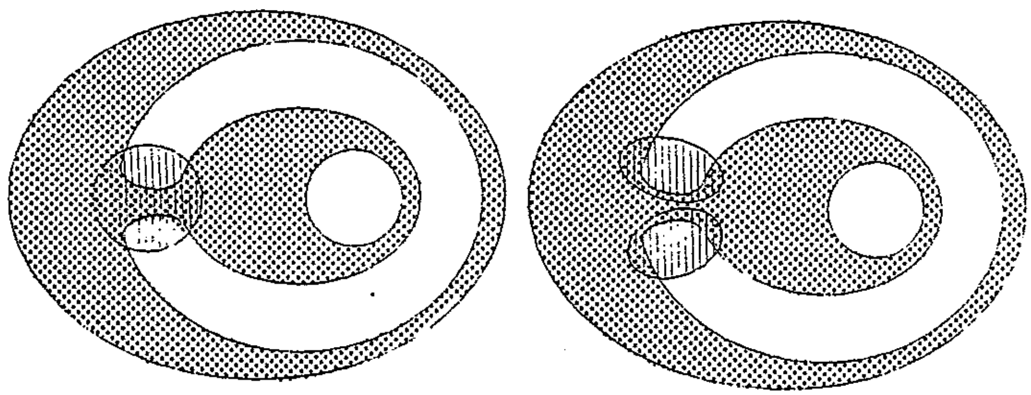


another view of the previous picture

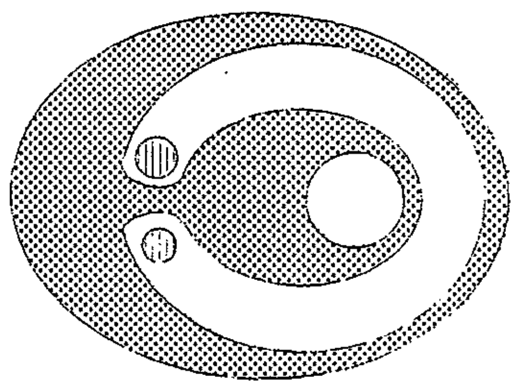
Now, let's apply the axioms to prove the theorem stated above in this case:



A covering by deformation of the previous picture



An excision applied to the previous picture



A covering by deformation of the previous picture

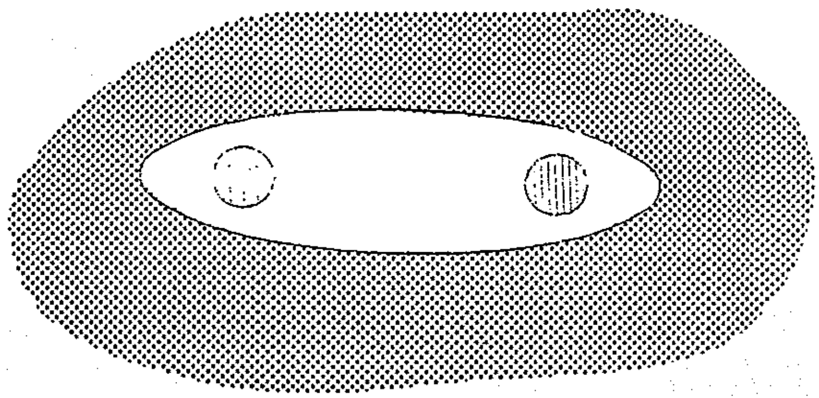
We have arrived: This pair is $(\sim B, S(\mu - 1))$, the outside of a 2-ball and a tubular neighborhood of a zero sphere.

Exercise 2.6. Construct a proof of the theorem in general, following the same sequence of steps as above.

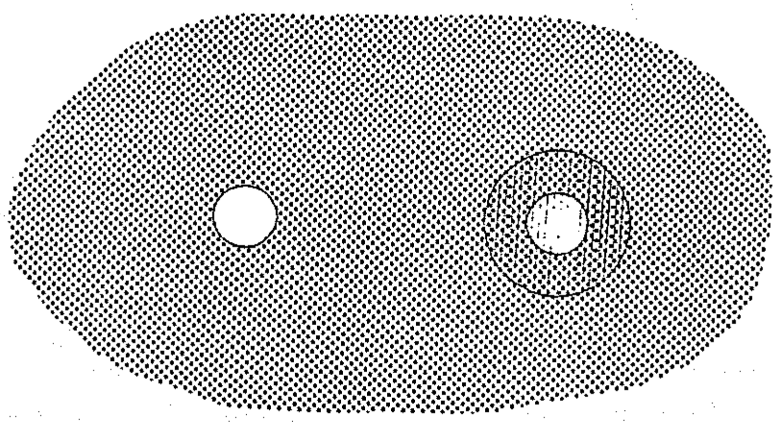
Theorem 2.9 *For any Fary functor $\mathcal{F}$,*

$$\mathcal{F}^i(X,\ {}^{\sim B}_{S(\mu-1)}) = \mathcal{F}^{i+\mu}(X,\ {}^{\sim B}_{\phi})$$

Let's prove this result in the situation where B is a 2-ball and μ is 1. We use the exactness axiom. Consider the following pairs:



$(\sim B, S(0))$



$(\sim S(0), G)$

Exercise 2.7. Show that $\mathcal{P}^i(X, {}^{\sim B}_{G}) = 0$ for all i. Deduce the theorem above in this case.

Exercise 2.8. Prove the theorem above in general. Imitate the calculation of the relative homomology of disk mod a sphere from topology.

Remark. One implication of the above development is that the axioms for a perverse sheaf are not independent or minimal. It would have been possible to replace the dimension axiom by a weaker axiom that requires the same statement for a single value of $\bar{\sigma}$. (Perhaps the most elegant choice would be $\bar{\sigma} = 0$.

Chapter 3

Monodromy and the Homotopy Covering Category

In this chapter, we build up some of the most important ideas associated to perverse sheaves (and Fary functors). These are the ideas related to monodromy. Monodromy is a map induced by a continuous deformation. First we define the monodromy maps associated to a smoothly varying family of opposed pairs. Then, we define the homotopy covering categroy, we interpret perverse sheaf as a functor on the homotopy covering category, and show that the monodromy map is induced by a map in the homotopy covering category.

The ideas of this chapter will be used not only for perverse sheavse on a manifold, but for perverse sheaves in general. The reason for introducing them in the special case of perverse sheaves on a manifold is that the concepts and proofs in general differ from the concepts and proofs in this specail case only by some routine stratified technology.

3.1 Monodromy.

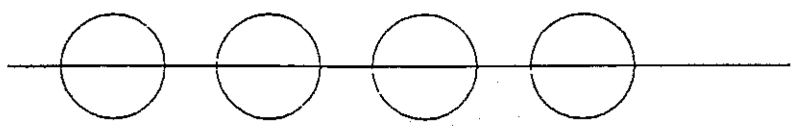
It is time to begin the theory of perverse sheaves on a manifold. Perhaps the most important construction is that of monodromy maps.

Suppose that we are given any smoothly varying family of opposed pairs $(R(t), G(t))$. (We make no assumptions that any of the pairs in the family covers any other.) Let us denote this parameterized family by the single symbol

σ. Suppose further that we are given any perverse sheaf $\mathcal{P}$ on X, not necessarily the homology perverse sheaf. In this section, we show that σ induces a map $\sigma^* : \mathcal{P}^i(X, {}^{R}_{G}) \longrightarrow \mathcal{P}^i(X, {}^{R'}_{G'})$ where $(R,G) = (R(0), G(0))$, and $(R', G') = (R(1), G(1))$. We call this map the *monodromy* map.

3.1.1 The idea

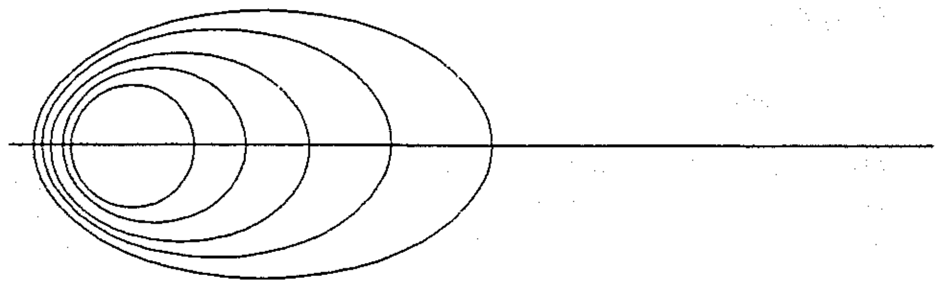
First, let's look at the intuitive idea. It is easier to visualize if we consider single smoothly enclosed sets rather than pairs. Consider a disk D_t that moves across the page horizontally from left to right.



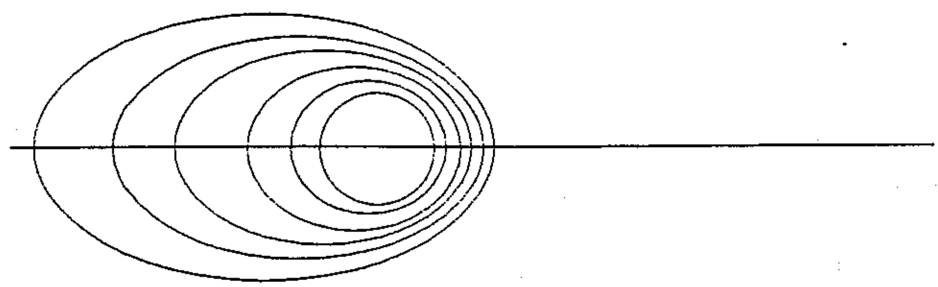
Successive positions of a moving disk D_t

We want to focus on the group $\mathcal{H}^1(X, {}^{\sim D_t}_{\phi}) = H_0(D_t)$. This is just generated by a point in the disk. It is easy see the intuitive idea of the monodromy map from the first disk to the last disk: just carry the point along as the disk moves. However, the axioms don't allow for this operation. Using the axioms, we can only compare the homology of two disks when one of them contains the other, and that doesn't happen here in this family at all.

The idea of the solution is to replace the disks by a family of baloons which alternately inflate and deflate as they move across the page. The motion looks like this:



The baloon inflates



The baloon deflates

Now in both the inflation and the deflation phases, there are maps between successive baloons. (In inflation, the maps go forward with time and in deflation, the maps go backward with time. Going backward doesn't cause any problems, because the maps are isomorphisms anyway.) In this way, we construct the monodromy map using only the abilities of a perverse sheaf.

3.1.2 The general construction

In order to construct the monodromy map, we choose a some data:

DEFINITION. *Monodromy data* for the family of opposed subsets $\sigma = (R(t), G(t))$ is the following:

- A partition of the interval $[0,1]$ into subintervals $[t_j, t_{j+1}]$ where $j \in \{0, 1, \ldots, k-1\}$ and $t_0 = 0$, $t_k = 1$, and $t_j \leq t_{j+1}$.
- For each $j \in \{0, 1, \ldots, k-1\}$, an opposed pair of subsets (R_j, G_j)such that for each $t \in [t_j, t_{j+1}]$, $(R(t), G(t))$ covers (R_j, G_j) by deformation.

Note that if for some $t \in [t_j, t_{j+1}]$, $(R(t), G(t))$ covers (R_j, G_j) by deformation, then for each $t \in [t_j, t_{j+1}]$, $(R(t), G(t))$ covers (R_j, G_j) by deformation.

Proposition 3.1 *Monodromy data exists for any smooth family σ of opposed subsets of X.*

Exercise 3.9. Prove this proposition.

DEFINITION. The monodromy map $\sigma^* : \mathcal{P}^i(X, {}^{R}_{G}) \longrightarrow \mathcal{P}^i(X, {}^{R'}_{G'})$ induced by the smooth family of opposed subsets σ and the monodromy data is the composition from top to bottom in the following diagram:

$$
\begin{array}{ccccc}
\mathcal{P}^i(X, {}^{R}_{G}) & = & \mathcal{P}^i(X, {}^{R(t_0)}_{G(t_0)}) & & \\
 & & & \searrow^{\approx} & \\
 & & & & \mathcal{P}^i(X, {}^{R_0}_{G_0}) \\
 & & & \swarrow^{\approx} & \\
 & & \mathcal{P}^i(X, {}^{R(t_1)}_{G(t_1)}) & & \\
 & & & \searrow^{\approx} & \\
 & & & & \mathcal{P}^i(X, {}^{R_1}_{G_1}) \\
 & & & \swarrow^{\approx} & \\
 & & \vdots & & \vdots \\
 & & & \swarrow^{\approx} & \\
 & & \mathcal{P}^i(X, {}^{R(t_{k-1})}_{G(t_{k-1})}) & & \\
 & & & \searrow^{\approx} & \\
 & & & & \mathcal{P}^i(X, {}^{R_{k-1}}_{G_{k-1}}) \\
 & & & \swarrow^{\approx} & \\
\mathcal{P}^i(X, {}^{R'}_{G'}) & = & \mathcal{P}^i(X, {}^{R(t_k)}_{G(t_k)}) & &
\end{array}
$$

Proposition 3.2 *The monodromy map σ^* is independent of the choice of the monodromy data, i.e. it depends only on the smooth family σ.*

The proof of this proposition will be deferred until some categorical developments are available. We want to define a category with the property that perverse sheaves are functors on that category, and such that the monodromy map is induced from a morphism in that category.

3.2 A homotopy category

The Eilenberg-Steenrod axioms for ordinary homology explicitly state that homology is functorial on the topological category: the category of topological spaces and continuous maps. However, one of the axioms is the homotopy axiom. This axiom guarantees that homology passes to a functor on the homotopy category, which is a sort of quotient category of the topological category.

In this section, we carry out a similar development for perverse sheaves. First we describe the category of pairs on which they are defined. Then we define the quotient category that the homotopy axiom guarantees that they pass to.

3.2.1 The category of opposed pairs and coverings

DEFINITION. The *category of opposed pairs and coverings* $C(X)$ is the following:

- The objects are opposed pairs (R, G) in X.
- Given two objects (R, G) and (R', G'), there is either one morphism or no morphisms from (R, G) to (R', G'). If $(R, G) \sqsupseteq (R', G')$, there is one morphism (called the *covering*); otherwise there are no morphisms.

The fact that $C_X$is a category is exactly the statement that $\sqsupseteq$ is a partial order relation. The functoriality axiom asserts that for any perverse sheaf $\mathcal{P}$, the symbol $\mathcal{P}^i(X, \,\vdots\,)$ is a functor from this category to finite dimensional vector spaces and linear transformation.

3.2.2 The homotopy category of opposed pairs

Let D be the set of coverings by deformation. It is a subset of the set of morphisms of C_X. The homotopy axiom says that coverings by deformation induce invertible maps, i.e. that the functor $\mathcal{P}^i(X, \,\vdots\,)$ takes any element of D to an invertible linear transformation.

Proposition 3.3 *There exists a category $C_X[D^{-1}]$ with a functor $F : C_X \longrightarrow C_X[D^{-1}]$ with the following universal property: For every functor $G : C_X \longrightarrow L$ such that for every morphism f in D, $G(f)$ is invertible, there exists a unique functor $E : C_X[D^{-1}] \longrightarrow E$ such that $E \circ F = G$.*

This universal property for $C_X[D^{-1}]$ is just the definition of the *localization* of the category C_X at the set of morphisms D. Its existence was proved, by actual construction, using the "calculus of fractions" by Gabriel and Zisman [GZ] p. 6.

DEFINITION. The ***homotopy category of opposed pairs*** is the category $C_X[D^{-1}]$, i.e. the category of opposed pairs and coverings localized at the set of coverings by deformation.

As usual, the homotopy category of opposed pairs is uniquely characterized by its universal property, and therefore the definition above is complete. However, it will be useful to have the actual construction of it from Gabriel and Zisman. We recall it (specialized to our case):

CONSTRUCTION. The homotopy category of pairs $C_X[D^{-1}]$ is constructed as follows: The objects are opposed pairs (R, G) in X. For brevity in this construction, we will notate such a pair by A or A_i. A morphism from A to A' is an equivalence classes of strings of opposed pairs of the following type

$$A = A_1 \sqsupseteq A_2 \overset{D}{\sqsubseteq} A_3 \overset{D}{\sqsubseteq} A_4 \sqsupseteq A_5 \sqsupseteq A_6 \overset{D}{\sqsubseteq} A_7 \sqsupseteq A_8 \overset{D}{\sqsubseteq} A_9 \overset{D}{\sqsubseteq} A_{10} \sqsupseteq A_{11} = A'$$

where for every two adjacent opposed pairs A_i and A_{i+1} in the string, either A_i covers A_{i+1} (either by deformation or not), or else A_{i+1} covers A_i by deformation (notated $A_i \overset{D}{\sqsubseteq} A_{i+1}$. We don't exclude that $A_i = A_{i+1}$ (in which case both are true). We also don't exclude the case that the string has only one pair A, and no coverings, in it.

The equivalence relation is generated by the following elementary equivalence relations, which are to be applied on segments of the strings. (There is some redundancy in this list.)

- The substring $A = A$ is equivalent to the substring A.
- The substring $A \overset{D}{\sqsubseteq} A_1 \sqsupseteq A$ is equivalent to $A = A$.
- The substring $A \sqsupseteq A_1 \overset{D}{\sqsubseteq} A$ is equivalent to $A = A$.
- The substring $A_1 \sqsupseteq A_2 \sqsupseteq A_3$ is equivalent to $A_1 \sqsupseteq A_3$.
- The substring $A_1 \overset{D}{\sqsubseteq} A_2 \overset{D}{\sqsubseteq} A_3$ is equivalent to $A_1 \overset{D}{\sqsubseteq} A_3$.
- The substring $A_1 \overset{D}{\sqsubseteq} A \sqsupseteq A_2$ is equivalent to $A_1 \sqsupseteq A' \overset{D}{\sqsubseteq} A_2$.

The composition in the category is induced by gluing strings together.

$$\left(A_6 \overset{D}{\sqsubseteq} A_7 \sqsupseteq A_8 \overset{D}{\sqsubseteq} A_9 \overset{D}{\sqsubseteq} A_{10} \sqsupseteq A_{11}\right) \circ \left(A_1 \sqsupseteq A_2 \overset{D}{\sqsubseteq} A_3 \overset{D}{\sqsubseteq} A_4 \sqsupseteq A_5 \sqsupseteq A_6\right) =$$

$$= A_1 \sqsupseteq A_2 \overset{D}{\sqsubseteq} A_3 \overset{D}{\sqsubseteq} A_4 \sqsupseteq A_5 \sqsupseteq A_6 \overset{D}{\sqsubseteq} A_7 \sqsupseteq A_8 \overset{D}{\sqsubseteq} A_9 \overset{D}{\sqsubseteq} A_{10} \sqsupseteq A_{11}$$

The identity of an object A in the category is the string A (or $A = A$) itself.

Proposition 3.4 *The functor $\mathcal{P}^i(X, \,\bullet\,)$ passes to a functor on the homotopy category of opposed pairs.*

This is true, of course, by the homotopy axiom and the universal property of $C_X[D^{-1}]$.

Exercise 3.10. Give an alternative proof of this proposition by showing directly that the functor $\mathcal{P}^i(X, \,\bullet\,)$ induces a map on a string as in the construction of $C_X[D^{-1}]$, and that if two strings are equivalent, then the two induced maps are equal.

The category of opposed pairs and coverings has at most one morphism between two given objects. This is not true for its localization, the homotopy category of opposed pairs, as shown by the example studied in the next section.

3.3 Back to monodromy

Theorem 3.5 *Any deformation σ of opposed pairs induces a morphism $\tilde{\sigma}$ in the homotopy category of opposed pairs. The map induced by $\tilde{\sigma}$ on the functor $\mathcal{P}^i(X, \,\bullet\,)$ is the monodromy homomorphism σ^*. The morphism $\tilde{\sigma}$ is an isomorphism.*

PROOF-CONSTRUCTION. In fact, the choice of monodromy data for σ gives the following morphism $\tilde{\sigma}$ in the homotopy category of opposed pairs

$$(R, G) = R(t_0), G(t_0)) \overset{D}{\sqsubseteq} (R_0, G_0) \sqsupseteq R(t_1), G(t_1)) \overset{D}{\sqsubseteq} (R_1, G_1) \sqsupseteq \cdots$$

$$\cdots \overset{D}{\sqsubseteq} (R_{k-1}, G_{k-1}) \sqsupseteq R(t_k), G(t_k)) = (R', G')$$

The definition of the monoromy morphism was exactly the morphism induced by $\tilde{\sigma}$

We now come back to the proof of the independence of the monodromy map form the choice of the monodromy data. This proof was postponed from the last section.

DEFINITION. A choice of monodromy data D *refines* another choice D_1 under these conditions:

- Each subinterval S of $[0, 1]$ in the partition for D is contained entirely within a single subinterval S_1 of the partition for D_1 .

- The opposed pair of subspaces indexed by S given by D covers the opposed pair indexed by S_1 given by D_1.

Exercise 3.11. Show that if D refines D_1, then the lift $\tilde{\sigma}_1$ of σ to the homotopy covering category induced by D is the same as the lift $\tilde{\sigma}_2$ of σ to the homotopy covering category induced by D_1.

Exercise 3.12. Show that, given any two choices of monodromy data D_1 and D_2 for σ, there exists a choice of monodromy data D which refines both D_1 and D_2.

3.3.1 Composition of monodromy maps

If $\sigma = (R(t), G(t))$ and $\sigma' = (R'(t), G'(t))$ are two smooth families of opposed subsets of X, then we can define the composed family $\sigma \circ \sigma'$ as usual by executing first σ then σ'. Namely $\sigma \circ \sigma' = (R(2t), G(2t))$ if $t \leq 1/2$ and $\sigma \circ \sigma' = (R'(2t-1), G'(2t-1))$ if $t \geq 1/2$. We define a constant family to be the family $(R(t), G(t)) = (R, G)$.

Proposition 3.6 *$(\tilde{\sigma} \circ \tilde{\sigma}') = \widetilde{\sigma' \circ \sigma}$ so $(\sigma \circ \sigma')^* = \sigma'^* \circ \sigma^*$. If σ is a constant family, then $\tilde{\sigma}$ is the identity, so σ^* is the identity.*

3.3.2 Homotopy invariance of monodromy.

A smoothly varying family σ of opposed pairs may be thought of intuitively as a smooth paths in the "manifold" of all opposed pairs. We want to study homotopies between two such paths with the same end points. Instead of attempting to make the "manifold" of all opposed paths rigorous, we will proceed as follows:

DEFINITION. An *n-parameter smoothly varying family* of smoothly enclosed subsets of X is a family $R(t)$ for $t \in \boldsymbol{R}^n$ of subsets such that there is a smooth function $X \times \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ notated $(x, t) \mapsto f_t(x)$, with no critical points having critical value a, such that $R(t) = f_t^{-1}((-\infty, a]) = X_{f_t \leq a}$. An *n-parameter family of opposed pairs* in X is a pair $(R(t), G(t))$ of n-parameter smoothly varying families of smoothly enclosed subsets such that $R(t)$ and $G(t)$ are opposed for each value to t.

Now suppose we have two (1-parameter) families $\sigma = (R(t), G(t))$ and $\sigma' = (R'(t), G'(t))$ of opposed pairs such that $(R(0), G(0)) = (R(0), G(0)) = (R, G)$ and $(R(1), G(1)) = (R(1), G(1)) = (R', G')$.

DEFINITION. A *homotopy* between σ and σ' is a 2-parameter family $(\bar{R}(u), \bar{G}(u))$ of opposed pairs together with two maps $f, f' : \boldsymbol{R} \longrightarrow \boldsymbol{R}^2$ such that

$f(0) = f'(0), f(1) = f'(1), (R(t), G(t)) = (\tilde{R}(f(t)), \tilde{G}(f(t)))$, and $(R'(t), G'(t)) = (\tilde{R}(f'(t)), \tilde{G}(f'(t)))$. The families σ and σ' are said to be ***homotopic*** if there exists a homotopy between them.

Proposition 3.7 *If σ and σ' are homotopic, then $\tilde{\sigma} = \tilde{\sigma'}$ so $\sigma^* = \sigma'^*$.*

Exercise 3.13. Prove this proposition. HINT: The homotopy between σ and σ' may be regarded as a map from a two dimensional space into the "manifold" of opposed pairs. Choose a fine cell decomposition of that two dimensional space.

3.4 The local system associated to a perverse sheaf on a manifold.

The most basic application of the monodromy map is the construction of a local system $L(\mathcal{P})$ over X for any perverse sheaf $\mathcal{P}$ on X. We will carry out this construction in this chapter.

3.4.1 The associated local system

First, we choose a family of balls B_x one for each $x \in X$ such that "B_x is centered at x, and it varies smoothly with x". The usual way to do this is to choose a Riemannian metric on X and to choose a smooth function $\epsilon(x) > 0$. Then B_x is the set of all points of X of distance $\leq \epsilon(x)$ from x as measured by the metric. If $\epsilon(x)$ is chosen small enough, then for any $x \in X$, B_x will be homeomorphic to a ball, and for any smooth map $f : \mathbf{R}^k \longrightarrow X$, the family $B(t) = B_{f(t)}$ will be a smooth family of smoothly enclosed subsets of X.

Next, we define a functor F from the fundamental groupoid of X to the homotopy covering category of X as follows: For any smooth path $f : \mathbf{R}^k \longrightarrow X$, the functor F applied to f will be $\tilde{\sigma}$ where σ is the family $(R(t), G(t)) = (\sim B_{f(t)}, \phi)$, the pair consisting of the complement of the ball $B(t)$ and the empty set.

Exercise 3.14. Show that this is a functor. Show that if we choose two different families of balls with the required properties, then the two functors constructed using these two families are naturally equivalent. (If you want to refresh your memory on what a natural equivalence of functors is, you may find it in any category theory book.)

Recall from appendix 7.2 that a local system on X can be defined as functor from the edge path groupoid of X to the category of finite dimensional vector spaces.

DEFINITION. If $\mathcal{P}$ is any perverse sheaf on a manifold X (of real dimension n), then the associated local system is the functor $\mathcal{P}^{n/2} \circ F$, where F is the functor from the fundamental groupoid to the homotopy covering category defined above.

Exercise 3.15. Show that the associated local system is functorial, i.e. that it is the map on objects of a functor $\mathbf{F}$ from the category of perverse sheaves to the category of local systems.

3.4.2 A generalization

It will be convenient later to have a generalization of this construction which constructs local systems for more general families of opposed pairs in X.

DEFINITION. A smooth family $(R(s), G(s))$, $s \in S$ of opposed pairs parameterized by a manifold S is a pair $(\tilde{R}, \tilde{G})$ of smoothly enclosed subsets of $S \times X$ such that for each s in S, the pair $(\pi_1^{-1}(s) \cap \tilde{R}, \pi_1^{-1}(s) \cap \tilde{G}) = (R(s), G(s))$ is an opposed pair in X, where π_1 is projection on the first factor.

Now, we define a functor F from the fundamental groupoid of S to the homotopy covering category of X as follows: For any smooth path $f : \mathbf{R}^k \longrightarrow S$, the functor F applied to f will be $\tilde{\sigma}$ where σ is the family $(R(f(t)), G((f(t)))$

Exercise 3.16. Show that this is a functor.

DEFINITION. For each integer i, the i^{th} local system induced by the smooth family $(R(s), G(s))$, $s \in S$ of opposed pairs parameterized by S is the functor $\mathcal{P}^i \circ F$, where F is the functor from the fundamental groupoid to the homotopy covering category defined above.

The fiber of this local system of this local system over $s \in S$ is $\mathcal{P}^i(X, \frac{R(s)}{G(s)})$.

3.4.3 The structure theorem for the category of perverse sheaves on a manifold

Recall the definition of homology with coefficients in a local system, from appendix 3.

DEFINITION. If L is a local system over X, then the *ordinary homology perverse sheaf with coefficients in* L, notated $\mathcal{L}$ is defined by the equation $\mathcal{L}^i(X, \frac{R}{G}) = H_{n-i}(X - R, G \cap (X - R); L)$.

Exercise 3.17. Show that is a functor $\mathbf{G}$ which associates to every local system L the homology perverse sheaf $\mathcal{L}$ with coefficients in L.

As promised in the introduction to this chapter, the category of perverse sheaves on a manifold is equivalent to an *a priori* much simpler category. In fact, every perverse sheaf on a manifold is a homology perverse sheaf with coefficients in some local system L.

Theorem 3.8 *The category of perverse sheaves on a manifold X is equivalent to the category of local systems on X. The functors* **F** *from perverse sheaves to local systems and* **G** *from local systems to perverse sheaves (described above) provide the equivalence.*

PROOF: The following exercises.

Exercise 3.18. Show that $\mathbf{F} \circ \mathbf{G}$ is the identity.

So the essential point is to find a natural transformation from $\mathbf{G} \circ \mathbf{F}$ to the identity. (At first sight, $\mathbf{G} \circ \mathbf{F}$ and the identity appear to have little to do with each other.)

Exercise 3.19. Imitate the technique of section 2.5 to show the following: Suppose $f : X \longrightarrow R$ is a smooth proper function with only one critical point $p \in X$ with critical value in the interval $[b, a]$, and assume that f has a Morse singularity at p . Then $\mathcal{L}^{-\bar{\sigma}}(X, \genfrac{}{}{0pt}{}{X_{f \geq a}}{X_{\leq b}})$ may be canonically identified with L_p where $\bar{\sigma}$ is on half of the signature of the Hessian quadratic form of F at p. (A priori, the identification is only well defined up to sign. In order to fix the sign, it is necessary to fix an "orientation" of the handle for f at p.)

Exercise 3.20. Now complete the proof, using the small complex defined in section 4.7 of the next chapter.

3.5 *Some remarks on the homotopy covering category.

In this section, we develop a little of the theory of the homotopy covering category. This material will not be used elsewhere in these notes. The object is to show that the homotopy covering category is a rich structure, and perhaps that it is worth being studied further.

3.5.1 Sets of small dimension.

DEFINITION. A smoothly enclosed subset A of X is said to have *Morse dimension* $\leq d$ if it is compact and there exists a Morse function $f : X \longrightarrow R$ with the property that $A = X_{f \leq a}$ and all of the critical points of f with critical values $\leq a$ have Morse index $\leq d$.

Exercise 3.21. Show, using Morse theory, that a smoothly enclosed subset of Morse dimension $\leq d$ has the homotopy type of a finite CW-complex of dimension $\leq d$.

I suggest that the reader who is not familiar with embedding techniques for finite CW-complexes should assume that $d = 1$ in the following series of exercises. Then a CW-complex of dimension d is just a finite graph.

Let X be a manifold and let R be a fixed smoothly enclosed subset. Let $C_d(X, R)$ be the full subcategory of the homotopy covering category of X whose objects are (R, G) where G has Morse dimension d, and R and G are disjoint.

Exercise 3.22. Show that if the dimension of X is greater than $2d$, and if R is the complement of a disk in X, then $C_d(X, R)$ is equivalent to the category of CW-complexes of dimension $\leq d$ and homotopy classes of maps.

So, for X of large enough dimension, finite homotopy theory can be embedded within the homotopy covering category.

Exercise 3.23. Let X be a compact $K(\pi, 1)$ manifold of dimension $> 2d$, and let G be a set of Morse dimension $\leq d$. Let $\mathcal{A}$ be the automorphism group of (ϕ, G) considered as an object of the homotopy covering category of X. Let A be the automorphism group of G in the category of finite CW-complexes and homotopy classes of maps. Then we have a short exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{A} \longrightarrow \pi \longrightarrow 0$$

Exercise 3.24. Show that if X is any compact manifold of dimension $> d$, and G has Morse dimension $\leq d$, then the automorphism group of (ϕ, G) in $C_d(X, \phi)$ has a subgroup $\mathcal{A}$ which fits into an exact sequence as above. If X is the 2-sphere and G is the annulus, show that $\mathcal{A}$ is not the whole automorphism group of (ϕ, G) in $C_d(X, \phi)$ by displaying an automorphism a not in $\mathcal{A}$.

3.5.2 Fary functors

Exercise 3.25. Show that any for Fary functor $\mathcal{F}$, $\mathcal{F}^i$passes to a functor on the homotopy covering category.

Exercise 3.26. Take X to be the 2-sphere and G to be the annulus, as above. Give an example of a Fary functor and an integer i so that the $\mathcal{F}^i$ takes every automorphism in $\mathcal{A}$ of (ϕ, G) to the identity automorphism, but it takes that automorphism a to an element different from the identity. HINT: Use the Fary functor that comes from the Hopf map from the 3-sphere to the 2-sphere.

Exercise 3.27. Show that the phenomenon of the last exercise cannot happen if the Fary functor in question is a perverse sheaf. (In fact, every perverse sheaf on X takes every automorphism of (ϕ, G) to the identity automorphism.

3.5.3 Further localization

The category that is of most fundamental interest for perverse sheaves and Fary functors is not $C[D^{-1}]$, the homotopy covering category, but rather a further localized category.

Exercise 3.28. Show that for any Fary functor $\mathcal{F}$, the functor $\mathcal{F}^i$ passes to a functor on $C[D^{-1}][E^{-1}]$ where E is the set of maps which induce isomorphisms by reason of excision.

The most mathematically natural definition of a perverse sheaf would start with the notion of a functor on this category. æ

Chapter 4

Perverse Sheaves

In this chapter, we finally come to the general definition of a perverse sheaf. Organizationally, we will follow the structure of Chapter 2 . The reason is to emphasize that the notion of a perverse sheaf is a very natural outgrowth of an attempt to generalize Morse theory to stratified spaces.

Notational conventions. Throughout this chapter, X will be a complex analytic manifold. We will denote by n the real dimension of X, so its complex dimension is $n/2$. We will assume given a Whitney stratification $X = \bigcup X_\alpha$ of X by complex analytic submanifolds.

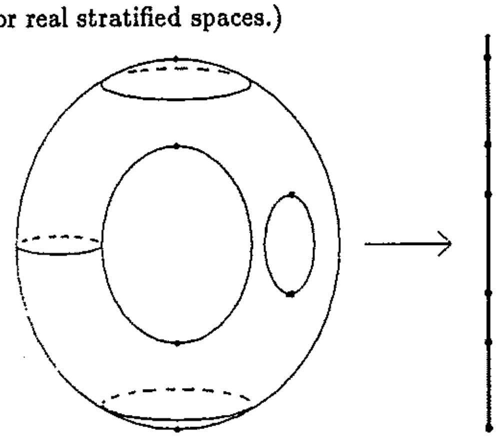
4.1 Stratified Morse Theory

There is a very natural extension of Classical Morse Theory on a manifold to a theory on a stratified space. This extension is called Stratified Morse Theory. It is treated in full in [GM6]. The following is a summary of some of the most salient points.

4.1.1 What is a critical point?

DEFINITION. If $f : X \longrightarrow \mathbf{R}$ is a smooth real valued function on X, then a *critical point* of f is a point x in a stratum $X_\alpha \subseteq X$ such that all of the partial derivatives of the restriction of f to X_α vanish at x, i.e. $d(f|X_\alpha)(x) = 0$.

This definition of a critical point depends on the stratification; if X is restratified, then the set of critical points will change. For example, if we stratify our torus by making a circle on it one stratum and making the rest of it the other stratum, then we have six critical points. (This is a real stratification, not a complex one. The definitions of critical points and Morse singularities work equally well for real stratified spaces.)



Critical points of Stratified Morse Theory.

Also, any zero dimensional stratum of X is automatically a critical point by this definition, no matter what we choose for f.

A *critical value* of f is a real number $v \in \mathbf{R}$ such that $v = f(x)$ for some critical point x.

Just as in classical Morse theory, if $c \in \mathbf{R}$ is any real number, we define the *truncation* of X by f at c, notated $X_{f<c}$ to be the subset of X where f takes values less than or equal to c, i.e. $X_{f<c} = f^{-1}((-\infty, c))$. If c is not a critical value of f, then $X_{f<c}$ is the interior of a smooth manifold with smooth boundary. The boundary lies in X in a way that is transverse to all of the strata X_α of X.

4.1.2 What happens between the critical values?

As with classical Morse theory, we want to study how the homology groups of $X_{f<c}$ changes as c varies. Now, however, we are interested also in intersection homology groups.

Theorem 4.1 (SMT Part A). *Choose any stratum X_α of X, choose any local system L over X_α, and choose any perversity p. Let Y be the closure of X_α, and let $Y_0 = X_\alpha$. Suppose that $f : X \longrightarrow \mathbf{R}$ is smooth and $f|Y$ is proper. As c varies in the open interval between two adjacent critical values, the intersection* *homology groups of $X_{f<c} \cap Y$ with coefficients in L remain constant. More specifically, if $b < c$ and if there is no critical value in the closed interval $[b, c]$, then*

$$I^pH_i(X_{f<c} \cap Y, X_{f<b} \cap Y; L) = 0$$

So, by the long exact sequence for homology, the map induced by inclusion

$$I^pH_i(X_{f<b} \cap Y; L) \longrightarrow I^pH_i(X_{f<c} \cap Y; L)$$

is an isomorphism.

In fact, $X_{f<b}$ and $X_{f<c}$ are homeomorphic by a stratum preserving homeomorphism, so this theorem follows from the topological invariance of intersection homology. This part of SMT holds for any Whitney stratified space, whether or not it is complex analytic.

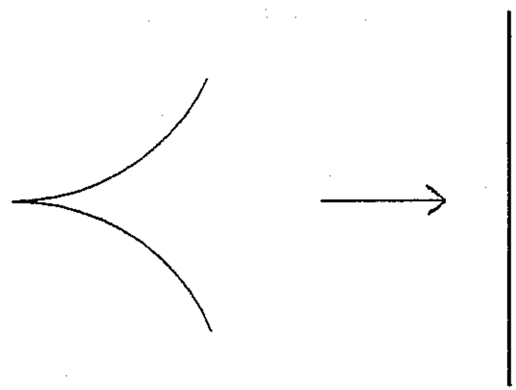
4.1.3 What is a Morse singularity?

Now we need the analogue of the Morse Index of a function. Consider a critical point $x \in X_\alpha$ of a smooth function $f : X \longrightarrow \mathbf{R}$. The Hessian $\mathcal{H}$ of f at x is the Hessian of $f|X_\alpha$, the restriction of f to the stratum X_α.

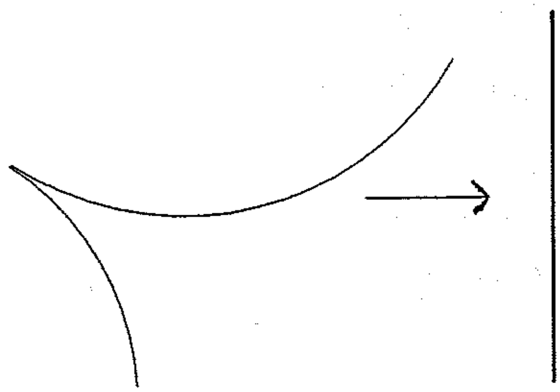
DEFINITION. The function f is said to have a *Morse singularity* at x if:

- The Hessian quadratic form of f at x is nondegenerate (in other words, if $f|X_\alpha$ has a Morse singularity at x in the classical sense), and
- For any stratum X_β containing X_α in its closure, and for any sequence of points $x_1, x_2, x_3, \ldots$ in X_β converging to x such that the limit of the sequence of tangent spaces $\lim_{i\to\infty} T_{x_i}X_\beta = \tau \subseteq T_xX$ exists, we have that the covector df does not annihilate τ, i.e. $df|\tau \neq 0$.

So the definition of a Morse singularity has two conditions, one restricting the behavior of f along the stratum containing x and the other restricting the behavior of f "normal to" the stratum containing x. This second condition being less familiar, we illustrate it with a pair of pictures of a stratification of a real two dimensional manifold X represented by the plane of the page.



The singularity of this height function at the point stratum is not Morse



The singularity of this height function at the point stratum is Morse

4.1.4 What happens at a critical value?

DEFINITION. The *signature* of f at x is the signature of the Hessian $\sigma = \sigma(\mathcal{H})$. The *Morse Index* of f at x is the Morse index of the Hessian $\mu = \mu(\mathcal{H})$ (see section 2.1 for the definitions of these). The signature σ being even, we can also define the half signature $\hat{\sigma} = \sigma/2$.

Theorem 4.2 (SMT Part B). *Let p be a perversity that is close to middle. Suppose that $f : X \longrightarrow \mathbf{R}$ is smooth and $f|Y$ is proper. If $b < c$ are not critical values, if there is only one critical point x with critical value v in the closed interval $[b, c]$, and if f has a Morse singularity at x with signature σ, then*

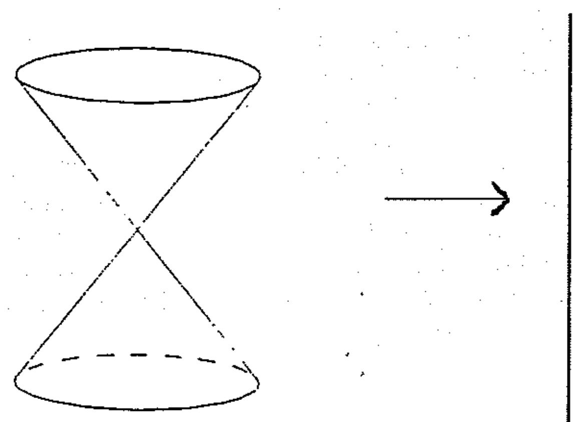
$$I^p H_i(X_{f<c} \cap Y, X_{f<b} \cap Y; L) = 0 \text{ if } i \neq (\dim Y + \sigma)/2$$

So, by the long exact sequence for homology, we have that as a varies from b to c, the degree $(\dim Y + \sigma)/2$ *intersection homology Betti number of* $X_{f<a} \cap Y$ *may be increased, and the degree* $(\dim Y + \sigma)/2 - 1$ *Betti number of* $X_{f<a} \cap Y$ *may be decreased, and all of the others will be unchanged.*

Remarks. The main content of the theorem is that there is only one degree i in which $I^pH_i(X_{f<c} \cap Y, X_{f<b} \cap Y; L)$ can be nonzero. This is a strict analogue of classical Morse theory. This result would be false without the assumption that the perversity p is close to middle. The degree in which it can be nonzero is most conveniently expressed in terms of the signature σ of the Hessian. It could, of course, also be expressed in terms of the Morse index μ since $(\dim Y + \sigma)/2 = (\dim Y - \dim X_\alpha)/2 + \mu$, however this expression is less elegant since it refers explicitly to the dimension of the stratum X_α containing x.

The group $I^pH_{(\dim Y+\sigma)/2}(X_{f<c} \cap Y, X_{f<b} \cap Y; L)$ may have any dimension as a rational vector space. It is a very interesting invariant of the singularity of Y at x (and the singularity of the local system). So in this way the theorem is not an exact analogue of classical Morse theory.

Exercise 4.1. Show that part B of SMT is false for real algebraic stratified spaces, even if they have a real algebraic stratification with only even codimensional strata. Consider the example of the classical cone $x^2 + y^2 = z^2$ where the function f is projection onto the z-axis. (Consider all three close to middle perversities.)



SMT Part B is false for this space and this Morse function

Exercise 4.2. The topological type of the singularity in the example of the cone above is the same as the topological type of the singularity of Example 1 of Chapter 1. Yet example 1 of Chapter 1 is homeomorphic to a complex algebraic subvariety of the complex projective plane (it is given by $(x)(x^2 + y^2 + w^2) = 0$ with respect to homogeneous coordinates). So SMT Part B should hold. How do you explain the discrepancy?

Exercise 4.3. Show that part B of SMT is false for perversities that are not close to middle, even if Y is a complex algebraic variety. Consider the example of a nonsingular algebraic surface Y (it has real dimension 4) and a Morse function f which has no critical point at $x \in Y$. Now restratify Y so that $\Sigma = Y_1 = x$, without changing the f. Use the perversities $p(Y_1) = -1$ or $p(Y_1) = 3$, which are just beyond the close to middle range.

As with classical Morse theory, there is much more to be said. Functions with only Morse singularities are called *Morse functions.* Morse functions in this sense also form an open dense set in the space of all real valued functions, with the appropriate topology, so any function can be approximated by a Morse function. Again, Morse functions with distinct critical values are the structurally stable functions, i.e. the functions with the property that sufficiently nearby functions have the same topological type, where topological type includes data about the stratification.

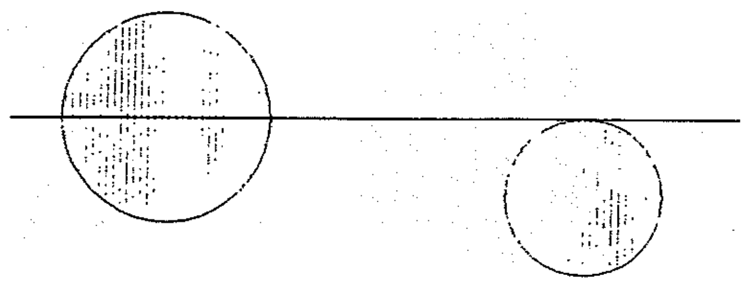
4.2 Opposed pairs of smoothly enclosed subsets.

4.2.1 Smoothly enclosed subsets.

The reader interested in quickly finding out what a perverse sheaf is may begin reading here (after reviewing the notational conventions at the beginning of the chapter).

DEFINITION. A subset of X is called *smoothly enclosed* if it is of the form $X_{f\leq c}$ for some smooth function f and for some value c which is not a critical value of f. We will call the choice of such an f and such a c a *presentation* of the smoothly enclosed set. If $G = X_{f\leq c}$ is a smoothly enclosed subset of X, then the interior of G is notated G^0. If $G = X_{f\leq c}$ is a presentation of G, then $G^0 = X_{f<c}$, the truncation of X by f at c. Smoothly enclosed subsets of X are just n dimensional submanifolds with boundary whose boundary is smooth and is transverse to the all of the strata X_α of X. In the following example, X is

the plane of the page and the horizontal line is a stratum of X



A smoothly enclosed subset   Not a smoothly enclosed subset

If R is a smoothly enclosed subset of X, then the closure of the complement of R, is denoted by $\sim R$ and called the *complimentary* smoothly enclosed set. For example, if $R = X_{f \leq c}$, then $\sim R$ is the subset of X where f takes values greater than or equal to c, which we denote by $X_{f \geq c}$. The set $\sim R$ is also smoothly enclosed, since $\sim R = X_{-f \leq -c}$.

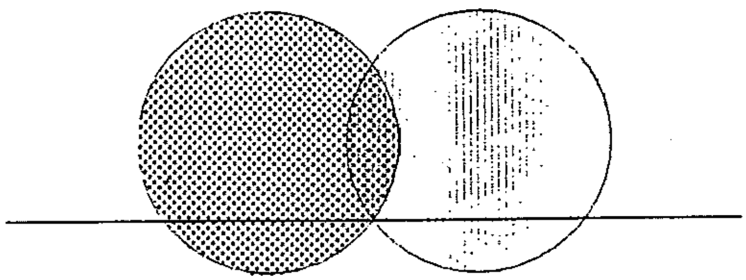
DEFINITION. A *smoothly varying family* of smoothly enclosed subsets of X is a family $R(t)$ for $t \in \boldsymbol{R}$ of subsets such that there is a smooth function $X \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ notated $(x,t) \mapsto f_t(x)$ with no critical points having critical value a, such that $R(t) = f_t^{-1}((-\infty, a]) = X_{f_t \leq a}$.

4.2.2 Opposed pairs

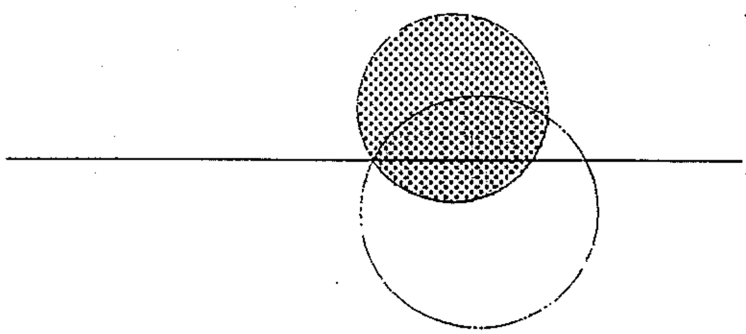
DEFINITION. Two smoothly enclosed subsets R and G are said to be *opposed* under the following conditions:

- Suppose $R = X_{f \leq a}$ and $G = X_{g \leq b}$. Then for no $p \in X$ does it happen that $f(p) = a$, $g(p) = b$, and $d(f|X_\alpha)(p)$ is a positive multiple of $d(g|X_\alpha)(p)$, where X_α is the stratum of X containing p.
- The subset $X - (R^0 \cup G^0)$, called the *support* of the opposed pair, is compact.

In the following examples, we assume that the horizontal line is a stratum X_α of X



These sets R and G are opposed



These sets R and G are not opposed

DEFINITION. Suppose that (R, G) and (R', G') are two opposed pairs in X. We say that (R, G) *covers* (R', G') , and we write $(R, G) \sqsupseteq (R', G')$ if both $R \supseteq R'$ and $G \subseteq G'$.

DEFINITION. A *smoothly varying family* of opposed pairs in X is a family $(R(t), G(t)), t \in \mathbf{R}$ such that $R(t)$ and $G(t)$ separately are smoothly varying families of smoothly enclosed subsets of X and such that for all t, the pair $(R(t), G(t))$ is opposed.

DEFINITION. Suppose that (R, G) and (R', G') are two opposed pairs in X. We say that (R, G) covers (R', G') *by deformation* if there is a smoothly varying family of opposed pairs in X is a family $(R(t), G(t)), t \in \boldsymbol{R}$ such that $(R, G) = (R(0), G(0))$, $(R', G') = (R(1), G(1))$, and $(R(t), G(t)) \sqsupseteq (R(1), G(1))$ for all $t \leq 1$.

4.3 Definition of a perverse sheaf

As always in this chapter, we have selected a stratification of X. The perverse sheaves that we are defining here are usually called perverse sheaves on X *constructible with respect to the stratification.* This is the general perverse sheaf, because every perverse sheaf is constructible with respect to some stratification.

4.3.1 The definition

DEFINITION. A *Perverse sheaf* on a stratified complex manifold X is a device $\mathcal{P}$ which does the following three things:

- $\mathcal{P}$ assigns to each opposed pair (R, G) in X and to each integer i a finite dimensional vector space over the rationals $\mathcal{P}^i(X, \,^R_G)$.
- Whenever (R, G) and (R', G') are two opposed pairs in X such that $(R, G) \sqsupseteq (R', G')$, then $\mathcal{P}$ gives a map denoted $\mathcal{R}^*$

  $$\mathcal{R}^* : \mathcal{P}^i(X, \,^R_G) \longrightarrow \mathcal{P}^i(X, \,^{R'}_{G'})$$

  called the *restriction* map.
- Whenever (R, C) and $(\sim C, G)$ are opposed subsets of X, $\mathcal{P}$ gives a map denoted ∂^*

  $$\partial^* : \mathcal{P}^i(X, \,^R_C) \longrightarrow \mathcal{P}^{i+1}(X, \,^{\sim C}_G)$$

  called the *coboundary homomorphism.*

The device $\mathcal{P}$ will qualify as a perverse sheaf if it satisfies the following axioms called the *modified Eilenberg - Steenrod axioms*:

- **Functoriality.** For any opposed pair (R, G), the map

  $$\mathcal{R}^* : \mathcal{P}^i(X, \,^R_G) \longrightarrow \mathcal{P}^i(X, \,^R_G)$$

  is an isomorphism. For any triple of opposed pairs (R, G), (R', G'), and (R'', G'') such that $(R, G) \sqsupseteq (R', G') \sqsupseteq (R'', G'')$, we have a diagram of restriction maps which commutes:

$$\begin{array}{ccccc} & & \mathcal{P}^i(X, \frac{R'}{G'}) & & \\ & \nearrow & & \searrow & \\ \mathcal{P}^i(X, \frac{R}{G}) & & \longrightarrow & & \mathcal{P}^i(X, \frac{R''}{G''}) \end{array}$$

- **Naturality.** Whenever (R,G), $(\sim C,G)$, (R',G'), and $(\sim C',G')$ are all opposed pairs, and $(R,C) \sqsupseteq (R',C')$ and $(\sim C,G) \sqsupseteq (\sim C',G')$then the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}^i(X, \frac{R}{C}) & \xrightarrow{\partial^*} & \mathcal{P}^{i+1}(X, \frac{\sim C}{G}) \\ \downarrow & & \downarrow \\ \mathcal{P}^i(X, \frac{R'}{C'}) & \xrightarrow{\partial^*} & \mathcal{P}^{i+1}(X, \frac{\sim C'}{G'}) \end{array}$$

  where the vertical arrows are restriction maps.

- **Exactness.** Whenever (R,C), $(\sim C,G)$, and (R,G) are opposed subsets of X with $(R,G) \sqsupseteq (R,C)$ and $(\sim C,G) \sqsupseteq (R,G)$, then the following sequence is exact:

$$\cdots \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{C}) \xrightarrow{\partial^*} \mathcal{P}^i(X, \tfrac{\sim C}{G}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{R}{G}) \longrightarrow \cdots$$

- **Excision.** Whenever (R,G) and (R,G') are two opposed pairs in X such that $(R,G) \sqsupseteq (R,G')$, if $G \cap \sim R = G' \cap \sim R$, then $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R}{G'})$ is a isomorphism. Dually, whenever (R,G) and (R',G) are two opposed pairs in X such that $(R,G) \sqsupseteq (R',G)$, if $R \cap \sim G = R' \cap \sim G$, then $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R'}{G})$ is a isomorphism.
- **Homotopy.** Suppose (R,G) covers (R',G') by deformation. Then the map $\mathcal{R}^* : \mathcal{P}^i(X, \frac{R}{G}) \longrightarrow \mathcal{P}^i(X, \frac{R'}{G'})$ is an isomorphism.
- **Dimension.** Let $f : X \longrightarrow \boldsymbol{R}$ is a smooth proper function with only one critical point $p \in X$ with critical value in the interval $[b,a]$, and assume that f has a Morse singularity at p . Then $\mathcal{P}^i(X, \frac{X_{f \geq a}}{X_{f \leq b}}) = 0$ for unless $i = -\bar{\sigma}$ where $\bar{\sigma}$ is on half of the signature of the Hessian quadratic form of F at p (see definitions in section 1).

Exercise 4.4. Show that, for any perverse sheaf $\mathcal{P}$, if $R \cup G = X$, then $\mathcal{P}^i(X, \frac{R}{G}) = 0$.

4.3.2 Fary functors

DEFINITION. A *Fary functor* is a device that does the same three things and satisfies all of the axioms for a perverse sheaf except for the last one, the Dimension axiom, and which satisfies instead the following additional axiom:

- There exists a number N such that $\mathcal{P}^i(X, \frac{R}{G}) = 0$ unless $-N \leq i \leq N$.

Thus a perverse sheaf is a Fary functor satisfying the dimension axiom.

4.3.3 Verifying that a Fary functor is a perverse sheaf.

Suppose you have something that gives the three structures $\mathcal{P}^i(X, \frac{R}{G})$, $\mathcal{R}^*$, and *partial**, you suspect is a perverse sheaf, and you would like to check the axioms. All of the axioms but the Dimension axiom, i.e. the Fary functor axioms, will often be easy to check, because your "something" will probably be defined in terms of a chain complex with some homotopy invariance property. The hard axiom to check is likely to be the dimension axiom. The theorem below reduces checking the dimension axiom to a finite amount of work.

DEFINITION. The opposed pair (R, G) has *pure degree* k if it has a presentation $(R, G) = (X_{f \geq r}, X_{f \leq g})$ for $g < r$ with the property that all of the critical points of f with critical values between g and r have half the signature $\bar{\sigma} = -k$. Suppose (R, G) has pure degree, and K is a connected component of a stratum in X. Then we say that K is *relevant* for (R, G) if for some presentation $(R, G) = (X_{f \geq r}, X_{f \leq g})$ as above, at least one of the critical points of f with critical value between g and r lies in K. (It is a fact of Stratified Morse Theory, which we won't need here, that if such a critical point lies in K for some such function f, then it is true for any function f such that $(R, G) = (X_{f \geq r}, X_{f \leq g})$.)

Exercise 4.5. Show from the axioms that if (R, G) has pure degree k, then for any perverse sheaf $\mathcal{P}$, we have the vanishing $\mathcal{P}^i(X, \frac{R}{G}) = 0$ unless $i = k$.

Theorem 4.3 *Suppose that $\mathcal{P}$ satisfies all of the axioms for a perverse sheaf except (possibly) the last one, and so is a Fary functor. Suppose that we have a finite collection of opposed pairs $(R_1, G_1), (R_2, G_2), (R_3, G_3), \ldots$ with pure degree $k_1, k_2, k_3, \ldots$ such that every connected component K of every stratum in X is relevant for at least one of the pairs (R_j, G_j). Then $\mathcal{P}$ is perverse if for each j, $\mathcal{P}^i(X, \frac{R_j}{G_j})$ vanishes for i not equal to k_j.*

4.4 The category of perverse sheaves

The set of perverse sheaves on a fixed complex manifold X with a fixed stratification forms a category $P(X)$. The properties and applications of the category structure of $P(X)$ is the most important aspect of perverse sheaf theory.

4.4.1 The definition of the category structure

In order to define the category structure, we need to tell what a morphism is.

DEFINITION. Given two perverse sheaves $\mathcal{P}'$ and $\mathcal{P}$, a *morphism* ρ from $\mathcal{P}'$ to $\mathcal{P}$ also called a *homomorphism* or a *map of perverse sheaves* from $\mathcal{P}'$ to $\mathcal{P}$ is the following set of data:

- For each opposed pair (R, G) in X, a vector space homomorphism

$$\rho_\star : \mathcal{P}'^i(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^i(X, \tfrac{R}{G})$$

subject to the following requirements:

- Whenever $(R, G) \sqsupseteq (R', G')$, then the following square commutes:

$$\begin{array}{ccc} \mathcal{P}'^i(X, \tfrac{R}{G}) & \xrightarrow{\mathcal{R}^*} & \mathcal{P}'^i(X, \tfrac{R'}{G'}) \\ \downarrow \rho_\star & & \downarrow \rho_\star \\ \mathcal{P}^i(X, \tfrac{R}{G}) & \xrightarrow{\mathcal{R}^*} & \mathcal{P}^i(X, \tfrac{R'}{G'}) \end{array}$$

- Whenever (R, C) and $(\sim C, G)$ are opposed pairs in X, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}'^i(X, \tfrac{R}{C}) & \xrightarrow{\partial^*} & \mathcal{P}'^{i-1}(X, \tfrac{\sim C}{G}) \\ \downarrow \rho_\star & & \downarrow \rho_\star \\ \mathcal{P}^i(X, \tfrac{R}{C}) & \xrightarrow{\partial^*} & \mathcal{P}^{i-1}(X, \tfrac{\sim C}{G}) \end{array}$$

4.4.2 The fundamental theorem about the category of perverse sheaves, part I

Now we state the fundamental theorem about the category of perverse sheaves:

Theorem 4.4 Fundamental theorem on the category of perverse sheaves, part I *The morphisms defined above make the set of perverse sheaves into a category. This category is abelian.*

Exercise 4.6. Prove that perverse sheaves with these morphisms a category, and that the category is additive. (Now would be a good time to refresh your memory on what an additive category is and what and abelian category is.)

4.4.3 How to see the abelian category structure.

The fact that perverse sheaves form an abelian category is one of the great mysteries of nature. Perhaps the reason that perverse sheaves were not discovered until 1980 is that this seems result so implausible.

How can we see the abelian category structure directly from the definition? The most naive guess would be that a sequence of perverse sheaves is exact if and only if the sequence of induced maps on any opposed pair (R, G) is exact. This is false. An example is given in an exercise in the next section. However, there is a true statement in this direction.

Theorem 4.5 *The sequence*

$$\mathcal{P}'' \xrightarrow{\rho'} \mathcal{P}' \xrightarrow{\rho} \mathcal{P}$$

is exact, in the abelian category of perverse sheaves if and only if for all integers k and for all opposed pairs (R, G) that have pure degree k, the following induced sequence is exact:

$$\mathcal{P}''^k(X, \tfrac{R}{G}) \xrightarrow{\rho'_*} \mathcal{P}'^k(X, \tfrac{R}{G}) \xrightarrow{\rho_*} \mathcal{P}^k(X, \tfrac{R}{G})$$

In fact, to guarantee the exactness of the sequence of perverse sheaves, it suffices to check the exactness of the induced sequence on a finite collection of opposed pairs with pure degree $(R_1, G_1), (R_2, G_2), (R_3, G_3), \ldots$ provided every connected component K of every stratum in X is relevant for at least one of the pairs (R_j, G_j).

Exercise 4.7. Show the "only if" part of this theorem directly from the axioms. (In the language of abelian categories, this is the statement that $\mathcal{P}^k(X, \tfrac{R}{G})$ is an exact functor if (R, G) has pure degree k.) Use the fact that a short exact sequence of chain complexes gives rise to a long exact sequence on homology.

Exercise 4.8. Construct a nontrivial local system L over the circle S^1 with the property that L fits into a short exact sequence

$$0 \longrightarrow Q \longrightarrow L \longrightarrow Q \longrightarrow 0$$

Show that neither $H_0(S^1, \bullet)$ nor $H_1(S^1, \bullet)$ gives an exact sequence when applied to this short exact sequence of local systems. For any continuous map $m : X \longrightarrow S^1$ of a complex manifold to S^1, show that the short exact sequence of local systems on S^1 induces a short exact of local systems on X and hence a short exact sequence of perverse sheaves on M. (Use the theorem above to check the exactness of the sequence of perverse sheaves.) Finally, show that the functor $\bullet^{(\dim X)/2}(X, \tfrac{\phi}{\phi})$ applied to this short exact sequence of perverse sheaves does not give a exact sequence of vector spaces.

4.5 The intersection homology perverse sheaf

As remarked in the introduction to this report, the most important examples of perverse sheaves are the intersection homology perverse sheaves.

4.5.1 Definition of the intersection homology perverse sheaf

DEFINITION. An *enriched stratum* is a pair (X_α, L) consisting of

1. A stratum X_α of X.
2. A local system L on X_α.

In the event that an enricned stratum has been chosen, we will denote X_α also by the symbol Y_0, and we will denote the closure of X_α by Y.

DEFINITION Let (X_α, L) be an enriched stratum and let p be a perversity. The corresponding *intersection homology perverse sheaf* $I^pC = I^pC(X_\alpha, L)$ is the following:

- I^pCassigns to any pair of opposed smoothly bounded subsets R and G of X and to any integer i the rational vector space $I^pC^i(X, \substack{R\\G})$, according to the following rule:

$$\begin{aligned} I^pC^i(X, \substack{R\\G}) &:= I^pH_{(\dim Y)/2-i}((X-R)\cap Y, G^0\cap(X-R)\cap Y; L) \\ &= I^pH_{(\dim Y)/2-i}(((X-R)\cup G^0)\cap Y, G^0\cap Y; L) \end{aligned}$$

  The two expressions on the right are equal to each other by excision.

- The restriction map

$$\mathcal{R}^* : I^pC^i(X, \substack{R\\G}) \longrightarrow I^pC^i(X, \substack{R'\\G'})$$

  is the map

$$I^pH_{n-i}(X-R, G^0\cap(X-R)) \longrightarrow I^pH_{n-i}(X-R', G'^0\cap(X-R'))$$

  induced by the inclusion of pairs.

- The coboundary homomorphism

$$\partial^* : I^pC^i(X, \substack{R\\C}) \longrightarrow \mathcal{P}^{i+1}(X, \substack{\sim C\\G})$$

  is just the homomorphism

$$\begin{aligned} &I^pH_{n-i}(X-R, C^0\cap(X-R)) \xrightarrow{\partial_*} I^pH_{n-i-1}(C^0\cap(X-R) \longrightarrow I^pH_{n-i-1}(C) = \\ &= I^pH_{n-i-1}((X-\sim C) \longrightarrow I^pH_{n-i-1}((X-\sim C), G^0\cap(X-\sim C) \end{aligned}$$

  where the unlabeled maps are induced by inclusions of pairs.

In the event that the perversity is the middle perversity m, then we will abbreviate the notation $I^mC(X_\alpha, L)$ to $IC(X_\alpha, L)$. In the event that the local system L is the trivial local system Q, then we will abbreviate $I^pC(X_\alpha, L)$ to $I^pC(X_\alpha)$.

The proof that the intersection homology perverse sheaf, as defined above, satisfies all the modified Eilenberg-Steenrod axioms but the last one (Dimension) is very similar to the proof that the ordinary homology perverse sheaf satisfies them on a manifold. It uses stratification theory to produce isotopies and the topological invariance of intersection homology to show that intersection homology is preserved by those isotopies. The reader familiar with [GM2] and [GM7] will have no difficulty in constructing the proof; we will omit it. The fact that it satisfies the Dimension axiom is, of course, essentially what was stated as the theorem SMT part B in section 3.1. The proof, in [GM9], is difficult. We will illustrate it in several examples.

4.5.2 The fundamental theorem on the category of perverse sheaves, part II

The enriched pair (X_α, L) is said to be *irreducible* if L is an irreducible local system over X_α. (Recall that an irreducible local system on X_α must be zero on all but one of the connected components of X_α.)

Theorem 4.6 (Fundamental theorem on the category of perverse sheaves, part II) *The irreducible perverse sheaves, as determined by the abelian category structure, are exactly the intersection homology sheaves formed from data (X_α, L, p) where L is an irreducible local system and p is the middle perversity. If X is compact (or if there are finitely many strata in X) the category $P(X)$ of perverse sheaves is Artinian, i.e. every object has a finite composition series whose successive quotients are irreducible.*

4.5.3 Properties of intersection homology sheaves.

We give some properties of intersection homology sheaves that relate them to the abelian category structure of the category of perverse sheaves.

The first exercise below shows that the intersection homology sheaf cannot be irreducible unless the local system is irreducible. For middle perversity, this is the only obstruction to irreducibility of the intersection homology sheaf, as shown by the fundamental theorem above.

If we are given a map of the local system L to the local system L', we have an induced map on intersection homology sheaves $I^pC(X_\alpha, L) \longrightarrow I^pC(X_\alpha, L')$ induced by the map of chain complexes. This provides our first source of examples of maps of perverse sheaves.

Exercise 4.9. Suppose we fix a stratum X_α and a perversity p that is close to middle. Show that the functor that takes a local system L on X_α to the intersection homology sheaf $I^pC(X_\alpha, L)$ associated the enricned stratum (X_α, L) is an exact functor (i.e. it takes short exact sequences of local systems to short exact sequences of perverse sheaves). You should use the characterization of exact sequences of the last section.

If the perversity p is less than the perversity q, we have a map $I^pC(X_\alpha, L) \longrightarrow I^qC(X_\alpha, L)$ induced by the inclusion of chain complexes. We call this the canonical map. This provides our second source of examples of maps of perverse sheaves.

Exercise. Show that if p is less than the middle perversity and if q is more than the middle perversity, then the image of the map $I^pC(X_\alpha, L) \longrightarrow I^qC(X_\alpha, L)$ is $I^mC(X_\alpha, L)$. Use the theorem above, and also the fact that any local system has a composition series.

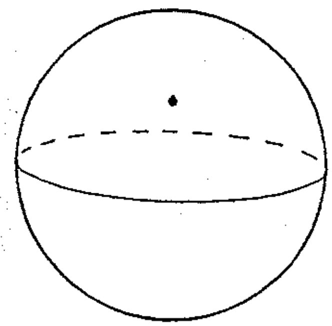
At the moment, these theorems and exercises (which are more exercises in homological algebra than in geometry) are meant to intrigue the reader. There is no way that these results could be geometrically clear or even plausible at this stage, since the abelian category structure is itself obscure.

4.6 Examples.

The examples are the real stuff of the subject. It is strongly recommended that the reader not skip the exercises in this section.

4.6.1 The Riemann sphere.

We may stratify $X =$ the Riemann sphere (i.e. the complex projective line) by a single point X_1 and the rest of it $X_2 = X - X_1$. This is a special case of example 5 of chapter 1.



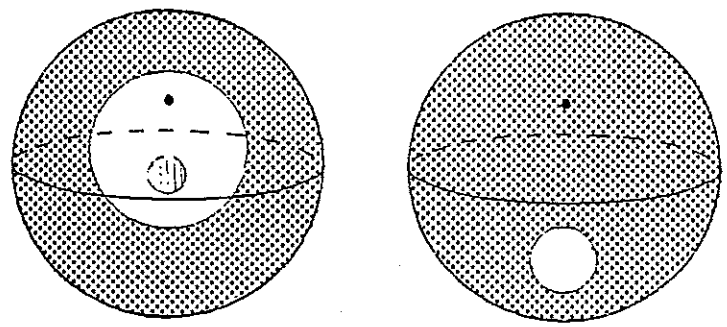
The Riemann sphere stratified with one point stratum

This is the simplest of all examples of stratified spaces, except for the case that there is only one stratum. Nevertheless, it is interesting. By the Beilinson-Bernstein correspondence, the category of perverse sheaves on this stratified space is equivalent to an important category in representation theory: the category $\mathcal{O}$, of Bernstein-Gelfand-Gelfand (see [Mi]).

There are two irreducible enriched strata for this stratification of X: (X_1, Q) and (X_2, Q). These yield a supply of four interesting perverse sheaves on X given by intersection homology sheaves: $I^{-1}C(X_2, Q)$, $I^0C(X_2, Q)$, $I^1C(X_2, Q, 1)$, and $I^pC(X_1, Q)$. Here -1, 0, and 1 represent the close to middle perversities for $Y = \overline{X_2}$ whose value on $\Sigma = X_1$ is -1, 0, and 1 respectively. As usual, Q represents the trivial local system with fiber the rationals. If $Y = X_1$, there is only one perversity; we call it p.

We choose two opposed pairs that have pure degree: Let R and C be two disjoint disks in X_2.

Exercise 4.10. Show that the opposed pair (R, C) has pure degree 0, and that the pair $(\sim C, \phi)$ has pure degree 1.



Two opposed pairs of pure degree

Exercise 4.11. Show that the values of the four perverse sheaves above on (R, C) in degree 0 are $\boldsymbol{Q}, 0, \boldsymbol{Q}$, and $\boldsymbol{Q}$ respectively. Draw geometric cycles representing the generators. Show that the values on $(\sim C, \phi)$ are $\boldsymbol{Q}, \boldsymbol{Q}, \boldsymbol{Q}$, and 0 respectively. Therefore, by Theorem 4.3, we have verified (without recourse to Theorem 4.2) that the intersection homology sheaves are perverse sheaves.

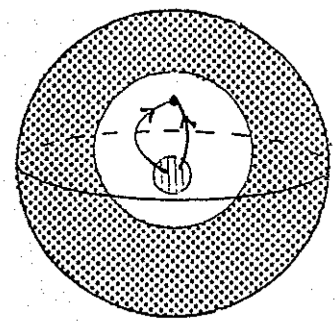
Now, according to Theorem 4.5, in order to check exactness of any sequence, it suffices to check exactness of its value on these two opposed pairs.

Exercise 4.12. Show that the canonical maps $I^{-1}C(X_2, Q) \longrightarrow I^0C(X_2, Q)$ is surjective, and the canonical map $I^0C(X_2, Q) \longrightarrow I^1C(X_2, Q)$ is injective (using Theorem 4.5 but without using Theorem 4.6).

The perverse sheaf $I^pC(X_1, Q)$ is a rather simple one. If the point X_1 is in the support of (R, G) (which is $X - (R^0 \cup G^0)$, then its value on (R, G) is $\boldsymbol{Q}$; otherwise its value on (R, G) is 0. We would like now to define two maps

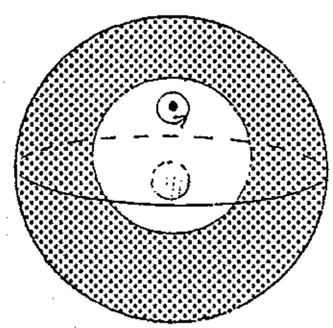
$$I^1C(X_2, Q) \xrightarrow{\rho} I^pC(X_1, Q) \xrightarrow{\rho'} I^{-1}C(X_2, Q)$$

To define ρ' we have only to say what ρ'_* does on a pair (R, G) where X_1 is contained neither in R nor in G. On such a pair (R, G), ρ'_* gives the element of $\boldsymbol{Z} \subset \boldsymbol{Q}$ which counts how (with multiplicities) how many cycles come in to the point, minus how many come out of the point.



ρ'_* of this element is $2 \in I^0C^0(X_1, Q)(X, \frac{R}{G})$

The map ρ on the pair (R, G) associates to $1 \in Q$ the class containing the cycle that runs once around the point X_1 counterclockwise.



$\rho_*(1)$

Remark. If we identify $I^{-1}H_*$ with relative homology, as explained in 1.4.1 , the map ρ'_* is identified with the boundary map. After the appropriate dualities, the map ρ_* may be identified with the coboundary map (see [G2]).

Exercise 4.13. Show that ρ' is injective and that ρ is surjective. Compute the kernel and the cokernel of the composed map $\rho' \circ \rho$.

These maps and their multiples and compositions turn out to be all of the maps between the objects that we have found so far. (Try to find more maps!)

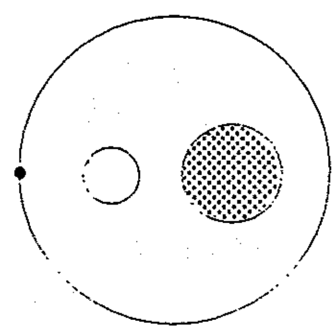
However, there is another perverse sheaf which is not a direct sum of intersection homology sheaves, and which is very interesting.

DEFINITION. The *largest indecomposable* perverse sheaf $\mathcal{P}$ on X is constructed as follows: Take a closed 2-disk D, and let q be a point on its boundary. Map the disk to X by collapsing the boundary to the point X_1 and otherwise mapping homeomorphically. Call this map $\gamma : D \longrightarrow X$. Now define $\mathcal{P}$ by:

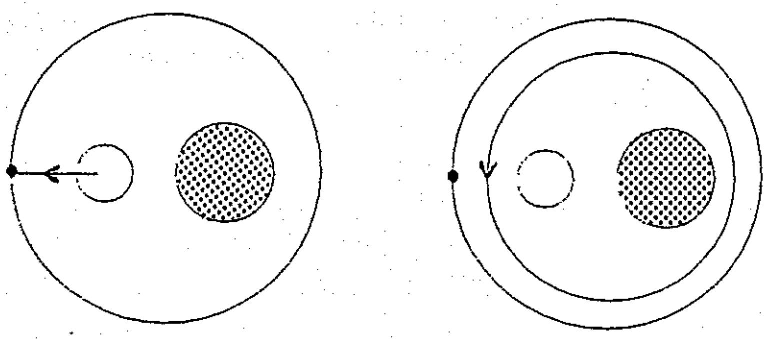
$$\mathcal{P}^i(X, {}^{R}_{G}) = H_{1-i}((D - \gamma^{-1}R), (q \cup \gamma^{-1}(G^0 - R))).$$

We need to check that this is a perverse sheaf.

Exercise 4.14. Show that $\mathcal{P}$ satisfies all of the axioms for a perverse sheaf except the Dimension axiom. By Theorem 4.3, to complete the verification that $\mathcal{P}$ is a perverse sheaf, it suffices to compute its values on (R, C) and $(\sim C, \phi)$. Let's look at (R, C). By definition we need to compute the homology groups of $H_{1-i}((D - \gamma^{-1}R), (q \cup \gamma^{-1}(G^0 - R))) = H_{1-i}((D - \gamma^{-1}R), (q \cup \gamma^{-1}G^0))$. Here is a picture of the disk D with $\gamma^{-1}R$, q, and $\gamma^{-1}G^0$ drawn in it.



Only the first homology group ($i = 0$) is nonzero, the that one is two dimensional generated by the following two cycles:



Exercise 4.15. Complete the verification that $\mathcal{P}$ is a perverse sheaf.

Exercise 4.16. Construct nonzero maps of perverse sheaves as follows:

$$\begin{array}{ccccc} & & I^0C(X_2,Q) & & \\ & \nearrow & & \searrow & \\ I{-}1C(X_2,Q,) & \longrightarrow & \mathcal{P} & \longrightarrow & I^1C(X_2,Q) \\ & \nwarrow & \uparrow\downarrow & \swarrow & \\ & & I^pC(X_1,Q) & & \end{array}$$

(The four diagonal arrows have already been constructed.)

Exercise 4.17. Show that the following sequences are exact:

$$0 \longrightarrow I^0C(X_2,Q) \longrightarrow I^1C(X_2,Q) \longrightarrow I^pC(X_1,Q) \longrightarrow 0$$

$$0 \longrightarrow I^pC(X_1,Q) \longrightarrow I^{-1}C(X_2,Q) \longrightarrow I^0C(X_2,Q) \longrightarrow 0$$

$$0 \longrightarrow I^{-1}C(X_2,Q,) \longrightarrow \mathcal{P} \longrightarrow I^pC(X_1,Q) \longrightarrow 0$$

$$0 \longrightarrow I^pC(X_1,Q) \longrightarrow \mathcal{P} \longrightarrow I^1C(X_2,Q) \longrightarrow 0$$

Assertion (without proof). The diagram above gives the whole story on this category in the following sense: Every perverse sheaf is a direct sum of objects selected from these five, and every map among these five is a multiple of one of the maps in the diagram above or of their compositions. (In this case, we were lucky that the strata were simply connected so we didn't have any local systems to contend with. There aren't many categories small enough to have a finite list of indecomposables. This is rare even among categories of perverse sheaves arising from a stratifications with simply connected strata.)

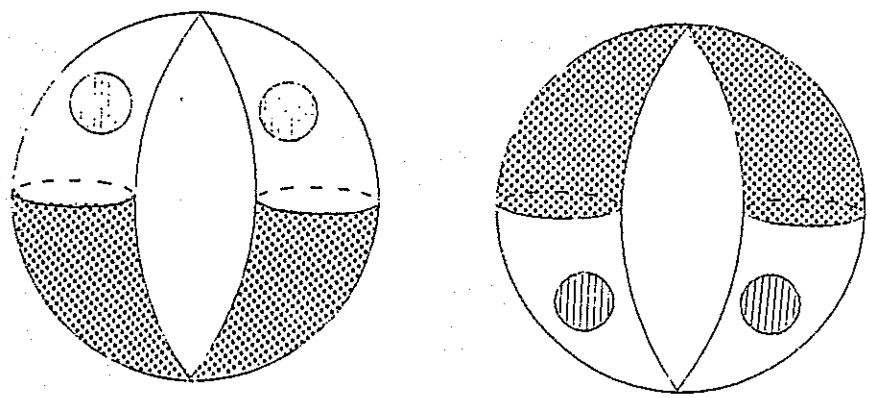
Exercise 4.18. Another perverse sheaf on X could be constructed similarly to $\mathcal{P}$ but with a finite collection of points $q_1, q_2, q_3, \ldots$ in place of q. Verify the assertion in this case by showing that this perverse sheaf is a direct sum of copies of the five indecomposables above.

Exercise 4.19. Show that $\mathcal{P}$ is both projective and injective in this category. What are the other indecomposable projectives and injectives?

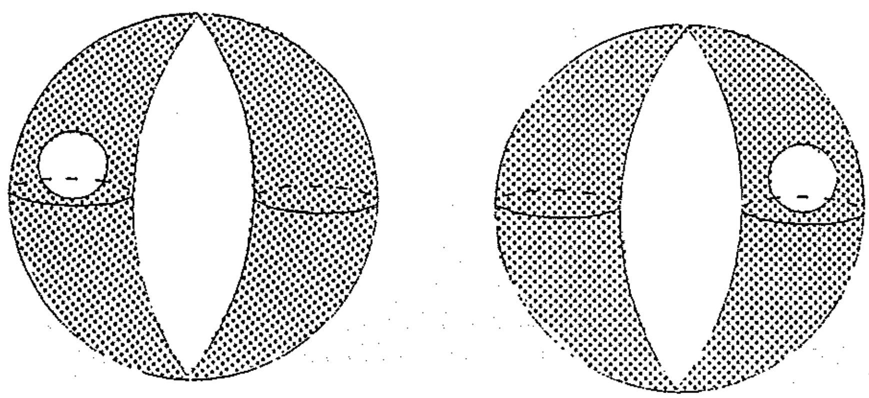
4.6.2 Two Riemann spheres joined at two points

Recall Example 1 from Chapter 1. This example is an algebraic curve in the complex projective plane. It can be given by the equation $(x)(x^2+y^2-w^2)=0$ with respect to homogeneous coordinates x, y, w in the plane. (The real affine picture is the unit circle cut by a vertical line: ϕ.) Therefore, we can take X to be the complex projective plane stratified by three strata: X_1 is the two points $(0,1,1)$ and $(0,-1,1)$, X_2 is the rest of the curve $x(x^2+y^2-w^2)=0$, and X_3 is the rest of the complex projective plane. We are interested in $Y=\overline{X_2}$.

Exercise 4.20. Show that the following four pairs of sets are intersections of Y with pairs of pure degree, and that each one has one of the four connected components of strata of Y as its relevant connected component. (This exercise requires that you look at the definition of a Morse function carefully, and that you write equations of the Morse function, unless you are adept at visualizing the four dimensional space X. As an example of the subtlety, if we omitted one of the two components of the green set in the degree 0 examples, it would not be of pure degree any longer.)



Two pairs of pure degree 1. (The green set is empty in each case.)



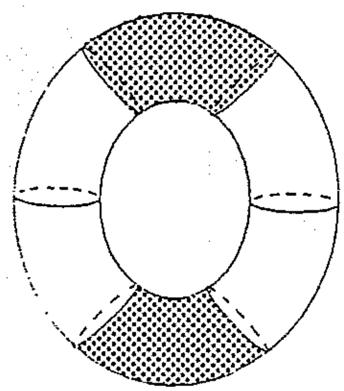
Two pairs of pure degree 0.

Examples of perverse sheaves are the three perversities of intersection homology from Example 1 of Chapter 1, and the intersection homology sheaves supported on the points. Further examples come from perverse sheaves on the Riemann sphere stratified with one point, discussed at length in Example 1 above, considering the Riemann sphere as one of the spheres of Y. (The sheaf will be zero on the other sphere.) Since the stratum X_2 is not simply connected, further examples come from intersection homology with coefficients in a nontrivial local system. But there are still other interesting examples:

Exercise 4.21. Show that the ordinary homology Fary functor is a perverse sheaf here (even though Y is singular. In general, the ordinary homology Fary functor is a perverse sheaf whenever Y is a complete intersection, which our Y

is since it is given by one equation. It is even enough that Y should be a local complete intersection.)

Exercise 4.22. Here is another perverse sheaf: Take the 2-torus T^2 with two disjoint annuli on it.



The torus with two disjoint annuli

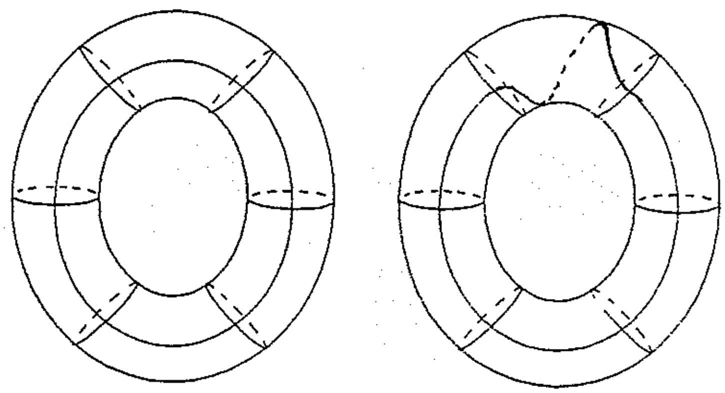
Map the torus onto Y by shrinking each annulus to a point. Call this map $\gamma : T^2 \longrightarrow Y$. Now define $\mathcal{P}$ by:

$$\mathcal{P}^i(X, \tfrac{R}{G}) = H_{1-i}((D - \gamma^{-1}R), \gamma^{-1}(G^0 - R)).$$

Exercise 4.23. Verify that this is a perverse sheaf.

(This perverse sheaf is an example of the "nearby cycles" or "$R\Psi$" perverse sheaf from algebraic geometry, as in section 5.8 . Again, an analogue of it exists whenever Y is a complete intersection.)

Exercise 4.24. . The perverse sheaf $\mathcal{P}$ has a beautiful automorphism given by the Dehn twist. The Dehn twist $D : T^2 \longrightarrow T^2$ is an automorphism of the torus which is the identity everywhere except inside one of the two annuli, and which, but which is not isotopic to the identity. The following picture shows the torus with a circle drawn on it both before and after the Dehn twist.



Before After

The Dehn twist

Now considered as an automorphism ρ of the perverse sheaf $\mathcal{P}$, the Dehn twist acts in this way:

$$\begin{array}{ccc} \mathcal{P}^i(X, {R \atop G}) & = & H_{1-i}((D - \gamma^{-1}R), \gamma^{-1}(G^0 - R)) \\ \downarrow \rho_* & & \downarrow D \\ \mathcal{P}^i(X, {R \atop G}) & = & H_{1-i}((D - \gamma^{-1}R), \gamma^{-1}(G^0 - R)) \end{array}$$

Exercise 4.25. Show that ρ as defined here is indeed an invertible homomorphism of perverse sheaves. Find the kernel and cokernel of the map (Identity $- \rho$).

4.7 Self-indexing Morse functions and the small chain complex

4.7.1 Disjoint pairs

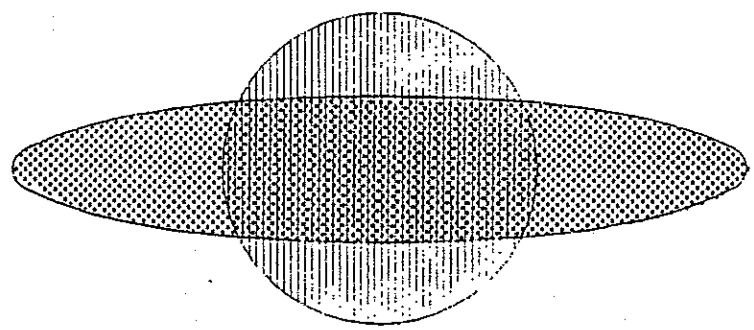
A pair (R, G) of smoothly enclosed subsets of X is called *disjoint* if $A \cap B = \phi$. Clearly, any disjoint pair is opposed.

Proposition 4.7 *Given any opposed pair* (R, G) *in* X, *there is an* associated disjoint pair (R', G') *such that* $(R, G) \sqsupseteq (R', G')$ *and, for any perverse sheaf* $\mathcal{P}$ *on* X, *we have*

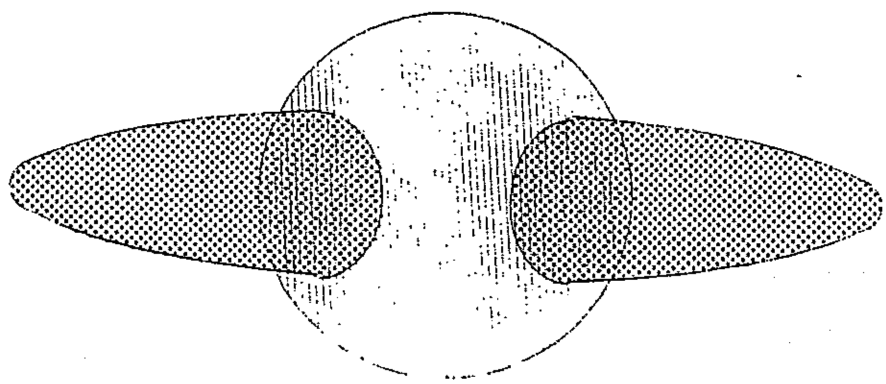
$$\mathcal{R}^* : \mathcal{P}^i(X, {R' \atop G'}) \longrightarrow \mathcal{P}^i(X, {R \atop G})$$

is an isomorphism.

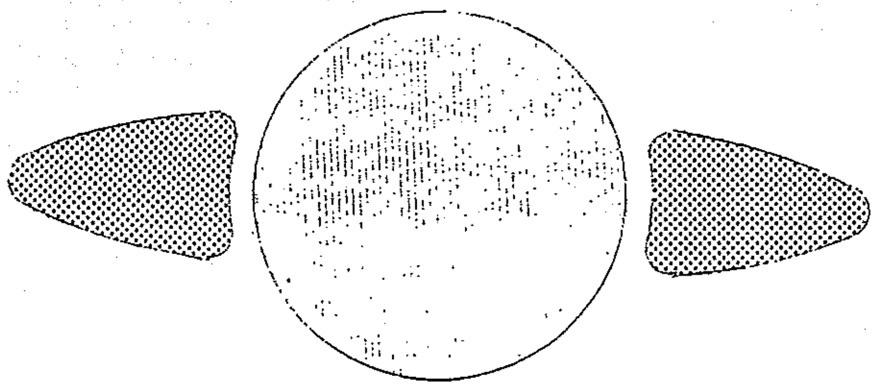
PROOF. The following picture provides the proof:



An opposed pair (R, G)



A pair that is equivalent by excision



The associated disjoint pair (R', G'), equivalent by homotopy.

The point of disjoint pairs is this: By the above proposition, to know a perverse sheaf completely it suffices to know it on all disjoint pairs. Our goal in this section is to give an explicit cochain complex whose cohomology computes the value of a perverse sheaf on a disjoint pair.

4.7.2 Self indexing Morse functions

It is easy to see that any disjoint pair is of the form $(X_{f\geq r}, X_{f\leq g})$ for some smooth function f where $r = (n+1)/2$ and $g = -(n+1)/2$. (The reason for this bizarre choice of r and g will become apparent.) We want to show that we can choose f to be of a very special form.

DEFINITION. A real valued function f on X is called *self indexing and Morse in* $[g, r]$ if g and r are not critical values, and for every critical point x whose critical value is in $[g, r]$, the critical value of x is $-\bar{\sigma}$, where $\bar{\sigma}$ is half of the signature of the f at x.

Theorem 4.8 *For any disjoint pair in X, there exists a function f which is self indexing and Morse in $[g, r]$ such that the given disjoint pair is $(X_{f\geq r}, X_{f\leq g})$.*

This is a stratified analogue of a theorem of Smale, which asserts the same thing on a manifold. This theorem is not as easy to prove as you might first think: If you start naively deforming the critical values of critical points in a random way, you are forced to add new critical points. The proof is similar to the proof of Smale: If two critical points have critical values that are out of order for this theorem, you must find "handle" going up from the lower critical point and a "handle" going down from the upper critical point. Then you must find an isotopy to make these two handles disjoint. We will not give a proof here, since it involves very heavy use of stratification theory. For Smale, this theorem was the starting point of his work on classifying differentiable structures. For us, it will provide a way to see that the category of perverse sheaves is abelian.

DEFINITION. Given a perverse sheaf $\mathcal{P}$ on X, a disjoint pair (R, G) in X, and a function f which is self-indexing and Morse in $[g, r]$, so that $(R, G) = (X_{f \geq r}, X_{f \leq g})$, we define the *associated complex* to be the chain complex

$$\begin{array}{c} \vdots \\ \uparrow \partial^* \\ \mathcal{P}^{i+1}(X, {}^{R(i+1)}_{G(i+1)}) \\ \uparrow \partial^* \\ \mathcal{P}^{i}(X, {}^{R(i)}_{G(i)}) \\ \uparrow \partial^* \\ \mathcal{P}^{i-1}(X, {}^{R(i-1)}_{G(i-1)}) \\ \uparrow \partial^* \\ \vdots \end{array}$$

where $R(i) = X_{f \geq i+1/2}$ and $G(i) = X_{f \leq i-1/2}$.

Theorem 4.9 *The associated complex is a chain complex. There is a canonical isomorphism between the i^{th} cohomology of the associated complex and $\mathcal{P}^i(X, {}^{R}_{G})$.*

The proof of this theorem is essentially the same as the proof of the theorem that the cohomology of a CW-complex is given by the cohomology of the associated small algebraic chain complex, starting from the Eilenberg- Steenrod axioms of cohomology.

The associated cochain complex is completely functorial in the perverse sheaf. If we have a map of perverse sheaves $\rho : \mathcal{P}' \longrightarrow \mathcal{P}$, then it clearly induces a map of associated chain complexes

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow \partial^* & & \uparrow \partial^* \\
\mathcal{P}'^{i+1}(X, {}^{R(i+1)}_{G(i+1)}) & \xrightarrow{\rho_*} & \mathcal{P}^{i+1}(X, {}^{R(i+1)}_{G(i+1)}) \\
\uparrow \partial^* & & \uparrow \partial^* \\
\mathcal{P}'^{i}(X, {}^{R(i)}_{G(i)}) & \xrightarrow{\rho_*} & \mathcal{P}^{i}(X, {}^{R(i)}_{G(i)}) \\
\uparrow \partial^* & & \uparrow \partial^* \\
\mathcal{P}'^{i-1}(X, {}^{R(i-1)}_{G(i-1)}) & \xrightarrow{\rho_*} & \mathcal{P}^{i-1}(X, {}^{R(i-1)}_{G(i-1)}) \\
\uparrow \partial^* & & \uparrow \partial^* \\
\vdots & & \vdots
\end{array}$$

Since this is a map of chain complexes, its kernel also forms chain complex whose group at each degree is the kernel of ρ_*. Similarly the cokernel forms a chain complex.

DEFINITION. Given a map of perverse sheaves $\rho : \mathcal{P}' \longrightarrow \mathcal{P}$, the *cokernel* perverse sheaf is the perverse sheaf that assigns to each disjoint pair (R, G) the cohomology of the cokernel complex of the map of associated chain complex, and the *kernel* perverse sheaf is the perverse sheaf that assigns to each disjoint pair (R, G) the cohomology of the kernel complex of the map of associated chain complex.

The hard part of proving that the category of perverse sheaves is abelian, Theorem 4.4, is showing that the kernel and cokernel as defined above do in fact satisfy the axioms for perverse sheaves. The fact that they are categorical kernels and cokernels is then easy to see.

4.8 Monodromy.

Just as in the last chapter, we have the following results about monodromy and the homotopy covering chapter.

- For any perverse sheaf $\mathcal{P}$, the functor $\mathcal{P}^i(X, {}^{\bullet}_{\bullet})$ on the category of opposed pairs and coverings passes to a functor on the homotopy covering category.
- A deformation of opposed pairs $\sigma = (R(t), G(t)), t \in \mathbf{R}$ induces a morphism $\tilde{\sigma} : (R(0), G(0)) \longrightarrow (R(1), G(1))$ in the homotopy covering category. The morphism $\mathcal{P}^i(X, {}^{R(0)}_{G(0)}) \longrightarrow \mathcal{P}^i(X, {}^{R(1)}_{G(1)})$ induced by $\tilde{\sigma}$ is called the monodromy map σ^*.
- Two deformations which are homotopic induce the same morphism in the homotopy covering category, and hence the same monodromy.

- A smooth family $(R(s), G(s))$ of opposed pairs parameterized by a manifold S induces a local system over S whose fiber at $s \in S$ is $\mathcal{P}^i(X, {}^{R(s)}_{G(s)})$.

The constructions and proofs of these statements may be read from Chapter 3 with no changes.

There are two types of families of opposed pairs whose local systems are particularly important.

4.8.1 Local systems on strata.

Let $S = X_\alpha$ be a stratum of X. We choose a family of balls B_x one for each $s \in X_\alpha$ such that "B_s is centered at s, it varies smoothly with s, and its boundary is always transverse to the stratification". We do this by choosing a Riemannian metric on X and choosing a smooth function $\epsilon(s) > 0$ on X_α. Then B_s is the set of all points of X of distance $\leq \epsilon(s)$ from $s \in X_\alpha$ as measured by the metric. If $\epsilon(s)$ is chosen small enough, then for any $s \in S$, B_s will be homeomorphic to a ball, and family B_s will be a smooth family of smoothly enclosed subsets of X. (These statements may be proved using the technique of moving the wall from Stratified Morse Theory [GM6].)

DEFINITION. The i^{th} stalk homology of the perverse sheaf $\mathcal{P}$ on X at the point $s \in S \subset X$ is $\mathcal{P}^i(X, {}^{\sim B_s}_{\phi})$. The i^{th} costalk homology of the perverse sheaf $\mathcal{P}$ on X at the point $s \in S \subset X$ is $\mathcal{P}^i(X, {}^{\phi}_{\sim B_s})$.

Exercise 4.26. . Show that the i^{th} stalk homology and the i^{th} costalk homology of $\mathcal{P}$ at $s \in X$ is independent of the choice of B_s in its definition.

DEFINITION. Let $\mathcal{P}$ be a perverse sheaf over X, and let $S = X_\alpha$ be a stratum of X. The i^{th} *stalk homology local system* $\mathbf{H}^i_S \mathcal{P}$ over S is the local system associated to the family of opposed pairs $(\sim B_s, \phi)$ parameterized by S whose fiber over $s \in S$ is the i^{th} stalk homology of the perverse sheaf $\mathcal{P}$ at the point s. The i^{th} *costalk homology local system* $\mathbf{H}^S_i \mathcal{P}$ over S is the local system associated to the family of opposed pairs $(\phi, \sim B_s)$ parameterized by S whose fiber over $s \in S$ is the i^{th} costalk homology of the perverse sheaf $\mathcal{P}$ at the point s.

Exercise 4.27. . Show that the dimension of the i^{th} stalk homology of $\mathcal{P}$ at $s \in X$ depends only on the connected component of the stratum that s lies in. Similarly for the dimension of the i^{th} costalk homology.

Exercise 4.28. Show that the Grothendieck K-group of the category of perverse sheaves on X is the same as the product over all strata X_α of the K-group of the category of local systems on X_α. Show furthermore that two perverse sheaves $\mathcal{P}$ and $\mathcal{P}'$ give the same element in the K-group if and only if for each stratum X_α,

$$\sum(-1)^i \mathbf{H}^i_S \mathcal{P} = \sum(-1)^i \mathbf{H}^i_S \mathcal{P}$$

where the alternating sum is taken in the K group of the group of local systems on S. HINT: Use the theorem describing the irreducible perverse sheaves.

4.8.2 Some ordinary sheaf theory

The point of this report is to remove ordinary sheaf theory form the subject of perverse sheaves. However, it is only fair to point out here that the i^{th} stalk homology of $\mathcal{P}$ forms a constructible sheaf, and the i^{th} costalk homology of $\mathcal{P}$ forms a constructible cosheaf.

DEFINITION. A *restricted path* in a stratified space X is a continuous path $\phi : [0,1] \longrightarrow X$ with the property that if $t < t'$ then the stratum containing t' is in the closure of the stratum containing t.

In other words, restricted paths keep going deeper and deeper into the singularities of the stratification.

DEFINITION. Two restricted paths are *restricted homotopic* if they are homotopic through restricted paths.

DEFINITION. The *restricted fundamental groupoid* of X is the category that has as its objects points x in X and as morphisms from x to x' the set of homotopy classes of restricted paths from x to x'. Composition is defined as usual.

Note that even if X is connected, there will be objects with no morphisms between them.

DEFINITION. A *constructible sheaf* on X is a contravariant functor form the restricted fundamental groupoid of X to the category of vector spaces over $\boldsymbol{Q}$. A *constructible cosheaf* on X is a covariant functor form the restricted fundamental groupoid of X to the category of vector spaces.

Exercise 4.29. Determine the invertible morphisms in the restricted fundamental groupoid of X. Show that any constructible sheaf (or constructible cosheaf) over X determines a local system over every stratum X_α of X.

Exercise 4.30. Define a constructible sheaf over X whose local system over $S = X_\alpha$ is $\mathbf{H}^i_S \mathcal{P}$. Define a constructible cosheaf over X whose local system over $S = X_\alpha$ is $\mathbf{H}^S_i \mathcal{P}$.

Exercise 4.31. Find a stratified space X, a point $x \in X$, an integer i and an exact sequence of perverse sheaves over X such that the associated sequence on the i^{th} stalk homology at x is not exact.

4.9 The microlocal stalks of a perverse sheaf

Now we come to a much more important class of local systems associated to a perverse sheaf. The stalk homology local systems of the last section, however interesting, are not exact functors on the category of perverse sheaves. Therefore, they are not the true stalks of the perverse sheaf. In this section, we describe some groups associated to a cotangent vector in X rather than a point in X. These groups are the true stalks. We call them the *microlocal stalks* of $\mathcal{P}$.

4.9.1 Some micro-local geometry

Micro-local geometry is, roughly speaking, geometry in the cotangent bundle T^*X of X.

Let $V = X_\alpha$ be a stratum of X. The *conormal bundle* $T^*_V X$ of V is the set of covectors $\xi \in T^*X$ over points y in V which annihilate the tangent space $T_y V$ of V.

Exercise 4.32. Prove that $T^*_V X$ is a submanifold of T^*X, and that the dimension of $T^*_V X$ is always $n = \dim X$, independent of the dimension of V.

DEFINITION. The *regular part* Λ^*_V of the conormal bundle $T^*_V X$ of V is the part of the conormal bundle to V that is not contained in the closure of the conormal bundle to any other stratum $X_\beta \neq X_\alpha$ of X.

Exercise 4.33. Let $\{df\}$ be the section of the cotangent bundle to X which associates the differential $df(x)$ to every point x in X. Show that a proper function $f : X \longrightarrow$ *reals* is Morse if and only if $\{df\}$ meets the conormal bundle to each stratum only transversely and only in its regular part,

DEFINITION. Let $S = \Lambda^*_V$, the regular part of the conormal bundle $T^*_V X$ of V. A *good family* of opposed pairs parameterized by S is a disjoint smooth family $(R(s), G(s)), s \in S$ that is disjoint and so that for each $s \in S$, $(R(s), G(s)) = (X_{f_s \geq r}, X_{f_s \leq g})$ for a family of functions $f_s : X \longrightarrow \boldsymbol{R}$ such that f_s has only one critical point p with critical value in $[g, r]$, and $df_s(p) = s$ and the signature of f_s at p is $-d/2$ where d is the (real) dimension of V (i.e. the Morse index of f_s at p is zero).

DEFINITION. The *conormal stalk local system* of the perverse sheaf $\mathcal{P}$ at the stratum V in X is the local system on $S = \Lambda^*_V$ whose fiber at $s \in S$ is

$\mathcal{P}^{d/2}(X, {}^{R(s)}_{G(s)})$where $(G(s), R(s))$ is any good family. The *stalk* of $\mathcal{P}$ at $s \in S$ is the fiber $\mathcal{P}^{d/2}(X, {}^{R(s)}_{G(s)})$

Proposition 4.10 *A good family of opposed pairs always exists. The stalk local system is canonically independent of the choice of the good family.*

The proof used Stratified Morse Theory, and is contained in [GM6]. The uniqueness part is a consequence of the fact that the Morse data is Normal Morse Data cross Tangential Morse Data, but since the Morse index is zero, the Tangential Morse Data has a canonical generator.

Exercise 4.34. Give the construction part of the proof.

Chapter 5

Formal Properties of Perverse Sheaves

In this chapter, we will summarize some of the formal properties of perverse sheaves.

5.1 Pushforwards and direct sums of Fary functors

As usual, $X = \bigcup X_\alpha$ will be a analytically Whitney stratified complex analytic space. Suppose that $Z = \bigcup Z_\alpha$ is another such stratified space.

DEFINITION. A *Morse stratified map* f from Z to X is a continuous map $f : Z \longrightarrow X$ which has the following properties:

- If (R, G) is any opposed pair in X, then $(f^{-1}(R), f^{-1}(G))$ is an opposed pair in Z.
- If (R_t, G_t) is any deformation of opposed pairs in X, then $(f^{-1}(R_t), f^{-1}(G_t))$ is a deformation of opposed pairs in Z.

Proposition 5.1 *If f is any proper complex analytic map from Z to X, then there are refinements of the stratifications of X and Z so that, with the refined stratifications, the map f is Morse stratified.*

PROOF. If f is proper and complex analytic, then a theorem of Thom and Mather shows that, after a refinement of the two stratifications, the map can be made "weakly stratified". Weakly stratified maps are Morse stratified (see [GM6]).

5.1.1 Pushforwards

DEFINITION. If $f : Z \longrightarrow X$ is Morse stratified, and $\mathcal{F}$ is a Fary functor on Z (for example, $\mathcal{F}$ may be a perverse sheaf), then the *pushforward Fary functor* $Rf_*\mathcal{F}$ is the Fary functor which assigns to the opposed pair (R,G) in X the value

$$Rf_*\mathcal{F}^i(X,\ {}^R_G) = \mathcal{F}^i(X,\ {}^{f^{-1}R}_{f^{-1}G})$$

Exercise 5.1. Show that the pushforward Fary functor $Rf_*\mathcal{F}$ is a Fary functor on X. Give an example to show that even if $\mathcal{F}$ is a perverse sheaf, $Rf_*\mathcal{F}$ need not be a perverse sheaf.

Exercise 5.2. Show that the Leray Spectral Sequence for the map f can be defined solely in terms of the Fary functor $Rf_*\mathcal{H}$, where $\mathcal{H}$ is the ordinary homology Fary functor on Z. In other words, show that for any Fary functor $\mathcal{F}$, there is a "Leray" spectral sequence abutting to $\mathcal{F}^*(X,\ {}^{\phi}_{\phi})$ which, when applied to $Rf_*\mathcal{H}$, gives the Leray spectral sequence for the map f. (This is the aspect of Fary functors that Fary himself had in mind. Fary, a student of Leray, claimed that this was close to the way Leray originally conceived of the Leray spectral sequence.)

5.1.2 Direct sums

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two Fary functors. We define the *direct sum* $\mathcal{F} \oplus \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ by the

$$(\mathcal{F} \oplus \mathcal{G})^i(X,\ {}^R_G) = \mathcal{F}^i(X,\ {}^R_G) \oplus \mathcal{G}^i(X,\ {}^R_G)$$

where the restriction maps of the direct sum are simply the direct sums of the respective restriction maps; likewise the coboundary maps of the direct sum are the direct sums of the coboundary maps.

At the risk of over-stressing the point, we wish to emphasize that anything constructed out of the direct sum Fary functor by an additive process is determined by the two Fary functors separately. Thus, the stalk homology is the direct sum of the stalk homologies; likewise for the costalk homology; likewise for the stalks. The monodromy maps are the direct sums of the separate monodromy maps.

Exercise 5.3. Show that the whole Leray spectral sequence for the Fary functor $\mathcal{F} \oplus \mathcal{G}$ is the direct sum of the Leray spectral sequence for $\mathcal{F}$ and the Leray spectral sequence for $\mathcal{G}$.

5.2 The decomposition theorem

We now return to the Decomposition Theorem, this time from the point of view of Fary functors. The was treated in a more elementary way in section 1.7 .

Theorem 5.2 *Let $f : Z \longrightarrow X$ be a proper complex algebraic map which we assume to be Morse stratified. Let Z' be the closure of a stratum of Z. Then there exists a unique finite list of irreducible enriched strata $E_\beta = (X_\beta, L_\beta)$ (X_β is a stratum of X, L_β is an irreducible local system over X_β) and for each enriched stratum E_β there is a unique Laurent polynomial $\phi^\beta(v) = \sum_{j=-N}^{N} \phi_j^\beta v^j$ with non-negative coefficients ϕ_j^β such that we have the direct sum decomposition of Fary functors*

$$Rf_\star IC(Z', Q) = \bigoplus_{\beta)} \bigoplus_{j\in Z} T^j IC(X_\beta, L_\beta) \otimes Q^{\phi_j^\beta} \quad .$$

Furthermore, if the map f is projective, then each finite Laurent series satisfies $\phi_j^\beta = \phi_{-j}^\beta$ and, for $j \geq 0$, $\phi_j^\beta \geq \phi_{j+2}^\beta$.

Explanation of notations $Rf_\star$ and $\oplus$ are explained in the last section. $\mathcal{T}$ is the shift functor: $(T^j \mathcal{F})^i(X, \frac{R}{G}) = \mathcal{F}^{i+j}(X, \frac{R}{G})$.

Exercise 5.4. Show that a much stronger uniqueness statement holds: Show that the list of enriched strata E_k and the finite Laurent series $\phi^\beta(v) = \sum_{j=-N}^{N} \phi_j^\beta v^j$ are determined once we know the stalk homology local systems of $Rf_\star IC(Z', Q)$. In fact, using the Artin-Schrier theorem for local systems, show that for any Fary functor $\mathcal{F}$, there is a unique finite list of enriched strata $E_\beta = (X_\beta, L_\beta)$ where X_β is a stratum of X, L_β is an irreducible local system over X_alpha, and there is a unique Laurent polynomial $\phi^\beta(v) = \sum_{j=-N}^{N} \phi_j^\beta v^j$ such that $\mathcal{F}$ and the formal sum

$$\sum_\beta \sum_{j\in Z} \phi_j^\beta T^j IC(X_\beta, L_\beta)$$

have the same stalk homology local systems for each stratum X_α of X. (Here the equality is to be taken in the free group generated by the irreducible local systems.) What is the first miracle here is that the local systems are all irreducible and the coefficients of ϕ^β are all non-negative.

Exercise 5.5. Consider the map from the Riemann Sphere (complex projective 1-space) to itself which has degree 2 and which branches at four points. (Show topologically that there is such a map.) Apply the decomposition theorem to this map to verify the calculation of Example 2 in section 1.2 .

Exercise 5.6. Prove Deligne's theorem that if X and Z are nonsingular, and if $f : Z' \longrightarrow X$ is a topological fibration, then the Leray-Serre spectral sequence for f collapses at E^2.

5.3 Borel-Moore-Verdier Duality.

DEFINITION. If $\mathcal{P}$ is a perverse shear on X, then the *Borel-Moore-Verdier dual* $\mathcal{DP}$ of $\mathcal{P}$ is the perverse sheaf obtained from $\mathcal{P}$ as follows:

1. $\mathcal{DP}^i(X, {}^R_G)$ is the vector space dual of $(\mathcal{P}^{-i}(X, {}^G_R))^*$
2. (R,G) and (R',G') are two opposed pairs in X such that $R \supseteq R'$ and $G \subseteq G'$ then $\mathcal{R}^{*\mathcal{DP}} : \mathcal{DP}^i(X, {}^R_G) \longrightarrow \mathcal{DP}^i(X, {}^{R'}_{G'})$ is the adjoint of $\mathcal{R}^{*\mathcal{P}}$: $\mathcal{P}^{-i}(X, {}^{G'}_{R'}) \longrightarrow \mathcal{P}^{-i}(X, {}^G_R)$
3. If (R,C) and $(\sim C, G)$ are opposed subsets of X, $\partial^{*\mathcal{DP}} : \mathcal{DP}^i(X, {}^R_C) \longrightarrow \mathcal{DP}^{i+1}(X, {}^{\sim C}_G)$ is the adjoint of $\partial^{*\mathcal{P}} : \mathcal{P}^{-i-1}(X, {}^G_{\sim C}) \longrightarrow \mathcal{P}^{-i}(X, {}^C_R)$

In summary, the Borel-Moore-Verdier dual $\mathcal{DP}^i(X, {}^R_G)$ is defined by dualizing the vector space, switching the roles of R and G, negating i, and taking the adjoints of $\mathcal{R}^*$ and ∂^*.

Exercise 5.7. Check that $\mathcal{DP}$ is a perverse sheaf if $\mathcal{P}$ is, i.e. that the modified Eilenberg-Steenrod axioms are self dual.

Proposition 5.3 *If p and p^* are dual close to middle perversities for the closure of the stratum X_α, and L and L^* are dual local systems on X_α, then the Borel-Moore-Verdier dual to $I^pC(X_\alpha, L)$ is $I^{p^*}C(X_\alpha, L^*)$.*

In particular, if L is self dual and m is the middle perversity, then the sheaf $IC(X_\alpha, L)$ is Borel-Moore-Verdier self dual.

Exercise 5.8. Suppose that x is a point in X. Show that the i^{th} stalk homology at x of a perverse sheaf is the $(-i)^{\text{th}}$ costalk homology of the Borel-Moore-Verdier dual perverse sheaf.

Generalization. The Borel-Moore-Verdier dual is defined for a general Fary functor by the same formulas. The formula for the dual of an intersection homology with near middle perversity extends to intersection homology with arbitrary perversity.

Exercise 5.9. Show that the dual of the pushforward of a Fary functor is the pushforward of the dual.

5.4 Perverse sheaves are locally defined

Perverse sheaves form something that is called a "stack". This means that to know a perverse sheaf locally everywhere is the same as to know it globally. Rather than going through the formalism of stacks, we will give an equivalent statement.

Let $U = \{U_\gamma\}$ be an open cover of X. We call an opposed pair (R, G) in X *U small* if for some γ, $R^0 \cup G^0 \cup U_\gamma = X$, i.e. the support of (R, G) is contained in a single set U_γ of the open cover. Two opposed pairs (R_1, G_1) and (R_2, G_2) in X are *U comparable* if for some single γ, both $R_1^0 \cup G_1^0 \cup U_\gamma = X$ and $R_2^0 \cup G_2^0 \cup U_\gamma = X$, i.e. their supports are contained in a single open set U_γ.

DEFINITION. A *U small* perverse sheaf on a stratified manifold on X is a device $\mathcal{P}$ which does the following three things:

- $\mathcal{P}$ assigns to each U small opposed pair (R, G) in X and to each integer i a finite dimensional vector space over the rationals $\mathcal{P}^i(X, \frac{R}{G})$.
- Whenever (R, G) and (R', G') are two U small opposed pairs in X that are U comparable such that $(R, G) \sqsupseteq (R', G')$, then $\mathcal{P}$ gives a map $\mathcal{R}^*$.

$$\mathcal{R}^* : \mathcal{P}^i(X, \tfrac{R}{G}) \longrightarrow \mathcal{P}^i(X, \tfrac{R'}{G'})$$

- Whenever (R, C) and $(\sim C, G)$ are U small opposed subsets of X that are U comparable, $\mathcal{P}$ gives a map denoted ∂^*

$$\partial^* : \mathcal{P}^i(X, \tfrac{R}{C}) \longrightarrow \mathcal{P}^{i+1}(X, \tfrac{\sim C}{G})$$

such that all of the modified Eilenberg-Steenrod axioms hold whenever they make sense.

Exercise 5.10. Give a precise statement of the following, and prove it: "To give a U small perverse sheaf on X is the same thing as to give a perverse sheaf on each open set U_γ, and to give an identification of the two perverse sheaves on $U_\gamma \cap U_{\gamma'}$, subject to the condition that on $U_\gamma \cap U_{\gamma'} \cap U_{\gamma''}$ the identifications are compatible."

Obviously, to any perverse sheaf on X and to any open cover U, we can assign a U small perverse sheaf by simply restricting the data. This gives us a forgetful functor from perverse sheaves to U small perverse sheaves.

Theorem 5.4 *The category of perverse sheaves is equivalent to the category of U small perverse sheaves. The equivalence is given by the forgetful functor.*

Exercise 5.11. Prove the theorem, using the idea that the small complex of the last chapter can always be reconstructed from a small perverse sheaf.

Exercise 5.12. Define a U small Fary functor. Show that the statement corresponding to the theorem above for Fary functors is false. Hint: Use the example of $Rf_*\mathcal{F}$ where $\mathcal{F}$ is the constant perverse sheaf on the 3-sphere, and f is the Hopf fibration.

5.5 The logarithmic and sublogarithmic intersection homology perverse sheaves

So far, only the middle intersection homology perverse sheaf has had a special role in the theory of perverse sheaves. However, the logarithmic and sublogarithmic ones are also very important.

DEFINITION. Let X_α be a stratum of X. Then $P(X, X_\alpha)$ is the full subcategory of the category $P(X)$ of perverse sheaves on X whose support lies on the closure of X_α.

If $\mathcal{P}$ is in $P(X, X_\alpha)$, then the restriction of $\mathcal{P}$ to X_α is a perverse sheaf on a manifold, so it is just a local system. We call this local system $\mathcal{P}|X_\alpha$.

Theorem 5.5 *For any perverse sheaf $\mathcal{P}$ in $P(X, X_\alpha)$, there is a unique map from $I^s C(X_\alpha, \mathcal{P}|X\alpha)$ to $\mathcal{P}$ which induces the identity map on X_α. Likewise, there is a unique map from $\mathcal{P}$ to $I^l C(X_\alpha, \mathcal{P}|X\alpha)$ which induces the identity map on X_α.*

Exercise 5.13. Rephrase this, showing that $I^s C(X_\alpha, \mathcal{P}|X\alpha)$ and $I^l C(X_\alpha, \mathcal{P}|X\alpha)$ give left and right adjoint functors to the functor that restricts an object in $P(X, X_\alpha)$ to X_α.

Note that, by either part of this theorem, there is a unique map form $I^s C(X_\alpha, \mathcal{P}|X\alpha)$ to $I^l C(X_\alpha, \mathcal{P}|X\alpha)$ which restricts to the identity map on X_α. We call this the *canonical* map.

Theorem 5.6 *The image of the canonical map $I^s C(X_\alpha, \mathcal{P}|X\alpha) \longrightarrow I^l C(X_\alpha, \mathcal{P}|X\alpha)$ is the middle perversity intersection homology sheaf $IC(X_\alpha, \mathcal{P}|X\alpha)$.*

The proof of these statements will be omitted (although they could be done within the present context). They are in [BBD].

Exercise 5.14. Show using, the theorem above, that the irreducible perverse sheaves are the middle perversity intersection homology sheaves (with irreducible coefficients).

Exercise 5.15. Show further that every perverse sheaf has a finite composition series whose successive quotients are irreducible perverse sheaves. (Work by induction, starting with the largest strata.)

5.6 Perverse homology

The notion of a Fary functor is a lot more general than the notion of a perverse sheaf. A number of exercises and a lot of theory have been devoted to showing that perverse sheaves have some nice properties that a general Fary functor does not have.

The idea of this section is to show that to an arbitrary Fary functor, we can associate a sequence of perverse sheaves called the ***perverse homology sheaves*** of the Fary functor.

Recall that to determine a perverse sheaf, it is enough to give its values on disjoint pairs (R, G).

DEFINITION. Let $\mathcal{F}$ be a Fary functor. Then the j^{th} *perverse cohomology sheaf* ${}^p\mathrm{H}^j\mathcal{F}$ of $\mathcal{F}$ is the perverse sheaf that for each disjoint pair (R, G) in X the value of ${}^p\mathrm{H}^j\mathcal{F}^i(X, \frac{R}{G})$ is the i^{th} homology of the chain complex constructed in the following way: Choose a function f which is self-indexing and Morse in $[g, r]$, so that $(R, G) = (X_{f\geq r}, X_{f\leq g})$, and take the chain complex

$$\begin{array}{c} \vdots \\ \uparrow \partial^* \\ \mathcal{F}^{i+j+1}(X, \frac{R(i+1)}{G(i+1)}) \\ \uparrow \partial^* \\ \mathcal{P}^{i+j}(X, \frac{R(i)}{G(i)}) \\ \uparrow \partial^* \\ \mathcal{P}^{i+j-1}(X, \frac{R(i-1)}{G(i-1)}) \\ \uparrow \partial^* \\ \vdots \end{array}$$

where $R(i) = X_{f\geq -i+1/2}$ and $G(i) = X_{f\leq -i-1/2}$.

We will omit the verification that this, in fact, gives a perverse sheaf. (At least the dimension axiom is clear.)

Exercise 5.16. Show that for any opposed pair (R, G), there is a spectral sequence whose E^2 term is the direct sum over i and j of all of the groups ${}^p\mathrm{H}^j\mathcal{F}^i(X, \frac{R}{G})$, and which abuts to the direct sum over k of $\mathcal{F}^k(X, \frac{R}{G})$.

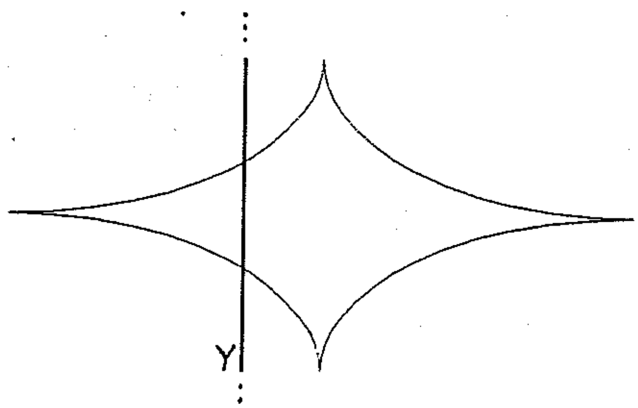
***Remark.** The notion of perverse homology was inspired by the following analogy: A perverse sheaf is to a sheaf, as a Fary functor is to a complex of sheaves, as the perverse homology sheaves are to the usual homology sheaves.

5.7 Restrictions

Suppose that X is a complex Whitney stratified complex manifold. Let $\theta : Y \hookrightarrow X$ be the inclusion of a complex submanifold of X.

DEFINITION. The inclusion θ is *normally nonsingular* if Y is transverse to each stratum X_α of X.

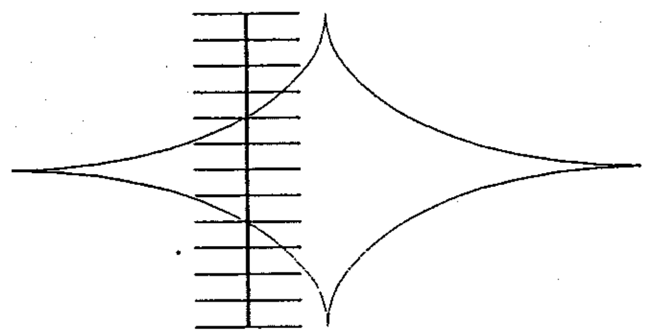
In this case, the decomposition $Y = \bigcup_\alpha (Y \cap X_\alpha)$ is a complex Whitney stratification of Y.



A normally nonsingular inclusion

If $\theta : Y \hookrightarrow X$ is a normally nonsingular inclusion of a complex submanifold of X, then there is a way of lifting opposed pairs (R, G) in Y to X. First, we choose a differentiable tubular neighborhood $\pi : T \longrightarrow Y$ of Y. (Note that it is impossible in general to choose this in such a way that the projection π

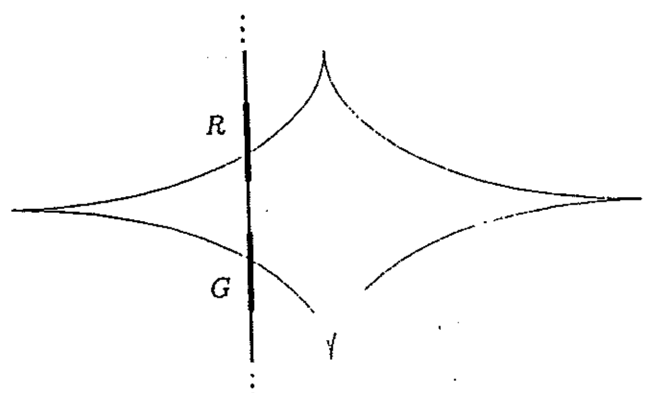
preserves strata, although the smallest counterexample is in three dimensions.)



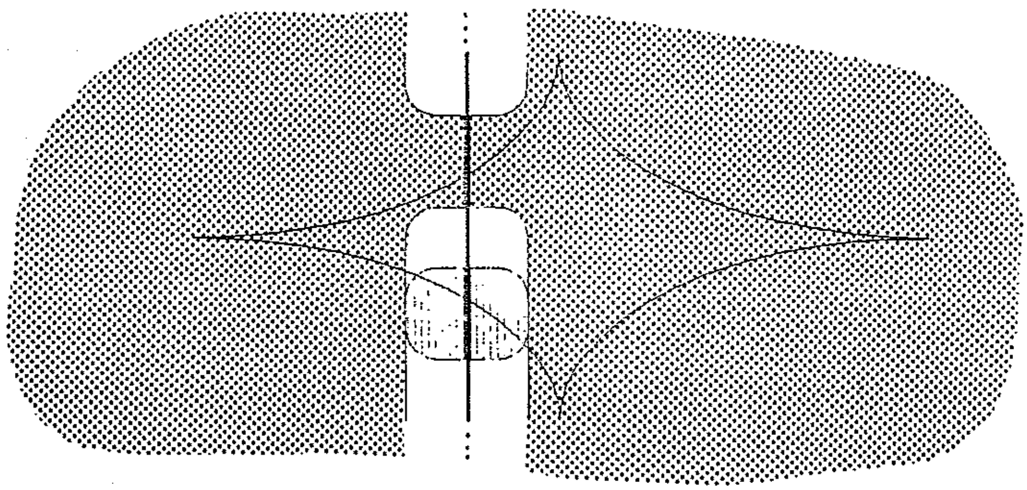
A tubular neighborhood T

Then we choose a distance function $\rho : T \longrightarrow \boldsymbol{R}^{\geq 0}$ which measures the distance away from Y.

DEFINITION. If (R, G) is an opposed pair in Y, then $(R(\epsilon), G(\epsilon))$ is the pair in X defined by the following pictures:



The opposed pair (R, G) in Y



The opposed pair $(R(\epsilon), G(\epsilon))$ in X

Exercise 5.17. Make this definition precise. Show that for every opposed pair (R, G) in Y, there exists an $\epsilon > 0$ so that $(R(\epsilon), G(\epsilon))$ is an opposed pair in X. (The choice of ϵ will depend on (R, G). The danger is that the $R(\epsilon)$ or $G(\epsilon)$ will not be smoothly enclosed.)

DEFINITION. Suppose that $\mathcal{F}$ is a Fary functor on X and that $\theta : Y \hookrightarrow X$ is a normally nonsingular inclusion of a complex submanifold of X. Then $\theta^*\mathcal{F}$ is the Fary functor on Y defined as follows: Given an opposed pair (R, G) on Y,

$$\theta^*\mathcal{F}^i(Y, {}^R_G) = \mathcal{F}^{i+\mathrm{codim}Y/2}(X, {}^{R(\epsilon)}_{G(\epsilon)})$$

for small enough ϵ.

Exercise 5.18. Complete the definition. Show that for small enough ϵ, the expression on the right hand side does not depend on ϵ. Show how to define the restriction maps and the coboundary maps.

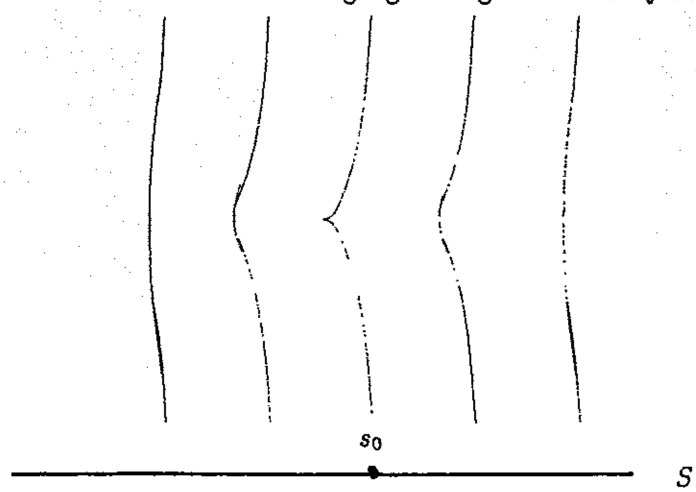
Exercise 5.19. Show that if $\mathcal{F}$ is perverse, then $\theta^*\mathcal{F}$ is perverse.

5.8 Vanishing cycles and nearby cycles

In this section, we will consider the following situation: Suppose that $f : X \longrightarrow S$ is a flat map of nonsingular complex algebraic varieties. We think of it as a family of algebraic varieties $f^{-1}(s)$ parameterized by points $s \in S$. (We do not assume f to be proper.) Suppose that $\mathcal{P}$ is a perverse sheaf on X, and that $s_0 \in S$ is a point. Assume that X and S are stratified so that f is a Morse stratified mapping, so that $\mathcal{P}$ is constructible with respect to the stratification

$X = \bigcup_\beta X_\beta$ and that s_0 is a single stratum of S. Let S_0 be the largest stratum of S and let B_{s_0} be a small ball centered around s_0, as in section 4.8.1 .

The fiber $f^{-1}(s_0)$ over s_0 is affectionately known as *the singular fiber*, and the fiber over any point $s \in S_0$ is known as *the general fiber*. The following picture of the situation has been serving algebraic geometers for years:



We will sketch the construction a perverse sheaf over $f^{-1}(s_0)$: the "sheaf of nearby cycles" $R\Psi(\mathcal{P})$, and an action of the fundamental group $\pi_1(S_0 \cap G_{s_0})$ on it.

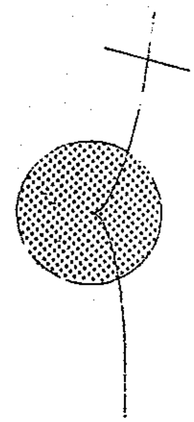
5.8.1 The retraction r

To begin with, we will assume that f is proper. This assumption will be removed later.

DEFINITION. A *normal slice* to a stratum X_α at $x \in X_\alpha$ is a submanifold $N \subseteq X$ which intersects X_α transversely at x and at no other point. (It follows that $\dim X_\alpha + \dim N = \dim X$.) A *round normal slice* to X_α at x is the intersection $N \cap B_x$, where B_x is a ball around centered at x, as in 4.8.1.

(Technical note: The limit on the size of the ball B_x required in order for the intersection to be considered "round" will depend on N. It must be small enough that if we restratify a neighborhood of x in X so that N is a union of strata, then B_x is smoothly enclosed, and so is every round ball centered at x that is smaller than B_x.)

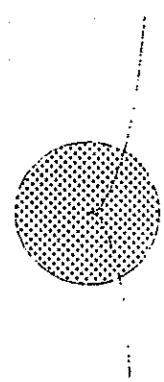
Proposition 5.7 ([GM6]) *There is a stratum preserving homeomorphism between any two round normal slices through X_α at x.*



Round normal slices through two points in $f^{-1}(s_0)$

We refine the stratification of X if necessary so that the special fiber $f^{-1}(s_0)$ is a union of strata X_α of X. Then, there is a map r from a neighborhood U of $f^{-1}(s_0)$ to $f^{-1}(s_0)$ with the property that the inverse image of any point x is a round normal slice through a point x' on the same stratum as x. This map is constructed rigorously in [G1] (see also [GM9]). The following is the idea of the construction.

First, choose the "smallest" stratum X_1 of $f^{-1}(s_0)$, i.e. let X_1 be any closed stratum. Choose any differentiable tubular neighborhood $\pi : T_1 \longrightarrow X_1$ of X_1. Its fibers will be normal slices. Let U_1 be the points x in T such that the distance from x to $\pi_1(x)$ is $\leq \epsilon_1$, for very small ϵ_1.

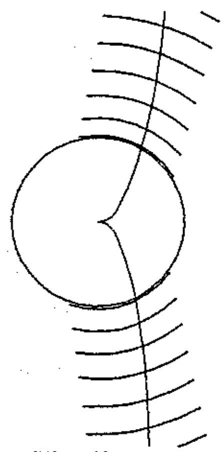


The neighborhood U_1

Next, choose any stratum X_2 of $f^{-1}(s_0)$ that has no stratum other than X_1 in its closure. Choose any differentiable tubular neighborhood $\pi_2 : T_2 \longrightarrow$

$(X_2 - U_1^0)$ of $X_2 - U_1^0$, where U_1^0 is the interior of U_1. We demand the following condition:

- If a point $x \in X_2$ lies in the boundary $U_1 - U_1^0$ of U_1, then the whole fiber $\pi_2^{-1}(x)$ lies in the boundary of U_1, and π_1 projects the whole fiber $\pi_2^{-1}(x)$ to a single point $\pi_1(x)$.



The fibers of π_2

Just as before, the fibers of π_2 will be normal slices to X_2. Let U_2 be the points x in T_2 such that the distance from of distance from x to $\pi_2(x)$ is $\leq \epsilon_2$, for ϵ_2 very much smaller than ϵ_1.

We proceed in a similar way, always insisting that the fiber $\pi_j^{-1}(x)$ be contained in all of the boundaries of previous tubular neighborhoods U_l that contain x.

Now, we define U to be the union of all of the U_l. We would like to define r to be π_j, for whichever one applies. The only problem is that this map would not be continuous. In our example above, its image would look like this:

Image of the bad definition of r

which is disconnected, whereas U is connected. The solution, of course, is to perform a stretching operation. For example, in our case we take a map m_2 from $X_2 - U_1^0$ to $\overline{X_2}$ with the property that it maps points far away from U_1 to themselves, and it maps $X_2 - U_1$ diffeomorphically onto X_2.

The map m_2

Now, we define r to be π_1 on U_1 and $m \circ \pi_2$ on U_2. This map is continuous, and it has the required properties. The general construction is similar ([G1]).

5.8.2 The perverse sheaves $R\Psi(\mathcal{P})$

Consider a point s in $A = B_{s_0} \cap S_0$. If the ball B_{s_0} has been chosen small enough, then the restriction $(r|f^{-1}(s)) : f^{-1}(s) \longrightarrow f^{-1}(s_0)$ will be proper whenever s

is in $B_{s_0} \cap S_0$. Let us denote by r also the restriction $r|f^{-1}(s)$ and by θ the inclusion $f^{-1}(s) \hookrightarrow X$.

DEFINITION. $R\Psi\mathcal{P} = Rr_*\theta^*\mathcal{P}$

Theorem 5.8 ([GM9]) *The sheaf $R\Psi\mathcal{P}$ is perverse.*

In the event that f is not proper, we can reduce to the case when f is proper by factoring f by a composition $X \hookrightarrow \bar{X} \longrightarrow S$ where X is dense in $\bar{X}$ and $\bar{X} \longrightarrow X$ is proper. Then we carry out the construction of r for $\bar{X} \longrightarrow S$. We replace $f^{-1}(s)$ in the construction by the subset of it that maps to $f^{-1}(s_0)$ under r.

Exercise 5.20. Extend this construction to give a perverse sheaf on $A \times f^{-1}(s_0)$ whose restriction to $s \times f^{-1}(s_0)$ is the perverse sheaf $R\Psi\mathcal{P}$ constructed above.

Exercise 5.21. Imitate the monodromy constructions of Chapter 3 to give a monodromy map of $\pi_1(A, s)$ on $R\Psi\mathcal{P}$.

Chapter 6

Applications of Intersection Homology and Perverse Sheaves

Topology was originally meant to be applied to other domains of mathematics. At the end of the introduction to the monumental 1895 paper that founded modern algebraic topology (then called *analysis situs*), Poincaré describes why he is doing it:

il y a donc une *Analysis situs* à plus de trois dimensions, comme l'ont montré Riemann et Betti.

Cette science nous fera connaître ce genre de relations, bien que cette connaissance ne puisse plus être intuitive, puisque nos sens nous font défaut. Elle va ainsi, dans certains cas, nous rendre quelques-uns des services que nous demandons d'ordinaire aux figures de Géométrie.

Je me bornerai à trois exemples.

La classification des courbes algébriques en genres repose, d'après Riemann, sur la classification des surfaces fermées réelles, faite au point de vue de l'*Analysis situs*. Une induction immédiate nous fait comprendre que la classification des surfaces algébriques et la théorie de leurs transformations birationnelles sont intimement liées à la classification des surfaces fermées réelles de l'espace à cinq dimensions au point de vue de l'*Analysis situs*. M. Picard, dans un

Mémoire couronné par l'Académie des Sciences, a déjà insisté sur ce point.

D'autre part, dans une série de Mémoires insérés dans le *Journal de Liouville*, et intitulés : *Sur les courbes définies par les équations différentielles*, j'ai employé l'*Analysis situs* ordinaire à trois dimensions à l'étude des équations différentielles. Les mêmes recherches ont été poursuivies par M. Walther Dyck. On voit aisément que l'*Analysis situs* généralisée permettrait de traiter de même les équations d'ordre supérieur et, en particulier, celles de la Mécanique céleste.

M. Jordan a déterminé analytiquement les groupes d'ordre fini contenus dans le groupe linéaire à *n* variables. M. Klein avait antérieurement, par une méthode géométrique d'une rare élégance, résolu le même problème pour le groupe linéaire à deux variables. Ne pourrait-on pas étendre la méthode de M. Klein au groupe à *n* variables ou même à un *groupe continu quelconque* ? Je n'ai pu jusqu'ici y parvenir, mais j'ai beaucoup réfléchi à la question et il me semble que la solution doit dépendre d'un problème d'*Analysis situs* et que la généralisation du célèbre théorème d'Euler sur les polyèdres doit y jouer un rôle.

Je ne crois donc pas avoir fait une œuvre inutile en écrivant le présent Mémoire; je regrette seulement qu'il soit trop long; mais, quand j'ai voulu me restreindre, je suis tombé dans l'obscurité; j'ai préféré passer pour un peu bavard.

(The apology at the end is for taking nearly 100 pages to write the paper which gave us the fundamental group, Poincaré duality, the topological invariance of the Euler characteristic, the Poincaré conjecture, the first workable version of Riemann and Betti's homology, and a preview of DeRham cohomology.) Poincaré's assertion made through three examples, is that topology should be applied to algebraic geometry, to differential equations and analysis, and to Lie group theory. These are, in fact, the areas where intersection homology and perverse sheaves have been successfully applied (just as earlier topological techniques have bourn out Poincaré's prediction).

The following is an "annotated bibliography" style summary of some of the applications of intersection homology and perverse sheaves. Aside from the traditional excuses for being so brief ("ignorance of the author", "lack of space"), I have the excuse that there have been recent and excellent survey articles on many of these subjects. (See the items marked with a $\star$ in the bibliography).

Applications to algebraic geometry and number theory Intersection homology and perverse sheaves, which have been defined here by clearly transcendental methods (Morse theory !), have purely algebraic meanings. This is because intersection homology can be constructed, and perverse sheaves can be defined, using only sheaf theoretic operations. These operations make purely algebraic sense for l-adic sheaves. (Recall from the introduction that we have avoided this sheaf theoretic approach in this report, for various reasons.)

As a consequence of the fact that intersection homology has a purely algebraic definition, it makes sense for algebraic varieties in characteristic p. Several of the applications of intersection homology hinge on this fact.

Zeta functions Suppose that a compact nonsingular algebraic variety Y is defined over the integers. Then it has reductions Y_p of Y mod p for every prime p, which are algebraic varieties over GF_p, the finite field with p elements. The Frobenius automorphism acts on the l-adic cohomology of Y_p with certain eigenvalues. The Hasse-Weil zeta function of Y is a way of encoding all these eigenvalues into a complex analytic function. The Poincaré duality of ordinary cohomology on Y translates into the functional equation of the Hasse-Weil zeta function. For each k, there are a finite number of points of Y_p which are rational over GF_{p^k}. The Weil conjectures, proved by Deligne, show that knowledge of the Hasse-Weil zeta function of Y is equivalent to knowledge of these finite numbers. A central problem of arithmetic algebraic geometry is understanding these Hasse-Weil zeta functions.

Now suppose that Y is singular. Deligne in his proof of the Weil conjectures had already isolated the properties of "sheaves" on Y that still make them give good Hasse-Weil zeta functions on Y: He called these "punctually pure" sheaves [D]. These turned out to be the intersection homology sheaves with coefficients in a local system of geometric origin. Their Hasse-Weil zeta function, similarly determined by the eigenvalues of the Frobenius automorphism, still satisfies the functional equation. They still count points of Y_p which are rational over GF_{p^k}, although now they count them with multiplicities which depend on the singularities. Therefore, a modernized generalization of the goal of arithmetic algebraic geometry mentioned above is to understand the Hasse-Weil zeta functions of intersection homology sheaves.

This generalization of the problem to singular spaces is especially useful for the following reason: The varieties for which one has a chance understanding the Zeta functions in the near future are Baily-Borel compactifications of modular varieties. These varieties are singular. (In fact, if you will take the testimony of someone who has spent many years studying singularities, they have the most complicated and interesting singularities of any spaces that I have met.) It is a major goal of the "Langlands school" to study the Hasse-Weil zeta functions of the intersection homology of the Baily-Borel compactifications of modular varieties. See [BL] for the first result in this direction. (It is fortunate that the proof [SS], [Lo] of the Zucker conjecture [Z2] relates these same intersection homology groups, via L^2-cohomology, to representation theory.)

Another relation between automorphic forms and perverse sheaves is contained in some conjectural constructions of the Langlands correspondence in the function field case [Lau],[Gi].

Other applications to algebraic geometry arise simply because perverse sheaves have such nice algebraic properties, e.g. [Br2],[Ki2],Ki3],[Z4].

Applications to representation theory

Weyl groups The first application of intersection homology to representation theory was to construct in a geometric way a basis for the Iwahori-Hecke algebra, which is a q-analogue of the Weyl group [KL1], [KL2]. This can be seen to arise from the Decomposition Theorem. The basis elements of the Hecke algebra are intersection homology sheaves. The multiplication is by a convolution which involves a pushforward, which preserves intersection homology sheaves by the decomposition theorem [Sp]. This basis has many remarkable properties.

Another application to representations of the Weyl group arises because the nearby cycle functor furnishes perverse sheaves. Take the adjoint quotient map of the Lie algebra of a complex reductive Lie group. This is flat, so the theory applies. Take nearby cycles of the constant sheaf at the most singular fiber. The fundamental group that acts on this, as in section 5.8, acts through the Weyl group. The nearby cycles sheaf turns out to be semisimple, and it has one isotypical component for each irreducible representation of the Weyl group. In this way, the irreducible representations of the Weyl group are parameterized by intersection homology sheaves on the singular fiber. This is the Springer parameterization. See [L6] and [BM].

Finite dimensional representations of algebraic Lie groups. Lusztig and independently Kashiwara have recently found a canonical basis for any finite dimensional representation of a complex algebraic Lie group. Lusztig's procedure goes by finding an intersection homology sheaf basis for the q-analogue of the positive part of the universal enveloping algebra. Here, multiplication is given by convolution, which preserves intersection homology sheaves by the decomposition theorem, just as for the Iwahori- Hecke algebra described above. The canonical basis for the positive part of the universal enveloping algebra projects to the desired basis in the finite dimensional representation. See [Lu5].

Infinite dimensional representations of Lie algebras. Several interesting categories of infinite dimensional representations of reductive Lie algebras turn out to be equivalent to the category of perverse sheaves on the Flag manifold of the group, stratified with a particular stratification. The first result in this direction came from the proof [BK], [BB] of the Kazhdan-Lusztig conjecture [KL1]. The proof in [BB] included many more cases, for example the case of Harish-Chandra modules. With this equivalence of categories, questions about the representations are reduced to questions in topology. See, for example, [MirV].

Representations of finite Chevalley groups. A finite Chevalley group is the set of rational points of an algebraic variety over some finite field. Therefore, it makes sense to talk about a perverse sheaf or an intersection homology on it. Lusztig has found some intersection homology sheaves called Character sheaves which are particularly interesting [L4], [L7], [MS]. They produce class functions on the Chevalley group by the trace of the action of Frobenius on their stalk cohomology. In some cases, these functions are the characters of the group. In other cases, the matrix relating them to the characters is nearly diagonal.

Quantum groups. The q-analogue of the positive part of the universal enveloping algebra mentioned above is a quantum group. It would be nice to find an intersection homology construction of the whole quantum group, not just the positive part. This has been done for $SL(n)$ in [BLM]

Applications to analysis In some sense, both intersection homology and perverse sheaves grew out of analysis. In the case of intersection homology, Cheeger and Zucker had already considered complexes of differential forms that turn out to give intersection homology. This was for spaces with metrically conical singularities in the case of Cheeger (all compact stratified spaces can be realized in this way), and for curves with the Poincaré metric, in the case of Zucker [C1],[C2],[Z1]. Now, the relations between L^2 cohomology and intersection homology is a central idea, as indicated in Chapter 1.

In the case of perverse sheaves, Kashiwara and Kawai had made a deep study of the properties of "holonomic $\mathcal{D}$-modules with regular singularities" which correspond to maximally overdetermined systems of partial differential equations [KK],[K1],[K2],[Me]. This is an Abelian category on a complex analytic manifold X. This category is the same as the category of perverse sheaves on X. In fact, in this case this category led to the discovery of perverse sheaves.

In both cases, the analytic object was already known to be interesting, but it wasn't clear that it was a topological invariant.

The relation of perverse sheaves to holonomic $\mathcal{D}$-modules is a step of the proof of the relation to infinite dimensional Lie algebra representations.

Applications to topology Perverse sheaves and intersection homology are topology. Still, one can ask about their applications to more classical topology.

One application is to construct interesting bordism theories [GP], [Pa], [S]. In [Pa], the cobordism theory constructed can be used to prove the Hauptvermutung. Another subject is the classification of stratified spaces, in the spirit of Surgery theory [CS2], [CW],[Qu1], see [W]. One of the results of this is another disproof of the integral Hodge conjecture. In this subject, the restriction to rational coefficients that we have taken here completely misses the point. Other coefficient rings are important [GS], [CS].

An important theme in some of this work is the L class of a singular space that was the very first application of intersection homology (in 1974, [GM3]), and its generalizations.

Applications to combinatorics Any sufficiently rational polyhedron determines an algebraic variety called a "toric" variety. The fact that the Hard Lefschetz Theorem holds for the intersection homology of the toric variety gives new results about the possible combinatorics of the polyhedra [St].

Chapter 7

Appendices

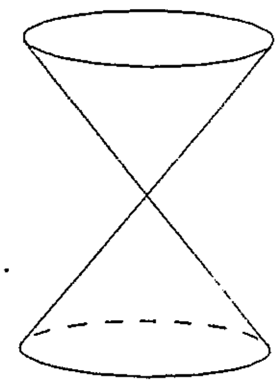
7.1 Appendix 1: Stratified spaces.

7.1.1 What should a singular space be?

This section is designed to give an introduction to stratification theory sufficient for the purposes of these notes. A much more complete survey of stratification theory is contained in [GM6], Chapter 1.

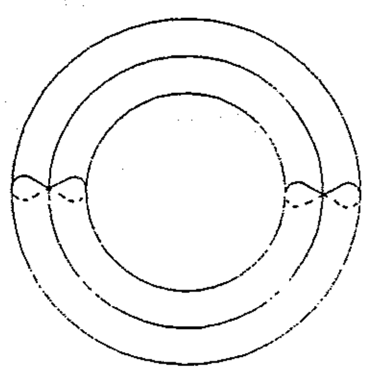
We want to include singular spaces in the class of spaces in which we do geometry. As described in the Introduction, one reason to be interested in singular spaces is that singular spaces are naturally associated to many mathematical objects of central importance. Anlther reason is that it is natural to consider topological spaces of "finite type", i.e. topological spaces that require finitely much data to specify them. Topological spaces of "finite type" include singular spaces.

One example of the type of singular spaces that interest us is the cone



The cone $x^2 = y^2 = z^2$.

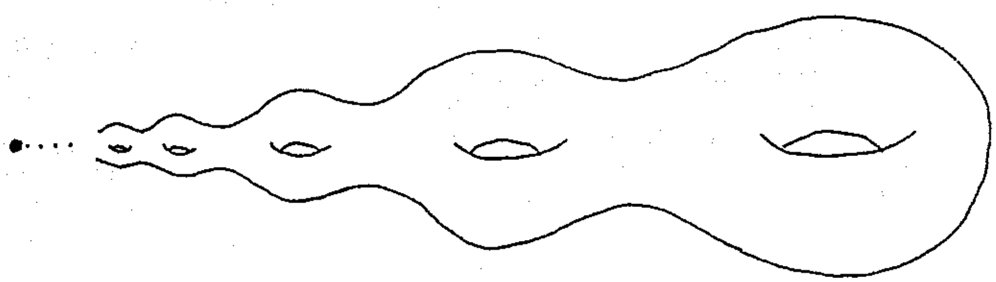
which has a point singularity, i.e. point at which it is not locally a manifold, at the origin. Another example is the figure 8 crossed with the circle S^1



$8 \times S^1$

which has a singularity along a circle.

However we don't want to consider arbitrary topological spaces, because they can be too complicated. For example, we don't want to consider a cantor set, since it has "infinite topological type". Also, we don't want to consider the one point compactification of the surface of infinite genus



The one point compactification of the surface of infinite genus.

even though it has only one "singularity", because it has "infinite topological type" at that singularity. In some sense, the quest for a definition of a stratified space is the same as the quest for the definition of a space of "finite topological type".

One can see from these examples that a good definition of stratified spaces is needed, and it is not obvious how to make one. In fact, giving a workable definition was historically a very difficult problem, and it very possible that the definitions we have today will not turn out to be the best ones.

The first idea is not to define a singular space in the abstract, but rather to define a singular subspace of a finite dimensional manifold X. This poses no loss of generality, because if a space does not admit an embedding in a manifold, we would declare it to be of "infinite topological type" and we would throw it away.

The second idea is to explicitly give ourselves, as part of the definition, a finite disjoint decomposition of our space into smooth manifolds of various dimensions called *strata.* (The singular of strata is *stratum.*) The decomposition of X into the strata is called the *stratification* of X. Certainly the two examples given above have such a decomposition. The cone has a stratification which consists of the origin, a zero dimensional manifold, union the rest, which is a two dimensional manifold. The $8 \times S^1$ has a stratification which is a circle, the singularity set, union the rest, which is also a two dimensional manifold.

Why do we want to consider only spaces V that admit a decomposition into manifolds? The intuitive answer is found by considering the group of all self homeomorphisms of V. Certainly if V is to be of "finite topological type", then this group should have finitely many orbits. It is these orbits that should be the natural strata of V. That the orbits of this group should be manifolds results from the meta-mathematical principle that a space of "finite topological type" whose group of self-homeomorphisms acts transitively must be a manifold. I don't know a precise mathematical statement that realizes this meta-mathematical principle, but I expect that there is one.

The third idea is this: Rather than defining a stratified subspace of X, we will define a stratification of X itself. Then a stratified space will be any locally closed union of strata of a stratification of a manifold X. To take the second example of a stratified space above, the three sphere S^3 can be stratified with three strata: the two involved in the example itself and rest of and the rest of S^3, which is a three dimensional stratum.

7.1.2 The definition of a Whitney stratified space.

So we want axioms to determine when a decomposition of a manifold X into a disjoint union of finitely many submanifolds is a stratification. These axioms are somewhat technical:

DEFINITION. A *Whitney stratification* of a manifold X is a disjoint decomposition $X = \bigcup_\alpha X_\alpha$ of Xinto a submanifolds (which are not necessarily connected, but which must each have a fixed dimension for all of their connected components) that satisfies the following four axioms:

1. *Local finiteness.* The decomposition is locally finite, i.e. every point $x \in X$ has a neighborhood U with the property that $U \cap X_\alpha$ is empty for all but a finite number of strata X_α.
2. *The axiom of the frontier.* If one stratum X_α has a non-empty intersection with the closure $\overline{X_\beta}$ of another stratum X_β, then X_α lies entirely within $\overline{X_\beta}$.
3. *Whitney's condition A.* Suppose that X_α lies in the closure of X_β. Suppose that $x_1, x_2, x_3, \ldots$ is a sequence of points in X_β which converges to a point y in X_α. Suppose further that the sequence of tangent spaces $T_{x_1}X_\beta$, $T_{x_2}X_\beta$, $T_{x_3}X_\beta, \ldots$ converges, as a sequence of subspaces of the tangent space TX to X, to a "limiting tangent space" $\tau \subseteq T_yX$. Then the tangent space T_yX_α is contained in the limiting tangent space τ.
4. *Whitney's condition B.* Suppose that X_α lies in the closure of X_β. Suppose that $x_1, x_2, x_3, \ldots$ is a sequence of points in X_β which converges to a point y in X_α, and that $y_1, y_2, y_3, \ldots$ is a sequence of points in X_α which also converge to y. Suppose as before that the sequence of tangent spaces $T_{x_1}X_\beta$, $T_{x_2}X_\beta$, $T_{x_3}X_\beta, \ldots$ converges, as a sequence of subspaces of the tangent space TX to X, to $\tau \subseteq T_yX$. Suppose further that the sequence of secant lines $\overline{x_1y_1}, \overline{x_2y_2}, \overline{x_3y_3}, \ldots$ converges to a limiting line $l \subseteq T_yX$. Then the limiting line l is contained in the limiting tangent space τ.

To make sense of the secant lines $\overline{x_i y_i}$ in Whitney's condition B, choose a local coordinate system around y. Also, the convergence of subspaces $T_{x_i}X_\beta$ can be taken to be convergence in a Grassmannian manifold, after choosing local coordinates. The truth or falsehood of the conditions is independent of the local coordinate system chosen.

If X is a complex manifold, then a *complex Whitney stratification* is a decomposition into complex submanifolds which satisfies the four conditions. In this case, it doesn't matter whether the tangent spaces and secant lines are taken to be complex or real.

7.1.3 Examples

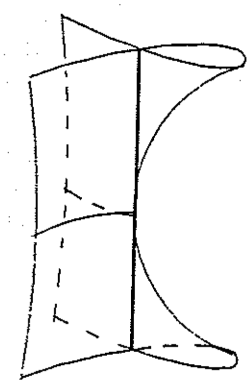
It is not at all apparent at first glance why Whitney conditions are the right ones. In fact, it doesn't even seem to have even been apparent to Whitney when he introduced them in 1965. It is only over the 25 intervening years that it has become clear that Whitney stratifications are in fact a very natural thing. However some intuition can be gained from considering some examples.

The reader can verify for himself, using mental pictures, that the cone and the $8 \times S^1$ satisfy the Whitney conditions. The one point compactification of the surface of infinite genus satisfies the Whitney condition A, but not the Whitney condition B. To see this, take y to be the point stratum X_α, and take $y_i = y$. It is possible to find a sequence of points x_i in the two dimensional stratum X_β with the property that the secant line $\overline{x_i y_i}$ is always perpendicular to the tangent space $T_{x_i}X_\beta$. Then Whitney's condition B fails for this choice, so the stratification is not a Whitney stratification.

The first procedure that almost everybody thinks of for producing a decomposition of a space Y into strata which are manifolds is the following. Let the largest stratum be all points of Y at which it is nonsingular. Let Y_1 be what is left when the largest stratum is deleted. Let the next stratum be the set of points at which Y_1 is a manifold. Let Y_2 be what is left when that stratum is deleted from Y_1. And so on.

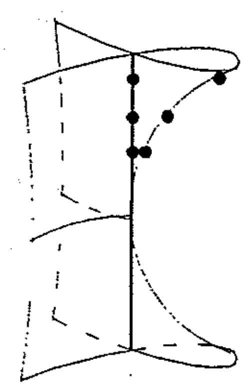
This procedure does give a decomposition into manifolds for all of the spaces in which we are interested. The trouble is that this decomposition is too course. The following example, due to Whitney himself, shows this. It is called the

Whitney Cusp.



The Whitney Cusp $y^2 = x^2(x - z^2)$

In this case, if we stratify it using procedure outlined above, we get is the z-axis ($= Y_1$) as one stratum, and the rest of it as the other stratum. For this potential stratification, the Whitney condition B fails if we take the points x_i and y_i as pictured. The secant lines $\overline{x_i y_i}$ are always parallel to the x-axis, so l is the x-axis. However, τ is the $y - z$ plane.



Failure of Whitney's condition B

What we have learned is that this is not a Whitney stratification of the Whitney Cusp. We wouldn't want it to be a stratification either. The group of self homeomorphisms is not transitive on the z-axis, since the origin looks different from every other point.

It does have the following Whitney stratification, however. We take the origin as one stratum, the rest of the z-axis as a second stratum, and the rest of the Whitney cusp as a third stratum (in addition to the strata outside of the Whitney cusp itself). In other words, Whitney's condition B forces us to consider the point at the origin as a separate stratum. This is good, since a neighborhood of that point looks different from neighborhoods of the other points on the z-axis. In other words, the origin is a separate orbit of the group of all homeomorphisms of the Whitney cusp.

7.1.4 Theorems about Whitney stratifications.

We will give without proof some of the standard results about Whitney stratifications. We will assume that X is a differentiable manifold.

Theorem 7.1 Transitivity of the homeomorphism group. *Let $X = \bigcup X_\alpha$ be a Whitney stratification of X. Let x and y be two points in the same connected component of a single stratum. Then there is a self homeomorphism h of X which takes each stratum X_α into itself and which takes x to y.*

Theorem 7.2 Tubular neighborhoods of strata. *For each stratum X_α there exists a* tubular neighborhood $T_\alpha \supset X_\alpha$, *a* distance function $\rho : T_\alpha \longrightarrow \mathbf{R}^{\geq 0}$ *and a* projection $\pi : T_\alpha \longrightarrow X_\alpha$ *with following properties*

1. *The projection $\pi : T_\alpha \longrightarrow X_\alpha$ is a projection of a fiber bundle, which restricts to the identity $X_\alpha \longrightarrow X_\alpha$.*
2. *The map $\rho : T_\alpha \longrightarrow \mathbf{R}^{\geq 0}$ Takes X_α to 0 and takes $T_\alpha - X_\alpha$ to $\mathbf{R}^{>0}$. The restriction $T_\alpha - X_\alpha \longrightarrow \mathbf{R}^{>0}$ is a projection of a fiber bundle.*
3. *For any $\beta \neq \alpha$, the restricted map $(\pi \times \rho) : (X_\beta \cap T_\alpha) \longrightarrow (X_\alpha \times \mathbf{R}^{>0})$ is a submersion.*

The next theorem is a refinement of this picture: it gives a local picture, near a point p in X_α of what the data of the last theorem look like.

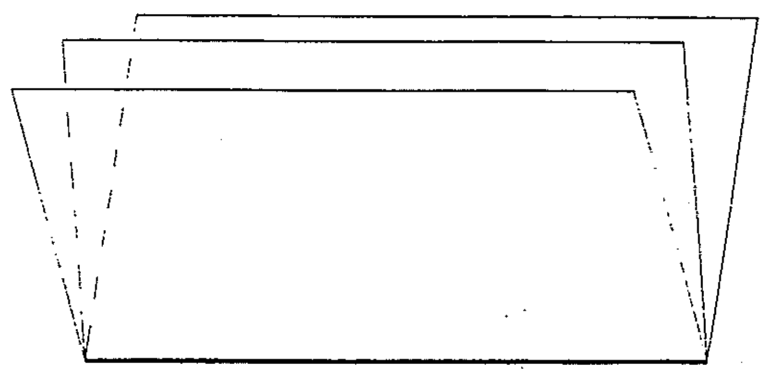
Before stating the next theorem, we note the following way to build Whitney stratifications. Suppose we are given a Whitney stratification $S = \bigcup S\alpha$ of the $(n-1)$-sphere S. We consider S to be embedded in $\mathbf{R}^n$ in the usual way as a sphere centered around the origin. Then $\mathbf{R}^n$ has a Whitney stratification, which we call the *conical stratification* , consisting of the following strata: For each S_α we take CS_α, set of all positive multiples of points in S_α. In addition, we take the origin. (Our first example, the cone in $\mathbf{R}^3$ was such an example.)

Theorem 7.3 Local Structure Theorem. *Let p be a point of a k-dimensional stratum X_α of a Whitney stratification $X = \bigcup X_\alpha$ of X (which has dimension n). Then there exists a stratification $S = \bigcup S_\alpha$ of the $(n-k-1)$-sphere S with* *the following property. Let* $R^k \times R^{n-k}$ *be stratified by the product stratification, where* R^k *is stratified as one stratum and* R^{n-k} *is stratified by the conical stratification, starting with the stratification of* S*. Then there is a homeomorphism from* $R^k \times R^{n-k}$ *to a neighborhood of* p *in* X*, taking strata to strata, and taking* $(0,0)$ *to* p*. The restriction of this homeomorphism to each stratum can be taken to be a diffeomorphism. Furthermore, this neighborhood can be chosen so that within it, the projection* π *is the projection of* $R^k \times R^{n-k}$ *onto the first factor* R^k*, and the distance function* ρ *is the distance in* $R^k \times R^{n-k}$ *from the subspace* R^k*.*

The stratification of the sphere S in this theorem is called the *link* stratification at p.

Exercise 7.1. Describe the link stratification for each point p in each of the three stratifications of R^3 given as examples earlier.

Let's see what the Local Structure Theorem 7.3 says about the neighborhood of a k dimensional stratum B in a $(k+1)$- dimensional stratum M. It says that there is a finite set F of points with the property that there is a homeomorphism from $R^{n-i} \times$ Cone$F$ to a neighborhood U of p in $M \cup B$ which is a diffeomorphism on M and on B. This is the "pages of a book" picture. In the following picture, k is 1 and n is 3.



The "pages of a book" picture

Theorem 7.4 Triangulability. *Assume that* X *is compact. Let* $X = \bigcup X_\alpha$ *be a Whitney stratification of* X*. Then there is a continuous triangulation of* X *such that each stratum* X_α *is the union of interiors of simplices.*

It follows that any closed union of strata is homeomorphic to a simplicial complex. So, up to homeomorphism, compact Whitney stratified spaces are just simplicial complexes. This shows that they are really spaces of "finite topological type", i.e. spaces requiring finitely much information to encode their topology. Conversely, every triangulated space is homeomorphic to a Whitney stratified space in some manifold.

7.1.5 Subanalytic subsets

There is a class of subsets of real analytic manifolds that is clearly the most beautiful one. This is the class of ***subanalytic*** subsets.

DEFINITION. The class of ***subanalytic subsets*** of analytic manifolds is the smallest class of subsets containing the satisfying the following conditions:

- If X is an analytic manifold, then X itself is a subanalytic set.
- If $f : X \longrightarrow Y$ is an analytic map of analytic manifolds, then the inverse image of a subanalytic subset of Y is a subanalytic subset of X.
- If $f : X \longrightarrow Y$ is a proper analytic map of analytic manifolds, then the image of a subanalytic subset of X is a subanalytic subset of Y.
- The class of subanalytic subsets of X is closed under finite unions, intersections, and differences.
- If X has a locally finite cover by open sets U_i such that $V \cap U_i$ is subanalytic for each U_i, then V is subanalytic.

If V is locally (in X) cut out by a finite number of equations $f_i(x) = 0$, inequalities $g_i(x) \geq 0$, and strict inequalities $h_i(x) > 0$ for analytic real valued functions f, g, h then V is subanalytic. (In fact, it satisfies the stronger property of being semianalytic.) In particular, any analytic subvariety of X is subanalytic. Osgood's example is subanalytic. (Osgood's example is the cone in $\mathbf{R}^3$ on a real analytic but not real algebraic curve c in the sphere $S \subset \mathbf{R}^3$. Osgood's example is not semianalytic.

Exercise 7.2. Verify that all of the examples in the previous paragraph are subanalytic.

7.1.6 Existence of Whitney stratifications

Theorem 7.5 Existence of Whitney stratifications. *Suppose that a $V_1, V_2, \ldots, V_k$ is a finite collection of analytic (or subanalytic) subsets of X. Then there exists a Whitney stratification $X = \bigcup X_\alpha$ of X such that each subset V_i is itself the union of some of the X_α.*

If X is complex analytic and the V_k are complex analytic subvarieties, then the stratification may be taken to be a complex Whitney stratification.

The next theorem is useful in constructing the stratifications that actually come up in the applications of perverse sheaves.

Theorem 7.6 Orbits of algebraic groups. *Let G be an algebraic group acting algebraically on X with only finitely many orbits. Then the decomposition $X = \bigcup X_\alpha$ of X into orbits of G is a Whitney stratification.*

7.2 Appendix 2. Local systems.

Local systems are an old and classical mathematical objects. They have many guises. One guise from algebra is a locally trivial sheaf. Another guise from differential geometry is a vector bundle with a flat connection. However, it may be useful to review them in purely topological language.

7.2.1 Local systems as covering spaces with extra algebraic structure

We fix a field K. For us, it will usually be the rational numbers, but sometimes the real numbers or the complex numbers.

DEFINITION. An *i-dimensional local system* on a topological space X is a topological space L, a map $l : L \longrightarrow X$, and, for each point p in X, a k vector space structure on $l^{-1}(p)$ with the following property: Every point $p \in X$ has a neighborhood U such that there is a homeomorphism $h : U \times K^i \longrightarrow l^{-1}(U)$ such that $l \circ h$ is projection on the first factor, and for each $x \in U$ the vector space structure on $l^{-1}(x)$ is induced by h from the one on $\{x\} \times K^i$.

In this definition, the topology on K is taken to be the discrete topology.

So a local system is a covering space (with infinitely many sheets, one for each point in K^i) with some extra algebraic structure.

For example suppose that X is the circle S^1. Fix a non-zero element m of K. We consider X as the identification space obtained by gluing the two ends of the closed interval $[0, 1]$ to each other. The local system L is the identification space obtained from $K \times [0, 1]$ by gluing $\{0\} \times a$ to $\{1\} \times ma$ for every $a \in K$. The projection is induced by the projection of $K \times [0, 1]$ on the first factor. This is called the local system over S^1 *with monodromy m*. In the event that m is -1, it is sometimes called the *Mobius* local system.

The local systems over a given topological space X form a category. If $l : L \longrightarrow X$ and $l' : L' \longrightarrow X$ are two local systems over X, then a morphism m from L to L' is a continuous map $m : L \longrightarrow L'$ with the following properties:

1. $l = l' \circ m$
2. For all points p in X, the restricted map $l^{-1}(p) \longrightarrow l'^{-1}(x)$ is a linear transformation.

If $\sigma : [0,1] \longrightarrow X$ is a path from one point p in X to another point q, then there is a *monodromy homomorphism* $\mu_\sigma : l^{-1}(p) \longrightarrow l^{-1}(q)$. This is defined just as in covering space theory as follows: Given $a \in l^{-1}(p)$, there is a unique lifted map $\tilde{\sigma} : [0,1] \longrightarrow L$that satisfies $\sigma = l \circ \tilde{\sigma}$ and takes the value a at 0. Then $\mu_\sigma(a) = \tilde{\sigma}(1)$, the value of that lifted map at $1 \in [0,1]$.

For example, if L is the local system over the circle S^1 with monodromy m, and if $p = q$ and σ is the path that goes around the circle once, then μ_σ is multiplication by m.

Exercise 7.3. Show that, just as in ordinary covering space theory, the monodromy map of σ depends only on the homotopy class of σ considered as a homotopy class of paths from p to q.

7.2.2 Local systems as representations of the fundamental groupoid

We want to formalize the collection of all of these monodromy maps. We do this as follows:

DEFINITION. The *fundamental groupoid* of X is the following category: It has an object O_p for every point p of X. If p and q are points of X, there is a morphism from O_p to O_q for every homotopy class of paths from p to q. Composition of morphisms is compositions of paths, i.e. following first one path then the second. The identity morphism on O_p is the homotopy class containing the path that stays constantly at p.

Exercise 7.4. Show that every morphism in the fundamental groupoid is an isomorphism. How many isomorphism classes of objects does the fundamental groupoid have?

DEFINITION. A *representation* of the fundamental groupoid of X is just a functor F from the fundamental groupoid to the category of finite dimensional K-vector spaces and linear transformations.

Note that for every morphism σ of the fundamental groupoid, $F(\sigma)$will automatically be an invertible map of vector spaces. The set of representations of the fundamental groupoid form a category. If F and F' are two representations, then a homomorphism from F to F' is a linear map $h_p : F(O_p) \longrightarrow F'(O_p)$ for each point $p \in X$ such that for each morphism σ from O_p to O_q, we have

$h_q \circ F(\sigma) = F'(\sigma) \circ h_p$. (This is called a natural transformation between the two functors.)

Given a local system $l : L \longrightarrow X$ over X, we can associate to it a representation of the fundamental groupoid of X in the following manner: The vector space $F(O_p)$ is just $l^{-1}(p)$. The linear transformation $F(\sigma)$ is just the monodromy μ_σ. A morphism of local systems clearly induces a morphism of representations of the fundamental groupoid.

Proposition 7.7 *If X is locally simply connected, then the functor from local systems to representations of the fundamental groupoid is an equivalence of categories.*

Exercise 7.5. Prove this.

Exercise 7.6. Suppose that X is locally simply connected. Let **n** be the category whose objects are sets with two elements and whose morphisms are bijections. Show that there is a one to one correspondence between:

1. functors from the fundamental groupoid of X to the category **n** and
2. n-sheeted covering spaces of X.

All of the local systems that are of interest to us are local systems over manifolds, which are all locally simply connected. We will speak of a local system either in the language of representations of the fundamental groupoid or in the language of spaces with extra structure mapping to X.

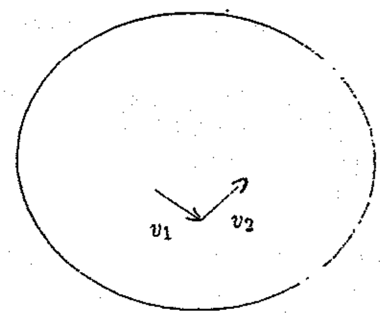
If we pick a base point p in X, then the full subcategory of the fundamental groupoid of X consisting of the object O_p together with all of its self-morphisms, is the fundamental group $\pi_1(X,p)$. A representation of the fundamental groupoid clearly restricts to a representation of the fundamental group. In this way, we obtain a functor from the category of all local systems on X to the category of representations of the fundamental group of X.

Proposition 7.8 *If X is locally simply connected and connected, and a base point p of X has been chosen, then the category of representations of the fundamental group $\pi_1(X,p)$ of X is equivalent to the category of local systems over X.*

7.2.3 Orientations.

Let M be a differentiable manifold of dimension i and let p be a point in M. A *frame* of M at p is an ordered basis for the tangent space T_pM, say $v_1, v_2, \ldots, v_i$

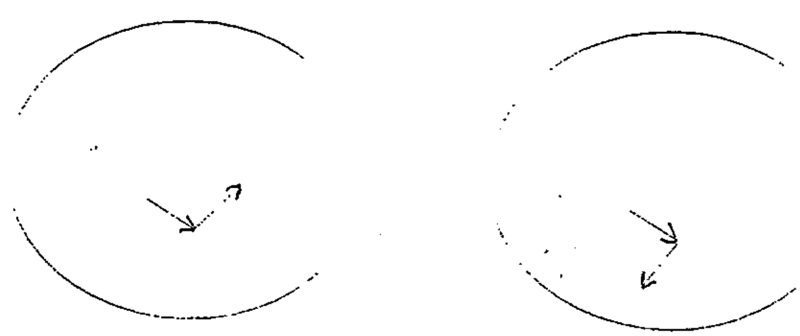
where the v_i are linearly independent. We can visualize a frame this way:



A frame at p in M

Clearly the set of frames of M at p forms a differentiable manifold diffeomorphic to the manifold of all invertible $i \times i$ matrices. A connected component of this manifold is called an *orientation* of M at p. (In the special case that M is zero dimensional, then an orientation of M at p is, by definition, either the symbol $+$ or the symbol $-$.)

Exercise 7.7. Show that there are exactly two orientations of M at p, by showing that the set of $i \times i$ matrices has exactly two connected components, for each $i > 1$.



The two orientations of M at p

The orientation double cover If σ is a path from a point p to a point q in M, then σ induces a map G_σ from the set of orientations at p to the set of orientations at q as follows: Given an orientation $\mathcal{O}$ at p, pick a frame $v_1, v_2, \ldots, v_i \in T_pM$ which represents that orientation (i.e. which lies in that connected component). Then pick any continuous variable (moving) frame $v_1(t), v_2(t), \ldots, v_i(t)$ in the tangent space $T_{\sigma(t)}M$ for $t \in [0,1]$. Then the frame $v_1(1), v_2(1), \ldots, v_i(1) \in T_qM$ represents $G_\sigma(\mathcal{O})$.

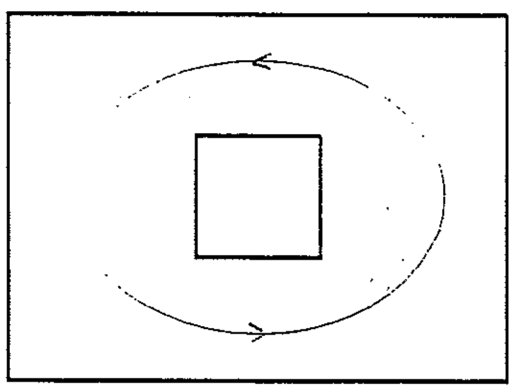
Exercise 7.8. Let **2** be the category whose objects are sets with two elements and whose morphisms are bijections. Show that there is a functor from the fundamental groupoid of M to the category **2** that takes $p \in M$ to the set of orientations of M at p and takes σ to G_σ.

DEFINITION. The covering space of M associated to the functor on the fundamental groupoid constructed in the last exercise is called the *orientation double cover* of M. We will denote it by $\omega : \tilde{M} \longrightarrow M$. Its fiber $\omega^{-1}(p)$ over p consists of the two possible orientations of M at p.

7.3 Geometric chains

7.3.1 Introduction

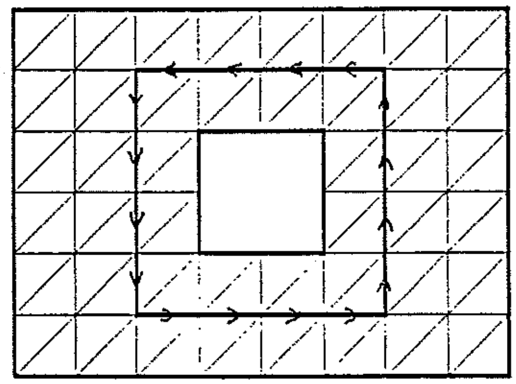
The 1895 paper of Poincaré already contains a version of homology defined by geometric cycles. A geometric i-cycle in a space X is some sort of i-dimensional surface in X with an orientation. For example, the following is a geometric cycle for $H_1(A)$ where A is the annulus in the plane.



A geometric cycle representing a generator of $H_1(A)$

Poincaré was rather vague about what kind of surfaces he was considering, although we may assume from what he writes that he intended them to be analytic.

More recently, it has became standard to define homology using simplices, either simplices in simplicial complexes or singular simplices in more general spaces.



A simplicial cycle

This trend is unfortunate from the point of view of developing intuition, since homology theory looks like it hinges on special properties of simplices. (In an alternative version cubes are used, but the impression still stands.) Perhaps one reason for the ubiquity of topology based on simplices is that after Poincaré that mathematicians started to worry about just what was meant by an i-dimensional surface in X. It is not enough for the purposes of homology to define an i-dimensional surface in X to be an submanifold, since using submanifolds you get the wrong groups. (You get cobordism.) Some singularities have to be allowed. But what singularities? Pathological examples of surfaces are known. Simplices were used as a means of obtaining a rigorous theory.

Now, nearly 100 years later, it is possible finally to realize Poincaré's original intent, since the theory of subanalytic sets has been developed (see section 7.1.5 above). It is true, however, that this does not give a short-cut to homology theory, since the development of subanalytic sets itself involves more technical difficulties than the development of singular homology theory does. In this appendix, we will assume that we are given a nice class of subsets of X, and we will show how to develop a homology theory based on them. The construction of such a class will be relegated to more technical treatises.

7.3.2 A good class of subsets

Let us fix a manifold X.

DEFINITION. A *good* class $\mathcal{C}$ of subsets of X is a class of subsets with the following properties:

1. If a subset S of X is in $\mathcal{C}$, then X has a Whitney stratification such that S is a union of strata, and each stratum is in $\mathcal{C}$. More generally, if $S_1, S_2, \ldots, S_k$ is a finite list of subsets of X that are in $\mathcal{C}$, then there is a Whitney stratification of X such that each set S_i is a union of strata and each stratum is in $\mathcal{C}$.
2. The class $\mathcal{C}$ is closed under finite set theoretic operations: unions, intersections, and differences.
3. The closure of a subset in $\mathcal{C}$ is in $\mathcal{C}$.

The main examples of good classes of subsets

Of the following list of examples, the first one may be the most intuitive whereas the second one is the most useful. The third example is a scholium.

In the whole development of intersection homology and perverse sheaves in this report, the reader has the choice of taking cycles in the first class or cycles in the second class. All of the theorems are true in either case.

The first class was used in [GM1], the original exposition of intersection homology. The advantage of it is that the geometric results about it which are needed can be proved fairly quickly by thinking about it. Most readers will have to take theorems about the second class on faith, since their proofs are long and hard.

The advantage of the second class is that it is canonical for the spaces that we are most interested in: complex analytic manifolds. For the first class, one must choose a triangulation in every stratified situation before proceeding.

1. First choose a smooth triangulation of X so that X has a piecewise linear structure as well as a smooth structure. Let $\mathcal{C}$ be all piecewise linear subspaces of X with respect to this piecewise linear structure. Then $\mathcal{C}$ is a good class of subsets. (A subspace of X is piecewise linear if its restriction to each simplex in X is a union of pieces each of which is cut out by finitely many equalities $f_i(x) = 0$ inequalities $g_j(x) > 0$ where the functions f_i and g_j real valued linear functions.)

2. Again fix a real analytic structure on X. Let $\mathcal{C}$ be all sub-analytic subsets of X with respect to this real analytic structure. Then $\mathcal{C}$ is a good class of subsets. (See section 7.1.5.)

3. Choose a smooth triangulation of X. Let $\mathcal{C}$ be all unions of interiors of simplices of X. Then $\mathcal{C}$ is a good class of subsets.

Unfortunately, if X is just endowed with a differentiable structure and no other structure, there seems to be no useful good class of subsets.

DEFINITION. A set $S \subset X$ has *pure dimension k* if for some Whitney stratification of S, we have that S is the closure of the union of its k dimensional strata.

Exercise 7.9. Show that the statement "S is the closure of the union of its k dimensional strata" is independent of the stratification of S chosen.

7.3.3 Geometric chains

Geometric chains will be defined to be equivalence classes of geometric prechains, relative to an equivalence relation.

Geometric prechains

We fix a manifold X, a good class of subsets $\mathcal{C}$ of X, and a local system L over X.

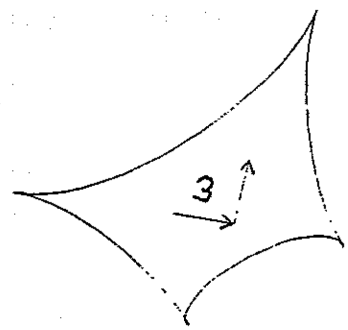
DEFINITION. A *geometric degree k prechain C in X (relative to $\mathcal{C}$) with coefficients in L* is the following data:

1. A closed subset S of X, called the *presupport* of C, which is in the class $\mathcal{C}$ and has pure dimension k.
2. A Whitney stratification $S = \bigcup S_\alpha$ of S such that each stratum is in $\mathcal{C}$ and S is a union of strata.
3. Over each k-dimensional stratum S_α contained in C, a map $c : \tilde{S_\alpha} \longrightarrow L|S_\alpha$ of the orientation cover $\tilde{S_\alpha}$ to the restriction of L to S_α with the following property: The map c takes the two orientations of S_α at p, i.e. the two points in $\omega^{-1}(p)$, to points in the fiber $l^{-}(p)$ of L over p to a pair of elements of the fiber of L over p which are negatives of each other. (It is not ruled out that c takes both points to zero.) The map c is called the *multiplicity map* of C.

Remarks on the definition.

The key point of the definition is that the two orientations are taken by c into negatives of each other in L.

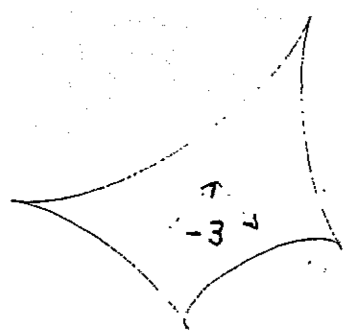
A useful way to represent the data provided by the map c by the following pair of choices: We may pick an orientation of S_α at p, and also pick an element an element of the fiber of L over p called the *multiplicity* of the chain. This pair of choices determines c, at least on the connected component of S_α containing p, by the requirement that c must map the chosen orientation to the multiplicity. Such a pair of choices gives a useful way to draw geometric cycles. For example, if S_α is a plane and L is the trivial local system with fiber Q, then this is a picture of a geometric prechain:



A geometric prechain

The number 3 represents the multiplicity. (If the connected component of S_α containing p is not simply connected, then not all choices of an orientation and multiplicity at p will work. The twisting of the orientation double cover must be compatible with the twisting of the local system.)

The fact that c maps the two orientations into elements of the fiber of L which are negatives of each other means, for example, that these two choices of orientation and multiplicity are identified for the purposes of defining a prechain:



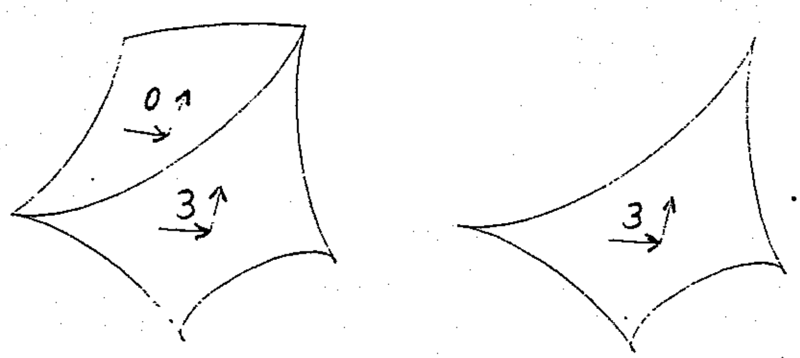
Choices of orientation and multiplicity that give the same prechain.

This corresponds to the usual idea in topology that chains with opposite orientations should be considered as negatives of each other.

The equivalence relation on geometric prechains

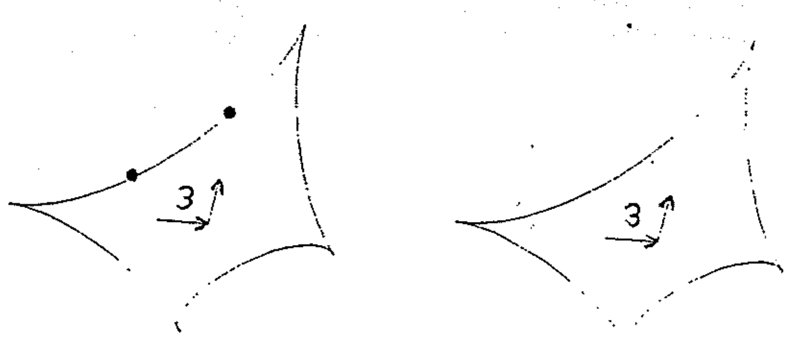
There are three reasons why two different prechains may give what we want to consider as the same chain:

1. The multiplicity of a stratum might be zero (i.e. c might map $\tilde{S}_\alpha$ into the zero section of L). In this case, an equivalent chain could be obtained by deleting the stratum S_α from S.



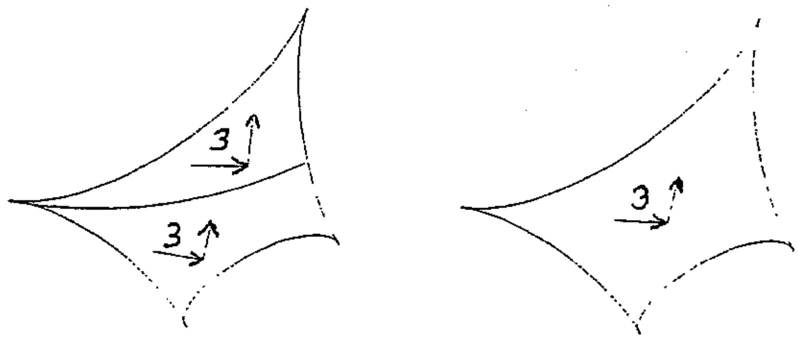
Two geometric prechains that represent the same chain

2. The stratifications of S might differ. This breaks down into two subcases. 2a. First the stratifications might differ in a way that all of the k- dimensional strata are the same.



Two more geometric prechains that represent the same chain

2b. Finally, the k-dimensional strata might be different. In this case, we want to consider the chains to be the same if on the intersection of the k-dimensional strata of the first stratification and the k-dimensional strata of the second stratification, the multiplicity maps c agree.



Two more geometric prechains that represent the same chain

As usual in such situations in mathematics, there are two solutions to the non-uniqueness problem. Either we let a geometric chain be an equivalence class of geometric prechains, or we can locate a canonical geometric chain in each equivalence class. This gives two possible definitions of geometric chains. As often happens, the one which is easier to visualize is the canonical choice, but the one which is easier to deal with rigorously is the equivalence relation.

Suppose that we have two geometric prechains C and C' (with presupports S and S', stratifications $\bigcup S_\alpha$ and $\bigcup S'_\beta$ and multiplicity maps c and c'). Form a two new geometric prechains $\bar{C}$ and $\bar{C}'$ as follows: The presupport for each of them is $S \cup S'$. The stratification for each of them is some common refinement of the stratifications for C and for C'. The multiplicity map $\bar{c}$ for $\bar{C}$ on a k-dimensional stratum T of the common refinement is defined in this way: If T is contained in one of the strata S_α for C, then it is the restriction of the multiplicity map c for the stratum S_α. Otherwise, $\bar{c}$ is zero. Likewise, the multiplicity map $\bar{c}'$ for $\bar{C}'$ is defined on T: If T is contained in one of the strata S'_β for C', then it is the restriction of the multiplicity map c' for the stratum S'_β. Otherwise, $\bar{c}'$ is zero.

DEFINITION. The geometric prechains C and C' are said to be *equivalent* if $\bar{C}$ and $\bar{C}'$ as constructed above are equal.

Exercise 7.10. Show that this is an equivalence relation.

DEFINITION. A *geometric chain* is an equivalence class of geometric prechains.

DEFINITION. The *support* of a geometric k-chain C is the closure in X of the union of all of the k dimensional strata on which the multiplicity map is nonzero for any geometric prechain representing C.

Exercise 7.11. Show that the support is well defined, i.e. it is the same for all geometric prechains representing C.

Geometric chains form a vector space

DEFINITION. The sum $C + C'$of two geometric prechains C and C' (with presupports S and S', stratifications $\bigcup S_\alpha$ and $\bigcup S'_\beta$ and multiplicity maps c and c') is the geometric prechain constructed as follows: The presupport of $C + C'$ is $S \cup S'$. The stratification of $C + C'$ is a common refinement of the stratifications for C and for C'. The multiplicity map $\bar{c}$ for $C+C'$ on a stratum T of the common refinement is defined in this way: If T is contained in both a stratum for C and a stratum for C', then $\bar{c} = c + c'$. If T is contained in a stratum for C but not in a stratum for C', then $\bar{c} = c$. If T is contained in a stratum for C' but not in a stratum for C, then $\bar{c} = c'$.

If r is a scaler in the field over which the local system l is defined, then multiplication of a geometric chain by r simply multiplies all of the multiplicity maps by r.

Exercise 7.12. Show that the equivalence class of $C + C'$ does not vary if C or C' is changed to an equivalent prechain, so that the operation $+$ passes to an operation on geometric chains.

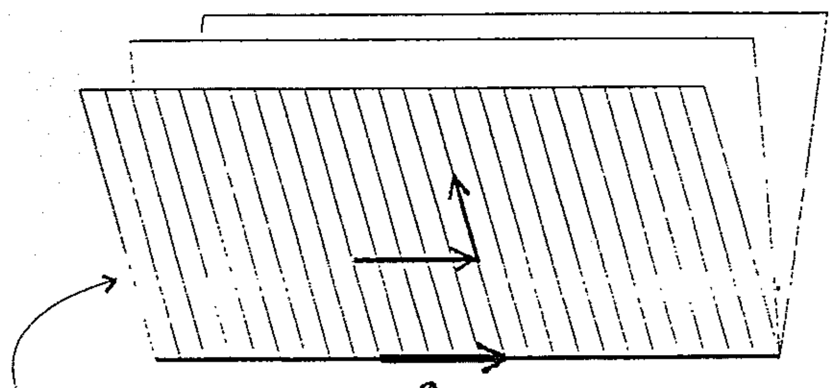
This makes the set of geometric chains into an abelian group. The zero element of the group is the geometric chain represented by the empty prechain (or equivalently by any prechain all of whose multiplicities are zero).

7.3.4 The boundary of a geometric chain

A geometric k-chain C with coefficients in L, determines a geometric $(k-1)$-chain∂C with coefficients in L called the *boundary* of C. In this section, we will define the boundary.

First, we need an auxiliary idea.

The induced orientation on a boundary Now suppose that we have a stratified space X and that M is a i- dimensional stratum of X. Let's consider a stratum $(i-1)$-dimensional stratum B in the closure of M. For any point p in B, we have a neighborhood U of p in $B \cup M$ with the property that U has a "pages of a book" decomposition $U = \boldsymbol{R}^{i-1} \times \text{Cone} F$ for a finite set F, as described just after Theorem 7.3. Then for each page $\boldsymbol{R}^{i-1} \times \text{Cone} f$ where $f \in F$, an orientation $\mathcal{O}$ of that page determines an orientation ind$\mathcal{O}$ on B called the *induced orientation*, described as follows. Let π be the projection of the page to $R^{i-1} \subset B$. Choose a frame $v_1, v_2, \ldots, v_i$ in $R^{i-1} \times \text{Cone} f$ representing $\mathcal{O}$ with the property that the differential $d\pi(v_i) = 0$ and $d\rho(v_i) > 0$. (Under these conditions, we say that v_i is *outward pointing.*) Then the induced orientation is represented by the frame $d\pi(v_1), d\pi(v_2), \ldots, d\pi(v_{i-1})$.



Exercise 7.13. Show that the induced orientation from a page is well defined.

DEFINITION. The *boundary* ∂C of a k-dimensional geometric prechain C is a $(k-1)$-dimensional geometric prechain defined as follows:

- The presupport of ∂C, is the closure of the union of all of the strata of C of dimension less than $k-1$.
- The stratification of ∂C is the restriction of the stratification of C.
- Over each $(k-1)$-dimensional stratum S_α contained in ∂C, the multiplicity map $c : \tilde{S}_\alpha \longrightarrow L|S_\alpha$ is obtained in this way: Given p in S, there is a neighborhood U of p in X with the property that the presupport of C in U looks like "pages of a book" (see the example after Theorem 7.3). Fix an orientation $\mathcal{O}'$ of X_α at p. For each page P_j, choose the orientation $\mathcal{O}_j$ of P_j that induces $\mathcal{O}'$ on S_α at p.Let $m_j \in L$ be the multiplicity of the page P_j in the cycle C. Then the multiplicity of ∂C at p with respect to $\mathcal{O}'$ is $\sum m_j$. (The elements m_j makes sense as elements in the stalk of L over p because U is contractible, so we can uniquely compare elements in the stalks over different points.)

  Exercise 7.14. Show that the definition of ∂C is independent of the choice of the prechain in the equivalence class. (The most interesting case to consider is when the chain C is restratified by introducing new $(k-1)$-dimensional strata.

7.3.5 Geometric homology.

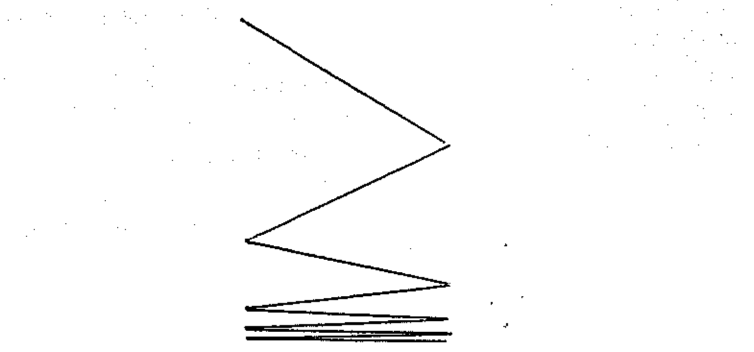
Now we have constructed a chain complex of geometric chains.

$$\cdots \xrightarrow{\partial} C_{k+1}(X,L) \xrightarrow{\partial} C_k(X,L) \xrightarrow{\partial} C_{k-1}(X,L) \xrightarrow{\partial} \cdots$$

What we are interested in, for the purposes intersection homology, is a slight generalization of this. Take a stratification of X by strata that are in the good class of subsets.. Pick one of these strata and call it Y_0. Let L be a local system over Y_0. We now get a good class of subsets of Y_0 just by taking all sets in Y_0 which are in the good class of subsets of X. Let $C_k(Y_0, L)$ be the geometric k-chains in Y_0 constructed with respect to this good class of subsets. (We do not rule out that Y_0 may equal X, in which case we get the previous definition back again.) Once more, we have a chain complex

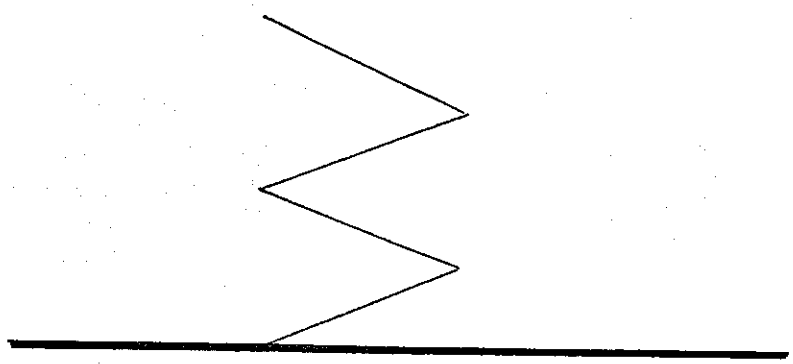
$$\cdots \xrightarrow{\partial} C_{k+1}(Y_0,L) \xrightarrow{\partial} C_k(Y_0,L) \xrightarrow{\partial} C_{k-1}(Y_0,L) \xrightarrow{\partial} \cdots$$

Remark. There is an important point about geometric chains C in Y_0. That is that the closure in X support of C will still be in teh good class. (This is because the good class in Y_0 was taken as the restriction to Y_0 of the good class in X.) For example, in the following picture, let Y_0 be the complement of the horizontal line. Then the chain shown will not be a geometric chain in Y_0, even though it looks like one everywhere in Y_0.



This is not a geometric chain in Y_0

Whereas, this one will.



This is a geometric chain in Y_0 .

As usual in this situation, we want to take its homology.

DEFINITION. The *geometric homology* of Y_0 with coefficients in L with respect to the good class of subsets $\mathcal{C}$ is the homology of this complex.

Proposition 7.9 *For all three good classes of subsets mentioned in 7.3.2 , the geometric homology of Y_0 is the usual homology of Y_0 with closed support.*

For the third example, geometric homology is just simplicial homology. For the first example, it is the limit of simplicial homology under refinement of the triangulation.

For the second example, which is the important one from our point of view, the theorem is in [H].

Exercise 7.15. Construct a good class of subsets for which the geometric homology is not the usual homology.

Exercise 7.16. Show that for any class of subsets, the geometric homology maps to the usual homology.

Remark. There probably is no canonical good class of subsets on a differentiable manifold that uses only the differentiable structure in its definition, and such that geometric homology with respect to it gives the ordinary homology. However, Goresky constructed a version of geometric homology for a differentiable manifold [G2]. He used Whitney stratified sets. These are not closed under unions, unless the two sets satisfy some sort of transversality. Therefore, addition, for example, is not defined for geometric chains. He defined addition in homology by using transversality.

Bibliography

The references that are broad survey papers and books are marked with a ⋆. One reason that we have neglected certain subjects here is that these sources exist for them.

[B] Beilinson, A.: How to glue perversee sheaves, Springer Lect. Notes in Math. 1289 (1987) 42-51.

[BB] Beilinson A., Bernstein, J.: Localisation des G-modules, C.R. Acad. Sci. Paris 292 (1981) 15-18.

⋆[BBD] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux Pervers, Astérisque 100 (1983).

[BLM] Beilinson, A., Lusztig, G., MacPherson, R.: A geometric setting for the quantum deformtion of GL_n, Duke Math. Jour. 61 (1990).

⋆[Bo1] Borel, A.: et al, *Intersection cohomology*, Progress in Mathematics 50, Birkhäuser (1984).

⋆[Bo2] Borel, A.: et al, *Algebraic $\mathcal{D}$- modules* , Perspectives in Mathematics 2, Academic Press (1987).

[BM] Borho W., MacPherson, R.: Representations dex groupes de Weyl et homologie d'intersection pour les variétés nilotents, C.R. Acad. Sc. Paris 292 (1981) 707-710.

[BGM] Brasselet, J.-P., Goresky, M., MacPherson, R.: to appear in Am. J. Math.

⋆[Br1] Brylinski, J.-L.: (Co)-homologie d'intersection et faisceaux pervers, Séminaire Bourbake 585, Astérisque 92-93 (1982), 129-158.

[Br2] Brylinski, J.-L.: Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier, et sommes trigonométriques, Astérisque 140-141 (1986) 3-134.

[BrK] Brylinski, J.-L.: Kashiwara, M.: Kazhdan-Lusztig chonjecture and holonomic systems, Inv. Math. 64 (1981) 387-410.

[BL] Brylinski, J.-L.: Labesse, J.-P., Cohomologie d'intersection et fonctions L de certaines variétés de Shimura, Ann. Sci. Ecole Norm. Sup. 17 (1984) 217-252.

⋆[BZ] Brylinski, J.-L., Zucker, S.: Recent results in Hodge theory, preprint.

[CS] Cappell, S., Shaneson, J.: Singular spaces, characteristic classes, and intersection homology, (1990) preprint.

[CS] Cappell, S., Shaneson, J.: Stratifiable maps and topological invariants, (1990) preprint.

[CW] Cappell, S., Weinberger, S.; Classification of certain stratified spaces, to appear.

[CKS] Cattani, E., Kaplan, A., Schmid, W.: L^2 and intersection cohomologies for a polarizable variation of Hodge structure, Inv. Math. 87 (1987) 217-252.

[C1] Cheeger, J.: On the spectral geometry of spaces with cone-like singularities, Proc. Nat. Acad. Sci. 76 (1979) 159-322.

[C2] Cheeger, J.: On the Hodge theory of Riemannian pseudomanifolds, Geometry of the Laplace Operator, Proc. Symp. pure Math. 36 (1980) 91- 146.

[C2] Cheeger, J.: Spectral geometry of singular Riemannian spaces, J. Diff. Geom. 18 (1983), 575-657.

⋆[CGM] Cheeger, J.: Goresky, M., MacPherson, R.: L^2 cohomology and intersection homology of singular algebraic varieties, *Seminar on differential geometry*, S.T. Yau (ed), Ann. of Math. Studies 102, Princeton Univ. Press, Princeton NJ (1982) 303-340.

[CGL] Cohen, D., Goresky, M., Lizhen, J.: On the Künneth formula for intersection cohomology, Trans. Am. Math. Soc., to appear.

[D] Deligne, P.: La conjecture de Weil II, Publ. Math. I.H.E.S., 52 (1980) 137-252.

[FM] Fulton W., MacPherson, R.: *A categorical framework for the study of singular spaces*, Memoirs of the American Math. Soc., 243 (1981).

[GZ] Gabriel, P., Zisman, M.: *Calculus of Fractions and Homotopy Theory*, Springer Ergebnisse (1967).

[GeM] Gelfand, S., MacPherson, R.: Verma modules and Schubert cells: a dictionary, Séminaire d'Algèbre, Springer Lect. Notes in Math. 924 (1982) 1-50.

⋆[GeMa] Gelfand, S., Manin, Yu.: Methods of homological algebra I: introduction to cohomology theory and derived categories, Springer (1991).

[Gi] Ginsburg, V.: Perverse sheaves on loop groups and Langlands duality, preprint.

[G1] Goresky, M.: Triangulation of stratified objects, Proc. Amer. Math. Soc. 72 (1978) 193-200.

[G2] Goresky, M.: Whitney stratified chains, Trans. Amer. Math. Soc. 261 (1981) 175-196.

⋆[GM1] Goresky, M., MacPherson, R.: Intersection homology theory, Topology 19 (1983) 135-162.

[GM2] Goresky, M., MacPherson, R.: Intersection homology II, Inv. Math. 71 (1983) 77-129.

[GM3] Goresky, M., MacPherson, R.: La dualité de Poincaré pour les espaces singuliers, C.R. Acad. Sci. Paris 284 (1977) 1549-1551.

Indian J Math 42 221-231

[GM4] Goresky, M., MacPherson, R.: Simplicial intersection homology, Inv. Math. 84 (1986) 432-33.

[GM5] Goresky, M., MacPherson, R.: Lefschetz fixed point theorem for intersection homology, Comment. Math. Helv. 60 (1985), 366- 391.

*[GM6] Goresky, M., MacPherson, R.: On the topology of complex algebraic maps, prodeedings of International Congress of Algebraic Geometry, La Rabida, Springer Lect. Notes in Math. 961 (1981) 119-129.

*[GM7] Goresky, M., MacPherson, R.: *Stratified Morse Theory*, Springer Ergebnisse 3 Folge, 14 (1988).

Top 6675 Arcata, Calif, 1981

[GM8] Goresky, M., MacPherson, R.: Stratified Morse theory, Proceedings of Symposia in Pure Mathematics 40, Amer. Math. Soc., (1983) 517-533

[GM9] Goresky, M., MacPherson, R.: Morse theory and intersection homology, Analyse et Topologie sur ler Espaces Singuliers, Astérisque 101 (1983) 135-192

[GP] Goresky, M., Pardon, W.: to appear in Topology.

[GS] Goresky, M., Segal, P.: Linking pairings on singular spaces, Comm. Math. Helf. 58 (1983) 96-110.

[H1] Hardt, R.: Topological properties of subanalytic sets, Trans. Amer. Math. Soc., 211 (1975) 57-70.

[H2] Hardt, R.: Triangulation of subanalytic sets and proper light subanalytic maps, Inv. Math. 38 (1977) 207-217.

[Hi] Hironaka, H.: Subanalytic sets, *Number theory, algebraic geometry, and commutative algebra*, volume in honor of A. Akizuki, Kcinokunya Tokyo (1973) 453-493.

[HP] Hsiang, W.-C., Pati, V.: L^2-cohomology of normal algebraic surfaces I, Inv. Math. 81 (1985) 395-412.

[K1] Kashiwara, M.: On the maximally overdetermined systems of linear differential equations I, Publ. R.I.M.S. Kyoto 10 (1975) 563-579.

[K2] Kashiwara, M.: Faisceaux constructibles et systèmes holonomes d'equations aux derivées partielles linéaires à points singuliers reguliers, Séminaire Goulaouic-Schwartz 19, 1979-80.

[KK1] Kashiwara, M., Kawai, T.: On the holonomic systems of linear differential equatins (systems with resular singularities) III, Publ. of R.I.M.S., Kyoto 17 (1981) 813-979.

[KK2] Kashiwara, M., Kawai, T.: The Poincaré lemma for a variation of polarized hodge structure, Proc. Japan Acad. 61 (1985) 164-167.

[KS1] Kashiwara, M., Schapira, P.: Microlocal study of sheaves, Astérisque 128 (1985).

*[KS2] Kashiwara M., Schapira, P.: *Sheaves on manifolds*, Springer Grundlehren 292 (1990).

[KL1] Kazhdan, D., Lusztig, G.: Representations of coxeter groups and Hecke algebras, Inv. Math. 53 (1979) 165-174.

[KL2] Kazhdan, D., Lusztig, G.: Schubert varieties and Poincaré duality, Proc. Symp. Pure Math. 36 (1980) 185-203.

[Kin] King, H. Topological invariance of intersection homology without sheaves, Topology 20 (1985) 229-234.

⋆[Ki1] Kirwan, F.: *An introduction to intersection homology theory*, Pitman Research Notes in Math. 187 (1988).

[Ki2] Kirwan, F. Rational intersection cohomology of quotient varieties, I: Inv. Math. 86 (1986) 471-505; II: Inv. Math. 90 (1987) 153-167.

[Ki3] Kirwan, Intersection homology and torus actions, J. Amer. Math Soc. 1 (1988) 385-400.

⋆[K] Kleiman, S.: The development of intersection homology, *A century of mathematics in America, part II*, Amer. Math. Soc. (1989) 543-585.

[Lau] Laumon, G., Correspondance de Langlands géométrique pour les corps de fonctions, Duke Math. j. 54 (1987) 309- 360.

[La] Lazzeri, F.: Morse theory on singular spaces, Singularités à Cargèse, Astérisque 7-8 (1973).

[Loj] Lojasiewicz, S.: Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa 18 (1964) 449-474.

[Lo] Looijenga, E.: L^2-cohomology of locally symmetric varieties, Comp. Math. 67 (1988) 3-20.

⋆[L1] Lusztig, G.: Intersection cohomology methods in representation theory, Plenary address to the International Congress of Matheamticians, Kyoto, 1990, preprint.

[L2] Lusztig, G.: Representations of finte Chevalley groups, A.M.S. Regional Conf. in Math. 39 (1978).

[L3] Lusztig, G.: Intersection cohomology complexes on a reductive group, Inv. Math. 75 (1984) 205-272.

[L4] Lusztig, G.: Character Sheaves, Adv. in Math. I, 56 (1985) 193-237; II, 57 (1985) 226-265; III, 57 (1985) 266-315; IV, 59 (1986) 1-63; V, 61 (1986), 103-155.

[L5] Lusztig, G.: Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990) 447-498.

[L6] Lusztig, G.: Green polynomials and singularities of unipotent classes, Adv. in Math. 42 (1981) 169-178.

⋆[L] Lusztig, G.: Introduction to character sheaves, Proc. Symp. Pure Math. 40 (1987) 165-179.

[LV] Lusztig, G., Vogan, D.: Singularities of closures of K-orbits on a flag manifold, Inv. Math. 71 (1983) 365-379.

⋆[M] MacPherson, R. Global questions in the topology of singular spaces, Plenary address to the the International Congress of Mathematicians, Warsaw 1983, PWN-Polish Scientific Publishers, Warsaw and Elsevier Science Publishers B.V. Amsterdam (1984) 213-235.

[MV] MacPherson, R., Vilonen, K.: Elementary construction of perverse sheaves, Inv. Math. 84 (1986) 403-436.

⋆[MS] Mars, J., Springer, T.: Character Sheaves, Astérisque 173-174 (1989) 111-198.

[Ma] Mather, J.: Notes on topological stability, (1970) available from J. Mather, Dept. of Mathematics, Princeton Universiy.

[Me] Mebkhout, Z.: Sur let problème de Hilbert-Riemann, C.R. Acad. Sci Paris (1980) 415-417.

⋆[Mi Milicec, D. *$\mathcal{D}$-modules and representation theory*, to appear.

[MiV] Mirkovic, I., Vilonen, K. Characteristic varieties of character sheaves, inv. Math. 93 (1988) 405-418.

[MirV] Mirollo, R., Vilonen, K. Bernstein-Gelfand-Gelfand reciprocity on perverse sheaves, Ann. Sci Ecole Norm. Sup. 120 (1985), 272-307.

[Pa] Pardon, W.: Cycles for $L_*(\mathbb{Z})$, Comm. Math. Helf. (1989).

[Pi] Pignoni, R.: Density and stability of Morse functions on a stratified space, Ann. Scuola Norm. Sup. Pisa 4 (1979) 592-608.

[Po] Poincaré, H.: Analysis situs, J. de l'Ecole Polytechnique 1 (1895) 1-121.

[Q1] Quinn, F. Homotopically stratified spaces, Jour. Amer. Math. Soc. 1 (1988) 441-449.

[Q2] Quinn, F. Resolution of homology manifolds, Inv. Math. 72 (1983) 267-284.

[S1] Saito, M.: Modules de Hodge polarisables,

[S2] Saito, M.: Mixed Hodge modules, R.I.M.S. preprint no. 585 (1987).

⋆[S3] Saito, M.: Introduction to mixed Hodge modules, preprint to appear in Proc. Conf. Pure Math. Sundance.

[Sa1] Saper, L.: L^2-cohomology and intersection homology of certain algebraic varieties with isolated singularities, Inv. Math. 82 (1985) 207-255.

[Sa2] Saper, L.: L^2-cohomology of isolated singularities, preprint.

[SS] Saper, L., Stern, M. L^2-cohomology of arithmetic varieties, Annals of Mathematics (1990).

[Si] Siegel, P.: Witt spaces: a cycle theory for KO-homology at odd primes, Amer. J. of Math. (1983) 1067-1105.

⋆[Sp] Springer, T. Quelques applications de la cohomologie d'intersection. Séminaire Bourbake 589, Astérisque 92-93 (1982) 249-274.

[St] Stanley, R. Generalized H-vectors, intersection homology of toric varieties, and related results, Adv. Sudies in Pure math. 11 (1987) 187-213.

[Te] Teissier, B.: Sur la triangulation des morphismes sous- analytiques, Publ. Math. I.H.E.S. 70 (1989).

[T] Thom, R.: Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. 75 (1969) 240-284.

[Tr] Trotman, D.: Comparing regularity conditions on stratifications, Proc. Symp. Pure Math. 40 (1983) 575-585.

⋆[We] Weinberger, S.: *Topological classification of stratified spaces*, University of Chicago, 1990.

[W] Whitney, H.: Local properties of analytic varieties, *Differentiable and Combinatorial Topology*, S. Cairns, ed., Princeton University Press, Princeton NJ (1965) 205-244.

[Z1] Zucker, S.: Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric, Ann. of Math. 109 (1971) 415-476.

[Z2] Zucker, S.: L_2-cohomology of warped products and arithmetic groups, Inv. Math. 70 (1982) 169-218.

[Z3] Zucker, S.: L^2-cohomology and intersection homology of locally symmetric varieties II, Compositio Math. 59 (1986) 339- 398.

[Z4] Zucker, S. The Hodge structure on the intersection homology of varieties with isolated singularities, Duke Math. Jour. 55 (1987) 603-616.