Chap. III : Obstruction Theory

In this chapter we consider the following problem:
L is a subcomplex of a CW complex K, and \( f : L \rightarrow Y \). We would like to extend this map to a map of \( K \rightarrow Y \). Since \( K \) is a CW complex, the obvious thing to do is to extend the map skeleton-by-skeleton and on each skeleton, cell-by-cell. So the crucial special case is where \( L = S^m \) and \( K = e^{m+1} \).

We see that \( f \) need not extend if \( \varphi_m(Y) \neq 0 \) and in fact, \( f \) will extend if and only if \( f \) represents the trivial element of \( \varphi_m(Y) \). Using this observation we will define below the obstruction to extending a map from the \( m \) skeleton to the \( m+1 \) skeleton.

In general, we define a map \( f : L \rightarrow Y \) to be \( m \)-extendible if there exists an extension of \( f \) mapping \( \overline{K}^m = K^m \cup L \rightarrow Y \). If we assume, as we will from now on, that \( Y \) is pathwise connected, then the following is obvious:

1. Lemma: Every map \( L \rightarrow Y \) is \( 1 \)-extendible.

We define the extension index of a map \( f : L \rightarrow Y \) to be the least upper bound of the set of integers \( m \) such that \( f \) is \( m \)-extendible.

2. Lemma: Homotopic maps of \( L \rightarrow Y \) have the same extension index.

Proof: For any \( m \), this follows from the homotopy extension theorem applied to the pair \( (\overline{K}^m, L) \).
3. Lemma— Let \( f : (K', L') \rightarrow (K, L) \) and \( e : Y \rightarrow Y' \),
then if \( f : L \rightarrow Y \) is \( m \)-extensible, then so is \( e f g : L' \rightarrow Y' \).

Proof: This is obvious, if \( g \) is cellular, and if \( g \) is not we can find \( g_1 : (K', L') \rightarrow (K, L) \) where \( g_0 = g \) and \( g_1 \) is cellular, by the cellular approximation theorem (applied twice). \( e f g \sim e f g_1 \) and hence by the previous lemma if \( e f g_1 \)
is \( m \)-extensible then so is \( e f g \).

Now assume that \( K \) and \( Y \) are pathwise connected and that \( Y \) is \( m \)-simple. Consider a map \( f : \overline{K}^m \rightarrow Y \). To define the obstruction we will require the following form of the relative Hurewicz theorem, which we will not prove:

Theorem— If \( m \geq 2 \) and \( (X, A) \) is an \( (m-1) \) connected CW pair, then the Hurewicz homomorphism \( m : H_m(X, A, x_0) \rightarrow H_m(X, A) \) is an epimorphism and its kernel is the subgroup generated by elements of the form \( \omega - \omega x \), where \( \omega \in H_m(X, A, x_0) \) and \( x \in H_1(A, x_0) \).

By lemma 1, we can assume that \( m \geq 1 \). By theorem 22 of chap. 1, \( (X, \overline{K}^m) \) is a connected and hence by considering the homotopy sequence of the triple \((K, \overline{K}^m, K)\) we get that \( w_m(K, \overline{K}^m) = 0 \) for \( m > m > 1 \) and from this and the homotopy sequence of the pair \((K, \overline{K}^m)\) we get that \( w_1(K, \overline{K}^m) = 0 \). We use that \( w_0(\overline{K}^m) \rightarrow w_0(K) \) is mono, which, since \( m \geq 1 \), is true.

Definition— If \( f : \overline{K}^m \rightarrow Y \), with \( m \geq 1 \), and \( K \) and \( Y \) pathwise connected, with \( Y \) \( m \)-simple, then we define \( e^{m+1} \sigma \in C^{m+1}(K, L; w_0(Y)) \) by the switchback:

\[
\begin{array}{ccc}
\omega & \rightarrow & H_{m+1}(K, \overline{K}^m, K_0) \\
\downarrow & & \downarrow \\
\omega & \rightarrow & H_{m+1}(\overline{K}^m, \overline{K}^m)
\end{array}
\]
We note that the kernel of \( K \) is of the form \( \{ \omega \in \omega \} \) by the Hurewicz theorem. \( f_\# (\omega; \omega) = f_\# (\emptyset; \omega; \omega) = 0 \) since \( Y \) is \( m \)-simple. Hence the switchback does define a homomorphism.

Intuitively, what we wish the obstruction to be is the following: Let \( \sigma \) be a cell of \( \text{I}^{m+1}_K = \text{I}^{m+1}_L \), with attaching map \( i_\sigma : (e^{m+1}, S^m) \rightarrow (\overline{\mathbb{R}} \cup \sigma, \overline{\mathbb{R}}) \). In order, that \( f \) extend to \( \overline{\mathbb{R}} \) \( \sigma \), it is necessary and sufficient that \( \text{cls} f(i_\sigma; S^m) \) of \( w_m(Y) \) = 0. Thus, we would like to define \( c_f^{m+1} \) on the \( m+1 \) cells of \( K \), by \( c_f^{m+1}(\sigma) = \text{cls} f(i_\sigma; S^m) \). We can make this meaningful by using the decomposition of chap. II theorem 4, i.e. use the switchback:

\[
\sum_\sigma w_m(e^{m+1}_\sigma; S^m_\sigma) \xrightarrow{\Sigma f} \sum_\sigma H_{m+1}(e^{m+1}_\sigma; S^m_\sigma) \xrightarrow{\Sigma \iota} H_{m+1}(\overline{\mathbb{R}}^{m+1}, \overline{\mathbb{R}}^m)
\]

To prove these two definitions are the same, pick a base point \( p_\sigma \) in each \( S^m_\sigma \), and let \( w_\sigma \) be a path in \( \overline{\mathbb{R}}^m \) joining \( i_\sigma(p_\sigma) \) and \( k_\sigma \). Then the fact that the following diagram commutes proves the two definitions equivalent.

\[
\begin{array}{ccc}
\sum_\sigma w_m(e^{m+1}_\sigma; S^m_\sigma, p_\sigma) & \xrightarrow{\Sigma f} & \sum_\sigma H_{m+1}(e^{m+1}_\sigma; S^m_\sigma) \\
\downarrow \Sigma \omega \tau & & \downarrow \Sigma \iota \\
\sum_\sigma w_m(\overline{\mathbb{R}}^{m+1}, \overline{\mathbb{R}}^m, p_\sigma) & \xrightarrow{\Sigma \omega \tau(c\iota; S^m)} & H_{m+1}(\overline{\mathbb{R}}^{m+1}, \overline{\mathbb{R}}^m)
\end{array}
\]

\[\xrightarrow{\Sigma \omega \tau(c\iota; S^m)} \xrightarrow{\Sigma f} w_m(\overline{\mathbb{R}}, k_\sigma) \xrightarrow{\Sigma f} w_m(Y)\]

From the second definition the following is obvious:

4. Theorem: \( f : \overline{\mathbb{R}}^{m+1} \rightarrow Y \), extends iff \( c_f^{m+1} = 0 \).
5. Theorem—\( S_{\gamma}^{n+1} = 0 \), i.e. \( \gamma_S^{n+1} \) is a cocycle.

Proof: We show that \( \gamma_S^{n+1} \otimes 0 = 0 \). This follows from the diagram:

\[
\begin{array}{cccc}
\omega_{n+2}(K^{n+2}, \overline{K}^{n+2}, k_e) & \xrightarrow{h} & H_{n+2}(K^{n+2}, \overline{K}^{n+2}) & \\
\downarrow & & \downarrow & \\
\omega_{n+1}(K^{n+2}, k_e) & \xrightarrow{h} & H_{n+1}(K^{n+1}) & \\
\downarrow & & \downarrow & \\
\omega_{n+1}(K^{n+2}, \overline{K}^{n+2}, k_e) & \xrightarrow{h} & H_{n+1}(K^{n+1}, \overline{K}^{n+1}) & \\
\omega & \downarrow & & \\
\omega_n(K^n, k_e) & \xrightarrow{f} & \omega_n(Y) & \\
\end{array}
\]

Since the column on the left contains two successive terms from the homotopy sequence of \((K^{n+1}, \overline{K}^{n+1})\), it follows that the composite is zero.

Because of our convenient global definition, the following naturality theorem is obvious.

6. Theorem—If \( \phi : (K; L) \rightarrow (K, L) \) is cellular then \( \gamma^{n+1}_{\phi^\#} = \phi^\# \gamma^{n+1}_{\phi} \).

We also note the following lemma:

7. Lemma—If \( f, f' : \overline{K}^n \rightarrow Y \) are homotopic, then \( \gamma^{n+1}_f = \gamma^{n+1}_{f'} \).

Now we consider a slightly different situation.

Let \( \phi_0, \phi_1 : \overline{K}^n \rightarrow Y \) and let \( h_t : \overline{K}^{n-1} \rightarrow Y \) be a homotopy between \( \phi_0 |_{\overline{K}^{n-1}} \) and \( \phi_1 |_{\overline{K}^{n-1}} \). We now recall that \( C^*(K; L, \omega_{n}(Y)) \) is isomorphic to \( C^*(K; L, \omega_{n}(Y)) \otimes C^*(1) \). \( C^*(1) \) is free on three generators \( 0, 1, \) and \( I \), with \( S \cdot 0 = -1 \) and \( S \cdot 1 = 1 \). Define

\[
F : \overline{K}^{n-1} \cup R^n \rightarrow Y \text{ by } h_t \text{ on } \overline{K}^{n-1} \cup R^n \text{ for } i = 0, 1.
\]

\[
c_{\gamma}^{n+1} = c_{\gamma}^{n+1} \otimes 0 - c_{\gamma}^{n+1} \otimes 1 \text{ is thus an element of } C^{n+1}((K, L) \times I).
\]
However, more than that, it pulls back to \( C^{n+1}(K,L; \tau(I,\hat{1}); l_\omega(Y)) \).

Consider the map: \( C^{n+1}(K,L; xI) \xleftarrow{j} C^{n+1}(K,L; xI; l_\omega(Y)) \)
\( \xrightarrow{k} C^n(K,L; \omega(Y)) \otimes C^1(I,\hat{1}) \xrightarrow{\gamma} C^n(K,L; \omega(Y)) \), where \( j \) is the inclusion and \( k \) is the composite of \( C^1(I,\hat{1}) \xleftarrow{j} C^0(\hat{1},1) \xrightarrow{\alpha} C^0(0) = \mathbb{Z} \), tensored with the identity on \( C^n(K,L) \). We define \( d^n(\omega_0, \omega_1; \hat{h}_t) = (-1)^{n+1} k \cdot j^{-1}(c^{n+1}_\omega - c^{n+1}_\omega \otimes 0 - c^{n+1}_\omega \otimes 1) \), which is thus an element of \( C^n(K,L; \omega(Y)) \). It is easily verified that the meaning of the difference cochain is as follows: let \( \sigma \) be a cell of \( I^m \), then \( (-1)^{n+1} d^n(\omega_0, \omega_1; \hat{h}_t)(\sigma) \) is represented by the map of \( S^m - \partial (e^m \times I) = e^m \times I \cup S^{m-1} \times I \), which is \( \omega_0 \) on \( e^m \times 0 \), \( \omega_1 \) on \( e^m \times 1 \), and \( h_t(\sigma \times S^{m-1}) \) on \( S^{m-1} \times I \). From this it follows that the analogue of theorem 4 holds:

10. Theorem: \( h_t \) extends to a homotopy of \( \omega_0 \) and \( \omega_1 \) iff \( d^n(\omega_0, \omega_1; \hat{h}_t) = 0 \).

11. Theorem: For every \( n \)-cochain \( c \) in \( C^n(K,L; \omega(Y)) \), \( \omega_0: K^m \rightarrow Y \) and \( h_t: K^{m-1} \rightarrow Y \) with \( h_t = g^t|K^{m-1} \), there exists \( \omega_1: K^m \rightarrow Y \) such that \( h_t = g^t|K^{m-1} \) and \( \omega = d^n(\omega_0, \omega_1; \hat{h}_t) \).
Proof: Let the \( m \)-sphere \( S = (e^x x_0) \cup (S^{x-1} x_1) \cup (e^x x_1) \).
There exists a map \( f : S \to Y \) representing \((-1)^{m+1} o(\sigma)\) (ie. \( c(1_\sigma) \)) where \( g \) is a chosen fixed generator of \( H^m(S, S^{x-1}) \),
which since \((e^x x_0) \cup (S^{x-1} x_1)\) is contractible, we can assume
= \( g \circ i_\sigma \) on \( e^x x_0 \) and \( h_t \circ i_\sigma \) on \( S^{x-1} x_1 \). Define \( g_1 = \{ h_t(x) \mid x \in \overline{K^{x-1}} \} \).
It is obvious that \( d(g_0, g_1; h_t) = c \).

The following are two easy, but important naturality
theorems:

12. Theorem- \( d^m(g_0, g_2; h_t) = d^m(g_0, g_1; h_t) + d^m(g_1, g_2; h_t) \) (where \( h_t \) is shorthand for the homotopy \( j_t \) with \( j_{t_0} = h_{t_0} \) and \( j_t = h_{t+1} \)).

13. Theorem- If \( \phi : (K', L') \to (K, L) \) is cellular,
then \( d^m(g_0, g_1; h_t) \) = \( \phi^* d^m(g_0, g_1; h_t) \).

Our first important application of these constructions
is the following extension theorem of Eilenberg:

14. Theorem- If \( f : \overline{K^x} \to Y \), then \( f \circ \sigma \) = 0 in \( H^{m+1}(K, L; \pi(Y)) \) iff there exists a map \( g : \overline{K^{x+1}} \to Y \) such that \( g \circ \overline{K^{x-1}} = f \circ \overline{K^{x-1}} \).

Proof: We let \( d(g_0, g_1; h_t) = d(g_0, g_1) \) if \( h_t = g_0 \circ \overline{K^{x-1}} \)
for all \( t \).
If \( \sigma \) exists, as described, then \( c^x \) = 0
and hence \( c^x \) is the coboundary of \( d^x(g_0, g) \).
If \( c^x \) is homologous to 0, then there exists
\( c \) in \( C^x(K, L; \pi(Y)) \) such that \( c^x \) = \( \sigma \). By theorem 11, there
exists \( g : \overline{K^x} \to Y \) such that \( d(f, g) = c \) and hence, by theorem 10,
\( c^x = 0 \) and hence \( g \) extends to \( \overline{K^{x+1}} \).
This result has the following important corollary:

15. Theorem—If $Y$ is $r$-simple and $H^{r+1}(K;L;w_r(Y)) = 0$ for every $r : m \leq r < m+1 \leq \infty$, then the $m$-extensibility of $f : L \to Y$ implies its $m$-extensibility. (Where $\infty$-extensibility means that $f$ can be extended to all of $K$.)

Related to the extension problem, that we have been discussing, is the homotopy problem. Given two maps $f, g : K \to Y$, such that $f|L = g|L$ for a subcomplex $L$, we would like to determine whether or not these two maps are homotopic, rel $L$.

Our method is again one of extending the homotopy skeleton by skeleton. If $h_t : \overline{K^{m-1}} \to Y$ is a homotopy of $f|\overline{K^{m-1}}$ with $g|\overline{K^{m-1}}$ (rel $L$), then $d^n(f,g;h_t) \in C^n(K;L;w_n(Y))$ as before (assuming $Y$ is $m$-simple). However, now we note that $c^n_f - c^n_g = 0$ since $f$ and $g$ are defined on $K$. Hence, we have that $d^n(f,g;h_t)$ is by definition $(-1)^{m-1} (k \gamma j^{-1})(c^n_F)$ and $\bigotimes d^n(f,g;h_t) = 0$, i.e. the difference cochain is, in this case, a cocycle. By a direct application of theorem 14, we get the following analogue, with the corollary, analogous to theorem 15:

16. Theorem—$\bigotimes d^n(f,g;h_t) = 0$ iff there exists a homotopy $k_t : \overline{K^n} \to Y$, such that $k_0 = f|\overline{K^n}, k_1 = g|\overline{K^n}$, $k_t|\overline{K^{n-2}} = h_t|\overline{K^{n-2}}$.

17. Theorem—If $Y$ is $r$-simple and $H^r(K;L;w_r(Y)) = 0$ for every $r : m < r < m+1 \leq \infty$, if $f|\overline{K^n}$ is homotopic to $g|\overline{K^n}$ (rel $L$), then $f|\overline{K^n}$ is homotopic to $g|\overline{K^n}$ (rel $L$).
We now consider a specialization of the theory which we have been developing, called primary obstruction theory. We now assume that $Y$ is $m$-connected. If $m > 2$, then $Y$ is simply connected and hence $r$-simple for all $r$. For the case $m = 1$, we also assume that $Y$ is $1$-simple, i.e. that $w_1(Y)$ is abelian.

Given a map $f : L \rightarrow Y$, with $L$ a subcomplex of $K$, then, by theorem 15, $f$ is $m$-extendible. Furthermore, by theorem 17, any two such extensions $f$ and $f'$ of a $m$-homotopy, i.e. $f \upharpoonright L^{m-1}$ is homotopic to $f' \upharpoonright L^{m-1}$ (rel $L$). If $h_t : L^{m-1} \rightarrow Y$ is such a homotopy, then by theorem 10, $d(h_t) = \partial (c_{m+1} - c_{m+1}^{'}) (-1)^{m}$. This shows that $c_{m+1}$ and $c_{m+1}^{'}$ are cohomologous, and thus represent the same element of $H^{m+1}(K, L; w_m(Y))$. This unique element determined by $f$ is called the primary obstruction of $f$ and is denoted by $\omega^{m+1}(f)$.

Pick a base point $y_0$ of $Y$, and let $0 : L \rightarrow y_0$ in $Y$. By theorem 17, $f \upharpoonright L^{m-1}$ and $0 \upharpoonright L^{m-1}$ are homotopic and hence by the homotopy extension theorem, $f$ is homotopic to a map of $L \rightarrow Y$ which when restricted to $L^{m-1} = 0 \upharpoonright L^{m-1}$. Hence, by theorem 7, we can assume that $f \upharpoonright L^{m-1} = 0 \upharpoonright L^{m-1}$, since the two homotopic maps have the same primary obstruction elements.

We then define $\kappa^m(f) = d(f, 0) \in H^m(L; w_m(Y))$. If we let $\delta^* : H^m(L; w_m(Y)) \rightarrow H^{m+1}(K, L; w_m(Y))$ be the coboundary homomorphism of the pair $(K, L)$, then we have the following:

18. Theorem: $\omega^{m+1}(f) = \delta^* \kappa^m(f)$. 
Proof: Since \( f(L^{n-1}) = y_0 \), it follows that \( f \) has an extension \( f^* : \overline{K^n} \rightarrow Y \) such that \( f^*(\overline{K^n} - L) = y_0 \). If we let \( \Theta^* : \overline{K^n} \rightarrow Y \) denote the constant map such that \( \Theta^*(\overline{K^n}) = y_0 \), then \( S * \nu^m(f) = S * \nu^m(f, \Theta^*) \) is represented by \( S \partial(f^* \Theta^*) \) which equals \( c_{f*}^{n+1} \) by theorem 10, and \( c_{f*}^{n+1} \) represents \( c_{f|L}^{n+1} \).

Akin to this theorem, is the corresponding result for homotopy. If \( f, g : K \rightarrow Y \), with \( f|L = g|L \), we can consider the elements of \( H^m(K, L; \mathbb{Z}_n(Y)) \) represented by cocycles of the form \( d(f, g; h_t) \) with \( h_t \) a homotopy rel \( L \).

Since these correspond under \( \sim_{l+1} \) to elements \( c_{f|L}^{n+1} \), it follows from the above that these cocycles are all cohomologous and hence represent a single element \( \omega^m(f, g) \in H^m(K, L; \mathbb{Z}_n(Y)) \), the primary obstruction to homotopy (rel \( L \)) of the pair \((f, g)\). If \( j : K \hookrightarrow (K, L) \) is the inclusion map, then we obtain an induced map, \( j^* : H^m(K, L; \mathbb{Z}_n(Y)) \rightarrow H^m(K; \mathbb{Z}_n(Y)) \).

19. Theorem: \( \kappa^m(f) - \kappa^m(g) = j^* \omega^m(f, g) \).

Proof: We may assume that \( f(K^{n-1}) = g(K^{n-1}) = y_0 \) by altering \( f \) and \( g \) by a homotopy if need be. The result follows from the fact, by theorem 12, that \( d^m(f, g) - d^m(g, \Theta) = d^m(f, g) \), where \( \Theta \) is the constant map.

From these results we get the primary extension theorems and primary homotopy theorems.

An element of the cohomology group \( H^m(L; G) \) is said to be extendible over \( K \) if it is contained in the image of the homomorphism \( i^* : H^m(K; G) \rightarrow H^m(L; G) \) induced by the inclusion map \( i : L \hookrightarrow K \).
20. Theorem—For a given map \( f : L \rightarrow Y \), the following statements are equivalent:

1. \( f \) is \( m+1 \) extendible over \( K \).
2. \( \omega^m(f) = 0 \).
3. \( \kappa^n(f) \) is extendible over \( K \).

Proof: The equivalence of (1) and (2) follows from theorem 14. The equivalence of (2) and (3) follows from theorem 18 and the exactness of the cohomology sequence of the pair \((K,L)\).

21. Corollary—If \( Y \) is \( r \)-simple and \( H^{r+1}(K,L;w_{\tau}(Y)) = 0 \) for every \( r \) satisfying \( m < r < \dim(K-L) \), then a necessary and sufficient condition for a given map \( f : L \rightarrow Y \) to have an extension is that the characteristic element \( \kappa^n(f) \) is extendible over \( K \).

Proof: Follows from theorems 20 and 15.

22. Theorem—For any two given maps \( f, g : K \rightarrow Y \) such that \( f|L = g|L \), the following are equivalent:

1. \( f \) and \( g \) are \( m \)-homotopic rel \( L \).
2. \( \omega^m(f,g) = 0 \)

If we also have \( H^{m-1}(L;w_{\tau}(Y)) = 0 \), we can add:

3. \( \kappa^n(f) = \kappa^n(g) \).

Proof: That (1) and (2) are equivalent follows from theorem 16. The equivalence of (2) and (3) given the added condition again follows from the cohomology sequence of the pair and theorem 19.

Note that the extra condition is always fulfilled if \( L = \varnothing \).
23. Corollary— If \( Y \) is \( r \)-simple and \( H^r(K;L,w_r(Y)) = 0 \) for each \( r \) satisfying \( m < r < \dim(K-L)+1 \), and \( H^{m-1}(L;w_m(Y)) = 0 \), then a necessary and sufficient condition for a given pair of maps \( f, g : K \rightarrow Y \) with \( f|L = g|L \), to be homotopic rel \( L \) is that 
\[ \chi^L(f) = \chi^L(g) . \]

Proof: Follows from theorems 22 and 17.

With these theorems as motivation, we now examine \[ \chi^L(f) \] more closely, for \( f : K \rightarrow Y \) and \( Y \) \( m-1 \)-connected and \( m \)-simple. To calculate \( \chi^L(f) \), we replace \( f \) by a map \( g \) which is homotopic to \( f \) and such that \( g(|K^{m-1}|) = y_e \), and then 
\[ \chi^L(f) = [d(z,0)] , \]
where \( d(z,0) \) is defined as follows:

We define \( \bar{F} : K^{m-1}xI \cup Kx0 \cup Kx1 \rightarrow Y \) by \( \bar{F}(x,t) = y_e \) for \( x \in K^{m-1} \) or \( t = 1 \), and \( \bar{F}(x,0) = g(x) \). \( F \) is a map defined on the augmented \( m \)-skeleton of the pair of complexes: \( (KxI,Kx0 \cup Kx1) \) or, letting \( I = [0,1] \), the pair \( Kx(I,\bar{I}) \). Thus, \( c_{F}^{m+1} \in c_{n+1}(Kx(I,\bar{I});w_m(Y)) \), i.e.

\[ c_{F}^{m+1} : H_{m+1}(K^{m-1}xI \cup Kx1,K^{m-1}xI \cup Kx1) \rightarrow w_m(Y) \]

or preceding by an excision inclusion, it maps: \( H_{m+1}(K^{m-1}xI \cup Kx1,K^{m-1}xI \cup Kx1) \rightarrow w_m(Y) \).

\[ (-1)^{n+1}d(z,0) \] is this map preceded by the composite function:

\[ H_m(K^n,K^{n-1}) \rightarrow H_m((K^n,K^{n-1})x0) \leftarrow H_m(K^n,K^{n-1}) \rightarrow H_m(K^n,K^{n-1}) \oplus H_0(0) \rightarrow H_m(K^n,K^{n-1}) \oplus H_1(I,\bar{I}) \rightarrow H_{m+1}((K^n,K^{n-1})x(I,\bar{I})) . \]

This is, more or less, a review of the definition. However, we assert that the above map is the same as the simpler composite:

\[ H_m(K^n,K^{n-1}) \leftarrow H_m(Kx0 \cup K^{n-1}xI,K^{n-1}xI) \rightarrow H_m(K^{n-1}xI \cup Kx1,K^{n-1}xI \cup K^{n-1}xI) \rightarrow H_m(K^{n-1}xI \cup Kx1) \leftarrow H_{m+1}((K^n,K^{n-1})x(I,\bar{I})) , \]

where \( q \) is the restriction of the projection of \( KxI \) onto \( K \).

This result comes from the diagram:
\[
H_n(K^n, K^{n-1}) \otimes H_1(I, I) \xrightarrow{\gamma} H_{n+1}((K^n, K^{n-1})x(I, I))
\]

Where the following identities are easily verified:

\[q_*k_* = 1, \ j_*k_* = i_* + i_*, \ q_*j_*^{-1}i_1 = 0\gamma, \ \gamma' = i_* \gamma' i_*^{-1}.\]

Now using the naturality for \(\gamma\) that was proved in chapter 2, we have:

\[(i_* + i_1)(\gamma'_0 + \gamma'_1)(\varnothing_0 + \varnothing_1)\gamma_0^{-1} = \varnothing_0.\]

Therefore, \(q_*j_*^{-1}i_* \gamma' = q_*j_*^{-1}i_* + i_* \gamma'_1(\varnothing_0 + \varnothing_1)\gamma_0^{-1} = q_*j_*^{-1}i_* \gamma'_1 \varnothing_0 \gamma_0^{-1} = q_*j_*^{-1}i_* \gamma'_1 \varnothing_0 \gamma_0^{-1} = q_*j_*^{-1}i_* \gamma'_1 \varnothing_0 \gamma_0^{-1} = q_*j_*^{-1}i_* \gamma'_1 \varnothing_0 \gamma_0^{-1}.\]

The latter is an isomorphism and hence the former is. The result that we want is the equality of the inverse maps.

Using this identification we prove the following:

24. Lemma: \(d(s, 0)\) is given by the switchback:

\[H_n(K^n, K^{n-1}) \leftarrow w_n(K^n, K^{n-1}) k_2 \rightarrow w_n(y, y_0) \rightarrow w_n(Y).\]

Proof: This follows from the diagram:

\[
\begin{array}{c}
\xrightarrow{w_{n+1}((K^n, K^{n-1})x(I, I))} H_{n+1}((K^n, K^{n-1})x(I, I)) \\
\xrightarrow{q_0} H_n(K^n, K^{n-1}x I) \\
\xrightarrow{w_n(K^n, K^{n-1}x I)} H_n(K^n, K^{n-1}x I) \\
\xrightarrow{w_n(y, y_0)} w_n(K^n, K^{n-1}) \\
\xrightarrow{d(s, 0)} H_n(K^n, K^{n-1})
\end{array}
\]


We next consider the situation that arises when $Y$ is itself a CW complex. In that case we can assume that $K = Y$ and look at the privileged map $l_Y$. $\varphi^*(l_Y)$ is called the fundamental class of $Y$, and is written $\varphi^*(Y)$. Its importance in the classification of maps comes from the following theorem, which follows directly from theorem 13:

25. Theorem—If $f : K \to Y$, then $\varphi^*(f) = f^* \varphi^*(Y)$.

If $Y$ is $n$-connected and $n \geq 2$, then we have the following result:

26. Theorem—The image of $\varphi^*(Y)$ under the isomorphism:

$H^n(Y; w_n(Y)) \xrightarrow{\cong} \text{Hom}(H_n(Y), w_n(Y)) \xrightarrow{\text{Hom}(\text{id})} \text{Hom}(w_n(Y), w_n(Y))$ is $(-1)^{n+1} w_n(Y)$.

Proof: This follows from lemma 24 and the commutativity of the following diagram, where $i : Y \to Y$ is homotopic to $l_Y$ and $i(Y^{n-1}) = Y_e$ is a closed in celluler portion in $Y$.

\[
\begin{align*}
\varphi^*(Y) &\xrightarrow{\cong} H^*(Y) \\
\varphi^*(Y) &\xrightarrow{\cong} H^*(Y) \\
\varphi^*(Y, Y_e) &\xrightarrow{\cong} H^*(Y^n, Y^{n-1})
\end{align*}
\]

The case where we would expect the most success with primary obstruction theory, would be the case where the primary obstructions are the only ones.

We say that a CW complex is a $K(w, n)$ if the homotopy groups of the space are 0 except for dimension $n$, where the homotopy group $w_n$ is just $w$. If $n \neq 1$, then $w$ must, of course, be abelian. We are only interested in these notes in the case $w$ abelian, since if $n = 1$, we will want 1-simplicity.
If \(X\) and \(Y\) are topological spaces, then we shall denote by \([X, Y]\) the set of homotopy classes of maps of \(X\) to \(Y\).

27. Theorem— If \(X\) is a CW complex, then there is a natural, one-to-one, onto correspondence:

\[ [X, K(w, n)] \rightarrow H^n(X; w), \]

By the correspondence: \(\tilde{f} \mapsto f^* \otimes^w(K(w, n))\).

Proof: If \(c\) is a cocycle representing an element of \(H^n(X; w)\) then by theorem 11, there exists a map \(f : X \rightarrow K(w, n)\) with \(f(x^{n-1}) = k_c\), a base point of \(K(w, n)\), and such that if \(Q\) is the constant map \(Q(x) = k_c\), then \(d(f, Q) = 0\). \(f = \delta c = 0\), and hence \(f\) extends to \(X^{n+1}\), and hence to all of \(X\) by theorem 15. Hence, \(f\) represents an element of \([X, K(w, n)]\) and \(\tilde{f} \mapsto c\), by theorem 25. The correspondence is hence onto.

The correspondence is easily seen to be one-to-one. For if \(f^* \otimes^w(K(w, n)) = g^* \otimes^w(K(w, n))\), then by theorem 15, \(\otimes^w(f) = \otimes^w(g)\) and hence \(f\) is homotopic to \(g\) by theorem 23.

The correspondence is obviously natural in \(X\).

Now as to the existence of \(K(w, n)\)'s. We note that since \(X\) and the singular complex \(S(X)\) have the same homotopy groups, the restriction that the \(K(w, n)\)'s be CW complexes is no restriction at all in an existence proof. We will explicitly construct \(K(w, n)\) for \(n \geq 2\) and \(w\) abelian.

It follows that \(K(w, 1)\)'s exist for \(w\) abelian, since \(w_1(\Omega K(w, n)) = w_{n+1}(K(w, n))\), where \(\Omega K(w, n)\) is the loop space of \(K(w, n)\).

Present \(w\) as \(F/R\) for \(F\) a free abelian group on a set of generators \(I\) and \(R\) a subgroup of \(F\). Let \(X\) be a wedge of \(n\)-spheres \(S_i\) for \(i \in I\). Since \(n > 1\), \(w_1(X) = \begin{cases} F & i = n \\ 0 & i < n \end{cases}\)
Under the correspondence of $w_n(X)$ with $F$, the elements of a set of generators for $R$ are represented by a collection of maps $f_i : S^i \to X$. We define $X^{n+1}$ by attaching an $n+1$-cell with each of these maps. We assert that $w_i(X^{n+1}) = \begin{cases} 0 & i = m \\ 1 & i < m \end{cases}$ and to prove this, it obviously suffices to show that $w_{m+1}(X^{n+1}, X) \to w_m(X)$ has image precisely $R$. The map is certainly onto $R$ and that it is included in $R$ follows from the fact that $w_{m+1}(X^{n+1}, X)$ is generated by elements of the form $i_{e_{m+1}}$ where $i_{e_{m+1}} : (e_{m+1}, S^m) \to X^{n+1}$ is the attaching map of the cell $e_{m+1}$. This in turn follows from the Hurewicz theorem.

We now proceed inductively, to construct $X^{n+r+1}$ by attaching $n+r+1$ cells using a generating set for $w_{n+r}(X^{n+r})$ and thus killing the $n+r$th homotopy group. It is easily seen that $X^{n+r}$ is a $K(w, m)$.

We now show that any two $K(w, m)$'s are homotopically equivalent.

28. Theorem-- If $X$ and $Y$ are CW complexes such that each is a $K(w, m)$ (same $w$, same $m$), then $X$ is homotopically equivalent to $Y$.

Proof: Let $f$ represent the element of $[X, Y]$ such that $f \ast \alpha^X(Y) = \alpha^Y(X)$, and let $g$ represent the element of $[Y, X]$ such that $g \ast \alpha^Y(X) = \alpha^Y(Y)$. Since $(fg) \ast \alpha^X(Y) = \alpha^X(X)$ and $(gf) \ast \alpha^Y(X) = \alpha^Y(Y)$, it follows from theorem 27$^2$ that $gf$ and $fg$ are homotopic to $l^X_Y$ and $l^Y_X$, respectively. Therefore $X$ is homotopically equivalent to $Y$. 

Finally, we prove a theorem which extends Whitehead's theorem (chap. 1, theorem 27).

29. Theorem—Let \( f : X \to X' \) be a continuous mapping of connected, \( n \)-simple topological spaces, such that \( f_* : \pi_i(X, x_0) \to \pi_i'(X', x_0') \) is an isomorphism for \( 1 \leq i < n+1 \leq \infty \), then if \( K \) is any \( n \)-dimensional CW complex, the induced map:

\[
f_* : [K, X] \to [K, X'],
\]

is one-to-one and onto.

Proof: By replacing \( X' \) by the mapping cylinder \( M' \), if necessary, we can assume that \( f \) is an inclusion.

\( f_* \) is onto: Let \( h : K \to X' \) represent an element of \([K, X']\). Since the relative groups \( \pi_i(X', X) = 0 \) for \( 1 \leq i < n+1 \), corollary 25 of chapter 1 implies that there exists \( h' : K \to X \)
which is homotopic to \( h \) in \( X' \). This implies that \( f_* \) is onto since \( f_\ast \text{cls } h' = \text{cls } h \).

\( f_* \) is one-to-one: Let \( h_0', h_1' : K \to X \) such that such that there is a homotopy \( H : K \times I \to X \) such that \( H_0 = h_0' \) and \( H_1 = h_1' \), i.e., \( f_\ast \text{cls } h_0' = f_\ast \text{cls } h_1' \). Again applying corollary 25 of chapter 1, we may assume that \( H((K \times I)^{\infty}) \subset X \).

Thus, the restriction of \( H \) is a map \( H' : K \times 0 \cup K^{\infty-1} \times I \cup K \times I \to X' \), which we would like to extend to a map of \( K \times I \) into \( X \). Consider the obstruction to such an extension: \( c^{n+1}_H \in C^{n+1}(K \times I; w_m(X)) \).

\[
f_\#: C^{n+1}(K \times I; w_m(X)) \to C^{n+1}(K \times I; w_m(X'))
\]
is an isomorphism. Since \( f_\ast c^{n+1} = c^{n+1} = 0 \), we have that \( c^{n+1}_H = 0 \) and hence \( H' \) extends as required. Thus, \( \text{cls } h_0' = \text{cls } h_1' \) and \( f_* \) is therefore one-to-one.
We now consider the concept of fiber space or fibration, due to Serre. For reference, we refer the reader to the treatment in Hu, chapter III.

A map \( p : E \to B \) is said to satisfy the covering homotopy property with respect to a space \( (X, A) \), if for every homotopy \( h_t : X \to B \) and every map \( f : X \to E \) such that \( pf = h_0 \), there exists a homotopy \( f_t : X \to E \) such that \( f_0 = f \) and \( pf_t = h_t \). A map \( p \) which satisfies the covering homotopy property for every (finitely) triangulable space \( X \), is said to be a fibration and the triple \( (E, B, p) \) is a fiber space.

A strengthening of this property is given by the covering homotopy extension property (CHEP). A map \( p : E \to B \) is said to satisfy the covering homotopy extension property, the CHEP, with respect to a pair \( (X, A) \) of spaces if for every homotopy \( h_t : X \to B \) and map \( f : X \to E \), and partial homotopy \( g_t : X \to E \) such that \( g_0 = f|A \) and \( pg_t = h_t|A \), there exists a homotopy \( f_t : X \to E \) such that \( f_0 = f \), \( pf_t = h_t \) and \( f_t|A = g_t \). The two properties are related by the following theorem:

1. Theorem—For a map \( p : E \to B \), the following are equivalent:
   
   i) \( p \) is a fibration

   ii) For each \( m \geq 0 \), \( p \) has the covering homotopy property with respect to the \( m \)-cell \( e^m \).

   iii) For each \( m \geq 0 \), \( p \) has the CHEP with respect to the pair \( m \)-cell mod boundary, \( (e^m, S^{m-1}) \).
iv) If has the CHEP with respect to every CW-pair (K,L).

v) If (K,L) is a CW-pair such that L is a strong deformation retract of K and if \( f : K \to B \) and \( g : L \to E \)
such that \( pg = f \mid L \), then \( g \) has an extension \( g' : K \to E \)
such that \( pg' = f \).

Proof: 1) --> ii) A fortiori.

ii) --> iii) see Hu pages 63–64.

iii) --> iv) Using the standard Zorn's lemma argument we can extend \( g_t : L \to E \) to a maximal pair
\((g^{t'}_t, L')\) with \( L' \) a subcomplex of \( K \) containing \( L \) and
\( g^{t'}_t : L' \to E \) such that \( g^{t'}_t = f \mid L' \), \( g^{t'}_t \mid L = g_t \) and \( pg^{t'}_t = h^{t'}_t \mid L' \).

We assert that by maximality \( L' = K \). If not, then let \( \sigma \) be a cell of smallest dimension of \( K-L' \). To extend
\( g^{t'}_t \) to \( \sigma \), it suffices to extend \( g^{t'}_t f_{t'_\sigma} : S^{m-1} \to E \) (where
\( m = \text{dimension } \sigma \)), to a homotopy \( f_t : e^m \to E \) such that
\( f_0 = f_{t'_\sigma}, pf_t = h_{t'_\sigma} \), and this extension exists by iii.

iv) --> v) Let \( k_t : K \to K \) be a strong deformation retraction of \( K \) to \( L \), i.e. \( k_t \) retracts \( K \) to \( L \) and \( k_t(x) = x \) if
\( x \in L \) or \( t = 1 \). Let \( \Upsilon = \triangledown k_0 : K \to E, \hat{\Upsilon}_t = f k_t : E \to B \)
and \( \tilde{\Upsilon}_t = g : L \to E \). If \( \Upsilon_t : K \to E \) is the homotopy
produced by an application of the CHEP, then \( \Upsilon_1 \) is the
required extension \( g' \).

v) --> i) If \( X \) is triangulable (in fact, if \( X \) is a CW complex) then \((X \times I, X \times 0)\) is a CW pair with \( X \times 0 \) a
strong deformation retract of \( X \times I \), whence i) from v).

Note that either the proof that v) --> i) or
property iv) specified down to the case \( L = \emptyset \), prove
that a fibration has the covering homotopy property
with respect to every CW complex.
We will assume from now on that the base is pathwise connected, from which it easily follows that p is onto. If we choose a base point \( b_0 \) in \( E \), we define \( p^{-1}(b_0) = F \) to be the fiber of \( p \). Choosing a base point \( x_0 \) in \( F \), we get (see Hu page 152) the homotopy sequence of the fibering, \( p \), which is an exact sequence:

\[
\cdots \rightarrow \pi_{n+1}(B, b_0) \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \rightarrow \pi_n(B, b_0) \rightarrow \cdots
\]

With a base point chosen, we shall sometimes write a fiber space as \( E \xrightarrow{p} B \), with the base point \( b_0 = p(F) \).

Given a fibering \( p : E \rightarrow B \) and a map \( f : B' \rightarrow B \) we define \( f^*E = \{ (x, y) \in B' \times E : f(x) = p(y) \} \) which is the topological "pull-back" of \( p \) and \( f \), i.e. given a space \( X \) and maps \( g_1 : X \rightarrow B' \) and \( g_2 : X \rightarrow E \) such that \( fg_1 = pg_2 \), then there exists a map \( h : X \rightarrow f^*E \) such that the following diagram commutes (\( h(x) = (g_1(x), g_2(x)) \) and is obviously unique):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & f^*E \\
\downarrow{g_1} & & \downarrow{g_2} \\
B' & \xrightarrow{f} & B \\
\end{array}
\]

Using this property, it is easily proved that the triple \( (f^*E, B', f^*)_1 \) is a fiber space (where \( q_1 \) and \( q_2 \) are the projections of \( f^*E \) to \( B' \) and \( E \), respectively). It is called the fiber space induced by \( f \).

We call two fiber spaces over the same base, isomorphic if there is a homeomorphism of spaces between them which commutes with projections, i.e.:
In general, if the map is understood, we shall refer to E as the fiber space, and B as the base space. We shall say E is isomorphic to E' and mean p is isomorphic to p'.

We leave to the reader the easy proofs that if $g : B^n \rightarrow B^n$, then $g^*(f^*E)$ is isomorphic to $(fg)^*E$, and that if $f$ is the inclusion of a subset into B, then $f^*E$ is isomorphic to $p^{-1}(B')$.

Note that if $f : (B', b'_0) \rightarrow (B, b_0)$, then $q_2 : (f^*E, F') \rightarrow (E, F)$ where $F'$ and F are the fibers over $b'_0$ and $b_0$, respectively. Since $F' = b'_0 x F$, with the restriction of $q_2$ just projection on the second factor, we see that $q_2|F' : F' \rightarrow F$ is a homeomorphism and we will usually identify the two fibers by this isomorphism.

From now on, we will assume that every fiber space has fixed base points chosen in E and B such that p is base-point preserving, i.e., the base-point of E is in the fiber over the base-point of B, and we shall refer to the fiber over the base-point of B, as the fiber of p, usually denoted F. We shall summarize the statement: "p : E \rightarrow B is a fiber space with fiber F" by writing p : $E \xrightarrow{F} B$. For convenience we shall refer to all the base-points as w.

We now examine the notion of principle fiber space.

Definition—A fiber space $E \xrightarrow{F} B$ is called a principle fiber space if: i—F is an H-space, i.e., there exists a map $\mu : FxF \rightarrow F$ such that $\mu$ is a homotopy unit for $\mu$, that is, if $\varphi : FxF \rightarrow F$ is the folding map sending $(x, w)$ and $(w, x)$ to $x$, then the following commutes up to a homotopy; rel $\mu w$.
\[ F \xrightarrow{\xi} F \]
\[ \cap \quad \mu \]
\[ F \xrightarrow{\eta} F \]

and ii- there exists a map \( \gamma : F \times E \rightarrow E \) which commutes with projection, i.e. the following commutes:

\[ F \times E \xrightarrow{\pi_2} E \]
\[ \mu \]
\[ F \xrightarrow{\eta} E \]

and are such that the following commute up to homotopy; rel \( \times x \):

\[ \times x E \]
\[ \downarrow \gamma \]
\[ E \]
\[ \mu \downarrow \]
\[ F \xrightarrow{\eta} E \]

The motivating example is where \( E \) is the space of paths based at \( \ast \) in \( B \) with the usual action \( \bigwedge B \times E \rightarrow E \).

The important property that we will need is given by the following lemma:

2. Lemma- If \( f^*E \) is the fiber space induced by the map \( f : (B, \ast) \rightarrow (B, \ast) \) and \( E \) is a principle fiber space over \( B \), then \( f^*E \) is a principle fiber space over \( B' \).

Proof: We must define \( \gamma' : F x^*E \rightarrow f^*E \). By the pullback property of \( f^*E \), it suffices to define maps into \( E \) and \( B' \) such that the two maps agree when composed with the map into \( B \). These two maps are: \( F x^*E \rightarrow f^*E \rightarrow B' \) and \( F x^*E \rightarrow F x E \xrightarrow{\gamma'} E \). The commutativity (up to a homotopy where required) of the three diagrams is an easy exercise, using the corresponding diagrams for \( \gamma' \).

On homotopy, \( \gamma_* \) corresponds to addition of elements of the two homotopy groups. Formally, we have the lemma:
3. Lemma.- Let \( F \xrightarrow{i} E \xrightarrow{\pi} B \) be a principle fiber space with map \( \gamma : F \times B \to E \), then the following commutes:

\[
\begin{align*}
\pi \circ (F \times B) & \xrightarrow{\gamma \times \gamma} \pi \circ (E) \\
& \xrightarrow{i_* \circ l_*} \pi \circ (F) \circ i_* + \pi \circ (B) \circ l_*
\end{align*}
\]

Proof: The result follows from the fact that we will produce a diagram that commutes up to homotopy with the top and bottom maps representing the two results: let \( x : S^q \to F \) and \( y : S^q \to E \) represent elements of \( \pi_q(F) \) and \( \pi_q(E) \), respectively. The following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
S^q & \xrightarrow{\Delta} & S^q \times S^q \\
\cup & \xrightarrow{\bigcup} & \xrightarrow{\bigcup} \\
\cup & \xrightarrow{\bigcup} & \xrightarrow{\bigcup} \\
& \xrightarrow{\pi} & F \times E \\
& \xrightarrow{\gamma} & E
\end{array}
\]

Where \( e \) is the map pinching the equator of \( S^q \) to \( x \) and \( \Delta \) is the diagonal map.

That the triangle on the left commutes up to homotopy is a standard result of homotopy theory. It is a special case of the following lemma, which we include for completeness:

4. Lemma.- Given a space \( X \) with base point \( x \), we define the suspension of \( X \), \( \Sigma X = X \cup X \cup X \cup X \). Let \( e : \Sigma X \to \Sigma X \times \Sigma X \) by \( e(x,t) = \begin{cases} (x,2t)x & 0 < t < \frac{1}{2} \\ (x,2t-1)x & \frac{1}{2} < t < 1 \end{cases} \) and \( \Delta : \Sigma X \to \Sigma X \times \Sigma X \) be the diagonal map, then the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\Sigma X & \xrightarrow{\Delta} & \Sigma X \times \Sigma X \\
& \xrightarrow{e} & \Sigma X \cup \Sigma X \cup \Sigma X \cup \Sigma X
\end{array}
\]

Proof: The required homotopy is given by:

\[
H((x,t),s) = (x, \min((1+s)t,1))x, (1- \min((1+s)(1-t),1))x
\]
We now return to the problem which motivated the previous chapter. Assume that we have a complex $K$ and subcomplex $L$ and a mapping $f : L \rightarrow X$. We would like to extend $f$ to a mapping of $K$ into $X$. The method considered in chapter three was to extend $f$ skeleton by skeleton, that is to say, use the skeletons $K^n$ as "approximations" to $K$ and to extend $f$ as far as is possible, using these approximations. We now introduce the notion of the Postnikov system which "approximates" $X$ instead of $K$, and approximates by using mappings which preserve homotopy groups instead of inclusions.

We define the Postnikov system of a space $X$ to be a diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f_n} & X_n \\
\downarrow & & \downarrow p_n \\
X_{n-1} & \xrightarrow{f_{n-1}} & X_n
\end{array}
$$

where $f_n^*: \pi_i(X, x) \rightarrow \pi_i(X_n, x)$ is an isomorphism, $i \leq n$, and where $\pi_i(X_n, x) = 0$ for $i > n$. (We assume that $X$ is $1$-connected from now on), and where $p_n$ is a fiber space map.

For $X$ a CW complex we can construct this system inductively using the same method of attaching cells to kill homotopy groups, that was used in the construction of $K_{(n, m)}$ at the end of chapter three. That is, we can find a space $Y$ such that $X \left(\pi_i Y, \pi_{i+1} X \right)$ and $\pi_{i+1} Y = 0$ for $i > n$.

It follows that the inclusion of $X$ into $Y$ induces an isomorphism of homotopy groups for $i \leq n$. By theorem 15 of chapter 3, $f_{n-1} : X \rightarrow X_{n-1}$ extends to a map of $Y \rightarrow X_{n-1}$. Thus, we get, by induction, a diagram like the above except that the $p_n$'s needn't be fiber maps. However, there is a method of "replacing" any map by a fiber map, by replacing the domain.
space by a homotopy equivalent space. By thus "replacing" the $p_n$'s by fiber maps, inductively, we can construct directly the Postnikov system for $X$.

There is another construction which is less geometrical but is rather cohomological. In addition to being easier to handle, this method generalizes the Postnikov system of a space to the Moore–Postnikov system of a map. However, before we give this construction, we must digress to introduce the homotopy groups of a map, and similarly homology and cohomology groups.

Given a map $f : X \rightarrow Y$, we have defined the mapping cylinder $M_f$ and we have the following properties.

$X$ (identified with $X \times 0$) and $Y$ are subsets of $M_f$ and the inclusion of $Y$ into $M_f$ is a homotopy equivalence, with inverse $r_f : M_f \rightarrow Y$ defined by $r_f(x, t) = f(x)$ and $r_f(y) = y$. We have the following commutative diagram, with the row being the homotopy sequence of the pair and hence exact:

$$
\cdots \xrightarrow{w_n(X)} w_n(M_f) \xrightarrow{\pi_{n-1}} w_n(M_f, X) \xrightarrow{\partial} w_{n-1}(X) \cdots
$$

We assume that $X$ and $Y$ are 1-connected so that we need not worry about base points. We define $w_n(M_f, X) = w_n(f)$ and replace $w_n(M_f)$ by $w_n(Y)$ as shown to obtain the exact homotopy sequence of the map:

$$
\cdots \xrightarrow{w_n(X)} w_n(Y) \xrightarrow{\pi_{n-1}} w_n(f) \xrightarrow{\partial} w_{n-1}(X) \cdots
$$

Similarly, if we define $\pi_n(f; G) = \pi_n(M_f, X; G)$ and $H^n(f; G) = H^n(M_f, X; G)$ then we obtain exact homology and cohomology sequences of $f$. 
... $H_n(X;G) \xrightarrow{f_*} H_n(Y;G) \rightarrow H_n(f_*G) \xrightarrow{\partial} H_{n-1}(X;G) ...$

... $H^n(X;G) \leftarrow H^n(Y;G) \leftarrow H^n(f_*G) \leftarrow H^{n-1}(X;G) ...$

If $f$ is a fiber map, then we also have the homotopy sequence of the fiber space:

... $w_n(x) \xrightarrow{f_*} w_n(y) \xrightarrow{\partial} w_{n-1}(F) \rightarrow w_{n-1}(x) ...$

It is obvious, by exactness of the two sequences that $w_i(F) = 0$ iff $w_i(x) = 0$ for any $i$. We will require a sharper isomorphism result only in the following special case:

5. Lemma—Let $p : E \xrightarrow{p} B$ be a fiber space, with $F$ $m-1$ connected and $(E\times\mathbb{R})$ having the homotopy type of a CW pair. Then the following is an isomorphism: $w_n(F) \xrightarrow{\partial} w_{n+1}(p)$. Where $cF$ is the cone on $F$, and $(cF,F) \rightarrow (M_p,E)$ is the inclusion $(x,t) \rightarrow (x,t)$ for $x$ in $F$. We also assume that $E$ is 1-connected.

Proof: Since $cF$ is contractible, $w_{n+1}(cF,F) \xrightarrow{\partial} w_n(F)$ is an isomorphism, and since $F$ is $m-1$ connected, the pairs $(cF,F)$ and $(M_p,E)$ are $m$-connected and hence the Hurewicz homomorphisms are isomorphisms for $m+1$, and thus, it suffices to prove that the inclusion induces an isomorphism of homology:

$H_{n+1}(cF,F) \rightarrow H_{n+1}(p)$.

$(B,\mathbb{R})$ has the homotopy type of a CW pair $(K,L)$. Since $L$ is contractible, $(K,L)$ has the same homotopy type as $(K/L,\mathbb{R})$ and hence we can assume that $L = \mathbb{R}$. Let $h : (B,\mathbb{R}) \rightarrow (K,\mathbb{R})$ and $k : (K,\mathbb{R}) \rightarrow (B,\mathbb{R})$ be two maps such that $kh$ is homotopic to $w_B$ rel $\mathbb{R}$. By the homotopy extension property for the pair $(K,\mathbb{R})$, there exists a neighborhood $U$ of $\mathbb{R}$ in $K$ and a homotopy $r_t : K \rightarrow K$ (rel $\mathbb{R}$) such that $r_0 = 1_K$ and $r_1(U) = \mathbb{R}$. If we let $r_t^*$ be the composite of the homotopy of $kh$ with $1_B$ followed by $kr_t h$.
then \( r_t : B \rightarrow B \) (rel \( \ast \)) and \( r_0' = 1_B \) while \( r_1'(U') = \ast \), where \( U' = h^{-1}(U) \). We lift the homotopy \( r_t' : E \rightarrow B \) to a homotopy \( s_t \) with \( s_0 = 1_E \) using the covering homotopy property. Let \( V = p^{-1}(U') \). Note that \( s_1 : V \rightarrow F \). Let \( V_p \) be the subset of \( M_p \) with \( (x,t) \in V_p \) whenever \( x \in V \). Define \( s : (V_p, V) \rightarrow (cF, F) \) by \( s(x,t) = (w_1(x), t) \). We then factor the inclusion of \( (cF, F) \) into \( (M_p, E) \), into the inclusions: \( (cF, F) \cong (V_p, V) \hookrightarrow (M_p, E) \).

\( i_2 \) is an excision and hence induces isomorphisms of homology groups. On the other hand, \( s_1 \) is homotopic to \( 1_{(cF, F)} \) and hence \( s_1 i_2s_1 = \text{identity} \), and further, \( i_2s_1 \) is homotopic to \( i_2 \) and hence \( i_2s_1 i_2s_2 = i_2 \). Since \( i_2 \) is an isomorphism it follows that \( i_2s_1 = \text{identity} \). Thus, \( i_1 \) and \( i_2 \) are isomorphisms and hence, so is the composite.

If \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y' \), and there are maps \( g_0 : X \rightarrow X' \) and \( g_1 : Y \rightarrow Y' \) such that \( g' \circ g_0 = g_1 \circ f \), and map is induced on mapping cylinders which in turn induces homomorphisms of the homotopy, homology and cohomology sequences.

Let \( f : X \rightarrow B \), with \( X \) and \( B \) 1-connected, \((X, \beta')\) and \((B, \omega)\) having the homotopy type of CW pairs. The Moore-Postnikov system of the map \( f \) is the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow f' & & \downarrow \gamma \\
X_{m-1} & \xrightarrow{f_{n-1}} & X_{m-1}
\end{array}
\]

With: (i) \( X_0 = B, f_0 = f \); each \((X_{m}, \omega)\) having the homotopy type of a CW pair, (ii) \( w_{i+1}(f_n) = 0 \) \( i < m \), (iii) \( p_n \) is a fiber map, with fiber a \( K(\omega, n) \).

Construction by induction, with \( n = 0 \) from (i).

Let \( \omega = w_{m+1}(f_{n-1}) \) (note that the lower homotopy groups vanish),

Let \( J : \text{Hom}(\omega, \pi) \rightarrow \text{Hom}(H_{n+1}(f_{n-1}, \omega)) \hookrightarrow H^{n+1}(f_{n-1}, \omega) \rightarrow H^{n+1}(X_{m-1}) \)
By theorem 27 of chapter 3, we can find a map $k : X_{n-1} \rightarrow K(\mathbb{W}, n+1)$ such that $k^*(\gamma(K(\mathbb{W}, n+1))) = \cup (1_\mathbb{W})$. We pick a base point $\ast$ of $K(\mathbb{W}, n+1)$ to be the image of the basepoint of $X_{n-1}$ under $k$ (the base point of $X_{n-1}$ is the image of the base point of $X$ under $f_{n-1}$). We let $\gamma_1 : \mathbb{P} \rightarrow K(\mathbb{W}, n+1)$ be the space of paths based at $\ast$, i.e., $\mathbb{P} = \zeta(0) : I \rightarrow K(\mathbb{W}, n+1)$, with $\zeta(0) = \ast \gamma$ and $\gamma_1(\zeta) = \zeta(1)$. $\gamma_1$ is a fiber map, and we define $X_n$ and $p_n$ as the induced fiber space over $X_{n-1}$. The diagram is:

$$
\begin{array}{c}
X_n \rightarrow \mathbb{P} \\
\downarrow p_n \downarrow \gamma_1 \\
X \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{k} K(\mathbb{W}, n+1)
\end{array}
$$

That $X_n$ has the homotopy type of a CW complex is a combination of two results. Milnor has proved that $\mathbb{P}$ has the homotopy type of a CW complex (1; thm. 3) and from this, a result of Stasheff (2; prop. 12) implies that $X_n$ has, too. (See notes for the sharpening of Stasheff's result.)

The fiber of $p_n$ is the same as the fiber of $\gamma_1$ which is $\cup K(\mathbb{W}, n+1)$ and is thus a $K(\mathbb{W}, n)$ (note: same $\mathbb{W}$).

Thus, our construction will be complete when we define $f_n$, a lifting of $f_{n-1}$ to $X_n$ so that $w_{i+1}(f_n) = 0$ for $i \leq n$.

First, we note that liftings of $f_{n-1}$ do exist, for $f_{n-1}^*k^* \gamma(K(\mathbb{W}, n+1)) = f_{n-1}^* \gamma(1_\mathbb{W}) = 0$, and hence $k f_{n-1}$ is homotopic to the constant map, by theorem 27 of chapter 3. Since the constant map can be lifted to $\mathbb{P}$, the covering homotopy property implies that $k f_{n-1}$ can be lifted to $\mathbb{P}$, and hence, using the pullback definition of $X_n$, $f_{n-1}$ can be lifted to $X_n$. Let $f' : X \rightarrow X_n$ be such a lifting. We consider the homotopy groups $w_i(f')$. 
We will, ambiguously, let \( p_n \) also represent the map \( M' \rightarrow M' \) induced by \( p_n \) and \( 1_X \cdot p_n \), then induces a map of the homotopy sequence of \( f' \) into that of \( f'_{n-1} \):

\[
\begin{array}{ccc}
w_{i+1}(X) & \xrightarrow{\text{id}} & w_{i+1}(X) \\
\downarrow & & \downarrow \\
w_{i+1}(F) & \xrightarrow{w_{i+1}(f')} & w_{i+1}(X_{n-1}) \\
\downarrow & & \downarrow \\
w_i(X) & \xrightarrow{\text{id}} & w_i(X) \\
\downarrow & & \downarrow \\
w_i(F) & \xrightarrow{w_i(f')} & w_i(X_{n-1}) \\
\end{array}
\]

The columns are the homotopy sequences of the two maps and the rows are the homotopy sequence of the fiber space where \( F \), a \( K(w_2, n) \), is the fiber of \( p_n \).

Thus, for \( i \leq n-2 \), the two columns are isomorphic by the five-lemma, and hence \( w_{i+1}(f') = 0 \).

We note that for \( i \leq n-1 \), \( w_{i+1}(f'_{n-1}) = 0 \) and \( w_i(F) = 0 \), and hence \( w_{i+1}(p_n) = 0 \), and thus \( f'_{n-1} : w_{i+1}(f'_{n-1}) \rightarrow w_{i+1}(p_n) \) is trivially an isomorphism for \( i \leq n-1 \). We show that with the additional assumption that \( f'_{*} : w_{n+1}(f'_{n-1}) \rightarrow w_{n+1}(p_n) \) is an isomorphism, we can prove that \( w_n(f') = 0 \) and \( w_{n+1}(f') = 0 \), and thus we can let \( f_n = f' \).

\( i = n-1 \): In the bottom square of the above diagram the top, bottom and right-hand maps are isomorphisms and hence the left-hand map, \( f'_{*} : w_{n-1}(X) \rightarrow w_{n-1}(X_n) \) is an isomorphism. Furthermore, if we consider \( f' \) mapping the sequence of the map \( f_{n-1} \) into that of the map \( p_n \), our assumption gives us, by the 5-lemma, that \( f'_{*} : w_{n}(X) \rightarrow w_{n}(X_n) \) is an isomorphism and hence, by exactness, \( w_{n}(f') = 0 \).

\( i = n \): It suffices, by exactness to show that the map
$f'_*: w_{n+1}(X) \rightarrow w_{n+1}(\mathbb{X}_m)$ is epi, which is an easy diagram chase of the diagram; (it is actually a special case of the weak 4-lemma):

\[
\begin{array}{c}
w_{n+2}(f_{n-1}) \downarrow f'_* \downarrow \\
w_{n+1}(X) \downarrow f'_* \downarrow \\
w_{n+1}(X_{m-1}) \downarrow \text{id} \downarrow \\
w_{n+1}(f_{n-1}) \downarrow f'_* \downarrow \\
\end{array}
\]

So the problem is reduced to finding a particular lifting $f_*$ so that $f_{n*}: w_{n+1}(f_{n-1}) \rightarrow w_{n+1}(p_n)$ is an isomorphism. Our method will be as follows: first, we will find a likely looking isomorphism of the two groups; we will then find a mapping of $X$ into the fiber $F$ which will "represent" the difference between $f'_*$ and this isomorphism; and finally, we will "twist" $f'$ by this map, using the principle fiber space structure induced by $P$, and $\gamma_1$.

Consider the map: $w_{n+1}(p_n) \xrightarrow{k_n} w_{n+1}(\gamma_1) \xleftarrow{\cdots} w_{n+1}(K(w,n+1))$.

The second map is an isomorphism since $P$ is contractible and the first map factors into $w_{n+1}(p_n) \xleftarrow{\cdots} w_{n+1}(dF,F) \rightarrow w_{n+1}(\gamma_1)$ each map of which is an isomorphism by lemma 5. Call this map $\omega: w_{n+1}(p_n) \rightarrow w_{n+1}(f_{n-1})$. Now consider the diagram:

\[
\begin{array}{c}
\text{Hom}(w_{n+1}(p_n),w) \xrightarrow{R_k} \text{Hom}(H_{n+1},w) \xleftarrow{\cdots} H^{n+1}(p_n;w) \rightarrow H^{n+1}(X_{m-1};w) \\
\text{Hom}(w_{n+1}(\gamma_1),w) \xrightarrow{R_k} \text{Hom}(H_{n+1},w) \xleftarrow{\cdots} H^{n+1}(\gamma_1;w) \\
\text{Hom}(w_{n+1}(K(w,n+1),w) \xrightarrow{R_k} \text{Hom}(H_{n+1},w) \xleftarrow{\cdots} H^{n+1}(K(w,n+1);w) \\
\end{array}
\]

We will let $\overline{\omega}$ be the element of $H^{n+1}(p_n;w)$ which $\omega$ maps into by the above succession of Hom of the Hurewicz map and the universal coefficient map. Similar notation for other homomorphisms will be used, thus, for example, $\varGamma_* = \kappa(K(w,n+1))$. 

This implies that $\overline{\omega} \rightarrow \nu(1_w)$, by commutativity and the definition of $k$. We can consider the element of the cohomology group:

$$\Gamma_w - f^* \overline{\omega} \left( = l_w - \omega f^*_* \right) \in H^{n+1}(f_{n-1}; w),$$

and we assert that it can be pulled back to $H^n(X; w)$. From the diagram:

$$\begin{array}{ccl}
H^{n+1}(X; w) & \xrightarrow{\nu(1_w)} & H^{n+1}(f_{n-1}; w) \\
6 & \xleftarrow{f^*} & H^{n+1}(p_n; w) \\
H^n(X; w) & \xrightarrow{\overline{\omega}} & H^n(X; w)
\end{array}$$

we note that $\Gamma_w - f^* \overline{\omega}$ goes into $\nu(1_w) - \nu(1_w)$ in $H^{n+1}(X_{n-1}; w)$ and this $= 0$ and so the element in question pulls back to some member of $H^n(X; w)$, which, by theorem 27 of chapter 3, is represented by some continuous function $g : X \rightarrow F$, since the fiber is a $K(p_m, n)$. For the definition of $\prec(K(p_m, n))$ we use the following map as an identification of $w_n(K(p_m, n))$ with $w_{n+1}(f_{n-1})$:

$$w_n^* \xrightarrow{\beta} w_{n+1}(cF, F) \rightarrow w_{n+1}(p_n) \xrightarrow{\overline{\omega}} w_n,$$

each map of which is an isomorphism. Applying theorem 26 of chapter 3, again, we note that $\beta \circ \overline{\omega} = \overline{\omega} \circ \beta = (l_w - \omega f^*_*)$ and hence $\overline{\omega} = l_w - \omega f^*_*$.

By lemma 2, $p_m : X_n \rightarrow X_{n-1}$ is a principal fiber space, with composition $\gamma : FxX_n \rightarrow X_n$. So from the lifting $f'$ of $f_{n-1}$ and the map $g$, we can form a new lifting $\gamma(gxf')$, which we call $f'_n$. The crucial lemma that we require is:

6. **Lemma**: Let $i : F(\pi X_n)$, then $f^*_n \circ w_{n+1}(f_{n-1}) \rightarrow w(p_n)$ is the sum $f^*_n + (ig)_n$.

**Proof**: This lemma is just a relativization of lemma 3.

The proof follows a similar diagram. If $\rho : (s^{n+1}, s^n) \rightarrow (M_{f_{n-1}})$ represents an element of $w_{n+1}(f_{n-1})$, then we have the diagram:
\[ e_{n+1} \xrightarrow{\Delta} e_{n+1} \bigcup e_{n+1} \xrightarrow{\beta} (M_{p_nq_2}^{F_nX_n}) \xrightarrow{\gamma} (M_{p_n, \iota}^{X_n}) \]

Where \( q_2 : F_nX_n \to X_n \) is the projection, \( \iota \) is the constant map

\[ F \to \ast (= p_n \sqcup F) \]

and \( \beta \) is the folding map of \( A \sqcup A \to A \) for any \( A \).

In this diagram, which commutes up to homotopy, the top map represents \( \gamma(gxf')\Delta \iota \), and the bottom map represents \( f' \iota + \iota \gamma \).

Armed with this lemma, we can show that \( f_{n \ast} \) is an isomorphism as required. We first consider just what \( (ig)_{n \ast} \) is.

It factors as shown:

\[ w_{n+1}(f_{n-1}) \xrightarrow{\iota_\ast} w_{n+1}(cF, F) \xrightarrow{\iota_\ast} w_{n+1}(p_n) \]

\[ \partial \downarrow \quad \partial \]

\[ w_n(X) \xrightarrow{\iota_\ast} w_n(F) \]

\[ f_{n \ast} = f'_{\ast} + (ig)_{n \ast} \text{ by the lemma. Hence, } \omega f_{n \ast} = \omega f'_{\ast} + \omega (ig)_{n \ast}, \]

but the above diagram shows that \( \omega (ig)_{n \ast} \)

is just \( \iota_\ast \partial \) followed by the map identifying \( w_n(F) \) with \( w_{n+1}(f_{n-1}) \).

This, in turn, we know to be equal to \( l_w - \omega f'_{\ast} \). It follows that \( \omega f_{n \ast} = l_w \) and hence \( f_{n \ast} \) is the isomorphism \( \omega^{-1} \).

The construction is thus completed.

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Bibliography

The following books have been referred to in the text by the name of the author(s).


