Chapter II

Monoid Complexes and Production of Semi-Simplicial Complexes.

In this chapter we shall consider special properties of Kan complexes which have a multiplicative structure, and shall then begin the consideration of the problem of constructing new semi-simplicial complexes from such a complex.

Definition 1.1: A semi-simplicial complex Γ is a monoid complex if

1) Γ_q is monoid with identity for $q \in Z^+$,

2) $\partial_i : \Gamma_{q+1} \longrightarrow \Gamma_q$, and $s_i : \Gamma_q \longrightarrow \Gamma_{q+1}$

are homomorphisms which send identity elements into identity elements.

We will denote by e_q the identity of Γ_q.

Γ is a group complex if Γ is a monoid complex and each Γ_q is a group. When each Γ_q is abelian, Γ will be called an abelian monoid complex, or an abelian group complex, as the case may be. If $x \in \Gamma_q$, the inverse of x will be denoted by $\overline{x}$.

Example 1: Let G be a topological group, and let Γ be the total singular complex of G. If $u,v : \Delta_q \longrightarrow G$ are singular q-simplexes, define $(u.v) : \Delta_q \longrightarrow G$ by $(u.v)(t_0,\dots,t_q) = u(t_0,\dots,t_q)\, v(t_0,\dots,t_q)$. It is easily verified that Γ is a group complex, and that Γ is abelian if and only if G is abelian.

Example 2: Let X be a topological space. A path in X is a pair (f,r) where r is a non-negative real number, and f : $[0,r] \longrightarrow X$ is a map ($[0,r]$ denotes the closed interval from 0 to r). A loop is a path (f,r) such that $f(0) = f(r)$. Topologize the set of all paths in X by using as a subbasis for the topology the sets W(C,V,U) defined as follows:

1) C is a compact subset of [0,1]
2) V is an open subset of R^+ (the non-negative real number),
3) U is an open subset of X
4) $W(C,V,U) = \{(f,r) \mid (f,r)$ is a path in X, $r \in V,\ f(rC) \subset U\}$.

Now let $x \in X$, and let E(X,x) be the space of paths in X which begin at x. Define $p : E(X,x) \longrightarrow X$ by $p(f,r) = f(r)$; Then (E(X,x),p,X) is a fibre space in the sense of Serre [1]. i.e. the covering homotopy theorem holds for finite complexes. The proof is the same as that of Serre, in which normalized paths $f : [0,1] \longrightarrow X$ are used. Further the space E(X,x) is contractible, and has as fibre $\Omega(X,x)$, the space of loops in X based at x. Define $(f,r)(g,s) = (h,r+s)$ where

$$h(t) = \begin{cases} f(t) & 0 \leq t \leq r \\ g(t-r) & r \leq t \leq r+s \end{cases} \quad \text{if } (f,r),(g,s) \in \Omega(X,x).$$

It is easily verified that $\Omega(X,x)$ is a monoid with identity, and that if Γ is the total singular complex of $\Omega(X,x)$, then Γ is a monoid complex when multiplication is defined as in the preceding examples by point-wise multiplication of q-simplexes.

Theorem 2.2: If Γ is a group complex, then Γ is a Kan complex.

Proof: To prove the proposition it suffices to show that Γ satisfies the extension condition. Suppose therefore that $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1} \in \Gamma_q$, and $\partial_i x_j = \partial_{j-1} x_j$ for $i < j$, $i, j \neq k$. We must find an $x \in \Gamma_{q+1}$ such that $\partial_i x = x_i$ for $i \neq k$.

We first show that there exists $u \in \Gamma_{q+1}$ such that $\partial_i u = x_i$ for $i < k$. This is trivial if $k = 0$; if $k > 0$ we define $u^r \in \Gamma_{q+1}$ by induction on r such that $\partial_i u^r = x_i$ for $i \leq r$. First let $u^0 = s_0 x_0$; then $\partial_0 u^0 = x_0$. Now if $r < k-1$, set $y^r = s_{r+1}((\partial_{r+1} \bar{u}^r) x_{r+1})$, $u^{r+1} = u^r y^r$. Now by an easy calculation it follows that $\partial_i y^r = e_q$ for $i \leq r$, and $\partial_{r+1} y^r = (\partial_{r+1} \bar{u}^r) x_{r+1}$, using the fact that $\partial_i u^r = x_i$ for $i \leq r$. Therefore we deduce that $\partial_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally let $u = u^{k-1}$, and we have $\partial_i u = x_i$ for $i < k$.

Now we shall show by induction on r that there exists an element $x^r \in \Gamma_{q+1}$ such that $\partial_i x^r = x_i$ for $i < k$ and for $i > q-r+1$. For $r = 0$ let $X^0 = u$. Suppose x^r is defined and $r \leq q - k$. Let $z^r = s_{q-r} ((\partial_{q-r+1} \bar{x}^r) x_{q-r+1})$, $x^{r+1} = x^r z^r$. A simple calculation shows that $\partial_i z^r = e_q$ if $i < k$ and $i > q-r+1$ and $\partial_{q-r+1} z^r = (\partial_{q-r+1} \bar{x}^r) x_{q-r+1}$. It follows that $\partial_i x^{r+1} = x_i$ for $i < k$ and for $i > q-r$. Finally if we take for x the element x^{q-k+1}, we have $\partial_i x = x_i$ for $i \neq k$. Thus the proof of the theorem is complete.

<u>Definition 2.3</u>: The monoid complex Γ is a <u>monoid complex with homotopy</u> if it is a Kan complex.

We shall denote $\pi_q(\Gamma, e_0)$ by $\pi_q(\Gamma)$.

<u>Proposition 2.4</u>: If Γ is a monoid complex with homotopy and $x, y \in \Gamma_q$ are elements such that $\partial_i x = \partial_i y = e_{q-1}$ for $i = 0, .., q$, then $[x], [y] \in \pi_q(\Gamma)$, and $[x][y] = [xy]$.

<u>Proof</u>: Consider the element $z = s_q x\, s_{q-1} y$ Now $\partial_i z = e_{q-1}$ for $i < q-1$, $\partial_{q-1} z = y$, $\partial_q z = xy$, and $\partial_{q+1} z = x$. In view of the definition of addition in the homotopy groups, the result is proved.

<u>Proposition 2.5</u>: If Γ is a monoid complex with homotopy, then $\pi_1(\Gamma)$ is abelian.

<u>Proof</u>: Let $x, y \in \Gamma_1$ be such that $\partial_i x = \partial_i y = e_0$, $i = 0, 1$. Let $w = s_0 y s_1 x$. Then $\partial_0 w = y$, $\partial_1 w = yx$, $\partial_2 w = x$. Therefore $[x][y] = [yx]$; but $[yx] = [y][x]$ by the preceding proposition, and the proof is complete.

The two preceding propositions are the analogues of the classical theorems that the group operations in the homotopy groups of a topological group come~~s~~ from the group operation in group, and that the fundamental group of a topological group is abelian (cf. e.g. [2]).

If Γ is a group complex, we wish to define the homotopy group of Γ in an alternative fashion.

Definition 2:6 If Γ is a group complex, define

$$\widetilde{\pi}_q(\Gamma) = \bigcap_{j=0}^{q-1} \text{kernel } \partial_i = \Gamma_q \longrightarrow \Gamma_{q-1}, \text{ and}$$

$$\widetilde{\pi}(\Gamma) = \Sigma_q \widetilde{\pi}_q(\Gamma).$$

Proposition 2:7 If Γ is a group complex, then

1) $\partial_{q+1}(\widetilde{\pi}_{q+1}(\Gamma)) \subset \widetilde{\pi}_q(\Gamma)$

2) $\partial_{q+1}(\widetilde{\pi}_{q+1}(\Gamma))$ is a normal subgroup of Γ_q,

3) image $\partial_{q+1} : \widetilde{\pi}_{q+1}(\Gamma) \longrightarrow \widetilde{\pi}_q(\Gamma)$ is contained in kernel $\partial_q : \widetilde{\pi}_q(\Gamma) \longrightarrow \widetilde{\pi}_{q-1}(\Gamma)$ for $q > 0$.

Proof: Let $x \epsilon \widetilde{\pi}_{q+1}(\Gamma)$. Now $\partial_i \partial_{q+1} x = \partial_q \partial_i x = e_{q-1}$ for $i \leq q$, and this implies 1) and 3).

Suppose $z \epsilon \Gamma_q$; consider $s_q z\, x\, s_q \overline{z}$.
Since $\partial_i(s_q z\, x\, s_q \overline{z}) = \partial_i(s_q z\, s_q \overline{z}) = e_q$ for $i \leq q$,
Therefore $s_q z\, x\, s_q \overline{z} \epsilon \widetilde{\pi}_{q+1}(\Gamma)$.
Since $\partial_{q+1}(s_q z\, x\, s_q \overline{z}) = z \partial_{q+1} x \overline{z}$, 2) follows.

The preceding proposition implies that $\widetilde{\pi}(\Gamma)$ is a chain complex (not necessarily abelian) with respect to the last face operator.

Definition 2.8: If Γ is a group complex, define

$$\pi_q{}'(\Gamma) = H_q(\widetilde{\pi}(\Gamma)).$$

<u>Proposition 2.9</u>: If Γ is a group complex,

$$\pi_q(\Gamma) = \pi_q'(\Gamma).$$

<u>Proof</u>: An element of $\pi_q(\Gamma)$ is represented by $x \in \Gamma_q$ such that $\partial_i x = e_{q-1}$ for $i = 0, \ldots, q$. However, such an element x also represents an element of $\pi_q'(\Gamma)$. Suppose $[x] = [y] \in \pi_q(\Gamma)$. Then there exists $z \in \Gamma_{q+1}$ such that $\partial_i z = e_q$ for $i < q$, $\partial_q z = x$, $\partial_{q+1} z = y$. Now $s_q\bar{x} \cdot z \in \tilde{\pi}_{q+1}(\Gamma)$, and $\partial_{q+1}(s_q\bar{x} \cdot z) = \bar{x}y$. Therefore $[x] = [y] \in \pi_q'(\Gamma)$, and there is a natural map of $\pi_q(\Gamma)$ into $\pi_q'(\Gamma)$. Further it is evident that this map is onto, and it is a homomorphism by proposition 2.4: Suppose now that $[x] = 0 \in \pi_q'(\Gamma)$. Then there exists $z \in \tilde{\pi}_{q+1}(\Gamma)$ such that $\partial_i z = e_q$ $i \leq q$ and $\partial_{q+1} z = x$. This means that $[x] = 0 \in \pi_q(\Gamma)$, and the proof is complete.

<u>Proposition 2.10</u>: A group complex Γ is minimal if and only if $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero for all q.

<u>Proof</u>: Suppose that Γ is minimal; then if $x, y \in \Gamma_{q+1}$, and $\partial_i x = \partial_i y$ for $i = 0, \ldots, q$, it follows that $\partial_{q+1} x = \partial_{q+1} y$. Now if $x \in \tilde{\pi}_{q+1}(\Gamma)$, then $\partial_i x = e_q = \partial_i e_{q+1}$ for $i \leq q$; hence, since Γ is minimal, $\partial_{q+1} x = \partial_{q+1} e_{q+1} = e_q$, and $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero.

Suppose now that $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \longrightarrow \tilde{\pi}_q(\Gamma)$ is zero for all q, and that $x, y \in \Gamma_{q+1}$ are elements such that $\partial_i x = \partial_i y$ for $i \neq k$. Then $\partial_i x \bar{y} = e_q$ for $i \neq k$. If $k = q+1$, let $z = x\bar{y}$; if $k = q$, let $z = (s_q \partial_q x \bar{y})(y \bar{x})$; while if $k < q$, let $z = (s_q \partial_k x \bar{y})(s_{q-1} \partial_k y \bar{x})$. Then $\partial_i z = e_q$ for $i \neq q+1$, and $\partial_{q+1} z = \partial_k x \bar{y}$. But $z \in \pi_{q+1}(\Gamma)$; therefore by hypothesis $\partial_{q+1} z = e_q$, so that $\partial_k x = \partial_k y$, and the proof is complete.

In order to define the explicit complexes $K(\pi, n)$ of Eilenberg-MacLane ([3],[4],[5]) it is convenient to recall the definition of the standard alternating cochain complex for the q-simplex Δ_q with coefficients in the abelian group π. The n-dimensional cochain group $C^n(\Delta_q; \pi)$ is the group of functions u defined on (n+1)-tuples $(m_0, \ldots, m_n)$ of integers such that $0 \leq m_0 \leq \ldots \leq m_i \leq m_{i+1} \leq \ldots \leq m_n \leq q$ with values in π, such that $u(m_0, \ldots, m_n) = 0$ if $m_i = m_{i+1}$ for some $i \leq n$. $\delta : C^n(\Delta_q; \pi) \longrightarrow C^{n+1}(\Delta_q; \pi)$ is defined by

$$\delta u(m_0, \ldots, m_{n+1}) = \sum_{j=0}^{n+1} (-1)^j u(m_0, \ldots, m_{j-1}, m_{j+1}, \ldots, m_n)_{+1}$$

Then $Z^n(\Delta_q; \pi)$ (the group of n-cocycles with coefficients in π) is the kernel of

$$\delta : C^n(\Delta_q; \pi) \longrightarrow C^{n+1}(\Delta_q; \pi).$$

<u>Notation</u>: Let $\lambda^i : \{0, \ldots, q\} \longrightarrow \{0, \ldots, q+1\}$ be defined by $\lambda^i(j) = j$ for $j < i$, and $\lambda^i(j) = j+1$ for $j \geq i$. Further, let $\eta^i : \{0, \ldots, q+1\} \longrightarrow \{0, \ldots, q\}$

be defined by $\eta^i(j) = j$ for $j \leq i$, $\eta^i(j) = j-1$ for $j > i$.

Definition 2.11: If π is an abelian group, define $K_q(\pi,n) = Z^n(\Delta_q;\pi)$. Further, define $\partial_i : K_{q+1}(\pi,n) \longrightarrow K_q(\pi,n)$ by $\partial_i u(m_0,\dots,m_n) = u(\lambda^i(m_0),\dots,\lambda^i(m_n))$, and $s_i : K_q(\pi,n) \longrightarrow K_{q+1}(\pi,n)$ by $s_i u(m_0,\dots,m_n) = u(\eta^i(m_0),\dots,\eta^i(m_n))$

Let $K(\pi,n) = \bigcup K_q(\pi,n)$

Theorem 2.12: If π is an abelian group, then

1) $K(\pi,n)$ is an abelian group complex,

2) $\pi_q(K(\pi,n)) = 0$ for $q \neq n$,

3) $\pi_n(K(\pi,n)) = \pi$.

4) $K(\pi,n)$ is minimal

Proof: The verification of u is routine, so that only 2),3) and 4) will be verified. First notice that $K_q(\pi,n) = 0$ for $q < n$. Therefore, $\tilde{\pi}_q(K(\pi,n)) = 0$ for $q < n$. Further since $Z^n(\Delta_n,\pi) = \pi$, we have that $\tilde{\pi}_n(K(\pi,n)) = K_n(\pi,n) = \pi$. Suppose now $u \in \tilde{\pi}_q(K(\pi,n))$ and $q > n$. Then $\partial_0 u = 0$; i.e. $u(m_{0+1},\dots,m_{n+1}) = 0$ whenever $(m_0,\dots,m_n)$ is a sequence of integers such that $0 \leq m_0 \leq \dots \leq m_n \leq q-1$. This means $u(m_0,\dots,m_n) = 0$ unless $m_0 = 0$. Therefore we only need consider sequences $(0,m_1,\dots,m_n)$. However, $\partial_i u = 0$, or in other words

$u(0,m_1+1,\ldots,m_n+1) = 0$, but this implies that $u(0,m_1,\ldots,m_n) = 0$ unless $m_1 = 1$. Continuing in this fasion we see that $u(m_0,\ldots,m_n) = 0$ unless $m_i = 1$, for $i = 0,\ldots,n$. Then since u is a cocycle, $\delta u(0,\ldots,n+1) = \Sigma_{j=0}^{n+1}(-1)^j u(0,\ldots,j-1,j+1,\ldots,n+1) = 0$; thus $u = 0$, and $\tilde{\pi}_q(\Gamma) = 0$ for $q \neq n$. This implies 2) and 3). Statement 4) follows from Proposition 2.10, and the proof is complete.

<u>Definition 2.13</u>: <u>A twisted Cartesian product</u> is a triple (F,B,E) such that

1) F,B, and E are semi-simplicial complexes,

2) $E_q = \{(a,b) | a \in F_q, b \in B_q\}$, $q \geq 0$,

3) if $(a,b) \in E_{q+1}$, $\partial_i(a,b) = (\partial_i a, \partial_i b)$ for $i > 0$,

4) if $(a,b) \in E_q$, $s_i(a,b) = (s_i a, s_i b)$, and

5) if $p : E \longrightarrow B$ is the map defined by $p(a,b) = b$, then $p\partial_0 = \partial_0 p$.

F is called the fibre of the twisted Cartesian product, B the base, and E the total complex. Usually, but not always, the map p will be a fibre map.

E is the <u>Cartesian product</u> [6] of F and B if (F,B,E) is a twisted Cartesian product and $\partial_0(a,b) = (\partial_0 a, \partial_0 b)$ for $(a,b) \in E_{q+1}$, all q. In this case E is denoted by FxB. Also, the elements of E in any twisted Cartesian product will sometimes be written a x b.

If Γ is a monoid complex, and if (Γ,B,E) is a twisted Cartesian product, then Γ acts on the left of E according to the rule $a'.(a,b) = (a'a,b)$ for $a,a' \epsilon \Gamma_q, b \epsilon B_q$. The twisted Cartesian product is said to be compatible with the left action of Γ if $\partial_0(a,b) = \partial_0 a.\partial_0(e_{q+1},b)$ for $(a,b) \epsilon E_{q+1}$. It will invariably be assumed that if a twisted Cartesian product has for fibre a monoid complex Γ, then the structure is compatible with the left action of Γ.

Example 1: Let A,B be topological spaces, $S(A),S(B)$ the total singular complexes of A and B respectively. Let $A \times B$ be the Cartesian product of A and B as topological spaces, and let $p_1:A \times B \longrightarrow A, p_2:A \times B \longrightarrow B$ be the projections. Then p_1 induces a semi-simplicial map which we shall still denote $p_1:S(A \times B) \longrightarrow S(A)$, and p_2 induces $p_2:S(A \times B) \longrightarrow S(B)$. It is easy to verify that the map $p':S(A \times B) \longrightarrow S(A) \times S(B)$ defined by $p'(y) = (p_1(y),p_2(y))$ is an isomorphism of semi-simplicial complexes.

Example 2: Let E be the total space of a principal fibre bundle with fibre a topological group G and base space B. Assume that G acts on the left of E. Denote the total singular complexes of E,B, and G by $S(E),S(B)$, and $S(G)$ respectively. Since G acts on the left of E, $S(G)$ acts on the left of $S(E)$. Let $\phi:S(B) \longrightarrow S(E)$ be a pseudo-cross section, i.e. $\phi(\partial_i) = \partial_i\phi$ for $i > 0$, and $\phi s_i = s_i\phi$. Define $\psi:S(G) \times S(B) \longrightarrow S(E)$ by $\psi(a,b) = a \cdot \phi(b)$ for $a \epsilon S(G)_q, b \epsilon S(B)_q$. Now ψ is a 1:1 correspondance, is compatible with ∂_i for $i > 0$, and with s_i for all i. Consequently if $S(E)$ is identified with $S(G) x S(B)$ as a set by means of ψ we see that $(S(G),S(B),S(E))$ is a twisted Cartesian product. In other words, to make the total singular complex of a principal fibre bundle into the total complex of a twisted Cartesian product it suffices to choose a pseudo-cross section, and this can be done for any fibre map.

Definition 2.14: If Γ is a monoid complex, a twisted Cartesian product (Γ,B,E) is said to satisfy the condition (W) if

1) B_0 has one element, and

2) the map ϕ of B_{q+1} into E_q defined by $\phi(b) = \partial_0(e_{q+1},b)$ is a 1:1 correspondance.

Theorem 2.15: If

1) Γ, Γ' are monoid complexes,

2) $f: \Gamma \longrightarrow \Gamma'$ is a map of monoid complexes,

3) (Γ,B,E) and (Γ',B',E') are twisted Cartesian products, the latter satisfying the condition (W), then there is a unique map $g: E \longrightarrow E'$ such that

4) $g(e_q x B_q) \subset e_q x B'_q$, and

5) $g(a,b) = f(a).g(e_q,b)$ for $(a,b) \in E_q$.

Proof: Suppose that we have such a map g. Denote by g_q the induced map of E_q into E'_q. Then

$g_0(e_0,b) \in e_0 x B'_0$; but B'_0 has one element, so that g_0 is uniquely determined. Let $S: E'_q \longrightarrow e_{q+1} x B'_{q+1}$ denote the inverse of ∂_0. Since $g: e_q x B_q \longrightarrow e_q x B'_q$, we have $g_{q+1}(e_{q+1},b) = S\partial_0 g_{q+1}(e_{q+1},b) = Sg_q\partial_0(e_{q+1},b)$. Consequently there is at most one such map g; but the above formulas have defined a function g such that $\partial_0 g = g\partial_0$ and $g(e_q x B_q) \subset e_q x B'_q$. It remains to verify that $\partial_{i+1} g = g\partial_{i+1}$ and that $s_i g = g s_i$.

If $b \in B_1$, we observe that $\partial_1 g(e_1,b) = (e_0,b')$, where b' is the unique element of B'_0. Further $g\partial_1(e_1,b) = g(e_0,\partial_1 b) = (e_0,b')$. Suppose now that $\partial_i g = g\partial_i$ for $i \leq j$. Then for $b \in B_{q+2}$, $\partial_{j+1} g(e_{q+2},b) = \partial_{j+1} Sg\partial_0(e,b) = S\partial_j g\partial_0(e,b) = Sg\partial_j\partial_0(e,b) = S\partial_0 g\partial_{j+1}(e,b) = g\partial_{j+1}(e,b)$.

Now $S: e_q x B'_q \longrightarrow e_{q+1} x B'_{q+1}$ is 1:1 into; but since $\partial_0 s_0 =$ identity, S is equal to s_0. Therefore $s_0 g(e,b) = s_0 Sg\partial_0(e,b) = SSg\partial_0(e,b) = Sg(e,b) = Sg\partial_0 s_0(e,b) = gs_0(e,b)$.

Finally, $s_{i+1} Sg\partial_0(e,b) = Ss_i g\partial_0(e,b) = Sgs_i\partial_0(e,b)$ by inductive hypothesis, and $Sgs_i\partial_0(e,b) = Sg\partial_0 s_{i+1}(e,b) = gs_{i+1}(e,b)$.

This completes the proof.

<u>Corollary 2.16:</u> If (Γ,B,E) and (Γ,A,D) are twisted Cartesian products satisfying the condition (W), and $g: E \longrightarrow D$,

$g': D \longrightarrow E$ are the maps of the preceding theorem, induced by the identity map of Γ, then $g'g$ and gg' are the identity maps of E and D.

We have now shown the essential uniqueness of twisted Cartesian products satisfying the condition (W), but it remains to prove existence. This will be done after the manner of MacLane [7].

Definition 2.17: Let Γ be a monoid complex.
Let $W_0(\Gamma) = \Gamma_0$, $W_{q+1}(\Gamma) = \Gamma_{q+1} + W_q(\Gamma)$, $\overline{W}_0(\Gamma)$ a set consisting of one element, and $\overline{W}_{q+1}(\Gamma) = \Gamma_q + \overline{W}_q(\Gamma)$.

Now in $W(\Gamma) = \bigcup_q W_q(\Gamma)$ define

1) $\partial_0(a,b) = \partial_0 a.b$, $\partial_1(a,b) = a$, where $a \in \Gamma_1, b \in \Gamma_0$;
2) $\partial_0(a,b) = \partial_0 a.b$ where $a \in \Gamma_{q+1}, b \in W_q(\Gamma)$, for $q > 0$;
3) $\partial_{i+1}(a,b) = (\partial_{i+1} a, \partial_i b)$;
4) $s_0(a,b) = (s_0 a, e_{q+1}, b)$, noting that $W_{q+2}(\Gamma) = \Gamma_{q+2} + \Gamma_{q+1} + W_q(\Gamma)$;
5) $s_{i+1}(a,b) = (s_{i+1} a, s_i b)$.

Theorem 2.18: If Γ is a monoid complex, then $(\Gamma, \overline{W}(\Gamma), W(\Gamma))$ is a twisted Cartesian product satisfying the condition (W).

The proof of this theorem is straightforward, and is left to the reader.

We remark that the notation here is somewhat different from that of [5], in that we consider only semi-simplicial complexes and not FD complexes, and that $\overline{W}$ corresponds to the W of [5].

If X is a Kan complex, and x is a point of X, it was shown in chapter 1 that there is a fibre space $(E(X,x),p,X)$ with fibre $\Omega(X,x)$ such that $\partial^{\#}: \pi_q(X,x) \longrightarrow \pi_{q-1}(\Omega(X,x),s_0(x))$ is an isomorphism for $q > 0$. If Γ is a monoid complex with homotopy, we shall always choose the base point to be $e_0 \in \Gamma_0$, and we shall denote $E(\Gamma,e_0)$ by $E(\Gamma)$, and $\Omega(\Gamma,e_0)$ by $\Omega(\Gamma)$.

Suppose now that Γ is a group complex such that $\tilde{\pi}_0(\Gamma) = \Gamma_0 = 0$. Then $E_q(\Gamma) = \Gamma_{q+1}$, and $0 \longrightarrow \Omega(\Gamma)_q \xrightarrow{i} E_q(\Gamma) \xrightarrow{p} \Gamma_q \longrightarrow 0$ is exact; but the homomorphism $s_0:\Gamma_q \longrightarrow \Gamma_{q+1}$ induces a homomorphism $u:\Gamma_q \longrightarrow E_q(\Gamma)$ such that pu is the identity. Therefore $E_q(\Gamma)$ is a split extension of Γ_q by $\Omega(\Gamma)_q$. this means that we may identify the set $E(\Gamma)$ with the set $\Omega(\Gamma) \times \Gamma$, the identification being compatible with the degeneracy operators s_i, and also with the face operators $\partial_i, i > 0$. Consequently we have the following

Theorem 2.19: If Γ is a group complex such that $\tilde{\pi}_0(\Gamma) = 0$, then $(\Omega(\Gamma),\Gamma,E(\Gamma))$ is a twisted Cartesian product satisfying the condition (W).

Proof: We need only verify that the twisted Cartesian product satisfies the condition (W). We have, however, that $\Gamma_0 = \tilde{\pi}_0(\Gamma)$ has one element. Further if $S:E_q(\Gamma) \longrightarrow E_{q+1}(\Gamma)$ is the homomorphism induced by $s_0:\Gamma_{q+1} \longrightarrow \Gamma_{q+2}$, then $\partial_0 S$ is the identity; but the image of S is just the subgroup identified with $e_{q+1} \times \Gamma_{q+1}$, and the result is proved.

By the preceding theorem we have, therefore, that Γ is in a natural 1:1 correspondence with $\overline{W}(\Omega(\Gamma))$. However Γ is a group complex, and therefore in general has more structure than $\overline{W}(\Omega(\Gamma))$.

Suppose now that Γ is a commutative monoid complex. Then the multiplication map of $\Gamma \times \Gamma \longrightarrow \Gamma$ is a map of monoid complexes. This induces by the preceding theorem a map $W(\Gamma \times \Gamma) \longrightarrow W(\Gamma)$. However $W(\Gamma \times \Gamma)$ may be identified in a natural manner with $W(\Gamma) \times W(\Gamma)$. Now $W_q(\Gamma) = \Gamma_q + \cdots + \Gamma_0$, and the map $W(\Gamma) \times W(\Gamma) \longrightarrow W(\Gamma)$ is given by $(x_q, \ldots, x_0) \times (y_q, \ldots, y_0) \longrightarrow (x_q y_q, \ldots, x_0 y_0)$. Thus $W(\Gamma)$ is a commutative monoid complex. Further, $\overline{W}(\Gamma)$ is also a commutative monoid complex, and as a monoid, $W_q(\Gamma) = \Gamma_q + \overline{W}_q(\Gamma)$. Therefore if Γ is a commutative monoid complex, we shall always mean by $W(\Gamma)$ and $\overline{W}(\Gamma)$ the commutative monoid complexes whose structure has just been described. Notice that if Γ is an abelian group complex, then $W(\Gamma)$ and $\overline{W}(\Gamma)$ are abelian group complexes.

Now if Γ is an abelian group complex and $\widetilde{\pi}_0(\Gamma) = 0$, then $E_q(\Gamma)$ is the direct sum $E_q(\Gamma) = \Omega_q(\Gamma) + \Gamma_q$. Further, the map $E(\Gamma) \times E(\Gamma) \longrightarrow E(\Gamma)$ given by the multiplication is just the map induced by $\Omega(\Gamma) \times \Omega(\Gamma) \longrightarrow \Omega(\Gamma)$. Therefore, in this case we may identify $E(\Gamma)$ and $W(\Omega(\Gamma))$, and Γ and $\overline{W}(\Omega(\Gamma))$, not only as semi-simplicial complexes, but as abelian group complexes.

Theorem 2.20: If Γ is a minimal abelian group complex such that $\pi_q(\Gamma) = 0$ for $q \neq n$, and $\pi_n(\Gamma) = \pi$, then Γ is naturally isomorphic to $K(\pi, n)$.

Proof: Since $\Omega^n(\Gamma)$ is an abelian group complex, $\overline{W}(\Omega^n(\Gamma))$ is also; we may thus iterate the $\overline{W}$ construction, setting $\overline{W}^1 = \overline{W}$, $\overline{W}^n = \overline{W}(\overline{W}^{n-1})$. Then since $\Gamma_q = 0$ for $q < n$,

$$\Gamma = \overline{W}^n(\Omega^n(\Gamma)).$$

Now $\Omega^n(\Gamma)$ is a minimal abelian group complex with one homotopy group π in dimension 0. Therefore if we prove the theorem for dimension 0, it will follow for dimension n by the above formula, since $K(\pi, n) = \overline{W}^n(K(\pi, 0))$.

Suppose that $n = 0$. Then since Γ is minimal, $\Gamma_0 = \pi$, and $\tilde{\pi}_q(\Gamma) = 0$ for $q > 0$. Further $\Omega(\Gamma)$ is minimal, and $\pi_q(\Omega(\Gamma)) = 0$ for all q. Therefore $\Omega(\Gamma)_q = 0$ for all q. This means that if $x \in \Gamma_{q+1}$, $\partial_0 x = e_q$ and $\partial_1 \cdots \partial_{q+1} x = e_0$, then $x = e_{q+1}$. Suppose then that $x \in \Gamma_{q+1}$, and let $y = x s_0 \partial_0 \bar{x}$, $z = s_0 \partial_0 x$. Then $yz = x$, $\partial_0 y = e_q$, $\partial_1 \cdots \partial_{q+1} y_1 = (\partial_1 \cdots \partial_{q+1} x)(\partial_1 s_0 \partial_1 \cdots \partial_q \partial_0 \bar{x}) = (\partial_1 \cdots \partial_{q+1} x)(\partial_1 \cdots \partial_{q+1} \bar{x}) = e_0$. Therefore $y = e_{q+1}$, and $x = z$. In other words if $x \in \Gamma_{q+1}$, then $x = s_0 \partial_0 x$, and therefore $\partial_0 : \Gamma_{q+1} \longrightarrow \Gamma_q$ is an isomorphism. Consequently $\Gamma_q \approx \pi$ for all q and the mappings $\pi \longrightarrow \pi$ induced by either

s_0 or ∂_0 are the identity. However $\partial_1 s_0$ is the identity, and thus the mapping $\pi \longrightarrow \pi$ induced by ∂_1 is the identity. Continuing in this manner we see that the mappings $\pi \longrightarrow \pi$ determined by either $\partial_i : \Gamma_{q+1} \longrightarrow \Gamma_q$ or $s_i : \Gamma_q \longrightarrow \Gamma_{q+1}$ are the identity. This proves the theorem.

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Chapter 2, Appendix A

Abelian group complexes.

Abelian group complexes have very special properties; we have already seen in the first part of this chapter that there is a unique minimal abelian group complex with the abelian group π for its n-th homotopy group, and with all other homotopy groups zero. Essentially all other abelian group complexes are products of such complexes. This will be proved here only for minimal abelian group complexes, but it will be proved later in studying cohomology operations that this is true in general.

Before dealing with minimal abelian group complexes, it will be convenient to clear up a small point. In chapter I, appendix C, it was shown that there was, up to isomorphism, a unique minimal complex with a single non zero homotopy group π in dimension n. We know therefore that such a complex is isomorphic as a semi-simplicial complex with the explicit complex $K(\pi, n)$. We now see that the multiplication in $K(\pi, n)$ is determined by the fact that it has a single homotopy group π in dimension n, and that it is minimal.

<u>Theorem:</u> If X is a minimal complex, π an abelian group, $n \in Z^+$, and $\pi_q(X) = 0$ for $q \neq n$, $\pi_n(X) = \pi$, then there is a unique multiplication in X such that $X_n \cong \pi_n(X)$, and X is a group complex.

<u>Proof</u>: X_q has only one element if $q < n$. Therefore

the multiplication is determined in dimension k, where $k \leq n$. Suppose now that the multiplication is given in X_q for $q \leq k$, $k \geq n$, and we want to define a multiplication in X_{k+1}. Let $x, y \in X_{k+1}$; we want the product of x and y to be an element $z \in X_{k+1}$ such that $\partial_i z = \partial_i x \cdot \partial_i y$, $i = 0, \ldots, k+1$. There is a unique such z since $\pi_{k+1}(X) = 0$ and X is minimal. Therefore, we define $x \cdot y = z$. It is now easy to verify the group axioms using the uniqueness of z.

Now let us turn to the decomposition of minimal abelian group complexes.

<u>Theorem:</u> If Γ is a minimal abelian group complex, then $\Gamma = \mathop{X}\limits_{n=0}^{\infty} K(\pi_n(\Gamma), n)$.

<u>Proof:</u> Since Γ is minimal, we have $\Gamma_0 = \pi_0(\Gamma)$. Further recall that $K_q(\pi_0(\Gamma), 0) \cong \pi_0(\Gamma)$, and that under this isomorphism all face and degeneracy operators correspond to the identity homomorphism. Now define $\phi : \Gamma \longrightarrow K(\pi_0(\Gamma), 0)$ by $\phi_q : \Gamma_q \longrightarrow K_q(\pi_0(\Gamma), 0)$ is the composite of $\partial_0^q : \Gamma_q \longrightarrow \Gamma_0$, and $s_0^q : \Gamma_0 = K_0(\pi_0(\Gamma), 0) \longrightarrow K_q(\pi_0(\Gamma), 0)$. ϕ is a homomorphism, since ∂_0 and s_0 are such, and we need only show that it commutes with ∂_i and s_i. We have $\partial_i s_0^q \partial_0^q = s_0^{q-1} \partial_0^q$ for $i \leq q$, and $s_0^{q-1} \partial_0^{q-1} \partial_i = s_0^{q-1} \partial_0^q$ for $i \leq q-1$, so that $\phi_{q-1} \partial_i x = \partial_i \phi_q x$, $i \leq q-1$. Further $s_0^{q-1} \partial_0^{q-1} \partial_q = s_0^{q-1} \partial_1 \partial_0^{q-1}$. Now since Γ is minimal, if $x, x' \in \Gamma_1$, and $\partial_0 x = \partial_0 x'$, then $\partial_1 x = \partial_1 x'$. This means

however that $\partial_1 x = \partial_1 s_0 \partial_0 x = \partial_0 x$ for $x \in \Gamma_1$, and that for $x \in \Gamma_q$, $s_0^{q-1}\partial_0^{q-1}\partial_q x = s_0^{q-1}\partial_0^{q} x$. Hence we also have $\phi_{q-1}\partial_q x = \partial_q \phi_q x$, and ϕ is a map of group complexes.

Let $\lambda : K(\pi_0(\Gamma), 0) \longrightarrow \Gamma$ be defined by $\lambda_0 : K_0(\pi_0(\Gamma), 0) \longrightarrow \Gamma_0$ is the identity, and $\lambda_q = s_0^q \lambda_0 \partial_0^q$. It is easily verified that λ is a map of group complexes, and $\phi\lambda$ is the identity. Consequently, letting $\Gamma' =$ kernel ϕ, we have $\Gamma \simeq K(\pi_0(\Gamma)) x \Gamma'$.

Now we are in a position to proceed by induction. First, $\widetilde{\pi}_0(\Gamma') = 0$. Therefore $\Gamma' = \overline{W}(\Omega(\Gamma'))$ by theorem 2.19. However, by what we have already proved $\Omega(\Gamma') \simeq K(\pi_0(\Omega), 0) x \Omega'$, and $\Gamma' \simeq K(\pi_1(\Gamma), 1) x \overline{W}(\Omega')$ since $\pi_0(\Omega) = \pi_1(\Gamma)$, and $\overline{W}(K(\pi_0(\Gamma), 0)) = K(\pi_1(\Gamma), 1)$. The remaining details of the induction will be left to the reader, and the theorem is now considered proved.

Although we are not yet ready to prove that every abelian group complex has the same homotopy type as a product of $K(\pi, n)$'s, we will prove a key fact in this proof, namely that for abelian group complexes there is a natural map of *homology* into *homotopy*.

Lemma: If Γ is an abelian group complex, $x \in \widetilde{\pi}_q(\Gamma)$, $\partial_q x = e_{q-1}$, $y \in \Gamma_{q+1}$, and $\prod_{j=0}^{q+1}(\partial_j y)^{\sigma(j)} = x$ where $\sigma(j) = (-1)^j$, then there exists $z \in \widetilde{\pi}_{q+1}(\Gamma)$ such that $\partial_{q+1} z = x$.

Proof: Let $y^0 = y s_0 \partial_0 \overline{y}$. Then $\prod_{j=0}^{q+1}(\partial_j y^0)^{\sigma(j)} =$

$$x\, s_0 \partial_0 \Big(\prod_{j=2}^{q+1} \partial_j y^{\sigma(j+1)} \Big) = x\, s_0\, \partial_0 (\bar{x}\ \partial_0 y\ \partial_1 \bar{y}\,) = x.$$

Suppose now that $r < q$, and we have defined y^r so that $\partial_i y^r = e_q$ for $i \leq r$, and $\prod_{j=0}^{q+1} (\partial_j y^r)^{\sigma(j)} = x$. Let $y^{r+1} = y^r s_{r+1} \partial_{r+1} \bar{y}^r$. It is not difficult to verify that $\partial_i y^{r+1} = e_q$ for $i \leq r+1$, and $\prod_{j=0}^{q+1} (\partial_j y^{r+1})^{\sigma(j)} = x$. Let $z = (y^q)^{\sigma(q+1)}$, and the result follows.

Definition: If Γ is an abelian group complex, define $\partial : \Gamma_q \longrightarrow \Gamma_{q+1}$ by $\partial x = \prod_{j=0}^{q+1} (\partial_j x)^{\sigma(j)}$.

Define $\pi_q^{\#}(\Gamma)$ to be kernel $\partial : \Gamma_q \longrightarrow \Gamma_{q-1}$ modulo image $\partial : \Gamma_{q+1} \longrightarrow \Gamma_q$.

Let $\phi : \pi_q(\Gamma) \longrightarrow \pi_q^{\#}(\Gamma)$ be the natural map.

Proposition: If Γ is an abelian group complex, then

$$\phi : \pi_q(\Gamma) \xrightarrow{\approx} \pi_q^{\#}(\Gamma).$$

Proof: By the preceding lemma ϕ is monomorphism. To prove that ϕ is an epimorphism suppose that $x \in \Gamma_q$, and $\prod_{j=0}^{q} (\partial_j x)^{\sigma(j)} = e_{q-1}$. Let $y^0 = x\, s_0 \partial_0 \bar{x}$.

Now $\prod_{j=0}^{q} (\partial_j y^0)^{\sigma(j)} = \prod_{j=2}^{q} (s_0 \partial_0 \partial_j \bar{x}\,)^{\sigma(j)}$, and $e_{q-2} = \partial_0 e_{q-1} = \prod_{j=2}^{q} \partial_0 (\partial_j x)^{\sigma(j)}$. Consequently $\prod_{j=0}^{q} (\partial_j y^0)\ ^{(j)} = e_{q-1}$.

Notice that $\prod_{j=0}^{q+1} \partial_j(s_0 x)^{\sigma(j)} =$ $s_0 \prod_{j=2}^{q+1} \partial_{j-1} x^{\sigma(j)} = s_0\partial_0 x = x\overline{y}^0$.

Therefore, $[x] = [y^0] \in \pi_q^{\#}(\Gamma)$, and $\partial_0 y^0 = e_{q-1}$.

Now proceed inductively to find y^q such that $\partial_i y^q =$ e_{q-1} $i \leq q$, and $[x] = [y^q]$. Then y^q represents an element of $\pi_q(\Gamma)$, and the proof is complete.

__Theorem:__ If Γ is an abelian group complex, then there is a map

$\lambda : H_q(\Gamma) \longrightarrow \pi_q(\Gamma)$ such that if $\mu : \pi_q(\Gamma) \longrightarrow H_q(\Gamma)$ is the natural map of homology into homotopy, then $\lambda\mu$ is the identity.

__Proof:__ There is a natural map of $c_q(\Gamma) \longrightarrow \Gamma_q$ which sends $r \circ x$ into x^r for $x \in \Gamma_q$. This gives rise to a chain map of $C(\Gamma) \longrightarrow \Gamma$ or a homomorphism $\lambda^{\#} : H(\Gamma) \longrightarrow \pi^{\#}(\Gamma)$. We now have a commutative diagram

$$\begin{tikzcd}
\pi_q(\Gamma) \arrow[r, "\mu"] \arrow[dr, "\phi", "\approx"'] & H_q(\Gamma) \arrow[d, "\lambda^{\#}"] \\
& \pi_q^{\#}(\Gamma)
\end{tikzcd}$$

Letting $\lambda = \phi^{-1}\lambda^{\#}$, the proof is complete.

__Errata:__ p. 1C-7, __Theorem (Poincaré):__ isomorphism $\phi' : \pi_1(X,x)/[\pi_1(X,x), \pi_1(X,x)] \longrightarrow H_1(X)$.

The construction FK

John Milnor

§1. Introduction

The reduced product construction of Ioan James [5] assigns to each CW-complex a new CW-complex having the same homotopy type as the loops in the suspension of the original. This paper will describe an analougous construction proceding from the category of semi-simplicial complexes to the category of group complexes. The properties of this construction FK are studied in §2.

A theorem of Peter Hilton [4] asserts that the space of loops in a union $S_1 \vee \dots \vee S_r$ of spheres splits into an infinite direct product of loops spaces of spheres. In §3 the construction of FK is applied to prove a generalization (Theorem 4) of Hilton's theorem in which the spheres may be replaced by the suspensions of arbitrary connected (semi-simplicial) complexes.

The author is indebted to many helpful discussions with John Moore.

§2. The construction.

It will be understood that with every semi-simplicial complex there is to be associated a specified base point.

Let K be a semi-simplicial complex with base point b_o. Denote $S_o^n b_o$ by b_n. Let FK_n denote the free group generated by the elements of K_n with the single relation $b_n = 1$. Let the face and degeneracy operations ∂_i, s_i in $FK = \cup FK_n$ be the unique homomorphisms which carry the generators k_n into $\partial_i k_n$, $s_i k_n$ respectively. Thus each complex K determines a group complex FK.

It will be shown that FK is a loop space for EK, the suspension of K. (Definitions will be given presently.)

Alternatively let $F^+K_n \subset FK_n$ be the free monoid (=associative semi-group with unit) generated by K_n, with the same relation $b_n = 1$. Then the monoid complex F^+K is also a loop space for EK. This construction is the direct generalization of James' construction. (See Lemma 4.)

The <u>suspension</u> EK of the semi-simplicial complex K is defined as follows. For each simplex k_n, other than b_n, of K there is to be a sequence (Ek_n), $(s_o Ek_n)$, $(s_o^2 Ek_n)$, ... of simplexes of EK having dimensions $n+1$, $n+2$, ... In addition there is to be a base point (b_o) and its degeneracies (b_n). The symbols $(s_o^i Eb_n)$ will be identified with (b_{n+i+1}). The face and degeneracy operations in EK are given by

$$\partial_j(Ek_n) = (E\,\partial_{j-1} k_n) \qquad (j > 0, n > 0)$$

$$s_j(Ek_n) = (E s_{j-1} k_n) \qquad (j > 0)$$

$$\partial_0(Ek_n) = (b_n), \qquad \partial_1(Ek_0) = (b_0)$$

$$s_0(Ek_n) = (s_0 Ek_n).$$

The face and degeneracy operations on the remaining simplexes $(s_0^i Ek_n) = s_0^i(Ek_n)$ are now determined by the identities

$$\partial_j s_0^i = \begin{cases} s_0^i \partial_{j-1} & (j > i) \\ s_0^{i-1} & (j \leq i \neq 0) \end{cases}$$

$$s_j s_0^i = \begin{cases} s_0^i s_{j-1} & (j > i) \\ s_0^{i+1} & (j \leq i). \end{cases}$$

It is not hard to show that this defines a semi-simplicial complex. The following lemma will justify calling it the suspension of K. Recall that the suspension of a topological space A with base point a_0 is the identification space of $A \times I$ obtaining by collapsing $(A \times \dot{I}) \cup (a_0 \times I)$ to a point.

Lemma 1. *The geometric realization* $|EK|$ *is canonically homeomorphic to the suspension of* $|K|$.

(For the definition of realization see [6.] In fact the required homeomorphism is obtained by mapping the point $(|k_n, \delta_n|, 1-t)$ of the suspension of $|K|$, where δ_n

has barycentric coordinates $(t_o,\ldots,t_n)$ into the point $|(EK_n), \delta_{n+1}| \in |EK|$, where δ_{n+1} has barycentric coordinates $(1-t, t\,t_o,\ldots,t\,t_n)$.

Next the space of loops on a semi-simplicial complex K will be discussed. If K satisfies the Kan extension condition then ΩK can be defined as in [7]. This definition has two disadvantages:

(1) Many interesting complexes do not satisfy the extension condition. In particular EK does not.

(2) There is no natural way (and in some cases[1] no possible way) of defining a group structure in ΩK.

The following will be more convenient. A group complex G, or more generally a monoid complex, will be called a <u>loop space</u> for K if there exists a (semi-simplicial) principal bundle with base space K, fibre G, and with contractible total space T.

(By a <u>principal bundle</u> is meant a projection p of T onto K together with a left translation $G \times T \longrightarrow T$ satisfying

$$(g_n \cdot g'_n) \cdot t_n = g_n \cdot (g'_n \cdot t_n)$$

where $g_n \cdot t_n = t_n$ if and only if $g_n = 1_n$ and where $g_n \cdot t_n = t'_n$ for some g_n if and only if $p(t_n) = p(t'_n)$. A complex is called <u>contractible</u> if its geometric realization is contractible. This is equivalent to requiring that the integral homology groups and the fundamental group be trivial.)

---

[1] Let K be the minimal complex of the n-sphere, $n > 2$. Then it can be shown that there is no group complex structure in ΩK having the correct Pontrjagin ring.

The existence of such a loop space for any connected complex K has been shown in recent work of Kan, which generalizes the present paper. The following Lemma is given to help justify the definition.

Lemma 2. *If* K *satisfies the extension condition, and the group complex* G *is a loop space for* K, *then there is a homotopy equivalence* $\Omega K \longrightarrow G$.

The proof is based on the following easily proven fact (compare [7] p. 2-10): Every principal bundle can be given the structure of a twisted cartesian product. That is one can find a one-one function

$$\eta : G \times K \longrightarrow T$$

satisfying $\partial_i \eta = \eta \partial_i$ for $i > 0$ and $s_i \eta = \eta s_i$ for all i, where $\partial_0 \eta$ is given by an expression of the form

$$\partial_0 \eta (g_n k_n) = \eta((\partial_0 g_n) \cdot (\tau k_n), \partial_0 k_n).$$

(For this assertion the fibre must be a monoid complex satisfying the extension condition.) Thus the bundle is completely described by G and K together with the "twisting function" $\tau : K_n \longrightarrow G_{n-1}$; where τ satisfies the identities

$$s_i \tau = \tau s_{i+1} \quad (i \geq 0), \qquad \partial_i \tau = \tau \partial_{i+1} \quad i \geq 1,$$

$$\tau s_0 k_n = 1_n, \qquad (\partial_0 \tau k_n) \cdot (\tau \partial_0 k_n) = \tau \partial_1 k_n.$$

Now a map $\bar{\tau} : \Omega K_{n-1} \longrightarrow G_{n-1}$ is defined by $\bar{\tau}(k_n) = \tau(k_n)$. From the definition of ΩK and the above identities it follows that $\bar{\tau}$ is a map. From the homotopy sequence of the bundle it is easily verified that $\bar{\tau}$ induces isomorphisms of the homotopy groups, which proves Lemma 2.

To define a principal bundle with fibre FK and base space EK it is sufficient to define twisting functions $\tau : EK_{n+1} \longrightarrow FK_n$. These will be given by

$$\tau(Ek_n) = k_n, \qquad \tau(s_0^i Ek_{n-i}) = 1_n \quad (i > 0).$$

Theorem 1. FK *is a loop space for* EK. *In fact the twisted cartesian product* $\{FK, EK, \tau\}$ has a con*tractible total space*.

It is easy to verify that τ satisfies the conditions for a twisting function. Hence we have defined a twisted cartesian product, and therefore a principal bundle. Let T denote its total space. Note that T may be identified with FK x EK except that ∂_0 is given by

$$\partial_0(g_n, (Ek_{n-1})) = (\partial_0 g_n \cdot k_{n-1}, (b_{n-1}))$$

$$\partial_0(g_n, (s_0^i Ek_{n-i-1})) = (\partial_0 g_n, (s_0^{i-1}(Ek_{n-i-1})) \quad (i \geq 1).$$

It will first be shown that the homology groups of T are trivial. This will be done by giving a contracting homotopy S for the chain complex C(T).

Lemma 3. *Let* G *be the free group on generators* x_α. *Then the integral group ring* ZG *has as basis* *(over Z) the elements* $gx_\alpha - g$, *where* g *ranges over all elements of* G; *together with the element* 1.

The proof is not difficult. Now define S by the rules

$$S(1_n,(b_n)) = \begin{cases} 0 & (n \text{ even}) \\ (1_{n+1},(b_{n+1})) & (n \text{ odd}) \end{cases}$$

$$S[(g_n \cdot k_n,(b_n)) - (g_n,(b_n))]$$

$$= \sum_{i=0}^{n} (-1)^i [(s_i g_n,(s_0^i E \partial_0^i k_n)) - (s_i g_n,(b_{n+1}))]$$

$$S[(g_n,(s_0^{r-1} E k_{n-r})) - (g_n,(b_n))]$$

$$= \sum_{j=r}^{n} (-1)^j [(s_j g_n,(s_0^j E \partial_0^{j-r} k_{n-r})) - (s_j g_n,(b_{n+1}))]$$

where g_n ranges over all elements of the group FK_n.

It follows easily from Lemma 3 that the elements for which S has been defined form a basis for $C(T)$, providing that k_n, k_{n-r} are restricted to elements other than b_n, b_{n-r}. However the above rules reduce to the identity $0 = 0$ if we substitute $k_n = b_n$ or $k_{n-r} = b_{n-r}$. This shows that S is well defined.

The necessary identity $Sd + dS = 1 - \varepsilon$, where $dx_n = \sum_{i=0}^{n} (-1)^i \partial_i x_n$ and where $\varepsilon : C(T) \longrightarrow C(T)$ is the augmentation $(\varepsilon \sum \alpha_i (g_0, b_0) = \sum \alpha_i (1_0, b_0))$ can now be verified by direct computation. Since this computation is rather long it will not be given here.

Proof that $|T|$ is simply connected. A maximal tree in the CW-complex $|T|$ will be chosen. Then $\pi_1(|T|)$ can be considered as the group with one generator corresponding to each 1-simplex not in the tree, and one relation corresponding to each 2-simplex.

As maximal tree take all 1-simplexes of the form $(s_0 g_0, (Ek_0))$. Then as generators of $\pi_1(|T|)$ we have all elements $(g_1, (Ek_0))$ such that g_1 is non-degenerate. The relation $\partial_1 x = (\partial_2 x) \cdot (\partial_0 x)$ where $x = (s_1 g_1, (s_0 Ek_0))$ asserts that

$$\begin{aligned} (g_1, (Ek_0)) &= (g_1, (b_1)) \cdot (s_0 \partial_0 g_1, (Ek_0)) \\ &= (g_1, (b_1)). \end{aligned}$$

From the 2-simplex $(s_0 g_1, (Ek_1))$ we obtain

$$\begin{aligned} (g_1, (E\partial_0 k_1)) &= (s_0 \partial_1 g_1, (E\partial_1 k_1)) \cdot (g_1 k_1, (b_1)) \\ &= (g_1 k_1, (b_1)). \end{aligned}$$

Combining these two relations we have

$$(g_1, (b_1)) = (g_1 k_1, (b_1)),$$

from which it follows easily that

$$(g_1, (b_1)) = 1$$

for all g_1. In view of the first relation, this shows that $|T|$ is simply connected, and completes the proof of theorem 1.

The following theorem shows that FK is essentially unique.

Theorem 2. *Any principal bundle over* EK *with any group complex* G *as fibre is induced from the above bundle by a homomorphism* $FK \longrightarrow G$.

Proof: We may assume that this bundle is a twisted cartesian product with twisting function $\tau:(EK)_{n+1} \longrightarrow G_n$. Define the homomorphism $\bar{\tau}:FK \longrightarrow G$ by $\bar{\tau}(k_n) = \tau(Ek_n)$. Since $\bar{\tau}(b_n) = \tau(Eb_n) = \tau(s_o(b_n)) = 1_n$ this defines a homomorphism. It is easy to verify that $\bar{\tau}$ commutes with the face and degeneracy operations, and induces a map between the two twisted cartesian products.

Corollary. *If* G *is also a loop space for* EK *then there is a homomorphism* $FK \longrightarrow G$ *inducing an isomorphism between the Pontrjagin rings*.

This follows easily using [7], IV Theorem B.

Analogues of theorems 1 and 2 for the construction $F^+(K)$ can be proved using exactly the same formulas. The following shows the relationship between $F^+(K)$ and the construction of James.

Lemma 4. *If* K *is countable then the realization* $|F^+K|$ *is homeomorphic to the reduced product of* $|K|$.

In fact the product $(k_n, k'_n, k''_n, \ldots) \longrightarrow k_n \cdot k' \cdot k'' \cdot \ldots$ maps $K \times \ldots \times K$ into F^+K. Taking realizations we obtain a map $|K| \times \ldots \times |K| \longrightarrow |F^+K|$. From these maps it is easy to define a map of the reduced product of $|K|$ into $|F^+K|$, and to show that it is a homeomorphism.

§3. A theorem of Hilton

If A, B are semi-simplicial complexes with base points a_o, b_o let $A \vee B$ denote the subcomplex $A \times [b_o] \cup [a_o] \times B$ of $A \times B$. Let $A \divideontimes B$ denote the complex obtained from $A \times B$ by collapsing $A \vee B$ to a point. The notation $A^{(k)}$ will be used for the k-fold "collapsed product" $A \divideontimes \cdots \divideontimes A$.

The free product $G \ast H$ of two group complexes is defined by $(G \ast H)_n = G_n \ast H_n$. There is clearly a canonical isomorphism between the group complexes $F(A \vee B)$ and $FA \ast FB$.

Lemma 5. <u>The complex</u> $F(A \vee B)$ <u>is isomorphic (ignoring group structure) to</u> $FA \times F(B \vee (B \divideontimes FA))$.

In fact we will show that $F(A \vee B)$ is a split extension:

$$I \longrightarrow F(B \vee (B \divideontimes FA)) \longrightarrow F(A \vee B) \longrightarrow FA \longrightarrow I.$$

The collapsing map $A \vee B \xrightarrow{c} A$ induces a homomorphism c' of $F(A \vee B)$ onto FA. Denote the kernel of c' by F'. The inclusion $A \xrightarrow{i} A \vee B$ induces a homomorphism $i' : FA \longrightarrow F(A \vee B)$. Since $c'i'$ is the identity it follows that $F(A \vee B)$ is a split extension of F' by FA.

We will determine this kernel F'_n for some fixed dimension n. Let a, b, ϕ range over the n-simplexes other than the base point of A, B, and FA respectively. Then $F(A \vee B)_n$ is the free group $\{a, b\}$ and F'_n is the normal subgroup generated by the b. By the Reidemeister-Schreier theorem (see [8]) F'_n is freely generated by the elements $w(a)bw(a)^{-1}$ where $w(a)$ ranges over all elements of the free group $\{a\} = FA_n$. Thus

$$F'_n = \{b, \phi b \phi^{-1}\}.$$

Now setting $[b,\phi] = b\phi b^{-1}\phi^{-1}$ and making a simple Tietze transformation (see for example [1]) we obtain

$$F'_n = \{b, [b,\phi]\}.$$

Identify $[b,\phi]$ with the simplex $b \divideontimes \phi$ of $B \divideontimes F(A)$. Then we can identify F'_n with $F(B \vee (B \divideontimes FA))$. Since this identification commutes with face and degeneracy operations, this proves Lemma 5.

Lemma 6. <u>The group complex</u> $F(B \divideontimes FA)$ <u>is isomorphic to</u>

$$F((B \divideontimes A) \vee (B \divideontimes A \divideontimes FA)).$$

The inclusion $A \longrightarrow FA$ induces a homomorphism

$$F(B \divideontimes A) \longrightarrow F(B \divideontimes FA).$$

A homomorphism

$$F(B \divideontimes A \divideontimes FA) \longrightarrow F(B \divideontimes FA)$$

is defined by

$$b \divideontimes a \divideontimes \phi \longrightarrow (b \divideontimes a)(b \divideontimes \phi a)^{-1}(b \divideontimes \phi).$$

(This is motivated by the group identity $[[b,a],\phi] = [b,a][b,\phi a]^{-1}[b,\phi]$).

Combining these we obtain a homomorphism

$$F(B \divideontimes A) \ast \tau F(B \divideontimes A \divideontimes FA) \longrightarrow F(B \divideontimes FA)$$

which is asserted to be an isomorphism.

Using the same notation as in Lemma 5 and identifying $b \divideontimes a \divideontimes \phi$ with $[[b,a],\phi]$ it is evidently sufficient to prove the following.

Lemma 7. *In the free group* $\{a,b\}$ *the subgroup freely generated by the elements* $[b,\phi]$ *is also freely generated by the elements* $[b,a]$ *and* $[[b,a],\phi]$.

The proof consists of a series of Tietze transformations. Details will not be given.

As a consequence of Lemma 6 we have:

Lemma 8. *For each* m *the group complex* $F(B \divideontimes FA)$ *is isomorphic to*

$$F(B \divideontimes A) * F(B \divideontimes A \divideontimes A) * \cdots * F(B \divideontimes A^{(m)}) * F(B \divideontimes A^{(m)} \divideontimes FA).$$

Proof by induction on m. For $m=1$ this is just Lemma 6. Given this assertion for the integer $m-1$ it is only necessary to show that $F(B \divideontimes A^{(m-1)} \divideontimes FA)$ is isomorphic to $F(B \divideontimes A^{(m)}) * F(B \divideontimes A^{(m)} \divideontimes FA)$. But this follows immediately from Lemma 6 by substituting $B \divideontimes A^{(m-1)}$ in place of B.

Theorem 3. *If* A *and* B *are semi-simplicial complexes with* A *connected, then there is an inclusion homomorphism*

$$F\left(\bigvee_{i=1}^{\infty} B \divideontimes A^{(i)}\right) \longrightarrow F(B \divideontimes F(A))$$

which is a homotopy equivalence.

Proof. Every element of $F(\bigvee_{i=1}^{\infty} B \divideontimes A^{(i)})$

is already contained in

$$F(\vee_{i=1}^{m} B \divideontimes A^{(i)}) = F(B \divideontimes A) \ast \cdots \ast F(B \divideontimes A^{(m)})$$

for some m. Hence by Lemma 8 it may be identified with an element of $F(B \divideontimes FA)$. Since A is connected, the "remainder term" $B \divideontimes A^{(m)} \divideontimes FA$ has trivial homology groups in dimensions less than m. From this it follows easily that the above inclusion induces isomorphisms of the homotopy groups in all dimensions.

Remark. The complex B may be eliminated from Theorem 3 by taking B as the sphere S^o, and noting the identity $S^o \divideontimes K = K$.

Combining theorem 3 with Lemma 5 we obtain the following

Corollary. If A is connected then there is a homotopy equivalence

$$F(A) \times F(\vee_{i=0}^{\infty} B \divideontimes A^{(i)}) \subset F(A \vee B).$$

This corollary will be the basis for the following.

Theorem 4. Let $A_1, \ldots, A_r$ be connected complexes. Then $F(A_1 \vee \cdots \vee A_r)$ has the same homotopy type as a weak infinite cartesian product $\prod_{i=1}^{\infty} F(A_i)$ where each $A_i, i > r$, has the form

$$A_1^{(n_1)} \divideontimes \cdots \divideontimes A_r^{(n_r)} .$$

The number of factors of a given form is equal to the Witt number

$$\phi(n_1,\ldots,n_r) = \frac{1}{n}\sum_{d|\delta} \frac{\mu(d)(n/d)!}{(n_1/d)!\cdots(n_r/d)!}$$

where $n = n_1+\cdots+n_r$, $\delta = GCD(n_1,\ldots,n_r)$.

Proof. For $n=1,2,3,\ldots$ define complexes A_i, to be called "basic products of weight n" as follows, by induction on n. The given complexes $A_1,\ldots A_r$ are the basic products of weight 1. Suppose that

$$A_1,\ldots,A_r,\ldots,A_\alpha$$

are the basic products of weight less than n. To each $i=1,\ldots,r,\ldots,\alpha$ assume we have defined a number $e(i)< i$, where $e(1) = \cdots = e(r) = 0$. Then as basic products of weight n take all expressions $A_i \divideontimes A_j$ where weight A_i + weight $A_j = n$ and $e(i)\leq j < i$. Call these new complexes $A_{\alpha+1},\ldots,A_\beta$ in any order. If $A_h = A_i \divideontimes A_j$ define $e(h) = j$. (For this disussion we must consider complexes such as $(A\divideontimes B)\divideontimes C$ and $A\divideontimes(B\divideontimes C)$ to be distinct!) This completes the construction of the A_i.

For each $m \geq 1$ define

$$R_m = F(\bigvee_{\substack{h \geq m \\ e(h)<m}} A_h).$$

Thus $R_1 = F(A_1\vee \cdots \vee A_r)$.

Lemma 9. *There is a homotopy equivalence*

$$F(A_m) \times R_{m+1} \subset R_m .$$

Note that $R_m = F(A_m \vee B)$, where $B = \bigvee_{\substack{h>m \\ e(h)<m}} A_h$.

By the corollary to theorem 3 there is a homotopy equivalence

$$F(A_m) \times F(\bigvee_{i=0}^{\infty} B \divideontimes A_m^{(i)}) \subset F(A_m \vee B) = R_m .$$

Substituting in the definition of B and using the distributive law

$$(A \vee B) \divideontimes C = (A \divideontimes C) \vee (B \divideontimes C),$$

the second factor of the first expression becomes

$$F(\bigvee_{i=0}^{\infty} \bigvee_{\substack{h>m \\ e(h)<m}} A_h \divideontimes A_m^{(i)}).$$

But (filling in parentheses correctly) this is just

$$F(\bigvee_{\substack{h>m \\ e(h)\leq m}} A_h) = R_{m+1} ,$$

which proves Lemma 9.

Now it follows by induction that there is a homotopy equivalence

$$F(A_1) \times F(A_2) \times \cdots \times F(A_m) \times R_{m+1} \subset R_1 = F(A_1 \vee \cdots \vee A_r).$$

This defines an inclusion of the weak infinite cartesian product $\prod_{i=1}^{\infty} F(A_i)$ into R_1. Since $A_1, \ldots, A_r$ are connected, it follows easily that the "remainder terms" R_m are k-connected where $k \longrightarrow \infty$ as $m \longrightarrow \infty$. From this it follows that the above inclusion map induces isomorphisms of the homotopy groups in all dimensions. This proves the first part of theorem 4.

Let $\phi(n_1,\ldots,n_r)$ denote the number of A_h having the form $A_1^{(n_1)} \divideontimes \cdots \divideontimes A_r^{(n_r)}$. To compute these numbers consider the free Lie ring L on generators $\alpha_1,\ldots,\alpha_r$. Corresponding to each "basic product" $A_h = A_i \divideontimes A_j$ define an element $\alpha_h = [\alpha_i, \alpha_j]$ of L, for $h = r+1, r+2, \ldots$. Then the elements α_h obtained in this way are exactly the standard monomials of M. Hall [2] and P. Hall [3]. M. Hall has proved that these elements form an additive basis for L.

The number of linearly independent elements of L which involve each of the generators $\alpha_1,\ldots,\alpha_r$ a given number $n_1,\ldots,n_r$ of times has been computed by Witt [9]. Since his formula is the same as that in theorem 4, this completes the proof.

In conclusion we mention one more interesting consequence of theorem 3.

Theorem 5. _If_ A _is connected then the complex_ EFA _has the same homotopy type as_ $\bigvee_{i=1}^{\infty} E\,A^{(i)}$.

The proof is based on the following lemma, which depends on Theorem 1.

Lemma 10. _If_ A _is connected, there is a homotopy equivalence_

$$EA \subset \overline{W}FA .$$

In fact the inclusion is defined by

$(s_0^i E a_n) \longrightarrow s_0^i (a_n, 1_{n-1}, \ldots, 1_0)$. It is easily verified that this is a map, and that it induces a map of the twisted cartesian product T into the twisted cartesian product W. Since both total spaces are acyclic, it follows from [7], IV Theorem A that the homology groups of EA map isomorphicly into those of $\bar{W}FA$. Since both spaces are simply connected, this completes the proof of Lemma 10.

Now from Theorem 3 we have a homotopy equivalence

$$\bar{W}F(\bigvee_{i=1}^{\infty} A^{(i)}) \subset \bar{W}FFA.$$

In view of Lemma 10, and the identity

$$E(A \vee B) = EA \vee EB,$$

this completes the proof.

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Chapter 3. Acyclic Models

Acyclic models.[1]

If $\mathcal{A}$ is a category and $\mathfrak{M}$ a subset of the objects of $\mathcal{A}$, we shall denote by $\mathcal{A}^{\mathfrak{M}}$ the set of mappings in $\mathcal{A}$ with domain in $\mathfrak{M}$.

Definition 3.1: The quadruple$(\mathcal{A}, \mathfrak{M}, \alpha, \beta)$ will be called a category with models if $\mathcal{A}$ is a category, $\mathfrak{M}$ a certain subset of the objects of $\mathcal{A}$, called the set of models, and α, β are functions of $\mathcal{A}^{\mathfrak{M}}$ into itself such that

0) $\alpha(1\ (M)) = \beta(1\ (M)) = 1(M),\ M \in \mathfrak{M}$

1) $\beta(u)\alpha(u) = u.$

2) $\alpha(\beta(u)) = \beta(\alpha(u)) = 1(M)$ where $M =$ domain $\beta(u)$ $=$ range $\alpha(u)$.

3) $\beta(f\ u) = \beta(f\beta(u))$ where f is a mapping of $\mathcal{A}$ such that domain $f =$ range u.

4) $\alpha(f\ u) = \alpha(f\beta(u))\alpha(u)$, where f means the same as in 3);

where $u \in \mathcal{A}^{\mathfrak{M}}$ throughout.

Notice that 3) implies $\beta(\beta(u)) = \beta(u)$ and 1) and 2) imply $\alpha(\alpha(u)) = \alpha(u)$.

Assumption: For the rest of this section, $(\mathcal{A}, \mathfrak{M}, \alpha, \beta)$ is a fixed category with models; it will usually be denoted by $\mathcal{A}$; "object" will mean "object of $\mathcal{A}$", and "mapping", "mapping of $\mathcal{A}$ ".

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[1] The theory of acylic models was introduced by Eilenberg and MacLane [1]. The version given here is a part of [2].

Definition 3.2: For any object A, S(A) will denote the set of mappings $u : M \longrightarrow A$ with $M \in \mathfrak{M}$, such that $\alpha(u) = 1(M)$.

Notation 3.3: For the rest of this paper, Λ will denote a fixed commutative ring with unit element; $\mathcal{G}_\Lambda$ the category of Λ-modules and Λ-homomorphisms.

Definitions 3.4: If $K : \mathcal{a} \longrightarrow G_\Lambda$ is a covariant functor, and $u : M \longrightarrow A$ an element of $\mathcal{a}^{\mathfrak{M}}$, we shall denote by $K(M,u)$ the module $K(M)$ with u added as an indexing symbol; the elements of $K(M, u)$ will be denoted by (k, u), where $k \in K(M)$; $(k, u) + (k', u) = (k + k', u)$, $\lambda(k, u) = (\lambda k, u)$ if $\lambda \in \Lambda$. We define the natural isomorphisms

$$\begin{tikzcd} K(M) \arrow[r, shift left, "i(u)"] & K(M, u) \arrow[l, shift left, "j(u)"] \end{tikzcd}$$

by $i(u)k = (k, u)$; $j(u)(k, u) = k$.

We now define a new functor $\hat{K} : \mathcal{a} \longrightarrow \mathcal{G}_\Lambda$ as follows:

$\hat{K}(A) = \Sigma_{u \in S(A)} K(M, u)$ for any object A.

$\hat{K}(f) \mid K(M, u) = i(\beta(fu)) K(\alpha(fu)) j(u)$ for any map $f : A \longrightarrow B$; thus $\hat{K}(f) \mid K(M, u) : K(M, u) \longrightarrow K(M', \beta(fu))$ where $M' =$ domain $\beta(fu)$; clearly $\beta(fu) \in S(B)$, as required.

Next, we define a natural transformation of functors $\Gamma_K : \hat{K} \longrightarrow K$ by

$\Gamma_K(A) \mid K(M, u) = K(u)\, j(u)$ for any object A; the necessary naturality condition is easily verified.

The functor K is said to be *representable* if there is a natural transformation of functors $\chi_K : K \longrightarrow \hat{K}$ such that $\Gamma_K \chi_K : K \longrightarrow K$ is the identity.

Notations and Conventions 3.5: Let $d\mathcal{G}_\Lambda$ denote the category of differential Λ-modules and admissible homomorphisms; in other words, an object of $d\mathcal{G}_\Lambda$ is a pair $(G \ , d_G)$ such that $G = \Sigma_{n \geq 0} G_n$, a direct sum of Λ-modules, d_G is a Λ-endomorphism of G such that $d_G d_G = 0$, $d G_n \subset G_{n-1}$ for $n > 0$ and $d_G G_0 = 0$. A mapping $f : (G, d_G) \longrightarrow (F, d_F)$ of $d\mathcal{G}_\Lambda$ is a Λ-homomorphism $f : G \longrightarrow F$ such that $d_F f = f d_G$. Usually we shall denote (G, d_G) simply by G, and d_G, indiscriminately, by d. The elements of G_n will be called *n-dimensional.* For every object (G, d) we define the k-skeleton (G^k, d), itself an object of $d\mathcal{G}_\Lambda$, by setting $G^k_n = G_n$ for $n \leq k$ and $G^k_n = 0$ for $n > k$, and using for d the natural restriction. In the category $d\mathcal{G}_\Lambda$, homology is defined as usual; we write $Z(G) =$ kernel d_G, $B(G) =$ image d_G, $H(G) = Z(G)/ B(G)$, $Z_n(G) = Z(G) \cap G_n$, $B_n(G) = B(G) \cap G_n$, $H_n(G) = Z_n(G)/B_n(G)$ so that $H(G) = \Sigma_{n \geq 0} H_n(G)$. Note that H and H_n can be regarded as covariant functors $d\mathcal{G}_\Lambda \longrightarrow \mathcal{G}_\Lambda$; the definition of $H(f)$, $H_n(f)$ being evident. The natural transformation $G_0 \longrightarrow H_0$ will be indiscriminately denoted by ε .

Definition 3.6: If $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ is a covariant functor, define $K^n : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ by $K^n(A) = (K\ (A))^n$ for any object

A and $K^n(f) = K(f) \mid K^n(A)$ for any map $f : A \longrightarrow B$. Further, define $K_n : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ by $K_n(A) = (K(A))_n, K_n(F) \mid K_n(A) = K(F) \mid K_n(A)$. We say that K is *representable* if K_n is representable for every $n \geq 0$; this is the same as saying that K is representable when regarded as a functor $K : A \longrightarrow \mathcal{G}_\Lambda$.

Notations 3.7: By $\widehat{\mathfrak{M}}$ we denote the subcategory of $\mathcal{A}$ the objects of which are those of $\mathfrak{M}$, and the maps all maps of the type $\alpha(u)$, or compositions of such maps.

Let $K, L : \mathcal{A} \longrightarrow \mathcal{G}_\Lambda$ be two functors and $U : K|\widehat{\mathfrak{M}} \longrightarrow L|\widehat{\mathfrak{M}}$ a natural transformation; then U determines a natural transformation $\hat{U} : \hat{K} \longrightarrow \hat{L}$ by $\hat{U} | K(M, u) = i(u)\ U(M)\ j(u)$ 1) (cf 1.4); so that $\hat{U} \mid K(M, u) : K(M, u) \longrightarrow L(M, u)$. If U is the restriction of $T : K \longrightarrow L$, i.e. $U = T \mid \widehat{\mathfrak{M}}$, we shall write $\hat{U} = \hat{T}$; and in this case we have $T\Gamma_K = \Gamma_L \hat{T}$.

This last remark is applied, for a functor $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$, to $d : K \longrightarrow K$; we thus obtain $\hat{d} : \hat{K} \longrightarrow \hat{K}$ such that $\hat{d}^2 = 0$, $d\Gamma_K = \Gamma_K \hat{d}$; and accordingly we can (and shall) regard $\hat{K}$ as a functor $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$.

Definition 3.8: A covariant functor $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ will be said to be a cyclic on models if there exist natural transformations[2] of functors.

$$\eta : H_0 K \mid \widehat{\mathfrak{M}} \longrightarrow K_0 \mid \widehat{\mathfrak{M}}, \qquad U : K \mid \widehat{\mathfrak{M}} \longrightarrow K \mid \widehat{\mathfrak{M}}$$

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1) Note that we use i(u), j(u) indiscriminately. In this formula j(u) is related to K, i(u) to L.

2) Here $d\mathcal{G}_\Lambda$ is considered only as a category of Λ-modules; i.e. $U(M)$ is a homomorphism of Λ-modules, but does not preserve gradation nor commute with d.

such that $U K_n|\hat{\mathfrak{M}} \subset K_{n+1}|\hat{\mathfrak{M}}$ and, writing $U_n = U|(K_n|\hat{\mathfrak{M}})$, the following are satisfied:

(1) $$d\,U_0 = 1 - \eta\varepsilon$$

(2) $$dU_n + U_{n-1}\,d = 1\ (K_n|\hat{\mathfrak{M}}) \quad \text{for} \quad n > 0$$

(3) $$U_0\eta = 0$$

where $\varepsilon : K_0 \longrightarrow H_0K$ is the natural transformation. Notice that for $M \in \mathfrak{M}$, any element $h \in H_0\,K\,(M)$ is of the form εk where $k \in K_0(M)$. Now, by the above

$$\varepsilon\eta\varepsilon\, k = \varepsilon(\,1 - d\,U_0)k = \varepsilon\, k$$

so that condition (1) implies

(4) $$\varepsilon\eta = 1$$

Lemma 3.9: If $K : \mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ is acyclic on models, there are natural transformations of functors $\hat{\eta} : H_0\hat{K} \longrightarrow \hat{K}_0$, $U : \hat{K} \longrightarrow \hat{K}$ such that $\hat{U}\hat{K}_n \subset \hat{K}_{n+1}$ and, writing

$\hat{U}_n = U|\,\hat{K}_n$,

(1) $$\hat{d}\,\hat{U}_0 = 1 - \hat{\eta}\,\hat{\varepsilon}$$

(2) $$\hat{d}\,\hat{U}_n + \hat{U}_{n-1}\,\hat{d} = 1 \quad \text{if} \quad n > 0$$

(3) $$\hat{U}_0\,\hat{\eta} = 0$$

(4) $$\hat{\varepsilon}\,\hat{\eta} = 1.$$

This is immediate from 1.8.

Notation 3.10: By $\overline{\mathfrak{M}}$ denote the sub-category of $\mathcal{A}$ the objects of which are all those of $\mathfrak{M}$, and the mappings all mappings having models as domain and range. $\hat{\mathfrak{M}}$ and $\overline{\mathfrak{M}}$ have the same objects; but $\overline{\mathfrak{M}}$ has more mappings.

Theorem 3.11: Let K, L : $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ be covariant functors and let $T : H_0K|\overline{\mathfrak{M}} \longrightarrow H_0L|\overline{\mathfrak{M}}$ be a natural transformation of functors; let K be representable and L acyclic on models. Then there is a natural transformation of functors $\Phi : K \longrightarrow L$ such that $\Phi \mid (K_0|\overline{\mathfrak{M}})$ induces T; Φ will be called "an extension of T".

Proof: T induces $\hat{T} : H_0\hat{K} \longrightarrow H_0\hat{L}$. Since L is acyclic, we have transformations $\hat{U} : \hat{L} \longrightarrow \hat{L}$, $\hat{\eta} : H_0\hat{L} \longrightarrow \hat{L}_0$ satisfying the conditions of 1.9. We define $\Phi_0 : K_0 \longrightarrow L_0$ by $\Phi_0 = \Gamma_L \hat{\eta}\hat{T}\hat{\varepsilon}\chi_K$ and $\Phi_1 : K_1 \longrightarrow L_1$ by $\Phi_1 = \Gamma_L \hat{U}_0 \hat{\Phi}_0 \hat{d}\chi_K$. (cf. 1.7). Then $d\Phi_1 = d\Gamma_L \hat{U}_0 \hat{\Phi}_0 \hat{d}\chi_K = \Gamma_L \hat{d}\hat{U}_0 \hat{\Phi}_0 \hat{d}\chi_K = \Gamma_L(1 - \hat{\eta}\hat{\varepsilon})\hat{\Phi}_0 \hat{d}\chi_K = \Gamma_L \hat{\Phi}_0 \hat{d}\chi_K = \Phi_0\Gamma_K \hat{d}\chi_K = \Phi_0 d$, since $\hat{\varepsilon}\hat{\Phi}_0\hat{d} = 0$; in fact $\varepsilon\Phi_0 d|\overline{\mathfrak{M}} = 0$. For restricting everything to the category $\overline{\mathfrak{M}}$, we have $\varepsilon\Phi_0 d = \varepsilon\Gamma_L\hat{\eta}\hat{T}\hat{\varepsilon}\chi_K d = \Gamma_{HL}\hat{\varepsilon}\hat{\eta}\hat{T}\hat{\varepsilon}\chi_K d = \Gamma_{HL}\hat{T}\hat{\varepsilon}\chi_K d = T\Gamma_{HK}\hat{\varepsilon}\chi_K d = T\varepsilon\Gamma_K\chi_K d = T\varepsilon d = 0$.

We proceed by induction: if Φ_k is defined, so is $\hat{\Phi}_k$, and we write $\Phi_{k+1} = \Gamma_L\hat{U}_K\hat{\Phi}_k \hat{d}\chi_K$; and verify

$$\begin{aligned} d\Phi_{k+1} &= d\Gamma_L \hat{U}_k \hat{\Phi}_k \hat{d}\chi_K \\ &= \Gamma_L \hat{d}\hat{U}_k\hat{\Phi}_k \hat{d}\chi_K \\ &= \Gamma_L (1 - \hat{U}_{k-1}\hat{d})\hat{\Phi}_k \hat{d}\chi_K \\ &= \Gamma_L \hat{\Phi}_k \hat{d}\chi_K \\ &= \Phi_k \Gamma_K \hat{d}\chi_K \\ &= \Phi_k d, \text{ as required.} \end{aligned}$$

Further notice that on $\widetilde{\mathfrak{M}}$ we have

$$\varepsilon\,\Phi_0 = \varepsilon\Gamma_L\hat{\eta}\hat{T}\hat{\varepsilon}\chi_K = \Gamma_{HL}\hat{\varepsilon}\hat{\eta}\hat{T}\hat{\varepsilon}\chi_K = \Gamma_{HL}\hat{T}\hat{\varepsilon}\chi_K$$

$= T\Gamma_{HK}\hat{\varepsilon}\chi_K = T\varepsilon\Gamma_K\lambda_K = T\varepsilon$, and so Φ is an extension of T.

<u>Definition 3.12:</u> Let K, L : $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ be covariant functors and let $\Phi, \Phi' : K \longrightarrow L$ be natural transformations. A <u>homotopy</u> V between Φ and Φ' is a natural transformation of functors $V : K \longrightarrow L$ such that $VK_n \subset L_{n+1}$ and $dV + Vd = \Phi - \Phi'$.

<u>Theorem 3.13:</u> If K, L : $\mathcal{A} \longrightarrow d\mathcal{G}_\Lambda$ are covariant functors, $T : H_0K \mid \widetilde{\mathfrak{M}} \longrightarrow H_0L \mid \widetilde{\mathfrak{M}}$ is a natural transformation of functors, K is representable and L acyclic on models, and if Φ, Φ' are extensions of T (cf. 1.7), then there is a homotopy V between Φ and Φ'.

Proof: Since Φ, Φ' are both extensions of T, we must have $\hat{\varepsilon}\hat{\Phi}_0 = \hat{\varepsilon}\hat{\Phi}'_0 = \hat{T}\hat{\varepsilon}$. We define

$$V_0 = \Gamma_L\hat{U}_0(\Phi_0 - \Phi'_0)\chi_K$$

where U, η again are the functors appropriate to L.

$$\begin{aligned} \text{Then}\quad dV_0 &= \Gamma_L\hat{d}\,\hat{U}_0(\hat{\Phi}_0 - \hat{\Phi}'_0)\chi_K \\ &= \Gamma_L(1 - \hat{\eta}\hat{\varepsilon})(\hat{\Phi}_0 - \hat{\Phi}'_0)\chi_K \\ &= \Gamma_L(\hat{\Phi}_0 - \hat{\Phi}'_0)\chi_K \\ &= \Phi_0 - \Phi'_0 \end{aligned}$$

as required. Now, we proceed inductively. Let $V_0, \ldots, V_k$ with all the necessary properties be defined. Then, in particular

$$d(\phi_{k+1} - \phi'_{k+1} - V_k d) = d\phi_{k+1} - d\phi'_{k+1} - dV_k d$$

$$= (\phi_k - \phi'_k - dV_k)d$$

$$= V_{k-1}\, d\, d = 0$$

whence $\hat{d}(\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d}) = 0$. Now we define

$$V_{k+1} = \Gamma_L \hat{U}_{k+1}\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$

Then

$$dV_{k+1} = \Gamma_L\, \hat{d}\, \hat{U}_{k+1}(\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$

$$= \Gamma_L(1 - \hat{U}_k\, \hat{d})\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K.$$

$$= \Gamma_L\, (\hat{\phi}_{k+1} - \hat{\phi}'_{k+1} - \hat{V}_k\, \hat{d})\chi_K$$

$$= \phi_{k+1} - \phi'_{k+1} - V_k\, d\ ,$$

as required.

Combining 1.11 and 1.13 we get

Theorem 3.14: If K, L.: $\mathcal{A} \longrightarrow d\mathfrak{g}_{\Lambda}$ are covariant representable functors which are acyclic on models, and if $T : H_0K | \overline{\mathfrak{M}} \longrightarrow H_0L | \overline{\mathfrak{M}}$ is a natural equivalence, then there is a unique natural equivalence $\phi_* : HK \longrightarrow HL$ such that $\phi_* | (H_0K | \overline{\mathfrak{M}}) = T$, and such that ϕ_* is induced by an extension of T

Now let $\mathcal{A}$ be the category of semi-simplicial complexes and maps. The model objects are to be the semi-simplicial complexes Δ_q (cf. appendix 1A), and α and β are defined as follows. If $u: \Delta_q \longrightarrow X$, let $x = u(0,\dots,q) \in X_q$. If x is non-degenerate, define $\alpha(u): \Delta_q \longrightarrow \Delta_q$ to be the identity, and $\beta(u) = u\colon \Delta_q \longrightarrow X$. Suppose that x is degenerate; then $x = s_{i_r} \cdots s_{i_1} y$, where y is non-degenerate and $i_r > \cdots > i_1$.

(1) Define $\beta(u)\colon \Delta_{q-r} \longrightarrow X$ to be the map determined by $\beta(u)(0,\ldots,q-r) = y$. Then

$$\beta(u)\ (s_{i_r} \cdots s_{i_1}(0,\ldots,q-r)\ ) = x.$$

(2) Define $\alpha(u)\colon \Delta_q \longrightarrow \Delta_{q-r}$ to be the map determined by $\alpha(u)(0,\ldots,q) = s_{i_r} \cdots s_{i_1}(0,\ldots,q-r)$.

It is easily verified that α and β satisfy the axioms and are uniquely defined, so that $\mathcal{A}$ is a category with models.

Let $d\mathcal{G}$ be the category of differential modules over the integers (taking Λ as the ring of integers in 3.5). We define functors $C, C_N\colon \mathcal{A} \longrightarrow d\mathcal{G}$ as follows. Let $C_q(X)$ be the free abelian group having the elements of X_q as generators; and set $C(X) = \sum C_q(X)$. The homomorphism $\partial\colon C_{q+1}(X) \longrightarrow C_q(X)$ is determined by $\partial x = \sum_0^q (-1)^i \partial_i x$, $x \in X_{q+1}$. Let $D_q(X)$ be the free abelian group having the degenerate elements of X_q as generators, and set $C_q(X)_N = C_q(X)/D_q(X)$, $C(X)_N = \sum C_q(X)_N$. Now $\partial(D_q(X)) \subset D_{q-1}(X)$; for

$$\partial s_i x = \sum_{j<i} (-1)^j \partial_j s_i x + (-1)^i \partial_i s_i x + (-1)^{i+1} \partial_{i+1} s_i x + \sum_{j>i+1} (-1)^j \partial_j s_i x$$

$$= \sum_{j<i} (-1)^j s_{i-1} \partial_j x + \sum_{j>i+1} s_i \partial_{j-1} x,$$

since the two middle terms are equal. Therefore ∂ induces a homomorphism $\partial\colon C_{q+1}(X)_N \longrightarrow C_q(X)_N$. It follows in the usual manner that $\partial\partial = 0$ in both cases, which completes the definition of C and C_N. C is called the <u>chain functor,</u>

C_N the <u>normalized</u> <u>chain</u> <u>functor</u>.

We now wish to show that C and C_N give the same homology. There is a natural transformation of functors $\Phi: C \longrightarrow C_N$ such that $\Phi(X): C(X) \longrightarrow C(X)_N$ is the projection onto the factor group. In order to obtain a homotopy inverse for Φ, we shall show that both C and C_N are representable and acyclic on models, and shall then apply theorems 3.11 and 3.13.

To show that C is representable, we define a natural transformation $\chi_c: C \longrightarrow \hat{C}$ as follows. Recall that $C_q(X)$ is free abelian, and let $x \in X_q$ be a generator. There is a unique map $u: \Delta_q \longrightarrow X$ such that $u(0,\dots,q) = x$. Let Δ_{q-r} be the domain of $\beta(u)$. Then $\chi_c(X)(x) = (\alpha(u)(0,\dots,q), \beta(u)) \in (C(\Delta_{q-r}), \beta(u)) \subset \hat{C}(X)$. Since $\Gamma\chi_c =$ identity, C is representable.

Now the homomorphism $\chi(X): C(X) \longrightarrow \hat{C}(X)$ carries $D(X)$ into the subgroup generated by degenerate simplexes, and hence induces a homomorphism $\chi'(X): C(X)_N \longrightarrow (\widehat{C(X)}_N)$. It is easy to verify that $\chi': C_N \longrightarrow (\hat{C}_N)$ is a natural transformation of functors, and that $\Gamma\chi' =$ identity, so that C_N is also representable. To show that C and C_N are acyclic on models, define

$S: (\Delta_q)_r \longrightarrow (\Delta_q)_{r+1}$ by $S(m_0,\dots,m_r) = (0,m_0,\dots,m_r)$.

Then S has the properties

$$\partial_0 S(m_0,\dots,m_r) = (m_0,\dots,m_r)$$

$$\partial_{i+1} S = S\,\partial_i$$

$$s_{i+1} S = S\, s_i$$

$$s_0 S = S^2$$

Let $x \in (\Delta_q)r, r > 0$. Then

$$\partial S x = \sum_{0}^{r+1} (-1)^i \partial_i S x = x + \sum_{1}^{r+1} (-1)^i S \partial_{i-1} x,$$ so that $\partial S x + S \partial x = x$. If $x \in (\Delta_q)_0$, then $\partial S x = x - (0)$.

Now suppose that $h: \Delta_{q+1} \longrightarrow \Delta_q$ is a map in the category $\hat{\mathfrak{M}}$. Since h is a simplicial map onto Δ_q, we need only define it on the vertices, and it has the form $h(j) = \begin{cases} j & \text{for } j \leq i \\ j-1 & \text{for } j > i \end{cases}$ for some $i \leq q$. Then clearly $S h = h S$. Since any map in $\hat{\mathfrak{M}}$ is a composition of maps of the form of h, S commutes with the maps of $\hat{\mathfrak{M}}$.

We define a natural transformation of functors $U : C | \hat{\mathfrak{M}} \longrightarrow C | \hat{\mathfrak{M}}$ as follows. The homomorphism[1] $U(\Delta_q) : C(\Delta_q) \longrightarrow C(\Delta_q)$ is determined by $U(\Delta_q)(x) = S(x)$ for $x \in X_q, x \neq (0)$; $U(\Delta_q)(0) = 0$. The fact that S commutes with the maps of $\hat{\mathfrak{M}}$ implies that U is a natural transformation of functors. Define $\eta : H_0 C / \hat{\mathfrak{M}} \longrightarrow C_0 / \hat{\mathfrak{M}}$ as follows: $H_0(\Delta_q)$ may be considered in a natural manner as a free group on the generator (0), and $\eta(\Delta_q) : H_0(\Delta_q) \longrightarrow C_0(\Delta_q)$ is determined by $\eta(\Delta_q)(0) = (0) \in C_0(\Delta_q)$. η is clearly a natural transformation of functors.

The conditions satisfied by S insure that U satisfies the conditions of (3.8), and hence C is acyclic on models.

Since S carries degenerate simplexes into degenerate simplexes, it induces a homomorphism $S : C_r(\Delta_q)_N \longrightarrow C_{r+1}(\Delta_q)_N$,

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[1] of modules; $U(\Delta_q)$ does not preserve gradation nor commute with d. Cf. footnote on p. 3-4 .

and the transformation $U': C_N/\hat{\mathfrak{m}} \longrightarrow C_N/\hat{\mathfrak{m}}$ in which $U'(\Delta_q) = S:C(\Delta_q)_N \longrightarrow C(\Delta_q)_N$ is a natural transformation of functors. The conditions on S insure that U' satisfies the conditions of (3.8), and hence C_N is also acyclic on models. Let H denote the homology functor obtained from the chain functor C, H_N that obtained from C_N.

Theorem 3.15 : $\Phi : C \longrightarrow C_N$ induces a natural equivalence $\Phi' : H \longrightarrow H_N$.

Proof: $H_0|\bar{\mathfrak{m}} = (H_N)_0|\bar{\mathfrak{m}}$, so that in theorems (3.11) and (3.13) we may take T to be the identity. By 3.11 we have natural transformations of functors

$$\begin{tikzcd} C \arrow[r, shift left, "\phi"] & C_N \arrow[l, shift left, "\psi"] \end{tikzcd}$$

which induce the identity on $H_0|\bar{\mathfrak{m}} = (H_N)_0\bar{\mathfrak{m}}$. The composition $\psi\phi$ is a natural transformation of C into itself which induces the identity on $H_0|\bar{\mathfrak{m}}$; therefore by (3.13), $\psi\phi$ is homotopic to the identity transformation of C. Similarly $\phi\psi$ is homotopic to the identity transformation of C_N. Hence ϕ induces a natural equivalence $\Phi' : H \longrightarrow H_N$. But by (3.13) Φ is homotopic to ϕ, and hence also induces Φ'. This completes the proof of the theorem.

Consider the category $\mathfrak{a} \times \mathfrak{a}$, having as objects pairs (K,L) of semi-simplicial complexes, and as maps pairs

$(f,g):(K,L) \longrightarrow (P,Q)$, where $f:K \longrightarrow P, g:L \longrightarrow Q$ are maps. The models are to be pairs (Δ_p, Δ_q) of models from $\mathcal{A}$. We give three methods for defining degeneracy in $\mathcal{A} \times \mathcal{A}$, and thus turning it into a category with models. Let $(u,v):(\Delta_p, \Delta_q) \longrightarrow (K,L)$ be a map in $\mathcal{A} \times \mathcal{A}$:

(i) ("Tensor product"): $\alpha(u,v) = (\alpha u, \alpha v)$; $\beta(u,v) = (\beta u, \beta v)$.

(ii) ("Cartesian product") $\alpha(u,v)=(1,1)$ $\beta(u,v) = (u,v)$, unless $p = q$; in this case, let $u(0,\ldots,p) = a \in K$, $v(0,\ldots,p) = b \in L$. Then $a \times b = s_{i_r} \ldots s_{i_1}(a' \times b')$, where $i_r > \ldots > i_1$ and $a' \times b'$ is non-degenerate in $K \times L$; furthermore, this decomposition is unique. Define $\alpha(u,v) = (\bar{u},\bar{v}):(\Delta_p, \Delta_p) \longrightarrow (\Delta_{p-r}, \Delta_{p-r})$, where $\bar{u} = \bar{v}$ is determined by $\bar{u}(0,\ldots,p) = s_{i_r} \ldots s_{i_1}(0,\ldots,p-r)$, and $\beta(u,v) = (u',v'):(\Delta_{p-r}, \Delta_{p-r}) \longrightarrow (K,L)$, where u' and v' are determined by $u'(0,\ldots,p-r) = a'$, $v'(0,\ldots,p-r) = b'$.

(iii) If neither of the above systems of degeneracy is postulated, we assume that $\mathcal{A} \times \mathcal{A}$ has no degeneracy; i.e. $\alpha(u,v) = (1,1)$, $\beta(u,v) = (u,v)$.

We wish to determine the relation between the two functors $C_N^{\otimes}, C_N^{\times} : \mathcal{A} \times \mathcal{A} \longrightarrow dg$ defined by

$$C_N^{\otimes}(K,L) = C(K)_N \otimes C(L)_N$$

$$C_N^{\times}(K, L) = C(K \times L)_N$$

(1) $C_N^{\otimes}$ is representable using tensor product degeneracies. For, $C(K)_N \otimes C(L)_N$ is free abelian, and a typical generator is $\sigma \otimes \tau$, where $\sigma \in K_p$, $\tau \in L_q$ are non-degenerate. Let $u: \Delta_p \longrightarrow K, v: \Delta_q \longrightarrow L$ be the unique maps determined by $u(0,\ldots,p) = \sigma, v(0,\ldots,q) = \tau$. Define a natural transformation of functors $\chi : C_N^{\otimes} \longrightarrow \hat{C}_N^{\otimes}$ by

$$\chi(K,L)(\sigma \otimes \tau) = ((0,\ldots,p) \otimes (0,\ldots,q),(u,v) \in (C(\Delta_p)_N \otimes C(\Delta_q)_N,(u,v)) \subset \hat{C}_N^{\otimes}(K,L).$$

Clearly $\Gamma\chi$ = identity, so that χ is a representation.

(2) C_N^{x} is representable using Cartesian product degeneracies : $C(K,L)_N$ is a free abelian group, and a typical generator is a non-degenerate simplex $\sigma x \rho$, where $\sigma \in K_p, \rho \in L_p$. Let u,w be the maps corresponding to σ, ρ respectively. Define a natural transformation of functors $\chi : C_N^{x} \longrightarrow \hat{C}_N^{x}$ by $\chi(K,L)(\sigma x \rho) =$

$$((0,\ldots,p)x(0,\ldots,p),(u,w)) \in (C(\Delta_p x \Delta_p)_N,(u,w)) \subset \hat{C}_n^{x}(K,L)$$

Then $\Gamma\chi$ = identity, so that χ is a representation.

(3) $C_N^{\otimes}$ is acyclic on models, using either system of degeneracy. Consider first the tensor degeneracies. $H_0(C(\Delta_p)_N \otimes C(\Delta_q)_N$ is an infinite cyclic group cyclic group, for which we may take as generator the class of $(0) \otimes (0)$. $\eta : H_0 C_N^{\otimes} | \hat{\mathcal{M}} \longrightarrow (C_N^{\otimes})_0 | \hat{\mathcal{M}}$ is then defined by $\eta(\Delta_p, \Delta_q)((0) \otimes (0)) = (0) \otimes (0)$. Recall that we defined a contracting homotopy $U' : C(\Delta_q)_N \longrightarrow C(\Delta_q)_N$; we may also define a contracting homotopy $U : C(\Delta_p)_N \otimes C(\Delta_q)_N \longrightarrow C(\Delta_p)_N \otimes C(\Delta_q)_N$

by

$$U(\sigma \otimes \tau) = U'\sigma \otimes \tau + \eta\varepsilon(\sigma) \otimes U'\tau \ .$$

Then $\partial U + U\partial = 1 - \eta\varepsilon$, and $U\eta = 0$. U commutes with the homomorphisms induced by maps of $\hat{\mathfrak{m}}$, and thus defines a natural transformation of functors. Hence, by definition (3.8), $C_N^{\otimes}$ is acyclic on models.

Using cartesian product degeneracies, the corresponding category $\hat{\mathfrak{m}}$ is a subcategory of that obtained from tensor product degeneracies; hence U commutes with the induces homomorphisms in this case also, and $C_N^{\otimes}$ is again acyclic on models.

(4) $C_N^{\times}$ is acyclic on models, using either system of degeneracy. $H_0(C(\Delta_p \times \Delta_q)_N)$ is cyclic infinite, generated by the class of $((0) \times (0))$, and $\eta_\times : H_0 C_N^{\times} | \hat{\mathfrak{m}} \to (C_N^{\times})_0 | \hat{\mathfrak{m}}$ is defined by $\eta_\times(\Delta_p, \Delta_q)((0) \times (0)) = ((0) \times (0))$. Define $S_\times : (\Delta_p \times \Delta_q)_r \to (\Delta_p \times \Delta_q)_{r+1}$ by $S_\times((m_0, \ldots, m_r) \times (\ell_0 \ldots \ell_r)) = (0, m_0, \ldots, m_r) \times (0, \ell_0, \ldots, \ell_r)$. $S_\times$ induces $U_\times : C_r(\Delta_p \times \Delta_q)_N \to C_{r+1}(\Delta_p \times \Delta_q)_N$ such that $\partial_{r+1} U_\times = U_\times \partial_r$ for $r \geq 1$. Hence $\partial U_\times + U_\times \partial = 1 - \eta_\times \varepsilon_\times$, and $U_\times \eta_\times = 0$. Using tensor product degeneracies, it is clear that $U_\times$ commutes with the homomorphisms induced by maps of $\hat{\mathfrak{m}}$; by the argument of the previous paragraph, the same holds true using Cartesian product degeneracies. Hence $C_N^{\times}$ is acyclic on models in either case.

Now, using tensor product degeneracies so that $C_N^{\otimes}$ is representable, we apply theorem 3.11 with

$T: H_0 C_N^{\otimes} | \overline{\mathcal{M}} \longrightarrow H_0 C_N^{x} | \overline{\mathcal{M}}$ the natural equivalence defined by $T(\Delta_p, \Delta_q)((0) \times (0)) = ((0) \times (0))$, to obtain a natural transformation of functors

$$\nabla : C_N^{\otimes} \longrightarrow C_N^{x}$$

Similarly, using Cartesian product degeneracies and the equivalence $T' : H_0 C_N^{x} | \overline{\mathcal{M}} \longrightarrow H_0 C_N^{\otimes} | \overline{\mathcal{M}}$ defined by $T'(\Delta_p, \Delta_q)((0) \times (0)) = ((0) \otimes (0))$, we obtain a natural transformation of functors

$$f : C_N^{x} \longrightarrow C_N^{\otimes} \quad .$$

Thus ∇f is a natural transformation of the functor C_N^{x} into itself. If we use the system of Cartesian product degeneracies, then C_N^{x} is representable; and since ∇f induces the transformation $TT' = 1$ in $H_0 C_N^{x} | \overline{\mathcal{M}}$, by theorem 3.13 there is a homotopy between ∇f and the identity transformation of C_N^{x}. The fact that such a homotopy is (by definition) natural will be used in later proofs. By a completely similar argument, using tensor product degeneracies, we see that $f \nabla$ is homotopic to the identity transformation of $C_N^{\otimes}$, so that ∇ and f are equivalences.

We now wish to find the explicit formulae for ∇ and f, as determined by (3.11). Throughout let u be the map corresponding to $a \in k_r$, v the map corresponding to $b \in L_s$. We first consider ∇ .

Dimension 0: Let $a \in K_0$, $b \in L_0$. Then

$$\nabla(a \otimes b) = \Gamma_x \hat{\eta}_x \hat{T} \hat{\varepsilon} \chi(a \otimes b) = \Gamma_x \hat{\eta}_x \hat{T} \hat{\varepsilon}((0) \otimes (0), (u,v))$$
$$= \Gamma_x((0) \times (0), (u,v)) = a \times b \ .$$

Dimension 1: case 1: Let $a \in K_1$ be non-degenerate, and let $b \in L_0$. Then

$$\begin{aligned}\nabla(a \otimes b) &= \Gamma_x \hat{U}_x \hat{\nabla} \hat{\partial} \chi(a x b) = \Gamma_x \hat{U}_x \hat{\nabla} \hat{\partial}(\,(0,1) \otimes (0), (u,v)\,) \\ &= \Gamma_x \hat{U}_x \hat{\nabla}(\,(1) \otimes (0) - (0) \otimes (0), (u,v)) = \Gamma_x \hat{U}_x(\,(1) x (0) - (0) x (0), (u,v)\,) \\ &= \Gamma_x(\,(0,1) x (0,0), (u,v)\,) = a \; x \; s_0 b.\end{aligned}$$

case 2: Let $a \in K_0$, and let $b \in L_1$ be non degenerate. Then in a similar fashion

$$\nabla(a \otimes b) = s_0 a \, x \, b\,.$$

Dimension 2: case 1: Let $a \in K_1, b \in L_1$ be non-degenerate. Then

$$\begin{aligned}\nabla(a \otimes b) &= \Gamma_x \hat{U}_x \hat{\nabla} \hat{\partial} \chi(a \otimes b) = \Gamma_x \hat{U}_x \hat{\nabla} \hat{\partial}(\,(0,1) \otimes (0,1), (u,v)\,) \\ &= \Gamma_x \hat{U}_x \hat{\nabla}(\,(1) \otimes (0,1) - (0) \otimes (0,1) - (0,1) \otimes (1) + (0,1) \otimes (0), (u,v)) \\ &= \Gamma_x \hat{U}_x(\,(1,1) x (0,1) - (0,0) x (0,1) - (0,1) x (1,1) + (0,1) x (0,0), (u,v)) \\ &= \Gamma_x(\,(0,1,1) x (0,0,1) - (0,0,1) x (0,1,1), (u,v)\,) \\ &= s_1 a \; x \; s_0 b - s_0 a \; x \; s_1 b.\end{aligned}$$

Similarly we have

case 2: Let $a \in K_0, b \in L_2$ be non-degenerate. Then

$$\nabla(a \otimes b) = s_1 s_0 a \; x \; b\,.$$

case 3: Let $a \in K_2, b \in L_0$ be non-degenerate. Then

$$\nabla(a \otimes b) = a \; x \; s_1 s_0 b.$$

The general formula, which we shall not prove, is the following. If (μ, ν) is a (p,q)-shuffle (cf. appendix 1A), let $\sigma(\mu, \nu)$ be the sign of the permutation $(\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$ of the integers $(0, 1, \ldots, p+q-1)$.

Then for $a \in K_p, b \in K_q$ both non-degenerate,

$$(3.16) \qquad \nabla(a \otimes b) = \sum_{(\mu,\nu)} \sigma(\mu,\nu) s_{\nu_q} \cdots s_{\nu_1} a \; x \; s_{\mu_1} \cdots s_{\mu_1} b\,,$$

the sum being taken over all (p,q)-shuffles.

We now consider $f\colon C(K \times L)_N \longrightarrow C(K)_N \otimes C(L)_N$

Dimension 0: Let $a \in K_0, b \in L_0$. Then, with the appropriate meanings of the functors in this case,

$$f(a \times b) = \Gamma \hat{\eta} \hat{T} \hat{\xi}_x \chi(a \times b) = \Gamma \hat{\eta} \hat{T} \hat{\varepsilon}_x((0) \times (0),(u,v) = \Gamma((0) \otimes (0),(u,v)) = a \otimes b$$

Dimension 1: Let $a \times b$ $(K \times L)$, be non-degenerate. Then

$$\begin{aligned} f(a \times b) &= \Gamma \hat{U} \hat{f} \hat{\partial}_x \chi_x(a \times b) = \Gamma \hat{U} \hat{f} \partial_x((0,1) \times (0,0),(u,v)) \\ &= \Gamma \hat{U} \hat{f}((1) \times (1) - (0) \times (0),(u,v) = \Gamma \hat{U}((1) \otimes (1) - (0) \otimes (0),(u,v)) \\ &= \Gamma(\ (0,1) \otimes (1) + (0) \otimes (0,1),(u,v)\ ) \\ &= \Gamma(\ (0,1) \otimes \partial_0(0,1) + (\partial_1(0,1)) \otimes (0,1),(u,v)\ ) \\ &= a \otimes \partial_0 b + (\partial_1 a) \otimes b. \end{aligned}$$

Dimension 2: Let $a \times b \in (K \times L)_2$ be non-degenerate. Then

$$\begin{aligned} f(a \times b) &= \Gamma \hat{U} \hat{f} \hat{\partial}_x \chi_x(a \times b) = \Gamma \hat{U} \hat{f} \hat{\partial}_x((0,1,2) \times (0,1,2),(u,v)) \\ &= \Gamma \hat{U} \hat{f}((1,2) \times (1,2) - (0,2) \times (0,2) + (0,1) \times (0,1),(u,v)\ ) \\ &= \Gamma \hat{U}((1,2) \otimes (2) + (1) \otimes (1,2) - (0,2) \otimes (2) - (0) \otimes (0,2) + (0,1) \otimes (1) + \\ &\qquad\qquad (0) \otimes (0,1),(u,v)) \\ &= \Gamma(\ (0,1,2) \otimes (2) + (0,1) \otimes (1,2) + (0) \otimes (0,1,2),(u,v)) \\ &= \Gamma((0,1,2) \otimes \partial_0^2(0,1,2) + \partial_2(0,1,2) \otimes \partial_0(0,1,2) + \partial_2 \partial_1(0,1,2) \otimes (0,1,2),(u,v)) \\ &= a \otimes \partial_0^2 b + \partial_2 a \otimes \partial_0 b + \partial_2 \partial_1 a \otimes b. \end{aligned}$$

The general formula for f, which we shall not prove is the following, where $\tilde{\partial}$ denotes the last face operator in any situation: let $a \times b \in (K \times L)_p$; then

$$(3.17) \qquad f(a \times b) = \sum_{i=0}^{p} (\tilde{\partial})^i a \otimes (\partial_0)^{p-i} b.$$

Note that this is the formula for the Alexander-Cech-Whitney cup product; it is not symmetric with respect to permuting K and L. It is routine to verify that

$$(3.18) \qquad f\nabla = \text{identity}$$

<u>Lemma 3.19:</u> ∇ is associative; i.e. the following diagram commutes, where the isomorphism is the natural one:

$$\begin{tikzcd}
(C(K)_N \times C(L)_N) \times C(M)_N \arrow[r, "\nabla\otimes 1"] \arrow[dd, "\approx"'] & C(K\times L)_N \otimes C(M)_N \arrow[dr, "\nabla"] & \\
& & C(K\times L\times M)_N \\
C(K)_N \otimes (C(L)_N \otimes C(M)_N) \arrow[r, "1\otimes\nabla"] & C(K)_N \otimes C(L\times M)_N \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

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