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ALGEBRAIC TOPOLOGY

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Introduction. One of the main problems of algebraic topology is that of classifying spaces and classifying continuous transformations of one space into a second. The spaces with which we shall deal are "nice" spaces; they often are manifolds or spaces which can be triangulated (i.e., can be broken up into a finite number of arcs, triangles, tetrahedra, etc... Two spaces will be put in the same class if they are homeomorphic, or, more generally, if they have the same homotopy type, in the sense described in §2 below. Two transformations of one space into another will be put in the same class if they are homotopic; that is, if the first transformation can be continuously deformed into the second.

Actually, one develops two methods of classification--homotopy and homology. Let, for instance, C_1 and C_2 be two continuous images of a circle in a space Y. Then C_1 and C_2 are homotopic if C_2 (regarded as a rubber-band) can be slipped continuously through the space to the position of C_1; while C_1 and C_2 are homologous if C_1 and C_2 are together the "boundary" of a two-dimensional piece of the space Y. After the concepts are developed, we will be able to attach to each space X a homotopy group $\pi_q(X)$ in every dimension

$q = 1, 2, \ldots$ and a homology group $H_q(X)$ in every dimension 0, 1, 2, ... Furthermore, to each transformation f of X into Y we shall associate a definite homomorphism

$$f_*: \pi_q(X) \rightarrow \pi_q(Y)$$

between the corresponding homotopy groups, and a similar induced homomorphism

$$f_*: H_q(X) \rightarrow H_q(Y)$$

for the homology groups in each dimension. These groups and homomorphisms are thus algebraic invariants associated with topological objects; their study and exploration is the main object of this course.

Chapter 1

THE FUNDAMENTAL GROUP

1. Homotopy. The classification of the continuous transformations of a space X into a space Y depends essentially upon the notion of homotopy. Intuitively speaking, two continuous maps of X into Y are said to be homotopic if it is possible to continuously deform the first into the second. To formulate this precisely, we imagine that this deformation takes place in a unit time interval. The deformation can then be regarded as a continuous map defined in the cartesian product XxI of the space X and the unit interval I, $0 \leq t \leq 1$ on the real t-axis.

DEFINITION: Two continuous maps f_o, f_1: $X \longrightarrow Y$ are homotopic (in symbols, $f_o \simeq f_1$) if and only if there is a continuous map F: $X \times I \longrightarrow Y$ of the cartesian product of X by the unit interval $I = [0,1]$ on the t-axis into the space Y such that

$$F(x,0) = f_o(x) \qquad F(x,1) = f_1(x). \tag{1.1}$$

The condition (1.1) states that the homotopy $F(x,t) \in Y$ starts, for t = 0, with the initial map f_o and ends, for t = 1, with the final map f_1.

To illustrate, it is convenient to use the identity map $i = i_X$ of any space X on itself (with $i(x) = x$ for $x \in X$), and the constant maps c: $X \longrightarrow Y$, which carry every point of X into one and the same point of Y. If X is the unit interval $0 \leq x \leq 1$, the identity map i: $X \longrightarrow X$ is homotopic to the constant map c: $X \longrightarrow X$ with $c(x) = 0$

for $x \in X$, the requisites for a homotopy $F: X \times I \to X$ being defined as

$$F(x,t) = (1-t)x, \qquad 0 \leq x \leq 1, \qquad 0 \leq t \leq 1.$$

Then $F(x,0) = x$, $F(x,1) = 0$. This homotopy deforms each point of X at uniform velocity toward the point $x = 0$. A similar argument shows that the identity map of an open interval, as of the whole real axis, is homotopic to a constant map. Spaces X for which the identity map $i_X: X \to X$ is homotopic to a constant map of X into X are said to be contractible. Thus intervals on the real axis are contractible spaces.

If S' is the circle, regarded as the set of complex numbers $\underline{z}$ of absolute value 1, then for each integer $\underline{n}$ the function $f_n(z) = z^n$ defines a continuous map of S' on S' which "wraps S' $\underline{n}$ times around itself". It is intuitively clear that two such maps of f_n and f_m with different integers $\underline{m}$ and $\underline{n}$ cannot be homotopic. This will be subsequently proved, together with the fact that any map $g: S' \to S'$ is homotopic to exactly one of the maps f_n. This means that we can associate with any $\underline{g}$ the number $\underline{n}$ of times which $\underline{g}$ wraps the circle around itself; this number is known as the Brouwer degree of $\underline{g}$. A similar result holds for the maps of the n-sphere on itself.

THEOREM 1.1. The relation of homotopy between maps $f: X \to Y$ is reflexive, symmetric and transitive.

PROOF: The relation is reflexive; for any $f: X \to Y$ is homotopic to itself under the manifestly continuous homology F defined by

$$F(x,t) = f(x), \qquad 0 \leqq t \leqq 1, \qquad x \in X.$$

The relation is symmetric, for if $F: f_0 \simeq f_1$, then $G: f_1 \simeq f_0$, where G is the homotopy defined by

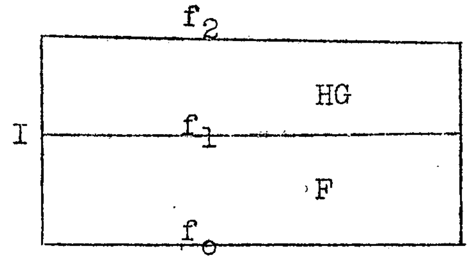
$$G(x,t) = F(x, 1-t), \qquad 0 \leqq t \leqq 1.$$

Here G is continuous, for it is the composite Foh, where h is the continuous map

$$h(x,t) = (x, 1-t), \qquad x \in X, \qquad 0 \leqq t \leqq 1$$

of XxI into itself.

The relation is transitive, for if $F: f_0 \simeq f_1$ and $G: f_1 \simeq f_2$ then we can define a homotopy $H: f_0 \simeq f_2$ by the scheme indicated in the diagram (for λ the unit interval); the bottom edge of ΛxI is mapped by f_0, the top by f_2, the middle segment by f_1, and the two



halves by the given homotopies F and G, squeezed down. Formally, we define H by

$$H(x,t) = F(x,2t), \qquad x \in X, \qquad 0 \leqq t \leqq 1/2,$$
$$H(x,t) = G(x,2t-1), \qquad x \in X, \qquad 1/2 \leqq t \leqq 1.$$

The two definitions agree at the common points with t = 1/2, for $F(x, 2.1/2) = F(x,1) = f_1(x) = G(x,0) = G(x,2.1/2 - 1)$. Also $H(x,0) = F(x,0) = f_0(x)$; $H(x,1) = G(x,1) = f_2(x)$, and H is continuous, by the previous continuity theorem, since it is compounded from two continuous functions on the two closed subsets $X\times[0,1/2]$ and $X\times[1/2,1]$ of $X\times I$.

THEOREM 1.2. If $f_0 \simeq f_1: X \longrightarrow Y$ and $g_0 \simeq g_1: Y \longrightarrow Z$, then the composite maps g_0f_0 and g_1f_1 of X into Z are homotopic.

PROOF: In view of the transitivity of the relation of homotopy, it will suffice to prove the special cases $g_0f_0 \simeq g_0f_1$ and $g_0f_1 \simeq g_1f_1$. For the first of these, we are given a homotopy $F: f_0 \simeq f_1$; the composite $H = g_0 \circ F$ is a continuous map of $X\times I$ to Z, and $H(x,0) = g_0(F(x,0)) = g_0f_0(x)$, $H(x,1) = g_0(F(x,1)) = g_0f_1(x)$. Hence $H: g_0f_0 \simeq g_0f_1$. For the second, we are given a homotopy $G: g_0 \simeq g_1: Y \longrightarrow Z$. Define a continuous map K of $X\times I$ to Z by setting

$$K(x,t) = G(f_1(x),\ t), \qquad x \in X, \qquad 0 \leqq t \leqq 1.$$

Then $K(x,0) = G(f_1(x),\ 0) = g_0(f_1(x))$ and $K(x,1) = G(f_1(x),\ 1) = g_1(f_1(x))$. Hence $K: g_0f_1 \simeq g_1f_1: X \longrightarrow Z$, as required.

COROLLARY 1.3. If Y is contractible, then any two maps $f_0,\ f_1: X \longrightarrow Y$ are homotopic.

PROOF: By the assumed contractibility, the identity map $i: Y \longrightarrow Y$ is homotopic to a constant map $c: Y \longrightarrow Y$. By the theorem $f_0 = if_0 \simeq cf_0: X \longrightarrow Y$, and similarly $f_1 \simeq cf_1: X \longrightarrow Y$. But cf_0 and cf_1 will map all points of X into one and the same point of Y, hence $cf_0 = cf_1$ and, by transitivity, $f_0 \simeq f_1$.

If $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$, we denote by $f\times f'$ the map of $X\times X'$ into $Y\times Y'$ defined by

$$(f\times f')(x,x') = (f(x),\ f'(x')), \qquad x \in X, \qquad x' \in X'.$$

THEOREM 1.4. If $f_0 \simeq f_1 : X \to Y$ and $f'_0 \simeq f'_1 : X' \to Y'$ then $f_0 x f'_0 \simeq f_1 x f'_1 : X x X' \to Y x Y'$.

PROOF: We are given homotopies $F: f_0 \simeq f_1$ and $F': f'_0 \simeq f'_1$. The required homotopy H, as a mapping of $(X x X')xI$ onto YxY', is defined by

$$H(x,x',t) = (F(x,t), F'(x',t)), \quad 0 \leq t \leq 1, \; x \in X, \; x' \in X'.$$

Clearly $H(x,x',0) = (F(x,0), F'(x',0)) = (f_0(x), f'_0(x')) = (f_0 x f'_0)(x,x')$ with a similar result at $t = 1$. The same theorem holds for cartesian products of more than two spaces.

COROLLARY 1.5. The cartesian product of two contractible spaces is contractible.

PROOF: By assumption, the identity maps i, i' of X and X', respectively, are homotopic to constant maps c, c'. By the Theorem, $ixi' \simeq cxc' ; XxX' \to XxX'$. But ixi' is the identity map of XxX', and cxc' is a constant map, carrying all of XxX' into the point $(c(X), c'(X'))$.

From the example above it now follows that Euclidean n-space, and any (open or closed) n-dimensional cube Ix...xI is contractible.

Example 1.6. Let X be the unit circular disc with its center point removed and $d: X \to X$ the mapping which carries each point of X radially onto a point on the circumference C of the disc. Then d is homotopic to the identity map. Indeed, if the points of X are labelled by polar coordinates (r,Θ), then X is homeomorphic to the cartesian product $(0,1]xS'$ of the half-open interval J; $0 < r \leq 1$ with the circle S'. The identity map of $(0,1]$ is homotopic to the constant map c with $c(r) = 1$. Hence by the theorem, the identity map is $i = i_j \times i_{S'}$, and $c \times i_{S'}$ is exactly the map d. The homotopy $F: i \simeq d$ is given explicitly by

$$F(r,z,t) = ((1-t)r+t,\ z), \quad 0 < r \leq 1, \quad z \in S'.$$

Note that during the whole homotopy, the points on the circumference ($r = 1$) stay fixed.

More generally, let S be a subspace of X. A mapping $f: X \longrightarrow X$ with $f(X) \subset S$ and $f(s) = s$ for $s \in S$ is called a <u>retraction</u> of X onto S'. The subspace is said to be a <u>deformation retract</u> of X if there is a map (a "deformation retraction") $F: X \times I \longrightarrow X$ with

$$(1.2) \qquad F(x,0) = x, \quad F(x,1) \in S, \quad F(s,t) = s, \quad (s \in S,\ 0 \leqq t \leqq 1).$$

These equations state that F establishes a homotopy of the identity i_X with a retraction $f(x) = F(x,1)$ of X onto S, and that the points of S are not moved by the homotopy. Thus in particular the circumference of a circular disc with center removed is a deformation retract.

One may also show that a circular ring (the set of points between one or two concentric circles in the plane) has either of these circles as a deformation retract. We cite without proof the formal results

<u>THEOREM 1.7</u>. If T is a deformation retract of S and S a deformation retract of X, then T is a deformation retract of X.

<u>THEOREM 1.8</u>. If S, S' are deformation retracts of X, X', respectively, then $S \times S'$ is a deformation retract of $X \times X'$.

2. Homotopy Type. Two spaces which are homeomorphic are topologically indistinguishable. For the purposes of algebraic topology, it is convenient to have a still wider classification of spaces.

DEFINITION: Two spaces X and Y are of the same homotopy type if there are maps $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ such that both the compositions fog and gof are homotopic to identity maps; i.e., such that

$$(2.1) \qquad i_Y \simeq fog: Y \longrightarrow Y, \qquad i_X \simeq gof: X \longrightarrow X.$$

A map f for which there exists such a g is called a homotopy equivalence (of X to Y), and g is homotopic inverse of f.

A homeomorphism $f: X \longrightarrow Y$ is trivially a homotopy equivalence, with f^{-1} as a homotopy inverse; hence homeomorphic spaces have the same homotopy type. That the concept is much wider than homeomorphism is illustrated by the

THEOREM 2.1. If $S \subset X$ is a deformation retract of X, then S and X have the same homotopy type.

In particular, the punched circular disc (Example 1.6) has the same homotopy type as its circumference.

PROOF: We shall show that the injection $k: S \longrightarrow X$ with $k(s) = s$ is a homotopy equivalence. By hypothesis, there exists a deformation retraction $F: X \times I \longrightarrow X$ with properties (1.2). Define $f: X \longrightarrow S$ by $f(x) = F(x,1)$. Since $F(s,1) = s$ for $s \in S$, $(fok)(s) = s$, hence fok is trivially homotopic to the identity. The other composite

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The relation "X has the same homotopy type as Y" is manifestly reflexive and symmetric. For the transitivity of this relation consider spaces X, Y, Z and maps

$$f,g,h,k: \quad X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$$

with $i_Y \simeq fog$, $i_X \simeq gof$, $i_Y \simeq koh$, $i_Z \simeq hok$. Then using Theorems (1.1) and (1.2),

$$(hof)o(gok) = ho(fog)ok \simeq hoi_Y ok = hok \simeq i_Z$$

with a similar argument for the other composite.

The invariants of a topological space defined in algebraic topology are invariants of the homotopy type, in the sense that they turn out to be the same for two spaces of the same homotopy type.

3. Arcwise Connectivity. We now turn to the definition of the fundamental group of a space, as perhaps the simplest example of a group associated with a space.

We denote the unit interval on the s-axis, by

$$I_s = \{\text{all real } s,\ 0 \leqq s \leqq 1\}$$

with a similar notation for I_t. A path (also called an arc) in the topological space X is a continuous map $\xi: I_s \to X$ of the unit interal into the space. Note that a path is not the set of points $\xi(I_s)$ in the space X, but is the mapping which associates each value of the parameter s between 0 and 1 in a continuous fashion with the point $\xi(s)$ of this set $\alpha(I_s)$--in other words, a path is not a curve, but a parametrical curve. We call the point $\xi(0) = p$ the <u>start</u> of the path and the point $\xi(1) = q$ the <u>end</u> of the path, and speak of ξ as a path from p to q.

If ξ is a path from p to q and η is a path from q to a third point r, then the <u>product</u> $\xi\eta$ is the path from p to r obtained by traversing first the path ξ, then the path η. Formally, the map $\xi\eta : I_s \longrightarrow X$ is defined by the conditions

$$(3.1)\qquad \begin{aligned} (\xi\eta)(s) &= \xi(2s) && 0 \leqq s \leqq 1/2, \\ (\xi\eta)(s) &= \eta(2s-1) && 1/2 \leqq s \leqq 1; \end{aligned}$$

in other words, the first half of the interval I_s is mapped by ξ (squeezed down to a shorter interval) and the second half of I_s is mapped by η. The product $\xi\eta$ of two paths ξ and η is defined <u>only</u> when the end of the first path ξ coincides with the start of the second path (as in the definition above).

In manipulating this product, it is convenient to replace the unit interval I_s by other closed intervals $J_s = [s_1, s_2]$ on the s-axis. J_s is homeomorphic to I_s under the explicit (affine) mapping $\Theta_{J|I} : J_s \longrightarrow I_s$ defined by

$$(3.2)\qquad \Theta(s) = (s-s_1)/(s_2-s_1), \qquad s_1 \leqq s \leqq s_2,$$

and we say that the path $\xi : I_s \longrightarrow X$ can be "shrunk" to the map $\xi' : J_s \longrightarrow X$ given as $\xi' = \xi\Theta$. The product $\xi\eta$ is then described as the path

$$0 \xrightarrow{\ \xi'\ } 1/2 \xrightarrow{\ \eta'\ } 1 \qquad \longrightarrow X$$

obtained by "hitching together" the paths ξ, η, and shrinking each to an interval of length 1/2.

The <u>inverse</u> ξ^{-1} of a path ξ is the original path traversed backwards; thus if ξ is a path from p to q, then ξ^{-1} is the path from q to p defined by

$$(3.3) \qquad \xi^{-1}(s) = \xi(1-s), \qquad 0 \leqq s \leqq 1.$$

One proves easily that $(\xi\eta)^{-1} = \eta^{-1}\xi^{-1}$.

A space X is said to be <u>arcwise connected</u> if, for each pair of points p and q in X there is a path joining p to q. A space X which is not arcwise connected may be decomposed uniquely into arcwise connected components, as follows. The relation "there exists a path from p to q" on the points p, q of X is a reflexive, symmetric, and transitive relation. It is reflexive because the constant path is a path from p to p, symmetric because if ξ joins p to q, then ξ^{-1} joins q to p; and transitive because if ξ joins p to q and η joins q to r, then $\xi\eta$ joins p to r. Subdivide the space X into its equivalence classes C with respect to this relation; in other words, place two points p and q of X in the same class C if and only if there is a path in X from p to q. Then each subspace C is an arcwise connected space, for every point on a path ξ from p to q is clearly joined by a path (namely, part of the path ξ) to p, hence the path ξ in X is also a path in the subspace C. Call the sets C the <u>arc-components</u> of X.

We have proved

PROPOSITION 3.1. Any space X is the union of its (disjoint) arc-components. Any arc-component of X is arcwise connected, and any arcwise connected subspace of X is contained in an arc-component of X.

The requirement of arcwise connectivity is stronger than that of connectivity, defined as usual in terms of a decomposition of X (262, §12).

PROPOSITION 3.2. An arcwise connected space X is connected.

PROOF: If X is not connected, there is a decomposition $X = U \cup V$ of X into disjoint non-void subsets U and V both open in X (and hence both closed in X). Choose points $p \in U$ and $q \in V$. By assumption, there is a path $\varphi : I_s \longrightarrow X$ joining p to q. Since φ is continuous, the inverse images $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are open subsets of the interval I_s, which are disjoint and which yield a decomposition $I_s = \varphi^{-1}(U) \cup \varphi^{-1}(V)$. This contradicts the fact that the interval I_s is connected (262, Theorem 12.3).

The converse of Proposition 3.1 is not true, however.

PROPOSITION 3.3. A connected open subset U of Euclidean n-space is arcwise connected.

PROOF: We first show that any arc-component C of U is open. Let p be a point of C. Since U is open in E^n, there is for any $p \in U$ a positive ε such that U contains the set S of all points q of E^n at distance less than ε from p. The straight line segment from p to any such point q lies entirely in S, hence in U; it may be regarded as a path from p to q in U. Its presence shows that q lies in the same arc-component C of U as does p. Thus this arc-component C contains the whole of S, showing that C is open in E^n, and hence in U.

If now the connected open set U is not arcwise connected, it has two or more arc-components. Let C be one, and D the union of all the other components. By the statement just proved, C and D are both open in U, and $U = C \cup D$ is a decomposition of U, contrary to the assumed connectivity of U.

An n-dimensional manifold M is defined to be a topological i.e. locally Euclidean space in which every point p has an open neighborhood V homeomorphic to the interior of the unit n-sphere (in Euclidean space). The argument of Proposition 3.3 will also show that a connected manifold is arcwise connected.

The properties of connectivity are relevant to algebraic topology because the higher homology and homotopy groups of a space may be regarded as measures of the "higher-dimensional" connectivity of that space.

<u>PROPOSITION 3.4.</u> Any contractible space X is arcwise connected.

<u>PROOF</u>: Since X is contractible, there is by definition a map $F: X \times I_E \to X$ such that $F(x,0) = x$, and $F(x,1) = q_o$, a fixed point of X. Let p be any point of X. The function $F(p,t)$, with p fixed, thus defines a path in X from p to q_o. It follows that any two points of X are connected by a path, as required.

4. <u>The Algebra of Paths</u>. Two paths $\wp_0$, $\wp_1$: $I_s \longrightarrow X$ in λ are said to be homotopic (relative to $0,1 \in I_s$) if the first path can be deformed continuously into the second, <u>leaving the end points fixed</u> during the deformation. In other words, there must exist a continuous mapping

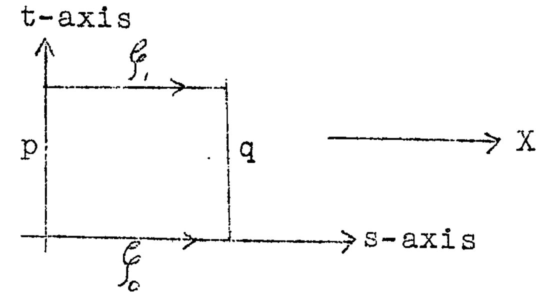
$$(4.1) \qquad F: I_s \times I_t \longrightarrow X, \qquad \text{such that}$$

$$(4.2) \qquad F(s,0) = \wp_0(s), \qquad F(s,1) = \wp_1(s), \qquad 0 \leqq s \leqq 1,$$

$$(4.3) \qquad F(0,t) = F(0,0), \qquad F(1,t) = F(1,0), \qquad 0 \leqq t \leqq 1.$$

We then write $F: \wp_0 \simeq \wp_1$ (rel 0,1), although we shall frequently drop the addendum "relative to 0 and 1". Clearly, two such homotopic paths must start at the same point p, and end at the same point q. Note in particular that this type of <u>relative homotopy</u> (during which the images of s = 0 and s = 1 stay put) is more restrictive than the <u>free homotopy</u> of the maps $\wp_0$, $\wp_1$: $I_s \longrightarrow X$ defined as in §1.

If we consider that the deformation takes place in unit time t, we may regard F(s,t), for fixed t, as the deformed position of the path at time t. Thus condition (4.2) states that the deformation starts with the path $\wp_0$ and ends with $\wp_1$, while (4.2) states that each path during the deformation is a path from p to q. The homotopy F may be pictured in the following way as a map of the unit square $I_s \times I_t$ into λ:

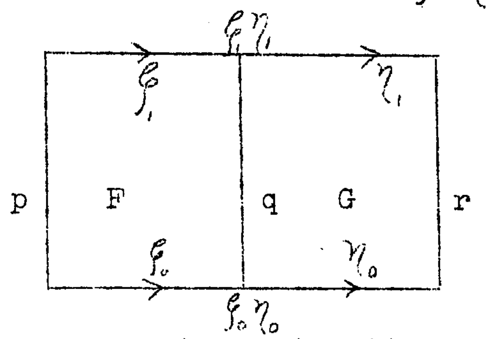


where the letters indicate that the left edge is mapped by the (constant mapping) into p, the right edge into q, and the top and bottom are mapped according to the given paths ξ_1 and ξ_0, respectively.

Since the unit interval I_t is homeomorphic to any other closed interval $J_t = \{\text{all } t, \ t_0 \leqq t \leqq t_1\}$, we may in the definition (4.2) replace I_t by any such interval J_t, replacing $t = 0$ by $t = t_0$ and $t = 1$ by $t = t_1$.

For paths ξ_0, ξ_1 in X the relation " $\xi_0 \simeq \xi_1$, rel (0,1)" is reflexive, symmetric and transitive; the proof is given exactly as in Theorem 1, keeping one eye peeled for the fixed end points. A <u>path-class</u> $x = \{\xi\}$ is an equivalence class of paths under this homotopy relation; that is, it consists of all paths ξ homotopic to a fixed path ξ_0 (rel 0,1). Each path class x has a definite starting point p and a definite end point q (namely; the start and the end of any path $\xi \in x$).

The multiplication of paths induces a multiplication of path-classes. Indeed, if ξ_0 and ξ_1 are homotopic paths from p to q, and η_0, η_1 homotopic paths from q to r, then the products $\xi_0 \eta_0$, $\xi_1 \eta_1$ are homotopic paths from p to r; for the given homotopies $F: \xi_0 \simeq \xi_1$, $G: \eta_0 \simeq \eta_1$ yield a homotopy $H: \xi_0 \eta_0 \simeq \xi_1 \eta_1$ described by the figure



or the (equivalent) equations (c.f. (3,1)).

$$H(s,t) = F(2s,t) \qquad 0 \leqq s \leqq 1/2, \qquad 0 \leqq t \leqq 1$$
$$H(s,t) = G(2s-1,\ t) \qquad 1/2 \leqq s \leqq 1, \qquad 0 \leqq t \leqq 1.$$

Without ambiguity we may define the product $x.y$ of two path classes x and y as the class containing the product $\rho\eta$ of any representative ρ of x by any representative η of y; i.e.,

$$\{\rho\}\{\eta\} = \{\rho\eta\}$$

Similarly, for the inverse, we observe that $\rho_0 \simeq \rho_1$ implies $\rho_0^{-1} \simeq \rho_1^{-1}$, hence that the inverse of a class x may be defined as the class of the inverse of any representative of x.

For each point p in X, we define the unit path ε_p as the constant path at p; i.e., as the path $\varepsilon_p : I_s \longrightarrow X$ with $\varepsilon_p(s) = p$ for all s. The unit path class e_p at p is the class of all paths homotopic to ε_p (rel 0,1).

THEOREM 4.1. The homotopy classes x of paths in a topological space X, under the operations of forming the product xy, the inverse x^{-1}, and the units e_p for $p \in X$, constitute an algebraic system with the following properties:

i) Each path-class x has a start p and an end q in X, and

(4.4) $$e_p x = x = x e_q$$

ii) Each path-class e_p starts and ends at p

iii) The product xy is defined if and only if the end of x is the start of y; the product then starts where x does, and ends where y does.

iv) Given x from p to q, y from q to r and z from r to p'

(4.5) $$x(yz) = (xy)z.$$

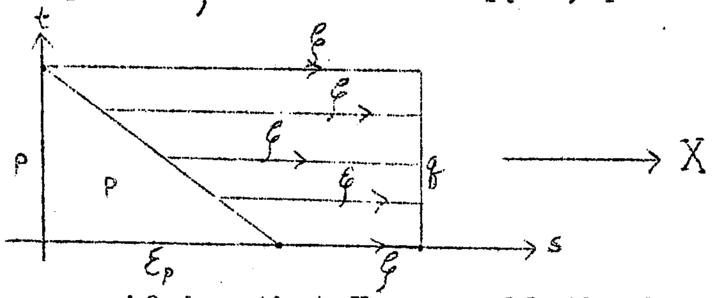
v) If x starts at p and ends at q, then x^{-1} starts at q and ends at p, and

$$(4.6) \qquad xx^{-1} = e_p \qquad\qquad x^{-1}x = e_q$$

These properties are reminiscent of the group axioms; they assert that the classes of paths form a somewhat more general type of algebraic system known as a groupoid.

The only point of interest is the demonstration of the homotopies implied by the equations (4.4), (4.5) and (4.6).

For (4.4), take paths $\xi \in x$, $\varepsilon_p \in e_p$; we must then show that the product $\varepsilon_p \xi$ is homotopic to ξ. The homotopy H, pictured by



is defined explicitly by specifying that H maps all the lower left triangle into p,

$$H(s,t) = p, \qquad 2s \leqq 1-t,$$

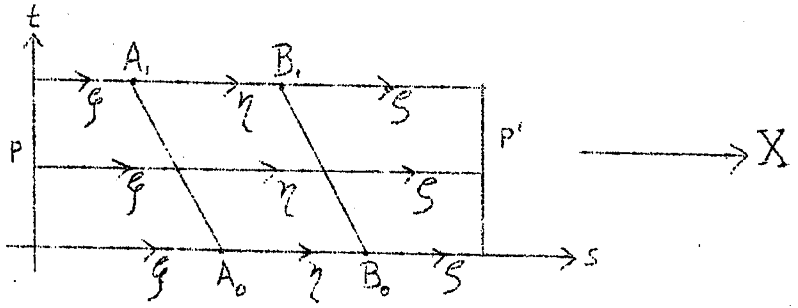
and that each horizontal line in the remainder of the square is mapped by the mapping ξ, suitably shrunk:

$$H(s,t) = \xi(2s-1+t/1+t) \qquad 2 \geqq 2s \geqq 1-t.$$

The two definitions agree when $2s = 1-t$; since each partial H is clearly continuous, the whole H is continuous (e.g., the piecewise-continuity theorem).

To prove (4.5), choose paths $\xi \in x$, $\eta \in y$, $\zeta \in z$. Then the product x(yz) is represented by the path $\xi(\eta\zeta)$, which by definition consists of ξ' (shrunk to $0 \leqq s \leqq 1/2$), η' (shrunk to

$1/2 \leq s \leq 3/4$) and ζ' (shrunk to $3/4 \leq s \leq 1$). The other product is represented by $(\rho\eta)\zeta$, with ρ shrunk to $0 \leq s \leq 1/4$, etc. We require the homotopy pictured by



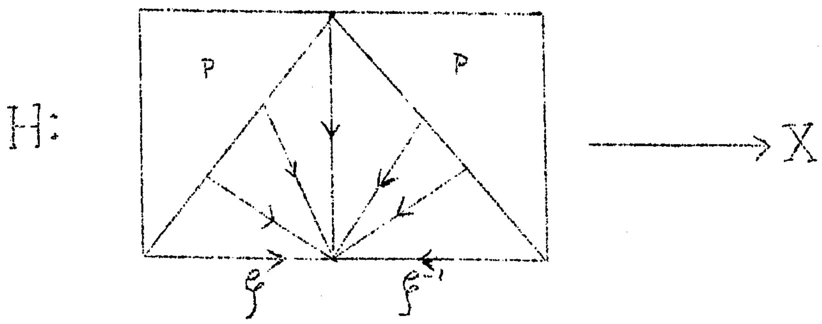
and described (precisely) as follows: subdivide the unit (s,t)-square by lines A_oA_1 and B_oB_1, where

$$A_o = (0, 1/2) \qquad B_o = (0, 3/4),$$
$$A_1 = (1, 1/4), \qquad B_1 = (1, 1/2).$$

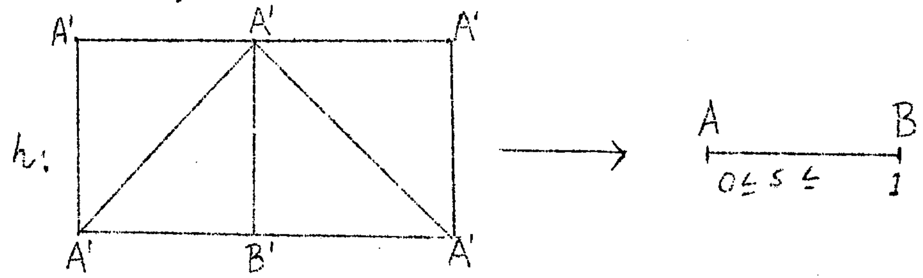
Map A_oA_1 into q, B_oB_1 into r. Map each horizontal interval of the section to the left of A_oA_1 by ρ (suitably shrunk), map each horizontal interval in the middle section by η', and each interval in the right section by ζ.

The explicit formula is easily obtainable and entirely uninteresting.

For the first half of (4.2) we give only the picture of the homotopy H as



in which each indicated directed segment is mapped by a (shrunken) ρ into X, while the remaining points are mapped into p. We may also describe H as the composite $H = \xi \circ h$, where h is a mapping of the square into the interval, defined so that



each point A' (or B') is mapped on A (or B), each segment A'B' (or AA') is mapped linearly on AB (or on A), with the triangles appropriately collapsed (see chap 2).

Any continuous map f: $X \longrightarrow Y$ induces a corresponding homomorphism of the groupoid of path classes of X into the groupoid of path classes in Y. Indeed, if $\xi: I_s \longrightarrow X$ is a path in X, the composite map $f \circ \xi: I_s \longrightarrow Y$ is a path in Y, while if $F: \xi_0 \simeq \xi_1$ (rel 0,1) is a homotopy between paths in X, the composite $f \circ F$ is a homotopy $f_0 \circ \xi_0 \simeq f_0 \circ \xi_1$ between the corresponding paths in Y. Thus, if $x = \{\xi\}$ is any homotopy class of paths in X, we may define a homotopy class $f_*(x)$ of paths in Y unambiguously by the equation

$$f_*(\{\xi\}) = \{f \circ \xi\} \qquad \xi \text{ a path in } X,\ f: X \longrightarrow Y. \tag{4.7}$$

We call $f_*(x)$ the class-composition of f and the class x.

<u>THEOREM 4.2</u>. A continuous map $f: X \longrightarrow Y$ induces by the "composition" (4.7) a transformation $x \longrightarrow f_*(x)$ of path classes in X into path classes in Y, which is a homomorphism in the sense that the following properties all hold.

$$f_*(xy) = (f_*x)(f_*y) \qquad \text{if } xy \text{ is defined,} \tag{4.8}$$

$$f_*(x^{-1}) = (f_*x)^{-1}, \tag{4.9}$$

(4.10) $\quad f_* e_p = e_{f(p)} \qquad p \in X$

If also $g: Y \to Z$, then $(fog)_* = f_* o g_*$.

PROOF: The assertions (4.8), (4.9) and (4.10) follow at once from the identities

(4.8') $\quad fo(\xi\eta) = (fo\xi)(fo\eta), \qquad \xi \in x, \quad \eta \in y,$

(4.9') $\quad fo\xi^{-1} = (fo\xi)^{-1}, \qquad \xi \in x$

(4.10') $\quad fo\varepsilon_p = \varepsilon_{f(p)}, \qquad p \in X,$

for representative paths in the given path classes. Each of these identities is proved directly by the relevant definitions. Thus, in (4.9'), by the definition (3.7) of ξ^{-1},

$$[fo\xi^{-1}](s) = f[\xi^{-1}(s)] = f[\xi(1-s)]$$
$$= (fo\xi)(1-s) = (fo\xi)^{-1}(s),$$

for all s in the unit interval I_s. The final assertion of the theorem is immediate, by the definition (4.7).

The algebraic system of homotopy classes of paths is "too big" --it contains a unit element e_p for each point $p \in X$, and many other elements besides. We now reduce this system to a smaller one, the Fundamental group of X--also called the Poincaré group of X.

5. <u>The Fundamental Group</u>. Assume now that X is an arcwise connected space, and choose a point $p_o \in X$, to be called the <u>base point</u> of X. Theorem (4.1) then shows that the homotopy class x of those paths which both start and end at p_o is a group, with $e_{p_o} = 1$ as identity, and x^{-1} as inverse.

DEFINITION 5.1. The fundamental group $\pi_1(X, p_o)$ of an arcwise connected space X relative to a chosen base point $p_o \in X$ is the group of homotopy classes of paths in X starting and ending at p_o, under the multiplication induced by the product $\xi\eta$ of two paths.

Paraphrasing the definition, we may say that an element of $\pi_1(X, p_o)$ is a closed path ξ in X, starting and ending at p_o; that two such paths are equal (as elements of the fundamental group) if one can be deformed continuously into the other, holding the end points at p_o during this deformation, and that the product $\xi\eta$ of two paths is the path obtained by following first ξ, then η.

THEOREM 5.2. If p_o and q_o are two points in an arcwise connected space X, the fundamental groups $\pi_1(X, p_o)$ and $\pi_1(X, q_o)$ with base points at p_o and q_o, respectively, are isomorphic. Specifically, each homotopy class u of paths from q_o to p_o yields an isomorphism

(5.1) $\quad C_u\colon \pi_1(X, p_o) \longrightarrow \pi_1(X, q_o)$

given by the formula ("conjugation")

(5.2) $\quad C_u(x) = u\,x\,u^{-1} \qquad x \in \pi_1(X, p_o)$

PROOF: If x is a path class from p_o to p_o, uxu^{-1} is a path class from q_o to q_o; hence C_u does map $\pi_1(X, p_o)$ into $\pi_1(X, q_o)$. By the laws for the algebra of classes of paths,

$$C_u(x_1x_2) = u(x_1x_2)u^{-1} = ux_1e_px_2u^{-1} = (ux_1u^{-1})(ux_2u^{-1})$$
$$= [C_ux_1][C_ux_2]$$

Hence C_u is a group homomorphism. If u^{-1} is the path class inverse to u, then $C_{u^{-1}}$ is a homomorphism of $\pi_1(X,q_o)$ into $\pi_1(Y,p_o)$, and

$$C_{u^{-1}}(C_u x) = C_{u^{-1}}(uxu^{-1}) = u^{-1}uxu^{-1}u = x.$$

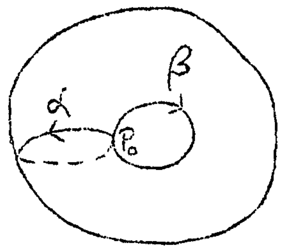
Therefore $C_{u^{-1}} o C_u$, and likewise $C_u o C_{u^{-1}}$, is the identity homomorphism, so that C_u is an isomorphism onto, as asserted in (5.1).

If u, v are two paths in X with uv defined, one readily shows that

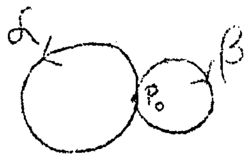
$$(5.3) \qquad C_u o C_v = C_{uv}.$$

Because the isomorphism C_u may depend upon the choice of the class y u it is usually unsatisfactory to speak of "the" fundamental group of a space, without specifying the base point to be used in its definition. Because this isomorphism C_u cannot be defined unless there is at least one path from p_o to q_o, the fundamental group of a space X is not defined unless the space is arcwise connected--indeed, the various arc components of a general space will usually have essentially fundamental groups.

We are not yet in a position to effectively determine the fundamental groups of sample spaces, but we may state without proof that the fundamental group of the circle is an infinite cyclic group, with generator the path determined by the mapping $s \rightarrow e^{2\pi is}$ of the interval onto the circle (regarded as the set of complex numbers of absolute value 1). The fundamental group of the torus is the free abelian group with two generators a and b, given by paths shown below.



In general the fundamental group is not abelian; this is the case, for instance with the fundamental group of the space obtained by joining two circles at a point



The paths α, β do not commute.

<u>THEOREM 5.3.</u> If X and Y are arcwise connected spaces and p_o a point of X, each continuous $f: X \longrightarrow Y$ induces a homomorphism

$$(5.3) \qquad f_*: \pi_1(X,p_o) \longrightarrow \pi_1(Y,f(p_o))$$

of the fundamental group of X at p_o onto the fundamental group of Y, at $f(p_o)$. Here f_* is defined, for each class x of paths at x, by class composition as in (4.7). If two maps f_o, $f_1: X \longrightarrow Y$ with $f_o(p_o) = f_1(p_o)$ are homotopes in such a way that $f_o(p_o)$ is left fixed during the homotopy, the induced homomorphisms f_{o*} and f_{1*} are identical.

<u>PROOF</u>: Theorem (4.2) yields the homomorphism (5.3) at once. As for the second assertion, we are given a homotopy $F: f_o \simeq f_1$; that is, a mapping $F: X \times I_t \longrightarrow Y$ with

$$F(p,0) = f_o(p) \qquad F(p,1) = f_1(p), \qquad p \in X$$

with the special property that $f_o(p_o)$ stays fixed during the homotopy; i.e., that $F(p_o,t) = f_o(p_o) = f_1(p_o)$, $0 \leqq t \leqq 1$.

Let ξ be any path from p_o to p_o in X, in a path-class x. Then $f_{o*}(x)$ is by the definition (4.7) the class of the path $f_o \circ \xi$ in Y, and $f_{1*}(x)$ is the class of the path $f_1 \circ \xi$ in Y. We need show only that these paths in Y are homotopic (rel 0,1); the requisite homotopy is the map $G: I_s \times I_t \longrightarrow Y$ defined by the equations

$$G(s,t) = F(\xi(s),t), \qquad 0 \leqq s \leqq 1, \qquad 0 \leqq t \leqq 1.$$

Indeed, $G(s,0) = F(\xi(s),0) = f_o(\xi(s)) = (f_o \circ \xi)(s)$, so the homotopy G starts with the path $f_o \circ \xi$, and for similar reasons ends with the path $f_1 \circ \xi$. During the homotopy the starting point (s = 0) stays put, for $G(0,t) = F(\xi(0),t) = F(p_o,t) = f_o(p_o)$, with a similar result for s = 1. Hence $G: f_o \circ \xi \simeq f_1 \circ \xi$, rel (0,1).

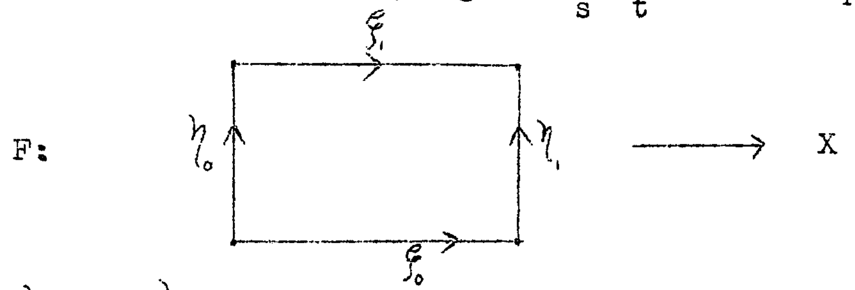
6. <u>The Wandering Base Point</u>. We wish to extend Theorem (5.3) to homotopies F which do not leave the base point $f_o(p_o)$ fixed in Y. This requires a lemma on the "free" homotopy of paths.

<u>LEMMA 6.1</u>. If $F: \xi_o \simeq \xi_1 : I_s \longrightarrow X$ is a <u>free</u> homotopy between the paths ξ_o, ξ_1 in X, and if the paths traced out in X by the end points of the interval I_s under the homotopy are denoted by $\eta_o, \eta_1 : I_s \longrightarrow X$, then the product $\eta_o \xi_1 \eta_1^{-1}$ is defined, and $\xi_o \simeq \eta_o \xi_1 \eta_1^{-1}$ (rel 0,1).

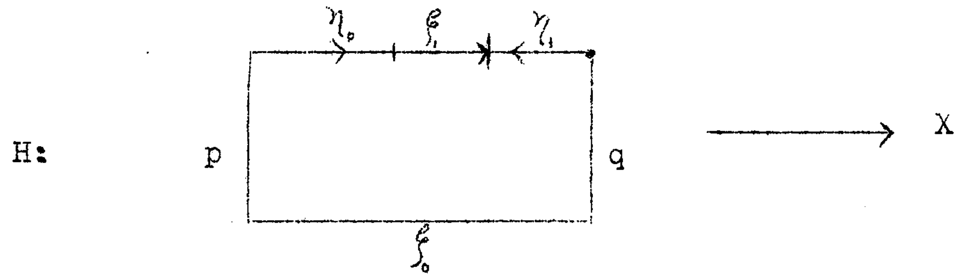
<u>PROOF</u>: The paths η_o, η_1 are defined by the equations

$$\eta_o(s) = F(0,s), \quad \eta_1(s) = F(1,s), \qquad 0 \leqq s \leqq 1.$$

The given homotopy F is a mapping of $I_s x I_t$ onto X represented as

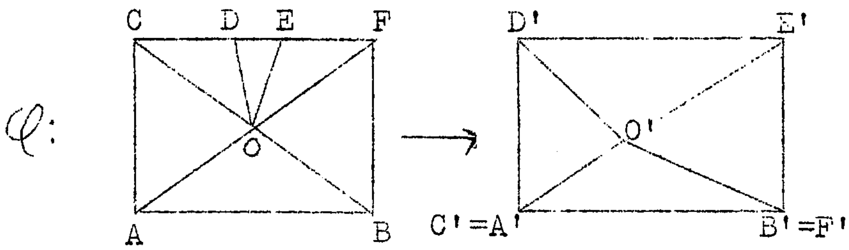


The paths η_0 and η_1 are the paths represented by the map F cut down to the vertical sides of the square, while ξ_0, ξ_1 are the paths represented by F on bottom and top, respectively. We wish to construct a new homotopy H represented by a figure



where p and q denote the start and the end of ξ_0, respectively. This can be done by defining H as the composite $F \circ \varphi$, where φ is a continuous map $\varphi: I_s x I_t \longrightarrow I_s x I_t$ of the square onto itself which carries the bottom identically onto the bottom, each side onto the corresponding end point of the bottom, and the top onto the top and sides. Such a map φ may be constructed as indicated in the figure.

(6.1)



Each labelled vertex is mapped onto the corresponding primed vertex, each segment is shrunk to the corresponding primed segment, and each triangle is mapped (in an affine manner) on the triangle with the corresponding vertices. In particular, the triangle ACO is to be collapsed upon the segment $A'C'O' = A'O'$. The general principles underlying the construction of such affine maps will be discussed in the next chapter.

We also need a classification of the homomorphisms of one group G onto another group. Each (fixed) element $g \in G$ determines the <u>inner automorphism</u> $C_g: G \to G$ defined by the formula

$$(6.2) \qquad C_g(x) = gxg^{-1}, \qquad x \in G.$$

If 1 is the identity element of g, i_G the identity automorphism of G, one readily proves that

$$(6.3) \qquad C_1 = i_G, \qquad C_{g^{-1}} = (C_g)^{-1}, \qquad C_{g_1 g_2} = C_{g_1} \circ C_{g_2}.$$

Furthermore, for any homomorphism $\gamma: G \to H$ one has

$$(6.4) \qquad \gamma \circ C_g = C_{(\gamma g)} \circ \gamma : G \to H,$$

$$\text{for } (\gamma \circ C_g)(x) = \gamma(C_g x) = \gamma(gxg^{-1}) = (\gamma g)(\gamma x)(\gamma g)^{-1}$$
$$= C_{\gamma g}(\gamma x) = (C_{\gamma g} \circ \gamma)(x), \quad x \in G.$$

Two homomorphisms $\gamma_o, \gamma_1: G \to H$ are said to be <u>conjugates</u> if there is an inner automorphism C_h of H such that $\gamma_1 = C_h \circ \gamma_o : G \to H$. The formulas (6.3) show at once that this relation "γ_o is conjugate to γ_1" is reflexive, symmetric and transitive. Hence we may say that γ_o and γ_1 belong to the same <u>homomorphism class</u> $\{\gamma_o\}$ (of G into H) if γ_o and γ_1 are conjugate, in this sense. If C_g and C_h are inner automorphisms of G and H, respectively, then any $\gamma: G \to H$ is conjugate to the composite $C_h \circ \gamma \circ C_g$, for

$$C_h \circ \gamma \circ C_g = C_h \circ C_{(\gamma g)} \circ \gamma = C_{h(\gamma g)} \circ \gamma$$

by (6.4) and (6.3). If γ is an isomorphism of G onto H, so is any one of its conjugates. In particular, the conjugates of an automorphism $\gamma: G \to G$ of G are also automorphisms, so that we may speak of an automorphism class of G.

If $\gamma_0, \gamma_1: G \to H$ are conjugates, while $\beta_0, \beta_1: H \to K$ are also, the composites $\beta_0 \circ \gamma_0, \beta_1 \circ \gamma_1: G \to K$ are also conjugate homomorphisms of G into K. Hence we may form the composite $\{\beta_0\} \circ \{\gamma_0\} = \{\beta_0 \gamma_0\}$ of the homomorphism classes.

<u>THEOREM 6.2</u>. If p_0 and q_0 are two points in an arcwise connected space X, then the isomorphisms $C_u: \pi_1(X, p_0) \to \pi_1(X, q_0)$ between the fundamental groups of X at these two base points by classes u of paths from q_0 to p_0 are all conjugate.

<u>PROOF</u>: Let u, v be two classes of paths from q_0 to p_0. The product vu^{-1} is a path class from q_0 to q_0, hence an element of $\pi_1(X, q_0)$. Using the definition (5.2) of the isomorphism C_v one has, for each $x \in \pi_1(X, p_0)$,

$$C_v(x) = vxv^{-1} = vu^{-1}uxu^{-1}uv^{-1} = (vu^{-1})(uxu^{-1})(vu^{-1})^{-1}$$

$$= C_{vu^{-1}}(C_u x).$$

This asserts that $C_v = C_{vu^{-1}} \circ C_u$ is conjugate to C_u, q.e.d.

Now, if X and Y are arcwise connected spaces with base points p_0, q_0 respectively, each $f: X \to Y$ and each path class u in Y from q_0 to fp_0 determines homomorphisms

$$\pi_1(X, p_0) \xrightarrow{f_*} \pi_1(Y, fp_0) \xrightarrow{C_u} \pi_1(Y, q_0).$$

The composite homomorphisms

(6.5) $\qquad \varphi = C_u \circ f_*: \pi_1(X, p_0) \to \pi_1(Y, q_0)$ defined by

(6.6) $\qquad \varphi(x) = u(f_* x)u^{-1}, \qquad x \in \pi_1(X, p_0)$

is called a homomorphism induced by f on the fundamental groups. By Theorem 6.2, different classes of u yield conjugate homomorphisms φ. Hence f induces a unique class of homomorphisms $\pi_1(X,p_o) \longrightarrow \pi_1(Y,q_o)$.

THEOREM 6.3. If X and Y are arcwise connected spaces with base points p_o; q_o respectively, then (freely) homotopic maps $f_o, f_1 : X \to Y$ induce the same class of homomorphisms of $\pi_1(X,p_o)$ into $\pi_1(Y,q_o)$.

PROOF: Let $F: X x I_t \to Y$ be a (free) homotopy $F: f_o \simeq f_1$, so that

$$(6.7) \quad F(p,0) = f_o(p), \quad F(p,1) = f_1(p), \quad p \in X.$$

Choose a path from q_o to $f_o(p_o)$ in Y, with path class $u = \{\mu\}$. The homomorphism $\varphi_o = C_u o f_{o*}$ induced by f_o is defined as in (6.6), for the class of any path ρ from p_o to p_o in X, by

$$(6.8) \quad \varphi_o\{\rho\} = \{\mu(f_o o \rho)\mu^{-1}\}.$$

As before, we compound the homotopy F with the map ρ to get a homotopy $G: I_s x I_t \to Y$ by

$$G(s,t) = F(\rho(s),t) \qquad 0 \leq s \leq 1, \qquad 0 \leq t \leq 1.$$

This homotopy starts at t = 0, with the path $f_o o \rho$, and ends with the path $f_1 o \rho$, for

$$G(s,0) = F(\rho(s),0) = f_o(\rho(s)), \qquad G(s,1) = f_1(\rho(s)).$$

During the homotopy G the end points, s = 0 and s = 1, trace out identical paths $\nu_o = \nu_1 = \nu$,

$$\nu_o(s) = F(\rho(0),s) = F(p_o,s), \qquad 0 \leq s \leq 1$$

$$\nu_1(s) = F(\rho(1),s) = F(p_o,s).$$

In fact ν is a path from $f_o(p_o)$ to $f_1(p_o)$. Therefore, by Lemma 6.1

$$f_o o \rho \simeq \nu(f_1 o \rho)\nu^{-1}$$

Hence, by associativity

$$\mu(f_o \circ \xi)\mu^{-1} \simeq (\mu\nu)(f_1 \circ \xi)\nu^{-1}\mu^{-1}$$

The left side yields $\varphi_o\{\xi\}$, as in (6.8). Since $\mu\nu$ determines a path class w from q_o, through $f_o(p_o)$, to f_1p_1, the right side yields the homomorphism

$$\varphi_1\{\xi\} = \{(\mu\nu)(f_1 \circ \xi)(\mu\nu)^{-1}\} = C_w(f_{1*}\{\xi\})$$

which is the homomorphism $C_w \circ f_{1*}$, one of the homomorphisms induced by f_1. Then $\varphi_o = \varphi_1$, hence they do lie in the same homomorphism class.

COROLLARY 6.4. If X,Y,Z are arcwise connected spaces with base points p_o, q_o, r_o respectively, with maps $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ inducing homomorphisms

$$\pi_1(X,p_o) \xrightarrow{\varphi} \pi_1(Y,q_o) \xrightarrow{\psi} \pi_1(Z,r_o)$$

on the fundamental groups, then the composite $\psi \circ \varphi$ is one of the homomorphisms induced by gof.

PROOF: By definition (6.5), φ and ψ are given by formulas

$$\varphi(x) = u(f_*x)u^{-1} \qquad x \in \pi_1(X,p_o)$$
$$\psi(y) = v(g_*y)v^{-1} \qquad y \in \pi_1(Y,q_o),$$

where u,v are path classes in Y,Z from q_o to fp_o, and r_o to $g(q_o)$, respectively. Then

$$\psi(\varphi(x)) = v\left(g_*[u(f_*x)u^{-1}]\right)v^{-1} = [v(g_*u)](g_*f_*x)[v(g_*u)]^{-1},$$

where g_*u is a path class in Z from $g(q_o)$ to gfp_o. The product $v(g_*u)$ is thus a path class in Z from r_o to gfp_o; since $g_*(f_*x) = (gf)_*x$, this formula states precisely that $\psi \circ \varphi$ is one of the homomorphisms induced by gof, q.e.d.

<u>THEOREM 6.5</u>. If the arcwise connected spaces X and Y have the same homotopy type, their fundamental groups are isomorphic.

<u>PROOF</u>: The assumption that X and Y have the same homotopy type means that there are continuous maps

$$f: X \longrightarrow Y, \qquad g: Y \longrightarrow X$$

with homotopies $fog \simeq i_Y$, $gof \simeq i_X$. Choose base points in X and Y, and induced homomorphisms φ, ψ on the corresponding fundamental groups. By Corollary 6.4, $\varphi o \psi$ is one of the homomorphisms induced by the identity map; hence (Theorem 6.3), $\varphi o \psi$ is conjugate to the identity homomorphism $i: \pi_1(X, p_o) \longrightarrow \pi_1(X, p_o)$. Therefore $\varphi o \psi$ is an isomorphism onto. The same holds for $\psi o \varphi$. It follows that both φ and ψ are isomorphisms onto.

7. <u>Alternative Description of the Fundamental Groups</u>. An element x of the fundamental group $\pi_1(X, p_o)$ is represented by a closed path in X, starting and ending at p_o. Instead of regarding this path as a continuous image of a unit segment, in which both end points are mapped into p_o, we can regard it as a continuous image of a circle S^1, in which some fixed point on the circle is mapped into p_o. This process yields a second explicit definition of the fundamental group of a space.

Specifically, take the circle S^1 as the set of complex numbers $z = e^{i\theta}$ of absolute value 1, and take the fixed point (north pole) on the circle to be the point $z = 1$, and consider maps

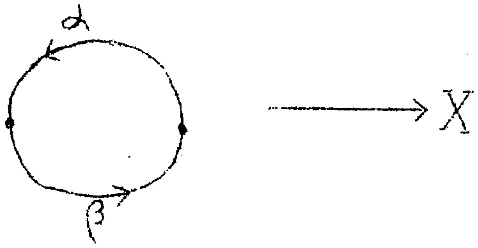
$$\alpha : S^1 \longrightarrow X \text{ with } \alpha(1) = p_o.$$

We use homotopies $F: \alpha_o \simeq \alpha_1$ which keep this north pole $z = 1$ fixed; i.e., mapping $F: S^1 x I_t \longrightarrow X$ with

$$F(z,0) = \alpha_0(z), \quad F(z,1) = \alpha_1(z), \quad F(1,t) = p_0$$

for $z \in S^1$ and $0 \leqq t \leqq 1$. This relation of homotopy is reflexive, symmetric and transitive, so that we may speak of the homotopy classes $a = \{\alpha\}$ of such maps.

The product $\alpha . \beta$ of two such "circular" paths α and β in X at p_0 is defined as indicated by the figure



which is to indicate that the upper half of the circle is mapped "by α", the lower half of the circle "by β". Explicitly, $\alpha . \beta$ is the continuous map of S^1 into X defined by

$$(\alpha\beta)(e^{i\theta}) = \alpha(e^{2i\theta}), \qquad 0 \leqq \theta \leqq \pi$$
$$= \beta(e^{2i\theta - \pi i}), \quad \pi \leqq \theta \leqq 2\pi$$

Then $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$ implies $\alpha_0\beta_0 \simeq \alpha_1\beta_1$, so that one may define the product of two homotopy classes α and β as $\{\alpha\} . \{\beta\} = \{\alpha\beta\}$, without ambiguity. It can then be proved directly that these homotopy classes form a group under this composition.

<u>PROPOSITION 7.1.</u> The group of homotopy classes of circular paths in X at p_0, as described above, is isomorphic to the fundamental group $\pi_1(X, p_0)$.

We shall exhibit an explicit "canonical isomorphism", and henceforth use this particular isomorphism to identify the fundamental group with the group of "circular" paths defined above. This isomorphism is obtained by taking a standard map m of the unit interval into the circle, with both end points sent to the pole of the circle, as follows

$$m: I_s \longrightarrow S^1 \quad \text{with } m(s) = e^{2\pi i s}$$

Then any circular path $\alpha : S^1 \longrightarrow X$, $\alpha(1) = p_o$, determines in composition a "linear" path $\rho : I_s \longrightarrow X$ as $\rho = \alpha o m$. Furthermore, any path $\rho: I_s \longrightarrow X$ starting and ending at the point p_o has the form $\rho = \alpha o m$ for some α. Explicitly, define $\alpha(e^{i\theta}) = \rho(\theta/2\pi)$ for $0 \leq \theta \leq 2\pi$; since $\rho(0) = \rho(1) = p_o$, there is no ambiguity at the point p_o. Thus $\alpha \longrightarrow \alpha o m = \rho$ defines a one-one correspondence between linear paths at p_o and circular paths at p_o. Two paths α_o, α_1 are homotopic if and only if the corresponding ρ_o, ρ_1 are homotopic, for every homotopy $F: I_s x I_t \longrightarrow X$ (leaving the end points fixed at p_o) can be represented uniquely as

$$F = F' o (m x i_I), \qquad F': S^1 \times I_t \longrightarrow X$$

with a homotopy F', in terms of the map

$$m x i_I : I_s x I_t \longrightarrow S^1 x I_t .$$

This correspondence $\alpha \longrightarrow \rho$ yields the desired isomorphism.

This argument depends essentially upon the fact that the space obtained by identifying the end points of the interval I_s is homeomorphic to the circle; or, more exactly, that the mappings m and mxi_I are identification mappings. (see appendix).

A useful special case is the following.

<u>PROPOSITION 7.2</u>. A circular path $\alpha: S^1 \longrightarrow X$ with $\alpha(1) = p_o \in X$ represents the identity element 1 of $\pi_1(X, p_o)$ if and only if α is freely homotopic to a constant map (of S^1 into X).

PROOF: If α represents the identity of π_1, it is homotopic to the constant map of S^1 onto p_o, by the definition of the fundamental group of circular paths.

Conversely, suppose α is freely homotopic to the constant map γ of S^1 into some point q of X, and that $F: S^1 x I_t \longrightarrow X$ is this homotopy. Then

$$F(z,0) = \alpha(z), \quad F(z,1) = q, \quad z \in S^1.$$

Represent α by a linear path ρ, so that $\alpha \circ m = \rho: I_s \longrightarrow X$. We then have continuous mappings

$$I_s x I_t \xrightarrow{m x i} S^1 x I_t \xrightarrow{F} X$$

Their composite $G = F\circ(mxi)$, given by the formula $G(s,t) = F(m(s),t)$, is a free homotopy. It starts, at $t = 0$, with the path ρ, for

$$G(s,0) = F(m(s),0) = \alpha(m(s)) = \rho(s),$$

since $\rho = \alpha \circ m$. It ends with the constant path ε_q, for

$$G(s,1) = F(m(s),1) = q.$$

During the homotopy, the end points of I_s both trace out the same path η (from p to q), for

$$G(0,t) = F(m(0),t) = F(m(1),t) = G(1,t).$$

Hence by the wandering base point Lemma 6.1,

$$\rho \simeq \eta\varepsilon_q\eta^{-1} \simeq \eta\eta^{-1} \simeq \varepsilon_p.$$

Therefore ρ (and α) represent the identity in π_1.

The unit circle S^1 may be considered as the boundary of the circular disc, $D = \{\text{all complex numbers, } z;\ |z| \leq 1\}$ in the complex plane. An alternative version of the last result is

PROPOSITION 7.3. A circular path $\alpha: S^1 \longrightarrow X$ represents the identity element of the fundamental group of X if and only if

α can be extended to a continuous map $h: D \longrightarrow X$.

By the previous result, α represents 1 if and only if there is a homotopy $F: S^1 \times I_t \longrightarrow X$ with $F(z,0) = \alpha(z)$ and $F(z,1)$ constant. Thus F is in effect a mapping into X of the space obtained by identifying all the points (with $t = 1$) on the top circumference of the cylindrical segment $S^1 \times I_t$. This identification space is just the disc D. Specifically, we may use the map

$$n: S^1 \times I_t \longrightarrow D \text{ with } n(z,t) = (1-t)z$$

carrying the top circumference into 0, the bottom into the boundary of D. Then any extension $h: D \longrightarrow X$ of α to the disc yields a homotopy $F = h \circ n$, and any homotopy F has this form, for some h.

8. Simply Connected Spaces. Throughout this section, X is an arcwise connected space. Such a space is said to be <u>simply connected</u> if its fundamental group (taken at any base point p_0) reduces to the identity. Thus X will be simply connected if every path φ starting and ending at p_0 is homotopic to the constant path at p_0--holding both endpoints fixed during the homotopy. This condition can be formulated more liberally.

THEOREM 8.1. X is simply connected if and only if every circular path $: S^1 \longrightarrow X$ is freely homotopic to a constant map.

The proof is immediate, by Proposition 7.2. A similar application of Proposition 7.3 yields

THEOREM 8.3. A contractible space is (arcwise connected and) simply connected.

PROOF: X is contractible, hence the identity map $i_X: X \to X$ is homotopic to a constant map c. If $\alpha : S^1 \to X$ is any circular path in X, then the composites $\alpha = i_X \circ \alpha$ and $c \circ \alpha$ are freely homotopic. Since $c \circ \alpha$ is a constant map of S^1 to X, the result follows by Theorem 7.1.

In particular, it follows that any Euclidean space is simply connected. One may also show that a cartesian product of simply connected spaces is simply connected.

9. (Appendix) Identification Maps. Given a reflexive, symmetric and transitive relation R on the points of a space X, the quotient space X/R is formed by identification; its points are the R-equivalence classes $\{p\}$ of points p in X, the canonical projection ρ is the function carrying each point p into its equivalence class $\{p\}$, and a set V is open in X/R if and only if $\rho^{-1}(V)$ is open in X. We call this map an identification map. More generally any continuous $f: X \to Y$ is said to be an identification map if $f(X) = Y$ and if a set V in Y is open in Y if and only if $f^{-1}(V)$ is open in X.

One has the following "factorization" theorem.

THEOREM 9.1. If $f: X \to Y$ is an identification map, and $g: X \to Z$ is any continuous map into a third space Z such that

(9.1) $f(x_1) = f(x_2)$ implies $g(x_1) = g(x_2)$, $\quad x_1, x_2 \in X$

then there is one and only one continuous map $h: Y \to Z$ such that $g = h \circ f$.

PROOF: Since $f(X) = Y$, any $y \in Y$ has the form $y = f(x_1)$ for some x_1. We may define $h(y) = g(x_1)$; condition (9.1) states that there will be no ambiguity arising from the choice of x. Also $g = hof$.

To prove h continuous, let W be an open set in Z_1, and $V = h^{-1}(W)$ its inverse image in Y. Then by the definition of an identification space, V is open in Y if and only if

$$f^{-1}(h^{-1}(W)) = (hof)^{-1}(W) = g^{-1}(W)$$

is open in X. But g is continuous, so $g^{-1}(W)$ is indeed open in X, q.e.d.

The mapping $m: I_s \to S^1$ of the line segment on the circle is an identification map (as used in §7). This, and many similar results, may be derived from the following general theorem.

THEOREM 9.2. A continuous mapping $f: X \to Y$ of a compact space X onto a Hausdorff space Y is an identification mapping. More generally, if $f: X \to Y$, with Y Hausdorff, and if there is a compact subset S of X with $f(S) = Y$, then f is an identification map.

This result is a generalization of the familiar theorems that a one-one continuous map of compact space onto a Hausdorff space is a homeomorphism.

PROOF: We need only show that if A is a subset of Y such that $f^{-1}(A)$ is open in X, then A is open in Y. Take any point $a \in A$, and some point $x \in X$ with $f(x) = a$. For each point s in S with $f(s) \notin A$ there are disjoint open sets V_s, W_s in Y with $f(s) \in V_s$, $a \in W_s$. The $f^{-1}(V_s)$ are open sets of X; these sets, together with the open set $f^{-1}(A)$, cover S. Therefore S is covered by a finite number of them, say $f^{-1}(A)$ and $f^{-1}(V_{s_j})$, $j = 1,\dots,n$. The set

$W = W_{s_1} \cap \ldots \cap W_{s_n}$ is open in Y, and $a \in W$. If we can show that $W \subset A$, it will follow that A is open in Y. But if $W \subset A$ fails, there is a point y in W and not A. Since $f(S) = Y$, there is a point s of S with $f(s) = y$. Clearly $y \notin f^{-1}(A)$; hence s must lie in one of the sets $f^{-1}(V_{s_j})$. Therefore $y = f(s)$ is in one of the sets V_{s_j}, a contradiction to the fact that $y \in W \subset W_{s_j}$.

Chapter 2

POLYHEDRA

10. Affine Geometry. For the purposes of algebraic topology, it is convenient to consider spaces which can be built up from a finite number of points, intervals, triangles, tetrahedra, etc. Such a space will be called a polyhedron, and the triangles, tetrahedra, ... from which it is constructed will be termed simplices.

Let E be a Euclidean space; that is, a vector space over the field R of real numbers in which each pair of vectors p, q determines a real number (p,q) as inner product, with the usual properties

Symmetry: $(p,q) = (q,p)$

Linearity: $(xp + yq, r) = x(p,r) + y(q,r) \quad x,y \in R, \quad p,q,r \in E$

Definiteness: $(p,p) \geqq 0, \quad (p,p) = 0$ if and only if $p = 0$.

The number $|p| = (p,p)^{1/2}$ is the norm of p, and E is a metric space with respect to the distance function $\rho(p,q) = |p-q|$. In particular, E may be the n-dimensional space with n-tuples $p = (a_1, \ldots, a_n)$, $q = (b_1, \ldots, b_n)$ of numbers a_i, b_i in R as its vectors, and with inner product

$$(p,q) = a_1b_1 + \ldots + a_nb_n$$

Alternatively, E may be a Hilbert space.

We wish to study the affine geometry of E; that is, the geometry in which the position of the origin is neglected. More exactly, if E and E' are two Euclidean spaces, r' a vector in E', and T a linear transformation of E into E', then the transformation $A: E \longrightarrow E'$ defined by

$$(10.1) \qquad A(p) = T(p) + r', \qquad p \in E$$

is called an affine transformation. Thus an affine transformation is a linear transformation T, of E into E', followed by a translation in E', by the fixed vector r'. The composite of two affine transformations is again an affine transformation. The transformation A is non-singular if it is a one-one transformation of E onto E'; this will be the case if and only if the linear transformation T is non-singular. When this is the case, the transformation inverse to A is also an affine transformation. In particular, the non-singular affine transformations of E onto E constitute a group, the <u>affine group</u> of E. The affine geometry of E is the study of properties invariant under the affine group.

If E is finite dimensional, any affine transformation $A: E \longrightarrow E'$ is continuous, for A is the composite of a linear transformation T and a translation $p' \longrightarrow p' + r'$, and each of these functions is continuous.

The midpoint of the segment joining two distinct points p_0, p_1 of E is the point $q = (1/2)p_0 + (1/2)p_1$. The operation of forming the midpoint is invariant under affine transformations A, since $A(q)$ is the midpoint of the segment $A(p_0)$, $A(p_1)$. More generally, the point q dividing the segment p_0, p_1 in the ratio $(1-t):t$ is the point $q = t\,p_0 + (1-t)p_1$. As t varies through the real numbers, q traces out the line joining p_0 to p_1. In other words, any point on the line has a unique representation as $q = x_0p_0 + x_1p_1$, with scalars x_0, x_1 such that $x_0 + x_1 = 1$. We call the scalars (x_0, x_1) the barycentric coordinates of q relative to p_0, p_1.

A line, plane, hyperplane of E (<u>not</u> necessarily passing through the origin) is called an affine subspace of E. More exactly, a subset S of E is an <u>affine subspace</u> if it contains with any two points the line joining those points; that is, if $p_0, p_1 \in S$ implies $tp_0 + (1-t)p_1 \in S$, for any real t. If $p_0, \ldots, p_m$ are $m+1$ points of E, the intersection of all subspaces containing $p_0, \ldots, p_m$ is an affine subspace, called the subspace <u>spanned</u> by $p_0, \ldots, p_m$.

For two points p_0, p_1 of E the <u>segment</u> joining p_0 to p_1 is the set of all points $tp_0 + (1-t)p_1$ for each t with $0 \leqq t \leqq 1$. A subset C of E is <u>convex</u> if it contains with any two points p_0, p_1 all points of the segment joining p_0 to p_1. The intersection of convex subsets of E is a convex subset, hence we may again speak of the convex subset of E <u>spanned</u> by $m+1$ given points $p_0, \ldots, p_m$.

<u>PROPOSITION 10.1</u>. The affine subspace S of E spanned by $m+1$ points $p_0, \ldots, p_m$ consists exactly of those points p of E which can be represented as linear combinations of the form

$$p = x_0p_0 + \ldots + x_mp_m, \qquad x_0 + x_1 + \ldots + x_m = 1. \tag{10.2}$$

The convex subset C spanned by $p_0, \ldots, p_m$ consists of all points representable in the form (10.2) with non negative coefficients $x_i \geqq 0$, $i = 0 \ \ldots, m$.

Both results are proved by the same argument. We first show by induction on m that every such point q lies in S (or C, if all $x_i \geqq 0$). For $m = 0$, $p = p_o \in S = C$. For $m = 1$, p is in S (or C), by the definition of a subspace (convex set). For $m > 1$, set $t = x_o + \ldots + x_{m-1}$. If $t = 0$, then $q = 1.p_m$ is in S and C. Otherwise $x_m = 1-t$, and the point

$$(10.3) \qquad p' = (x_o/t)p_o + \ldots + (x_{m-1}/t)p_{m-1}$$

lies in S (or in C) by the induction assumption. Furthermore

$$(10.4) \qquad p = tp' + (1-t)p_m, \qquad x_m = 1-t.$$

Hence p lies in S (or C) by definition.

Secondly, the set of all points p of the form (10.2) constitutes an affine subspace. For if

$$(10.5) \qquad r = y_o p_o + \ldots + y_m p_m \qquad y_o + y_1 + \ldots + y_m = 1$$

is a second such point, and t is any real number, then

$$tp + (1-t)r = \sum_{i=0}^{m} [tx_i + (1-t)y_i] p_i,$$

where the sum of the coefficients is again 1. The same argument applies, mutatis mutandis, to show that the p's with $y_i \geqq 0$ constitute a convex subset. with [illegible]

If A is an affine transformation of E onto E', then for each point p of the form (10.2) one has

$$(10.5) \qquad A(p) = x_o Ap_o + x_1 Ap_1 + \dots + x_m Ap_m$$

Indeed, this result is immediate for a linear transformation, while for a translation $A(p) = p+r$ by a fixed vector r in the space E one has

$$A(p) = (\sum x_i p_i) + r = (\sum x_i p_i) + (\sum x_i)r$$
$$= \sum x_i(p_i + r) = \sum x_i A(p_i)$$

as required. It follows that an affine transformation carries affine subspaces and convex subsets of E onto affine subspaces and convex subsets of E', respectively, a conclusion which can also be deduced directly from the definitions.

The sequence $p_o, \dots, p_m$ of $m+1$ points in E is said to be <u>affine independent</u> if the vectors $p_1 - p_o, \dots, p_m - p_o$ are linearly independent. For an affine transformation A as in (10.1) one has $Ap_i - Ap_o = Tp_i - Tp_o = T(p_i - p_o)$; hence a non singular affine transformation carries affine independent points into affine independent points.

<u>PROPOSITION 10.2</u>. The sequence $p_o, \dots, p_m$ is affine independent in E if and only if every point in the subspace spanned by $p_o, \dots, p_m$ has a unique representation (10.2) in terms of $p_o, \dots, p_m$.

PROOF: Suppose first that the points are independent, but that some point p in the subspace has two representations $p = \sum x_i p_i = \sum x_i' p_i$, both with $\sum x_i = 1 = \sum x_i'$. Then $x_o' - x_o = (x_1 - x_1') + \dots + (x_m - x_m')$, and the zero vector has a representation

$$0 = \sum_{i=0}^{m} (x_i - x_i')p_i = \sum_{i=1}^{m} (x_i - x_i')p_i - (x_0' - x_0)p_0$$

$$= \sum_{i=1}^{m} (x_i - x_i')(p_i - p_0).$$

Since the vectors $p_i - p_0$ are independent, we conclude that $x_i = x_i'$, for $i = 1, \ldots, m$. Since $x_0 = 1 - (x_1 + \ldots + x_m)$, we also have $x_0 = x_0'$. The representation (10.2) is thus unique.

Secondly, suppose that the points $p_0, \ldots, p_m$ are affine dependent. Then there is a linear relation $\sum c_i(p_i - p_0) = 0$ with some coefficient, say c_1, not zero. By division, we can assume $c_1 = 1$. Then

$$p_1 = -c_2p_2 - \ldots - c_mp_m + (c_2 + \ldots + c_m - 1)p_0,$$

a representation in which the sum of the coefficients is 1. But p_1 has a second representation as $p_1 = 1.p_1$, hence the representation (10.2) is indeed not unique.

In the definition of affine independence, the first point p_0 played a special role. Since however the criterion for independence stated in Proposition (10.2) is independent of the order of the points p_i, it follows that the concept of independence does not depend on the order.

When the points $p_0, \ldots, p_m$ are affine independent the scalars $x_0, \ldots, x_m$ appearing in the representation (10.2) of points in the subspace spanned by $p_0, \ldots, p_m$ are called the <u>barycentric coordinates</u> of q relative to $p_0, \ldots, p_m$. Note that any m of these coordinates determine the remaining coordinate, in virtue of $x_0 + \ldots + x_m = 1$.

An inductive criterion for affine independence may be given as follows.

PROPOSITION 10.3. If points $p_o, \dots, p_m$ are affine independent, and q is an additional point, then $p_o, \dots, p_m, q$ are affine independent if and only if q does not lie in the affine space spanned by $p_o, \dots, p_m$.

PROOF: If q lies in the affine space spanned by $p_o, \dots, p_m$, then $q = \sum x_i p_i$; thus, with $q = 1.q$ gives two representations of q in terms of $p_o, \dots, p_m, q$; hence these points are dependent. Conversely, if the points $p_o, \dots, p_m, q$ are dependent, the vectors $p_1 - p_o, \dots, p_m - p_o, q - p_o$ are linearly dependent. Since the first m vectors here are independent, there is a relation $q - p_o = x_1(p_1 - p_o) + \dots + x_m(p_m - p_o)$, which gives a representation of q as $q = (1 - \sum x_i)p_o + \sum x_i p_i$, with $i = 1, \dots, m$. Since the sum of all coefficients is 1, this states that q lies in the affine space spanned by $p_o, p_1, \dots, p_m$.

THEOREM 10.4. If the affine independent points $p_o, \dots p_m$ span an affine subspace S of E, then for any $m+1$ points $q_o, \dots, q_m$ in a second Euclidean space E' there is one and only one affine transformation A of S into E' with $A(p_i) = q_i$, $i = o, \dots, m$. This transformation $A: S \to E'$ is continuous, and maps S onto the subspace S' of E' spanned by $q_o, \dots, q_m$. If $q_o, \dots, q_m$ are also affine independent, A is a homeomorphism of S to S'.

PROOF: Because of the explicit formula (10.5), there can be at most one such transformation. To show that one exists, let L be the linear subspace (of dimension either m or m+1) of E spanned by the vectors $p_o, \ldots, p_m$ of E. Since $p_1 - p_o$, ..., $p_m - p_o$ are independent vectors of L, there exists a linear transformation $T: L \longrightarrow E'$ with $T(p_i - p_o) = q_i - q_o$. Let r be the fixed vector $q_o - Tp_o$ in E', and define the affine transformation $A: L \longrightarrow E'$ by the equation $Ap = Tp + r$, for any p in L. Then, for $i \geq 1$

$$Ap_i = T(p_i - p_o + p_o) + r = (q_i - q_o) + Tp_o + r = q_i,$$

while $Ap_o = Tp_o + r = Tp_o + q_o - Tp_o = q_o$, so that A is the desired transformation. Because L is a finite dimensional vector space, T and hence A (cut down to S) is a continuous transformation. If the q's are also affine independent, there is a second continuous affine transformation $B: S' \longrightarrow S$ with $B(q_i) = p_i$. The composite $A \circ B$ is then the unique affine transformation $S' \longrightarrow S'$ with $(A \circ B)q_i = q_i$, hence $A \circ B$, and likewise $B \circ A$, is the identity. Hence A and B are homeomorphisms, and $A^{-1} = B$.

COROLLARY 10.5. If p is a point in the affine subspace S spanned by m+1 affine independent points $p_o, \ldots, p_m$, then the assignment to p of its i-th barycentric coordinate $\beta_i(p) = x_i$ is a continuous mapping β_i of S into the reals.

PROOF: The mapping β_i is identical with the affine mapping A of S into the reals with $A(p_i) = 1$, $A(p_j) = 0$ for $j \neq i$.

By using the mapping A into Euclidean m space with $A(p_o)$ the zero vector and $A(p_i)$ the i-th unit vector (0, ..., 1, ..., 0) we also prove

COROLLARY 10.6. The affine space S spanned by m+1 affine independent points $p_o, \ldots, p_m$ is homeomorphic to the m-dimensional Euclidean space E_m under an affine transformation $f(\sum x_i p_i) = (x_1, \ldots, x_m)$. Consequently the topology of S is determined by the metric

$$\rho\left(\sum_{i=o}^{m} x_i p_i, \sum_{i=o}^{m} y_i p_i\right) = \left[(x_1-y_1)^2 + \ldots + (x_m-y_m)^2\right]^{1/2}.$$

We have chosen to develop affine geometry, assuming vector geometry. It is possible to give an independent definition of an affine space S over the real numbers or, for that matter, as over any field F. One procedure would be to assume in the space S as primitive operation the formation of the weighted mean $x_o p_o + x_1 p_1$, with $x_o + x_1 = 1$, of any two points p_o and p_1, subjecting this operation to the appropriate algebraic laws. In this sense any affine subspace S of a Euclidean space E is (taken by itself) an affine space. An affine space S spanned by m+1 affine independent points has dimension m; as in the case of a vector space, this dimension does not depend on the particular choice of a basis $p_o, \ldots, p_m$.

11. Simplices. An m-dimensional affine simplex is a set determined by m+1 affine independent points $p_o, \ldots, p_m$ in a Euclidean space E. The open affine simplex

(11.1) $$s = \langle p_o, \ldots, p_m \rangle$$

consists of all points of E which have positive barycentric coordinates relative to $p_o, \ldots, p_m$; i.e., all points p of E of the form

$$p = x_o p_o + \ldots + x_m p_m, \qquad x_i > 0, \qquad x_o + \ldots + x_m = 1.$$

The <u>closed affine simplex</u>

(11.2) $$\bar{s} = |p_o, \ldots, p_m|$$

is the convex subset of E spanned by $p_o, \ldots, p_m$; it consists of all points of the form

$$p = x_o p_o + \ldots + x_m p_m, \qquad x_i \geqq 0, \qquad x_o + \ldots + x_m = 1.$$

In particular, a zero dimension simplex (closed or open) is a point, a 1-dimensional simplex is a line segment, and a 2-dimensional simplex is the interior of a triangle (with the boundary, if the simplex is to be closed).

Since the function $\beta_i(p) = x_i$ assigning to p its i-th barycentric coordinate is a continuous function on the affine space S spanned by the p_i, the set of points of S with $x_i > 0$ ($x_i \geqq 0$) is an open (closed) subset of S; therefore s, as the intersection of a finite number of open sets, is open in S, and $\bar{s}$ is likewise closed in S. If we regard S as a Euclidean space, as in Corollary (10.6), $\bar{s}$ is contained in the bounded subset of S with $0 \leqq x_i \leqq 1$, $i = 1, \ldots, m$. Hence any closed simplex $\bar{s}$ is a <u>compact</u> (metric) <u>space</u>. Note that an open simplex s is open in its space S but <u>not</u> necessarily in the whole Euclidean space; for example, a point is an "open" simplex.

The closed simplex $\bar{s}$ is the closure of s in S, and the open simplex s is the interior of $\bar{s}$ (i.e., is the largest open set of S contained in $\bar{s}$. Indeed, any closed subset of S containing s clearly contains $\bar{s}$. On the other hand, if V is an open subset of S contained in $\bar{s}$, and if some point p of V has $\beta_i(p) = 0$, then V must contain the inverse image under β_i of some neighborhood of zero, hence must contain points with negative i-th barycentric coordinate. Therefore V open and $V \subset \bar{s}$ implies $V \subset s$, and s is the interior of $\bar{s}$.

The simplex s or $\bar{s}$, given in a subset of an affine space, determines uniquely the set $\{p_o, \ldots, p_m\}$ of its vertices. Indeed a point q of $\bar{s}$ is one of the vertices if and only if, for every pair of points r_o, r_1 of $\bar{s}$, the line segment joining r_o to r_1 contains q if and only if $q = r_o$ or $q = r_1$ (proof as exercise).

Any subset of the vertices $p_o, \ldots, p_m$ determines an (open) simplex t called an open <u>face</u> of $s = < p_o, \ldots, p_m >$. Thus s itself is one of the faces, and the remaining faces have lower dimensions. A face t of dimension n is thus a simplex

$$t = < p_{i_o}, p_{i_1}, \ldots, p_{i_n} >, \quad i_o, i_1, \ldots, i_n \text{ distinct;}$$

it consists of all points $\sum x_i p_i$ with $x_{i_k} > 0$, $k = 0, \ldots, n$ and the remaining $x_i = 0$. The closed simplex $\bar{s}$ is thus the union of all the open faces of s. Closed faces are similarly defined.

We repeatedly use affine maps of one simplex into another. Given two closed simplices

$$\bar{s} = |p_o, \ldots, p_m|, \quad \bar{s}' = |q_o, \ldots, q_n|$$

in the same or different spaces, and a function f which maps each vertex p_i of s to one of the vertices $q_o = f(p_i)$ of s', we may construct, by Theorem 10.3, the affine map of the space spanned by the p_i into the space spanned by the q_j, with $A(p_i) = f(p_i)$. This map induces a continuous transformation

$$f_*: \bar{s} \longrightarrow \bar{s}'$$

of the first closed simplex into the second. If f maps the vertices p onto the vertices q, then f_* also carries the open simplex s into the open simplex s'; in general f_* maps the open simplex s onto an open face of s'. For example, one may in this fashion construct a map "collapsing" an n dimensional simplex upon one of lower dimension. On the other hand if m = n and f is a one-one mapping, then f^{-1} is defined, $(f^{-1})_* = (f_*)^{-1}$, and f_* is a homeomorphism. Hence any two closed m-simplices are homeomorphic. The same result holds for open simplices.

A polyhedron P is a finite set of open simplices s_1, s_2, ... all in the same affine space, such that

(i) $s_1 \neq s_2 \in P$ implies $s_1 \cap s_2 = \emptyset$ (the simplices are disjoint)

(ii) if $s \in P$ and t is an open face of s, then $t \in P$.

The topological space associated with the polyhedron is the union

$$|P| = \bigcup_{s \in P} s$$

of all the simplices of P. By (i) every point p of $|P|$ belongs to exactly one of the simplices s; we call this simplex the carrier of p.

If $s \in P$, we say that $\bar{s}$ is one of the closed simplices of P. Since each closed simplex $\bar{s}$ is the union of the open faces of s, and all these are included amongst the simplices of P, we may also write

$$|P| = \bigcup_{s \in P} \overline{s}$$

Thus $|P|$, as the union of a finite number of compact sets $\overline{s}$, is a compact set (and is closed in the affine space in which P lies). The <u>dimension</u> of P is the largest dimension of any one of its faces.

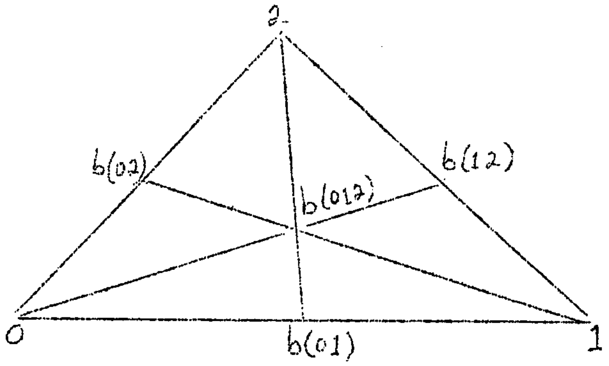
If two simplices s and t of P have two common faces u and v, then every vertex of u and every vertex of v is a vertex of s; hence these vertices taken together span a simplex w which is a face of s (and of t), and which has u and v as faces. Thus if s and t have any common faces, they have a "largest" common face w, of which all other common faces u are faces. Since each closed simplex $\overline{s}$ is the union of the faces of s, the intersection of two closed simplices $\overline{s}$ and $\overline{t}$ of P must be the union of their common faces; i.e., the closure of the largest common face. Hence

(i') The intersection of two closed simplices $\overline{s}_1$, $\overline{s}_2$ of P is void or a closed simplex of P which is a face of both $\overline{s}_1$ and $\overline{s}_2$.

In the presence of condition (ii), the requirement is equivalent to condition (i) of the definition of a polyhedron. Indeed, given (i'), disjoint simplices s_1 and s_2 are such that $\overline{s}_1 \cap \overline{s}_2$ is a common face $\overline{w}$. Unless this face is identical with $\overline{s}_1$ and $\overline{s}_2$, it is a proper face of one of them; so that the open simplices s_1 and s_2 are disjoint, as required by (i).

Each simplex s determines by itself a polyhedron P(s) consisting of all faces of s. Clearly $|P(s)| = \overline{s}$.

12. Barycentric Subdivision. The barycentric subdivision of a line segment is obtained by dividing it at its midpoint; the barycentric subdivision of a triangle is the set of six triangles into which the triangle is cut by its medians.



Here b(01) denotes the midpoint of the segment joining the point 0 to the point 1, and similarly b(012) denotes the centroid of the triangle 012. In terms of the barycentric coordinates, the medians have the equations $x_0 = x_1$, $x_0 = x_2$ and $x_1 = x_2$ in barycentric coordinates, so that the six 3-simplices of the subdivision are determined as follows.

$$\begin{array}{ll}
< 0,b(01),\ b(012) > , & x_0 > x_1 > x_2 > 0, \\
< 0,b(02),\ b(012) > , & x_0 > x_2 > x_1 > 0, \\
< 1,b(01),\ b(012) > , & x_1 > x_0 > x_2 > 0, \\
< 1,b(12),\ b(012) > , & x_1 > x_2 > x_0 > 0, \\
< 2,b(02),\ b(012) > , & x_2 > x_0 > x_1 > 0, \\
< 2,b(12),\ b(012) > , & x_2 > x_1 > x_0 > 0.
\end{array}$$

Each point of the original triangle is either a point of one of these open 3-simplices, or a point of one of the 12 open 2-simplices in the subdivision, or a vertex of the subdivision.

We may describe the simplices appearing in this subdivision inductively as (i) all the simplices t appearing in the subdivision of the three edges, (ii) all the simplices obtained by adjoining the vertex b(012) to the simplices t, (iii) the 0-dimensional simplex b(012).

In general, let

$$(12.1) \qquad b(p_0, \ldots, p_m) = \frac{1}{m+1}p_0 + \cdots + \frac{1}{m+1}p_m$$

denote the <u>barycenter</u> (= center of gravity) of the m+1 affine independent points $p_0, \ldots, p_m$. This barycenter does not depend on the order of the p_i, so that we may also call it the barycenter b(s) of the simplex s spanned by $p_0, \ldots, p_m$. In particular, the barycenter of a vertex is that vertex.

Every point p of the open simplex s lies on the segment joining the barycenter b(s) to some point q on one of the (proper) faces of s; that is, any $p \in \langle p_0, \ldots, p_m \rangle$ is either b(s), or can be uniquely represented in the form

$$(12.2) \qquad p = yq + (1-y)b(s) \quad , \qquad 0 < y < 1,$$

with q on some proper face of s. Indeed, let $p = \sum x_i p_i$, pick the smallest (or one of the smallest) barycentric coordinates x_k, set $y = \sum_i (x_i - x_k)$. Then, since $\sum x_i = 1$, $y = 1 - (m+1)x_k$, and since x_k is positive, $0 \leqq y < 1$. Set $z_i = (x_i - x_k)/y$, for each i = 0, ..., m. Then $\sum z_i = 1$ and

$$p = \sum x_i p_i = y \sum z_i p_i + x_k \sum p_i$$

$$= y(\sum z_i p_i) + (1-y) \sum \frac{1}{m+1} p_i = yq + (1-y)b(s)$$

where $q = \sum z_i p_i$ belongs to the face $< p_o, \ldots, \hat{p}_k, \ldots, p_m >$ spanned by the vertices p_i, with p_k omitted. The representation is unique, because q and y determine the z_i and hence the x_i.

If P(s) is the polyhedron consisting of all the faces of an m-dimensional simplex s, we define the first barycentric subdivision P(s)' as a polyhedron with $|P(s)'| = |P(s)| = \bar{s}$, by induction on the dimension m, as follows. If $s = < p_o >$ is a vertex, P(s)' = P(s) consists only of that vertex. If $s = < p_o, \ldots, p_m >$, with $m > 0$, then P(s)' consists of the following simplices

(i) The barycenter b(s) (a 0-dimensional simplex)

(ii) Every simplex t appearing in the barycentric subdivision of any proper face of s

(iii) The simplices $< t, b(s) >$ obtained by adjoining the vertex b(s) to any simplex t obtained in (ii).

In other words, the subdivision is made by subdividing the boundary and joining all these simplices to the barycenter.

To justify step (iii) of this construction, observe that t is contained in the affine space spanned by one of the (m-1) faces $< p_o, \ldots, \hat{p}_k, \ldots, p_m >$ of s, while the barycenter b(s) has a positive coordinate in p_k, hence is not in this space. Thus b(s) and the vertices of t are affine independent, according to Proposition 10.3, so that the simplex is well defined.

By the construction, it is clear that any face of P(s)' is itself one of the simplices of P(s)'. We must also show that $|P(s)'| = \bar{s}$. Since $\bar{s}$ is convex, any segment joining a point q of a simplex t to the barycenter b(s) is contained in $\bar{s}$; hence

$|P(s)'| \subset \bar{s}$. Conversely, any point p of $\bar{s}$ either is a point of some proper face of s, hence lies in one of the simplices t of (ii), or is a point of s. In the latter case either $p = b(s)$, or there is a unique representation (12.2), in which q must be a point of some simplex t in the barycentric subdivision of the boundary. Then q lies in the corresponding simplex $< t, b(s) >$ of (iii) above. The simplex of $P(s)'$ to which q belongs is thus uniquely determined, and hence the simplices of $P(s)'$ are disjoint. It follows that $P(s)'$ is a polyhedron.

<u>Each simplex</u> t <u>of the barycentric subdivision</u> $P(s)'$ <u>is contained in a simplex</u> s_1 <u>of</u> $P(s)$, <u>of dimension at least that of</u> t. This is immediate for the simplices (i) and (ii) above. If the simplex t of (iii) is contained in the simplex s_1 of the subdivision of the boundary, then the new simplex $< t, b(s) >$ is clearly contained in the simplex $< s_1, b(s) >$ of $P(s)'$,

By induction we also observe that the n-dimensional simplices of $P(s)'$ can be described explicitly as follows. Take simplices $s_o, \ldots, s_n$ of $P(s)$, each a proper face of the next, and form the simplex

$$(12.3) \qquad t = < b(s_o), b(s_1), \ldots, b(s_n) >$$

This is an n-dimensional simplex of $P(s)'$, contained in s_n, and all simplices of $P(s)'$ have this form. In particular, it follows that all the $(m+1)!$ m-simplices of $P(s)'$ may be found as follows. Take any permutation $q_o, \ldots, q_m$ of the vertices $p_o, \ldots, p_m$ of s, and form the simplex

$$(12.4)\quad u = < q_o, \frac{q_o+q_1}{2}, \frac{q_o+q_1+q_2}{3}, \ldots, \frac{q_o+q_1+\ldots+q_m}{m+1} >.$$

Alternatively, we may say that the vertices v of P(s)' are all barycenters $b(s_1)$ of faces s_1 of s, that these vertices are partially ordered by the relation $b(s_1) < b(s_2)$ if and only if s_1 is a proper face of s_2, and that $< v_o, v_1, \ldots, v_k >$ is a simplex of P(s)' if and only if $v_o < v_1 < \ldots < v_n$ in the partial order.

Since s lies in a metric space, we may define the <u>mesh</u> of $\bar{s}$ to be its diameter, and the <u>mesh</u> of the subdivision P(s)' to be the maximum diameter of any one of its simplices. A basic result is

<u>THEOREM</u> <u>12.1</u>. If s is an m-dimensional simplex, then

$$\text{mesh } (P(s)') \leqq \frac{m}{m+1} \text{ mesh } (s).$$

The proof depends on a Lemma.

<u>LEMMA</u> <u>12.2</u>. The diameter of a(closed) simplex $s = |p_o, \ldots, p_m|$ is the diameter of the set of its m+1 vertices $\{ p_o, \ldots, p_m \}$.

PROOF: If p and q are points of s, with $p = \sum x_i p_i$, the distance $\rho(p,q)$ is given by the norm $|p-q|$ of the vector p-q. By the triangle law

$$|p-q| = | \sum x_i p_i - q | = | \sum x_i p_i - (\sum x_i)q|$$

$$= | \sum x_i (p_i - q) | \leqq \sum | x_i (p_i - q) |$$

$$= \sum x_i |p_i - q| \leqq (\sum x_i) \operatorname{Max}_i |p_i - q| \leqq \operatorname{Max}_i |p_i - q|.$$

Using a similar expression $q = \sum y_j p_j$ we find $|p_i - q| \leq \operatorname{Max}_j |p_i - p_j|$. Hence $|p-q| \leq \operatorname{Max}_{i,j} |p_i - p_j|$. This maximum is by definition the diameter of the set of m+1 points $p_o, \ldots, p_m$. Since the maximum is attained for some pair of points p, q of $\bar{s}$, the result is established.

To compute the mesh of P(s)' we thus need only determine the maximum diameter of the set of vertices q_o, $(q_o+q_1)/2$, ..., $(q_o+\ldots+q_m)/(m+1)$ of one of the simplices (3). Now, for example,

$$|q_o - (q_o+q_1)/2| = (1/2)\ |2q_o - (q_o+q_1)| = (1/2)\ |q_o - q_1|$$

$$|q_o - (q_o+q_1+q_2)/3| = (1/3)\ |3q_o - (q_o+q_1+q_2)|$$

$$\leq (1/3)\ (|q_o-q_1| + |q_o-q_2|) \leq (2/3)\ \operatorname{Max}_i |q_o - q_i|.$$

In general, for the i-th and j-th vertices of (3), with $0 \leq i \leq j \leq m$, one has

$$|(1/i+1)(q_o+\ldots+q_i) - (1/j+1)(q_o+\ldots+q_j)|$$

$$= \frac{1}{(i+1)(j+1)}\ |(j+1)(q_o+\ldots+q_i) - (i+1)(q_o+\ldots+q_j)|.$$

The first sum involves (j+1)(i+1) terms $q_o, \ldots, q_i$; of these, q_o occurs i+1 times in the second sum , and these terms cancel. There remain j(i+1) differences, and by the triangle law, the result is then

$$\leq \frac{j(i+1)}{(i+1)(j+1)} \operatorname{Max}_{k,l} |q_k - q_l| = \left(\frac{j}{j+1}\right) \operatorname{Mesh} < q_o, \ldots, q_j >.$$

Since $j \leq m$, the factor $j/(j+1)$ is at most $m/(m+1)$, q.e.d.

The barycentric subdivision P' of any polyhedron P is defined to be the set of all simplices occurring in the barycentric subdivisions of simplices of P. The mesh of a polyhedron is the largest mesh of any one of its simplices.

THEOREM 12.3. The barycentric subdivision of a polyhedron P is a polyhedron P' with $|P'| = |P|$ and of the same dimension m as P. Each simplex of P' is contained in a unique simplex of P, of the same or larger dimension. If P has dimension m,

$$\text{mesh } P' \leq \frac{m}{m+1} \text{ mesh } P.$$

The only item requiring explicit proof is the statement that distinct simplices t_1, t_2 of P' are disjoint (required if P' is to be a polyhedron). But t_1, t_2 occur in the subdivision of simplices s_1, s_2 of P, and are therefore contained in faces r_1, r_2 of s_1, s_2, respectively. If $r_1 \neq r_2$, they are disjoint, hence $t_1 \subset r_1$ and $t_2 \subset r_2$ are disjoint. If $r_1 = r_2$, then both t_1 and t_2 occur in the subdivision $P(r)'$ of the same simplex $r = r_1$ of P, hence they are disjoint, by the facts already established for $P(s)'$.

The n-th barycentric subdivision $P^{(n)}$ is formed by iteration. Because of Theorem 12.1, we can always find, for given P, a barycentric subdivision with mesh less than any prescribed positive ε.

We presently need the following geometric fact.

LEMMA 12.4. Let t be an (m-1)-simplex in a barycentric subdivision of $P(s)^{(n)}$ of an m-simplex $\bar{s}$. Then either

(i) $t \subset s$, and t is a face of exactly two m-simplices of $P(s)^{(n)}$

or

(ii) $t \subset s_1$, where s_1 is an (m-1) face of s, and t is a face of exactly one m-simplex of $P(s)^{(n)}$.

Geometrically, it is clear that a simplex t will either be "inside" $\overline{s}$, in which case (i) obtains, or on the boundary of $\overline{s}$, in which case (ii) holds.

In any event, t is contained in some face of s of at least dimension m-1, so that we have either $t \subset s$ or $t \subset s_1$. The other assertion we prove by induction on n. For the case n = 1 of the first barycentric subdivision, the (m-1)-dimensional simplex t of $P(s)'$ is determined as in (12.3) by m simplices $s_0 \subset s_1 \subset \ldots \subset s_{m-1}$ of P(s), each properly contained in the next. If $\overline{s}_{m-1}$ does not contain all vertices of s, it omits exactly one vertex q_m, and

$$t = < q_0, (q_0+q_1)/2, \ldots, (q_0+\ldots+q_{m-1})/m > \subset s_{m-1}$$

is a face of exactly one m-simplex of $P(s)'$, namely, of the simplex formed by adjoining the vertex $(q_0+\ldots+q_m)/m+1$ to t. We thus have case (ii) of the Lemma. On the other hand, if $\overline{s}_{m-1}$ contains all vertices of s, then $t \subset s$ and exactly one of the simplices s_k has two vertices more than its predecessor, so that, after suitable labelling of vertices

$$s_i = < q_0, \ldots, q_i > \qquad i = 0,\ldots,k-1,$$

$$s_j = < q_0, \ldots, q_{j+1} > \qquad j = k,\ldots,m-1.$$

In particular the (k-1)st and the k-th vertices of t are

$$b(q_0,\ldots,q_{k-1}), \qquad b(q_0,\ldots,q_{k-1},q_k,q_{k+1})$$

If t is a face of some m-simplex t' of $P(s)'$, then t' must be obtained by adding exactly one new vertex to those of t. This vertex can be either

$$b(q_0,\ldots,q_k) \quad \text{or} \quad b(q_0,\ldots,q_{k-1}, q_{k+1}).$$

Thus $t \subset s$, and t is a face of exactly two m-simplices of $P(s)'$, as asserted in case (i) above.

Suppose now that the result has been established for $P(s)^{(n-1)}$, and let $u_1, u_2, \ldots$ be the m-simplices in $P(s)^{(n-1)}$. Each (m-1)-simplex t of the next subdivision $P(s)^{(n)}$ occurs in one or more of the subdivisions $P(u_k)'$. Then, by the result already established for a single barycentric subdivision, we have either

Case 1: $t \subset u_k$, and t is a face of two m-simplices of $P(u_k)'$

Case 2: t contained in some (m-1)-face w_1 of u_k, and t is a face of exactly one m-simplex of $P(u_k)'$.

If the first case occurs, u_k is the unique open simplex of $P(s)^{(n-1)}$ containing t. Since u_k is m-dimensional, $u_k \subset s$, hence $t \subset s$, and we have case (i) of the Lemma, with t on two m-simplices of $P(s)^{(n)} \supset P(u_k)'$. In the second case, the (m-1)-simplex w_1 of $P(s)^{(n-1)}$ containing t is uniquely determined. By the induction assumption, w_1 may be a face of two m-simplices u_k, u_ℓ of $P(s)^{(n-1)}$, or of just one, u_k. Under the first alternative, t is a face of one m-simplex of $P(u_k)'$ and of one m-simplex of $P(u_\ell)'$, and thus is a face of two m-simplices of $P(s)^{(n)}$. Furthermore, $t \subset w_1$, $w_1 \subset s$, hence $t \subset s$ and we have case (i) of the Lemma. Under the second alternative, $t \subset w_1$ and $w_1 \subset$ an (m-1)-face of s, and we have case (ii) of the Lemma. We must only observe that whenever t is a face of some m-simplex v of $P(s)^{(n)}$, then this m-simplex will occur in one of Cases 1 or 2 above. But v must then arise from the first subdivision of some u_k; since all the faces of v also occur in the subdivision, t must occur amongst them.

13. The Brouwer fixed point Theorem. Any topological space which, like the cartesian product of m closed intervals, is homeomorphic to a closed m-simplex $\bar{s}$ is called a closed m-cell. To illustrate the utility of the barycentric subdivision, we shall establish the Brouwer fixed point theorem for such cells.

THEOREM 13.1. Any continuous map f of the closed m-cell into itself has at least one fixed point p, with $f(p) = p$.

The proof depends upon the Sperner Lemma.

LEMMA 13.2. Let s be an m-simplex, and g a function mapping each vertex v of the barycentric subdivision $P(s)^{(n)}$ into a vertex g(v) of s, in such a fashion that, for each face s_1 of s,

$$v \in s_1 \text{ implies } g(v) = \text{a vertex of } s_1.$$

Then there is an m-simplex $< v_o, \ldots, v_m >$ of $P(s)^{(n)}$ such that $g(v_o), \ldots, g(v_m)$ are the vertices of s, in some order.

For the proof, we will say that a k-chain in any polyhedron P is a formal linear combination

$$c = t_1 + t_2 + \ldots + t_\ell$$

of k-dimensional simplices t_i of P, with coefficients integers mod 2. The boundary of a k-simplex is the (k-1)-dimensional chain given by the formula

$$(13.1) \qquad \partial < q_o, \ldots, q_k > = \sum_{i=o}^{k} < q_o, \ldots, \hat{q}_i, \ldots, q_k >,$$

where the $\hat{q}_i$ indicates that q_i is to be omitted.

The given function g on the vertices of $P(s)^{(n)}$ to those of $P(s)$ determines a mapping (also called g) of the k-chains of $P(s)^{(n)}$ into those of $P(s)$. This mapping is defined for a simplex of $P(s)^{(n)}$ as

$$g < v_o, \ldots, v_k > \;=\; < gv_o, gv_1, \ldots, gv_k > \text{ or } 0$$

according as the vertices $gv_o, \ldots, gv_k$ are distinct or not. The mapping g is extended to chains by linearity. The property

$$(13.2) \qquad \partial g\, c \;=\; g \partial c$$

is basic. Since both ∂ and g are linear, it suffices to prove this for the case in which the chain c is a simplex t = $< v_o, \ldots, v_k >$. If $gt \neq 0$, the proof is immediate by the definition of the boundary operator. If $gt = 0$, then two vertices gv_i and gv_j of s are identical; in this case the terms i and j in $g\partial t$ cancel, and the remaining terms are zero.

Now let $c = t_1 + \ldots + t_\ell$ be the m-dimensional chain of $P(s)^{(n)}$ consisting of all the m-simplices in this polyhedron. Then gc is an m-dimensional chain in $P(s)$; since there is only one m-simplex here, we must have

$$(13.3) \qquad gc = \mathcal{E} < p_o, \ldots, p_m >$$

where $\mathcal{E}$ is 0 or 1. If we can prove that $\mathcal{E} = 1$, we are done, because then at least one m-simplex t_i of $P(s)^{(n)}$ must have $gt_i \neq 0$, and in fact $gt_i = s$.

The proof that $\mathcal{E} = 1$ is by induction on m. For m = 0 it is trivial. Assume it true for m = 1, and observe that

m-1

$$\partial c = \partial t_1 + \partial t_2 + \ldots + \partial t_\ell$$

consists of (m-1)-simplices u of $P(s)^{(n)}$. The number of times a simplex u appears here is exactly the number of m-simplices of which u is a face. By Lemma 12.4, this number is 2 ($\equiv 0$) if $u \subset s$, and is 1 when u is contained in one of the (m-1)-dimensional faces of s. Therefore ∂c is exactly the formal sum of all the (m-1)-simplices occurring on the faces $< p_o, \ldots, \hat{p}_i, \ldots p_n >$ of s. Therefore, by the induction assumption we have

$$g \, \partial \, c = \sum_{i=o}^{m} < p_o, \ldots, \hat{p}_i, \ldots, p_m >$$

On the other hand, by (13.3), (13.2) and the definition of the boundary, we have

$$g \, \partial \, c = \partial \, gc = \varepsilon \sum_{i=o}^{m} < p_o, \ldots, \hat{p}_i, \ldots, p_m >.$$

Hence $\varepsilon = 1$, q.e.d.

Now to prove the Brouwer Theorem, consider any continuous map $f: \bar{s} \longrightarrow \bar{s}$, and write, in barycentric coordinates

$$f(x_o p_o + \ldots + x_m p_m) = y_o p_o + \ldots + y_m p_m .$$

Then y_i is a continuous function of the point $p = \sum x_i p_i$ of $\bar{s}$. Let A_i denote the closed set of all points p of the simplex for which $y_i \leqq x_i$. Since $\sum y_i = \sum x_i = 1$, every point p belongs to at least one of these closed sets A_i. It will suffice to prove that

(13.7) $\qquad A_o \cap A_1 \cap \ldots \cap A_m \neq O$,

for any common point p all A_i must have $y_i \leqq x_i$ for all i, hence $y_i = x_i$, so that p is fixed under the mapping f.

We have

$$(13.8) \qquad < p_{i_0}, p_{i_1}, \ldots, p_{i_k} > \subset A_{i_0} \cup A_{i_1} \cup \ldots \cup A_{i_k},$$

for if p is any point of the simplex here displayed, then its coordinates satisfy

$$x_{i_0} + x_{i_1} + \ldots + x_{i_k} = 1,$$

so that at least one of the coordinates y_{i_j} of f(p) is not larger than the corresponding x_{i_j}.

Now assign to each vertex v of $P(s)^n$ a vertex $g(v) = p_i$, in such fashion that the corresponding set A_i contains v. In particular, if v is contained in a proper face (13.8) of s, we take care to choose p_i as one of the vertices of that face. Then g satisfies the hypothesis of the Lemma. The conclusion asserts that there is an m-simplex $t = < v_o, \ldots, v_m >$ such that $g(v_o), \ldots, g(v_m)$ are the vertices $p_o, \ldots, p_m$. Thus each set A_i contains at least one point of the closed simplex $\bar{t}$, and for each point r of t we have

$$\rho(r, A_i) \leq \text{Mesh } P(s)^{(n)}, \quad i = o, \ldots, m.$$

Recall that Mesh $P(s)^{(n)} \longrightarrow 0$ as $n \rightarrow \infty$.

For each barycentric subdivision we can have such a point r_n in the corresponding t. If an infinite number of the points r_n are equal (say to r_o) then the distance from r_o to the closed sets A_i is less than any prescribed $\varepsilon > 0$, hence $r_o \in A_i$ for each i and (13.7) holds. If there are an infinite number of distinct points r_n, all in the compact metric space $\bar{s}$, they have a limit point r_o, and for this limit point the same result obtains.

The same argument proves the following Lemma, due to Knaster, Kuratowski and Mazurkiewicz:

LEMMA (K, K&M). If $F_o, \ldots, F_m$ are m+1 closed sets covering the closed m-simplex $\bar{s}$ in such fashion that for each face

$$< p_{i_o}, p_{i_1}, \ldots, p_{i_k} > \subset F_{i_1} \cup \ldots \cup F_{i_k},$$

then $F_o \cap \ldots \cap F_m \neq \emptyset$.

14. Simplicial Maps. The advantage of using simplices rather than cubes or other types of convex cells as building blocks for polyhedral spaces lies in the fact that a simplex is determined by the set of its vertices, and that every affine map of a simplex is determined by the images of the vertices under that map.

DEFINITION: If P and Q are (affine) polyhedra, a continuous map $f: |P| \longrightarrow |Q|$ is said to be simplicial if f, restricted to each closed simplex $\bar{s}$ of $|P|$, is an affine map of $\bar{s}$ onto some closed simplex of Q.

In order to formulate the sense in which such maps f are determined by the images of vertices, it is convenient to replace a polyhedron P by its schema V(P), which is the combinatorial object consisting of the set V(P) of all vertices of (simplices of) P, in which a set of vertices $\{p_o, \ldots, p_m\}$ is called a frame of V(P) if and only if they are the vertices of a simplex $< p_o, \ldots, p_m >$ of V. The schema of V(P) is an abstract simplicial complex, in the sense of the following definition.

DEFINITION; An abstract simplicial complex (ask) V is a finite set of objects v, w, called the vertices of V, together with a collection of sets $S = \{v_0, \ldots, v_m\}$ of these vertices, called the frames of V, subject to the conditions

(i) Any subset of a frame of V is a frame of V

(ii) The set $\{v\}$ consisting of any vertex v of V, taken by itself, is a frame of V.

If S, T are frames of V, the inclusion relation $S \subset T$ may be read "S is a piece of T".

The schema of a polyhedron is always an ask; conversely, every ask V is isomorphic to the schema of some affine polyhedron P. Indeed, we may replace the finite number of independent vertices v of V by the same number of points p in a suitable affine space so chosen that all the points p are affine independent. Then whenever $S = \{v_0, \ldots, v_m\}$ is a frame of V, the corresponding points $\{p_0, \ldots, p_m\}$ are affine independent, and thus span a simplex $s = < p_0, \ldots, p_m >$. The set of all these simplices constitute a polyhedron P, for by condition (ii) above any face of such a simplex is again a simplex of P, while each point of the affine space has at most one representation $\sum_j x_j p_j$ by barycentric coordinates in the given independent points, hence belongs to at most one simplex of P. Thus P satisfies both conditions in the definition of a polyhedron; its schema is manifestly isomorphic to the given ask V.

A somewhat sharper result can be obtained. If we define the dimension of V to be the largest integer n such that V has a frame with n+1 points, then we may choose the polyhedron P above to lie in an affine space of dimension 2n+1. For example, a graph (polyhedron or ask of dimension 1) can always be realized by rectilinear segments in 3-dimensional space, although it is known that 2-dimensional space does not always suffice, as in the case of the graph consisting of all the edges joining five distinct points in all possible ways.

An (abstract) simplicial map $\varphi: V \to W$ of one ask into another is simply a homomorphism of the algebraic system V into the system W; that is, it is a correspondence which assigns to each vertex v of V a vertex $\varphi(v)$ of W in such a fashion that any frame of V is mapped into a frame of W. Also, φ is an isomorphism if it is a one-one map of V into W, and its inverse is a simplicial map. The composite of simplicial maps is simplicial.

The main result on simplicial maps is

THEOREM 14.1. Any abstract simplicial map φ: $V(P) \to V(Q)$ on the schema of two polyhedra induces a unique simplicial map φ_*: $|P| \to |Q|$ for which $\varphi_*(p) = \varphi(p)$ for every vertex p of P. Every simplicial map f: $|P| \to |Q|$ has the form $f = \varphi_*$ for some (unique) abstract simplicial map φ.

We have already employed a few simple such maps φ_* in the construction of homotopies; e.g., in proving the lemma about the wandering base point.

PROOF: For each simplex $\bar{s} = < p_0, \ldots, p_m >$ of P the vertices $\varphi p_0, \ldots, \varphi p_m$ of Q are the vertices of some simplex $\bar{t}$ of Q, possibly with repetitions. There is a unique affine map $A: \bar{s} \to \bar{t}$ with $A(p_i) = \varphi(p_i)$ $i = o,\ldots,m$. Define $\varphi_*: |P| \to |Q|$ by putting together these various affine maps; i.e., set $\varphi_*(p) = A(p)$ whenever the point p lies in the closed simplex $\bar{s}$ of P. No ambiguity is induced, for if p lies in two closed simplices $\bar{s}_1$ and $\bar{s}_2$, it lies in their (greatest) common face and the two affine maps A_1 in $\bar{s}_1$ and A_2 in $\bar{s}_2$ agree on this common face, since they have the same effect upon the vertices of the face. Thus φ_* carries the set $|P|$ into $|Q|$, and it is continuous in each of the closed sets $\bar{s}$, which together cover $|P|$. By the previous continuity theorem (262 notes, Theorem 10.1), it follows that φ_* is continuous.

Consider now an arbitrary simplicial map $f: |P| \to |Q|$. By definition f carries each vertex (0-simplex) of P onto a vertex of Q, hence induces a map φ in the vertices of V(P) to those of V(Q). Each closed simplex $\bar{s} = |p_o, \ldots, p_m|$ of P is by assumption mapped onto some closed simplex $\bar{t} = |q_o,\ldots,q_n|$ of Q. Since the q_j are the only vertices of Q lying in $\bar{t}$, each one of the vertices p_i must be carried by f into some q_j; hence φ carries the vertices of a simplex s of P onto vertices of a simplex of Q. Thus φ is an (abstract) simplicial map of the schema of P to that of Q. The continuous map φ_* induced by φ is affine on each simplex of Q, and agrees with f on the vertices; hence $\varphi_* = f$.

<u>COROLLARY 14.2</u>. If two polyhedra P and Q have isomorphic schema, their spaces are homeomorphic, under a simplicial homomorphism.

PROOF: The isomorphism $\varphi: V(P) \to V(Q)$ between the schema and its inverse $\varphi^{-1}: V(Q) \to V(P)$, induce continuous maps $\varphi_*: |P| \to |Q|$, $(\varphi^{-1})_*: |Q| \to |P|$, and $\varphi_*(\varphi^{-1})_* = \text{identity} = (\varphi^{-1})_*\varphi_*$, by the uniqueness assertion of the theorem.

The last argument here depends implicitly on the proposition that abstract simplicial mappings $\varphi: V(P) \to V(Q)$ and $\psi: V(Q) \to V(R)$ on the schema of polyhedra P, Q, R induce simplicial maps with $(\psi\varphi)_* = \psi_*\varphi_*$.

15. <u>Nerves</u> <u>of</u> <u>Coverings</u>. As another example of an abstract simplicial complex, consider any covering $\mathfrak{A}$ of a topological space X by a finite number of non-empty sets A_i,

$$X = A_1 \cup \ldots \cup A_n.$$

(The case of a finite open covering will be especially useful). By the <u>nerve</u> $N(\mathfrak{A})$ of the covering $\mathfrak{A}$ we mean the ask in which the vertices are the sets A_i of the covering, and in which the vertices $A_{i_0}, \ldots, A_{i_m}$ belong to a frame of $N(\mathfrak{A})$ if and only if the intersection of the corresponding sets is non-empty

$$A_{i_0} \cap \ldots \cap A_{i_m} \neq \emptyset.$$

Any subset of these A's then has a non-void intersection, hence a subset of a frame is indeed a frame, as required in the definition of an ask.

If $f: X \to Y$ is a continuous map, each covering $\mathfrak{A}$ of Y by sets A_i determines a covering $f^{-1}\mathfrak{A}$ of X by the inverse image sets $f^{-1}(A_i)$, which will be open if the sets A_i are open. If f is a map of X onto Y, intersecting families of sets on Y are carried backwards into such on X, hence in this case f induces an abstract simplicial map $\varphi : N(\mathcal{O}\!\!\mathcal{l}) \longrightarrow N(f^{-1}(\mathcal{O}\!\!\mathcal{l}))$ on the nerves.

An important instance arises with polyhedra P. If p is any vertex of P, the star of p is the union

$$St(p) = \sum s, \quad p \text{ a vertex of } s$$

of all the open simplices of P which have p as vertex. The Star of p is an open set in P, although the open simplices s need not be open in P. Indeed, the complement of St(p) is the union of all open simplices t of which p is not a vertex. With each such t, every face of t is also one in which p is not a vertex. Thus the complement P - St(p) is the union of the closed simplices $\bar{t}$, which are closed in P, hence is closed. Therefore St(p) is open in P.

<u>THEOREM 15.1</u>. The nerve of the covering of a polyhedron $|P|$ by the stars of its vertices is an abstract simplicial complex isomorphic to the schema of P, under the correspondence sending each St(p) into the vertex p.

PROOF: By the definitions, the conclusion amounts to the assertion that, for distinct vertices $p_o, \dots, p_m$,

$$St(p_o) \cap \dots \cap St(p_m) \neq \emptyset \text{ if and only if } \langle p_o, \dots, p_m \rangle \in P.$$

If the stars displayed have a point p in common, the simplex s of P containing this point must lie in each $St(p_i)$, hence must have each p_i as one of its vertices. The vertices $p_o, \dots, p_m$ are thus those of some face of s.

Conversely, if $t = \langle p_o, \dots, p_m \rangle$ is a simplex of P, this simplex is contained in each $St(p_i)$, hence these stars have a non-void intersection.

A similar closed covering of a polyhedron may be defined by the barycentric stars. If p is a vertex of P, the <u>barycentric star</u> $B_{st}(p)$ is the set defined as the union of all those closed simplices t of the first barycentric subdivision P' such that p is one of the vertices of t. This star $B_{st}(p)$ is therefore a closed subset of $|P|$.

<u>THEOREM 15.2</u>. The barycentric stars of the vertices p of a polyhedron P constitute a closed covering of the space $|P|$. The nerve of this covering is isomorphic to the schema of P, under the correspondence $p \to B_{st}(p)$.

PROOF: Every point x of $|P|$ is contained in one of the open simplices

$$t = \langle b(s_o), \ldots, b(s_n) \rangle$$

of P', where $s_o, \ldots, s_n$ are simplices of P with $\bar{s}_o \subset \ldots \subset \bar{s}_n$. Either s_o is a single vertex p of P, and in this case the closed simplex $\bar{t}$ appears in the barycentric star of this vertex p, or we may choose p to be one of the vertices of s_o, and form the (larger) closed simplex

$$u = \langle p, b(s_o), \ldots, b(s_n) \rangle$$

of P'. Then $x \in t \subset \bar{u}$, and $\bar{u}$ is part of the barycentric star of p. Hence these stars cover $|P|$.

To show the asserted isomorphism on the nerve of this covering, we need only prove, for any vertices $p_o, \ldots, p_n$ of P, that

$B_{st}(p_o) \cap \ldots \cap B_{st}(p_n) \neq \emptyset$ if and only if $\langle p_o, \ldots, p_n \rangle \in P$.

Indeed, if $\langle p_o, \ldots, p_n \rangle = s$ is a simplex of P, then its barycenter b(s) is a point on each closed simplex $|p_i, b(s)|$ of P', and this simplex is contained in the barycentric star of p_i. Hence these barycentric stars have the point b(s) in common.

Conversely, it will suffice to prove that $B_{st}(p) \subset St(p)$ for then a collection of barycentric stars has a non-void intersection only if the corresponding open stars do. To prove $B_{st}(p) \subset St(p)$, consider any simplex

$$\bar{t} = |p, b(s_1), \ldots, b(s_n)|$$

in the barycentric star of p. Any point x of $\bar{t}$ is covered by some open face u of t, with last vertex $b(s_i)$, and this open face is contained in the open simplex s_i with p as one of its vertices. Then $x \in u \subset s_i \subset St(p)$, q.e.d.

16. The Plaster Theorem. We now turn to some additional theorems proved by the general type of method used for the Brouwer fixed point theorem. First some properties of coverings of compact metric spaces.

LEMMA 16.1. If $F_1, \ldots, F_n$ are closed sets in a compact metric space X, with

(16.1) $$F_1 \cap F_2 \cap \ldots \cap F_n = \emptyset$$

then there is a positive number $c > 0$ such that any point of X has distance at least c from some one of the sets F_i.

PROOF: For each i and each integer m let $U_{m,i}$ be the open subset of X consisting of all points x of X with distance from F_i greater than 1/m (the set $U_{m,i}$ is open because the distance function $\rho(x,F_i)$ is continuous). Since the intersection (16.1) is empty, every point x of X has positive distance from at least one of the closed sets F_i, hence must lie in at least one of the sets $U_{m,i}$. In other words, these sets constitute an open covering of X. Because X is compact, a finite number of sets $U_{m,i}$ will then cover X. Let c be the minimum value of $1/m$ occuring in any index of this finite covering. This c has the required property.

LEMMA 16.2. If $F_1, \ldots, F_n$ are closed sets in a compact metric space X, then there is a positive number $d > 0$ such that, for any point $x \in X$ and any subcollection $F_{e_1}, \ldots, F_{e_m}$ of the F's,

$$\rho(x, F_{e_j}) < d, \quad j = 1, \ldots, m \text{ implies } F_{e_1} \cap \ldots \cap F_{e_m} \neq \emptyset.$$

Also, if a subset A of X of diameter less than d meets each of $F_{e_1}, \ldots, F_{e_m}$, then $F_{e_1} \cap \ldots \cap F_{e_m} \neq \emptyset$.

PROOF: For each list $F_{h_1}, \ldots, F_{h_t}$ of sets F_i with an empty intersection $F_{h_1} \cap \ldots \cap F_{h_t} = \emptyset$ we may choose a positive number $c > 0$ with the property stated in Lemma 16.1, for this list. There are but a finite number of such lists; choose d as the minimum of the c's which arise. Then if $\rho(x, F_{e_j}) < d$ the sets $F_{e_1}, \ldots, F_{e_m}$ cannot be one of these lists, hence have a non-void intersection. The alternative conclusion is immediate, for any point x of A is at distance less than d from a point of A in F_{e_j}, $j = 1, \ldots, m$.

These lemmas lead up to

THEOREM 16.3. If $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a finite covering of a compact metric space X by open sets U_i, there is a positive number $d > 0$ such that every subset of X of diameter less than d is contained entirely within one of the sets U_i.

The conclusion of this Lemma asserts, in a precise fashion, that the open sets U_i must "overlap" if they are to cover X.

PROOF: Let $F_i = X - U_i$ be the complements of the sets U_i of the covering. Since the U_i cover X, it follows that $F_1 \cap \ldots \cap F_n = \emptyset$. Choose c as in Lemma 16.1. If a set $A \subset X$ has diameter less than c, and is contained in no one set U_i of the covering, then A meets every set F_i, and a point $x \in A$ is at distance less than c from every F_i, in contradiction to the conclusion of Lemma 16.1.

The least upper bound of all the numbers d having the property expressed in this theorem also has this property. This number is known as the <u>Lebesgue number</u> of the covering $\mathcal{U}$.

<u>COROLLARY 16.4</u>. If d is the Lebesgue number of the open covering $\mathcal{U}$, then for each point $x \in X$ we may choose $U_i \in \mathcal{U}$ such that $x \in U_i$ and $\rho(x, X - U_i) \geqq d/2$.

dim X = inf order 𝒞 where 𝒞 covers X.

The <u>order</u> of a finite covering $\mathcal{C} = \{C_1, \ldots, C_n\}$

$$X = C_1 \cup \ldots \cup C_n$$

of a space X is defined to be the maximum number k such that some k+1 sets C_i have a common point. Thus the order of $\mathcal{C}$ is exactly the dimension of the nerve of $\mathcal{C}$.

<u>THEOREM 16.5</u>. (The Lebesgue Plaster Theorem). If P is an n-dimensional affine complex, there exists a number $d > 0$ such that any closed covering of $|P|$ by sets of diameter less than d has at least the order n (= dimension of P).

PROOF: Since P consists of a finite number of simplices, of which at least one is of dimension n, it suffices to prove the theorem for the case of an n-simplex $\bar{s}$. For the (n-1)-dimensional closed faces $\bar{t}_o, \ldots, \bar{t}_n$ of $\bar{s}$ choose a number d as in Lemma 16.2. Now let $\{C_1, \ldots, C_r\}$ be any closed covering of $\bar{s}$, with each C_i of diameter less than d. The (n-1)-dimensional faces of $\bar{s}$ have no point in common; hence, by the choice of d, no set C_i can meet every closed face $\bar{t}_j$. Let A_j be the union of all the sets C_k which do not meet $\bar{t}_j$, but which meet every $\bar{t}_i$, for i < j. Then the closed sets $A_o, \ldots, A_n$ contain all the sets C_j and hence cover $\bar{s}$. Since no point of $\bar{t}_j$ is contained in A_j C_j we must have

$$\bar{t}_j \subset A_o \cup \ldots \cup \hat{A}_j \cup \ldots \cup A_n .$$

Let $u = < p_{i_o}, \ldots, p_{i_k} >$ be any face of $\bar{s}$, and j any index different from all the subscripts i appearing here. Then u is contained in $\bar{t}_j$, hence cannot meet A_j, for each such j. Therefore

$$< p_{i_o}, \ldots, p_{i_k} > \subset A_{i_o} \cup \ldots \cup A_{i_k} .$$

This is the hypothesis of the KKM Lemma. We conclude that $A_o \cap \ldots \cap A_n \neq \emptyset$. Any common point here is a point common to n+1 distinct sets C of the given covering, q.e.d.

A converse assertion is

<u>LEMMA 16.6</u>. If P is an n-dimensional affine complex, then for each positive ε there exists a closed (or an open) finite covering of P of order n in which each set has diameter less than ε .

PROOF: Take a sufficiently fine barycentric subdivision of P. The covering by open stars of vertices in this subdivision $P^{(m)}$ is then an open covering with the required property, while the covering by closed barycentric stars of vertices of $P^{(m)}$ is a closed covering with this same property, for the nerve of either covering is isomorphic to the schema of $P^{(m)}$, and hence of dimension n.

THEOREM 16.7. (Brouwer dimension theorem). If $m > n$, then Euclidean n-space contains no topological image of a closed m-simplex.

PROOF: Suppose to the contrary that f is a homeomorphism of the m-simplex $\bar{s}$ to a subset of E_n. The image $f(\bar{s})$ is a compact and hence bounded subset of E_n; it is therefore contained in a suitably large closed n-simplex $\bar{t}$ of E_n. Since $f: \bar{s} \to \bar{t}$ and its inverse are uniformly continuous, we may choose for each $\varepsilon > 0$ a $\delta > 0$ such that every set of diameter less than δ in $\bar{t}$ has its inverse image of diameter less than ε in $\bar{s}$. Take in particular a closed covering of $\bar{t}$ (say that by suitable closed barycentric stars, Lemma 16.6) in which the sets have diameter less than δ. The inverse images of the sets of this covering cover $\bar{s}$, have diameter less than ε, and form a covering of order $n < m$. This contradicts the plaster theorem.

The dimension of a compact metric space X is defined by the assertion that $\dim(X) \leq n$ if and only if X has closed coverings of arbitrarily small mesh and order $\leq n$. Lemma 16.6 and Theorem 16.5 assert that a polyhedron P with dimension n (defined as the maximum dimension of the simplices of P) has topological dimension n in the sense just defined.

17. Simplicial Approximation. The reduction of the study of continuous maps to the study of abstract maps depends upon the following definition of approximation.

DEFINITION: If P and Q are polyhedra, and $f\colon |P| \longrightarrow |Q|$ is a continuous map, then an (abstract) simplicial map $\varphi\colon V(P) \longrightarrow V(Q)$ is called a simplicial approximation of f if and only if, for each vertex p of P,

(17.1) $$f(St(p)) \subset St(\varphi(p)).$$

Note that the continuous map f between the spaces of the polyhedra is "approximated" by the abstract homomorphism φ between the schema of the polyhedra. We may replace this description by one in terms of the simplicial map φ_* induced by φ.

LEMMA 17.1. The map φ is a simplicial approximation to f if and only if, for each point $r \in |P|$ and each (open) simplex t of Q

(17.2) $$f(r) \in t \quad \text{implies} \quad \varphi_*(r) \in \bar{t}.$$

In other words, the two continuous maps $f, \varphi_*\colon |P| \longrightarrow |Q|$ must be such that any closed simplex of Q containing the f-image of a point also contains the φ_* image of that point. In particular, the distance from $f(r)$ to $\varphi_*(r)$ is less than the mesh of Q, and in this sense the two maps are not far apart.

PROOF: For each point r of $|P|$ we have simplices s in P, t in Q with

(17.3) $\quad r \in s = \langle p_o, \ldots, p_m \rangle \qquad f(r) \in t = \langle q_o, \ldots, q_n \rangle$

Also $r \in St(p_i)$, $i=o,\ldots,m$ and $f(r) \in St(q_j)$, $j=o,\ldots,n$, and these are the only stars in Q containing $f(r)$. The mapping φ must carry the vertices $p_o, \ldots, p_m$ of s to the vertices $\varphi p_o, \ldots, \varphi p_m$ which are (possibly with repetitions) vertices of a simplex of Q. If r is expressed by barycentric coordinates, then

$$r = \sum_{i=o}^{m} x_i p_i, \qquad \sum x_i = 1, \qquad x_i > 0, \qquad i = 0,\ldots,m.$$

Since the simplicial map φ_* is linear on $\bar{s}$, we have

(17.4) $\quad \varphi_*(r) = \sum_{i=o}^{m} x_i \varphi(p_i).$

Now suppose that φ is a simplicial approximation to f, as in (17.1). For each $r \in |P|$, as above, $r \in St(p_i)$, hence by (17.1) $f(r) \in St(\varphi(p_i))$. Since $St(q_i)$ are the only stars containing $f(r)$, by (17.3), each $\varphi(p_i)$ is one of the q_j, and (17.4) asserts that $\varphi_*(r)$ lies in the convex set spanned by $q_o,\ldots,q_n$; i.e., $\varphi_*(r) \in \bar{t}$, as required in (17.2).

Conversely, suppose that (17.2) holds. If p is any vertex of P, and $r \in St(p)$, then p is one of the p_i of (17.3), say p_o. By (17.4), the open simplex u of Q carrying $\varphi_*(r)$ has the vertices $\varphi(p_o),\ldots,\varphi(p_m)$. By (17.2), u must be a face of $\bar{t}$, hence the vertices $\varphi(p_i)$ must be among the q_j. In particular, $\varphi(p_o)$ is some q_j, and therefore $f(r) \in t \subset St(q_j) = St(\varphi(p_o))$, as required in (17.1).

From the definitions one may readily show that if maps $f\colon |P| \longrightarrow |Q|$ and $g\colon |Q| \longrightarrow |R|$ have simplicial approximations

$\varphi: V(P) \to V(Q)$, $\psi: V(Q) \to V(R)$, then the composite $\psi \circ \varphi$ is a simplicial approximation to gof.

As an illustration we cite the following instance of a simplicial approximation to the identity.

<u>LEMMA 17.2</u>. If P' is the first barycentric subdivision of a polyhedron P, and $\varphi: V(P') \to V(P)$ maps each barycenter b(s) of a simplex s of P into one of the vertices p of s, then φ is a simplicial approximation to the identity map of $|P'|$ to $|P|$.

PROOF: We must first observe that φ is indeed an abstract simplicial map. Any open simplex of P' has the form

$$t = < b(s_0), b(s_1), \ldots, b(s_n) >, \qquad s_i \text{ a face of } s_{i+1},$$

and is contained in the open simplex s_n of P. Under φ the vertices of t are all mapped into vertices of s_n, hence φ is simplicial. Furthermore, the star of the vertex b(s) in P' is the union of all simplices t above in which b(s) occurs, and each of these simplices $t \subset s_n$ is contained in the star (in P) of every vertex of s. Hence

$$St(b(s)) \subset St(\varphi(b(s))).$$

This asserts, according to the definition (17.1), that φ is a simplicial approximation to the identity.

We now specify more carefully the sense in which the "affine" map φ_* approximates to f.

<u>THEOREM 17.4</u>. If $\varphi: V(P) \to V(Q)$ is a simplicial approximation to f: $|P| \dashrightarrow |Q|$, then there is a homotopy F: $f \simeq \varphi_*$:

$|P| \longrightarrow |Q|$. For each point $r \in |P|$ with $f(r)$ in an open simplex t of Q, the image of r moves during the homotopy in the closed simplex t.

PROOF: We must define a mapping $F: |P| \times I \longrightarrow |Q|$. For each $r \in |P|$ the points $f(r)$ and $\varphi_*(r)$ lie in one and the same closed simplex t of Q. For each value of the parameter u, $0 \leq u \leq 1$, we then set

$$F(r,u) = (1-u)f(r) + u\varphi_*(r),$$ F(r,u)

in other words the image of $f(r)$ moves at uniform speed along the segment joining $f(r)$ to $\varphi_*(r)$. At the start of the homotopy $F(r,u) = f(r)$; at the end, $F(r,1) = \varphi_*(r)$. Furthermore, F is continuous because f and φ_* are, and the process of forming the weighted average is continuous. This homotopy satisfies all the stated conditions.

We now turn to the construction of simplicial approximations. It is convenient to speak of an <u>ordered</u> polyhedron P--a polyhedron in which the vertices have been so partially ordered that the vertices of any one simplex are linearly ordered. (Any linear order of the vertices will do this.)

<u>THEOREM 17.5</u>. If P and Q are polyhedra and f a map of $|P|$ to $|Q|$, then f has a simplicial approximation $\varphi: V(P) \longrightarrow V(Q)$ if and only if the image under f of each star of P is contained in at least one star of Q. When this is the case, and Q is ordered, there is a unique approximation φ satisfying the condition that, for each vertex p of P, $\varphi(p)$ is the first vertex of Q such that

(17.5) $\qquad f(St(p)) \subset St(\varphi(p))$.

PROOF: By the definition (17.1), the condition cited is necessary. Suppose conversely that this conclusion holds. Then the description by (17.5) above defines $\varphi(p)$ uniquely for each vertex p of P. This map φ will be an approximation if it is simplicial. But if $p_0,\ldots,p_m$ are vertices of an m-simplex of P, then

$$St(p_0) \cap St(p_1) \cap \ldots \cap St(p_m) \neq \emptyset.$$

A point x in this intersection must by (17.5) lie in every $St(\varphi(p_i))$. Hence the intersection of the latter stars is non-empty, and the vertices $\varphi(p_i)$ must therefore be vertices of a simplex of Q (possibly with repetitions). Therefore φ is indeed simplicial.

Using the order in Q, we may speak as in Theorem 17.5 of <u>the</u> simplicial approximation φ to f, when it exists. If R is a sub-polyhedron of Q, and f cut down to $|R|$ already has its unique simplicial approximation ψ ; $V(R) \longrightarrow V(Q)$, the approximation f on V(P) must therefore be an extension of the approximation ψ .

A simplicial approximation can always be found by subdivision, as follows

<u>THEOREM 17.6</u>. If P and Q are polyhedra, and f: $|P| \longrightarrow |Q|$ is continuous, there exists a repeated barycentric subdivision $P^{(n)}$ of P such that f has a simplicial approximation $\varphi : V(P^{(n)}) \longrightarrow V(Q)$.

PROOF: Let q range over the vertices of Q. Then the open sets St(q) cover $|Q|$; hence their inverse images $f^{-1}(St(q))$ are open sets covering $|P|$. Let $\varepsilon > 0$ be the Lebesgue number of this covering of the compact metric space $|P|$. By repeated barycentric subdivision, we can find $P^{(n)}$ with mesh less than $\varepsilon/2$. Then the diameter of any star of $P^{(n)}$ is less than ε, hence any St(p) in $P^{(n)}$ is contained in some $f^{-1}(St(q))$. This gives at once the necessary condition of Theorem 17.5.

One may also verify easily that if

$$f: |P| \longrightarrow |Q|, \qquad g: |Q| \longrightarrow |R|$$

have simplicial approximations

$$\varphi: V(P) \longrightarrow V(Q), \qquad \psi: V(Q) \longrightarrow V(R),$$

then the composite $\psi\varphi$ is a simplicial approximation of the composite gf.

18. Calculation of the Fundamental Group. Simplicial approximations reduce the determination of the fundamental group of the space of a (connected) polyhedron P to a strictly algebraic problem, dealing with (a finite number of) "edge paths" in P.

An edge in the polyhedron P is a symbol E = (pq), where p = q is a vertex of P, or p and q are the vertices of a 1-simplex of P. Call p the start of E, q the end of E. An edge path L in P is any finite formal product (or string) of edges $E_1, \dots, E_k$ in P such that the end of each E_i is the start of E_{i+1}, for $i = 1, \dots, k-1$. Then L has the form

(18.1) $\quad L = E_1 \dots E_k = (p_o p_1)(p_1 p_2) \dots (p_{k-1} p_k), \quad p_i$ vertices of P.

This edge path L starts at p_o, the start of E_1, and ends at p_k, the end of E_k. The product LM of two edge paths is formed by juxtaposition (string the edges of M behind those of L) and is defined if and only if M starts where L ends. Two edge paths L, L' are <u>equal</u> if and only if L' can be obtained from L by a finite number of certain rational moves; each move consists in replacing the edge path on either side of——— the following equation by the edge path on the other side:

$$L(pq)(qr)M = L(pr)M \tag{18.2}$$

<u>provided</u> p, q, r are (not necessarily distinct) vertices of a simplex of P. Here L and M denote arbitrary edge paths in P, which may be various; the move can be applied only when the product exhibited on one side of this equation is known to be defined, and this insures that the product on the other side is defined. The single rule (18.2) can be sent into second cases, according to possible equalities between p, q and r, thus

$$L(pq)(qr)M = L(pr)M, \qquad \langle pqr \rangle \in P, \tag{18.3}$$

$$L(pq)(qp)M = L(pp)M, \qquad \langle pq \rangle \in P, \tag{18.4}$$

$$L(pp)(pq)M = L(pq)M, \qquad \langle pq \rangle \in P, \tag{18.5}$$

$$L(pq)(qq)M = L(pq)M, \qquad \langle pq \rangle \in P, \tag{18.6}$$

$$L(pp)(pp)M = L(pp)M, \qquad \langle p \rangle \in P. \tag{18.7}$$

If L = L' and M = M', and LM is defined, then LM = L'M', hence the product of edge paths is well defined under this equality. It is trivial to verify, with this multiplication and equality, that the edge paths in P form a groupoid In particular, the edge paths starting and ending at a fixed vertex p_o of P form a group, the <u>edge path group</u> $\mathcal{E}(P,p_o)$.

The same definition will yield the edge path group $\mathcal{E}(V,p_o)$ of an abstract simplicial complex V. Exactly as in the case of the fundamental group, any edge path N from p_o to a second vertex q_o of V will yield an isomorphism $L \to NLN^{-1}$ of $\mathcal{E}(V,p_o)$ onto $\mathcal{E}(V,q_o)$, and the isomorphism is unique up to conjugates.

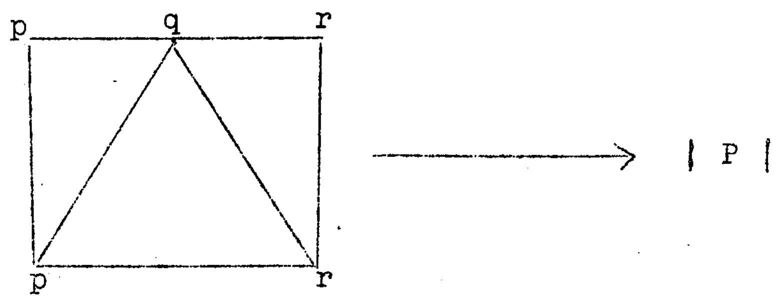
If $\varphi : V \to W$ is a simplicial map, the definition $\varphi(pq) = (\varphi p, \varphi q)$ yields a map which carries each edge of V into an edge of W; since φ carries frames of V into frames of W, this induces a homomorphism

(18.8) $$\varphi_* : \mathcal{E}(V,p_o) \to \mathcal{E}(W,\varphi p_o)$$

on the appropriate edge path groups.

Each edge path L from p_o to p_k in a polyhedron P determines an ordinary or continuous path class λL in $|P|$ from p_o to p_k. This path class may be described as that homotopy class which contains the path which follows the edges (1-simplices) of L in succession. Explicitly, let $E = (pq)$ be an edge in P, regard the unit interval I as (the space of) a polyhedron I with its ends 0 and 1 as the two vertices; construct the simplicial map $\alpha : I \to P$ with $\alpha(0) = p$, $\alpha(1) = q$ and the corresponding affine map $\alpha_* : |I| \to |P|$. Then α_* is a path in $|P|$ from p to q, and $\lambda(E)$ is defined as the class of paths homotopic (rel 0,1) to α_*. For a product of edges, define $\lambda(E_1 \ldots E_k)$ as $\lambda(E_1) \ldots \lambda(E_k)$.

This mapping λ carries equal edge paths into the same (continuous) path classes, for if p,q,r are the vertices of a simplex of P, we triangulate the square IxI as shown and con-



struct that simplicial map φ_*: IxI $\longrightarrow |P|$ which carries the vertices of this triangulation into the labelled vertices of P. This map is clearly a homotopy of $\lambda(pr)$ to $\lambda(pq)\lambda(qr)$, corresponding to the edge path equality (pr) = (pq)(qr) of (18.2).

By its very definition, the mapping λ is a homomorphism of the edge path group of P at p_o into the fundamental group at p_o. The basic result is

<u>THEOREM 18.1</u>. The mapping λ described above is an isomorphism of the edge path group of P at p_o onto the fundamental group of the space $|P|$ at the base point p_o:

$$\lambda: \ \mathcal{E}(P,p_o) \longleftrightarrow \pi_1(\ |P|,\ p_o\ ).$$

The proof depends upon simplicial approximations, and uses an auxiliary result. Let I be the unit interval, regarded as a polyhedron, I_k the polyhedron obtained by subdividing I into k equal intervals. Let the vertices of the polyhedron P be partially ordered.

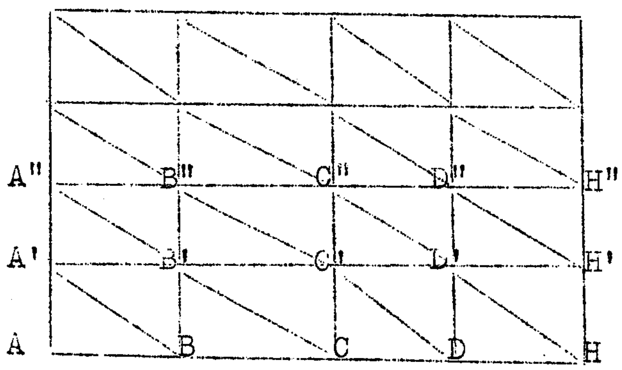
<u>LEMMA 18.2</u>. If β: $V(I) \longrightarrow V(P)$ is simplicial, then β_*: $|I| \longrightarrow |P|$ has a simplicial approximation φ: $V(I_k) \longrightarrow V(P)$, and φ_* represents the same edge path as β_*. In particular, if β_* is a constant mapping, so is φ_*.

PROOF: The given β maps the ends 0, 1 of I into vertices p, q of P. If $p \neq q$, we may assume (say) that $p < q$ in the order of the vertices of P. The interval I is subdivided by points i/k, $i=o,\ldots,k$. Since β_* carries I linearly onto $|pq|$, the star of each interval subdivision point i/k in I_k is the open interval for $(i-1)/k$ to $(i+1)/k$, hence is contained in the star of p in $|P|$. Thus the unique simplicial approximation φ must map i/k into p, for $i < k$, and 1 into q, and φ_* represents the edge path $(pp)^{k-1}(pq) = (pq)$. If $p = q$, φ carries each vertex of I_k into p.

Return now to the proof of the theorem. We first show that λ is a homomorphism onto $\pi_1(|P|, p_o)$. Let $\alpha : I \rightarrow |P|$ be any path at the base point p_o in the space $|P|$. Regard I as the space of a polyhedron with vertices 0, 1 and 1-simplex $< 0,1 >$. By the simplicial approximation theorem, there is a subdivision $I^{(m)}$ of I and a simplicial approximation $\varphi : V(I^{(m)}) \longrightarrow V(P)$ to α ; furthermore, α is homotopic to φ_*. The end point 0 of I has image $\alpha(0) = p_o$ contained in the 0-simplex $< p_o >$ of P; during the homotopy it does not leave the closure $|p_o| = < p_o >$ of this simplex; hence it stays fixed during the homotopy. The same argument applies to the endpoint 1. Hence $\alpha \simeq \varphi_*$ (rel 0,1). The path class of α is thus the path class of φ_*, and the latter is clearly the λ-image of an edge path (composed of 2^m edges arising from the 2^m intervals in the subdivision $I^{(m)}$).

Next we show that λ is an isomorphism into. Let L be an edge path with $\lambda(L)$ the identity class. If $L = E_1\ldots E_k$, $\lambda(L)$

is by definition represented by a path α in $|P|$ which arises from a simplicial mapping of the subdivision of I_k into k equal intervals into P. Hence there is a homotopy $F: I_u \times I_v \longrightarrow |P|$ which starts (v = 0) with the path α and which ends (v=1) with the constant map into p_o. We now subdivide the square $I_u \times I_v$ first by (k-1) equally spaced horizontal and vertical lines into smaller squares (this to match the subdivision already given on the base I_u) and then into more equal smaller squares, so small that the diameter of the image of any such square under F is less than half the Lebesgue number of the covering of $|P|$ by its barycentric stars. If each such square is cut into two triangles by a diagonal (see figure) we can regard the square



$I_u \times I_v$ as the space of a polyhedron Q with all the triangles (and their faces) as simplices of Q. The given homotopy F: $|Q| \longrightarrow |P|$ then has a simplicial approximation $\varphi: V(Q) \longrightarrow V(P)$. On each of the four sides of the square the map φ_* must, by Lemma 18.2, represent the same edge path as did F. In particular, since F maps top and lateral sides to p_o, so does φ_*, and φ_* on the bottom must represent the given edge path.

The given edge path L, represented by the simplicial mapping on the bottom of the square, can now be altered successively over each triangle of the square (figure above) as follows

$$ABCDH \longrightarrow A'BCDH \longrightarrow A'B'BCDH \longrightarrow A'B'CDH \ldots$$

$$\longrightarrow A'B'C'D'H' \longrightarrow A''B'C'D'H' \longrightarrow \ldots$$

Each of these alterations replaces one edge of a simplex (pqr) of P by the other two edges, or vice-versa. Hence, carried out in succession, they show that the given edge path L is equal, in the combinatorial sense, to the identity path $(p_o p_o)$. This completes the proof of the Theorem.

If P is a polyhedron, we let P^k denote its k-dimensional <u>skeleton</u>, that is, the polyhedron whose vertices are the vertices of P, and whose simplices are all simplices of dimension at most k in P. A simple argument will show

<u>THEOREM 18.3</u>. The space of a polyhedron P is connected if and only if the one-dimensional skeleton P^1 of P has the following property: any two vertices of P^1 can be joined by an edge path in P^1.

To define the edge path group one need only know which pairs of vertices in P belong to 1-simplices; to define equality we must also know which triples of vertices belong to two simplices. The edge path group of P, and hence the fundamental group of $|P|$, thus depend only on the two-dimensional skeleton P^2. Without using the edge path group, one can prove directly, by the technique of simplicial approximation:

<u>THEOREM 18.4</u>. Let P be a connected polyhedron, P^1 and P^2 one- and two-dimensional skeletons, $i_1: |P^1| \longrightarrow |P|$ and $i_2: |P^2| \longrightarrow$

$|P|$ the continuous maps given by the identity transformations. Then, for any vertex p_o of P, i_1 induces a homomorphism of $\pi_1(|P^1|, p_o)$ onto $\pi_1(|P|, p_o)$, and i_2 induces an isomorphism of

$$\pi_1(|P^2|, p_o) \cong \pi_1(|P|, p_o).$$

The edge path group $\mathcal{E}(P, p_o)$ actually depends only upon the schema $V(P)$. Much as in the case of the fundamental group, one can prove algebraically that the edge path groups at two vertices are isomorphic, the isomorphism being determined up to an inner automorphism. Also any simplicial map $\varphi: V(P) \longrightarrow V(Q)$ induces a homomorphism of $\mathcal{E}(P, p_o)$ into $\mathcal{E}(Q, \varphi p_o)$ in the obvious manner (map each edge (p,q) onto the edge $(\varphi p, \varphi q)$). This can be used to compute, not only the fundamental groups of $|P|$ and $|Q|$, but the homomorphism between them induced by a continuous map $f: |P| \longrightarrow |Q|$. Indeed, we know that the induced map may be obtained from any f_1 homotopic to f, so we may replace f by a simplicial approximation $\varphi: V(P) \longrightarrow V(Q)$, if possible. The induced map is then essentially the map induced by φ on the edge path groups. In this process it may be necessary to subdivide P, but from Theorem 18.1 it follows readily that P and $P^{(m)}$ have isomorphic edge path groups. This fact can also be established directly (i.e., algebraically).

19. Degrees.

THEOREM 19.1. The fundamental group of the circle is an infinite cyclic group, with the (class of) identity map of the circle on itself as generator.

PROOF: The circle is homeomorphic to the boundary of the two-simplex, hence may be regarded as a polyhedron $\dot{\Delta}_2$ with three vertices p,q,r and the two-simplices < pq >, < qr >, < pr >. We show that the edge path group $\mathcal{E} = \mathcal{E}(\dot{\Delta}_2, p)$ is infinite cyclic, with generator the edge path

$$L_1 = (pq)(qr)(rp).$$

To this end, define a homomorphism $f: \mathcal{E} \longrightarrow J$, with J the additive group of integers, by setting

$$f(qr) = 1, \qquad f(rq) = -1$$

and f(E) = 0 for any other edge E of $\dot{\Delta}_2$. The value of f on an edge path is then defined as

$$f(E_1 \ldots E_k) = f(E_1) + \ldots + f(E_k).$$

We must show first that f is well defined under the equality (18.2) of edge paths. Since there are no 2-simplices in $\dot{\Delta}_2$, this equality can remove an edge (qr) or (rq) only in the case (qr)(rq) = (qq); in this case f remains unaltered. By its very definition f is a homomorphism of $\mathcal{E}$ into J; since $f(L_1) = 1$, it is a homomorphism onto J, with L_1 mapped on the generator of J.

It remains only to calculate the kernel of f, to prove that f(L) = 0 implies L = (pp), the identity of $\mathcal{E}$. To this end represent an edge path L as a product of edges $E_1 \dots E_k$; and call this representation, <u>reduced</u> if it contains no edge (bb), for <u>b</u> a vertex of $\dot{\Delta}_2$ and no pair of edges (bc)(cb) in succession, for b and c vertices of $\dot{\Delta}_2$. By the rules (18.2) for equality, and by induction on the number of edges k in a representation, every edge path not (pp) clearly has a reduced representation, $E_1 \dots E_k$. The path starts at p, hence the initial edge must be either (pq) or (pr). In the first case, the next edge cannot be (qp), by the "reduced" condition, hence must be (qr), and the third edge must likewise be (rp). Thus $L = L_1 E_4 \dots E_k$, where E_4 must be (pq), and ultimately $L = L_1^s$ for some exponent s. In this case $f(L) = sf(L_1) \neq 0$. In the second case, when L starts in its reduced representation with (pr), we obtain $L = L_1^{-s}$. Thus any reduced path is either (pp) or a power of L_1, q.e.d.

Any continuous mapping $f: S^1 \to S^1$ of the circle onto itself induces a homomorphism

$$f_*: \pi_1(S^1) \longrightarrow \pi_1(S^1).$$

Since the fundamental group is abelian, there is a canonical isomorphism between the fundamental groups at any two base points, so we may neglect the base point. We <u>orient</u> the circle S^1 by choosing one of the two generators of the fundamental group. The induced homomorphism f_* is then a homomorphism of an infinite cyclic group with generator 1 into itself; such a homomorphism is completely determined by the map n.1 of the generator 1. We call n the <u>degree</u> of the continuous map f. Intuitively, the degree represents the number of times the map wraps the circle around itself, and is negative when it is wrapped around itself in the opposite direction.

THEOREM 19.2. Two continuous maps f_o, f_1: $S^1 \longrightarrow S^1$ of an oriented circle into itself are homotopic if and only if they have the same degree.

PROOF: We already know from Chapter 1 that homotopic maps have the same effect upon the (abelian) fundamental groups, hence have the same degree. Conversely, we may construct to each degree n a map f of that degree; for example, the map corresponding to the edge path L_1^n discussed above. It will then suffice to show that any f: $S^1 \longrightarrow S^1$ is homotopic to one of these standard maps (which will necessarily have the same degree). Let p be chosen as the base point on $S^1 = |\dot{\Delta}|$. By rotating all images f(x) on S^1, one can make f homotopic to a map carrying p into p. Then f represents an element of the fundamental group of S^1; hence, by Theorem 19.1, is homotopic to the map representing one of the edge paths L_1^n, q.e.d.

This notion of degree can be used to prove the fundamental theorem of algebra.

Chapter 3

HOMOLOGY THEORY

20. The singular complex of a space. Among the various methods of defining the homology groups of a topological space we choose the "singular" theory. For each dimension $q \geqq 0$ pick a "standard" closed q-simplex $|\Delta_q|$. For example, it is convenient to let 0 denote the origin in an (infinite dimensional) Hilbert space H, and 1,2,... orthogonal unit vectors in H. Let Δ_q denote the polyhedron with vertices all points $0,1,\ldots,q$ and simplices all faces of $< 0,\ldots,q >$. Then

$$|\Delta_q| = |0, \ldots, q|, \qquad q = 0, 1, \ldots$$

is the standard closed q-simplex. We regard Δ_q as an ordered polyhedron, with vertices 0, 1, ..., q in their natural order.

Now let X be an arbitrary topological space. A singular q-simplex or a q-cell of X is defined to be a continuous map $T: |\Delta_q| \longrightarrow X$. The abelian group $C_q(S(x))$ of singular q-chains of X is now defined as the free abelian group with the q-cells of X as its generators; in other words, a q-chain in X is a formal linear combination

$$(20.1) \qquad c_q = g_1T_1 + \ldots + g_hT_h, \qquad T_k \text{ a q-cell, } g_k \text{ an integer}$$

of a finite number of q-cells with integral coefficients g_k, and the sum of two q-chains is obtained by adding the coefficients of corresponding cells (a cell which does not appear has coefficient zero).

If $f: X \longrightarrow Y$ is any continuous map, each singular q-simplex T in X determines a singular q-simplex $fT: |\Delta_q| \longrightarrow Y$ in Y; we write $S_q(f)T = fT$ for this correspondence. This induces a unique homomorphism i.e. $S_q(f): T \to fT$

(20.2) $$S_q(f): C_q(S(X)) \longrightarrow C_q(S(Y))$$

of the chains of X into the chains of Y, according to the formula

$$S_q(f)\ (\sum g_i T_i) = \sum g_i\ [S_q(f)T_i].$$

If also $g: Y \longrightarrow Z$, it follows at once that

$$S_q(gf) = S_q(g)S_q(f): C_q(S(X)) \longrightarrow C_q(S(Z)).$$

In particular, if $X = M$ is a subset of an affine space, and $A: |\Delta_q| \longrightarrow M$ is a continuous map given by an affine map of $|\Delta_q|$ onto a closed simplex contained in M, then A is completely determined by the images $A(i) = p_i \in M$, and we write the symbol

(20.3) $$A = (p_o, \ldots, p_q)_M$$

for these "affine" singular simplices in M. The symbol $(p_o, \ldots, p_q)_M$ is defined whenever M is a subset of affine space containing the convex set (closed simplex of dimensions $\leq q$) spanned by $p_o, \ldots p_q$. In particular, $\beta_q = (0,1,\ldots,q)_{|\Delta_q|}$

thus denotes the "basic" singular simplex determined by the identity map of $|\Delta_q|$ onto itself; thus any singular simplex $T: |\Delta_q| \longrightarrow X$ can be represented uniquely as

(20.4) $$T = S(T)(0, 1, \ldots, q)_{|\Delta_q|} .$$

The <u>boundary</u> of a q-simplex T in X, with dimension $q > 0$, is the q-1 chain in X defined by the formula

(20.5) $$\partial_q T = S(T) \sum_{i=0}^{q} (-1)^i (0,\ldots,\hat{i},\ldots,q)_{|\Delta_q|} ,$$

where the symbol $\hat{i}$ indicates that the vertex i is to be omitted. More intuitively, the q-simplex Δ_q has q+1 faces of dimension q-1, which may be obtained by omitting in succession the vertices 0, 1, ..., i, ..., q of Δ_q. The mapping $T: |\Delta_q| \longrightarrow X$, when cut down to the i-th such face, will determine a singular (q-1)-simplex in X; indeed the symbol $S(T)\ (0,\ldots,\hat{i},\ldots,q)$ represents this simplex. The boundary is the chain formed by taking the alternating sums of these "faces" of T.

The boundary of any q-chain is defined by additivity as

$$\partial_q(\sum g_i T_i) = \sum g_i(\partial_q T_i).$$

Hence the boundary operation is a homomorphism

(20.6) $$\partial_q: C_q(S(X)) \longrightarrow C_{q-1}(S(X)).$$

In the definition of the boundary, the symbols $(0,\ldots,\hat{i},\ldots,q)_{|\Delta_q|}$ stand for the affine maps $\varepsilon_i |\Delta_{q-1}| \longrightarrow |\Delta_q|$ which carry the vertices $0,\ldots,q-1$ of Δ_{q-1} in order upon the vertices 0, ..., q of Δ_q, with i omitted. Thus $\partial_q T = S(T) \sum_{i=0}^{q} (-1)^i \varepsilon_i$. From this we derive the following very useful formula for the boundary of an affine singular simplex:

$$(20.7) \qquad \partial_q(p_0,\ldots,p_q)_M = \sum_{i=0}^{q} (-1)^i (p_0,\ldots,\hat{p}_i,\ldots,p_q)_M .$$

Indeed, if A is the affine map of (20.3), the definition of the boundary yields

$$\partial_q(p_0,\ldots,p_q)_M = \partial_q A = S(A) \sum_{i=0}^{q}(-1)^i \varepsilon_i = \sum_{i=0}^{q}(-1)^i (A\varepsilon_i).$$

But the composite $A\varepsilon_i : |\Delta_{q-1}| \longrightarrow M$ is an affine map carrying $0,\ldots,q-1$ in order upon the points $p_0,\ldots,\hat{p}_i,\ldots,p_q$ in M. Hence $A\varepsilon_i$ is represented by the symbols $(p_0,\ldots,\hat{p}_i,\ldots,p_q)_M$, which gives the formula (20.7).

The most important property of the boundary formula is

$$(20.8) \qquad \partial_{q-1}\partial_q = 0: C_q(S(X)) \longrightarrow C_{q-2}(S(X)), \qquad q \geqq 2;$$

in words: the boundary of the boundary of any chain is zero. Since the boundary is a homomorphism, it will suffice to prove that $\partial_{q-1}\partial_q T = 0$ for any singular q-cell in X. Now by the definition (20.5)

$$\partial_{q-1}\partial_q T = S(T) \sum_{i=0}^{q} (-1)^i \partial_{q-1}(0,\ldots,\hat{i},\ldots,q)_{|\Delta_q|} .$$

The boundaries on the right may be calculated by the rule (20.7), for the case $M = |\Delta_q|$. Upon splitting the resulting sum into two parts we have

$$\partial_{q-1}\partial_q T = \mathcal{S}(T)\left\{\sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^{i+j}(0,\ldots,\hat{j},\ldots,\hat{i},\ldots,q)\,|\Delta_q|\right.$$
$$\left. + \sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^{i+j-1}(0,\ldots,\hat{i},\ldots,\hat{j},\ldots,q)\,|\Delta_q|\right\},$$

for to omit vertex number $j-1 \geqq i$ in $(0,\ldots,\hat{i},\ldots,q)$ is to omit vertex number j from the original list of vertices. But the interchange of the letters i and j in the second double sum makes this sum equal to the first double sum, except for sign. The whole is thus zero.

Another useful rule is

$$(20.9)\qquad \partial_q S_q(f) = S_{q-1}(f)\partial_q : C_q(\mathcal{S}(X)) \longrightarrow C_q(\mathcal{S}(Y))$$

for any continuous $f: X \longrightarrow Y$. It again suffices to consider the effect of each homomorphism upon a singular simplex T in X. But

$$\partial_q S_q(f)T = \partial_q(fT) = \mathcal{S}(fT)\sum_{i=0}^{q}(-1)^i(0,\ldots,\hat{i},\ldots,q)\,|\Delta_q|$$
$$= \mathcal{S}(f)\mathcal{S}(T)\sum_{i=0}^{q}(-1)^i(0,\ldots,\hat{i},\ldots,q)\,|\Delta_q| = \mathcal{S}(f)\partial_q T.$$

The algebraic system $\mathcal{S}(X)$ consisting of the groups of singular chains and the boundary homomorphisms in all dimensions is known as the <u>singular complex</u> of the space X

$$\mathcal{S}(X):\qquad C_0(\mathcal{S}(X)) \longleftarrow C_1(\mathcal{S}(X)) \longleftarrow C_2(\mathcal{S}(X)) \longleftarrow \cdots \quad .$$

We shall often omit the subscript q in the symbol ∂_q for the boundary homomorphisms.

The homology groups of a space X are determined by this singular complex, in the fashion to be described below (§22).

21. Homology groups of a complex. By a complex (more fully, an abstract chain complex) K we mean any doubly infinite sequence of (additive) abelian groups $C_q = C_q(K)$ for all integral "dimensions" q, together with a sequence of homomorphisms $\partial_q : C_q \longrightarrow C_{q-1}$ such that $\partial_{q-1}\partial_q = 0$. Thus

$$K = \left\{ \cdots \xleftarrow{\partial} C_{q-1} \xleftarrow{\partial} C_q \xleftarrow{\partial} C_{q+1} \xleftarrow{\partial} \cdots \right\}.$$

Such a sequence which terminates (at either or both ends) may be extended to a doubly infinite sequence by the convention of adding all the remaining groups C_q as groups consisting of 0 alone and all the remaining homomorphisms as the zero homomorphisms. Thus, for example, the singular complex S(X) of a space is extended by defining the chain groups $C_{-n}(S(X))$ to be zero for $n > 0$; in particular, the boundary of a zero-dimensional chain is the zero chain of dimension -1.

A q-cycle z_q is a q-chain with boundary zero: $\partial z_q = 0$. The cycles constitute a subgroup $Z_q(K)$ of $C_q(K)$; in fact this subgroup is the kernel of $\partial_q : C_q \longrightarrow C_{q-1}$.

A q-boundary b_q is a q-chain which is the boundary $b_q = \partial c_{q+1}$ of some q+1 chain c_{q+1}. Any boundary is a cycle, for $\partial b_q = \partial\partial c_{q+1} = 0$. The q-boundaries constitute a subgroup $B_q(K)$ of $C_q(K)$, in fact this subgroup is the image of the homomorphism $\partial_{q+1} : C_{q+1} \longrightarrow C_q$.

The rule $\partial\partial = 0$ shows that the group B_q of boundaries is a subgroup of the group Z_q of cycles. The q-th homology group is defined as the factor group

$$H_q(K) = Z_q(K)/B_q(K).$$

In more detail, we may say that two q-chains c_q and c'_q are homologous if their difference is a boundary; in symbols

$$c_q \sim c'_q \text{ if and only if } c_q - c'_q = \partial c_{q+1}, \text{ some } c_{q+1}.$$

A chain homologous to a cycle is itself a cycle, and an element of $H_q(K)$ is a <u>homology class</u> or coset $z + B_q$, consisting of all cycles homologous to some fixed q-cycle z.

The singular homology groups of a space X are the homology groups $H_q(X) = H_q(S(X))$ of the singular complex. By the conventions above, these groups are all zero in dimensions $q < 0$. In general, these groups are measures of the connectivity of the space. To illustrate this in a simple case, we examine the zero-dimensional homology group of X. By our convention about adding zero chain groups in dimensions less than 0, any zero-dimensional chain of X is automatically a cycle; the zero-dimensional homology group is then

$$H_0(X) = C_0(S(X))/B_0(X(X)).$$

<u>THEOREM</u> <u>21.1</u>. If the space X has exactly m arc-components then $H_0(X)$ is isomorphic to the direct sum of m infinite cyclic groups.

PROOF: Let $X_1, \ldots, X_m$ be the m arc-components of X. Each 0-simplex T of X is a mapping of a standard $|\Delta_0|$ into X, hence is determined by the point $p = T(|\Delta_0|)$ of X, which must then belong to one of the m components of X. Break each 0-chain c_0 up into the parts belonging to these components, so that

$$c_0 = \sum_{k=1}^{m} (g_{k1}T_{k1} + \ldots + g_{kn_k}T_{kn_k})$$

where each T_{kj} is a 0-simplex in X_k. Then define the homomorphism α of $C_0(X)$ into the direct sum of m copies of the additive group J of integers by setting

$$\alpha c_0 = (g_{11} + \ldots + g_{1n_1}, \ldots, g_{m1} + \ldots + g_{m_1 n_m}).$$

In other words, add all the coefficients belonging to cells in any one component. Then α is clearly a homomorphism onto $J + \ldots + J$, m times. To complete the proof we need only show that the kernel of α is exactly the group $B_0(X)$ of boundaries, for then α induces an isomorphism of C_0/B_0 to the direct sum in question.

First, any boundary lies in the kernel of α. For a 1-cell of X is a map $T: |\Delta_1| \longrightarrow X$; since Δ_1 is an interval, T is an arc in X, which must then lie in some one of the arc components. The boundary ∂T consists of the two end points of this arc (regarded as 0-cells); thus has the α-image zero.

Conversely, it suffices to show that if $c = g_1T_1 + \ldots + g_nT_n$ is a 0-chain with simplices T_k all in the same component and $g_1 + \ldots + g_n = 0$, then c is a boundary. But choose any 0-simplex

T in this component, join the point T to the point T_i by an arc, represent each of these arcs as a 1-cell S_i, and calculate

$$\partial(g_1S_1+\dots+g_nS_n) = g_1T_1+\dots+g_nT_n - (g_1+\dots+g_n)T = c, \quad \text{q.e.d.}$$

The argument is also valid for any (infinite) number of arc components; $H_0(X)$ is then a weak direct sum of the same number of copies of J. In such cases, however, the singular homology groups of X are not of great interest. The group of singular chains of a space X is very "big"; the essential fact is that the homology groups for a decent space will be small; in fact, for the space of a polyhedron P, the homology groups are finitely generated, as will appear presently.

The argument of Theorem 21.1 will also prove that when X has m arc-components $X_1,\dots,X_m$, then

$$H_q(X) = H_q(X_1) + \dots + H_q(X_{m_1})$$

for all dimensions q.

If $K = \left\{ C_{q-1} \xleftarrow{\partial} C_q \right\}$ and $K' = \left\{ C'_{q-1} \xleftarrow{\partial'} C'_q \right\}$ are chain complexes, a chain transformation $\lambda : K \longrightarrow K'$ is, by definition, a sequence of homomorphisms

$$\lambda_q = C_q \longrightarrow C'_q \quad ,$$

such that

(21.1) $$\partial_q \lambda_q = \lambda_{q-1}\partial_q : C_q \longrightarrow C'_{q-1} \quad .$$

The situation is illustrated by the diagram

$$
\begin{tikzcd}
K: & C_0 & C_1 \arrow[l, "\partial"'] & C_2 \arrow[l, "\partial"'] & C_3 \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"'] \\
K': & C'_0 \arrow[from=u, "\lambda_0"] & C'_1 \arrow[l, "\partial"] \arrow[from=u, "\lambda_1"] & C'_2 \arrow[l, "\partial"] \arrow[from=u, "\lambda_2"] & C'_3 \arrow[l, "\partial"] \arrow[from=u, "\lambda_3"] & \cdots \arrow[l, "\partial"]
\end{tikzcd}
\qquad ; \tag{21.2}
$$

the condition (21.2) states that the two paths in each square of this diagram, from upper right to lower left, have the same result. If we regard a chain complex as a single algebraic system (composed of groups and homomopphisms between them), then a chain transformation $\lambda : K \longrightarrow K'$ is simply a homomorphism of the first system K into the second; the condition (21.1) is just the requirement that this homomorphism preserve the basic boundary operation in the system.

Each chain transformation $\lambda : K \longrightarrow K'$ induces for every dimension q a homomorphism

$$H_q(\lambda) : H_q(K) \longrightarrow H_q(K')$$

on the corresponding homology groups; we denote this induced homomorphism by $H_q(\lambda)$ or by λ_*, when there is no ambiguity. Specifically, if z_q is a q-cycle of K, then $\partial_q \lambda_q z_q = \lambda_{q-1} \partial_q z_q = \lambda_{q-1} 0 = 0$; hence λ carries cycles into cycles. Also, if b_q is a q-boundary of K, then $b_q = \partial c_{q+1}$ for some q+1-chain c_{q+1}, and $\lambda b_q = \lambda \partial c_{q+1} = \partial \lambda c_{q+1}$, hence λ carries boundaries into boundaries. The induced homomorphism on homology groups therefore is defined, for any homology class $z_q + B_q(K)$, as

$$\lambda_*(z_q + B_q(K)) = \lambda z_q + B_q(K') \in H_q(K');$$

the definition is independent of the choice of the cycle z_q used to represent the coset $z_q + B_q(K)$.

If $\lambda : K \longrightarrow K'$ and $\mu : K' \longrightarrow K''$ are chain transformations, their composite $\mu\lambda : K \longrightarrow K''$ is a chain transformation, and one readily proves that $H_q(\mu\lambda) = H_q(\mu)H_q(\lambda)$.

Specifically, if X and Y are spaces, any continuous map $f: X \longrightarrow Y$ induces the transformation $S_q(f)$ of singular q-chains of X into those of Y, as described in §20. The condition (20.9) asserts that the family of all $S_q(f)$ is a chain transformation $S(f): S(X) \longrightarrow S(Y)$. Hence we may speak of the induced homomorphisms $H_q(f) = f_*: H_q(X) \longrightarrow H_q(Y)$ on the homology groups. These homomorphisms provide an important tool for the algebraic classification of continuous maps.

22. The complex of a polyhedron. Our objective is to show that the singular homology groups of the space $|P|$ of a polyhedron P can be effectively computed. The idea is that of using simplicial approximations to replace the arbitrary continuous maps $T: |\Delta_q| \longrightarrow |P|$ by simplicial maps. There will result but a finite number of these simplicial singular simplices. The group of chains formed from such simplices is then a finitely generated abelian group, and the same will be true for the corresponding group of cycles, boundaries, and homology classes. The structure of the homology groups can thus be given by the fundamental theorem on finitely generated abelian groups.

Specifically, if P is a polyhedron, a q-cell of P will mean a singular q-simplex determined by a simplicial map $T: |\Delta_q| \longrightarrow |P|$. Any such singular simplex may be written, in the notation of §20, as $\sigma_q = (p_0, \ldots, p_q)_{|P|}$; we drop the subscript $|P|$. A q-chain of P is any chain containing only the cells σ_q; thus the group $C_q(P)$ of these q-chains is the free abelian group with generators the symbols $\sigma_q = (p_0, \ldots, p_q)$, where $p_0, \ldots, p_q$ is any ordered list of vertices (with possible repetitions) of a frame of P. The boundary of a cell is given by the formulas of §20 as

$$(22.1) \qquad \partial(p_0, \ldots, p_q) = \sum_{i=0}^{q} (-1)^i (p_0, \ldots, \hat{p}_i, \ldots, p_q),$$

and is again a q-chain. Hence we have associated with the polyhedron P a complex

$$K(P) = \left\{ C_0(P) \xleftarrow{\partial} C_1(P) \xleftarrow{\partial} C_2(P) \xleftarrow{\partial} \cdots \right\}$$

in which the groups of chains are finitely generated free groups. This complex is a subcomplex of the singular complex $S(|P|)$.

The homology groups of this complex are the simplicial homology groups $H_q(P)$.

If $\phi: P \longrightarrow Q$ is an abstract simplicial map (vertices to vertices, frames to frames), then the definition

$$K_q(\phi)(p_0, \ldots, p_q) = (\phi_{p_0}, \ldots, \phi_{p_q})$$

determines a homomorphism of $C_q(P)$ into $C_q(Q)$; by the boundary formula it follows that these homomorphisms commute with the boundary homomorphisms and hence yield a chain transformation $K(\phi): K(P) \longrightarrow K(Q)$. If ϕ_* denotes the continuous simplicial map $\phi_*: |P| \longrightarrow |Q|$ induced by ϕ, it follows from the definition that $K(\phi)$ is identical with $S(\phi_*)$, cut down to apply only to simplicial chains.

A still smaller complex may be formed by using a partial order of the vertices of P. Indeed, let P^o be an ordered polyhedron in this sense.

Use only the ordered cells $\sigma_q = (p_o, \ldots, p_q)$ in which the vertices $p_o, \ldots, p_q$ are abstract vertices of a frame of P, in the linear order given in P^o. The boundary (22.1) of such an ordered cell then consists again of such ordered cells; using the chains generated by such cells we again obtain a complex $K(P^o)$.

We shall prove that for any ordered polyhedron P^o the homology groups of $K(P^o)$, of $K(P)$, and of $S(|P|)$ in dimension q are all isomorphic. In particular, since $C_q(P)$ is a finitely generated abelian group, so is $H_q(P)$. It can therefore be written as the direct sum of finite and infinite cyclic groups, say in the form

$$H_q(P) = \sum_{i=1}^{\beta_q} J_i + \sum_{j=1}^{r} (J/m_jJ),$$

where J_i is a group isomorphic to J, and each J/m_jJ is a cyclic group of order m_j. The orders m_j can be so shown that each m_j divides m_{j+1}, $j = 1, \ldots, r-1$. With this choice the number β_q of infinite cyclic summands and the orders m_j are invariants of the group $H_q(P)$. We call β_q the q-th Betti numbers of P and the $m_1, \ldots, m_r$ the q-th torsion coefficients of P.

23. The complexes of a schema. These definitions of the complexes associated with P no longer depend upon the space $|P|$, but only upon the vertices of P and the arrangement of these vertices into frames. Hence the same formulas will define complexes and homology groups for an abstract simplicial complex V.

Specifically, a q-cell of V is a symbol $\sigma_q = (p_o, \ldots, p_q)$ consisting of q + 1 vertices of a frame of V, in some order but with possible duplications. The group $C_q(V)$ of q-chains is the free abelian group with the σ_q as generators, and the boundary homomorphism is again determined by the formula (22.1). The fact that $\partial\partial = 0$ is again proved by the same formal calculation as before; hence we have a complex K(V) associated with each ask V.

Similarly, let V^o be an ordered ask. For each q-dimensional frame of V^o we introduce a q-cell $\sigma_q = (p_o, \ldots, p_q)^o$ consisting of the vertices of the frame <u>in order</u>. If V has n_q frames of dimension q, there are then n_q such cells $\sigma_{q,i}$, $i = 1,\ldots,n_q$ in dimension q, the groups $C_q(V^o)$ of q-dimensional chains will then be the free group with these generators, and then with elements

$$c_q = g_1\sigma_{q1} + \cdots + g_n\sigma_{qn}, \qquad n = n_q.$$

The boundary homomorphism $\partial : C_q(V^o) \longrightarrow C_{q+1}(V^o)$, for $q > 0$ is again determined by setting

$$(23.1) \qquad \partial\,(p_o, \ldots, p_q)^o = \sum_{i=o}^{q} (-1)^i (p_o,\ldots,\hat{p}_i,\ldots,p_q)^o.$$

As before $\partial\partial = 0$, as is shown for example in low cases by

$$\begin{aligned}\partial\,(p_op_1p_2p_3)^o &= \partial\,[(p_1p_2p_3)^o - (p_op_2p_3)^o + (p_op_1p_3)^o - (p_op_1p_2)^o]\\ &= (p_2p_3)^o - (p_1p_3)^o + (p_1p_2)^o - (p_2p_3)^o + (p_op_3)^o\\ &\quad - (p_op_2)^o + (p_1p_3)^o - (p_op_3)^o + (p_op_1)^o - (p_1p_2)^o\\ &\quad + (p_op_2)^o - (p_op_1)^o = 0.\end{aligned}$$

We thus have a complex $K(V^o) = \left\{\partial : C_q(V^o) \longrightarrow C_{q-1}(V^o)\right\}$; its homology groups are the groups $H_q(V^o)$ of the ordered ask V^o. If P^o is an ordered polyhedron and V^o its schema, with the same order, then our construction is such that the complexes $K(P^o)$ and $K(V^o)$, and hence the corresponding homology groups, are isomorphic. We have also

THEOREM 23.1. The homology groups of $K(V^o)$ are independent of the chosen order of vertices in the ask V.

To prove this theorem, it is convenient to introduce symbols for certain q-chains in $K(V^o)$. Specifically, with any sequence $r_o, \ldots, r_q$ of $q + 1$ vertices of V, distinct or not, which all belong to a frame of V, we define a q-chain $(r_o, \ldots, r_q)^o$. If the $r_o, \ldots, r_q$ are not all distinct, set

$$(r_o, \ldots, r_q)^o = 0, \qquad \text{some } r_i = r_j, \quad i \neq j. \tag{23.2}$$

If the $r_o, \ldots, r_q$ are <u>distinct</u>, they can be placed in the standard order of V^o by a suitable permutation; we set

$$(p_{\pi o}, \ldots, p_{\pi q})^o = (\operatorname{sgn} \pi)\,(p_o, \ldots, p_q)^o, \tag{23.3}$$

where $\operatorname{sgn} \pi = \pm 1$ according as π is an even or an odd permutation. We assert that the boundary formula (23.1) is still valid for any symbol $(r_o, \ldots, r_q)^o$; i.e.,

$$\partial (r_o, \ldots, r_q)^o = \sum_{i=o}^{q} (-1)^i (r_o, \ldots, \hat{r}_i, \ldots, r_q)^o. \tag{23.4}$$

Case 1. ("Degenerate" cells). Some $r_j = r_k$ for $j < k$. The left hand side is then zero. On the right, all terms except possibly the j-th and the k-th are zero, since they have two entries the same. Terms j and k differ only in the fact that $r = r_j = r_k$ occurs in the first at position k, in the second at position j. But (k-j-1) transpositions will then bring them to agreement. According to the definition (23.3), these transpositions will alter the sign of the k-th term to $(-1)^k \cdot (-1)^{k-j-1} = (-1)^{j-1}$. Therefore the two terms cancel.

Case 2. ("Non-degenerate" cells). Since any permutation can be effected by successive transpositions of adjacent letters, it suffices to prove that if (23.4) holds, then it still holds after interchange of r_j with r_{j+1}. This follows by a simple calculation.

The group $C_q(V^o)$ has as its free generators those q-chains $(r_o, \ldots, r_q)^o$ in which the $r_o, \ldots, r_q$ are distinct and occur in the order of V^o. Since a generator may be replaced by itself or its negative, to give a new set of generators for the same free group, we may replace any one generator $(p_o, \ldots, p_q)^o$ by the q-chain $(p_{\pi o}, \ldots, p_{\pi q})^o$ of (23.3). In particular, given a new partial order of the vertices of V, we may choose each permutation π so that the new symbol $(p_{\pi o}, \ldots, p_{\pi q})^o$ has its vertices in the new order. Then since the boundary formula (23.4) for these new generators has the standard form, the complex $K(V^o)$ with these new generators is clearly isomorphic to the complex $K(V^{o'})$ derived directly from V' in its new order. This proves Theorem 23.1.

This symbolism also allows us to associate to each abstract simplicial map $\emptyset: V \longrightarrow W$ a chain transformation $K^o(\emptyset): K(V^o) \longrightarrow K(W^o)$. For any cell of $K(V^o)$, we set

$$K_q^o(\emptyset)(p_o, \ldots, p_q)^o = (\emptyset p_o, \ldots, \emptyset p_q)^o \in C_q(K(W^o)).$$

Since the same boundary formula (23.4) holds for all the symbols, it follows that $K_q^o(\emptyset)$ commutes with the boundary homomorphisms. Hence $K^o(\emptyset)$ is a chain transformation, and thus induces homomorphisms on the homology groups.

The homology groups of V may also be defined without choosing any one order of the vertices of V. For each q-frame $\{r_o, \ldots, r_q\}$ of V we choose an order of those vertices. We say that two orders determine the same orientation of the frame if the one can be obtained from the other by an even permutation; otherwise we have the opposite orientation. Now choose for each q-frame of V, one orientation determined by an order $r_o, \ldots, r_q$, and associate with each frame $\{r_o, \ldots, r_q\}$ a q-cell $\{r_o, \ldots, r_q\}'$. The boundary then is

$$(23.5) \quad \partial\{r_o, \ldots, r_q\}' = \sum_{i=o}^{q} (-1)^i \eta_i \{r_o, \ldots, \hat{r}_i, \ldots, r_q\}',$$

where $\eta_i = +1$ or -1 according as the order $r_o, \ldots, r_q$ of the original cell will induce an order $r_o, \ldots, \hat{r}_i, \ldots, r_q$ which agrees with or is opposite to the chosen orientation on the (q-1)-cell determined by $r_o, \ldots, \hat{r}_i, \ldots, r_q$. This approach using orientations is the classical one, and gives a complex isomorphic to $K(V^o)$.

24. Groups of Simplices and spheres. If P is any polyhedron lying in an affine space A, and t is a point not in that affine space, we may form the cone over P with vertex t as the set of all points on line segments joining t to a point of P. This cone is clearly also the space of a polyhedron Q, in which the simplices are (i) the simplices of P; (ii) the 0-simplex t; (iii) the simplices formed by adjoining the vertex t to any simplex of P. In particular, if P is the polyhedron determined by a q-simplex and its faces, then a cone over P is the polyhedron determined by a (q + 1)-simplex.

The same applies for an ask; if t is not a vertex of the ask V, the <u>cone</u> over V with vertex t is the abstract simplicial complex (V;t) with vertices t and the vertices of V, and with frames: (i) all frames of V; (ii) the 0-frame $\{t\}$; (iii) the frame found by adding the vertex t to any frame of V.

<u>THEOREM 24.1</u>. The integral homology groups of a cone (V;t) vanish in dimensions greater than 0, and the zero-dimensional homology group is infinite cyclic. This holds for both complexes K(V;t) and $K(V;t)^o$.

Geometrically, this is plausible, because any cycle on the cone bounds the chain of dimension one higher formed by "joining" the cycle to the vertex of the cone. To give an algebraic proof in the complex $K(V;t)^o$, order the vertices of (V;t) with t first. For any cell $(p_o, \ldots, p_q)^o$ of the cone; set

$$(24.1) \qquad D(p_o, \ldots, p_q)^o = (t, p_o, \ldots, p_q)^o$$

(the result is zero if t is one of the vertices $p_o, \ldots, p_q$). This definition of D for each of the free generators $(p_o, \ldots, p_q)^o$ of the group $C_q(V;t)^o$ of integral chains determines a homomorphism

$$(23.2) \qquad D_q\colon C_q(V;t) \longrightarrow C_{q+1}(V;t).$$

Furthermore, from the definition (23.1), if $q > 0$

$$\partial D(p_o, \ldots, p_q) = (p_o, \ldots, p_q) - \sum_{i=o}^{q} (-1)^i (t, p_o, \ldots, \hat{p}_i, \ldots, p_q).$$

The sum on the right is exactly D of the boundary of $(p_o, \ldots, p_q)$. Hence

$$\partial Dc + D\partial c = c, \qquad c \text{ any } q\text{-chain, } q > 0.$$

In particular, if c is a q-cycle, then $\partial c = 0$ and $c = \partial Dc$ is in fact a boundary. Hence $H_q(V;t) = 0$ for $q > 0$.

If $q = 0$,

$$\partial D(p) = \partial (t, p) = (p) - (t).$$

We define a homomorphism λ of C_o to the integers by setting $\lambda(p) = 1$. This equation then states that

$$\partial Dc = c - (\lambda c)(t), \qquad c \text{ a 0-chain.}$$

Clearly λ is a homomorphism $C_o \longrightarrow J$ mapping B_o to 0; if $\lambda c = 0$, the equation gives $c = \partial Dc$, so that c is a boundary. Thus λ induces an isomorphism of H_o to J.

Since any simplex is a cone over a simplex of dimension one lower we have

COROLLARY 24.2. The simplicial integral homology groups of an abstract simplex s are

$$H_o(s) \cong J, \qquad H_q(s) = 0, \qquad q > 0.$$

An n-dimensional sphere may be so triangulated as to be homeomorphic to the polyhedron obtained from an (n+1)-dimensional simplex s by deleting the simplex s itself from P(s). The homology groups of dimension less than n are not thereby altered. In dimension n the cycles of P(s) are exactly the boundaries of P(s). Since there is exactly one cell of dimension (n+1), and this cell is not a cycle, the cycles of P(s) form an infinite cyclic group, generated by the boundary of this one cell. Upon removal of the (n+1)-dimensional simplex s, these cycles can no longer bound. Hence

COROLLARY 24.3. The simplicial integral homology groups of an n-sphere S^n (triangulated as the boundary of an (n+1)-simplex) are

$$H_o(S^n) \cong J, \qquad H_n(S^n) \cong J, \qquad H_q(S^n) = 0, \quad 0 < q < n.$$

The same arguments apply to the complex $K(V;t)$; simply define $D\sigma$ for any cell $\sigma = (p_o, \ldots, p_q)$ of $K(V;t)$, with the p_i vertices of a frame in any order, by the same formula (24.1) with the superscript 0's erased. Since the formula for boundary remains the same, all the conclusions follow.

We can now prove

THEOREM 24.4. If V is any ask, with any order of vertices, then the homology groups of $K(V)$ and $K(V^o)$ are isomorphic.

This will justify the use of these two constructions; the first, $K(V)$, is closer to the singular theory; the second is smaller and hence better for computations.

Given any order of the vertices of V, define homomorphisms

$$\lambda': C_q(V) \longrightarrow C_q(V^o), \qquad \lambda : C_q(V^o) \longrightarrow C_q(V)$$

by setting, for the generators of the respective groups,

$$\lambda(p_o, \ldots, p_q)^o = (p_o, \ldots, p_q); \qquad \lambda'(r_o, \ldots, r_q) = (r_o, \ldots, r_q)^o,$$

where $p_o, \ldots, p_q$ are the vertices of a frame, in order; while $r_o, \ldots, r_q$ are the vertices (with possible repetitions) of some frame. Because of the character of the boundary formulas

$$\partial\lambda = \lambda\partial : C_q(V^o) \longrightarrow C_{q-1}(V),$$

$$\partial\lambda' = \lambda'\partial : C_q(V) \longrightarrow C_{q-1}(V^o).$$

Thus λ, λ' induce homomorphisms in the corresponding homology groups. Furthermore $\lambda'\lambda : C_q(V^o) \longrightarrow C_q(V^o)$ is the identity map. To treat the other composition $\lambda'\lambda$ it will suffice (see below) to construct a homomorphism

(24.3) $$D_q : C_q(V) \longrightarrow C_{q+1}(V)$$

in each dimension $q = 0, 1, \ldots,$ in such a fashion that

(24.4) $$\partial D_q c + D_{q-1}\partial c = \lambda\lambda' c - c$$

for any q-chain c of $C_q(V)$.

We now construct the homomorphism D_q by induction on the dimension q, and subject to the side condition that $D(r_o, \ldots, r_q)$ shall always be a chain involving only these vertices $r_o, \ldots, r_q$. In dimension 0 we interpret (24.4) to mean that $\partial D_o c = \lambda\lambda' c - c$ (∂c is zero). But for any 0-cell (r), $\lambda\lambda'(r) = (r)$; hence we set $D_o = 0$. Suppose now that D has been defined for $m < q$ to satisfy (24.4) and the side condition. For each q-cell $\sigma = (r_o, \ldots, r_q)$ consider the q-chain

$$c = \lambda\lambda'\sigma - D_{q-1}\partial\sigma - \sigma .$$

Its boundary is, by (24.4) for q-1

$$\begin{aligned}\partial c &= \partial\lambda\lambda'\sigma - \partial D_{q-1}(\partial\sigma) - \partial\sigma \\ &= \lambda\lambda'\partial\sigma - \lambda\lambda'\partial\sigma + D_{q-2}\partial(\partial\sigma) + \partial\sigma - \partial\sigma = 0.\end{aligned}$$

Hence c is a cycle. Furthermore it involves only the vertices $r_o, \ldots, r_q$. Hence c is a q-cycle on the (abstract) simplex with these vertices and is consequently the boundary of some (q+1)-chain c_{q+1}, which involves only these vertices. If we set

$$D\sigma = c_{q+1},$$

we again have the desired equation (24.4), and the side condition on D.

Now the chain transformations $\lambda: K(V^o) \longrightarrow K(V)$ and $\lambda': K(V) \longrightarrow K(V^o)$ induce homomorphisms

(24.5) $\lambda_*: H_q(K(V^o)) \longrightarrow H_q(K(V))$, $\lambda'_*: H_q(K(V)) \longrightarrow H_q(K(V^o))$

upon the homology groups. Since $\lambda'\lambda$ is the identity, so is the induced homomorphism $\lambda'_*\lambda_*: H_q(K(V^o)) \longrightarrow H_q(K(V^o))$. Consider the action of the homomorphisms $\lambda_*\lambda'_*$ upon the homology class of a q-cycle z. By (24.4), and $\partial z = 0$ we have $\lambda\lambda' z - z = \partial D_q z$. This states that $\lambda\lambda' z$ is homologous to z, and hence that $\lambda\lambda' z$ and z determine the same homology class. Thus the induced homomorphism $\lambda_*\lambda'_* = (\lambda\lambda')_*$ is the identity map of $H_q(K(V))$ on itself. It now follows that both λ_* and λ'_* in (24.5) are isomorphisms onto.

25. Computation of homology groups. A cell complex K is a chain complex in which the groups C_q of chains are the zero groups in negative dimensions q, and in which, for each dimension $q \geq 0$, there is given a set $\{\sigma, \tau, \cdots\}$ of q-chains (called cells) which are free generators of the abelian group C_q of q-chains. A cell complex is thus determined by giving the cells in each dimension, and for each cell a boundary formula for $\partial\sigma$ as a linear combination of the cells of dimension one lower; the condition $\partial\partial\sigma = 0$ must be satisfied. The singular complex S(X) of a space is a cell complex, with cells all $T: |\Delta_q| \longrightarrow X$. The simplicial complex $K(V^o)$ of an ordered ask V^o is a cell complex, with cells all $(p_o, \dots, p_q)^o$,

for p_i the vertices of a frame in proper order. The complex $K(V)$ is a cell complex with cells all symbols $(r_0, \ldots, r_q)$ determined by any ordered list of vertices from a frame of V.

In a cell complex K one may choose different cells (i.e., free generators) in C_q without altering the groups C_q of chains. Moreover, one may endeavor to obtain a simpler complex (with smaller chain groups) which will still have the same homology groups as K. A <u>subcomplex</u> L of K is a set of subgroups $C'_q \subset C_q(K)$, $q = 0, 1, \ldots$ such that $\partial_q(C'_q) \subset C'_{q-1}$. The subcomplex will be called <u>adequate</u> if, for all dimensions q,

(i) every q-cycle in C_q is homologous (in K) to a q-cycle in C'_q;

(ii) every q-cycle of C'_q which is the boundary of a chain of C_{q+1} is also the boundary of a chain of C'_{q+1}.

These two conditions clearly imply that the homology group $H_q(L)$ is isomorphic to the homology group $H_q(K)$, under the correspondence mapping each coset $z'_q + B'_q$ of Z'_q/B'_q into the coset $z'_q + B_q$ of $Z_q/B_q = H_q$.

For a cell complex K we give two simple rules for obtaining an adequate subcomplex.

<u>RULE 1.</u> If σ is a q-cell of K which is the boundary of a $(q+1)$-cell τ (with $\partial\tau = \sigma$), and which does not appear in the boundary of any other $(q+1)$-cell, then one may remove the cells σ, τ to obtain an adequate subcomplex.

PROOF: There can not be a $(q+2)$-cell ρ for which the boundary formula is $\partial\rho = n\tau + c$, where c is a chain not involving the cell τ, unless $n = 0$. For $\partial\partial\rho = n\partial\tau + \partial c = n\sigma + \partial c$. And by assumption ∂c cannot involve σ. Since $\partial\partial\rho = 0$, then $n = 0$. Hence deletion of the cells σ and τ will leave a subcomplex L, which will have the same groups of cycles and boundaries as K, except perhaps in dimensions q and q + 1.

In dimension q + 1 a cycle of K cannot involve the cell τ, since $\partial\tau = \sigma$ and does not appear in other boundaries. Hence the (q+1)-cycles of K are exactly those of L; (i) and (ii) above hold in this dimension.

A q-chain of K can be written in the form $c = n\sigma + c'$, where c' is a q-chain of L. Since $\partial c = n\partial\sigma + \partial c' = \partial c'$, c is a cycle if and only if c' is a cycle, and $c - c' = n\sigma = n\partial\tau$, hence $c \sim c'$ in K, and (i) holds. If a q-cycle c' of L is the boundary of a (q + 1)-chain d of K, this chain d cannot involve τ, hence lies in L. Thus (ii) holds, and the rule is established.

RULE 2. If σ is a q-cell of K which appears on the boundary of exactly 2 (q + 1)-cells τ_1, τ_2, of K, in the form

$$\partial\tau_1 = \sigma + c_1, \qquad \partial\tau_2 = -\sigma + c_2 ,$$

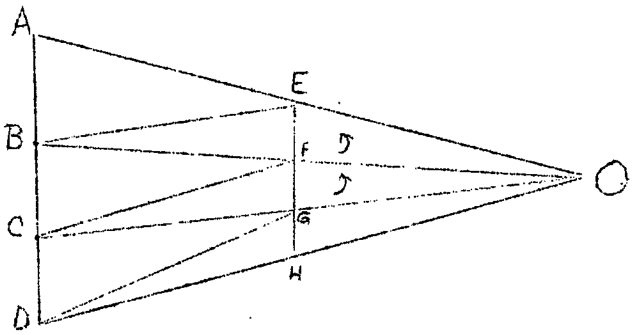
where c_1, and c_2 are chains not involving σ, then one may replace the cells τ_1 and τ_2 by $\tau_1 + \tau_2$ and suppress the cell σ to obtain an adequate subcomplex.

PROOF: We introduce a new set of free generators in C_{q+1} and C_q, replacing the cells τ_1 and τ_2 by $\tau_1 + \tau_2$ and τ_1 in C_{q+1}, and replacing σ in dimension q by $\sigma' = \sigma + c_1$. The new boundary formula reads

$$\partial(\tau_1 + \tau_2) = c_1 + c_2, \qquad \partial\tau_1 = \sigma'.$$

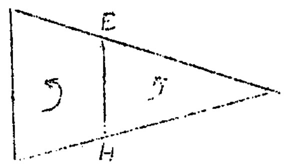
By Rule 1 we may then delete σ' and τ_1, q.e.d.

To illustrate these rules we calculate the homology groups of the two-dimensional manifolds M^2. Such a manifold is represented by a polygon P with 2m sides with affine identification of pairs of sides in such fashion that all the vertices are identified. First we triangulate this polygon, say as follows. Insert a vertex at the center of the polygon, subdivide each side by two new points, and join the center to all vertices on the periphery. Then subdivide each of the triangles thus formed as below:

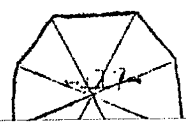


One readily verifies that this triangulation represents the manifold as a two-dimensional polyhedron. We compute the group of ordered simplicial chains; there being 18m 2-cells (9 for each of the 2m triangles above) and a large number of 0 and 1-cells. Note that each edge of the boundary appears twice, so that AB, for example, will appear on the triangle above, and on the other triangle. Orient each two cell (vertices always in clockwise order).

We now apply Rule 2 in each triangle, as above, to remove in succession the 1-cells corresponding to OF, OG, BE, BF, CF, CG, DG. The 0-cells (vertices), F, G, B, C now appear on exactly two boundaries, hence may also be removed by rule 2. The three 1-cells HG, GF, FE, are thereby combined into a single 1-cell, which is on the boundary



of exactly two 2-cells, with opposite signs. By rule 2, this one-cell may be removed (consolidating the two adjacent 2-cells). We now have a polygon with cells as indicated. All but two of the 1-cells joining 0 to the



boundary may be removed by Rule 2. Then Rule 2 removes 0, and then the two remaining such cells.

We are left with an adequate subcomplex containing

(i) one 0-cell (the single vertex).

(ii) m 1-cells (the edges of the original polygon, each appearing twice).

(iii) one 2-cell τ --the polygon.

For an orientable surface of genus p, represented by the symbol $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_p b_p a_p^{-1} b_p^{-1}$ we have m = 2p, and there are 2m 1-cells, which we denote by $a_1, b_1, \ldots, a_p, b_p$. The boundary formulas are

$$\text{1-cells} \quad \partial a_i = \rho - \rho = 0, \qquad \partial b_i = \rho - \rho = 0$$
$$\text{2-cells} \quad \partial \tau = a_1 + b_1 - a_1 - b_1 + \ldots + a_p + b_p - a_p - b_p = 0.$$

The two cycles are $n\tau$, for any integer n; the one cycles are all a_i, b_i, and their combinations, and the 0-cycles are $n\rho$. Thus no torsion is present and the Betti numbers are

$$\beta_0 = 1, \qquad \beta_1 = 2p, \qquad \beta_2 = 1.$$

For a non-orientable surface in the standard form $a_1 a_1 \ldots a_m a_m$ we have boundary formulas

$$\partial a_i = \rho - \rho = 0,$$
$$\partial \tau = a_1 + a_1 + \ldots + a_m + a_m = 2(a_1 + \ldots + a_m).$$

Change the one-cells to $a_1, \ldots, a_{m-1}, a_1 + \ldots + a_m$. These are all cycles, and twice the latter is homologous to 0. Hence

$$\beta_0 = 1, \qquad \beta_1 = m-1, \qquad \beta_2 = 0,$$

and there is one torsion coefficient $\tau_1 = 2$ in dimension 1.

26. <u>Chain Homotopies</u>. Let K and K' be chain complexes, and $\lambda, \mu : K \longrightarrow K'$ chain transformations. A chain homotopy $D: \lambda \simeq \mu$ is a family D of homomorphisms

(26.1) $$D_q : C_q(K) \longrightarrow C_{q+1}(K') \qquad \text{all } q$$

(raising dimensions by 1!), such that

(26.2) $$\partial D_q c_q + D_{q-1} \partial c_q = \lambda c_q - \mu c_q$$

for every c_q in $C_q(K)$. We have already had two examples; in § 24, p. 110, we defined for the complex of a cone, a chain homotopy between the identity and the zero chain transformation. At the end of § 24 we defined a chain homotopy D between $\lambda \lambda' : K(V) \longrightarrow K(V)$ and the identity. In the second instance, we used a special case of the following general result.

<u>THEOREM 26.1</u>. If $D: \lambda \simeq \mu : K \longrightarrow K'$, then the induced homomorphisms $\lambda_*, \mu_* \; H_q(K) \longrightarrow H_q(K')$ are identical.

In other words, chain homotopic mappings have the same effect upon homology groups.

<u>PROOF</u>: Let z_q be any q-cycle in K. Then, by (26.2),

$$\partial D_q z_q + D_{q-1} \partial z_q = \partial D_q z_q = \lambda z_q - \mu z_q \quad .$$

The cycles λz_q and λz_q are thus homologous in K'; in other words, the λ_* and μ_* images of the homology class of z_q are identical.

A chain transformation $\lambda : K \longrightarrow K'$ is called a <u>chain equivalence</u> (and K and K' are called chain equivalent), if there is a second chain transformation $\lambda' : K' \longrightarrow K$ such that $\lambda \lambda'$ is homotopic to the identity map of K' and $\lambda' \lambda$ homotopic to the identity map of K. The transformation λ' is then <u>called a homotopy inverse</u> of λ.

COROLLARY 26.2. A chain equivalence $\lambda : K \longrightarrow K'$ induces isomorphisms $\lambda_*: H_q(K) \longrightarrow H_q(K')$ of the homology groups of K onto those of K'.

Indeed, by the theorem, both $\lambda_* \lambda_*'$ and $\lambda_*' \lambda_*$ are the appropriate identity maps.

COROLLARY 26.3. If K is a chain complex with $C_q(K) = 0$ for $q < 0$, and if $D_q: C_q(K) \longrightarrow C_{q+1}(K)$ is a family of homomorphisms such that

$$\partial D_q c_q + D_{q-1} \partial c_q = c_q , \qquad q>0, \tag{26.3}$$

then $H_q(K) = 0$, for $q > 0$.

PROOF: Let c_q in (26.3) be a cycle; since $\partial c_q = 0$, this equation then asserts that c_q is a boundary of $D_q c_q$. We may also apply the Theorem directly, showing that K is chain equivalent to the subcomplex with the group $\partial D_o C_o$ on its only (non-trivial) chain group.

It is convenient to have a composition theorem for homotopies:

THEOREM 26.4. If $\lambda \simeq \mu : K \longrightarrow K'$ and $\lambda' \simeq \mu'$: $K' \longrightarrow K''$, then $\lambda' \lambda \simeq \mu' \mu : K \longrightarrow K''$.

PROOF: We have given homotopies $D: \lambda \simeq \mu$ and $D': \lambda' \simeq \mu'$, with

$$\partial D + D \partial = \lambda - \mu, \qquad \partial D' + D' \partial = \lambda' - \mu' .$$

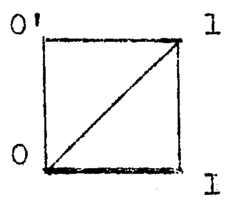
Then since λ' and μ are chain transformations (comutes with ∂) we have

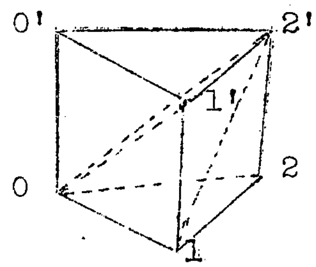
$$\partial \lambda' D + \lambda' D \partial = \lambda' \lambda - \lambda' \mu ,$$
$$\partial D' \mu - D' \mu \partial = \lambda' \mu - \mu' \mu .$$

We thus obtain the requisite homotopy $E: \lambda' \lambda \simeq \mu' \mu$ by setting

$$E_q = \lambda'_{q+1} D_q + D'_q \mu_q : C_q(K) \longrightarrow C_{q+1}(K'').$$

27. <u>Prisms</u>. Our aim is to reduce geometric homotopies to algebraic chain homotopies. We must first subdivide prisms; that is, cartesian products $|\Delta_q| \times I$ of a q-simplex by an interval. Typical subdivisions of $|\Delta_1| \times I$ into two triangles and of $|\Delta_2| \times I$ into three tetrahedra are pictured below.





This process can be carried out for any Δ_q, and may be described inductively as follows: subdivide all the lateral faces of $|\Delta_q| \times I$ by the previous step, and then subdivide $|\Delta_q| \times I$ itself by joining the leading vertex 0 of the base to the subdivisions of the lateral sides. This process, which may be described as that of "viewing" the previous subdivisions from the leading vertex 0, clearly yields the subdivision shown above for $|\Delta_2| \times I$. It also means that the subdivision process has a <u>consistency property</u>: the subdivision of $|\Delta_q| \times I$ on each lateral face is precisely that used for a prism $|\Delta_{q-1}| \times I$ of one dimension lower.

In the subdivision of $|\Delta_1| \times I$ the 2-simplices are $0\,1\,1'$ and $0\,0'\,1'$; for $|\Delta_2| \times I$ the 3-simplices are

$$0122', \qquad 0\,11'2', \qquad 00'1'2' \quad .$$

Similarly the simplices of maximum dimension in $|\Delta_q| \times I$ have the form $0\ 1\ 2\ \ldots\ i\ i'\ (+1)'\ \ldots\ q'$. Consider the chain formed by the alternating sum of these simplices of the subdivision,

(27.1) $\quad L(0\ 1\ 2\ \ldots\ q) = \sum_{i=0}^{q} (-1)^i (p\ 1\ 2\ \ldots\ i\ i'\ (i+1)',\ \ldots q').$

Since these simplices "fill up" the interior of the prism $|\Delta_q|\ I$ it is plausible to conjecture that the boundary of L should consist of the (i) the top os the prism; (ii) the base of the prism; (iii) the lateral sides of the prism, subdivided as by L. In other words

$$\partial L(0\ 1\ 2\ \ldots\ q) = (0'\ 1'\ \ldots\ q') - (0\ 1\ \ldots\ q) - L\partial(0\ 1\ \ldots q).$$

This homotopy L may be called the prismatic homotopy.

To establish the properties, we need not prove the (true) fact that the process of subdivision actually cuts the prism up into the simplices of a polyhedron. Instead, we may just regard the simplices $(0\ \ldots\ i\ i'\ \ldots\ q)$ appearing in L as affine singular simplices in the space $M = |\Delta_q| \times I$, regarded as a convex subset of a suitable affine space.

In manipulating the formula for L, it is convenient to use a join notation for affine simplices in M. Let $\sigma = (p_0\ \ldots\ p_q)$ be an affine q simplex of M and r a point of M. Then the join $\sigma \vee r$ is the affine $(q+1)$ simplex

(27.2) $\quad (p_0\ \ldots\ p_q) \vee r = (p_0\ \ldots\ p_q r).$

The join $c \vee r$, where c is a linear combination of affine simplices, is defined by linearity. It is easy to verify that

(27.3) $\quad \partial(\sigma \vee r) = (\partial\sigma) \vee r + (-1)^{m+1}\sigma.$

Now let $|\Delta_q| \times I$ be a prism. The <u>base</u> b and the <u>top</u> t of the prism are the continuous maps $b, t : |\Delta_q| \longrightarrow |\Delta_q| \times I$ obtained by the affine mappings carrying each vertex i of $|\Delta_q|$ into the corresponding vertices $(i,0)$ or $(i,1)$ on the prism. The prismatic homotopy L is a set of homomorphisms

$$L_n : C_n\,(K(\Delta_q)) \longrightarrow C_{n+1}(S(|\Delta_q| \times I))$$

defined by recursion on n by the equations $L_0(p) = (bp, tp)$,

(27.4) $$L_{q+1}(\sigma \vee r) = (L_q \sigma) \vee tr + (-1)^{q+1} b\sigma \vee br \vee tr.$$

This recursive definition agrees with the previous explicit formula (27.1), for it describes $L(0 \ldots q+1)$ as the simplices of $L(0 \ldots q)$ with the new vertex $(q+1)'$ adjoined, plus a new simplex $(0 \ldots q, q+1, (q+1)')$.

<u>LEMMA</u> <u>27.1</u>. For each Δ_q the associated prismatic homotopy satisfies

$$L: \; S(t) \simeq S(b) : K(\Delta_q) \longrightarrow S(|\Delta_q| \times I).$$

<u>PROOF</u>: We show $[\partial L + L\partial - S(t) + S(b)]\,\sigma = 0$, by induction in the dimension of σ. For $\dim \sigma = 0$

$$L(p) = \partial(bp, tp) = tp - bp, \qquad L\partial(p) = 0,$$

and the result is immediate. Now assume the result for a simplex σ of dimension q. Then by (27.4)

$$\partial L(\sigma \vee r) = (\partial L \sigma) \vee tr + (-1)^{q+2} L\sigma + (-1)^{q+1} \partial b\sigma \vee br \vee tr$$
$$+ b\sigma \vee tr - b\sigma \vee br.$$

Apply the induction assumption to the first term, which then becomes $t\sigma \vee tr - b\sigma \vee tr - L\partial\sigma \vee tr$, so that

$$\partial L(\vee r) = t(\sigma \vee r) - b\sigma \vee tr - (L\partial\sigma) \vee tr + (-1)^{q+2} L\sigma$$
$$+ (-1)^{q+1} \partial b\sigma \vee br \vee tr + b\sigma \vee tr - b(\sigma \vee r).$$

But $L\partial(\sigma \vee r)$ may be calculated from (27.3) and the definition (27.4) as

$$L\,\partial(\sigma \vee r) = (L\partial\sigma) \vee tr + (-1)^q b\,\partial\sigma \vee br \vee tr + (-1)^{q+1} L\sigma.$$

Adding these two equations, and cancelling, gives the desired result.

The prismatic homotopy L has a consistency property which is an analogue for the consistency property of the subdivision process:

LEMMA 27.2. If $\Psi: \Delta_p \longrightarrow \Delta_q$ is a simplicial map, $\Psi_*: |\Delta_p| \longrightarrow |\Delta_q|$ the induced continuous map and $i: I \longrightarrow I$ the identity, then the prismatic homotopies $L^{(p)}$ and $L^{(q)}$ belonging to Δ_p and Δ_q satisfy

$$S(\Psi_* \times i)\, L^{(p)} = L^q\, K(\Psi).$$

PROOF: We have the diagram

$$\begin{tikzcd} C_n(K(\Delta_p)) \arrow[r, "L^{(p)}"] \arrow[d, dash, "K(\Psi)"] & C_{n+1} S(|\Delta_p| \times I) \arrow[d, dash, "S(\Psi_* \times i)"] \\ C_n(K(\Delta_q)) \arrow[r, "L^{(q)}"] & C_{n+1}(S(|\Delta_q| \times I). \end{tikzcd}$$

The required commutativity for the diagram follows directly by applying the appropriate definitions; it is just a reflection of the fact that we have used the same "formula" to define L in all prisms.

28. The Cylinder Homotopy. The precise relation of geometric to algebriac homotopy may now be stated.

THEOREM 28.1. If $f_o \simeq f_1: X \to Y$ are homotopic continuous mappings, then the induced chain transformations $S(f_o)$, $S(f_1)$: $S(X) \to S(Y)$ are chain homotopic.

This theorem will imply for example that two spaces of the same homotopy types have isomorphic homology groups.

We first reduce this theorem to a special case, that of the cylinder $X \times I$ (I the unit interval) constructed over a space X. The continuous maps b_X, t_X: $X \to X \times I$ given by

$$b_o(x) = (x, 0), \qquad t(x) = (x, 1)$$

may be called the base and top of the cylinder. Clearly b and t are homotopic (as continuous mappings of X into $X \times I$).

LEMMA 27.2 For any cylinder $X \times I$, with base b and top t there is a chain homotopy

$$E: S(t) \simeq S(b) : S(X) \to S(X \times I).$$

This lemma implies the Theorem. For let $F: X \times I \to Y$ be a (continuous) homotopy between f_o, $f_1: X \to Y$. Then F at the start gives f_o, so $Fb = f_o$; likewise $Ft = f_1$. Define

$$D_F: C_q(S(X)) \to C_{q+1}(S(Y)) \qquad q = 0, 1 \cdots$$

by setting $D_F c = S(F)Ec$, where E is the "cylinder homotopy" of the Lemma. Then since $\partial E + E\partial = S(t) - S(b)$, we have

$$\begin{aligned} \partial D_F + D_F \partial &= \partial S(F)E + S(F)E\partial \\ &= S(F)(\partial E + E\partial) = S(F)(S(t) - S(b)) \\ &= S(Ft) - S(Fb) = S(f_1) - S(f_o). \end{aligned}$$

This asserts $D_F: S(f_1) \simeq S(f_o)$, as required.

We now construct the cylinder homotopy of the Lemma. Let $T: |\Delta_q| \to X$ be any singular q-simplex of X, $i: I \to I$ the identity map, $T \times i: |\Delta_q| \times I \to X \times I$ their product. We define the cylinder homotopy E on T as

(28.1) $\quad ET = S(T \times i)L(0\ 1 \cdots q)$,

where $L = L^{(q)}$ is the prismatic homotopy of Δ_q. If $\omega_q = (0\ 1 \cdots q)$ is the "basic" q-cell in Δ_q, we may express T as the image of ω_q under the mapping T, so that our definition takes the form $ES(T)\omega_q = S(T \times i)L\omega_q$. We assert that the same formula holds for any chain c in $K(\Delta_q)$:

(28.2) $\quad ES(T)c = S(T \times i)Lc$.

It will suffice to prove this for any r-cell $c = (i_0, \cdots, i_r)$ of $K(\Delta_q)$. Let $\Psi: \Delta_r \to \Delta_q$ be the simplicial map with $\Psi(j) = i_j$, $j = 0, \cdots, r$, and ω_r the basic r-cell in Δ_r. Then

$$c = K(\Psi)\omega_r, \qquad S(T)c = S(T)K(\Psi)\omega_r = S(T\Psi_*)\omega_r,$$

so that $S(T)c$ is the singular cell of X given by the mapping $T\Psi_*: |\Delta_r| \to X$. By the definition (28.1)

$$ES(T)c = S(T\Psi_* \times i)L\omega_r = S((T \times i)(\Psi_* \times i))L\omega_r.$$

Hence, by the consistency property of L (Lemma 27.2)

$$ES(T)c = S(T \times i)LK(\Psi)\omega_r = S(T \times i)Lc,$$

as stated in (28.2).

We now prove that E is the homotopy required in Lemma 28.2. For any cell $T = S(T)\omega_q$ of $S(X)$ we have

$$\begin{aligned}(\partial E + E\partial)S(T)\omega_q &= S(T \times i)L\omega_q + ES(T)\partial\omega_q \\ &= S(T \times i)\ L\omega_q + S(T \times i)L\partial\omega_q \\ &= S(T \times i)(S)t_\Delta) - S(b_\Delta))\omega_q,\end{aligned}$$

where t_Δ, b_Δ are base and top maps for $|\Delta_q| \times I$. But clearly

$(T\times i)t_{\Delta} = t_X T: |\Delta_q| \to X\times I$, and similarly for the base. Hence

$$(\partial E + E\partial)S(T)\omega_q = \{S(t_X) - S(b_X)\}S(T)\omega_q.$$

This asserts that $E: S(t_X) \simeq S(b_X)$, as desired.

The method of proof of this Lemma is typical for the construction of homotopies. Instead of constructing the homotopy E in the space X, the homotopy is first constructed in the simplex Δ_q and then carried into an arbitrary singular simplex T of X by the simplex T, considered as a mapping $T: |\Delta_q| \to X$. The original construction of the prismatic homotopy L in $|\Delta_q| \times I$ can actually be carried out without the explicit formula of §27; indeed the construction really depends merely upon the fact that $|\Delta_q| \times I$, as a convex set in affine space, has banishing homology groups in dimensions > 0.

THEOREM 28.2. The singular homology groups of a topological space $X = \{x\}$ consisting of a single point are

(28.3) $\quad H_0(\{x\}) \cong J, \qquad H_q(\{x\}) = 0, \; q > 0.$

Proof. The space $\{x\}$, regarded as a subset of affine space, is a convex subset; all its singular simplices T are thus affine simplices σ. Define a homotopy D

$$D = (-1)^{m+1}\,\sigma \vee x \qquad m = \dim \sigma .$$

Then by the join boundary formula of §27

$$\partial D\sigma + D\partial\sigma = \sigma \qquad (\dim \sigma > 0).$$

The conclusion follows by Corollary 26.3 (for H_0 the result is already known).

Much the same argument can be used to prove that the homology groups of any convex subset of affine space have the same values (28.3). One can also argue: the convex subset is contractible, hence has the homotopy type of a point, hence by Theorem 28.1, has the same homology groups as a point.

29. Barycentric subdivision. Let $X = | P |$ be the space of a polyhedron P. The singular complex $S(X)$ and its homology groups are already invariants of X, because they are defined directly in terms of the space X, without using the "triangulation" of X given by the polyhedron. The complex $K(P)$ determined by P is not an invariant, since it is constructed from the particular triangulation of X given by P, but on the other hand the groups of chains of $K(P)$ are finitely generated, and so the homology groups of $K(P)$ are computable. Our next main objective is to show that the simplicial homology groups of $K(P)$ are isomorphic to the singular homology groups of the space $| P |$. This will show, on the one hand, that the homology groups of $K(P)$ are invariants of the space $| P |$, and, on the other hand, that the singular homology groups of the space of a polyhedron are computable. The proof of this basic theorem uses barycentric subdivision.

Let P be a polyhedron, and $K(P)$ the associated simplicial complex, defined as on page 104, in which the q-cells are $\sigma_q = (p_0, \cdots, p_q)$, with the $p_0, \cdots, p_q$ the vertices of a frame s of P. Let $b(\sigma_q)$ denote the barycenter of that frame. The first barycentric subdivision P' of P (p.55) has all these $b(\sigma)$ as vertices.

Parallel to the geometric operation of barycentric subdivision we assign an algebraic operation, which maps each q-cell σ of $K(P)$ into the q-chain of $K(P')$ which consists of the cells of dimension q appearing in the geometric subdivision of the simplex of σ. Formally, we define a homomorphism

$$\beta = \beta_q : C_q(K(P)) \longrightarrow C_q(K(P'))$$

by induction on q, setting

(29.1) $\qquad \beta_0(p_0) = b(p_0)$

29.2) $\qquad \beta_q(\sigma) = (-1)^q(\beta_{q-1}\partial\sigma) \vee b(\sigma), \qquad \dim \sigma = q.$

(provided this join makes sense). This formula corresponds exactly to the geometric plan of subdividing σ by joining the barycenter of σ to the subdivision $\beta\partial\sigma$ on the boundary of σ. Explicitly, for $q = 1$ the formula gives

$$
\begin{aligned}
(29.3) \qquad \beta_1(p_o p_1) &= -\beta_o[(p_1) - (p_o)] \vee b(p_o p_1) \\
&= -[b(p_1) - b(p_o)] \vee b(p_o p_1) \\
&= (b(p_o), b(p_o p_1)) - (b(p_1), b(p_o p_1)).
\end{aligned}
$$

This states that the barycentric subdivision of an "edge" $(p_o p_1)$ consists precisely of the two edges into which it is cut by $b(p_o p_1)$. The reader should similary compute $\beta_2(p_o p_1 p_2)$ from (29.2) and (29.3), and observe that this result is the sum with signs of six terms like the six simplices displayed on page 52.

THEOREM 29.1 β is a chain transformation $\beta : K(P) \to K(P')$.

By induction on q, we prove simultaneously: (i) $\partial\beta_q = \beta_{q-1}\partial$ and (ii) $\beta_q\sigma$ is a q-chain on the barycentric subdivision of the sub-polyhedron σ of P. For $q = 0$, these facts are immediate. Given these results for $q - 1$, we first observe that $\beta_{q-1}\partial\sigma$, by (ii), is a $q - 1$ chain, each cell of which is a cell in the subdivision of some face of σ; hence the cell formed by joining it with $b(\sigma)$ lies in P', and indeed in the subdivision of σ. This proves (ii) for q. To prove (i), apply to (29.2) the boundary formula (27.3) for the join giving

$$
\begin{aligned}
\partial\beta_q\sigma &= (-1)^q \partial\beta_{q-1}\partial\sigma \vee b(\sigma) + \beta_{q-1}\partial\sigma \\
&= (-1)^{q-1}\beta_{q-2}\partial\partial\sigma \vee b\sigma + \beta_q\partial\sigma = \beta_{q-1}\partial\sigma
\end{aligned}
$$

for $\partial\beta_{q-1} = \beta_{q-2}\partial$ by induction.

The transformation $\beta = \beta_p$ is defined for every polyhedron. Let $\phi : P \to Q$ be an (abstract) simplicial map of one polyhedron into a second. The definition $\phi' b(p_o, \cdots, p_q) = b(\phi p_o, \cdots, \phi p_q)$

then yields a similar map $\phi' = P' \to Q'$. On the other hand, ϕ and ϕ' then induce chain transformations $K(\phi)$, $K(\phi')$ on the complexes, and we have a diagram

$$\begin{tikzcd} K(P) \arrow[r, "K(\phi)"] \arrow[d, "\beta_P"'] & K(Q) \arrow[d, "\beta_Q"] \\ K(P') \arrow[r, "K(\phi')"] & K(Q'). \end{tikzcd}$$

THEOREM 29.2. Commutativity holds in this diagram:

(29.4) $\quad K(\phi')\beta_P = \beta_Q K(\phi)$

The proof is again by induction on q:

$$K(\phi')\beta\sigma = (-1)^q K(\phi')\{\beta\partial\sigma \vee b\sigma\} = (-1)^q\{K(\phi')\beta\partial\sigma\} \vee (bK(\phi)\sigma)$$
$$= (-1)^q(\beta K(\phi)\partial\sigma) \vee \{bK(\phi)\sigma\} = \beta K(\phi)\sigma.$$

By definition, $K(P)$ and $K(P')$ are both subcomplexes of the singular complex $S(\,|P|\,)$, so that β may be regarded as a chain transformation $\beta : K(P) \to S(\,|P|\,)$. We show now that β is a chain homotopic to the identity $i : K(P) \to S(\,|P|\,)$, by defining a suitable homotopy

$$\gamma_q : C_q(K(P)) \to C_{q+1}(S(\,|P|\,))$$

by recursion, setting

(29.5) $\quad \gamma_0(p_0) = 0$

(29.6) $\quad \gamma_q(\sigma) = (-1)^{q+1}\{\sigma - \gamma_{q-1}\partial\sigma\} \vee b(\sigma) \qquad \dim\sigma = q$

For example, $\gamma_1(p_0p_1) = -(p_0, p_1, b(p_0p_1)) + (p_0, b(p_0p_1))$.

THEOREM 29.3. $\gamma : i \simeq \beta : K(P) \to S(\,|P|\,)$, and if $\phi : P \to Q$ is a simplicial map, then

(29.6a) $\quad \gamma K(\phi) = S(\phi_*)\gamma$.

Proof. We show, by induction on q, that

(i) $\quad \partial\gamma_q = -\gamma_{q-1}\partial + i - \beta_q$

(ii) $\quad \gamma_q\sigma$ is an affine q-chain in $|\sigma|$.

For $q = 0$, these results are immediate. If they are given for $q - 1$, then σ and $\gamma_{q-1}\partial\sigma$ are both affine $(q - 1)$-chains on the convex set $|\sigma|$, so that the join involved in the definition (29.6) makes sense, and proves (ii) for q. Also, using (i) for the subscript $q - 1$,

$$\begin{aligned}
\partial \gamma_q \sigma &= (-1)^{q+1}\{\partial\sigma - \partial\gamma_{q-1}\partial\sigma\} \vee b\sigma + \{\sigma - \gamma_{q-1}\partial\sigma\} \\
&= (-1)^{q+1}\{\beta_{q-1}\partial\sigma - \gamma_{q-2}\partial\partial\sigma\} \vee \partial\sigma + \{\sigma - \partial\gamma_{q-1}\partial\sigma\} \\
&= -\beta\sigma - 0 + \sigma - \gamma_{q-1}\partial\sigma
\end{aligned}$$

q.e.d.

The proof of condition (29.6) is much like that in Theorem 29.2.

We now transfer the operations β and γ to the singular theory. Any singular q-cell T in X a space X has the form $T = S(T)\omega_q$, where ω_q is the basic q-cell in Δ_q and $S(T) : S(|\Delta_q|) \to S(X)$. Define

$$BT = S(T)\beta\omega_q \in C_q(S(X))$$

$$\Gamma T = S(T)\gamma\omega_q \in C_{q+1}(S(X)).$$

THEOREM 29.4. The mapping $B : S(x) \to S(X)$ is a chain transformation, and Γ is a chain homotopy

(29.7) $\quad \partial\Gamma + \Gamma\partial = i - B,$

where i is the identity map $i : S(X) \to S(X)$. Also, if $f : X \to Y$, then

(29.8) $\quad B_Y S(f) = S(f) B_X \qquad \Gamma_Y S(f) = S(f)\Gamma_X,$

where B_X, B_Y denote the maps B for the spaces X, Y respectively.

Proof. The proof of (29.8) is immediate. We next show that B agrees with β, whenever β is defined. In other words, if P is a polyhedron, and σ a cell of P, then $\beta\sigma$, regarded as a singular chain in $C_q(|P|)$, is identical with $B\sigma$. Indeed $\sigma = (p_o, \cdots p_q)$, regarded as a singular cell $T : |\Delta_q| \to |P|$, is just the cell $T = \phi_*$, given by the simplicial map $\phi : \Delta_q \to |P|$ with $\phi(i) = p_i$. Therefore

$\sigma = S(\phi_*)\omega_q$ and the definition of B gives

$$B\sigma = S(\phi_*)\beta\omega_q.$$

Now $S(\phi_*)$ on the simplicial chain $\beta\omega_q$ of $\Delta_{q'}$ is identical with $K(\phi')$, hence by Theorem 29.2

$$B\sigma = K(\phi')\beta_\Delta\omega_q = \beta_P K(\phi)\omega_q = \beta_P\sigma$$

as asserted.

Now compute

$$\partial BT = \partial S(T)\beta\omega_q = S(T)\partial\beta\omega_q = S(T)\beta\partial\omega_q$$

By the fact just observed $\beta\partial\omega_q$ is $B_\Delta\partial\omega_q$; hence by (29.8)

$$\partial BT = S(T)B_\Delta\partial\omega_q = B_X S(T)\partial\omega_q$$
$$= B_X\partial S(T)\omega_q = B_X\partial T.$$

The proof of (29.7) is similar and again depends on the fact that Γ agrees with γ whenever γ is defined.

COROLLARY. 29.5. For any integer n, $B^n : S(X) \to S(X)$ is chain homotopic to the identity, and hence induces isomorphisms onto $(B^n)_* : H_q(X) \to (H_q(X)$ on the homology groups.

Proof. B^n is clearly a chain transformation. By Theorem 26.4, $B^n \simeq i$, and the explicit homotopy Γ_n can be written as

(29.8a) $\Gamma_n = \Gamma + B\Gamma +, \cdots, + B^{n-1}\Gamma.$

Since $\Gamma_n : B^n \simeq i$, $(B^n)_*$ is an isomorphism onto, by Theorem 26.1.

The advantage of B^n is that B^nT can be made to have an arbitrarily small diameter. This may be illustrated as follows. If $\mathcal{U}$ is any collection of open sets covering a space X, we call a singular simplex T of X, $\mathcal{U}$-small if $T(|\Delta_q|)$ is contained in one of the sets U of the covering, $\mathcal{U}$. Since any face of a "small" simplex is small, the singular chains of X which involve only $\mathcal{U}$-small simplices clearly constitute a subcomplex $S_{\mathcal{U}}(X)$. The homology groups of X can be computed from this complex, in the following sense:

THEOREM 29.6. If $\mathcal{U}$ is any open covering of X, then the identity injection $i : S_{\mathcal{U}}(X) \to S(X)$ induces an isomorphism of the homology groups of $S_{\mathcal{U}}(X)$ onto those of $S(X)$.

PROOF. We first observe that if T is $\mathcal{U}$-small, so are BT and ΓT. Indeed $\beta\sigma$ and $\gamma\sigma$ are chains lying on the set $|\sigma|$, hence BT and ΓT are, by their definitions, singular chains lying on the set $T(|\Delta|)$; thus if this set is "small", so are the chains BT and ΓT, and, for that matter, so are $B^n T$ and $\Gamma_n T$.

Now a continuous map $T : |\Delta_q| \to X$ carries the covering $\mathcal{U}$ by open sets U back into a covering of $|\Delta_q|$ by the open sets $T^{-1}U$. But $|\Delta_q|$ is a compact space, hence is covered by a finite number of these sets, say $T^{-1}U_1$, ...$T^{-1}U_{l_n}$. Also $|\Delta_q|$ is a metric space, hence this finite open covering has a Lebesgue number $\epsilon > 0$ such that any subset of $|\Delta_q|$ os diameter less than ϵ lies entirely in one of the sets $T^{-1}U_i$. By the refinement property of barycentric subdivision (Theorem 12.3), there is therefore an integer n such that each simplex of the n-th barycentric subdivision $\Delta_q^{(n)}$ has diameter less than ϵ. Since $\beta^n \omega_q$ is a chain in $K(\Delta_q^{(n)})$, each cell of $\beta^n \omega_q$ lies in one of the sets $T^{-1}U_i$. Hence each cell of $B^n T = S(T)\beta^n \omega_q$ lies in one of the sets U_i of the covering $\mathcal{U}$. In other words, we have found for each T an n such that $B^n T$ is $\mathcal{U}$-small. Since a singular chain c involves only a finite number of cells T, we can find for c an integer n (the maximum of the integers appropriate to its cells) such that $B^n c$ is $\mathcal{U}$-small.

In particular, let z be a cycle of $S(X)$, and choose

n so that $B^n z$ is in $S_{\mathcal{U}}(X)$. Then

$$z - B^n z = \partial \Gamma_n z + \Gamma_n \partial z = \partial \Gamma_n z.$$

This equation asserts that the cycle z is homologous to the small cycle $B^n z$, in other words

$$i : S_{\mathcal{U}}(X) \to S(X)$$

is a homomorphism onto for homology groups.

Now let z be a $\mathcal{U}$-small cycle which become a boundary in $S(X)$, so that $z = \partial c$. As before, pick n so that $B^n c$ is $\mathcal{U}$-small. Then

$$c = B^n c + \partial \Gamma_n c + \Gamma_n \partial c$$

and

$$z = \partial c = \partial B^n c + \partial\partial \Gamma_n c + \partial \Gamma_n \partial c = \partial(B^n c + \Gamma_n z),$$

which states that z is already the boundary of the small chain $B^n + \Gamma_n z$ (note that z-small implies $\Gamma_n z$ small). This means that i above is an isomorphism into for the homology groups, and completes the theorem.

The Axioms for Homology

30. Relative Homology groups. Let K be a chain complex; a set of subgroups $C'_q \subset C_q(K)$, one for each dimension, constitutes a sub-complex K' of K if each boundary homomorphism $\partial: C_q \longrightarrow C_{q-1}$ carries C'_q into C'_{q-1}:

$$\begin{array}{ccccccc}
\cdots & C_{q-2} & \xleftarrow{\partial} & C_{q-1} & \xleftarrow{\partial} & C_q & \cdots \\
 & \cup & & \cup & & \cup & \\
\cdots & C'_{q-2} & \xleftarrow{\partial} & C'_{q-1} & \xleftarrow{\partial} & C'_q & \cdots
\end{array}$$

The identity function i with $i(c') = c' \in C_q$ for each $c' \in C'_q$ is then a chain transformation $i : K' \longrightarrow K$, hence induces a homomorphism

(30.1) $$i_* : H_q(K') \longrightarrow H_q(K)$$

on the homology groups in each dimension. We wish to determine the kernel and the image of this homomorphism. The kernel consists of the homology classes of those cycles in K' which become boundaries of chains in the larger complex K; the image consists of the homology classes of those cycles z in K which are homologous (in K) to cycles lying in the subcomplex K'. These groups will be determined by using as auxiliaries the homology groups of the factor complex K/K' and certain "relative" homology groups of K modulo K'.

The factor complex K/K' is defined to be a chain complex with chain groups $C_q(K/K') = C_q(K)/C_q(K')$; since $\partial: C_q \longrightarrow C_{q-1}$ maps C'_q into C'_{q-1}, it induces a homomorphism

$$\bar{\partial}: C_q(K/K') \longrightarrow C_{q-1}(K/K')$$

defined on the coset $c + C'_q$ of each $c \in C_q$ by

$$\bar{\partial}(c + C_q(K')) = \partial c + C_{q-1}(K') .$$

Clearly $\bar{\partial}\bar{\partial} = 0$, so that K/K', with this boundary $\bar{\partial}$, is again a chain complex, and thus has homology groups $H_q(K/K')$ in each dimension. If we denote the canonical homomorphism $c \longrightarrow c + C_q(K')$ by $j: C_q(K) \longrightarrow C_q(K/K')$, then the definition of the boundary $\bar{\partial}$ reads $\bar{\partial} jc = j\partial c$. This asserts that j is a chain transformation $j: K \longrightarrow K/K'$, and hence that j induces homomorphisms

(30.2) $$j_*: H_q(K) \longrightarrow H_q(K/K') .$$

The groups $H_q(K/K')$ may also be described in terms of "relative" cycles and boundaries. A <u>relative</u> q-<u>cycle</u> of K modulo K' is a chain $c \in C_q$ such that $\partial c \in C'_{q-1}$ (i.e., such that $j\partial c = 0$). The set of all relative q-cycles is a subgroup $Z_q(K,K')$ of $C_q(K)$. A <u>relative</u> q-<u>boundary</u> of K modulo K' is a chain of C_q which can be written in the form

$$c = c' + \partial b, \qquad c' \in C_q(K'), \qquad b \in C_{q+1}(K) .$$

The set of all such relative boundaries is a subgroup $B_q(K,K')$ of $C_q(K)$, and is indeed the subgroup

$$B_q(K,K') = C_q(K') + B_q(K)$$

spanned by all q-chains of K' and all q-boundaries of K. Since $\partial(c' + \partial b) = \partial c' + \partial\partial b = \partial c' \in C_{q-1}(K')$, every relative boundary is a relative cycle, so that $B_q(K,K') \subset Z_q(K,K')$. The relative homology group of K modulo K' is then defined as

(30.3) $$H_q(K,K') \cong Z_q(K,K')/B_q(K,K') .$$

Lemma 30.1. The map $j(c) = c + C'_q$ induces an isomorphism onto

$$H_q(K,K') \cong H_q(K/K') .$$

Proof. Consider the map $j: C_q \longrightarrow C_q(K/K')$. By definition, c is a relative cycle if and only if jc is a cycle in K/K', and c is a relative boundary if and only if jc is a boundary in K/K'. Hence j induces an isomorphism

$$j: Z_q(K/K')/B_q(K/K') \longrightarrow Z_q(K/K')/B_q(K/K'),$$

as asserted in our lemma.

We shall use this isomorphism to identify the relative homology groups of K modulo K' with the homology groups of K/K'.

If c is a relative cycle, then ∂c is a q-1 chain of K' which is surely a cycle in K', since $\partial\partial c = 0$. (Indeed, ∂c is a boundary in K, but need not be a boundary in K'; recall our aim of studying those cycles of K' which become boundaries in K'). Thus $\partial : C_q \longrightarrow C_{q-1}$ induces a homomorphism

$$\partial : Z_q(K,K') \longrightarrow Z_{q-1}(K') \text{ .}$$

Under this homomorphism, a relative boundary $c = c' + \partial b$ is carried to $\partial c = \partial c' + \partial\partial b = \partial c'$, a boundary in K'. In other words, ∂ maps $B_q(K,K')$ into $B_{q-1}(K')$, and thus induces a map

(30.4) $$\partial_* : H_q(K,K') \longrightarrow H_{q-1}(K') \text{ .}$$

Upon combining the homomorphisms (30.3), (30.2), and (30.4) we have a sequence of homology groups and homomorphisms

(30.5) $$\cdots \longrightarrow H_q(K') \xrightarrow{i_*} H_q(K) \xrightarrow{j_*} H_q(K,K') \xrightarrow{\partial_*} H_{q-1}(K') \longrightarrow \cdots$$

which we call the relative homology sequence of $K \supset K'$. Its basic property is

Theorem 30.2. The relative homology sequence (30.5) of a complex K and a subcomplex K' is exact.

Here we use the

Definition. A sequence

$$\cdots \xrightarrow{\alpha_{p+1}} A_{p+1} \xrightarrow{\alpha_p} A_p \xrightarrow{\alpha_{p-1}} A_{p-1} \longrightarrow A_{p-2} \longrightarrow \cdots$$

of groups and homomorphisms is exact if for each p the kernel of $\alpha_p : A_p \longrightarrow A_{p-1}$ is equal to the image of $\alpha_{p+1} : A_{p+1} \longrightarrow A_p$.

In other words, for each $a \in A_p$, $\alpha_p a = 0$ if and only if $a = \alpha_{p+1} b$.

for some $b \in A_{p+1}$. Note that the requirement kernel $\alpha_p \supset$ Image α_{p+1} by itself means that $\alpha_{p+1}\alpha_p : A_{p+1} \longrightarrow A_{p-1}$ is zero. Hence we could equally well say that an exact sequence is a chain complex with all its homology groups zero.

The proof of Theorem 30.2 breaks up into three parts:

(i) Image i_* = Kernel j_* in $H_q(K)$,

(ii) Image j_* = Kernel ∂_* in $H_q(K,K')$,

(iii) Image ∂_* = Kernal i_* in $H_{q-1}(K')$.

For example, in (ii) we first prove that image $\subset$ kernel; i.e., that $\partial_* j_* = 0$. For, take a cycle z in $Z_q(K)$. Then $j_* z$ is the same cycle, considered now modulo $B_q(K,K')$, and ∂_* maps it onto the homology class of $\partial z = 0$ in K'. On the other hand, image $\supset$ kernel. For let c be a relative cycle with a homology class $\{c\} = c + B_q(K,K')$. If $\partial_*\{c\} = 0$, then ∂c must be a boundary in K', hence $\partial c = \partial b'$ for some $b' \in K'$. But $\partial b'$ is a relative boundary, and hence c, regarded as a relative cycle, is homologous to $z = c - b'$, with $\partial z = \partial c - \partial b' = 0$. In other words, the homology class of C in $H_q(K,K')$ is that of Z, the image $j_*\{z\}$ of a homology class from $H_q(K)$.

The proofs of the other parts (i) and (iii) above are similar.

It will be convenient to record the effect of chain transformation and chain homotopies upon the relative groups.

Theorem 30.3. Let $K \supset K'$, $L \supset L'$ be chain complexes. Then any chain transformation $\lambda : K \longrightarrow L$ such that $\lambda C_q(K') \subset C_q(L')$ for each q induces a chain transformation

$$\bar{\lambda} : K/K' \longrightarrow L/L' ,$$

with

$$\bar{\lambda}\,[C + C_q(K')\,] = \lambda C + C_q(L') .$$

In the diagram for the two relative homology sequences

$$(30.6)\quad \begin{tikzcd} \cdots \arrow[r] & H_q(K') \arrow[r, "i_*"] \arrow[d, "\lambda_*"] & H_q(K) \arrow[r, "j_*"] \arrow[d, "\lambda_*"] & H_q(K,K') \arrow[r, "\partial_*"] \arrow[d, "\bar{\lambda}_*"] & H_{q-1}(K') \arrow[r] \arrow[d, "\lambda_*"] & \cdots \\ \cdots \arrow[r] & H_q(L') \arrow[r, "i_*"] & H_q(L) \arrow[r, "j_*"] & H_q(L,L') \arrow[r, "\partial_*"] & H_{q-1}(L') \arrow[r] & \cdots \end{tikzcd}$$

commutativity holds in each square; for example

$$(30.7)\qquad \partial_* \lambda_* = \lambda_* \partial_* : H_q(K_1K') \longrightarrow H_{q-1}(L') .$$

Any chain homotopy $D: \lambda \simeq \mu: K \longrightarrow L$ between two such chain transformations, and such that $DC_q(K') \subset C_{q+1}(L')$ induces a chain homotopy

$$(30.8)\qquad \bar{D}: \bar{\lambda} \simeq \bar{\mu}: K/K' \longrightarrow L/L' .$$

The proof is immediate, by the various definitions. For example, to establish the commutativity (30.7), let c be any relative cycle of K modulo K' and $\{c\}$ its relative homology class. Then $\lambda_* \partial_* \{c\} = \lambda_* \{\partial c\} = \{\lambda \partial c\}$, and $\partial_* \bar{\lambda}_* \{c\} = \partial_* \{\lambda c\} = \{\partial \lambda c\}$; since $\lambda \partial = \partial \lambda$, the results are equal. Similarly in (30.8) we define $\bar{D}[c + C_q(K')]$ to be $Dc + C_{q+1}(L')$.

31. The Five Lemma.

In the manipulation of exact sequences, we frequently consider diagrams of groups and homomorphisms such as

$$(31.1)\quad \begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\gamma_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "\gamma_4"] & A_5 \arrow[d, "\gamma_5"] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

Lemma 31.1 (Five Lemma). In the diagram (31.1), assume that each row is an exact sequence (at A_2, A_3, A_4 and at B_2, B_3, B_4) and that commutativity holds in each rectangle (i.e., $\gamma_2\alpha_1 = \beta_1\gamma_1$, etc.). If

(i) $\quad \gamma_2(A_2) = B_2$, $\gamma_4(A_4) = B_4$ and γ_5 has kernel 0,

then $\quad \gamma_3(A_3) = B_3$. If

(ii) $\quad \gamma_1(A_1) = B_1$, γ_2 and γ_4 have kernel 0

then $\quad \gamma_3$ has kernel zero.

The conclusion is often quoted in the weaker but snappier form:

If $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ are isomorphisms, so is γ_3.

The proof of (i) is accomplished by the method of "diagram chasing".

We first show that $\gamma_3 A_3$ includes the kernel of β_3. Starting with k_3 in the kernel, we construct elements

$$\begin{tikzcd} a_2 \arrow[r] \arrow[d, dash] & a_3 \arrow[d, dash] \\ b_2 \arrow[r] & k_3 \end{tikzcd}$$

as follows. Since $\beta_3 k_3 = 0$, exactness at B_3 gives an element b_2 in B_2 with $\beta_2 b_2 = k_3$. Since $\gamma_2(A_2) = B_2$, there is then an a_2 in A_2 with $\gamma_2 a_2 = b_2$. Set $a_3 = \alpha_2 a_2$. Then $\gamma_3 a_3 = \gamma_3 \alpha_2 a_2 = \beta_2 \gamma_2 a_2 = k_3$, and thus $k_3 \in \gamma_3 A_3$, as asserted.

Now let b_3 be any element in B_3. We construct elements

$$\begin{tikzcd} a_3 \arrow[r] \arrow[d] & a_4 \arrow[r] \arrow[d] & 0 \arrow[d] \\ b_3 \arrow[r] & b_4 \arrow[r] & 0 \end{tikzcd}$$

as follows. Let $b_4 = \beta_3 b_3$. Then by exactness at B_4, $\beta_4 b_4 = \beta_4 \beta_3 b_3 = 0$. Since $\gamma_4(A_4) = B_4$, there is an element a_4 in A_4 with $\gamma_4 a_4 = b_4$. Then $\gamma_5 \alpha_4 a_4 = \beta_4 \gamma_4 a_4 = \beta_4 b_4 = 0$. Since γ_5 has kernel 0, $\alpha_4 a_4 = 0$. By exactness at A_4, there is an element a_3 with $\alpha_3 a_3 = a_4$. Then $\beta_3 \gamma_3 a_3 = \gamma_4 \alpha_3 a_3 = \gamma_4 a_4 = b_4 = \beta_3 b_3$. Hence $\beta_3(b_3 - \gamma_3 a_3) = 0$, so that $b_3 - \gamma_3 a_3$ is in kernel (β_3), hence in $\gamma_3(A_3)$ by the previous result. Therefore $b_3 = \gamma_3 a_3 + (b_3 - \gamma_3 a_3)$ is in $\gamma_3(A_3)$, q.e.d.

To prove (ii), start with an a_3 in A_3 with $\gamma_3 a_3 = 0$ and construct elements

$$\begin{tikzcd} a_1 \arrow[r] \arrow[d] & a_2 \arrow[r] \arrow[d] & a_3 \arrow[r] \arrow[d] & 0 \arrow[d] \\ b_1 \arrow[r] & b_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

as follows. Since $\gamma_3 a_3 = 0$, $\beta_3 \gamma_3 a_3 = \gamma_4 \alpha_3 a_3 = 0$. But γ_4 has kernel zero, hence $\alpha_3 a_3 = 0$. By exactness at A_3, there is an a_2 with $\alpha_2 a_2 = a_3$; set $b_2 = \gamma_2 a_2$. Then $\beta_2 b_2 = \beta_2\gamma_2 a_2 = \gamma_3\alpha_2 a_2 = 0$, so by exactness at B_2, there is a b_1 with $\beta_1 b_1 = b_2$. Since $\gamma_1(A_1) = B_1$, there is an a_1 with $\gamma_1 a_1 = b_1$. Then $\gamma_2\alpha_1 a_1 = \beta_1\gamma_1 a_1 = \beta_1 b_1 = b_2 = \gamma_2 a_2$, or $\gamma_2(\alpha_1 a_1 - a_2) = 0$. Since γ_2 has kernel 0, $a_2 = \alpha_1 a_1$. Therefore $a_3 = \alpha_2 a_2 = \alpha_2\alpha_1 a_1 = 0$, by exactness at A_2.

The result applies at once to the situation in Theorem 30.3, where we have chain complexes $K \supset K'$, $L \supset L'$ and a chain transformation $\lambda: K \longrightarrow L$ with $\lambda(K') \subset L'$, with the corresponding induced homomorphisms on the homology groups

(31.2) $\lambda_*: H_q(K) \longrightarrow H_q(L)$, $\lambda_*: H_q(K') \longrightarrow H_q(L')$, $\bar{\lambda}_*: H_q(K_1K') \longrightarrow H_q(L_1L')$.

The five Lemma for the diagram (30.6) yields

Corollary 31.2. If two of the three homomorphisms of (31.2) are isomorphisms for all q, so is the third.

32. <u>The homotopy axiom</u>. Let X' be a subspace of the topological space X. The singular complex S(X') is then subcomplex of S(X). Hence we obtain as in §30 the relative singular homology groups $H_q(S(X), S(X')$ which we write more simply as $H_q(X,X')$. The basic geometric picture is given by the observation that a relative cycle of X modulo X' is just a chain of X whose boundary lies in the subspace X'. In particular if X' is the empty subset 0, $H_q(X,0)$ is simply $H_q(X)$.

Let $Y \supset Y'$ be a second pair (space + subspace) and $f: X \longrightarrow Y$ a continuous map with $f(X') \subset Y'$. Then f induces the usual chain transformation $S(f): S(X) \longrightarrow S(Y)$, which carries S(X') into S(Y'). Therefore Theorem 30.3 applies to yield a diagram

$$
(32.1)\quad
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_q(X') & \xrightarrow{i_*} & H_q(X) & \xrightarrow{j_*} & H_q(X,X') & \xrightarrow{\partial_*} & H_{q-1}(X') & \longrightarrow \cdots \\
 & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
\cdots \longrightarrow & H_q(Y') & \longrightarrow & H_q(Y) & \longrightarrow & H_q(Y,Y') & \longrightarrow & H_{q-1}(Y') & \longrightarrow \cdots
\end{array}
$$

for the relative homology sequences of X, X' and of Y, Y'. Commutativity holds in each square of this diagram.

Theorem 32.1 (Homotopy Axiom). If $f, g: X \longrightarrow Y$ are continuous maps and X', Y' are subspaces of X, Y respectively with $f(X') \subset Y'$ and $g(X') \subset Y'$, while $F: X \times I \longrightarrow Y$ is a homotopy between f and g such that $F(x', t) \in Y'$ for every $x' \in X'$ and $t \in I$, then

$$(32.2)\qquad \bar{f}_* = \bar{g}_*: H_q(X,X') \longrightarrow H_q(Y,Y').$$

We already know, using the cylinder homotopy (Theorem 28.1) that $S'(f)$ and $S(g): S(X) \longrightarrow S(Y)$ are chain homotopic and hence by Theorem 26.1 that

$$f_* = g_*: H_q(X) \longrightarrow H_q(Y) .$$

The same results apply to $f_* = g_*: H_q(X') \longrightarrow H_q(Y')$. The current theorem extends this result to the relative homology groups. The essential hypothesis is the assertion that during the homotopy, images of points in the subspace X' of X are moved only through the subspace Y' of Y.

Proof. Use the cylinder homotopy of § 28:

$$D_F: S(g) - S(f): S(X) \longrightarrow S(Y) .$$

If $T: \Delta_q \longrightarrow X'$ is a singular simplex of the subspace X', then $S(f)T$ and $S(g)T$ are singular simplices of Y', by the hypothesis that $f'X' \subset Y'$, $g(X') \subset Y'$. Furthermore $D_F T$, by its construction, lies in $S(Y')$. Hence the hypotheses of Theorem 30.3 apply, and D_F induces a chain homotopy between the two chain transformations f and g of $S(X)/S(X')$ into $S(Y)/S(Y')$.

33. The excision axiom. The determination of singular homology groups by small simplices, as discussed in Theorem 29.6, also applies to the relative groups as follows.

Lemma 33.1. If $\mathcal{U}$ is a covering of the space X by open sets U, then the collection $\mathcal{U}'$ of open sets $U \cap X'$, for $U \in \mathcal{U}$, is an open covering of the subspace X' of X, and a singular simplex of X' is $\mathcal{U}$-small precisely when t is $\mathcal{U}'$-small. The identity injection $\varkappa: S_{\mathcal{U}}(X) \longrightarrow S(X)$ induces an isomorphism.

(33.1) $$\varkappa_*: H_q(S_{\mathcal{U}}(X)/S_{\mathcal{U}'}(X')) \cong H_q(S(X)/S(X'))$$

Proof. The statements of the first sentence are immediate consequences of the definition of "small" simplicies. We therefore have the situation of Theorem 30.3, with complexes $S_{\mathcal{U}}(X) \supset S_{\mathcal{U}}(X')$ and $S(X) \supset S(X')$ and a chain transformation $\varkappa$ of the first pair into the second. Furthermore the basic result (Theorem 29.6) on the sufficiency of small simplices shows that

$$\varkappa_*: H_q(S_{\mathcal{U}}(X)) \longrightarrow H_q(S(X)), \quad \varkappa_*: H_q(S_{\mathcal{U}'}(X')) \longrightarrow H_q(S(X'))$$

are isomorphisms. The 5-Lemma, in the final Corollary 31.2, then gives the conclusion.

Intuitively, the relative homology of a space X modulo a subspace X' should not depend on what happens "inside" the subspace X'. This will be expressed by a theorem which discusses the effect of "excising" a subset inside X'.

Theorem 33.2 (Excision axiom). Let $X \supset X' \supset A$ be spaces, with the closure of A contained in the interior of X', and let $X-A \supset X'-A$ denote the subspaces obtained by removing all points of A from X and X' respectively. Then the identity mapping k of X-A into X induces isomorphisms on the relative homology groups

(33.2) $$k_*: H_q(X-A, X'-A) = H_q(X,X') .$$

Proof. The hypothesis on the closure of A insures that X has an open covering $\mathcal{U} = \{U, V\}$ by the two sets

$$U = \text{interior } X', \quad V = X-\bar{A} .$$

Using simplices small with respect to this covering for X and each of its subspaces, the appropriate identity injections yield a diagram of quotient complexes

$$(33.3)\qquad \begin{array}{ccc} S_{\mathfrak{U}}(X - A)/S_{\mathfrak{U}}(X' - A) & \xrightarrow{\pi} & S(X - A)/S(X) \\ \Big\downarrow{\scriptstyle S(k)} & & \Big\downarrow{\scriptstyle S(k)} \\ S_{\mathfrak{U}}(X)/S_{\mathfrak{U}}(X') & \xrightarrow{\qquad\pi\qquad} & S(X)/S(X') \end{array}$$

clearly commutativeiy holds in this diagram ($\pi S_{\mathfrak{U}}(k) = S(k)\pi$). There is a corresponding diagram for the homology groups of these quotient complexes, which is again commutative. Our conclusion is to be that the right hand map k_* is an isomorphism for homology. By Lemma 33.1 the top and bottom maps π_* are isomorphisms for the homology diagram, hence the conclusion will follow if we show that S(k) induces an isomorphism for homology groups.

We shall prove a little more; to wit, that

$$(33.4)\qquad S_{\mathfrak{U}}(k) : S_{\mathfrak{U}}(X - A)/S_{\mathfrak{U}}(X' - A) \cong S_{\mathfrak{U}}(X)/S_{\mathfrak{U}}(X')$$

is an isomorphism for chain complexes (and hence certainly induces an isomorphism for homology groups). Now if L and M are subcomplexes of a chain complex K, we can define their intersection $L \cap M$ to be the subcomplex of K with chain groups $C_q(L \cap M) = C_q(L) \cap C_q(M)$ in each dimension q, and we can also define their <u>union</u> to be the subcomplex $L \cup M$ of K with chain groups $C_q(L \cup M) = C_q(L) \cup C_q(M)$, the subgroup of $C_q(K)$ spanned by $C_q(L)$ and $C_q(M)$. One of the basic isomorphism theorems for groups asserts that for subgroups L and M of an abelian group the identity injection provides an isomorphism

$$(33.5)\qquad k : L/L \cap M \cong (L \cup M)/M .$$

Exactly the same isomorphism is valid for subcomplexes L and M of a chain complex; indeed, the identity injection is clearly a chain transformation, and in each dimension it is an isomorphism for the chain groups, by the group theoretic theorem. We shall show that the derived isomorphism (33.4) is a special case of the isomorphism theorem (33.5).

Indeed $S_{\mathcal{U}}(X')$ is a subcomplex of $S(X)$ which is generated by certain of the free generators (singular simplices) of $S(X)$, and Lemma 33.1 asserts that

$$S_{\mathcal{U}}(X') = S_{\mathcal{U}}(X) \cap S(X'), \quad S_{\mathcal{U}}(X - A) = S_{\mathcal{U}}(X) \cap S(X - A).$$

Furthermore the subcomplexes $S(X')$ and $S(X - A)$ are free groups with generators the singular simplices T in X' and $X - A$, respectively; a little consideration shows that their intersection is exactly the subcomplex with generators the singular simplices T in $X' - A$, hence the subcomplex $S(X' - A)$. Therefore

$$(33.6) \qquad S_{\mathcal{U}}(X') \cap S_{\mathcal{U}}(X - A) = S_{\mathcal{U}}(X') \cap S(X' - A) = S(X' - A).$$

On the other hand the union $S_{\mathcal{U}}(X') \cup S_{\mathcal{U}}(X - A)$ is spanned by the $\mathcal{U}$-small singular simplices of X which lie either in X' or in $X - A$; but by the choice of $\mathcal{U}$ above every $\mathcal{U}$-small simplex is either in $U \subset X'$ or $V \subset X - A$, so that

$$(33.7) \qquad S_{\mathcal{U}}(X') \cup S_{\mathcal{U}}(X - A) = S_{\mathcal{U}}(X)$$

In view of (33.6) and (33.7), the projected isomorphism of (33.4) becomes exactly

$$S_{\mathcal{U}}(X - A)/(S_{\mathcal{U}}(X') \cap S_{\mathcal{U}}(X - A)) \longrightarrow (S_{\mathcal{U}}(X') \cup S_{\mathcal{U}}(X-A)/S_{\mathcal{U}}(X')$$

which is indeed a special case of the isomorphism theorem 33.5, q.e.d.