Don Anderson
Thesis

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§0 Introduction

In this paper, we describe a new form of K-theory which is intermediate between the K-theories based on the orthogonal and the unitary groups. We shall further show that this theory has finite dimensional approximations to its classifying space which are differentiable manifolds; in fact they are differentiable fiber bundles over the complex Grassman manifolds with fiber and structure group unitary groups. In this paper, we shall not go into either the question of characteristic classes for the new theory, nor that of internal cohomology operations. We shall rather concern ourselves with relations between the new theory and the older K-theories. These relations will have the form of long exact sequences, and can be used to obtain information about the real K-theory $KO^*(X)$ of a space from the complex K-theory $KU^*(X)$. In particular, we shall use this technique to prove a new theorem about $KO^*(B_G)$, where B_G is the classifying space of a compact Lie group G.

In defining our new theory, we do not use the Grothendieck construction as in [2], but take the concept of stable equivalence classes of bundles as the basis of our description. This is done to more easily establish the exact sequence of a pair. We assume that the reader is familiar with such a description of $KU^*(X)$; if not he can easily infer it from what follows in this paper.

§1 The Theory KC*

It X is a finite CW-complex, the group $\widetilde{KC}(X)$ will be defined in terms of complex vector bundles over X, together with equivalences between such bundles. In order that these equivalences may be differentiated from one another, our fiber bundles will be taken to be coordinate bundles in the sense of Steenrod [7]. All bundle maps will be coordinate bundle maps. Two bundle equivalences f_0 and f_1 will be said to be homotopic as bundle equivalences if there is an extention of f_0 and f_1 to a continuous family f_t of bundle equivalences defined for $0 \leq t \leq 1$.

By a real, complex, or quaternionic vector bundle, we shall mean a fiber bundle with fiber a real, complex, or quaterionic vector space, and structure group an orthogonal, unitary, or symplectic group, respectively.

It is well known that a complex vector bundle in our sense can be thought of as a real bundle E, together with an automorphism $J:E \longrightarrow E$ such that $J^2 = -I$, where I is the identity automorphism. A quaternionic vector bundle is given by a real bundle E with two automorphisms J_1, J_2 such that $J_1{}^2 = J_2{}^2 = -I$, and $J_1J_2 = -J_2J_1$. The complex conjugate to a complex bundle E with complex structure J is the same underlying real bundle with complex structure given by $-J$. Clearly, quaternionic bundles give rise to three types of quaternionic conjugates.

If E, E' are two complex vector bundles with complex structures given by J, J' respectively, a map of the real underlying bundles $f: E \longrightarrow E'$ is a map of complex bundles iff $fJ = J'f$. A self-conjugacy of a complex bundle E is an automorphism e such that $eJ = -Je$. A bundle admits a self-conjugacy iff it is equivalent to its conjugate.

If E is a real bundle, the complexification $E \otimes C_1$ of E is self-conjugate. C_1 is the trivial one dimensional complex line bundle. The complex structure on $\varepsilon_U(E) = E \otimes C_1$ is given by $1 \otimes J$, where J is the complex structure of C_1, and the self-conjugacy of $\varepsilon_U(E)$ is given by $1 \otimes c_1$, where c_1 is the conjugation of complex numbers. This is called the canonical self-conjugacy of $\varepsilon_U(E)$. If E is a quaternionic bundle with quaternionic structure given by J_1 and J_2, $\varepsilon_U(E)$ is the complex bundle with the same underlying real structure as E, with the complex structure given by J_1, and with canonical self-conjugacy given by J_2.

If E is a complex bundle, and e is a self-conjugacy, we might be led to conjecture that E could be split into a direct sum of the form $E = E' \oplus E''$, $e = e' \oplus e''$, where E' is the complexification of a real bundle, and E'' comes from a symplectic bundle in such a way that e' and e'' are the canonical self-conjugacies. We will eventually see that there is a counterexample to this conjecture on the three dimensional sphere, even if e is allowed to vary by a homotopy.

<u>Definition 1.1</u> A C-pair (E,e) on a finite CW-complex X is a pair whose first element is a self-conjugate complex vector bundle on X, and whose second element is a self-conjugacy of this bundle.

<u>Definition 1.2</u> If E,F are complex vector bundles, and $f:E \longrightarrow F$ is a map of complex vector bundles, $\overline{f}:\overline{E} \longrightarrow \overline{F}$ is the map which agrees with f on the underlying bundles, where $\overline{E}$, $\overline{F}$ denote the complex conjugates of the bundles E and F respectively.

<u>Definition 1.3</u> Two C-pairs (E,e) and (E',e') over a space X are equivalent if there exists a bundle equivalence $f:E \longrightarrow E'$ such that $e'f$ and $\overline{f}e$ are homotopic as bundle equivalences.

It is clear that given two C-pairs (E,e) and (F,f) over a space, we can form their direct sum by defining $(E,e)\oplus(F,f) = (E\oplus F, e\oplus f)$. The trivial n-dimensional C-pair (C_n,c_n) over a space X is the pair consisting of the trivial n-dimensional complex bundle C_n over X, together with the automorphism c_n given by the conjugation of complex vectors. Two C-pairs will be called stably equivalent if they become equivalent when suitable trivial C-pairs are added to each of them. The stable equivalence class of (E,e) is denoted by $\{E,e\}$.

<u>Definition 1.4</u> If X is a finite CW-complex, $\widetilde{K}C(X)$ is the abelian semigroup of stable equivalence classes of C-pairs over X. If A a subcomplex of X, and TA is the cone on A, $KC^{-n}(X,A) = \widetilde{K}C(\Sigma^n(X\cup TA))$, where Σ^n denotes reduced suspension (TA is not the reduced cone, so its vertex serves as a basepoint for $X\cup TA$).

For general facts about cohomology theories defined in this way, we refer the reader to [3] and [5].

Recall that $\widetilde{K}U(X)$ is the semigroup of stable equivalence classes of complex bundles over X. We can define a cohomology operation $\zeta: KC^{-n}(X,A) \longrightarrow KU^{-n}(X,A)$ by $\zeta\{E,e\} = \{E\}$. The image of ζ

is the kernel of another cohomology operation $\psi: KU^{-n}(X,A) \longrightarrow KU^{-n}(X,A)$ given by $\psi(x) = x - \bar{x}$. In Adams's notation, $\psi = \psi^1 - \psi^{-1}$. From Adams [1] we see that if $\pi_U: KU^{-n}(X,A) \longrightarrow KU^{-n-2}(X,A)$ is the periodicity isomorphism, $\pi_U\psi = \varepsilon_U\varepsilon_O\pi_U$, and $\pi_U{}^2\psi = \psi(\pi_U)^2$. ε_O deontes the operation which takes the real underlying bundle of a complex bundle, and ε_U is the complexification as before.

There is another operation which will be of interest to us. $\widetilde{KU}^{-1}(X) = [X,U]$. If $f: X \longrightarrow U(n)$, f determines an automorphism $a(f)$ of C_n. We define $\gamma(f)$ to be $\{C_n, c_n a(f)\} \in \widetilde{KC}(X)$. It is clear that every self-conjugacy of the trivial bundle is of the form $c_n a(f)$ for some f, so the image of γ is the kernel of ζ.

<u>Lemma 1.1</u> The sequence

$$\widetilde{KU}^{-1}(X) \xrightarrow{\gamma} \widetilde{KC}^0(X) \xrightarrow{\zeta} \widetilde{KU}^0(X) \xrightarrow{\psi} \widetilde{KU}^0(X)$$

is exact. Thus $\widetilde{KC}^0(X)$ is an abelian group.

<u>Theorem 1.1</u> In those dimensions in which KC^n is defined, KC satisfies all the Eilenberg-Steenrod axioms for a cohomology theory except the dimension axiom.

proof: All that needs verification is that there is an exact sequence for a pair. From the work of Dold [5], it suffices to show that for every CW-pair (X,A) the sequence $\widetilde{KC}(X \cup TA) \longrightarrow \widetilde{KC}(X) \longrightarrow \widetilde{KC}(A)$ is exact. Suppose that (E,e) is a C-pair on X such that its restriction to A is trivial. Then on A there exists a complex bundle equivalence $f: E \longrightarrow C_n$ such that $c_n f$ and $\bar{f}e$ are homotopic as bundle equivalences. Let F be the bundle over $X \cup TA$ which results from identifying E on X with C_n on TA along A by means of f. In the coordinate system of C_n on TA e becomes $\overline{f}ef^{-1}$. However, c_nf and $\overline{f}e$ are homotopic as bundle equivalences, so c_n and $\overline{f}ef^{-1}$ are. Using this homotopy and the paramater along the cone, we have an equivalence which agrees with e at the X end of the cone, and c_n at the vertex. Thus $\{E,e\}$ extends to $X\cup TA$.

For our next theorem, we need to know one additional fact about the homotopy commutativity of the unitary groups in one another. We can define two maps of $U(n)\times U(n)$ into $U(2n)$ by sending (a,b) into either $a\oplus b$ or $(ab)\oplus I_n$, where I_n is the $n\times n$ unit matrix. Notice that

$$\begin{pmatrix} ab & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

$U(2n)$ is arcwise connected, so any matrix is homotopic to any other. Thus, the multiplication given by $(ab)\oplus I_n$ is homotopic to that given by sending (a,b) to

$$\begin{pmatrix} a & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a\oplus b.$$

Thus we can represent sums in $\widetilde{KU}^{-1}(X) = [X,U]$ either by direct sums or by matrix products, and negatives by inverses.

<u>Theroem 1.2</u> For any finite CW-complex X, the sequence

$$\widetilde{KU}^{-1}(X) \xrightarrow{\psi} \widetilde{KU}^{-1}(X) \xrightarrow{\gamma} \widetilde{KC}^{0}(X)$$

is exact. Thus for any finite CW-pair we have a long exact sequence

$$\cdots \xrightarrow{\psi} KU^{-n-1}(X,A) \xrightarrow{\gamma} KC^{-n}(X,A) \xrightarrow{\zeta} KU^{-n}(X,A) \xrightarrow{\psi} \cdots \xrightarrow{\psi} KU^{0}(X,A).$$

Thus the groups $KC^{-n}(\text{point})$ are given by:

$$KC^{-4n}(\text{point}) = Z \qquad KC^{-4n-2}(\text{point}) = 0$$

$$KC^{-4n-1}(\text{point}) = Z_2 \qquad KC^{-4n-3}(\text{point}) = Z$$

and $\zeta: KC^{-4n}(\text{point}) \longrightarrow KU^{-4n}(\text{point})$ is an isomorphism all $n \geq 0$.

proof: Suppose that $f:X \longrightarrow U(n)$. The condition that $\gamma(f) = 0$ is that there exist a map $g:X \longrightarrow U(n)$ (increasing n if necessary) such that $c_n a(g)$ and $\overline{a(g)}c_n a(f)$ are homotopic as bundle equivalences. However, as may easily be seen by looking at complex vectors, $\overline{a(g)} = c_n a(\overline{g})\overline{c}_n$, so this is the same as requiring that $c_n a(g)$ and $c_n a(\overline{g})a(f) = c_n a(\overline{g}f)$ be homotopic as bundle equivalences. This is the same, however, as the condition that g be homotopic to $\overline{g}$ f, or, equivalently, that f be homotopic to $g(\overline{g}^{-1}) = \psi(g)$.

The second part of the argument follows from the first part, together with lemma 1.1 and the fact that $\psi\pi_U = \pi_U\varepsilon_U\varepsilon_O$, $\psi(\pi_U)^2 = (\pi_U)^2\psi$.

Cup Products in KC

We have just seen that the groups KC^{-n}(point) are periodic with period four. This will continue to be true if a point is replaced by any finite CW-complex. To show this, we must introduce cup products into our theory. We defint the products in $\widetilde{K}C(X)$ for any X, and then by the usual constructions (see, for example, [3] and [5]) we obtain cup products taking $KC^{-p}(X,A)\otimes KC^{-q}(X,B)$ into $KC^{-p-q}(X,A\cup B)$ for every CW-triad $(X;A,B)$.

If X is a space with a basepoint x, then $\widetilde{K}C^0(X)$ is by definition $KC^0(X,x) = \widetilde{K}C(X)$. However, whilethe group structure of $\widetilde{K}C^0(X)$ does not depend on the choice of basepoint, the ring structure does depend on the basepoint. If E is a bundle over X, we define dim(E) to be the dimension of the fiber of E over the basepoint x.

<u>Definition 1.5</u> If (E,e) and (E',e') are C-pairs on a space X which has a basepoint, then we define:

$$\{E,e\} \smile \{E',e'\} = \{E \otimes E', e \otimes e'\} - \dim(E)\{E',e'\} - \dim(E')\{E,e\}.$$

It is clear that this definition is independent of the choice of representatives for the stable classes.

<u>Theorem 1.3</u> Let X be a finite CW-complex with subcomplexes A, B, C, and D, and let $a \in KC^{-p}(X,A)$, $b \in KC^{-q}(X,B)$, $c \in KU^{-r}(X,C)$, and $d \in KU^{-s}(X,D)$. Then

$$\begin{aligned}
\zeta(a \smile b) &= \zeta(a) \smile \zeta(b)\\
\psi(\zeta(a) \smile c) &= \zeta(a) \smile \psi(c)\\
\gamma(\zeta(a) \smile c) &= a \smile \gamma(c)\\
\gamma(c) \smile \gamma(d) &= 0
\end{aligned}$$

Therefore, there exists a cohomology operation $\pi_C : KC^{-n}(X,A) \longrightarrow KC^{-n-4}(X,A)$ which is an isomorphism for all n and (X,A), given by $\pi_C(a) = a \smile \zeta^{-1}(\pi_U{}^2(1))$, where 1 denotes the unit of $KU^0(\text{point})$. If the isomorphism π_C is used to define the positive dimensional groups $KC^n(X,A)$, KC^* becomes a cohomology theory satisfying all the axioms of Eilenberg-Steenrod except the dimension axiom, and the exact sequence of Theroem 1.2 extends to all dimensions, giving an exact triangle for all (X,A):

$$\begin{tikzcd}
KU^*(X,A) \arrow[rr, "\psi"] & & KU^*(X,A) \arrow[dl, "\gamma"] \\
& KC^*(X,A) \arrow[ul, "\zeta"] &
\end{tikzcd}$$

proof: The first two equalities follow from the fact that the complex conjugation is a ring automorphism on KU^*. The fourth identity follows from the third by noticing that $\zeta\gamma = 0$, so $\gamma(c) \smile \gamma(d) = \gamma(\zeta\gamma(c) \smile d) = \gamma(0) = 0$. The rest follows from these three initial equalities and Theorem 1.2, together with the five lemma and the periodicity of KU^* [3]. Therefore, the only part which needs proof is the equality $\gamma(\zeta(a) \smile c) = a \smile \gamma(c)$.

By the usual sort of argument with smash products, suspensions, and diagonal maps, it suffices to consider the case when A and C are points, $a \in \widetilde{KC}^0(X)$, $c \in \widetilde{KU}^{-1}(X)$, where X is a finite CW-complex with basepoint. The operation γ has been defined in terms of the identification of $\widetilde{KU}^{-1}(X)$ with $[X,U]$, and cup products are defined in terms of the identification of $\widetilde{KU}^{-1}(X) = KU^{-1}(X)$ with $\widetilde{KU}(X \times S^1, X)$, where X is included in $X \times S^1$ in the usual fashion. The first thing which we must do is to relate these two descriptions so that we may relate γ to cup products. To simplifly this somewhat, notice that the projection of $X \times S^1$ onto X splits the exact sequence for KU^* of the pair $(X \times S^1, X)$, so that we may identify $\widetilde{KU}(X \times S^1, X)$ with kernel$(\widetilde{KU}(X \times S^1) \longrightarrow \widetilde{KU}(X))$, i. e. with the stable equivalence classes of bundles on $X \times S^1$ which are trivial along X.

The identification of these two descriptions will be given by a twisting function t which assigns to every bundle E over X and automorphism f of E a bundle t(E,f) over $X \times S^1$. Explicitly, this is done by taking the projection of $X \times [0,1]$ onto X, and taking the induced bundle $\pi_X{}^*(E)$ over $X \times [0,1]$. The restriction of this bundle along $X \times [0]$ is identified with the restriction to $X \times [1]$ by means of f. The resulting bundle on $X \times S^1$ is denoted by $t(E,f)$, and is called E twisted by f. If E is a complex bundle, and f is an automorphism of complex bundles, $t(E,f)$ is clearly a complex bundle. The twisting function t clearly satisfies the following properties:

i) $t(E \oplus E', f \oplus f') = t(E,f) \oplus t(E',f')$

ii) $t(E \otimes G, f \otimes 1) = t(E,f) \otimes \pi_X{}^*(G)$

iii) if f_0 and f_1 are homotopic as bundle automorphisms, then $t(E,f_0)$ is equivalent to $t(E,f_1)$

We can now identify $[X,U]$ with kernel$(\widetilde{K}U(X \times S^1) \longrightarrow \widetilde{K}U(X))$ as follows: if $f:X \longrightarrow U(n)$, and $a(f)$ is the automorphism of C_n which is induced by f, then f corresponds to $\{t(C_n, a(f))\}$. It is an elementary matter to see that this correspondence is well defined and is bijective. It is not hard to show that this is the usual identification.

To make our argument proceed more smoothly, we would like an explicit formula for $\gamma(\{t(E,f)\} - \{t(E,1)\})$ in the case when E is a self-conjugate but not necessarily trivial bundle. Since E is self-conjugate, there exists a self-conjugacy e of E. Since $\widetilde{K}C(X)$ is a group, there exists a C-pair (E',e') and an equivalence for some m of the form $h:E \oplus E' \longrightarrow C_m$ such that $c_m h$ and $\overline{h}(e \oplus e')$ are homotopic as bundle equivalences. $\gamma(\{t(E,f)\} - \{t(E,1)\}) = \gamma(\{t(E \oplus E', f \oplus 1)\}) = \gamma(\{t(C_m, h(f \oplus 1)h^{-1})\}) = \{C_m, c_m h(f \oplus 1)h^{-1}\}$.

This, however, is the same as $\{E\oplus E', \bar{h}^{-1}c_m h(f\oplus 1)h^{-1}h\} = \{E\oplus E', \bar{h}^{-1}c_m h(f\oplus 1)\}$. Since $c_m h$ and $\bar{h}(e\oplus e')$ are homotopic, this is the same as $\{E\oplus E', (e\oplus e')(f\oplus 1)\} = \{E\oplus E', (ef)\oplus(e')\}$. Therefore. $\gamma(\{t(E,f)\} - \{t(E,1)\}) = \{E,ef\} - \{E,e\}$.

We are now in a position to finish the proof of our theorem. Let a be $\{E,e\}$, and let c be $\{t(C_n,f)\}$. Then we have
$\zeta(a)\smile c = \{\pi_X{}^*(E)\otimes t(C_n,f)\} - n\{\pi_X{}^*(E)\} - \dim(E)\{t(C_n,f)\}$. Now $n\{\pi_X{}^*(E)\} = \{\pi_X{}^*(E)\otimes t(C_n,1)\}$. By properties i and ii of the twisting operation, we see that
$\zeta(a)\smile c = \{t(E\otimes C_n, 1\otimes f)\} - \{t(E\otimes C_n, 1\otimes 1)\} - \dim(E)\{t(C_n,f)\}$.
Applying γ, we see from our formula above that:
$\gamma(\zeta(a)\smile c) = \{E\otimes C_n, e\otimes(c_n f)\} - \{E\otimes C_n, e\otimes c_n\} - \dim(E)\gamma(c)$.
However, this is just $a\smile\gamma(c)$.

Cohomology Operations Relating K-theories

There are a number of cohomology operations relating the theories KO*, KSp*, KU*, and KC*. In order that we may fix notation, we will describe some of these operations in this section. Since there are so many operations to keep track of, some of our notation will do double duty, with subscripts describing the range of the operation, while the domain is understood. This should cause no serious difficulty, since the domain of the operations is usually clear from the context.

A concept which will be of use to us later is that of an operation which is a KO*-module map. For each of the theories

$K^* = KO^*$, KSp^*, KU^*, or KC^*, there is a well defined way in which $K^*(X,A)$ is a module over $KO^*(X)$ for all (X,A), since the tensor product of a real bundle with a symplectic bundle, a complex bundle, or a self-conjugate complex bundle is again a symplectic, complex, or self-conjugate bundle, respectively. If $K_1{}^*$ and $K_2{}^*$ are two of these theories, a cohomology operation α from $K_1{}^*$ to $K_2{}^*$ is said to be a KO^*-module map is for every (X,A) the homomorphism $\alpha: K_1{}^*(X,A) \longrightarrow K_2{}^*(X,A)$ is a $KO^*(X)$-module map. If $K_1{}^* = KO^*$, there is a one-one correspondence between KO^*-module maps and the elements of $K_2{}^*(\text{point})$ which is given by evaluating the operation on the unit $1 \in KO^0(\text{point})$; if $K_1{}^* = KSp^*$, there is a one-one correspondence between the KO^*-module maps to $K_2{}^*$ and the elements of $K_2{}^*(\text{point})$ given by evaluating the operation on the element $1_{Sp} \in KSp^{-4}(\text{point})$ which corresponds to 1 under the isomorphism $KO^n(X) = KSp^{n-4}(X)$ of the Bott periodicity.

<u>Definition 1.6</u> The two cohomology operations

$$\varepsilon_C: KO^n(X,A) \longrightarrow KC^n(X,A)$$

$$\varepsilon_C: KSp^n(X,A) \longrightarrow KC^n(X,A)$$

are the KO^*-module maps such that $\zeta\varepsilon_C = \varepsilon_U$, where

$$\varepsilon_U: KO^n(X,A) \longrightarrow KC^n(X,A)$$

is the complexification of real bundles described previously, and

$$\varepsilon_U: KSp^n(X,A) \longrightarrow KU^n(X,A)$$

is the operation which takes the first of the two quaternionic automorphisms of a symplectic bundle as the complex structure.

Remark: It is easy to see that the self-conjugacy part of $\varepsilon_C(E)$ is the canonical self-conjugacy described earlier.

Definition 1.7 The operations

$$\varepsilon_O: KU^n(X,A) \longrightarrow KO^n(X,A)$$
$$\varepsilon_O: KSp^n(X,A) \longrightarrow KO^n(X,A)$$

are both defined by taking the underlying real bundles of the complex or symplectic bundles involved.

Definition 1.8 The operations

$$\varepsilon_{Sp}: KO^n(X,A) \longrightarrow KSp^n(X,A)$$
$$\varepsilon_{Sp}: KU^n(X,A) \longrightarrow KSp^n(X,A)$$

are defined by:

a) if E is a real bundle, and Q_1 is the trivial one dimensional quaternionic bundle, $\varepsilon_{Sp}(E) = E \otimes Q_1$, where the quaternionic structure is that induced by Q_1.

b) if E is a complex bundle, $\varepsilon_{Sp}(E)$ has the same underlying real structure as $E \oplus \overline{E}$. If J defines the complex structure of E, $J_1(a,b) = (J(a),-J(b))$, $J_2(a,b) = (b,a)$, where (a,b) are elements of the underlying real bundle of $E \oplus E$, which is the same as that of $E \oplus \overline{E}$.

These operations enjoy certain well known properties, some of which we give here:

i) if E is real, $\quad \varepsilon_O \varepsilon_U(E) = E \oplus E$

$$\varepsilon_O \varepsilon_{Sp}(E) = E \oplus E \oplus E \oplus E$$

ii) if E is complex $\quad \varepsilon_U \varepsilon_O(E) = E \oplus \overline{E}$

$$\varepsilon_U \varepsilon_{Sp}(E) = E \oplus \overline{E}$$

In addition to these standard operations, we shall need another operation which is based on the idea of a twisted bundle, as introduced in the proof of Theorem 1.3. We repeat the definition here.

Definition 1.9 If F is a real bundle over a space X, and f is an automorphism of F, $t(F,f)$ is the bundle over $X \times S^1$ defined by taking the bundle $\pi_X^*(F)$ over $X \times [0,1]$ induced by the projection onto X, and identifying the restriction to $X \times [0]$ with the restriction to $X \times [1]$ by means of f. If (E,e) is a C-pair over X, we define $\tau(\{E,e\})$ to be $\{t(E,e)\} - \{t(E,1)\} \in \widetilde{KO}(X \times S^1)$, where e and the identity 1 are thought of as automorphisms of the real underlying bundle of E.

The operation τ defined above is a homomorphism from $\widetilde{KC}(X)$ to $\widetilde{KO}(X \times S^1)$. It is clear from the definition that for any C-pair (E,e) over X, $\tau(\{E,e\})$ is zero when restricted to either X or S^1. Therefore, the image of τ lies in kernel$(\widetilde{KO}(X \times S^1) \longrightarrow \widetilde{KO}(X \vee S^1))$, which is the same as $\widetilde{KO}(X \wedge S^1) = \widetilde{KO}^{-1}(X)$. By the usual sort of suspension argument, we obtain a cohomology operation $\tau: KC^n(X,A) \longrightarrow KO^{n-1}(X,A)$ for all (X,A) and all $n \leq 0$. Since τ is a KO^*-module map, and the periodicity in $KO^*(X,A)$ is defined by cup product with the generator of KO^{-8}(point), τ can be extended to positive dimensions. From the discussion of the twisting bundle in the proof of Theorem 2.3, it is immediate that $\tau\gamma = \varepsilon_O : KU^n(X,A) \longrightarrow KO^n(X,A)$. From this fact and a knowledge of ε_O and ε_U on the groups of a point, we can determine the action of τ and ε_C on the groups of a point.

Lemma 1.2 The homomorphism $\tau: KC^n(\text{point}) \longrightarrow KO^{n-1}(\text{point})$ is as follows:

$n \mod(8)$	1	0	-1	-2	-3	-4	-5	-6	-7
$KC^n(\text{point})$	Z	Z	Z_2	0	Z	Z	Z_2	0	Z
$\downarrow \tau$	$\times 2$	onto	$\cong$	0	$\cong$	0	0	0	$\times 2$
$KO^{n-1}(\text{point})$	Z	Z_2	Z_2	0	Z	0	0	0	Z

The homomorphism $\varepsilon_C: KO^n(\text{point}) \longrightarrow KC^n(\text{point})$ is as follows:

$n \mod(8)$	0	-1	-2	-3	-4	-5	-6	-7	-8
$KO^n(\text{point})$	Z	Z_2	Z_2	0	Z	0	0	0	Z
$\downarrow \varepsilon_C$	$\cong$	$\cong$	0	0	$\times 2$	0	0	0	$\cong$
$KC^n(\text{point})$	Z	Z_2	0	Z	Z	Z_2	0	Z	Z

proof: The action of τ is immediate from the relation $\tau\gamma = \varepsilon_0$, and the usual information about ε_0, except for the case $n = 0$. This, however, follows from the fact that if we take the trivial line bundle C_1 over a point, $t(C_1, c_1)$ is the Möbius band plus one trivial bundle.

The action of ε_C can be deduced from the fact that $\zeta\varepsilon_C = \varepsilon_U$, except in the case when $n = -1$. Let t be the generator of $KO^{-1}(\text{point})$. Then it suffices to show that $\varepsilon_C(t) \neq 0$. Both τ and ε_C are KO^*-module maps, so $\tau\varepsilon_C(t) = t \smile \tau\varepsilon_C(1) = t \smile t \neq 0$. ($t$, of course, is the "desuspension" of the stable class of the Möbius bundle on S^1.)

"Grassman Manifolds" for KC^*

Since KC^* is a cohomology theory defined for all finite CW-complexes, and since $KC^*(\text{point})$ is a countable group, by a result of E. Brown [4] there exists an Ω-spectrum $\underline{C} = \{\underline{C}^{(n)}\}$ and a natural equivalence $KC^n(X) = [X, \underline{C}^{(n)}]$. Similarly, the theory KU^* is representable by an Ω-spectrum $\underline{U}$, and the cohomology operations γ, ζ, and ψ can be represented by spectrum maps.

The classifying space B_U of the infinite unitary group is the connected component of the identity of the 0-dimensional element of the spectrum $\underline{U}$. The space B_U can be represented as the direct limit of Grassman manifolds $G_{m,n} = U(m+n)/U(m) \times U(n)$. Let L^m and R^n denote the m-plane and n-plane bundles associated to the two obvious $U(m)$ and $U(n)$ bundles over $G_{m,n}$. The manifold $G_{m,n}$ is a classifying space for KU^0 for spaces of dimension less than twice the minimum of m and n; the bundle L^m defines a map $\lambda^m : G_{m,n} \longrightarrow B_U$ which induces an isomorphism on homotopy groups up to dimension equal to twice the minimum of m and n. We can likewise produce "Grassman manifolds" for KC^0.

Let $C_{m,n}$ be the total space of the principal $U(m+n)$ bundle over $G_{m,n}$ whose associated $(m+n)$-plane bundle is $L^m \oplus \overline{R^n}$. $C_{m,n}$ is a principal $U(m+n)$ bundle with base space $G_{m,n}$, and is a differentiable fiber bundle. The space $C_{m,n}$ has defined on it a canonical C-pair which takes the place of the classifying bundle L^m.

If $\pi: C_{m,n} \longrightarrow G_{m,n}$ is the projection, the bundle $\pi^*(L^m \oplus \overline{R^n})$ is trivial, since the principal bundle to which it is associated has a section - the "diagonal" section. Dropping superscripts, we have an equivalence $\pi^*(L \oplus \overline{R}) \cong \pi^*(L \oplus R)$. These bundles are trivial, so we may take conjugates on either side. Thus, we see that $\pi^*(\overline{L} \oplus R) \cong \pi^*(L \oplus R)$, so we have an equivalence

$$\overline{\pi^*(L \oplus R \oplus L)} \cong \pi^*(\overline{L} \oplus \overline{R \oplus L}) \cong \pi^*(\overline{L} \oplus R \oplus L) \cong \pi^*(L \oplus R \oplus L).$$

Taking the bundle $\pi^*(L \oplus R \oplus L)$ together with this self-conjugacy, we obtain a C-pair over $C_{m,n}$, which defines a map of $C_{m,n}$ into the classifying space for $\widetilde{K}C^0(C_{m,n})$. The proof that this map induces an isomorphism on homotopy groups up to dimension equal to twice the minimum of m and n is left to the reader. It proceeds by comparing the fibration of $C_{m,n}$ over $G_{m,n}$ with the fibration associated to the cohomology operation ζ, and applying the fiber exact sequence for homotopy groups and the five lemma.

§2 Wood's Description of B_U and Bott's Exact Sequence

In his proof of the periodicity of the homotopy groups of the infinite orthogonal group O, Bott established a number of homotopy equivalences between the higher loop spaces of the infinite orthogonal group (or, rather, its classifying space), and certain well known homogeneous spaces. The inclusions of O(n) in U(n), U(n) in Sp(n), Sp(n) in U(2n), and U(n) in O(2n) give rise to inclusions of the corresponding infinite groups O in U, U in Sp, Sp in U, and U in O. Likewise, these inclusions define homogeneous U/O, Sp/U, U/Sp, and O/U. Bott's description of the higher loop spaces of B_O is as follows:

$$\begin{array}{ll} \Omega^1 B_O = O & \Omega^2 B_O = O/U \\ \Omega^3 B_O = U/O & \Omega^4 B_O = Z \times B_{Sp} \\ \Omega^5 B_O = Sp & \Omega^6 B_O = Sp/U \\ \Omega^7 B_O = U/Sp & \Omega^8 B_O = Z \times B_O \end{array}$$

This last equality, of course, expresses the periodicity of the groups $KO^n(X)$ for any finite complex X. In the usual fashion, there is a sequence of fibrations $U \longrightarrow O \longrightarrow O/U \longrightarrow B_U \longrightarrow B_O$. The above identifications give an exact sequence for any X

$$KU^{-1}(X) \longrightarrow KO^{-1}(X) \longrightarrow KO^{-2}(X) \longrightarrow KU^0(X) \longrightarrow KO^0(X).$$

This exact sequence has been used to compute $KO^*(X)$ from $KU^*(X)$ and the usual spectral sequence. By replacing X by its various suspensions, we get a sequence which extends to arbitrary negative dimensional K-groups, and this sequence is compatible with the periodicity, and so extends to all dimensions. This compatibility is not elementary. It is possible to identify all the cohomology operations which occur in this sequence, though this is not easy, either.

Recently, in some unpublished work, R. Wood has established a homotopy equivalence between the classifying space for the infinite unitary group and the component of the basepoint of the space of maps of the complex projective plane into the classifying space of the infinite orthogonal group. This equivalence is most easily expressed in terms of the K-theories, and this is the form which we shall consider. It will turn out that the classifying space for KC^0 bears a similar relationship to B_O, and from this we shall be able to prove the existence of many exact sequences relating the four forms of K-theory. We reproduce Wood's proof in its entirety. From his result, we shall be able to easily establish the existence of an exact sequence which looks very much like Bott's. It can actually be proven to be the same as Bott's, but we shall not need this fact.

We denote by $\eta: S^3 \longrightarrow S^2$ the Hopf map. The complex projective plane is denoted by P, and has cell structure $P = S^2 \cup_\eta e^4$. The inclusion map $i: S^2 \longrightarrow P$ induces in the usual fashion a sequence $S^2 \xrightarrow{i} P \xrightarrow{j} S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\Sigma i} \Sigma P \xrightarrow{\Sigma j} \ldots$ of cofibrations, where Σ denotes suspension. We shall refer to this as the cofibration sequence of P.

We shall not want to have to work with reduced cohomology groups, so we must describe the suspension in terms of relative groups. If X is a space, X^+ denotes the topological sum of X and a disjoint base point. The smash product $(X^+) \wedge Y$ is equal to the identification space $(X \times Y)/X$. Therefore, we see that the suspension isomorphism for absolute groups is of the form

$$\sigma^k : KO^n(X) \cong KO^{n+k}(X \times S^k, X).$$

Further, if Y is a finite CW-complex, we can define a new cohomology theory by setting $K\underline{Y}^n(X) = KO^n(X \times Y, X)$. It is clear that the classifying space for KY^* in any given dimension is the space of maps of Y into the appropriate higher loop space of B_O. Wood's theorem gives us an identification of $K\underline{P}^n(X)$ with $KU^{n-4}(X)$.

Definition 2.1 Let H be the (complex) Hopf bundle over P, and let $h = H - 1$. Define $W : KU^n(X) \longrightarrow K\underline{P}^{n+4}(X) = KO^{n+4}(X \times P, X)$ by $W(x) = \varepsilon_O(\pi_U^{-2}(x) \otimes h)$.

Theorem 2.1 (Wood) W is an isomorphism for all X.

Corollary 2.1 For any space X, we have an exact sequence

$$\cdots \xrightarrow{\varepsilon_U} KU^n(X) \xrightarrow{\varepsilon_O \pi_U^{-1}} KO^{n+2}(X) \xrightarrow{\tau\varepsilon_C} KO^{n+1}(X) \xrightarrow{\varepsilon_U} KU^{n+1}(X) \longrightarrow \cdots$$

proof: For the proof of the theorem, we need only know that W induces an isomorphism when we take X to be a point, since W is a transformation of cohomology theories [5]. The corollary will then follow by simple computations from the cofibration sequence of P. We take up the theorem first.

$\widetilde{KU}^{-2n}(P)$ is free on the two generators $\pi_U^n(h)$ and $\pi_U^n(h^2)$. Since $H \otimes \overline{H}$ is trivial, $(h+1)(\overline{h}+1) = 1$, so $\overline{h} = (1+h)^{-1} - 1$. Since $h^3 = 0$, $\overline{h} = -h + h^2$. Thus $\widetilde{KU}^{-2n}(P)$ is free on two generators, $\pi_U^n(h)$ and its complex conjugate. Therefore, the image of the map

$\varepsilon_O:\widetilde{KU}^{-2n}(P) \longrightarrow \widetilde{KO}^{-2n}(P)$ is a torsion free group on the one generator $\varepsilon_O(\pi_U{}^n(h))$ (here we use the fact that $\varepsilon_U\varepsilon_O(x) = x + \bar{x}$). Since $\widetilde{KU}^{-2n-1}(P) = 0$, W will be an isomorphism iff $\varepsilon_O:\widetilde{KU}^*(P) \longrightarrow \widetilde{KO}^*(P)$ is surjective, for if 1 denotes the generator of KU^0(point), $W(1) = \varepsilon_O(h)$, and more generally, $W(\pi_U{}^n(1)) = \varepsilon_O(\pi_U{}^n(1) \otimes h) = \varepsilon_O(\pi_U{}^n(h))$.

The spectral sequence of [3] can be generalized to any cohomology theory, and is natural with respect to stable cohomology operations between the two theories involved. If k^* is a cohomology theory, for a space X, the E_2 term is given by $E_2{}^{p,q} = H^p(X;k^q(\text{point}))$, where H^p denotes the ordinary (singular) cohomology group of dimension p. If h^* is another such theory, and $\alpha:h^*(X) \longrightarrow k^*(X)$ is a stable cohomology operation, α induces $\alpha_r:{}'E_r{}^{p,q} \longrightarrow {}''E_r{}^{p,q}$ such that $\alpha_2:H^p(X;h^q(\text{point})) \longrightarrow H^p(X;k^q(\text{point}))$ is induced by the coefficient homomorphism $\alpha:h^q(\text{point}) \longrightarrow k^q(\text{point})$ (${}'E_r$ refers to the spectral sequence associated to $h^*(X)$, ${}''E_r$ to that associated to $k^*(X)$). We now take X to be P, h^* to be KU^*, k^* to be KO^*, and α to be ε_O. The spectral sequence ${}'E_r$ collapses, i.e. ${}'E_2 = {}'E_\infty$, since there is no torsion [3]. There is one non-zero differential in the sequence ${}''E_r$. According to Thomas [8], $d_2 = Sq^2$ or $d_2 = Sq^2\rho_2$, where ρ_2 is reduction of integral classes mod 2. These descriptions of d_2 are valid, of course, only where they make sense; the other actions of d_2 must be zero since either the image or range group vanishes. The action of $\varepsilon_O:{}'E_3 \longrightarrow {}''E_3$ in those cases when not both of the groups in question are zero is as follows:

q mod(8)	0	-1	-2	-3	-4	-5	-6	-7
${}'E_3^{2,q}$	Z	0	Z	0	Z	0	Z	0
$\downarrow \varepsilon_0$	onto	0	onto	0	onto	0	0	0
${}''E_3^{2,q}$	Z	0	Z_2	0	Z	0	0	0
${}'E_3^{4,q}$	Z	0	Z	0	Z	0	Z	0
$\downarrow \varepsilon_0$	x2	0	0	0	onto	0	0	0
${}''E_3^{4,q}$	Z	0	0	0	Z	0	0	0

Since ${}'E_3 = {}'E_\infty$ and ${}''E_3 = {}''E_\infty$, from the surjection part of the five lemma, we see that $\varepsilon_0 : \widetilde{KU}^n(P) \longrightarrow \widetilde{KO}^n(P)$ is surjective if n is not congruent to 4 mod(8). It is easy to see that ${}'E_\infty^{4,-4}$ has its generator represented by $\pi_U^2(h^2) = \pi_U^2(h + \bar{h})$. From the spectral sequence above, $\varepsilon_0(\pi_U^2(h^2))$ generates a subgroup of $\widetilde{KO}^{-4}(P)$ of index 2. However, $\varepsilon_0(\pi_U^2(h^2)) = \varepsilon_0(\pi_U^2(h+\bar{h})) = 2\varepsilon_0(\pi_U^2(h))$. Thus, $\varepsilon_0(\pi_U^2(h))$ generates $\widetilde{KO}^{-4}(P)$. Thus $\varepsilon_0 : \widetilde{KU}^*(P) \longrightarrow \widetilde{KO}^*(P)$ is a surjection, so W is an isomorphism on the groups of a point, and so our theorem is proved.

To prove the corollary, the cofibration sequence of P gives us an exact sequence of the form

$$\cdots \xrightarrow{(1\times\pi)^*} KO^{n+4}(X\times S^4, X) \xrightarrow{(1\times j)^*} KO^{n+4}(X\times P, X) \xrightarrow{(1\times i)^*} KO^{n+4}(X\times S^2, X) \longrightarrow \cdots$$

Combining this sequence with suspension isomorphisms and the isomorphism W, we obtain a sequence in which the groups are the same as in the corollary. Further, it is clear that all the maps in the sequence are KO^*-module maps, since W is a composite of KO^*-module maps, and suspensions are KO^*-module maps, as are maps induced by topological maps. Taking X to be a point, we have an exact sequence $KO^0(\text{point}) \longrightarrow KO^{-1}(\text{point}) \longrightarrow KU^{-1}(\text{point}) = 0$. Thus, the map from $KO^n(X)$ to $KO^{n-1}(X)$ agrees with $\tau\varepsilon_C$ on $KO^0(\text{point})$, and since they are both KO^*-module maps, they are therefore the same. Similarly, the map $KO^*(X) \longrightarrow KU^*(X)$ is $\pm\varepsilon_U$. Taking the plus sign does not affect the exactness of the sequence, in either case.

The map $KU^n(X) \longrightarrow KO^{n+2}(X)$ is equal to $(\sigma^2)^{-1}(1\times i)^*W$. Now $(\sigma^2)^{-1}(1\times i)^*W(x) = (\sigma^2)^{-1}(1\times i)^*\varepsilon_O(\pi_U^{-2}(x)\otimes h) = \varepsilon_O(\sigma^2)^{-1}(\pi_U^{-2}(x)\otimes i^*(h))$. However, $i^*(h)$ is the usual generator of $\widetilde{K}U^0(S^2)$, and π_U is defined by $\pi_U(y) = (\sigma^2)^{-1}(y\otimes i^*(h))$, so we see that $(\sigma^2)^{-1}(1\times i)^*W = \varepsilon_O(\pi_U{}^{-1})$. This completes the proof of the corollary.

§3 The KC^* Analogue of Wood's Theorem and Applications.

KC^* is related to KO^* in much the way that KU^* is, and this relationship will allow use to derive exact sequences relating the various K-theories. In the next section we shall show the usefulness of one of these sequence in the problem of determining KO^* of the classifying space for a compact Lie group.

The cofibration sequence of a map $a:A \longrightarrow B$ is the sequence $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma A \xrightarrow{\Sigma a} \Sigma B \xrightarrow{\Sigma b} \Sigma C \xrightarrow{\Sigma c} \ldots$, where C is the mapping cone on B of a. Any composite of two successive maps in this sequence is a cofibration. This sequence is often called the "Puppe Sequence". We need one simple fact about this construction.

<u>Lemma 3.1</u> If
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "i'"] & B' \end{tikzcd}$$

is a commutative diagram of maps of finite CW-complexes, and if C is the mapping cone of i, C' is the mapping cone of i', A" is the mapping cone of a, B" is the mapping cone of b, with induced maps $c:C \longrightarrow C'$, $i'':A'' \longrightarrow B''$, then the mapping cones of c and of i" both have the homotopy type of the same space C". Further, the identifications of these two spaces with C" can be done in such a fashion that there is a homotopy commutative diagram of spaces and maps with rows and columns equivalent to the cofibration sequences of their first maps:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C \\ a\downarrow & & b\downarrow & & c\downarrow \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \\ a'\downarrow & & b'\downarrow & & c'\downarrow \\ A'' & \xrightarrow{i''} & B'' & \xrightarrow{j''} & C'' \end{array}$$

proof: The mapping cone of i" is equal to $B''\cup_{i''}TA''$, which is equal to $(B'\cup_b TB)\cup_{i'}T(A'\cup_a TA)$. The mapping cone of c is equal to $C'\cup_c C$, which is equal to $(B'\cup_{i'}TA')\cup_c T(B\cup_i TA)$. Since all of these spaces are finite CW-complexes, there is a natural identification of these two spaces with $B'\cup_b TB\cup_{i'}TA'\cup_{(i\times a)}T^2A$.

The composition square $\eta^2:S^4 \longrightarrow S^2$ is, by definition, the composition of the Hopf map η with its suspension. The mapping cone of η^2 is denoted by P'. P' will play a role for KC* similar to that which P plays for KU*. If we construct the commutative square

$$\begin{array}{ccc} S^4 & \xrightarrow{\Sigma\eta} & S^3 \\ \downarrow \text{identity} & & \downarrow \eta \\ S^4 & \xrightarrow{\eta^2} & S^2, \end{array}$$

the mapping cone of the identity map on S^4 is contractable. Therefore, from the last lemma, we see that there exists a map $u:\Sigma P \longrightarrow P'$ whose mapping cone has the homotopy type of P, and such that there is a commuting (up to homotopy) diagram of spaces and maps with rows the cofibration sequences of their first maps:

$$\begin{array}{ccccccccccc} S^3 & \xrightarrow{\eta} & S^2 & \xrightarrow{i} & P & \xrightarrow{j} & S^4 & \xrightarrow{\Sigma\eta} & S^3 & \xrightarrow{\Sigma i} & \cdots \\ \downarrow\Sigma i & & \downarrow a & & \downarrow & & \downarrow\Sigma^2 i & & \downarrow\Sigma a & & \\ \Sigma P & \xrightarrow{u} & P' & \xrightarrow{v} & P & \xrightarrow{w} & \Sigma^2 P & \xrightarrow{\Sigma u} & \Sigma P' & \xrightarrow{\Sigma v} & \cdots \end{array}$$

where the vertical map from P to itself is the identity, and i, j, and a are as defined earlier.

We now need an element in $\widetilde{K}C^2(P')$ which will replace the reduced Hopf bundle $h \in \widetilde{K}U^0(P)$. The following lemma describes an element with exactly those two properties which we shall need.

<u>Lemma 3.2</u> There exists an element $k \in \widetilde{K}C^2(P')$ such that:

$$\text{i)} \quad u^*(k) = \pm\sigma\gamma(h)$$

$$\text{ii)} \quad \zeta(k) = v^*(\pi_U^{-1}(h))$$

argument needs slight change $\exists k \ni u^*(k) = \pm\sigma\gamma(h)$ etc

proof: Since $\widetilde{K}C^3(S^5) = KC^2(\text{point}) = 0$, from the exact sequence for the pair (P', S^2), there exists an element $k \in \widetilde{K}C^2(P')$ such that $a^*(k) = \sigma^2(1) \in \widetilde{K}C^2(S^2)$, where $1 \in KC^0(\text{point})$ is the unit. We first show that $\zeta(k) = v^*(\pi_U^{-1}(h))$. Now $\widetilde{K}U^2(S^5) = 0$, so from the exact sequence $\widetilde{K}U^2(S^5) \xrightarrow{b^*} \widetilde{K}U^2(P') \xrightarrow{a^*} \widetilde{K}U^2(S^2)$, it suffices to show that $a^*\zeta(k) = a^*v^*(\pi_U^{-1}(h))$. However, from the diagram which preceeds this theorem, we see that $a^*v^* = i^*$. Thus we need only show that $\zeta(a^*(k)) = \pi_U^{-1}(i^*(h))$. From the definition of the periodicity of KU^*, $\pi_U^{-1}(i^*(h)) = \sigma^2(1)$, where $1 \in KU^0(\text{point})$ is the unit. Since ζ commutes with suspensions, and since ζ maps the unit in $KC^0(\text{point})$ to the unit in $KU^0(\text{point})$, we see that these two expressions are equal, so $\zeta(k) = v^*(\pi_U^{-1}(h))$.

The element $\sigma\gamma(h)$ lies in $\widetilde{K}C^2(\Sigma P) \cong \widetilde{K}C^1(P)$. Since $\widetilde{K}U^1(P) = 0$, we know that $\widetilde{K}C^1(P) \cong \widetilde{K}U^0(P)/\psi(\widetilde{K}U^0(P))$. Since we may take as generators for $\widetilde{K}U^0(P)$ the elements h and $h - \overline{h}$, and $\psi(h) = h - \overline{h}$, and $\psi(h - \overline{h}) = 2(h - \overline{h})$, we see that $\widetilde{K}C^1(P)$ is a free abelian group on the generator $\gamma(h)$. In view of the exact sequence $\widetilde{K}C^2(S^5) \xrightarrow{(\Sigma j)^*} \widetilde{K}C^2(\Sigma P) \xrightarrow{(\Sigma i)^*} \widetilde{K}C^2(S^3) \xrightarrow{T^*} \widetilde{K}C^2(S^4) = 0$, and the fact that $\widetilde{K}C^2(S^2) = Z_2$, to show that $u^*(k) = \pm\sigma\gamma(h)$, it suffices to show that $(\Sigma i)^* u^*(k) \neq 0$. Since we showed above that $(\Sigma i)^* u^* = \Pi^* a^*$, it suffices to show that $\Pi^* a^*(k) \neq 0$. We have a commuting diagram

$$\begin{tikzcd} Z = \widetilde{KO}^2(S^2) \arrow[r, "\Pi^*"] \arrow[d, "\varepsilon_C"] & \widetilde{KO}^2(S^3) = Z_2 \arrow[d, "\varepsilon_C"] \\ \widetilde{KC}^2(S^2) \arrow[r, "\Pi^*"] & \widetilde{KC}^2(S^3) \end{tikzcd}$$

From the results of Lemma 1.1, and the fact that ε_C commutes with suspensions, the two vertical maps are isomorphisms. In the proof of Wood's theorem, we saw that the upper map was the usual surjection of Z onto Z_2. By the definition of k, $a^*(k)$ is the generator of $\widetilde{KC}^2(S^2)$, so we see that $\Pi^* a^*(k) \neq 0$. This finishes the proof of the lemma.

<u>Definition 3.1</u> $W': KC^n(X) \longrightarrow KO^{n+5}(X \times P'.X)$ is defined by

$$W'(x) = \tau(\pi_C^{-1}(x) \otimes k)$$

<u>Theorem 3.1</u> For all X, W' is an isomorphism.

<u>Corollary 3.1</u> For all X, the sequence

$$\cdots \xrightarrow{\varepsilon_C} KC^n(X) \xrightarrow{\tau\pi_C^{-1}} KO^{n+3}(X) \xrightarrow{(\tau\varepsilon_C)^2} KO^{n+1}(X) \xrightarrow{\varepsilon_C} KC^{n+1}(X) \longrightarrow \cdots$$

is exact.

This theorem and its corollary are the KC^* analogues of Wood's theorem and the exact sequence of Bott.

proof of the theorem: From the results of theorems 1.3 and 2.1, it will suffice to show that the following diagram is commutative up to signs:

$$\begin{array}{ccccccc}
KU^n(X) & \xrightarrow{\psi} & KU^n(X) & \xrightarrow{\nu} & KC^{n+1}(X) & \xrightarrow{\zeta} & KU^{n+1}(X) \\
\downarrow W & & \downarrow W(\pi_U^{-1}) & & \downarrow W' & & \downarrow W \\
KO^{n+4}(X\times P,X) & \xrightarrow{\sigma^2(1\times w)^*} & KO^{n+6}(X\times P,X) & \xrightarrow{\nu^*} & KO^{n+6}(X\times P',X) & \xrightarrow{\sigma^{-1}(1\times u)^*} & KO^{n+5}(X\times P,X)
\end{array}$$

Since $w = (\Sigma^2 i)j$, $\sigma^2(1\times w)^*W = \sigma^2 W(W^{-1})(1\times j)^*(1\times\Sigma^2 i)^*W$. From the proof of Wood's theorem (or, more precisely, the proof of its corollary) we know that $(1\times\Sigma^2 i)^*W = \sigma^2\varepsilon_O(\pi_U^{-1})$, and that $(W^{-1})(1\times j)^* = \sigma^{-4}\varepsilon_U$. Thus $\sigma^2(1\times w)^*W = W\sigma^2\varepsilon_U\sigma^{-2}\varepsilon_O(\pi_U^{-1}) = W\varepsilon_U\varepsilon_O(\pi_U^{-1})$. Since the complex conjugation anticommutes with π_U^{-1}, $\varepsilon_U\varepsilon_O(\pi_U^{-1}) = \pi_U^{-1}\psi$. Thus we see that the first square commutes.

If $x \in KU^n(X)$, $(1\times v)^*W(\pi_U^{-1}(x)) = (1\times v)^*\varepsilon_O(\pi_U^{-2}(x)\otimes\pi_U^{-1}(h)) = \varepsilon_O(\pi_U^{-2}(x)\otimes v^*(\pi_U^{-1}(h)) = \varepsilon_O(\pi_U^{-2}(x)\otimes\zeta(k)) = \tau\gamma(\pi_U^{-2}(x)\otimes\zeta(k)) = \tau(\pi_C^{-1}(\gamma(x))\otimes k) = W'\gamma(x)$. Therefore the second square is commutative.

If $x \in KC^{n+1}(X)$, $\sigma^{-1}(1\times u)^*W'(x) = \sigma^{-1}\tau(\pi_C^{-1}(x)\otimes u^*(k)) = \pm\tau(\pi_C^{-1}(x)\otimes\gamma(h)) = \pm\tau\gamma(\pi_U^{-2}(\zeta(x))\otimes h) = \pm W\zeta(x)$. Thus the right hand square commutes up to sign.

From here, the proof of the corollary proceeds just as the proof of Bott's exact sequence did, and is left to the reader.

The natural thing to try at this point is to look at the maping cone P" of the composition cube $\eta^3: S^5 \longrightarrow S^2$ of the Hopf map η. The resulting cohomology theory turns out not to be new, but the space P" allows us to derive some exact sequences among the various K-theories by finding cofibrations relating P, P', and P". Two of these seem to be the most useful, and we shall describe them here.

First, we notice that from the cofibration sequence

$$S^5 \xrightarrow{\eta^3} S^2 \xrightarrow{e} P'' \xrightarrow{f} S^6 \longrightarrow \cdots$$

of η^3 we obtain an exact sequence

$$\cdots \xrightarrow{\sigma\eta^{3*}} \widetilde{KO}^n(S^6) \xrightarrow{f^*} \widetilde{KO}^n(P'') \xrightarrow{e^*} \widetilde{KO}^n(S^2) \xrightarrow{\sigma\eta^{3*}} \widetilde{KO}^{n+1}(S^6) \longrightarrow \cdots$$

From our earlier work, we know that $\sigma\eta^{3*} = \sigma^3(\tau\varepsilon_C)^3\sigma^{-2}$. However, since $(\tau\varepsilon_C)^3$ is a KO^*-module map, it is clear that it is zero, since $KO^{-3}(\text{point}) = 0$. Thus we have short exact sequences for all n: $0 \longrightarrow \widetilde{KO}^n(S^6) \xrightarrow{f^*} \widetilde{KO}^n(P'') \xrightarrow{e^*} \widetilde{KO}^n(S^2) \longrightarrow 0$. If 1 denotes the unit in $KO^0(\text{point})$, we define $r_1 = f^*(\sigma^6(1))$, and choose r_2 so that $e^*(r_2) = \sigma^2(1)$.

Definition 3.2 $W'': KO^n(X) \oplus KO^{n-4}(X) \longrightarrow KO^{n+2}(X \times P'', X)$ is defined by $W''(x,y) = (x \otimes r_2) + (y \otimes r_1)$.

Since W'' induces an isomorphism on the groups of a point, W'' is an isomorphism for all finite CW-complexes X.

From the commuting square

$$\begin{tikzcd} S^5 \arrow[r, "\eta^3"] \arrow[d, "\Sigma^2\eta"] & S^2 \arrow[d, "\text{identity}"] \\ S^4 \arrow[r, "\eta^2"] & S^2 \end{tikzcd}$$

we obtain a sequence of cofibrations

$$P'' \longrightarrow P' \longrightarrow \Sigma^3 P \longrightarrow \Sigma P'' \longrightarrow \Sigma P' \longrightarrow \cdots$$

Applying the various isomorphisms W, W', and W'', we obtain

Theorem 3.2 For any finite CW-complex X, there is an exact sequence

$$\begin{tikzcd} KC^*(X) \arrow[rr] & & KO^*(X) \oplus KSp^*(X) \arrow[dl] \\ & KU^*(X) \arrow[ul] & \end{tikzcd}$$

where the maps are given by:

i) $KU^n(X) \longrightarrow KC^n(X)$ is $\varepsilon_C\varepsilon_O$

ii) $KC^n(X) \longrightarrow KO^{n-1}(X)$ is $-\tau$

iii) $KC^n(X) \longrightarrow KSp^{n-1}(X)$ is the composition

$$KC^n(X) \xrightarrow{\pi_C} KC^{n-4}(X) \xrightarrow{\tau} KO^{n-5}(X) \xrightarrow{\cong} KSp^{n-1}(X)$$

iv) $KO^{n-1}(X) \longrightarrow KU^{n+1}(X)$ is $\pi_U{}^{-1}\varepsilon_U$

v) $KSp^{n-1}(X) \longrightarrow KU^{n+1}(X)$ is $\pi_U{}^{-1}\varepsilon_U$.

Since we have already investigated some of these maps in the two forms of the Bott sequence, and since all of the maps here are KO*-module maps, the proof is now straightforward and easy, and is left to the reader, as are the proofs of the next two theorems. The most useful sequence, for the purposes of our next section, is the exact sequence which arises from the commutative square:

$$\begin{tikzcd} S^5 \arrow[r, "\eta^3"] \arrow[d, "\Sigma\eta^2"'] & S^2 \arrow[d, "1"] \\ S^3 \arrow[r, "\eta"] & S^2 \end{tikzcd}$$

This square gives us a cofibration sequence of the form:

$$P'' \longrightarrow P \longrightarrow \Sigma^2 P' \longrightarrow \Sigma P'' \longrightarrow \Sigma P \longrightarrow \ldots$$

From this we can deduce another exact sequence.

<u>Theorem 3.3</u> For any finite CW-complex X, there is an exact sequence of the form:

$$\begin{tikzcd} KU^*(X) \arrow[rr] & & KO^*(X) \oplus KSp^*(X) \arrow[dl] \\ & KC^*(X) \arrow[ul] & \end{tikzcd}$$

where the maps are given by:

i) $KU^n(X) \longrightarrow KO^n(X)$ is ε_O

ii) $KU^n(X) \longrightarrow KSp^n(X)$ is $-\varepsilon_{Sp}$

iii) $KO^n(X) \longrightarrow KC^n(X)$ is ε_C

iv) $KSp^n(X) \longrightarrow KC^n(X)$ is ε_C

v) $KC^n(X) \longrightarrow KU^{n+1}(X)$ is $\pi_U^{-1}\varepsilon_U\tau$

We could continue on in this fashion and exhibit several more exact sequences, but these other sequences are somewhat complicated, and we shall not have use for them here. Using the fact that stably $2\eta = 0$, one can also derive sequences which involve the mod 2 theories.

There is some uncertainty as to the proper manner of extending a cohomology theory from the category of finite CW-pairs to the category of arbitrary CW-pairs. However, if we use the results of [4], we know that KC^* is a representable theory, and further, that the operation W' is representable by a map from the classifying space of KC^{-5} to the function space $(B_O)^{P'}$. Since this induces an isomorphism on the groups of a point, and the spaces involved all have the homotopy type of a CW-complex, it is a homotopy equivalence. Likewise, $(B_O)^P$ is a classifying space for KU^{-4}, and $(B_O)^{P''}$ is a classifying space for $KO^{-2} \oplus KSp^{-2}$. From the work of Borsuk we know that a cofibration $A \longrightarrow B \longrightarrow C$ induces a Hurewicz fibration $Y^C \longrightarrow Y^B \longrightarrow Y^A$ for any Y. However, this means that the maps which occur in our exact sequences can be extended to all CW-complexes, and the sequences will remain exact. This removes the unpleasantness which arises form the fact that the inverse limit functor is not exact, in general.

§4 The K-theory of Classifying Spaces

If G is a countable CW-group, and $\rho:G \longrightarrow U(n)$ is an n-dimensional representation of G, from the Milnor construction of classifying spaces we see that there is a map $\alpha(\rho):B_G \longrightarrow B_{U(n)}$ whose loop map is $\alpha:\Omega B_G \longrightarrow \Omega B_{U(n)}$. Further, if ρ_1 and ρ_2 are homotopic as representations, $\alpha(\rho_1)$ is homotopic to $\alpha(\rho_2)$. Since the unitary groups are connected, any two equivalent representations are homotopic as representations. Thus to every equivalence class of representations we have assigned an element of $KU^0(B_G)$, where KU^* is extended to all CW-complexes by $KU^0(X) = [X, Z \times B_U]$, etc, by composing $\alpha(\rho)$ with the inclusion of $B_{U(n)}$ in the component $[n] \times B_U$. If ρ is a representation, we write $\alpha(\rho)$ for the element of $KU^0(B_G)$ corresponding to the equivalence class of ρ.

The representations of G form a semiring under direct sum and tensor product. The representation ring RU(G) is defined to be the ring obtained by applying the Grothendieck construction to the semiring formed by the equivalence classes of representations of G into finite dimensional unitary groups. Since the direct sum addition and the loop addition are homotopic in B_U, we see that α can be regarded as a homomorphism $\alpha:RU(G) \longrightarrow KU^0(B_G)$. In general, α is neither injective nor surjective. However, if G is a compact Lie group, then in the proper topology, the image of α is dense. If G is also connected, α is an injection. These results are due to Atiyah and Hirzebruch [2], [3]. In this section we shall prove similar results for $KC^0(B_G)$, $KO^0(B_G)$, and $KSp^0(B_G)$, and in fact, we shall completely determine $KO^*(B_G)$. This will all be done, however, under the restriction that G is either a compact connected Lie group or a finite group. These are the only two cases where there are published proofs that α has as its image a dense subset of $KU^0(B_G)$. It seems likely that our results can be extended to the case where G is an arbitrary compact Lie group. From our techniques, the reader can easily see that our results hold for the product of a finite group and a compact connected Lie group.

The first thing which we must do to formulate our theorem is to describe the topology on $KU^*(B_G)$. If we let k^* stand for any cohomology theory defined on the category of CW-complexes, and if X is a CW-complex with n-skeleton X^n, $k^*(X)$ can be given the structure of a topological group if we take as the fundamental system of neighborhoods of zero the groups, $\text{kernel}(k^*(X) \longrightarrow k^*(X^n))$. The resulting topology will be called the inverse limit topology on $k^*(X)$. We denote the inverse limit functor by $\lim^0$. There is one non-zero right derived functor $\lim^1$ when we work in the category of abelian groups, and $\lim^0$ is left exact. The relationship between $k^*(X)$ and $\lim^0(k^*(X^n))$ has been described by Milnor for a class of theories k^* which satisfy one more axiom than the Eilenberg-Steenrod axioms. This axiom is called "additivity" by Milnor, and says that when applied to topological sums, k^* gives the direct product of the k^*'s of the individual components. Any representable theory clearly satisfies this condition, and any additive theory is representable [4]. Milnor's result [6] states that for an additive theory k^* and a CW-complex X, there is for all p an exact sequence:

$$0 \longrightarrow \lim{}^1(k^{p-1}(X^n)) \longrightarrow k^p(X) \longrightarrow \lim{}^0(k^p(X^n)) \longrightarrow 0$$

For a representable theory, of course, the surjectiveness of the map $k^*(X) \longrightarrow \lim^0(k^*(X^n))$ follows from the homotopy extention property for CW-pairs. The following is implicit in §3,4 [2]:

Proposition 4.1 A sufficient condition that $\lim^1(k^*(X^n)) = 0$ is that in the spectral sequence [5] which connects $H^*(X;k^*(\text{point}))$ with $k^*(X)$, for all (p,q) there exists an r such that $E_r^{p,q} = E_\infty^{p,q}$.

If both $H^*(X;Z)$ and $k^*(\text{point})$ are of finite type, it is clear from this proposition that a sufficient condition that $\lim^1(k^*(X^n)) = 0$ is that every element of $E_2^{p,q} = H^p(X;k^q(\text{point}))$ have some multiple which is an infinite cycle in the spectral sequence. If $k^* = KU^*$, we know from [3] that this will happen if $ch^p: KU^*(X)\otimes Q \longrightarrow H^p(X;Q)$ is onto for every p. If we take X to be the classifying space of a compact Lie group G, then $ch^p: RU(G)\otimes Q \longrightarrow H^p(B_G;Q)$ is surjective for all p (see [3] for the case when G is connected, and use this to conclude the same result for any compact Lie group). Thus, we see that for a compact Lie group G, $KU^*(B_G) = \lim^0(KU^*(B_G{}^n))$.

From Milnor's theorem, we can conclude that if $k^*(X) = \lim^0(k^*(X^n))$, then $k^*(X)$ is Hausdorff. If h^* is another cohomology theory, and $\lambda: k^*(X) \longrightarrow h^*(X)$ is a cohomology operation, λ is continuous, and if $h^*(X)$ is Hausdorff, the kernel of this map λ

is a closed subgroup of $k^*(X)$. We shall need this fact later.

The results of Atiyah and Hirzebruch on $KU^*(B_G)$ can be summarized as follows:

i) If $RU(G)^\wedge$ denotes the completion of $RU(G)$ in the inverse limit topology, then $\alpha: RU(G)^\wedge \longrightarrow KU^0(B_G)$ is an isomorphism.

ii) $KU^1(B_G) = 0$

iii) If $I(G)$ is the augmentation ideal of $RU(G)$, where the augmentation assigns to every representation its dimension, the $I(G)$-adic topology on $RU(G)$ agrees with the inverse limit topology.

To use these results to determine $KC^*(B_G)$ and $KO^*(B_G)$, we must know one purely algebraic fact about $RU(G)$.

<u>Lemma 4.1</u> Let $RC(G)$ be the subring of $RU(G)$ which is fixed under complex conjugation, and let $\psi: RU(G) \longrightarrow RU(G)$ by $\psi(x) = x - \overline{x}$. Then the sequence

$$0 \longrightarrow RC(G) \longrightarrow RU(G) \xrightarrow{\psi} RU(G) \xrightarrow{\varepsilon_U \varepsilon_O} RU(G)$$

is exact.

proof: Let R_1 be the subring of $RU(G)$ generated by the irreducible representations which are self-conjugate. Partition the remaining irreducible representations into two sets so that if a representation lies in one of the sets, its conjugate lies in the other. Let R_2 be the subgroup generated by one of these sets. Then $\overline{R}_2$ is the subgroup generated by the other, and $RU(G) = R_1 \oplus R_2 \oplus \overline{R}_2$. Any element x of $RU(G)$ is of the form $x = a - b$, where a and b are representations. Let $a = a_1 + a_2 + a_3$, and $b = b_1 + b_2 + b_3$, where $a_1, b_1 \in R_1$, $a_2, b_2 \in R_2$, and $a_3, b_3 \in \overline{R}_2$. Since $x + \overline{x} = \varepsilon_U \varepsilon_O(x)$, if $\varepsilon_U \varepsilon_O(x) = 0$, we have $a_1 = b_1$, and $a_2 + \overline{a}_3 - b_2 - \overline{b}_3 = 0$. This is true because RU(G) is the free abelian group generated by the irreducible representations of G. From these equalities, we see that $x = a_2 + a_3 - b_2 - b_3 = a_2 + a_3 - (a_2 + \overline{a}_3 - \overline{b}_3) - b_3 = \psi(a_3 + \overline{b}_3)$.

<u>Lemma 4.2</u> Let G be either a compact connected Lie group or a finite group, and let I(G) be the augmentation ideal of RU(G). Then, if RU(G) is given the I(G)-adic topology, and RC(G) is given the induced topology, the completion of the sequence of the last lemma

$$0 \longrightarrow RC(G)^{\wedge} \longrightarrow RU(G)^{\wedge} \xrightarrow{\psi} RU(G)^{\wedge} \xrightarrow{\varepsilon_U \varepsilon_O} RU(G)^{\wedge}$$

is exact ., where ^ denotes completion.

proof: RC(G) is a subring of RU(G), so RU(G) is an RC(G)-module; further, both ψ and $\varepsilon_U \varepsilon_O$ are RC(G)-module maps. Since completion is an exact functor on finitely generated modules over topological Noetherian rings, it suffices to show that RC(G) is Noetherian, that RU(G) is finitely generated over RC(G), and that if $IC(G) = I(G) \cap RC(G)$, that the I(G)-adic topology is the same as the IC(G)-adic topology on RU(G) and RC(G).

If G is finite, RU(G) is of finite rank, and therefore a finitely generated ring over Z. RC(G) is the subring fixed by Z_2 which acts on RU(G) by complex conjugation. Thus we are in the situation described in §4.1 of [3], and the conditions described above hold.

If G is a compact connected group with maximal torus T and Weyl group $W(G)$, the inclusion of T in G induces an isomorphism $RU(G) \cong RU(T)^{W(G)}$, where the superscript denotes the subring left fixed by the group $W(G)$. If Z_2 acts on $RU(T)$ by complex conjugation, the actions of Z_2 and $W(G)$ commute with one another, since Z_2 acts as automorphisms of the unitary groups, and $W(G)$ by automorphisms of T. Thus $RC(G) \cong RU(T)^{W(G) \oplus Z_2}$, so by §4.1 of [3] we see that $RU(T)$, and therefore $RU(G)$, is a finitely generated $RC(G)$-module, and the $IC(G)$-adic topology agrees with the $I(T)$-adic topology. However, on $RU(G)$, the $I(G)$-adic topology and the $I(T)$-adic topology agree, so the $I(G)$-adic topology agrees with the $IC(G)$-adic topology. Also, we have that $RC(G)$ is Noetherian, so we have satisfied the conditions of the first paragraph of the proof.

From what follows, it will be seen that all we need are the results of Atiyah and Hirzebruch on $KU^*(B_G)$ and the conclusion of the previous lemma. For example, the previous lemma is easily seen to be true for the product of a compact connected Lie group and a finite group. From now on, we shall assume that G is such that the above lemma is true for G.

<u>Theorem 4.1</u> $\zeta: KC^0(B_G) \longrightarrow KU^0(B_G)$ is an injection, and α maps $RC(G)^\wedge$ isomorphically onto the image of ζ. Thus $KC^0(B_G) \cong RU(G)^\wedge$. Further, $KC^1(B_G) \cong RU(G)^\wedge/\psi(RU(G)^\wedge)$ $\boxed{\cong KC^2(B_G),}$ and $KC^3(B_G) \cong RU(G)^\wedge/\varepsilon_U\varepsilon_0(RU(G)^\wedge)$.

$KC^2(B_G) \cong \psi(RU(G^\wedge)) \cong \ker \varepsilon_0$

proof: Since $KU^1(B_G) = 0$, from the exact sequence between KU^* and

KC^*, we have exact sequences:

$$0 \longrightarrow KC^0(B_G) \xrightarrow{\zeta} KU^0(B_G) \xrightarrow{\psi} KU^0(B_G) \xrightarrow{\gamma} KC^1(B_G) \longrightarrow 0$$

$$0 \longrightarrow KC^2(B_G) \xrightarrow{\zeta} KU^2(B_G) \xrightarrow{\psi} KU^2(B_G) \xrightarrow{\gamma} KC^3(B_G) \longrightarrow 0$$

Since $\pi_U \psi \pi_U^{-1} = \varepsilon_U \varepsilon_O$, the theorem follows from the last lemma.

<u>Corollary</u>: $KO^1(B_G) = 0$ and $KSp^1(B_G) = 0$

proof: From theorem 3.3, we have an exact sequence

$$0 \longrightarrow KO^1(B_G) \oplus KSp^1(B_G) \longrightarrow KC^1(B_G) \xrightarrow{\pi_U^{-1}\varepsilon_U\tau} KU^2(B_G).$$

Since $\tau\gamma = \varepsilon_O$, we have a commuting diagram with exact rows:

$$\begin{tikzcd}
RU(G)^\wedge \arrow[r, "\psi"] \arrow[d, equal] & RU(G)^\wedge \arrow[r, "\gamma\alpha"] \arrow[d, equal] & KC^1(B_G) \arrow[r] \arrow[d, "\pi_U^{-1}\varepsilon_U\tau"] & 0 \\
RU(G)^\wedge \arrow[r, "\psi"] & RU(G)^\wedge \arrow[r, "\pi_U^{-1}\alpha\varepsilon_U\varepsilon_O"] & KU^2(B_G) &
\end{tikzcd}$$

Thus, the vertical line is an injection, so $KO^1(B_G) \oplus KSp^1(B_G) = 0$.

<u>Corollary:</u> $\varepsilon_U : KO^0(B_G) \longrightarrow KU^0(B_G)$ and $\varepsilon_U : KSp^0(B_G) \longrightarrow KU^0(B_G)$ are both injections. Further, these groups are injected so that their images are closed subgroups.

proof: From the Bott sequence, we have exact sequences

$$0 = KO^1(B_G) \xrightarrow{\tau\varepsilon_C} KO^0(B_G) \xrightarrow{\varepsilon_U} KU^0(B_G) \longrightarrow KO^2(B_G)$$

$$0 = KSp^1(B_G) \longrightarrow KSp^0(B_G) \xrightarrow{\varepsilon_U} KU^0(B_G) \longrightarrow KSp^2(B_G)$$

Thus, we see that the maps are injections, and their images will be closed if $KO^2(B_G^n)$ and $KSp^2(B_G^n)$ have their $\lim^1$ equal to 0. Since $\varepsilon_O : KU^*(\text{point}) \otimes Q \longrightarrow KO^*(\text{point}) \otimes Q$ and $\varepsilon_{Sp} : KU^*(\text{point}) \otimes Q \longrightarrow KSp^*(\text{point}) \otimes Q$ are both onto, from our earlier discussion, we see that $\lim^1$ does vanish on these sequences.

Definition 4.1 $RO(G)$ is the subring of $RU(G)$ generated by those representations $\rho: G \longrightarrow U(n)$ whose image lies in the subgroup $O(n)$. $RSp(G)$ is the subgroup of $RU(G)$ generated by those representations $\rho: G \longrightarrow U(2n)$ whose image lie in the subgroup $Sp(n)$. $RO(G)$ and $RSp(G)$ are both given the topology induced from the inverse limit topology on $RU(G)$. Their closures (that is, their completions) in this topology are denoted by $RO(G)^{\wedge}$ and $RSp(G)^{\wedge}$ respectively.

Lemma 4.3 $\alpha: RU(G)^{\wedge} \longrightarrow KU^0(B_G)$ carries $RO(G)^{\wedge}$ into the subgroup $KO^0(B_G)$ and $RSp(G)$ into the subgroup $KSp(G)$.

proof: Given a representation $\rho: G \longrightarrow O(n)$, we obtain a map of B_G into $B_{O(n)}$, and thus an element of $KO^0(B_G)$. Since ε_U injects $KO^0(B_G)$ into $KU^0(B_G)$, α carries $RO(G)$ into $KO^0(B_G)$. Since $KO^0(B_G)$ is closed in $KU^0(B_G)$, α carries the closure of $RO(G)$ into $KO^0(B_G)$. Similarly, α carries $RSp(G)$ into $KSp^0(B_G)$, and thus carries the closure into $KSp^0(B_G)$.

Lemma 4.4 α maps $RO(G)^{\wedge}$ isomorphically onto $KO^0(B_G)$ and $RSp(G)^{\wedge}$ isomorphically onto $KSp^0(B_G)$.

proof: Let $RU(G) = R_1 \oplus R_2 \oplus \overline{R}_2$ be the decomposition described in the proof of lemma 4.1. Let $D = \{x + \overline{x} \mid x \in R_2\}$. Then $RC(G) = R_1 \oplus D$. Both $RO(G)$ and $RSp(G)$ are subgroups of $RC(G)$. D is clearly a subgroup of both $RO(G)$ and $RSp(G)$. An irreducible representation which is self-conjugate is either real or symplectic. Thus R_1 is a subgroup of $RO(G)+RSp(G)$. Therefore, $RC(G) = RO(G)+RSp(G)$. Since no irreducible representation is both real and symplectic, we have an exact sequence

$$RU(G) \xrightarrow{(\varepsilon_U\varepsilon_O, -\varepsilon_U\varepsilon_{Sp})} RO(G) \oplus RSp(G) \longrightarrow RC(G) \longrightarrow 0.$$

Completing, we have a sequence of order two. However, the map $RO(G)^\wedge \oplus RSp(G)^\wedge \longrightarrow RC(G)^\wedge$ is a surjection. To see this, let $N_n = \text{kernel}(RU(G) \xrightarrow{\alpha} KU^0(B_G{}^n))$ be the fundamental system of neighborhoods of 0, and let $M_n = RC(G) \cap N_n$. If $S_n = (RO(G) \cap M_n) + (RSp(G) \cap M_n)$, according to §3 of[2] we have a surjection $RO(G)^\wedge \oplus RSp(G)^\wedge \longrightarrow \lim^0(RC(G)/S_n)$. We have an exact sequence $\lim^0(RC(G)/S_n) \longrightarrow RC(G)^\wedge \longrightarrow \lim^1(M_n/S_n)$ coming from the exact sequences

$$0 \longrightarrow M_n/S_n \longrightarrow RC(G)/S_n \longrightarrow RC(G)/M_n \longrightarrow 0.$$

Since $RU(G)/N_n$ is a subgroup of $KU^0(B_G)$, it is a finitely generated abelian group. Thus $RO(G)/RO(G) \cap M_n$ and $RSp(G)/RSp(G) \cap M_n$ are finitely generated, so their image $RC(G)/S_n$ is finitely generated. Thus M_n/S_n is finitely generated. However, if $x \in M_n$, $2x = x + \overline{x}$ lies in S_n. Thus M_n/S_n is finite. However, from §3 of [2] we have $\lim^1(M_n/S_n) = 0$, since $\lim^0$ is exact on inverse sequences of finite groups. Thus, we see that $RO(G)^\wedge \oplus RSp(G)^\wedge \longrightarrow RC(G)^\wedge$ is onto. From theorem 3.3, we have a commuting diagram

$$\begin{array}{ccccccc}
RU(G)^\wedge & \longrightarrow & RO(G)^\wedge \oplus RSp(G)^\wedge & \longrightarrow & RC(G)^\wedge & & \\
\downarrow & & \downarrow & & \downarrow & & \\
KU^0(B_G) & \longrightarrow & KO^0(B_G) \oplus KSp^0(B_G) & \longrightarrow & KC^0(B_G) & \longrightarrow & 0
\end{array}$$

where the bottom row is exact. The left hand and the right hand vertical maps are isomorphisms, so the center map is onto. We already know that it is injective, so it is an isomorphism.

Corollary: The topology on $RO(G)$ induced from $RU(G)$ is the same as the inverse limit topology induced by $KO^0(B_G)$, and similarly for $RSp(G)$.

Theorem 4.2 Let G be a compact Lie group which satisfies the conclusion of lemma 4.2 Then α induces isomorphisms:

$$
\begin{array}{ll}
KO^{0}(B_G) = RO(G)^{\wedge} & KO^{-4}(B_G) = RSp(G)^{\wedge} \\
KO^{-1}(B_G) = RO(G)^{\wedge}/\varepsilon_O(RU(G)^{\wedge}) & KO^{-5}(B_G) = RSp(G)^{\wedge}/\varepsilon_{Sp}(RU(G)^{\wedge}) \\
KO^{-2}(B_G) = RU(G)^{\wedge}/RSp(G)^{\wedge} & KO^{-6}(B_G) = RU(G)^{\wedge}/RO(G)^{\wedge} \\
KO^{-3}(B_G) = 0 & KO^{-7}(B_G) = 0
\end{array}
$$

proof: This now follows immediately from the Bott sequence, which gives us exact sequences:

$$
\begin{array}{l}
0 \longrightarrow KO^{0}(B_G) \longrightarrow KU^{0}(B_G) \longrightarrow KO^{-6}(B_G) \longrightarrow 0 \\
\qquad KU^{-6}(B_G) \longrightarrow KO^{-4}(B_G) \longrightarrow KO^{-5}(B_G) \longrightarrow 0 \\
0 \longrightarrow KO^{-4}(B_G) \longrightarrow KU^{-4}(B_G) \longrightarrow KO^{-2}(B_G) \longrightarrow 0 \\
\qquad KU^{-2}(B_G) \longrightarrow KO^{0}(B_G) \longrightarrow KO^{-1}(B_G) \longrightarrow 0
\end{array}
$$

Since there are eight groups here, it is somewhat tedious to describe the ring structure of $KO^*(B_G)$, though this can be fairly easily done by knowing the ring structure of RO(G) and the module structure of RSp(G) over RO(G), together with a complete description of the ring structure of $KO^*(\text{point})$. To see how the elements of $KO^*(\text{point})$ act on representations, one can simply take G to be the trivial group, so that B_G is a point. This gives an unusual description of $KO^*(\text{point})$.

While the exact sequences described here will probably never replace spectral sequence arguments for the purpose of computing $KO^*(X)$, it would seem from the results of this section that they could be quite a help, especially in proving theorems about the real K-theory of whole classes of spaces.

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