Discrete Mathematics II Homework

Due December 11

II. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

Make a graph in which the vertices correspond to the animals and there is an edge connecting pairs of animals if one of them will eat the other. If we color the graph with a valid coloring and put each animal in a habitat corresponding to its color, then every animal will go in a habitat where it will not get eaten. So the minimal number of needed habitats is the chromatic number of the graph.

III. The mathematics department has six committees, each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time? The committees are $C_1 = \{Arlinghaus, Brand, Zaslavsky\}$, $C_2 = \{Brand, Lee, Rosen\}$, $C_3 = \{Arlinghaus, Rosen, Zaslavsky\}$, $C_4 = \{Lee, Rosen, Zaslavsky\}$, $C_5 = \{Arlinghaus, Brand\}$, $C_6 = \{Brand, Rosen, Zaslavsky\}$.

Let the C_i be vertices of a graph G and make an edge if the committees share a member. The vertices C_2, C_3, C_5, C_6 participate in a subgraph isomorphic to K_4 so $\chi(G) \geq 4$. In fact, this graph can be colored with 4 colors. For example: Color C_1 and C_2 red, C_3 blue, C_4 and C_5 yellow, and C_6 green.

IV. Construct the graphs associated to the maps shown. Then find the least number of colors needed to color the map so that no two adjacent regions have the same color.

The first graph can be colored with three colors. For example, let A (which touches all the other regions) be red. Let B and D be blue. Let C and E be green. F can then be either blue or green.

The second graph can be colored with only two colors. For example, color the vertices corresponding to regions A, C, and D red and the ones corresponding to B, E, and F blue.

Due Monday, December 9

2. Show that if a connected planar graph with at least 3 vertices has no circuits of length 3 then $|E| \leq 2|V| - 4$ (hint: modify the proof of Theorem 13.8.2, which we discussed in class)

Let $|E| = e$, $|V| = v$ and let the number of regions be r. Since there are no cycles of length 3, the degree of every region must be at least 4. So modifying the proof in the book, we have $2e = \sum deg(R) \geq 4r = 4(2 + e - v) = 8 + 4e - 4v$. So $-2e \geq 8 - 4v$, so $e \leq 2v - 4$.

3. Use the fact of the preceding problem to give another proof (different from the one from class) that $K_{3,3}$ is not planar.

Since $K_{3,3}$ is bipartite it cannot have cycles of odd length, so if it were planar we would have $e \leq 2v - 3$. But we have $v = 6$ and $e = 9$, which is impossible from the above result.

4. Suppose that G is a connected k-regular graph. For what k can you be sure that G is not planar?

We showed in class that connected planar graphs must have a vertex with degree ≤ 5, so if $k \geq 6$ then G cannot be planar.

Due Friday, December 6

II.a. Dirac's theorem does not apply because $n = 5$ and there are vertices with degree 2 which is less than $5/2$. Ore's theorem does not apply because there are non-adjacent vertex pairs whose degrees only add to 4. There is a Hamilton circuit - just run around the outsize.

b. The answers are the same as for part a.

c. $n = 5$ and all of the degrees are at least 3 so Dirac's theorem applies. Since the hypotheses of Dirac's theorem are satisfies, so are the hypotheses for Ore's theorem. From the theorems, there is a Hamilton cycle

d. $n = 6$. All vertices have degree 3 so both Dirac's and Ore's theorems apply and there is a Hamilton cycle.

III. **Can you find a simple graph with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n-1)/2$?**

Three vertices connected in a line will do.

Due Monday November 18

2. For which values of n is K_n bipartite? Explain your reasoning.

K_0, K_1, K_2 are bipartite (draw them and color). No K_n is bipartite for $n \geq 3$. To see this, we can note that if V_1 and V_2 are our two different vertex subsets then by the pigeonhole principle we would have to have at least two vertices in one of these sets if $n \geq 3$. But then since K_n is complete there is an edge between any two vertices, so two vertices in one of V_1 or V_2 would be connected by an edge. This is not allowed for a bipartite graph.

3. For which values of n is C_n bipartite? Explain your reasoning.

Let's choose a vertex and then as we go around the ring of C_2 label the vertices $e_1, e_2, \ldots, e_n$. For C_n to be bipartite, if $e_i \in V_1$ then $e_{i+1} \in V_2$ and vice versa. If n is even, then we can let $e_i \in V_1$ for all odd i and $e_i \in V_2$ for all even i. If n is odd and e_1 in V_1 then also e_n would be in V_1, which is not allowed for a bipartite graph since e_1 and e_n share an edge. The same problem occurs is $e_1 \in V_2$.

So C_n is bipartite if and only if n is even.

Which of the following graphs are bipartite?

The first graph is - for example put the center vertex in V_1 and all others in V_2

The second graph also is - for example put $b, d, e \in V_1$ and $a, c \in V_2$.

The third graph is not bipartite. The subgraph made up of b, c, f and the edges connecting them is a copy of K_3, so we can apply the argument above in problem 2 with these vertices.

Due November 11

2. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

The characteristic polynomial is x^3-7x-6. The easiest way to factor this is to observe by inspection that -1 is a root. Polynomial division then lets us factor into $(x+1)(x^2-x-6) = (x+1)(x-3)(x+2)$. So the general solution is $\alpha(-1)^n + \beta 3^n + \gamma(-2)^n$. Solving for the initial conditions gives $8(-1)^n + 4 \cdot 3^n + 3(-2)^n$

Due Friday November 1

2. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates. Hints: What is the formula for the midpoint? What has to hold for a pair of points for the midpoint to have integer coordinates?

The midpoint of (x_i, y_i) and (x_j, y_j) is $\left(\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}\right)$. This will be a point with integer coordinates if the x_i and x_j have the same parity (both even or both odd) and similarly for y_i and y_j. For a given point the options are (odd, odd), (odd, even), (even, odd), and (even, even). Since there are five points, the Pigeonhole Principle says that two must share a pair of parities. So these two will have a midpoint with integer coordinates.

3. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 p.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

Let a_i, $1 \le i \le 75$, be the number of matches in the first i hours. This is a strictly increasing sequence of numbers all between 1 and 125 inclusive. We also have the sequence $a_1 + 24, a_2 + 24, \ldots, a_{75} + 24$, which is strictly increasing and consists of numbers between 25 and 149 inclusive. Togethere these two sequences have 150 entries, so the pigeonhole principles says there must be a repeat. Since each of our two sequences is strictly increasing, a number in the first sequence must equal a number in the second sequence, so there are an a_j and a_i so that $a_j = a_i + 24$. So from hour i to hour j there are 24 matches exactly.

Due Monday October 28

1. Give Pascal's triangle up through n=10.

See `https://www.cut-the-knot.org/arithmetic/combinatorics/PascalTriangleProperties.shtml`

2. What is the coefficient of x^3y^7 in $(x+y)^{10}$?

120

Due Wednesday, October 16

2. Suppose the Math Department has 15 professors, the Computer Science Department has 10 professors, and the Engineering Department has 12 professors. How many ways are there to form a committee of 7 by choosing 3 professors from one of the departments and 2 from each of the others?

$C(15,3)C(10,2)C(12,2) + C(10,3)C(15,2)C(12,2) + C(12,3)C(15,2)C(10,2)$

Due Wednesday, October 4

2. Consider a 3x3x3 cube (like a Rubik's cube). Three ants named Abigail, Bobbie, and Cliff have decided to live in the cube, each in their own 1x1x1 cell, but they don't want any of their coordinate addresses to be the same. In other words no two can share the same x coordinate, y coordinate, or z coordinate for their cell. So, for example, if one ant lives in the top face of the cube, then no other ant can live in the top face. They don't like to look at each other. How many ways are there for the ants to live in the cube? (Note: you can tell the ants apart. You can also assume you can tell the cube locations apart, so symmetry of the cube doesn't matter.)

MY ORIGINAL SOLUTION HAD AN ERROR - HERE IT'S FIXED.

Abigail has $3 \times 3 \times 3 = 27$ choices for where to live. Once Abigail has chosen, there are only two options left for each coordinate, so Bobbie has $2 \times 2 \times 2 = 8$ choices. There is now only one of each coordinate left for Cliff. So the total number of possibilities is $27 \times 8 \times 1 = 216$.

3. Abigail, Bobbie, and Cliff have moved out of the cube from the last problem, and three indistinguishable termites have moved in. How many ways are there to put the termites in the cube so that no two share a coordinate, recognizing that once they're in there you can't tell them apart.

There is a 6-to-1 function from the set of the last problem to this problem that turns Abigail, Bobbie, and Cliff into termites (think about why 6). So the answer is $216/6 = 36$.

Due Wednesday, October 2

2. a. Suppose a class has 18 men and 18 women. The class will elect three class officers: President, Vice-President, and Treasurer. No person can hold more than one of these positions. The President and Treasurer must be the same gender. The President and Vice President must be different genders. How many ways can the officers be chosen?

There are 36 ways to choose the president, leaving 18 ways to choose the VP and 17 ways to choose the treasurer, so $36 \times 18 \times 17 = 11016$

b. Suppose the class instead has 17 men and 18 women. How many ways can the officers be chosen now? Hint: the generalized product rule will not work for this problem (think carefully about why).

Notice that the number of options for both VP and treasurer depend on whether the president is male or female. Since those are disjoint options, we can split into two cases using the sum rule. The number of options with a male president is $17 \times 18 \times 16 = 4896$. The number of options with a female president is $18 \times 17 \times 17 = 5202$. So altogether there are 10098 options.

Due Friday, September 27

2. How many injective (one-to-one) functions are there from a set with m elements to a set with n elements?

Let A and B be sets with $|A| = m$ and $|B| = n$. For $m \leq n$ we must assign to each element of A and element of B, with no repetitions allowed. So there are n choices for the first element of b, $n-1$ for the next choice and so on. So by the generalized product rule, the answer is $n \cdot (n-1) \cdot (n-2) \ldots (n-m+1)$.

Due Wednesday, September 25

2. 1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

By the product rule: $18 \cdot 325 = 5850$

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

By the sum rule, $18 + 325 = 343$

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

By the product rule: 4^{10}

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Now there are 5 choices for each question, so 5^{10}

II. Consider the equivalence relation on the real numbers from class with $R = \{(x, y) | x - y \in \mathbb{Z}\}$.

a) What is the equivalence class of 1 for this equivalence relation?

[1] is the set of real numbers x such that $x - 1$ is an integer. This is exactly the set of integers.

b) What is the equivalence class of 1/2 for this equivalence relation?

[1/2] is the set of real numbers x such that $x - 1/1$ is an integer. This is exactly the set of "half-integers," i.e. the decimal number of the form $X.5$ for $X \in \mathbb{Z}$. Alternatively, this is the set $\{\ldots, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5, \ldots\}$.

II. In class and in the text, you have seen that an equivalence relation on a set A determines a partition of A. Prove the converse: given a partition of A into pairwise disjoint subsets $A_1, A_2, \ldots, A_n$, define an equivalence relation on A whose equivalence classes are the given subsets A_i. (Hint: your relation won't be given in general by any kind of algebraic or arithmetic formula, rather you should describe in words how to tell if two elements of A are related.)

Define the relation $\sim$ by $x \sim y$ if x and y are both in the same subset A_i (i.e. if there exists a subset A_i with $x \in A_i$ and $y \in A_i$.

proof that this is an equivalence relation (technically I didn't ask for this, so let's make it extra credit if you did it):

$\sim$ is reflexive because x is certainly contained in the same A_i as itself.

$\sim$ is symmetric since if x is in the same subset A_i as y, then y is in the same subset A_i as x.

For transitivity, suppose $x \sim y$ and $y \sim z$. Suppose $x, y \in A_i$ and $y, z \in A_j$. Since $y \in A_i \cap A_j$ and since the subsets are a partition, we must have $A_i = A_j$ since otherwise $A_i \cap A_j = \emptyset$. Thus x and z are also in the same subset, so $x \sim z$.

Due Wednesday, September 18

Which of these relations on the positive integers are partial orders? Which are total orders?

skip

a) aRb **if** $a = b$

This is a partial order but not a total order since if $a \neq b$ then a and b are not comparable

b) aRb **if** $a \neq b$

This is not a partial order since it's not reflexive.

c) aRb **if** $a \geq b$

This is a total order

d) aRb **if** $a < b$

This is not a partial order since it's not reflexive

e) aRb **if** $b = a^n$ **for some integer** $n \geq 1$

This is a partial order but not a total order since, for example, 2 and 3 are not comparable

2. Which of the following graphs labeled 9, 10, 11 show partial orders. Note that 11 has "two pieces"?

Graph 11 is a partial orders. (Note: my original answer here was incorrect. Graph 10 is not transitive.)

3. Which of these pairs of elements are comparable in the poset given by the integers with aRb if a divides b? a) 5,15 b) 6,9 c) 16, 8 d) 7,7

a, c, and d are comparable pairs

4. Let $\mathbb{Z}$ denote the integers. Sort the following elements of $\mathbb{Z} \times \mathbb{Z}$ into the lexicographic order determined by the total order $\leq$ on $\mathbb{Z}$: (1,3) (2,4), (-2,3), (1,1), (2,6)

$(-2, 3) \leq (1, 1) \leq (1, 3) \leq (2, 4) \leq (2, 6)$

Due Monday, September 16

2. Consider the relation on the real numbers such that xRy if and only if $|x-y| \leq 2$. Describe R^+. You do not have to provide a proof.

xRy for all $x, y \in \mathbb{R}$

3. Let R be a relation on a set A.

a. Say precisely what it means for xR^+y to be true in terms of powers of R.

xR^+y if there is an positive integer k such that xR^ky.

b. Prove that $R+$ is transitive.

Suppose xR^+y and yR^+z. Then there are integers j and k (possibly different!!) so that x^ky and yR^jz. But then by the definition of composition we have $(x, z) \in R^j \circ R^k = R^{j+k}$. So $(x, z) \in R^+$.

c. Prove that R^+ is the smallest transitive relation on A that contains R. In other words, show that if S is another transitive relation on A such that $R \subset S$ then $R^+ \subset S$.

This shows that any other way of extending R to be a transitive relation must include R^+.

Suppose xR^+y. This means that $(x,y) \in R^k$ for some k. In other words, there's a walk of length k so that xRz_1, z_1Rz_2, ..., $z_{k-1}Ry$. Since $R \subset S$, this means that also xSz_1, z_1Sz_2, ..., $z_{k-1}Sy$. But S is transitive by assumption, so xSy. Thus $(x,y) \in R^+$ implies $(x,y) \in S$.

Due Friday, September 13

2. If R is the relation on the real numbers with xRy if and only if $|x-y| \leq 2$, show that xR^2y if and only if $|x-y| \leq 4$. Provide a detailed proof. Hint: you will need to use the triangle inequality.

Suppose xR^2y. Then there is a z such that xRz and zRy. In other words, $|x-z| \leq 2$ and $|z-y| \leq 2$. So, using the triangle inequality, $|x-y| = |x-z+z-y| \leq |x-z| + |z-2| \leq 2+2 = 4$. Thus $|x-y| \leq 4$.

Conversely, suppose $|x-y| \leq 4$. Let $z = \frac{x+y}{2}$. Then $|x-z| = |x-\frac{x+y}{2}| = |\frac{x-y}{2}| = \frac{|x-y|}{2} \leq 4/2 = 2$ and $|z-y| = |\frac{x+y}{2} - y| = |\frac{x-y}{2}| = \frac{|x-y|}{2} \leq 4/2 = 2$. So there is a z with xRz and zRy, so xR^2y is true.

4. Prove that if R is reflexive and transitive then $R^n = R$ for all $n \geq 1$. You may use the theorem from class as part of your proof.

In class we showed that R is transitive iff $R^n \subset R$ for all n. So if R is reflexive and transitive, then in particular it is transitive, so $R^n \subset R$ for all $n \geq 1$. So we only need to show that also $R \subset R^n$ for all $n \geq 1$. Suppose $(x,y) \in R$, we must show $(x,y) \in R^n$ for any $n \geq 1$. In other words, we must show that there are $z_1, z_2, \ldots, z_{n-1}$ so that xRz_1, $z_1Rz_2, \ldots, z_{n-1}Ry$, i.e. that there is an n-step walk in the graph of R from x to y. But we can just take all $z_i = x$. This works since xRx by the assumption R is reflexive, and we have assumed xRy. So our n-step walk is $x, x, \cdots, x, y$. Therefore $(x,y) \in R^n$.

5. Show that if R is reflexive then so is R^n for all $n \geq 1$. Hint: use induction.

For the base case, since x is reflexive xRx for all x. Suppose that R^{n-1} is reflexive; we will show xR^nx is reflexive. For any x, we have $xR^{n-1}x$ by induction assumption and xRx because x is reflexive. But then (x,x) is in the composition $R \circ R^{n-1}$ by definition of composition. So R^n is reflexive.

6. Suppose that R is antireflexive. Must R^2 be antireflexive? Why or why not?

No. For example, let $A = \{1,2\}$ and let $R = \{(1,2),(2,1)\}$, which is antireflexive. Then $R^2 = \{(1,1),(2,2)\}$, which is not antireflexive.

Due Monday, September 9

2. Give an example of a single relation that is all of the following: reflexive, symmetric, and transitive.

There are lots of answers to this. A simple one is $A = \{1\}$, $R = \{(1,1)\}$.

3. A relation R is called asymmetric if $(a,b) \in R$ implies that $(b,a) \notin R$. Give an example of an asymmetric relation on a non-empty set.

Again there are lots of examples. One example is $A = \{1,2\}$, $R = (a,b)$.

4. Must an asymmetric relation be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.

An asymmetric relation must be antisymmetric since it's forbidden to ever have both aRb and bRa. An antisymmetric relation need not be asymmetric since asymmetric does not allow self-loops while antisymmetric does (if R is asymmetric then $(a,a) \in R$ implies $(a,a) \notin R$, a contradiction if $(a,a) \in R$ for any a).

Due Wednesday, September 4:

I. List the ordered pairs in the relation R from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$, where $(a,b) \in R$ if and only if

a) $a = b$**,**

$R = \{(0,0),(1,1),(2,2),(3,3)\}$

b) $a + b = 4$**,**

$R = \{(1,3),(2,2),(3,1),(4,0)\}$

c) $a > b$**,**

$R = \{(1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\}$

d) $a|b$**,**

$R = \{(0,0),(1,0),(2,0),(3,0),(4,0),(1,1),(1,2),(1,3),(2,2),(3,3)\}$