Complex Analysis
Sketches of Solutions to Selected Exercises

Homework 1

2.1.a $(\sqrt{2}-i)-i(1-\sqrt{2}i)=\sqrt{2}-i-i+i^2\sqrt{2}=\sqrt{2}-i-i-\sqrt{2}=-2i$

2.1.b $(2,-3)(-2,1)=(2(-2)-(-3)1,2(1)+(-3)(-2))=(-1,8)$

2.2.a $Re(iz)=Re(i(x+iy))=Re(ix-y)=-y=-Im(z)$

2.2.b $Im(iz)=Im(i(x+iy))=Im(ix-i)=x=Re(z)$

2.4 $(1+i)^2-2(1+i)+2=1+2i+i^2-2-2i+2=0$, $(1-i)^2-2(1-i)+2=1-2i+i^2-2+2i+2=0$

2.5 $(x+iy)(a+ib)=xa-yb+i(xb+ya)=(az-by)+i(bx+ay)=(a+ib)(x+iy)$. The middle $=$ uses commutativity of real numbers.

2.8.a If (u,v) is an additive identity, i.e. a complex number such that $(x,y)+(u,v)=(x,y)$, then we can subtract (x,y) from both sides to get $(u,v)=(0,0)$.

2.8.b We know that $(1,0)$ is *a* multiplicative identity and that if $(x,y)\neq(0,0)$ then there is some $(x,y)^{-1}$ so that $(x,y)^{-1}(x,y)=(1,0)$. If (u,v) is also a multiplicative identity, then $(x,y)(u,v)=(x,y)$ for all (x,y). So we can suppose $(x,y)\neq(0,0)$ and multiply by $(x,y)^{-1}$. This results in $(u,v)=(1,0)$.

2.11 $z^2+z+1=(x+iy)^2+(x+iy)+1=x^2-y^2+2xyi+x+iy+1=x^2-y^2+x+1+i(2xy+y)$. If this is 0 then $x^2-y^2+x+1=0$ and $2xy+y=0$. Using the hint, we can divide the second equation by y to get $2x+1=0$ so $x=-1/2$. Plugging in to the other equation results in $y=\pm\sqrt{3}/2$

3.1a $\frac{1+2i}{3-4i}+\frac{2-i}{5i}=\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)}-i\frac{2-i}{5}=\frac{-5+10i}{25}+\frac{-2i-1}{5}=\frac{-1+2i-2i-1}{5}=-2/5$

3.5 Just multiply and simplify

5.1.a Picture problem

5.1.d Picture problem

5.5.a Circle of radius 1 centered at $1-i$

5.5.b Closed disk of radius 3 centered at $-i$

5.5.c Exterior of the open disk of radius 4 centered at $4i$

5.8 Multiply out the left side then factor.

5.9 Base case: $|z^1| = |z| = |z|^1$. Induction step: Suppose $|z^n| = |z|^n$ up to some $n \geq 1$. Then $|z^{n+1}| = |z^n z| = |z^n||z| = |z|^n|z| = |z|^{n+1}$. The second $=$ is Exercise 8. The third is the induction hypothesis.

6.1.a $\overline{\bar{z} + 3i} = \bar{\bar{z}} + \overline{3i} = z - 3i$

6.1.b $\overline{iz} = \overline{ix - y} = \overline{-y + ix} = -y - ix = -i(x - iy) = -i\bar{z}$

6.1.c $\overline{(2+i)^2} = \overline{3 + 4i} = 3 - 4i$

6.2.a $Re(\bar{z} - i) = Re(x - iy - i) = Re(x - i(y+1)) = x = Re(z)$. So this is the line $x = 2$

6.2.b $|2\bar{z} + i| = 2|\bar{z} + i/2| = 2|\overline{\bar{z} + i/2}| = 2|z - i/2|$. So the equation is the same as $|z - i/2| = 2$. So this is a circle of radius 2 centered at $i/2$.

6.3 Write out

6.10.a If z is real then $z = x + i0 = x - i0 = \bar{z}$. If $\bar{z} = z$, then $x - iy = x + iy$, so $2iy = 0$, so $y = 0$, so z is real.

6.10.b If z is real then $\bar{z}^2 = x^2 = z^2$. If z is pure imaginary then $\bar{z}^2 = (-iy)^2 = (iy)^2 = z^2$. Conversely, if $\bar{z}^2 = z^2$ then $(x+iy)^2 = (x-iy)^2$ so $x^2 - y^2 + i(2xy) = x^2 - iy^2 - i2xy$. So $2xy = -2xy$ or $4xy = 0$. This implies that either x or y is 0, so z is real or pure imaginary.

9.2.a $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

9.2.b $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

Homework 2

9.1.a $\frac{-2}{1+\sqrt{3}i} = \frac{-2+2\sqrt{3}i}{4}$, so $\tan\theta = -\sqrt{3}$. Therefore θ has the form $-\pi/3 + k\pi$ for some k. Since z is in quadrant II, $Arg(z) = 2\pi/3$

9.1.b $(\sqrt{3} - i)^6 = (4e^{-i\pi/6})^6 = 4^6 e^{-\pi i}$. So $Arg(z) = \pi$.

9.10a By de Moivre's formula, $\cos(3\theta) = Re((\cos\theta + i\sin\theta)^3)$. But

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= (\cos\theta + i\sin\theta)(\cos^2\theta - \sin^2\theta + i2\cos\theta\sin\theta)\\ &= \cos^3\theta - \cos\theta\sin^2\theta - 2\cos\theta\sin^2\theta + i[\cdots]\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i[\cdots]\end{aligned}$$

11.1.a $z = 2i = 2e^{i\pi/2}$, so the square roots are $\sqrt{2}e^{i\pi/4} = \sqrt{2}(\sqrt{2}/2 + i\sqrt{2}/2) = 1 + i$ and $-(1 + i) = -1 - i$

11.1.b $z = 1 - \sqrt{3}i = 2e^{-i\pi/3}$, so the square roots are $\pm\sqrt{2}e^{-i\pi/6} = \pm(\sqrt{2}(\sqrt{3}/2 - i/2)) = \pm(\sqrt{3} - i)/\sqrt{2}$

11.2 $z = -8i = 8e^{-i\pi/2}$. So $|z^{1/3}| = 2$ and $\arg(z^{1/3}) = -\pi/6 + k2\pi/3$. These are $2e^{-i\pi/6} = \sqrt{3} - i$, $2e^{i\pi/2} = 3i$ and $2e^{i7\pi/6} = 2(-\sqrt{3}/2 + -i/2) = -\sqrt{3} - i$.

11.3 $-8 - 8\sqrt{3}i = 16e^{-i2\pi/3}$. So $|z^{1/4}| = 2$ and $\arg(z^{1/4}) = -\pi/6 + k2\pi/4 = -\pi/6 + k\pi/2$. Converting back to rectangular coordinates gives (working counterclockwise $\sqrt{3} - i$, $1 + \sqrt{3}i$, $-\sqrt{3} + i$,$-1 - \sqrt{3}i$

11.4.a $(-1) = 1e^{i\pi}$, so $(-1)^{1/3} = e^{i\pi/3 + ik2\pi/3}$. In rectangular coordinates, these are $e^{i\pi/3} = 1/2 + i\sqrt{3}/2$, $e^{i\pi} = -1$, and $e^{i5pi/3} = e^{-i\pi/3} = 1/2 - i\sqrt{3}/2$

11.4.b $8^{1/6} = (8e^{i0})^{1/6} = \sqrt{2}e^{ik2\pi/6}$. These are $\sqrt{2}(1/2 \pm i\sqrt{3}/2)$, $\sqrt{2}(-1/2 \pm i\sqrt{3}/2)$, $\sqrt{2}$, and $-\sqrt{2}$.

5.2 $x \le |x|$ is true for all real numbers, so $Re(z) \le |Re(z)|$ and $Im(z) \le |Im(z)|$. For the others, $0 \le x^2 \le x^2 + y^2$, so, taking square roots, $|x| \le \sqrt{x^2 + y^2}$, so $Re(z) \le |z|$, and similarly for Im

5.3 Using the preceding exercise and the triangle inequality, $Re(z_1 + z_2) \le |z_1 + z_2| \le |z_1| + |z_2|$. From the alternative form of the triangle inequality, $|x_3 + z_4| \ge ||z_3| - |z_4||$. Since $|z_3| \ne |z_4|$, neither of these values is 0 (WHY?!). So $1/|x_3 + z_4| \le 1/||z_3| - |z_4||$. No multiply the two inequalities.

Show that a set is closed according to the book definition if and only if it is the complement of an open set. Let $S \subset \mathbb{C}$. Let I be the interior points of S, E the exterior points, and B the boundary points. From the book definitions, $I \subset S$ and $E \subset \mathbb{C} - S$, so I and E are disjoint. Furthermore, B is defined to be $\mathbb{C} - (E \cup I)$. So E, I, and B form a partition of $\mathbb{C}$ (they are pairwise disjoint and their union is $\mathbb{C}$). We also observe that boundary points are those points such that every ϵ-neighborhood intersects both S and $\mathbb{C} - S$. So, by symmetry, the boundary points of S are also the boundary points of $\mathbb{C} - S$. Furthermore, by the definition of E, if $z \in E$ then z contains a neighborhood in $\mathbb{C} - S = E$. So the points of E are the interior points of $\mathbb{C} - S$ and similarly the points of I are the exterior points of $\mathbb{C} - S$.

Putting this all together, we see that $B \subset S$ if and only if $\mathbb{C} - S = E$ if and only if $\mathbb{C} - S$ contains none of its boundary points if and only if $\mathbb{C} - S$ is open (by the book definition).

Note that this discussion also demonstrates the claim from class that a set S is open if and only if every point $z \in S$ has an ϵ-neighborhood also contained in S: The points that have this property are exactly the interior points of S, and so saying that every point of S has this property is the same as saying that $S = I$, which is the same as saying that S contains no boundary points, which is the book's definition of open.

12.1 a. Closed disk of radius 1 centered at $2-i$, not a domain; b. points outside of circle of radius 2 centered at $-3/2$, is a domain; c. Half play $y > 1$, is a domain; d. line $y = 1$, not a domain; e. 45 degree wedge from the origin, origin not included; not a domain; f. Half plane $x \leq 2$, not a domain

12.2 e

12.3 a

12.4 a. whole plane, b. whole plane, d. $\arg(z)$ is in $[-\pi/4, \pi/4]$ or $[3\pi/4, 5\pi/4]$

Homework 3

12.8 Suppose there is a boundary point z of S that is not contained in S. Since it's a boundary point, every ϵ-neighborhood of z contains a point of S. Since z is not in S, it must in fact be that every deleted ϵ-neighborhood of z contains a point of S. So z is an accumulation point of S. But this implies that $z \in S$, a contradiction. So all boundary points of S are in S and thus S is closed.

12.9 If z_0 is a point in a domain S, then since the domain is open there is an ϵ such that the disk $|z-z_0| < \epsilon$ is contained in S, so certainly the deleted neighborhood $0 < |z-z_0| < \epsilon$ is contained in S. Any smaller deleted neighborhood around z_0 is thus also contained in S, and any larger disk intersects S in at least one point that is not z_0. Thus every deleted neighborhood around z_0 intersects S, so z_0 is an accumulation point.

14.5 There are two such domains. One of them is bounded on the left by the right branch of the hyperbola $x^2-y^2 = 1$, on the right by the right branch of the hyperbola $x^2-y^2 = 2$, on the bottom by the top branch of the hyperbola $xy = 1/2$, and on the top by the right branch of the hyperbola $xy = 1$.

14.8 a) quarter disk in the first quadrant given by $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$, b) similar but with $0 \leq \theta \leq 3\pi/4$, c) upper half disk

Find an equation satisfied by all the complex numbers that are taken to the line $x = 1$ under the map $w = z^3$. Without looking it up or using any algebraic geometry you should be able to give a rough sketch of this set. $x^3 - 3xy^2 = 1$; picture discussed in class

18.1.a Let $z_0 = x_0 + iy_0$ and $z = x + iy$. Let $f(z) = Re(z) = x$. Let $\epsilon > 0$. We need to show that there is a δ such that $|Re(z) - x_0| = |x - x_0| < \epsilon$ whenever $0 \leq |z - z_0| \leq \delta$. Recall that for any complex number a we know that $|Re(a)| \leq |a|$. So in particular it is always true that $|x - x_0| \leq |z - z_0|$. So if we choose $\delta = \epsilon$ then $|z - z_0| < \epsilon = \delta$ implies that $|x - x_0| < \epsilon$ as desired.

18.1.c Fix any $\epsilon > 0$. We need to show that there is a δ such that $0 < |z - 0| = |z| < \delta$ implies $|\bar{z}^2/z - 0| = |\bar{z}^2/z| < \epsilon$. But from the properties of conjugates and moduli, $|\bar{z}^2/z| = |\bar{z}|^2/|z| = |z|^2/|z| = |z|$. So again we can take $\delta = \epsilon$, and if $|z| < \epsilon$, so is $|\bar{z}^2/z| = |z|$.

18.5 If $z = x$ then $f(z) = (x/x)^2 = 1$. If $z = iy$, then $f(z) = (iy/(-iy))^2 = (-1)^2 = 1$. But if $y = z$, then $f(z) = (\frac{x+ix}{x-ix})^2 = (\frac{2ix^2}{2x^2})^2 = -1$. So there are points arbitrarily close to 0 that evaluate to 1 and also points arbitrarily close to 0 that evaluate to -1. So the limit cannot exist.

18.6.b Suppose $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Let $\epsilon > 0$. Then there are a δ and Δ such that $0 < |z - z_0| < \delta$ implies $|f(z) - w_0| < \epsilon/2$ and $0 < |z - z_0| < \Delta$ implies $|F(z) - W_0| < \epsilon/2$. Choose δ_1 so that $0 < \delta_1 < \delta$ and $0 < \delta_1 < \Delta$. Then if $|z - z_0 < \delta_1|$, we have $|f(z) + F(z) - (w_0 + W_0)| = |f(z) - w_0 + F(z) - W_0| \leq |f(z) - w_0| + |F(z) - W_0| < \epsilon/2 + \epsilon/2 = \epsilon$. So δ_1 works. Since ϵ was arbitrary, this completes the argument.

Homework 4

A.II. By definition, $\lim_{z\to z_0} f(z) = w_0$ means that for all $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$. But these are exactly the same formulas as for showing that $\lim_{z\to 0}(f(z) - w_0) = 0$.

18.10.a We look at $\lim_{z\to 0} \frac{4/z^2}{((1/z)-1)^2}$. Multiplying by z^2/z^2 we get $\lim_{z\to 0} \frac{4}{(1-z)^2} = 4$.

18.10.b We use that $\lim_{z\to 1}(z - 1)^3 = 0$.

18.10.c We look at $\frac{(1/z)-1}{(1/z^2)+1} = \frac{z-z^2}{(1+z^2)}$, which goes to 0 as z goes to 0.

18.10.13 By definition, S is bounded if there is an R such that $|z| < R$ for all $z \in S$. So unbounded means that for all R there is a $z \in S$ with $|z| > R$. But this means precisely that every neighborhood of infinity (which as the form $|z| > R$) contains a point of S.

20.1 $\lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z} \lim_{\Delta z\to 0} 2z + \Delta z = 2z$

20.8.a We look at $\frac{\Delta w}{\Delta z} = \frac{Re(z+\Delta z) - Re(z)}{\Delta z} = \frac{Re(z) + Re(\Delta z) - Re(z)}{\Delta z} = \frac{Re(\Delta z)}{\Delta z}$. When Δz is real, this is 1. When Δz is pure imaginary this is 0. So there can't be any limit as $\Delta z \to 0$.

20.9 Note that we're only asked about the derivative at $z_0 = 0$. So in this case $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{\overline{\Delta z}^2/\Delta z}{\Delta z} = \frac{\overline{\Delta z}^2}{(\Delta z)^2}$. When Δz is real, this is $\frac{(\Delta z)^2}{(\Delta z)^2} = 1$. When Δz is pure imaginary, this is $\frac{(-\Delta z)^2}{(\Delta z)^2} = 1$. If $\Delta x = \Delta y$, this becomes $\frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \left(\frac{1-i}{1+i}\right)^2 = \left(\frac{(1-i)^2}{2}\right)^2 = \left(\frac{-2i}{2}\right)^2 = -1$. So there can't be a limit as $\Delta z \to 0$.

24.1.b $z - \bar{z} = 2iy$. So $v_y = 1 \neq 0 = -u_x$. So f is not differentiable.

24.1.c $u_x = 2$ and $v_y = 2xy$, so f can't be differentiable unless $xy = 1$. Also $u_y = 0$ and $-v_x = iy^2$, so f can't be differentiable unless $y = 0$. Since $y = 0$ and $xy = 1$ can't both happen, f is not differentiable.

24.1.d $f = e^x \cos y - ie^x \sin y$. So $u_x = e^x \cos y$ and $v_y = -e^x \cos y$. These are equal only if $\cos y = 0$. Also $u_y = -e^x \sin y$ and $-v_x = e^x \sin y$, and these are equal only when $\sin y = 0$. Since $\sin y$ and $\cos y$ cannot be 0 simultaneously, there are no points where f is differentiable.

24.2.b Check that the Cauchy-Riemann equations are satisfied for $f = u + iv$ and that the partial derivatives are all continous. Then we can write $f' = u_x + iv_x$ and perform the same check for this function. The result is that $f'' = u_{xx} + iv_{xx} = e^{-x} \cos y - ie^{-x} \sin(y) = f$.

24.3.a $1/z = \frac{x-iy}{x^2+y^2}$. So $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. So $u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, $u_y = \frac{-2xy}{(x^2+y^2)^2}$, $v_x = \frac{2xy}{(x^2+y^2)^2}$, $v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$. So the CR equations are satisfied everywhere the derivatives exist, but they do not exist at 0 (in fact $z = 0$ is not in the domain of the function). The partials are also continuous everywhere but at 0. So we have $f' = u_x + iv_x = \frac{y^2-x^2+2xyi}{(x^2+y^2)^2}$. We can recognize the top as $\overline{(-z^2)} = -\bar{z}^2$ and the denominator as $|z|^4 = z^2\bar{z}^2$. So the quotient is $-1/z^2$.

24.3.b $u = x^2$ and $v = y^2$, so $u_x = 2x$, $u_y = 0$, $v_x = 0$ and $v_y = 2y$. So CR implies $x = y$, and these are continuous everywhere, so df/dz is defined along $y = x$ where it equals $u_x + iv_x = 2x$.

24.4.a $f = \frac{1}{r^4}e^{-4\theta i} = \frac{1}{r^4}(\cos(4\theta) - i\sin(4\theta))$. So $u_r = \frac{-4}{r^5}\cos(4\theta))$, $u_\theta = \frac{-4}{r^4}\sin(4\theta))$, $v_r = \frac{4}{r^5}(\sin(4\theta))$ and $v_\theta = \frac{-4}{r^4}(\cos(4\theta))$. Thus $ru_r = v_\theta$ and $u_\theta = -rv_r$, and these are all continuous (for $r > 0$). So the derivative exists and is $e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{-4}{r^5}\cos(4\theta) + i\frac{4}{r^5}(\sin(4\theta))) = e^{-i\theta}\frac{-4}{r^5}(\cos(4\theta) - i\sin(4\theta)) = e^{-i\theta}\frac{-4}{r^5}e^{-4i\theta} = \frac{-4}{r^5}e^{-5i\theta} = \frac{-4}{z^5}$

24.4.b $u_r = -e^{-\theta}\sin(\ln r)/r$, $u_\theta = -e^{-\theta}\cos(\ln r)$, $v_r = e^{-\theta}\cos(\ln r)/r$, $v\theta = -e^{-\theta}\sin(\ln r)$. So polar CR is satisfied and the partials are continuous for $r > 0$ and $f' = e^{-\theta}(u_r + iv_r) = e^{-i\theta}(-e^{-\theta}\sin(\ln r)/r + ie^{-\theta}\cos(\ln r)/r) = \frac{-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)}{re^{i\theta}} = if(z)/z$.

Homework 5

26.1.c $u_x = e^{-y}\cos x$, $u_y = -e^{-y}\sin x$, $v_x = e^{-y}\sin x$, $v_y = e^{-y}\cos x$. So the Cauchy-Riemann equations hold and the partials are continuous everywhere in the plane. So f' is defined everywhere, and f is entire.

26.2.a $u_x = y$ and $v_y = 1$, so $u_x = v_y$ is only possible when $y = 1$. Thus no point in the plane can have a neighborhood on which f' is defined, so f cannot be analytic.

26.4.a The the numerator and denominator share no common factor and the denominator is 0 at $0, i, -i$, so f is not defined at these points. By the quotient rule, the derivative exists at all other points. So $0, i, -i$ are singular points and the function is analytic everywhere else.

26.6 It's simplest to use the polar form of the Cauchy-Riemann equations. We see that $u_r = 1/r$, $u_\theta = 0$, $v_r = 0$ and $v_\theta = 1$. So the polar Cauchy-Riemann equations are satisfied and the partials are continuous. Thus the function is defined on the domain (notice that the domain is carefully chosen so that θ is well defined without ambiguity.

Now, from our geometrical understanding of functions, z^2 takes the open first quadrant to the upper half plane and adding one then shifts everything one unit to the right. So, as the suggestion notes, $Im(z^2 + 1) > 0$ for z in the open first quadrant, i.e. $0 < \text{Arg}(z^2 + 1) < \pi$. Since this is in the domain of g, the composition rule tells us that $g(z^2 + 1)$ is analytic.

The rest follows from the CR formula for the derivative of g and from the chain rule.

26.7 If f is real valued and analytic on D, then $f = \bar{f}$, so $\bar{f}$ is also analytic on D. The result now follows from Example 3 in Section 26.

27.2 The formulas $u(x, y) = c_1$ and $v(x, y) = c_2$ determine curves that locally take the for $y = y(x)$. As noted in the suggestion, we have from the multivariable chain rule (differentiating $u(x, y)$ with respect to x) that $u_x + u_y \frac{dy}{dx} = 0$ and $v_x + v_y \frac{dy}{dx} = 0$.

Find all real values of a, b, c, d so that $ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic. Let $u = ax^3 + bx^2y + cxy^2 + dy^3$. Then $u_x = 3ax^2 + 2bxy + cy^2$ and $u_{xx} = 6ax + 2by$. Similarly, $u_y = bx^2 + 2cxy + 3dy^2$ and $u_{yy} = 2cx + 6dy$. So u is harmonic if and only if $6a = -2c$ and $2b = -6d$, i.e. $c = -3a$ and $b = -3d$. So any function of the form $u = ax^3 - 3dx^2y - 3axy^2 + dy^3$ is harmonic.

Show by hand that $u = x^3 - 3x^2y - 3xy^2 + y^3$ is harmonic. Find a v so that $f = u + iv$ is entire (hint: use the Cuachy-Riemann equations). $u_x = 3x^2 - 6xy - 3y^2$ and $u_{xx} = 6x - 6y$. $u_y = -3x^2 - 6xy + 3y^2$ and $u_{yy} = -6x + 6y$. So u is harmonic.

By the Cauchy-Riemann equations, to find our desired v we need $v_x = -u_y = 3x^2 + 6xy - 3y^2$ and $v_y = u_x = 3x^2 - 6xy - 3y^2$. Integrating v_x with respect to x, we see that we must have $v = x^3 + 3x^2y - 3xy^2 + g(y)$ for some function g depending only on y. Taking the y derivative of this, we must have $v_y = 3x^2 - 6xy + \frac{dg}{dy}(y)$. Comparing with the v_y we must have from the Cauchy-Riemann equations, we see that $\frac{dg}{dy}(y) = -3y^2$. So $g(y) = -y^3 + C$. So we see that $v = x^3 + 3x^2y - 3xy^2 - y^3 + C$ satisfies the Cauchy-Riemann equations with u, and everything in sight is continuous. So any such v works.

Homework 6

30.1.b $e^{\frac{2+\pi i}{4}} = e^{1/2}e^{\pi i/4} = \sqrt{e}(\cos \pi/4 + i \sin \pi/4) = \sqrt{e}(1/\sqrt{2} + i/\sqrt{2}) = \sqrt{e/2}(1 + i)$

30.3 $e^{\bar{z}} = e^x(\cos y - i\sin y)$. So $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = -e^x \sin y$, $v_y = -e^x \cos y$. If CR holds, then $\cos y = -\cos y$, so $\cos y = 1/2$ and $\sin y = -\sin y$, so $\sin y = 1/2$. Since $\sin y$ and $\cos y$ are never both $1/2$ for the same y, $e^{\bar{z}}$ cannot be differentiable anywhere.

30.6 $|e^{z^2}| = |e^{x^2-y^2+i2xy}| = |e^{x^2-y^2}||e^{i2xy}| = e^{x^2-y^2} = e^{x^2+y^2}e^{-y^2} = e^{|z^2|}e^{-y^2}$. Since $-y^2 \leq 0$ we hace $e^{-y^2} \leq 1$. So $|e^{z^2}| \leq e^{|z^2|}e^{-y^2}$.

30.8.a $e^z = -2$ implies that $e^x(\cos y + i\sin y) = -2$. Since -2 is real, $\sin y = 0$, so y is a multiple of π. Since $-2 < 0$, we must have $\cos y < 0$, which means that is an odd multiple of π, i.e. $y = (2n+1)\pi$. Then $e^z = -e^x = -2$, so $e^x = 2$, which implies that $x = \ln 2$.

30.11 As $x \to -\infty$, e^z moves toward 0 along a ray from the origin. As $y \to \infty$, e^z moves clockwise around a circle of radius e^x centered at the origin an infinite number of times.

38.7 for $|\sin z|^2$ Using (13), $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x \cosh^2 y - \sin^2 x \sinh^2 y + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y(\sin^2 x + \cos^2 x) = \sin^2 x + \sinh^2 y$.

Show that $\sin^2 z + \cos^2 z = 1$ $\sin^2 z + \cos^2 z = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 = \frac{e^{2iz}+e^{-2iz}-2}{-4} + \frac{e^{2iz}+e^{-i2z}+2}{4} = 4/4 = 1$

Show that $\overline{e^z} = e^{\bar{z}}$: $\overline{e^z} = \overline{e^x \cos y + ie^x \sin y} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}}$.

Show that $\overline{\sin z} = \sin \bar{z}$ **and** $\overline{\cos z} = \cos \bar{z}$**.** Using the last problem and basic properties of conjugates, $\overline{\sin z} = \frac{e^{-i\bar{z}}-e^{i\bar{z}}}{-2i} = \frac{e^{i\bar{z}}-e^{-i\bar{z}}}{2i} = \sin \bar{z}$. cos is similar.

33.1.a $|-ei| = e$ and $\arg(-ie) = -\pi/2 + 2\pi n$, so $Log(-ei) = \ln e + i\theta = 1 + i(-\pi/2)$

33.2.c $|-1+\sqrt{3}i| = 2$ and $\arg(-1+\sqrt{3}i) = 2\pi/3 + 2\pi n$, so $\log(-1+\sqrt{3}i) = \ln 2 + i(2\pi/3 + 2\pi n)$

33.4 With the chosen branch, $\log(i^2) = \log(-1) = \pi i$, while $2\log i = 2(i5\pi/2) = i5\pi$

33.9 In general, we have $e^z = e^{x+iy} = e^x e^{iy}$, so $|e^z| = e^x$ and $\arg(e^z) = y + 2\pi n$. If we use the branch $\alpha < \theta < \alpha + 2\pi$ for log, then we have that $\log(e^z) = \ln e^x + i\Theta$, where Θ is the value of $\arg(e^z) = y + 2\pi n$ with $\alpha < \Theta < \alpha + 2\pi$. But with the assumption, this is precisely y. So $\log e^z = x + iy = z$.

33.10.a By definition/branch cuts, $\text{Log}(z)$ is analytic so long as z does not lie on the part of the real axis with $x \leq 0$. Thus, since the composition of anlaytic functions is analytic where it is defined (and $\text{Log}\, z$ and $z - i$ are analytic) $\text{Log}(z-i)$ is thus analytic so long as $z - i$ is not on the non-positive x-axis, which is equivalent to z not being on the line $y = 1$ with $x \leq 0$.

33.11 One way to do this is by direct computation of derivatives. Alternatively, $\ln(x^2+y^2)$ is the real part of $2\log z$ for any branch cut. For any branch cut, $2\log z$ is analytic in its domain, and so its real part is harmonic in that domain, which covers all of the plane except one ray. If we choose a different branch cut, then we see that $\ln(x^2+y^2)$ is also harmonic on the ray, except for at 0. Since being harmonic is a local property and we have verified it at all non-zero points of the plane, the function is harmonic on $\mathbb{C}-\{0\}$.

Homework 7

34.3 Let $z_1 = -\sqrt{2}/2 + i\sqrt{2}/2$, and let $z_2 = -i$. Then $z_1/z_2 = -\sqrt{2}/2 - i\sqrt{2}/2$, so $\mathrm{Log}(z_1/1_z) = -i3\pi/4$. But $\mathrm{Log}(z_1) - \mathrm{Log}(z_2) = i3\pi/4 - (-i\pi/2) = i5\pi/4$.

36.1.a $(1+i)^i = e^{i\log(1+i)} = e^{i(\ln\sqrt{2}+i(\pi/4+2\pi n))} = e^{-\pi/4+2\pi n}e^{i\ln\sqrt{2}} = e^{-\pi/4+2\pi n}e^{i(\ln 2)/2}$

36.2.a Principal value of $(-i)^i = e^{i\,\mathrm{Log}(-i)} = e^{i(-\pi i/2)} = e^{\pi/2}$

36.2.c Principal value of $(1-i)^{4i} = e^{4i\,\mathrm{Log}(1-i)} = e^{4i(\ln\sqrt{2}-\pi i/4)} = e^{\pi+i2\ln(2)} = e^{\pi}(\cos(2\ln 2)+i\sin(2\ln 2))$

36.6 If a is real then $|z^a| = |e^{a\log z}| = |e^{a(\ln|z|)+ai\arg(z)}| = |e^{a\ln|z|}e^{ai\arg z}| = |e^{a\ln|z|}| = |e^{\ln|z|^a}| = ||z|^a|$. But if we take the principal value of $|z|^a$ then since $|z|$ is a positive real number we have $|z|^a = e^{a\,\mathrm{Log}|z|} = e^{a\ln|z|}$ as *real* numbers, and this is the usual real number $|z|^a$, which is also positive. So $||z|^a| = |z|^a$.

36.8.a Using principal values, we have $z^{c_1}z^{c_2} = e^{c_1\mathrm{Log}\,z}e^{c_2\mathrm{Log}\,z}$. As noted in Section 35, when we use principal values e^z is exactly the function we're used to working with, so, in particular, $e^{c_1\mathrm{Log}\,z}e^{c_2\mathrm{Log}\,z} = e^{c_1\mathrm{Log}\,z+c_2\mathrm{Log}\,z} = e^{(c_1+c_2)\mathrm{Log}\,z}$, which is the principal branch of $z^{c_1+c_2}$.

42.2.a $\int_0^1(1+it)^2\,dt = \int_0^1 1+2it-t^2\,dt = t+it^2-t^3/3|_0^1 = 1+i-1/3-(0) = \frac{2}{3}+i$

42.2.c $\int_0^{\pi/6} e^{i2t}dt = \frac{e^{i2t}}{2i}|_0^{\pi/6} = \frac{e^{i\pi/3-1}}{2i} = \frac{1}{2i}(\frac{-1}{2}+i\frac{\sqrt{3}}{2} = \frac{\sqrt{3}+i}{4}$

42.4 $\int_0^{\pi} e^{(1+i)x}\,dx = \frac{e^{(1+i)x}}{1+i}|_0^{\pi} = \frac{e^{(1+i)\pi}-1}{1+i} = \frac{-e^{\pi}-1}{1+i} = \frac{1}{2}(-e^{\pi}-1)(1-i)$. So $\int_0^{\pi} e^x\cos x\,dx = \frac{1}{2}(-e^{\pi}-1)$ and $\int_0^{\pi} e^x\sin x\,dx = \frac{1}{2}(e^{\pi}+1)$.

43.1.a Let $\tau = -t$. Then $d\tau = -dt$ and $\int_{-b}^{-a} w(-t)dt = -\int_{-b}^{-a} w(\tau)d\tau = \int_a^b w(\tau)d\tau$

46.1.a $z' = 2ie^{i\theta}$ so $\int_C \frac{z+2}{z}\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}2ie^{i\theta}\,d\theta = \int_0^{\pi} 2ie^{i\theta}+2i\,d\theta = 2e^{i\theta}+2i\theta|_0^{\pi} = -2+2\pi i-2 = -4+2\pi i$

46.2.b Take $z(t)=t$, $0\le t\le 2$. Then $\int_C z-1\,dz = \int_0^2 t-1\,dt = t^2/2-t|_0^2 = 2-2-0 = 0$

46.4 Our parametrization is $z = t+it^3$ but we need to consider two pieces, $-1\le t\le 0$ and $0\le t\le 1$. We also have $z' = 1+3t^2i$. The first part gives us $\int_{-1}^0 1(1+3t^2i)\,dt = t+t^3i|_{-1}^0 = -(-1-i) = 1+i$. The second part is $\int_0^1 4t^3(1+3t^2i)\,dt = \int_0^1 4t^3+12t^5i\,dt = t^4+2t^6i|_0^1 = 1+2i$. So altogether we have $1+i+1+2i = 2+3i$

46.5 Given any $z(t)$, $a \le t \le b$, we have $\int_C 1\,dz = \int_a^b \frac{dz}{dt}\,dt = z|_a^b = z(b) - z(a) = z_2 - z_1$

46.6 Note that the contour only has an endpoint on the branch cut. So we use $z' = ie^{i\theta}$ and the integral is $\int_0^\pi e^{i\,\mathrm{Log}\,z} ie^{i\theta} d\theta = \int_0^\pi e^{i(i\theta)} ie^{i\theta} d\theta = \int_0^\pi e^{-\theta} ie^{i\theta} d\theta = \int_0^\pi e^{i\theta-\theta} i d\theta = \int_0^\pi e^{(i-1)\theta} i d\theta = \frac{ie^{(i-1)\theta}}{i-1}|_0^\pi = \frac{ie^{\pi i - \pi} - i}{i-1} = \frac{-ie^{-\pi} - i}{i-1} = \frac{(-1-i)(-ie^{-\pi}-i)}{2} = \frac{(-1+i)(e^{-\pi}+1)}{2}$

Homework 8

43.5 Let $w(t) = f(z(t)) = u(x(t), y(t)) + iv(x(t), y(t))$. Then by the chain rule, $\frac{dw}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\frac{\partial v}{\partial x}\frac{dx}{dt} + i\frac{\partial v}{\partial y}\frac{dy}{dt}$. Writing $u_x, u_y, v_x, v_y, x', y'$ for the appropriate derivatives and using Cauchy-Riemann, this is $u_x x' + u_y y' + i(v_x x' + v_y y') = u_x x' - v_x y' + i(v_x x' + u_x y') = (u_x + iv_x)(x' + iy') = \frac{df}{dz}\frac{dz}{dt}$ as desired.

47.1.a On part of the circle of radius 2, $|z+4| \le |z|+4 = 2+4 = 6$ and $|z^3-1| \ge ||z|^3-1| = 7$. The length of the quarter circle is $2\pi \cdot 2/4 = \pi$. So by the Theorem in this section, the integral is $\le 6\pi/7$.

47.2 As noted in the suggestion, the minimum $|z|$ on the curve occurs at the midpoint $1/2 + i/2$ and so is $\sqrt{1/2}$. Thus $|1/z^4| \ge 1/(\sqrt{1/2})^4 = 4$. The length of the line segment is $\sqrt{2}$ by the Pythagorean theorem. So the integral is $\le 4\sqrt{2}$.

47.4 On the circle of radius R we have that $|2z^2-1| \le 2|z|^2+1 = 2R^2+1$ while $|z^4+5z^2+4| = |z^2+4||z^2+1| \ge ||z|^2-4|||z|^2-1| = (R^2-4)(R^2-1)$. So $|\frac{2z^2-1}{z^4+5z^2+4}| \le \frac{2R^2+1}{(R^2-4)(R^2-1)}$ and the integral is $\le \frac{2R^2+1}{(R^2-4)(R^2-1)}\pi R$. As $R \to \infty$ this expression goes to 0.

47.7 $|x + i\sqrt{1-x^2}\cos\theta| = \sqrt{x^2 + (1-x^2)\cos^2\theta}$. Since $1 - x^2 \ge 0$ by the assumption $|x| \le 1$, we have $(1-x^2)\cos^2\theta \le 1 - x^2$. So $|x + i\sqrt{1-x^2}\cos\theta| \le x^2 + 1 - x^2 = 1$. So $|(x + i\sqrt{1-x^2}\cos\theta)^n| = |x + i\sqrt{1-x^2}\cos\theta|^n \le 1^n = 1$. So the modulus of the integral is $\le \pi$ and therefore $|P_n(x)| \le 1$.

49.1 For n a nonnegative integer, z^n is entire with antiderivative $F(z) = \frac{z^{n+1}}{n+1}$ in the whole plane. So a contour integral from z_1 to z_2 of z^n is $F(z_2) - F(z_1) = \frac{z_2^{n+1} - z_1^{n+1}}{n+1}$.

49.2.b $2\sin(z/2)$ is an antiderivative of $\cos(z/2)$. So the integral is $2(\sin(\pi/2+i) - \sin(0)) = 2\sin(\pi/2+i) = \frac{e^{i(\pi/2+i)} - e^{-i(\pi/2+i)}}{i} = \frac{e^{-1+i\pi/2} - e^{1-i\pi/2}}{i} = \frac{e^{-1}i - e(-i)}{i} = e^{-1} + e$

49.3 Since the case $n = 0$ is omited, $(z - z_0)^{n-1}$ has antiderivative $\frac{(z-z_0)^n}{n}$ on the domain $\mathbb{C} - \{z_0\}$ (if $n > 0$ this is an antiderivative on all of $\mathbb{C}$). As long as a contour does not pass through z_0, the integrand is defined and continuous on the contour. So why the Theorem in Section 48, the integral is 0 when the contour is also closed.

49.4 As observed in the text, the branch of $z^{1/2}$ using $\pi/2 < \theta < 5\pi/2$ is defined at all points of any contour that lies below the real axis except for its endpoints at -3 and 3, and for all points on such a contour the values of this branch aree with those of the branch $0 < \theta < 2\pi$, except at the point 3. But as one point does not affect the value of the integral, we can use the $\pi/2 < \theta < 5\pi/2$ branch to compute. Furthermore, this branch of $z^{1/2}$ has antiderivative $z^{3/2}/(3/2)$ (with the same branch choice) on a domain containing the contour. So by the Theorem in Section 48, the contour integral is $\frac{2}{3}(3^{3/2} - (-3)^{3/2}$. For this branch, $3^{3/2} = (3e^{2\pi i})^{3/2} = 3^{3/2}e^{3\pi i} = -3^{3/2}$ while $(-3)^{3/2} = (3e^{\pi i})^{3/2} = 3^{3/2}e^{i3\pi/2} = -i3^{3/2}$. So the integral is $\frac{2/3}{(} - 3^{3/2} + i3^{3/2}) = 2(-\sqrt{3} + i\sqrt{3})$.

49.5 As in the suggestion, if we consider the branch of z^i with $-\pi/2 \arg z < 3\pi/2$ to write $z^i = e^{i \log z}$ then this agrees with $e^{i \operatorname{Log} z}$ on the entirety of any contour that lies about the real axis (except for its endpoints). So the integral is the same treating z^i either way. But now using our branch, z^i has antiderivative $z^{i+1}/(i+1)$ (using the same branch to define z^{i+1}) and so by the Theorem in Section 48, the integral is $(1^{i+1} - (-1)^{i+1})/(i+1)$. Using our branch, we compute $1^{i+1} = e^{(i+1)\log 1} = e^{(i+1)(0+0i)} = e^0 = 1$, while $(-1)^{i+1} = e^{(i+1)\log(-1)} = e^{(i+1)(\ln 1 + i\pi)} = e^{-\pi + i\pi} = -e^{-\pi}$. So the integral is $\frac{1+e^{-\pi}}{i+1} = \frac{1+e^{-\pi}}{2}(1-i)$

Homework 9

53.1.a This function is analytic except at $z = -3$. So it is analytic on and inside the circle, so the integral is 0.

53.1.c This function only fails to be analytic at $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. These points are outside the disk, so by Cauchy-Goursat the integral is 0.

53.1.f f fails to be analytic at the points $\{x + iy | x \leq -2, y = 0\}$. These don't intersection the unit disk, so by Cauchy-Goursat the integral is 0.

53.2.a This function is analytic except where $z = \pm\sqrt{1}3i$. So, for example, f is analytic on the domain $|z| > 0.6$. Thus the Theorem (more precisely the Corollary) from Section 53 applies.

53.3 $z_0 = 2 + i$ is in the interior of the rectangle described, and for all n the function $(z - 2 - i)^{n-1}$ is analytic on $\mathbb{C} - \{z_0\}$. So by the Corollary of Section 53, the integral on C_0 equals the integral on C. The result follows from the C_0 integral computations given.

53.6 We can't use Cauchy-Goursat here because the square root isn't well defined at 0, so the hypotheses of the theorem aren't satisfied. However, we compute the pieces. On the semicircle we use $z = e^{i\theta}$ so $z' = ie^{i\theta}$ and we have $\int_0^\pi e^{i\theta/2} ie^{i\theta} d\theta = i\int_0^\pi e^{i3\theta/2} d\theta = e^{i3\theta/2} 2/3 = (2/3)(e^{i3\pi/2} - 1) = (2/3)(-i - 1) = -2/3 - i2/3$. For the part on the positive x axis we have $\int_0^1 \sqrt{t}\, dt = (2/3)t^{3/2}|_0^1 = 2/3$. For the negative real part we have $\int_{-1}^0 \sqrt{-t} e^{i\pi/2}\, dt = i2/3$. Adding these three pieces, we get 0.

53.7 This comes from Green's Theorem. With $f = x - iy$, Green's Theorem (see (4) in Section 50) says that $\int_C u - iv dz = \iint_R v_x - u_y + i(u_x + v_y) dA = \iint_R 0 - 0 + i(1+1)\, dA = \iint R 2i\, dA = 2i \cdot \text{Area}(R)$.

Homework 10

57.1.a $2\pi i e^{-\pi i/2} = 2\pi i(-i) = 2\pi$

57.1.b $2\pi i \frac{\cos(0)}{0^2+8} = \pi i/4$

57.1.c $\frac{z}{2z+1} = \frac{z/2}{z+1/2}$ so the answer is $2\pi i(-1/4) = -\pi i/2$

57.1.e $2\pi i \frac{d}{dz}\tan(z/2) = 2\pi i \sec^2(z/2)/2$. So the answer is $\pi i \sec^2(x_0/2)$

57.2.a $z^2+4 = (z+2i)(z-2i)$. $2i$ is in the circle but $-2i$ isn't. So the integral is $2\pi i$ times $\frac{1}{z+2i}$ evaluated at $2i$. So the answer is $2\pi i/4i = \pi/2$

57.3 $g(2) = \int_C \frac{2s^2-s-2}{s-2}\,ds$, so $g(2) = 2\pi i(2(2)^2-2-2) = 2\pi i4 = 8\pi i$. If $|z|>3$ then the integrand is analytic on and inside the contour so $g(z)=0$.

57.5 If z_0 is outside the contour, both sides are 0. If z_0 is inside the contour and C is positively oriented then from the Cauchy formulas both sides are $f'(z_0)$, using that f analytic implies that f' is analytic. If C is negatively oriented, then the integrals reverse signs but are still equal.

57.7 From the integral formula, the integral is $2\pi i e^{a0} = 2\pi i$. In terms of θ, we let $z = e^{i\theta}$ so that $z' = ie^{i\theta}$. Then the integral is $\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)} ie^{i\theta}/e^{i\theta} d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta))d\theta$. Since $e^{a\cos\theta}\sin(a\sin\theta)$ is an odd function, its integral from $-\pi$ to π is 0. So we get $i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi i$. Also $e^{a\cos\theta}\cos(a\sin\theta)$ is even so its integral from $-\pi$ to 0 is equal to the integral from 0 to π. So $\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi$

57.10 Let z_0 be a point in the plane, and let C_R be the circle of radius R around z_0. The points on C_R have the form $z_0 + Re^{i\theta}$. Then on C_R we have $|f(z)| \leq A|z_0 + Re^{i\theta}| \leq A(|z_0|+R)$. So $f''(z_0) \leq 2A(|z_0|+R)/R^2$. As R goes to ∞, we see that $f''(z_0) = 0$. So $f'(z)$ is constant, i.e. $f'(z) = a_1$. Since f is an antiderivative of f', this implies $f(z) = a_1 z + C$. But $f(0)$ must be 0 for $|f(z)| \leq A|z|$ to hold, $f(z) = a_1 z$.

59.1 If f is entire, so is $e^f = e^{u+iv} = e^u e^{iv}$. So if u is bounded, then so is $|e^f| = e^u$. So then e^f is constant. So $|e^f| = e^u$ is constant. So $u = ln e^u$ is constant.

59.3 Let R be the region $|z| \leq 1$. Then if $f(z) = z$, $|f(z)| = |z| = 0$ at $z=0$, but $|f(z)| = |z| > 0$ for $z \neq 0$. So $|f|$ has a minimum in the interior of R.

59.8.a Just multiply and cancel

59.8.b By part a, we can write $z^k - z_0^k = (z-z_0)P_{k-1}(z)$, where P_{k-1} is a polynomial of degree $k-1$.

$$\begin{aligned} P(z) - P(z_0) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n - a_0 - a_1 z_0 - a_2 z_0^2 - \cdots - a_n z_0^n \\ &= a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z^n - z_0^n) \\ &= a_1(z-z_0) + a_2(z-z_0)P_1(z) + \cdots + a_n(z-z_0)P_{n-1}(z) \\ &= (z-z_0)(a_1 + a_2 P_1(z) + \cdots + a_n P_{n-1}(z) \\ &= (z-z_0)Q(z) \end{aligned}$$

When $P(z_0) = 0$ we see that $P(z) = (z-z_0)Q(z)$, as required.

Suppose $f(z)$ is entire and $|f(z)| \geq 1$ for all z. Show that f is constant. Since $|f(z)| \geq 1$, we have $|\frac{1}{f(z)}| \leq 1$, so $|\frac{1}{f(z)}|$ is bounded. It is entire since $f(z)$ is never 0. So $1/f(z)$ is constant by Liouville's Theorem. So $f(z)$ is also constant.

Let R be a closed bounded region of the plane. Suppose f and g are continuous on R and analytic in the interior of R. Show that if $f = g$ on the boundary of R then $f = g$ on all of R. Consider $f - g$, which is 0 on the boundary. By the corollary to the maximum modulus principle, the maximum of $|f - g|$ is on the boundary of R, so $|f - g|$ must be 0 on all of R. So $f = g$. Technically that corollary requires $f - g$ not be constant, but if $f - g$ is constant, then since it's 0 on the boundary it's 0 everywhere in R so again $f = g$.

What is the maximum of $|e^{iz^2}|$ on the disk $|z| \leq 1$. From the Maximum Modulus Principle, we know that the maximum must be on the boundary. So we consider $|e^{iz^2}| = |e^{i(x^2-y^2+2ixy)}| = e^{-2xy}$ on the circle. This will have its maximum where $-2xy$ has its maximum on the circle. Using Calc III methods (either parametrize the curve or use Lagrange multipliers), the maximum will be where $y = -x$ on the circle, i.e. $(1/\sqrt{2})(1 - i)$ and $(1/\sqrt{2})(-1 + i)$. Then $e^{-2xy} = e$.

61.1 We want $|(\frac{1}{n^2} + i) - i| = |\frac{1}{n^2}| < \epsilon$. This will be true so long as $n > 1/\sqrt{\epsilon}$

Show directly from the definitions that if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$. Let $S_m = \sum_{n=1}^{m} a_n$ and $T_m = \sum_{n=1}^{m} b_n$. Then by definition for all $\epsilon > 0$ there are M_1 and M_2 so such that $|S_m - A| < \epsilon$ for $m > M_1$ and $|T_m - B| < \epsilon$ for $m > M_2$. By taking the larger of M_1, M_2, we see there is a single M so that $|S_m - A| < \epsilon$ and $|T_m - B| < \epsilon$ simultaneously for $m > M$. Now let $U_m = S_m + T_m = \sum_{n=1}^{m}(a_n + b_n)$ (we can do this because these sums are finite). Then for $M > m$, $|U_m - (A + B)| = |S_m + T_m - (A + B)| = |S_m - A + T_m - B| \leq |S_m - A| + |T_m - B|$. If we choose M so that $|S_m - A|, |T_m - B| < \epsilon/2$ for $m > M$, then we have $|U_m - (A + B)| < \epsilon$ for $m > M$. This shows that U_m converges to $A + B$ as desired.

Homework 11

65.2.b $e^z = ee^{z-1} = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$. This applies for all $z \in \mathbb{C}$.

65.4 $\cos z = -\sin(z - \pi/2) = -\sum_{n=0}^{\infty} \frac{(z-\pi/2)^{2n+1}}{(2n+1)!}$. This holds everywhere in $\mathbb{C}$.

65.8.a $\cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}(\sum \frac{(iz)^n}{n!} + \sum \frac{(-iz)^n}{n!}) = \frac{1}{2}(\sum \frac{(iz)^n}{n!} + \sum \frac{(-)^n(iz)^n}{n!})$. Using that the terms are negatives of each other for n odd and equal to each other for n even this becomes $\frac{1}{2}(\sum \frac{2(iz)^{2n}}{(2n)!}) = \sum \frac{(iz)^{2n}}{(2n)!} = \sum \frac{(i)^{2n}z^{2n}}{(2n)!} = \sum \frac{(-1)^n z^{2n}}{(2n)!}$.

65.9 $f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{4n+2}}{(2n+1)!}$. Since this only has powers of the form z^{4n+2}, all of the terms $z^{4n}, z^{4n+1}, z^{4n+3}$ must have trivial coefficients, so $f^{(4n)}(0) = 0$ and similarly for the others. Notice that $4n+1$ and $4n+3$ together give all the positive odd integers, so we can restate that condition as vanishing for all $2n + 1$.

Find a Maclaurin series for $\frac{z^3}{z^2+16}$. On what set does this converge? $\frac{z^3}{z^2+16} = \frac{z^3}{16}\frac{1}{1-\frac{z^2}{-16}} = \frac{z^3}{16}\sum_{n=0}^{\infty}(\frac{z^2}{-16})^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+3}}{(16)^{n+1}}$. This converges when $|z^2/16| < 1$, i.e. when $|z| < 4$

Find a Taylor series for $\frac{z}{1-z}$ centered at $z = 3$. What is the region of convergence?

$$\begin{aligned}
\frac{z}{1-z} &= \frac{z-3+3}{-2-(z-3)} \\
&= \frac{z-3}{-2-(z-3)} + \frac{3}{-2-(z-3)} \\
&= \frac{z-3}{-2}\frac{1}{1-\frac{z-3}{-2}} + \frac{3}{-2}\frac{1}{1-\frac{z-3}{-2}} \\
&= \frac{z-3}{-2}\sum_{n=0}^{\infty}\left(\frac{z-3}{-2}\right)^n + \frac{3}{-2}\sum_{n=0}^{\infty}\left(\frac{z-3}{-2}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{z-3}{-2}\right)^{n+1} + \sum_{n=0}^{\infty}\frac{3\,(z-3)^n}{(-2)^{n+1}} \\
&= \sum_{n=1}^{\infty}\left(\frac{z-3}{-2}\right)^n + \frac{3}{-2} + \sum_{n=1}^{\infty}\frac{3}{-2}\frac{(z-3)^n}{(-2)^n} \\
&= \frac{-3}{2} + \sum_{n=1}^{\infty}\left(1+\frac{-3}{2}\right)\frac{(z-3)^n}{(-2)^n} \\
&= \frac{-3}{2} + \sum_{n=1}^{\infty}\frac{-1}{2}\frac{(z-3)^n}{(-2)^n} \\
&= \frac{-3}{2} + \sum_{n=1}^{\infty}\frac{(z-3)^n}{(-2)^{n+1}}
\end{aligned}$$

This converges for $|z-3| < 2$

65.10 $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$ so $\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots$ and $\frac{\sin z^2}{z^4} = z^{-1} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots$

65.11

$$\begin{aligned}\frac{1}{4z-z^2} &= \frac{1}{4z}\frac{1}{1-\frac{z}{4}} \\ &= \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}} \\ &= \frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}\end{aligned}$$

Find Laurent expansions about $z_0 = 0$ for $\frac{1}{z^3-4z}$ on the regions $0 < |z| < 2$ and $|z| > 2$. When $0 < |z| < 2$ we have $\frac{1}{z^3-4z} = \frac{-1}{4z}\frac{1}{1-(\frac{z}{2})^2} = \frac{-1}{4z}\sum_{n=0}^{\infty}(\frac{z^2}{4})^n = \sum_{n=0}^{\infty}\frac{-z^{2n-1}}{4^{n+1}}$. When $|z| > 2$, $\frac{1}{z^3-4z} = \frac{1}{z^3}\frac{1}{1-\frac{4}{z^2}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{4}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{4^n}{z^{2n+3}}$

68.5 (just for D_2) $f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{-1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$

Let C be the contour $|z| = 2$ oriented positively. Compute $\int_C z\cos(1/z)\,dz$. We have the Laurent series $z\cos(1/z) = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n} = z\cos(1/z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n-1}}$. So $b_1 = -1/2$. Thus the integral is $2\pi i(-1/2) = -\pi i$.

Homework 12

72.1 Differentiating both sides of the first equation, we get $\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n$. Differentiating again, $\frac{2}{(1-z)^3} = \sum_{n=1}^{\infty} n(n+1)z^{n-1} = \sum_{n=0}^{\infty}(n+1)(n+2)z^n$. This is all valid only within the circle of convergence $|z| < 1$

72.4 Using the Taylor series for $\cos z$, which is entire, $1 - \cos z = 1 - \left(\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty}\frac{z^{2n}}{(2n)!}$. So $(1-\cos z)/z^2 = \sum_{n=1}^{\infty}\frac{z^{2n-2}}{(2n)!}$. At $z = 0$, this is $1/2$, so this series represents $f(z)$ on all of $\mathbb{C}$. Since the series for $\cos z$ converges for all z, our new series also converges for all z (for each fixed z, multiplying by $\frac{1}{z^2}$ is multiplication by a constant and so doesn't affect the convergence). Since the series converges everywhere, it represents an entire function by the corollary in section 71.

72.6 Let C be a contour from 1 to z where z satisfies $|z-1| < 1$. Then we can integrate both sides along this contour. On the left we get $\int_C \frac{1}{w}\,dw = \operatorname{Log} w|_1^z = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z$. on the right we have $\int_C \sum_{n=0}^{\infty}(-1)^n(w-1)^n\,dw = \sum_{n=0}^{\infty}\int_C(-1)^n(w-1)^n\,dw = \sum_{n=0}^{\infty}(-1)^n\frac{(w-1)^{n+1}}{n+1}|_1^z = \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{n}$.

72.7 Away from $z = 1$ on the described domain, $f(z)$ is analytic by the analyticity of Log and the quotient rule. Inside the circle $|z-1| < 1$ (and only here), $f(z) =$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^{n-1}}{n} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{n+1}$. Note that this agrees with $f(1) = 1$, and the series absolutely converges inside the circle by comparison to the geometric series $\sum |z-1|^n$. So $f(z)$ is analytic at 1 also.

Show that if f is analytic in a domain except for finitely many singular points then they are all isolated singularities. Let $z_1, \ldots, z_n$ be the singular points. Without loss of generality, consider z_1. For each $i > 1$, let $R_i = |z_1 - z_i|$. Since there are finitely many singular points, there is a minimum $R = \min\{R_2, \ldots, R_n\}$. Now choose ϵ so that $0 < \epsilon < R$ and so that z_1 has an ϵ neighborhood in the domain (this is possible because domains are open). Then f is analytic on the deleted ϵ neighborhood around z_1, so z_1 is an isolated singular point. The argument is the same for the other singular points.

77.1.a $\frac{1}{z+z^2} = \frac{1}{z}\frac{1}{1+z} = \frac{1}{z}(1 - z + \cdots) = \frac{1}{z} - 1 + \cdots$ for $|z| < 1$. So the residue at 0 is 1.

77.1.b $z\cos(1/z) = z(1 - \frac{1}{2z^2} + \cdots) = z - \frac{1}{2z} + \cdots$ on the whole plane. So the residue is $-1/2$.

77.1.c $\frac{z-\sin z}{z} = \frac{1}{z}(z - (z - z^3/6 + \cdots)) = \frac{1}{z}(z^3/6 + \cdots) = z^2/6 + \cdots$. So the residue is 0.

77.2.a $\frac{e^{-z}}{z^2} = \frac{1}{z^2}(1 - z + z^2/2 + \cdots) = \frac{1/z^2}{-}\frac{1}{z} + \cdots$. So the integral is $2\pi i(-1) = -2\pi i$.

77.2.d There are singularities at 0 and 2. By partial fractions, $\frac{z+1}{z^2-2z} = \frac{-1}{2z} + \frac{3}{2(z-2)}$. At $z = 0$, the second function is analytic, so the residue is $-1/2$. At $z = 2$, the first function is analytic so the residue is $3/2$. So the integral is $2\pi i(-1/2 + 3/2) = 2\pi i$

Let C be the positively oriented circle $|z| = 5$. Compute $\int_C \frac{\sin z}{(z-\pi)^2}$. To find the residue at π, we need to expand $\sin z$ in powers of $z - \pi$. We use $\sin(z) = \sin(z - \pi + \pi) = \sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi = -\sin(z-\pi) = -(z-\pi) + (z-\pi)^3/6 - \cdots$. So $\frac{\sin z}{(z-\pi)^2} = -1/(z-\pi) + \cdots$. So the residue is -1 and the integral is $-2\pi i$.

77.3 Looking at $\frac{f(1/z)}{z^2}$ we have $\frac{1}{z^2}\frac{4/z-5}{(1/z)(1/z-1)} = \frac{1}{z^2}\frac{z(4-5z)}{1-z} = \frac{4-5z}{z(1-z)}$. The residue at 0 is $\frac{4-5(0)}{(1-0)} = 4$, so the integral is $8\pi i$.

77.4.a $\frac{1}{z^2}f(1/z) = \frac{1}{z^2}\frac{1/z^5}{1-1/z^3} = \frac{1}{z^7}\frac{1}{1-1/z^3} = \frac{1}{z^4}\frac{-1}{1-z^3} = \frac{-1}{z^4}\sum_{n=0}^{\infty} z^{3n}$. So the residue is -1 and the integral is $-2\pi i$.

77.4.b $\frac{1}{z^2}f(1/z) = \frac{1}{z^2}\frac{1}{1+1/z^2} = \frac{1}{z^2+1}$. This is analytic at 0 so the residue at 0 is 0. So the integral is 0.

77.7 Consider $P(1/z)/Q(1/z)$. The biggest power of $1/z$ in the expression is $1/z^m$. Multiplying top and bottom by z^m, we get a polynomial with non-zero constant term on the bottom and a polynomial with 0 constant term and linear term on the top. Dividing by z^2, the numerator is still a polynomial and the denominator is a polynomial with non-zero constant term. So $\frac{1}{z^2}\frac{P(1/z)}{Q(1/z)}$ is analytic at 0, so its residue at 0 is 0 and the integral is 0.

79.1.a The principal part is $\sum_{n=2}^{\infty} \frac{1}{z^{n-1}n!}$. This singularity is essential.

79.1.b Note that $z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1$. So $\frac{z^2}{z+1} = (z+1) - 2 + \frac{1}{z+1}$. So the principal part is $\frac{1}{z+1}$ and this is a simple pole.

79.1.c $\sin z/z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}$. So this is a removable singularity.

79.2b $\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}(1-\sum_{n=0}^{\infty} 2^n z^n/n!) = \frac{1}{z^4}(-\sum_{n=1}^{\infty} 2^n z^n/n!)$. So the principal part is $-2/z^3 - 2/z^2 - 8/6z$. So the pole has order 3 and residue $-4/3$.

Homework 13

Section 83.1 From the Maclaurin series we know that $\sin z$ has a zero of order 1. So by the Theorem in section 83, the residue is $\frac{1}{\cos 0} = 1$.

83.2 We have $q(z) = 1 - \cos z$, $q'(z) = \sin z$, and $q''(z) = \cos z$. So $q(0) = q'(0) = 0$ but $q''(0) = 1$. So 0 is a zero of order 2.

83.4.a We can write $z \sec z$ as $z/\cos z$. Since $f(z) = \cos z$ is 0 as $\pi/2 + n\pi$ but $f'(z) = -\sin z$ is not zero at this points, we see that $\cos z$ has zeros of degree 1 at these points. So from theorem, the residues are $\frac{z}{\sin z}$ evaluated at these points. For $z_n = \pi/2 + \pi n$ we have $\sin(z_n) = (-1)^n$, so the residues are $(-1)^n z_n$ as desired.

83.5.a $\tan z = \sin z/\cos z$ has singularities at $\pm\pi/2$. From the theorem in section 83, the residue at these points is $\sin z/(-\sin z) = -1$ evaluated at these points, so each is just -1. So by the residue theorem, the integral is $2\pi i(-1-1) = -4\pi$

Homework 14

81.1.a $f(z) = \frac{z+1}{(z-3i)(z+3i)}$. So at $3i$ we have a simple pole with $\phi(z) = \frac{z+1}{z+3i}$ and residue $\phi(3i) = \frac{1+3i}{6i}$. And $-3i$ we have a simple pole with $\phi(z) = \frac{z+1}{z-3i}$ and residue $\phi(-3i) = \frac{1-3i}{-6i}$.

81.1.c $f(z) = \frac{z^3}{2^3(z+1/2)^3}$. So we have a pole of order 3 at $-1/2$ with $\phi(z) = z^3/8$. So the residue is $\phi''(-1/2)/2 = -3/16$.

81.2.a $(-1)^{1/4}$ is not 0, so we have a simple pole at -1 with residue $(-1)^{1/4} = (e^{\pi i})^{1/4} = e^{\pi i/4} = \sqrt{2}/2 + i\sqrt{2}$

81.2.b $f(z) = \frac{\operatorname{Log} z}{(z+i)^2(z-i)^2}$, so we have a pole of order 2 at i with $\phi(z) = \frac{\operatorname{Log} z}{(z+i)^2}$. The residue is $\phi'(i)$. $\phi' = \frac{(z+i)^2(1/z)-(\operatorname{Log} z)(2z+2i)}{(z+i)^4}$ so $\phi'(i) = \frac{(2i)^2(1/i)-(\operatorname{Log} i)(4i)}{(2i)^4} = \frac{4i+2\pi}{16} = \frac{\pi+2i}{8}$

81.3.b Note that ϕ is not $\frac{1}{e^z-1}$ because this is undefined at 0. Rather we observe that $e^z - 1 = \sum_{n=1}^{\infty} z^n/n! = z\sum_{n=0}^{\infty} z^n/(n+1)!$. So $f(z) = \frac{1}{z^2 g(z)}$ where $g(z) = \sum_{n=0}^{\infty} z^n/(n+1)!$ or $\phi(z)/z^2$ with $\phi = 1/g$. Thus we have a pole of order 2 and the residue is $\phi'(0)$. Then $\phi'(0) = -g'(0)/g^2(0)$. From the series for g we have $g(0) = 1$ while, differentiating term by term, $g' = \sum_{n=1}^{\infty} nz^{n-1}/(n+1)!$. So $g'(0) = 1/2$. Thus $\phi'(0) = -1/2$.

81.5.a The contour only goes around the pole -4, where the residue is $1/4^3 = 1/64$. So the integral is $2\pi i/64 = \pi i/32$.

81.5.b The contour goes around both poles. The integral is $2\pi i$ times the sum of the residues. At -4 we have a simple pole with residue $1/(-4)^3 = -1/64$. At 0 we have a pole of order 3 and so need $\phi''(0)/2$. Here $\phi = 1/(z+4)$, so $\phi' = -1/(z+4)^2$ and $\phi''/2 = 1/(z+4)^3$. At 0 this is $1/64$. So the integral is $2\pi i(1/64 - 1/64) = 0$

83.11 By contradiction, assume there are an infinite number of zeros. Since the contour together with the region inside it constitute a closed bounded region, by the Bolzano-Weierstrass Theorem, the set of zeros must have an accumulation point, i.e. a point so that there is a zero in every deleted neighborhood.

This accumulation point cannot be a pole: In a neighborhood of a pole $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ for some ϕ that is analytic at z_0 and with $\phi(z_0) \neq 0$. But then if z_i is a sequence of zeros of f for which z_0 is a limit point we must have $\phi(z_i) = 0$ in this neighborhood. So by continuity we also have $\phi(z_0) = 0$, a contradiction.

So the accumulation point is not a pole but rather a point z_0 at which f is analytic and again by continuity $f(z_0) = 0$.

But now by (the contrapositive of) the theorem in section 82, this situation can only happen is f is identically zero in a neighborhood of the accumulation point. But then the accumulation point is a zero of infinite order (and/or on the boundary), contradicting the assumption that all the zeros have finite order and are interior to C.

86.1 Since the integrand is even, we start with the PV integral $\int_{-\infty}^{\infty} f(x)\,dx$. Letting $f(z) = 1/(z^2+1) = \frac{1}{(z-i)(z+i)}$, this has poles at $\pm i$. The residue at i is $1/2i = -i/2$. So integrating around a semi-circular contour, the PV integral will be $2\pi i(-i/2) = \pi$ and the desired integral will be $\pi/2$ if we can show that the integral around the top part of the contour goes to 0. Since $|z^2+1| \geq ||z|^2 - 1|$, on the semi circle of radius R we'll have $f(z) \leq \frac{1}{R^2-1}$. So the integral of the top semicircle is $\leq \frac{1}{R^2-1}\pi R$. This goes to 0 as R goes to infinity.

86.2 The procedure is essentially the same as in the preceding problem but now $f(z) = \frac{1}{(z-i)^2(z+i)^2}$. We have a pole of order 2 at i, and letting $\phi(z) = 1/(z+i)^2$, the residue will be $\phi'(i)$. Since $\phi' = -2/(z+i)^3$, we have $\phi'(i) = -2/(2i)^3 = -1/4i^3 = -1/-4i = -i/4$. So the PV integral will be $2\pi i(-i/4) = \pi/2$ and the desired integral will be $\pi/4$. For the top semicircle, we use that the maximum will be $\leq 1/(R^2-1)^2$ using the triangle inequality.

86.5 Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$. The poles in the upper half plane will be the simple poles at i and $2i$. The easiest way to find the residues here will be to use Theorem 2 in Section 83. In this case, $p/q' = \frac{z^2}{2z(z^2+4)+(z^2+1)2z} = \frac{z^2}{4z^3+1z} = fracz4z^2+10$. So the desired integral will be $\pi i(\frac{i}{4i^2+10} + \frac{2i}{4(2i)^2+10}) = \pi i(\frac{i}{6} + \frac{2i}{-6}) = \pi i(\frac{-i}{6}) = \pi/6$. The upper half circle integral is $\leq \frac{R^2}{(R^2-1)(R^2-4)}\pi R$, which goes to 0

86.9 First note that $1/(z^3+1)$ has simple poles at $e^{\pi i/3+2\pi ni/3}$. The only one of these in the contour is $e^{\pi i/3}$. Using the theorem from section 83, the residue there is $\frac{1}{3(e^{\pi i/3})^2} = \frac{1}{3e^{2\pi i/3}} = e^{-2\pi i/3}/3$. So the contour integral gives $2\pi i e^{-2\pi i/3}/3$. The integral along the circular arc of the contour is $\leq \frac{1}{R^3-1}\frac{2\pi R}{3}$ which goes to 0 as R goes to infinity. For the diagonal piece, letting the contour grow to infinity, we note that we can parameterize it in the wrong direction by $z(t) = te^{i2\pi/3}$, $0 \leq t < \infty$. Writing out the contour integral along this piece in the wrong direction using the parametrization we get $\int_0^\infty \frac{1}{(te^{i2\pi/3})^3+1} e^{i2\pi/3}\,dt = \int_0^\infty \frac{1}{t^3+1} e^{i2\pi/3}\,dt = e^{i2\pi/3}\int_0^\infty \frac{1}{t^3+1}\,dt$. Not that the integral here is the same one we're interested in evaluation and that we get along the positive x axis. However, the contribution to the contour is negative since we parameterized in the wrong direction. So altogether the contour integral has three parts: the circle part that goes to 0, the part $\int_0^\infty \frac{1}{x^3+1}\,dx$ and the diagonal part $-e^{i2\pi/3}\int_0^\infty \frac{1}{x^3+1}\,dx$.

Adding these up and using our residue computation from earlier, we have that $(1 - e^{i2\pi/3})\int_0^\infty \frac{1}{x^3+1} = 2\pi i e^{-2\pi i/3}/3$.

So the integral is $\frac{2\pi i e^{-2\pi i/3}}{3(1-e^{i2\pi/3})}$. This simplifies to $\frac{2\pi}{3\sqrt{3}}$ by basic arithmetic/trigonometry.

Homework 15

Let C be the curve —z—=2 oriented positively. For each of the following functions f(z), determine how many times (with sign) the image f(C) winds around the origin: 1) $f(z) = z^3$, winding number 3; 2) $f(z) = z^4/(z-1)^2$, winding number 2; 3) $f(z) = 1/(z^2+1)^2$, winding number -4

94.5 Suppose that f has a zero of degree m_k at z_k. Then near z_k, $f(z) = (z-z_k)^{m_k} g(z)$, where $g(z)$ is analytic at z_k and $f(z_k) \neq 0$. Then $f' = m_k(z-z_k)^{m_k-1}g(z) + (z-z_k)^{m_k}g'(z)$. Then $zf'/f = \frac{z(m_k(z-z_k)^{m_k-1}g(z)+(z-z_k)^{m_k}g'(z))}{(z-z_k)^{m_k}g(z)} = \frac{zm_k}{z-z_k} + \frac{zg'(z)}{g(z)}$. The second summand is analytic at z_k and, unless $z_k = 0$, the first has a simple pole at z_k with residue $m_k z_k$ (remember that if $\phi(z)$ is analytic at z_0 then the residue of $\phi(z)/(z-z_0)$ at z_0 is $\phi(z_0)$). If $z_k = 0$ then there is no pole, but this is consistent with the contribution to the formula being $m_k z_k = 0$. Since f is analytic, the only singularities of zf'/f are where $f(z) = 0$. So from the residue theorem, the integral is $2\pi i \sum m_k z_k$.

94.6.c On $|z| = 1$, $|4z^3| = 4$, while $|z^7 + z - 1| \leq |z|^7 + |z| + 1 = 3$. So the full polynomial has the same number of zeros in the circle as $4z^3$, which has 3.

94.7.a On $|z| = 2$, we have $|9z^2| = 36$ while $|z^4 - 2z^3 + z - 1| \leq 2^4 + 2\cdot 2^3 + 2 + 1 = 35$. So the full polynomial has as many zeros in the circle as $9z^2$, which has 2.

94.9 On the circle, $|cz^n| = |c|$ while $|e^z| = e^x \leq e^1 = e$. So since $e < |c|$ by assumption, cz^n and $cz^n - e^z$ have the same number of zeros in the circle, which is n.