Certifying solutions using $\alpha$-theory

Consider the polynomial system $f : \mathbb{C}^N \to \mathbb{C}^N$ with Newton’s method

$$N_f(x) = \begin{cases} x - Jf(x)^{-1}f(x) & \text{if } Jf(x)^{-1} \text{ exists} \\ x & \text{otherwise} \end{cases}$$

A point $x$ is an approximate solution of $f = 0$ with associated solution $\xi$ if, for every $k \geq 1$,

$$\|x_k - \xi\| \leq \left(\frac{1}{2}\right)^{2^k-1} \|x - \xi\|$$

where $x_0 = x$ and $x_k = N_f(x_{k-1})$. That is, the sequence $\{x_k\}_{k=0}^\infty$ immediately converges quadratically to $\xi$. 
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\[ \alpha(f, x) := \beta(f, x) \gamma(f, x), \]
\[ \beta(f, x) := \|x - N_f(x)\| = \|Df(x)^{-1}f(x)\|, \text{ and} \]
\[ \gamma(f, x) := \sup_{k \geq 2} \left\| \frac{Df(x)^{-1}D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}. \]

**Theorem (Smale, Shub-Smale, . . .)**

If $\alpha(f, x) < \frac{13-3\sqrt{17}}{4} \approx 0.1577$, then $x$ is an approximate root of $f$.

In this case, if $\xi$ is the associated root, then

\[ \|x - \xi\| \leq 2\beta(f, x). \]