Certifying solutions using α-theory

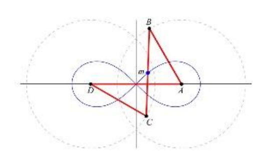
Consider the polynomial system $f : \mathbb{C}^N \to \mathbb{C}^N$ with Newton's method

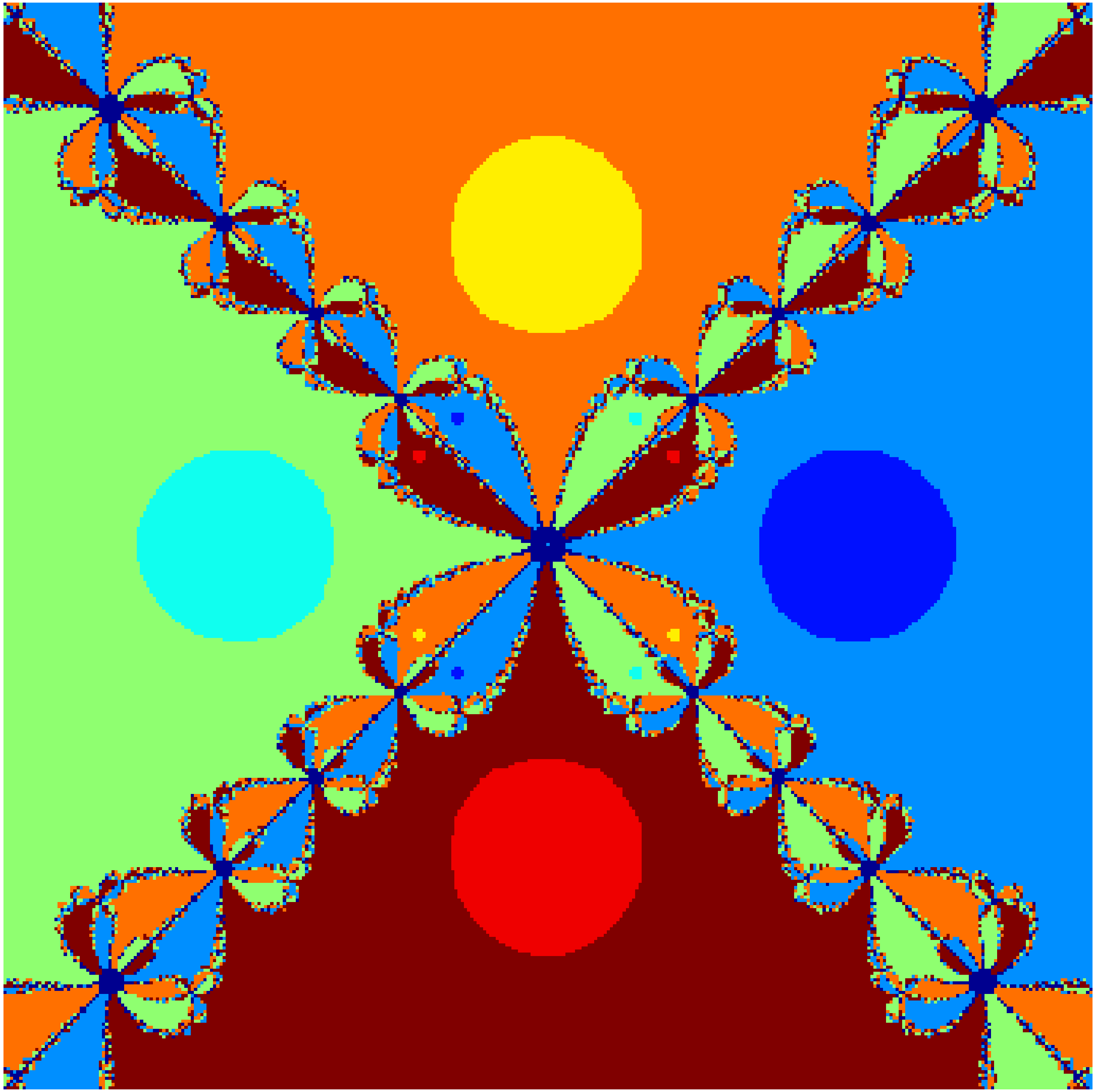
$$N_f(x) = \begin{cases} x - Jf(x)^{-1} f(x) & \text{if } Jf(x)^{-1} \text{ exists} \\ x & \text{otherwise} \end{cases}$$

A point x is an *approximate solution* of $f = 0$ with *associated solution* ξ if, for every $k \geq 1$,

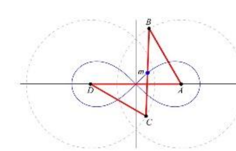
$$\|x_k - \xi\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \xi\|$$

where $x_0 = x$ and $x_k = N_f(x_{k-1})$. That is, the sequence $\{x_k\}_{k=0}^{\infty}$ immediately converges quadratically to ξ.





$z^4 - 1$



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$$
\begin{aligned}
\alpha(f,x) &:= \beta(f,x)\gamma(f,x)\,,\\
\beta(f,x) &:= \|x - N_f(x)\| = \|Df(x)^{-1}f(x)\|\,,\ \text{and}\\
\gamma(f,x) &:= \sup_{k\geq 2}\left\|\frac{Df(x)^{-1}D^kf(x)}{k!}\right\|^{\frac{1}{k-1}}\,.
\end{aligned}
$$

Theorem (Smale, Shub-Smale,. . .)

If $\alpha(f,x) < \frac{13-3\sqrt{17}}{4} \approx 0.1577$, then x is an approximate root of f. In this case, if ξ is the associated root, then

$$\|x - \xi\| \leq 2\beta(f,x).$$