Abstracts of Professor Cox's Talks

NSF/CBMS Regional Conference in the Mathematical Sciences
Applications of Polynomial Systems - June 4-8, 2018

The following description of planned lectures has been provided by Professor Cox. The talks are divided into five topics of two lectures each. Each such pair of lectures will be followed by a complementary lecture given by an expert in that subject area.

TOPIC I: ELIMINATION THEORY Elimination theory is an important part of symbolic algebra that can help one understand the structure of a system of polynomial equations. These two lectures will explore elimination theory from a historical point of view.

Lecture 1. Elimination Theory in the 18th and 19th Centuries. Although the origins of the subject trace back to the 17th century, our story begins about 100 years later with Bézout's 1779 book *Théorie Générale des Équations Algebriques* [3]. These days, Bézout's Theorem is usually regarded as part of enumerative algebraic geometry, but for Bézout and his contemporaries, elimination was at the heart of the matter. Although Bézout was not the first to state his theorem, his book studied many versions of the theorem, including ones for polynomials whose Newton polytopes are truncated simplices.

This lecture will also explore Minding's 1841 paper *Ueber die Bestimmung des Grades einer durch Elimination hervorgehenden Gleichung* [42], where he gave a formula for the number of solutions of $f(x, y) = g(x, y) = 0$ that can be seen as computing the mixed volume of two polygons by means of a mixed subdivision. Elimination also plays an important role in Kronecker's great 1882 paper *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* [40]. Another important development is the emergence of the theory of resultants.

Lecture 2. Elimination Theory in the 20th Century. The century began with the maturation of resultants (Netto [43]; Macaulay [41]) and the *Principle of Conservation of Number* (Severi [47]), which involves generic points and multiplicities. The first rigorous definitions of generic point and multiplicity used elimination theory and resultants (van der Waerden [51] and [52]).

As the century progressed, the development of algebra "eliminates from algebraic geometry the last traces of elimination theory" (Weil [55]), a trend amplified by the theory of schemes (Grothendieck and Dieudonné [28]). But the 1960s also witnessed the birth of Gröbner bases (Buchberger [7]) and computer algebra in general. A decade later, resultants made a comeback (Jouanolou [35]; Sederberg [44]) and exploded with the introduction of sparse resultants (Gel'fand, Kapranov and Zelevinsky [26]; Sturmfels [50]).

Two extended examples will give the flavor of these developments:

- The evolution of generic point, with definitions from van der Waerden in 1926–27, Weil in 1946, and Grothendieck and Dieudonné in 1961–67.

- The fundamental theorem of projective elimination theory, with proofs from van der Waerden's *Moderne Algebra* [53] and Hartshorne's *Algebraic Geometry* [31].

Follow-up Lecture. After Lectures 1 and 2, Carlos D'Andrea will lecture on elimination theory in the 21st century.

TOPIC 2: POLYNOMIAL SYSTEMS IN THE REAL WORLD These talks will discuss some numerical issues that can arise when trying to solve polynomial systems.

Lecture 3. Polynomial Systems via Numerical Linear Algebra. Basic facts from algebra and elimination theory become problematic in the presence of floating point numbers. This will be illustrated by examples involving the binomial theorem and a polynomial system from *Ideals, Varieties, and Algorithms* [16], where changing a coefficient from 2 to 2.1 or 2.01 turns a lovely solution via Gröbner bases into something much more complicated.

There are several paradigms for what is now called *numerical algebraic geometry.* The remainder of the lecture will explore the approach described in Stetter's *Numerical Polynomial Algebra* [49], focusing on systems with finitely many solutions over the complex numbers. Here are two of the main ideas:

- Solving such a polynomial system can be turned into an eigenvalue problem, allowing use of methods from numerical linear algebra.

- When the coefficients are floating point numbers, a solution will not satisfy the system exactly but will be regarded as valid provided it is an exact solution of a nearby system.

Other topics that will be discussed include real solutions, condition numbers, Gröbner bases, and border bases, all illustrated with numerous examples.

Lecture 4. Polynomial Systems via Homotopy Continuation. Another approach to numerical solving views an inexact system as a polynomial system whose coefficients are variables living in a parameter space. The idea is that the system has a stable behavior when the parameters are chosen generically in the parameter space.

Following [2, 48], the lecture will discuss homotopy continuation, which links the given system to a start system with known solutions, and explain the connection to Severi's Principle of Conservation of Number from 1912. Bézout's Theorem also enters when thinking about the solutions of the start system, and applying these ideas to some of Bézout's examples from 1779 leads to the idea of a polyhedral start system.

For systems with infinitely many solutions, it is common to represent a positive-dimensional component of the solution set using a *witness set*. As will be seen, this relates nicely to both elimination theory and the idea of a generic point.

Other topics that may be mentioned include the Stewart-Gough platform, recent work of Hauenstein and Sottile [32] on witness sets and Newton polyhedra, and Kukelova's award-winning construction [8] of numerical solvers in computer vision that takes an engineering approach to S-polynomials in the theory of Gröbner bases.

Follow-up Lecture. After Lectures 3 and 4, Jonathan Hauenstein will lecture on the research frontier in numerical algebraic geometry.

TOPIC 3: GEOMETRIC MODELING The interactions between geometric modeling, algebraic geometry, and commutative algebra are rich and varied. There are stories to tell about the relation between applied math and algebra, as well as some nontrivial commutative algebra.

Lecture 5. The Geometry and Algebra of Curve Parametrizations. In geometric modeling, curve parametrizations have many applications, including outline fonts and the design of automobiles, ships, and airplanes. Sometimes it is useful to have the equation of a parametrized curve, which is where elimination theory enters the picture.

In the mid 1990s, Sederberg and his coworkers introduced *moving lines* [45] and *moving curves* [46] to improve computational complexity and highlight the geometry. The surprise was the link between moving curves and syzygies, explored in [17], and more generally between moving curves and the Rees algebra, studied in [10, 12]. After introducing some of the key players and discussing the especially nice case of rational quartics [12], the lecture will survey some recent work with Iarrobino [14] and with Kustin, Polini, and Ulrich [15] about various ways to stratify the space of all parametrizations of a given degree.

Lecture 6. The Geometry and Algebra of Surface Parametrizations. Curves are nice, but surfaces are much more important in geometric modeling. They are also a lot harder – nice behavior in the curve case often becomes more complicated in the surface case. The lecture will begin with basepoints, a phenomenon not present in the curve case, and explain how basepoints (real or complex) influence the degree of the parametrized surface in $\mathbb{R}^3$.

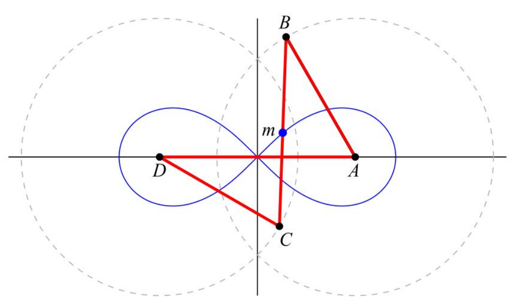
To give the flavor of how surface implicitization works, moving quadrics will be used to construct the equation of a basepoint-free rectangular surface parametrization [13]. There are also results when one allows nice basepoints [9], where "nice" means "local complete intersection." This gives a hint of the commutative algebra involved.

The adjective "rectangular" in the previous paragraph refers to the shape of the Newton polytope of the polynomials in the parametrization. When the Newton polytope is more complicated, various approaches are available. One can specify the Newton polytope in advance and use the *toric surface patches* introduced by Krasauskas [39]. Alternatively, one can make no assumptions in advance about the Newton polytope and develop methods that work for arbitrary parametrizations, as done in [6].

In the last decade, *matrix representation* has emerged as a useful tool that replaces the single (often complicated) equation of the surface with a matrix M of (often simpler) polynomials that drops rank precisely on the surface. This approach is taken in the paper [6] mentioned above. It will be seen that matrix representations can also be used to study surface intersections, following [1].

Follow-up Lecture. After Lectures 5 and 6, Hal Schenck will lecture on some recent interactions between geometric modeling and algebraic geometry.

TOPIC 4: GEOMETRIC CONSTRAINT THEORY This fascinating topic involves the study of configurations of rods and joints in $\mathbb{R}^2$ and $\mathbb{R}^3$. Here is a version of a linkage introduced by James Watt in 1784:



In this picture, $AD = BC = 2$, $AB = CD = \sqrt{2}$, A, D are fixed, and B, C rotate freely about their respective circles. The midpoint m of BC traces out the lemniscate shown above.

Given a rod and joint configuration, a key question is whether a given configuration is rigid, and if not, what motions are possible. Studying these questions involves Euclidean geometry, algebraic geometry, and combinatorics. And there are some great pictures.

Lecture 7. Configuration Spaces and Cayley-Menger Spaces. Given a graph $G = (V, E)$ and edge lengths $\ell : E \to \mathbb{R}_{>0}$, a realization in $\mathbb{R}^d$ is a function $\rho : V \to \mathbb{R}^d$ satisfying $|\rho(u) - \rho(v)| = \ell(uv)$ for all edges $uv \in E$. The set of all such bar and joint linkages is a real algebraic variety in the affine space $\mathbb{R}^{|V|}$. This construction has many variants, including constraints imposed by requiring that certain vertices map to preassigned points and certain angles between bars are specified in advance. The configuration space of the lemniscate linkage has two irreducible components, one of which is a resolution of singularities of the lemniscate.

A notable result of the 19th century was Kempe's 1875 proof [37] that any compact real algebraic curve in the plane can be generated by a linkage. From the modern point of view, the idea is (roughly) that any compact real algebraic variety is isomorphic to a suitably chosen configuration space, as proved by King [38] and Kapovich and Millson [36].

By construction, configuration spaces are invariant under the action of the affine isometry group. The quotient under this group action lives naturally in what is called a Cayley-Menger variety, studied in [4]. These varieties are determinantal and can be interpreted as secant varieties of Veronese embeddings. Some lovely algebraic geometry is happening here.

Lecture 8. Rigidity. A quadrilateral in the plane is not rigid – it is easily deformed without changing the edge lengths. On the other hand, a triangle in the plane is rigid. In general, the notion of rigidity has various flavors, including global rigidity, local rigidity, minimal rigidity, and infinitesimal rigidity. This lecture will introduce the key ideas, following White and Whitely [54]. Some of the tools that appear are what one might expect, such as the rigidity matrix whose kernel gives information about infinitesimal deformations, while others are more unexpected, such as the exterior algebra.

One goal of the lecture is to explain why the condition

$$|E| = d|V| - \tbinom{d+1}{2} \tag{1}$$

for realizations of $G = (V, E)$ in $\mathbb{R}^d$ is good from the rigidity point of view. As an illustration, suppose $d = 2$, so that (1) reduces to $|E| = 2|V| - 3$. In this case, a 1970 result of Laman states that a generic realization of G in the plane is minimally rigid if and only if every subgraph (V', E') of G satisfies $|E'| \leq 2|V'| - 3$ when $|E'| > 0$.

Time permitting, the lecture will conclude with a result of Borcea and Streinu [5], which says that if G satisfies (1) and has an infinitesimally rigid generic realization in $\mathbb{R}^d$ with $|V| \geq d + 1$, then the number of distinct realizations is at most

$$2 \prod_{k=0}^{|V|-d-2} \frac{\binom{|V|-1+k}{|V|-d-1-k}}{\binom{2k+1}{d}}.$$

The proof uses Bézout's Theorem on an appropriately chosen Cayley-Menger variety.

Follow-up Lecture. After Lectures 7 and 8, Jessica Sidman will lecture on current research in rigidity theory.

$\boxed{\text{TOPIC 5: CHEMICAL REACTION NETWORKS}}$ In modern biology, it is becoming increasingly more important to take a systems perspective when studying biological processes. This involves tools from many areas, including graph theory and algebraic geometry from mathematics.

Lecture 9. Mass Action Kinetics. The standard way of writing a chemical reaction leads to a system of ODEs. An example involving nitrogen N and oxygen O is

$$2NO + O_2 \xrightarrow{\kappa} 2NO_2 \tag{2}$$

with reaction rate κ. Applying the law of mass action leads to the system of ODEs

$$\begin{aligned} \tfrac{d[NO]}{dt} &= -2\kappa[NO]^2[O_2] \\ \tfrac{d[O_2]}{dt} &= -\kappa[NO]^2[O_2] \\ \tfrac{d[NO_2]}{dt} &= 2\kappa[NO]^2[O_2] \end{aligned} \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix}[NO]\\ [O_2]\\ [NO_2]\end{pmatrix} = \underbrace{\begin{pmatrix}2 & 0\\ 1 & 0\\ 0 & 2\end{pmatrix}}_{Y} \underbrace{\begin{pmatrix}- \kappa & 0\\ \kappa & 0\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}[NO]^2[O_2]\\ [N_2O]^2\end{pmatrix}}_{\Psi}, \tag{3}$$

where $[\cdots]$ denotes concentration, Ψ consists of monomials in the concentrations coming from the law of mass action, A is the transposed Laplacian of the weighted directed graph $\bullet \xrightarrow{\kappa} \bullet$ underlying (2), and Y records the integer coefficients appearing in (2).

This lecture will begin with the law of mass action (including its subtleties [29]) and the general form of (3) for the vector of concentrations $\mathbf{x}$, which following [22] is given by

$$\frac{d\mathbf{x}}{dt} = YA\Psi(\mathbf{x}). \tag{4}$$

The steady states of such a system can have biological significance and are positive real solutions of the system of polynomial equations

$$YA\Psi(\mathbf{x}) = \mathbf{0}. \tag{5}$$

Since reaction rates for individual reactions are typically unknown, they are usually regarded as parameters, similar to the numerical polynomial systems discussed in Lecture 4.

Methods used to study the system (5) include the matrix-tree theorem [29] and elimination theory [11]. About fifteen years ago, Gatermann noticed that the monomial map Ψ parametrizes a toric variety that ends up being a key player [23, 24, 25]. The lecture will conclude with a discussion of invariants of steady-state solutions, as studied in [20] and [30]. The paper [20] gives a splendid overview of this emerging application of algebraic geometry.

Lecture 10. Toric Dynamical Systems. In chemistry, each side of the reaction (2) is called a *complex*. These correspond to the vertices of the directed graph. The columns of the matrix Y in (4) and (5) record how each complex is built from the molecules appearing in the reactions. *Complex balancing* refers to the equation

$$A\Psi(\mathbf{x}) = \mathbf{0}. \tag{6}$$

Any solution of this equation gives a steady state solution of (4), and the system is said to be *complex balanced* if (6) has a real solution with positive entries. Intuitively, this means that there is a positive steady state solution such that the total incoming flow equals the total outgoing flow at each vertex (= complex) of the graph. The importance of complex balancing was identified in 1972 [34].

In recent years, the phenomenon of complex balancing has been observed for other systems, including recombination equations in population genetics and quadratic dynamical systems in computer science. Because of these connections outside of chemistry and the relation to toric geometry (via Gatermann), complex balanced systems were renamed in 2009 as *toric dynamical systems* [19].

For systems with reversible reactions and a property called *formally balanced,* complex balancing is equivalent to the existence of a positive solution of the equations

$$\kappa_{ij}\mathbf{x}^{y_i} = \kappa_{ji}\mathbf{x}^{y_j},$$

where y_i is the ith column of Y [21]. In general, equations of this sort define toric varieties (up to a twist), so that the toric connection is even more evident.

In 1972, Horn and Jackson [34] showed that in a complex balanced (= toric) system with weakly reversible reactions, each stoichiometric compability class contains a unique positive fixed point that is a local attractor. After some initial confusion, they realized that their proof did not show that it was a global attractor. This led to the formulation of the Global Attractor Conjecture (GAC) in 1974 [33]. Since then, the GAC has attracted a lot of attention in the research literature.

In 2015, Gheorge Craciun, one of the authors of [19], announced a proof [18] of the GAC. The strategy of [18] is to show that the conjecture is a consequence of a differential toric inclusion built from a suitable polyhedral fan. Craciun's proof is featured in the excellent article [27] in the July/August 2016 issue of SIAM News.

Follow-up Lecture. After Lectures 9 and 10, Alicia Dickenstein will lecture on the latest developments in chemical reaction network theory.

References

[1] T. Ba, L. Busé and B. Mourrain, *Curve/surface intersection problem by means of matrix representations*, in *Proceedings of the 2009 Conference on Symbolic Numeric Computation* (H. Kai and H. Sekigawa, Eds.), ACM, New York, 2009, 71-78.

[2] D. Bates, J. Hauenstein, A. Sommese and C. Wampler, *Numerically Solving Polynomial Systems with Bertini*, SIAM, Philadelphia, PA, 2013.

[3] E. Bézout, *Théorie générale des équations algebriques*, Ph.-D. Pierres, Paris, 1779.

[4] C. Borcea, *Point configurations and Cayley-Menger varieties*, (2002), `http://arxiv.org/abs/math/0207110`.

[5] C. Borcea and I. Streinu, *The number of embeddings of minimally rigid graphs*, Discrete Comput. Geom. **31** (2004), 287–303.

[6] N. Botbol and A. Dickenstein, *Implicitization of rational hypersurfaces via linear syzygies: a practical overview*, J. Symbolic Comput. **74** (2016), 493–512.

[7] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Ph.D. Thesis, University of Innsbruck, 1965.

[8] M. Bujnak, S. Kukelova and T. Pajdla, *Making minimal solvers fast*, 2013 IEEE Conference on Computer Vision and Pattern Recognition, 1506–1513.

[9] L. Busé, D. Cox and C. D'Andrea, *Implicitization of surfaces in $\mathbb{P}^3$ in the presence of base points*, J. Algebra Appl. **2** (2003), 189–214.

[10] L. Busé and J.-P. Jouanolou, *On the closed image of a rational map and the implicitization problem*, J. Algebra **265** (2003), 312–357.

[11] V. Bykov, A. Kytmanov, and M. Lazman M., with the assistance of M. Passare, *Elimination Methods in Polynomial Computer Algebra*, Mathematics and Its Applications **448**, Kluwer, Dordrecht, 1998.

[12] D. Cox, *The moving curve ideal and the Rees algebra*, Theoret. Comput. Sci. **392** (2008), 23–36.

[13] D. Cox, R. Goldman and M. Zhang, *On the validity of implicitization by moving quadrics for rational surfaces with no base points*, J. Symbolic Comput. **29** (2000), 419–440.

[14] D. Cox and A. Iarrobino, *Strata of rational space curves*, Comput. Aided Geom. Design **32** (2015), 50–68.

[15] D. Cox, A. Kustin, C. Polini and B. Ulrich, *A Study of Singularities on Rational Curves Via Syzygies*, Memoirs of the AMS, AMS, Providence, RI, 2013.

[16] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, Fourth Edition, Springer, New York, 2015.

[17] D. Cox, T. Sederberg and F. Chen, *The moving line ideal basis of planar rational curves*, Comput. Aided Geom. Design **15** (1998), 803–827.

[18] G. Craciun, *Toric differential inclusions and a proof of the global attractor conjecture*, preprint, 2015, `arXiv:1501.02860[math.DS]`.

[19] G. Craciun, A. Dickenstein, A. Shiu and B. Sturmfels, *Toric dynamical systems*, J. Symbolic Comput. **44** (2009), 1551–1565.

[20] A. Dickenstein, *Biochemical reaction networks: An invitation for algebraic geometers*, in *Mathematical Congress of the Americas* (J. de la Peña, J. A. López-Mimbela, M. Nakamura and J. Petean, eds.), Contemporary Mathematics **656**, AMS, Providence, RI, 2016, 65–83.

[21] A. Dickenstein and M. Pérez Millán, *How far is complex balancing from detailed balancing?*, Bulletin of Mathematical Biology **73** (2011), 811–828.

[22] M. Feinberg, *Mathematical Aspects of Mass Action Kinetics*, Chapter 1 in *Chemical Reactor Theory: A Review* (N. Amundson and L. Lapidus, eds.), Prentice-Hall, New York, 1977.

[23] K. Gatermann, *Counting stable solutions of sparse polynomial systems in chemistry*, in *Symbolic Computation: Solving Equations in Algebra, Geometry and Engineering* (E. Green, S. Hoşten, R. Laubenbacher and V. Powers, eds.), Contemporary Mathematics **286**, AMS, Providence, RI, 2001, 53–69.

[24] K. Gatermann and B. Huber, *A family of sparse polynomial systems arising in chemical reaction systems*, J. Symbolic Comput. **33** (2002), 275–305.

[25] K. Gatermann, M. Eiswirth and A. Sensse, *Toric ideals and graph theory to analyze Hopf bifurcations in mass action systems*, J. Symbolic Comput. **40** (2005), 1361–1382.

[26] I. Gel'fand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[27] E. Gross, M. Johnston and N. Meshkut, *Discussing the proof of the Global Attractor Conjecture*, SIAM News, July/August 2016, 10.

[28] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, IHES, Bures-sur-Yvette, 1961–1967.

[29] J. Gunawardena, *Theory and Mathematical Methods*, Comprehensive Biophysics **9** (2012), 243–265.

[30] J. Gunawardena, *Distributivity and processivity in multisite phosphorylation can be distinguished through steady-state invariants*, Biophys. J. **93** (2007), 3828–3834.

[31] R. Hartshorne, *Algebraic Geometry*, Springer, 1978.

[32] J. Hauenstein and F. Sottile, *Newton polytopes and witness sets*, Math. Comput. Sci. **8** (2014), 235–251.

[33] F. Horn, *The dynamics of open reaction systems*, in *Mathematical Aspects of Chemical and Biochemical Problems and Quantum Chemistry (Proc. SIAM-AMS Sympos. Appl. Math., New York)* (D. Cohen, ed.), SIAM-AMS Proceedings **8**, AMS, Providence, R.I., 1974, 125–137.

[34] F. Horn and R. Jackson, *General mass action kinetics*, Arch. Ration. Mech. Anal. **47** (1972), 81–116.

[35] J.-P. Jouanolou, *Singularités rationnelles du resultant*, in *Algebraic Geometry, Proceedings Copenhagen 1978* (K. Lønsted, ed.), Lecture Notes in Math. **732**, Springer, New York, 1979, 183–213.

[36] M. Kapovich and J. Millson, *Universality theorems for configuration spaces of planar linkages*, Topology **41** (2002), 1051–1107.

[37] A. Kempe, *On a general method of describing plane curves of the n-th degree by linkwork*, Proc. London Math. Soc. **7** (1875), 213–216.

[38] H. King, *Planar linkages and algebraic sets*, Turkish J. Math. **23** (1999), 33–56.

[39] R. Krasauskas, *Toric surface patches*, Adv. Comput. Math. **17** (2002), 89–113.

[40] L. Kronecker, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, J. Reine Angew. Math. **92** (1882), 1–122.

[41] F. S. Macaulay, *On some Formulæ in Elimination*, Proc. London Math. Soc. **35** (1903) 3–27,

[42] F. Minding, *Ueber die Bestimmung des Grades einer durch Elimination hervorgehenden Gleichung*, J. Reine Angew. Math. **22** (1841), 178–183.

[43] E. Netto, *Vorlesungen über Algebra*, Volume II, Teubner, Leipzig, 1900.

[44] T. Sederberg, *Implicit and Parametric Curves and Surfaces for Computer Aided Geometric Design*, Ph.D. Thesis, Purdue University, 1983.

[45] T. W. Sederberg, T. Saito, D. Qi and K. S. Klimaszewksi, *Curve implicitization using moving lines*, Comput. Aided Geom. Design **11** (1994), 687–706.

[46] T. W. Sederberg, R. N. Goldman and H. Du, *Implicitizing rational curves by the method of moving algebraic curves*, J. Symbolic Comput. **23** (1997), 153–175.

[47] F. Severi, *Sul principio della conservazione del numero*, Rendiconti del Circolo matematico di Palermo **33** (1912), 313–327.

[48] A. Sommese and C. Wampler, *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*, World Scientific, Singapore, 2005.

[49] H. Stetter, *Numerical Polynomial Algebra*, SIAM, Philadelphia, PA, 2004.

[50] B. Sturmfels, *On the Newton polytope of the resultant*, J. Algebraic Combin. **3** (1994), 207–236.

[51] B. L. van der Waerden, *Zur Nullstellentheorie der Polynomideale*, Math. Annalen **96** (1926), 183–208.

[52] B. L. van der Waerden, *Der Multiplizitätsbegriff der algebraischen Geometrie*, Math. Annalen **97** (1927), 756–774.

[53] B. L. van der Waerden, *Moderne Algebra*, Volume II, Springer-Verlag, Berlin, 1940 (Second Edition) and 1959 (Fourth edition, with new title *Algebra*).

[54] N. White and W. Whitely, *The algebraic geometry of stresses in frameworks*, SIAM. J. Algebraic Discrete Methods, **4** (1983), 481–511.

[55] A. Weil, *Foundations of Algebraic Geometry*, AMS, Providence, RI, 1946.