Is a generic framework associated $G = (V, E)$ minimally infinitesimally rigid in $\mathbb{R}^3$ if and only if

1. $|E| = 3|V| - 6$
2. If $V' \subset V$, and $E' \subset E$ is the set of edges induced on $V'$, then $|E'| \leq 3|V'| - 6$?

Figure: A counterexample: the double banana

Question
What kinds of frameworks do other constraint systems give?
Bodies and bars

**Definition**

A $d$-dimensional body-bar framework in $\mathbb{R}^d \subset \mathbb{P}^d$ is a finite collection of full-dimensional rigid bodies $B_1, \ldots, B_n$ connected at flexible joints by fixed-length bars by $m$ fixed-length bars.

Is a given framework of bodies and bars rigid or flexible?
Combinatorial rigidity theory for body-bar frameworks

**Figure:** Two rigid bodies in the plane connected by three bars.

**Figure:** Multigraph $G$ associated to a framework with 2 bodies joined by 3 bars.

**Theorem (Tay $^1$, White-Whiteley $^2$)**

*A generic framework associated to a multigraph $G(V,E)$ is minimally infinitesimally rigid in $\mathbb{P}^d$ if and only if*

- $|E| = k|V| - k$
- *If $V' \subset V$, and $E' \subset E$ is the set of edges induced on $V'$, then $|E'| \leq k|V'| - k$,*

*where $k = \binom{d+1}{2}$.*


Combinatorial rigidity theory for body-bar frameworks

Figure: Two rigid bodies in the plane connected by three bars.

Figure: Multigraph $G$ associated to a framework with 2 bodies joined by 3 bars.

Theorem (Tay\textsuperscript{3}, White-Whiteley\textsuperscript{4})

\textit{A generic framework associated to a multigraph $G(V, E)$ is minimally infinitesimally rigid in $\mathbb{P}^d$ if and only if $G$ can be decomposed as a union of $k$ edge-disjoint spanning trees where $k = \binom{d+1}{2}$.}

\textsuperscript{3}Tay, Rigidity of Multi-graphs. I. Linking Rigid Bodies in n-Space, 1984.

An aside: bar-joint rigidity theory in terms using spanning trees

**Theorem (Laman $^5$, Pollaczeck-Geiringer $^6$, Haas $^7$)**

*A generic framework associated to $G = (V, E)$ is minimally infinitesimally rigid in $\mathbb{R}^2$ if and only if adding any edge $e$ to $G$ (which may give a multigraph) results in a graph that can be decomposed into 2 edge-disjoint spanning trees.*

---

$Laman$, On graphs and the rigidity of plane skeletal structures, 1970  
$Pollaczeck-Geiringer$, Über die gliederung ebener fachwerk, 1927  
$Haas$, Characterizations of Arboricity in Graphs, 2002
Given \( n \) bodies in \( \mathbb{P}^d \) and \( m \) bars, we form an \( m \times kn \) rigidity matrix.

- \( k = \binom{d+1}{2} = \text{dim of group Euclidean motions} \).
- \( x_{ij} = \text{Pl"ucker coordinates of bar from vertex } i \text{ to vertex } j \).
- Study the generic minimally rigid case where \( m = kn - k \).
- Eliminate trivial motions by tying down a body (1)

\[
C_G := \det \begin{pmatrix}
... & ... & \cdots & x_{ij} & \cdots & -x_{ij} & \cdots & 0 \\
0 & \cdots & x_{ij} & \cdots & -x_{ij} & \cdots & 0 \\
... & ... & ... & ... & ... & ... & ...
\end{pmatrix}
\]

**Definition**

The polynomial \( C_G \) is the pure condition of \( G \).

---

Key idea of the proof (for 3-frame in $P^2$)

\[
\begin{pmatrix}
  a_1 & a_2 & a_3 & -a_1 & -a_2 & -a_3 & 0 & 0 & 0 \\
  b_1 & b_2 & b_3 & -b_1 & -b_2 & -b_3 & 0 & 0 & 0 \\
  c_1 & c_2 & c_3 & 0 & 0 & 0 & -c_1 & -c_2 & -c_3 \\
  d_1 & d_2 & d_3 & 0 & 0 & 0 & -d_1 & -d_2 & -d_3 \\
  0 & 0 & 0 & e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 \\
  0 & 0 & 0 & f_1 & f_2 & f_3 & -f_1 & -f_2 & -f_3 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Key idea of the proof: rearrange the columns

\[
\begin{pmatrix}
    a_1 & -a_1 & 0 \\
    b_1 & -b_1 & 0 \\
    c_1 & 0 & -c_1 \\
    d_1 & 0 & -d_1 \\
    0 & e_1 & -e_1 \\
    0 & f_1 & -f_1 \\
\end{pmatrix}
\begin{pmatrix}
    a_2 & -a_2 & 0 \\
    b_2 & -b_2 & 0 \\
    c_2 & 0 & -c_2 \\
    d_2 & 0 & -d_2 \\
    0 & e_2 & -e_2 \\
    0 & f_2 & -f_2 \\
\end{pmatrix}
\begin{pmatrix}
    a_3 & -a_3 & 0 \\
    b_3 & -b_3 & 0 \\
    c_3 & 0 & -c_3 \\
    d_3 & 0 & -d_3 \\
    0 & e_3 & -e_3 \\
    0 & f_3 & -f_3 \\
\end{pmatrix}
\]
**Key idea of the proof: expand via trees**

![Graph with nodes 1, 2, and 3 connected by edges labeled a, b, c, d, e, f.]

\[
\begin{pmatrix}
  a_1 & -a_1 & 0 \\
b_1 & -b_1 & 0 \\
c_1 & 0 & -c_1 \\
d_1 & 0 & -d_1 \\
e_1 & -e_1 & 0 \\
f_1 & -f_1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

Jessica Sidman  Mount Holyoke College  Polynomial methods and rigidity theory
More geometric constraints: body-cad frameworks

Figure: line-line coincidence constraint

Figure: line-plane angle constraint

Figure: point-plane distance constraint

Question: Does a given set of constraints specify an infinitesimally rigid structure?

Constraints may restrict more than one degree of freedom.

Some constraints only restrict rotational degrees of freedom, so generically there are “extra” zeroes in the rigidity matrix.

- line-line coincidence
- point-plane coincidence
- point-point distance

---

10Figure from Lee-St. John, S, Combinatorics and the Rigidity of CAD Systems, 2013.
Primitive cad graphs

$H = (V, R \cup B)^{11}$

- One vertex per body.
- One edge per row in rigidity matrix.
- Edges partitioned into angular and non-angular constraints.

---

11 Figure from Lee-St. John, S, Combinatorics and the Rigidity of CAD Systems, 2013.
A combinatorial characterization of body-cad rigidity

$G = (V, R \cup B)$ is an $[a, b]$-graph if $\exists B' \subset B$ such that

- $(V, R \cup B')$ is the disjoint union of $a$ spanning trees
- $(V, B \setminus B')$ is the disjoint union of $b$ spanning trees

**Theorem (Lee-St.John, S.)**

A body-and-cad framework is generically minimally infinitesimally rigid iff it is a

- $[1, 2]$-graph in 2D.
- $[3, 3]$-graph in 3D (point-point coincidences are omitted).

---

**Figure:** [1, 2]-frame

**Figure:** [3, 3]-frame

---

What about nongeneric frameworks?

- Frameworks designed to be useful may have nongeneric properties.

- Geiss-Schreyer\textsuperscript{13}: Realizations of the Stewart-Gough platform via finite fields.

\textbf{Figure:} (Geiss-Schreyer, Figure 2) Motions generate a curve of degree 12 and genus 7 in its canonical embedding.

\textsuperscript{13}Geiss-Schreyer, A family of exceptional Stewart-Gough mechanisms of genus 7
Nongeneric frameworks and the pure condition

- $C_G(x)$ is a nonzero polynomial if and only if $G$ is generically infinitesimally rigid.

- $V(C_G)$ is the set of frameworks associated to $G$ with infinitesimal motions.

What can we learn from studying $C_G$?

- Rigid substructures?
- Intuitive, geometric descriptions of special positions?
Writing the pure condition: brackets

\[ G = \begin{pmatrix} a & -a & 0 \\ b & -b & 0 \\ c & 0 & -c \\ d & 0 & -d \\ 0 & e & -e \\ 0 & f & -f \\ I_3 & 0 & 0 \end{pmatrix} \]

\[ C_G = \det \]

\[ C_G = [abe][cdf] - [abf][cde] = (a \land b) \lor (c \land d) \lor (e \land f) \]

via Grassmann-Cayley factorization

\[ ^{14}\text{White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.} \]
The rigidity matrix and the pure condition: body-cad

**Example**

“Tie down” a body by appending \( \dim \text{SE}(d) \) rows.

\[
M_T = \begin{bmatrix}
    a_1 & a_2 & a_3 & 0 & 0 & 0 \\
    b_1 & 0 & 0 & 0 & 0 & 0 \\
    c_1 & 0 & 0 & -c_1 & 0 & 0 \\
    d_1 & d_2 & d_3 & -d_1 & -d_2 & -d_3 \\
    0 & 0 & 0 & e_1 & e_2 & e_3 \\
    0 & 0 & 0 & f_1 & f_2 & f_3 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The pure condition \( = \det M_T \).

---

Farre et al, Algorithms for detecting dependencies and rigid subsystems for CAD, 2016
The structure of the pure condition: body-cad

\[ \text{det } M_T = \text{det } \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & -c_1 & 0 & 0 \\ d_1 & d_2 & d_3 & -d_1 & -d_2 & -d_3 \\ 0 & 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & 0 & f_1 & f_2 & f_3 \end{pmatrix} \]

\[ = [abc][def] - [abd][cef] \]

\[ = -[abd][cef] \]

\[ = b_1(a_2 d_3 - a_3 d_2)c_1(e_2 f_3 - e_3 f_2) \]
The pure condition: body-bar vs body-cad

Body-bar

- Each edge is in the support of one irreducible factor.
- Each factor is the pure condition of a graph minor.
- Circuits have the form isostatic plus one edge.

Body-cad

- Each edge is in the support of one irreducible factor.
- Irreducible factors may not be pure conditions of $[a, a], [b, b]$ or $[a+b, a+b]$ graphs.
- Circuits are mysterious!

---

17 Farre et al, Algorithms for detecting dependencies and rigid subsystems for CAD, 2016
The bar-joint pure condition

The tied-down rigidity matrix

\[
\begin{pmatrix}
  x_1 - x_2 & x_2 - x_1 & 0 & 0 & 0 \\
  x_1 - x_3 & 0 & x_3 - x_1 & 0 & 0 \\
  x_1 - x_4 & 0 & 0 & x_4 - x_1 & 0 \\
  0 & x_2 - x_3 & x_3 - x_2 & 0 & 0 \\
  0 & x_2 - x_5 & 0 & 0 & x_5 - x_2 \\
  0 & 0 & x_3 - x_4 & x_4 - x_3 & 0 \\
  0 & 0 & 0 & x_4 - x_5 & x_5 - x_4 \\
  x_1 - a & 0 & 0 & 0 & 0 \\
  x_1 - b & 0 & 0 & 0 & 0 \\
  0 & x_2 - c & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[\text{18 White, Whiteley, The algebraic geometry of stresses in frameworks, 1983.}\]
The pure condition of $G$

$$M_T = \begin{pmatrix}
    x_1 - a & 0 & 0 & 0 & 0 \\
    x_1 - b & 0 & 0 & 0 & 0 \\
    x_1 - x_2 & x_2 - x_1 & 0 & 0 & 0 \\
    0 & x_2 - c & 0 & 0 & 0 \\
    x_1 - x_3 & 0 & x_3 - x_1 & 0 & 0 \\
    x_1 - x_4 & 0 & 0 & x_4 - x_1 & 0 \\
    0 & x_2 - x_3 & x_3 - x_2 & 0 & 0 \\
    0 & x_2 - x_5 & 0 & 0 & x_5 - x_2 \\
    0 & 0 & x_3 - x_4 & x_4 - x_3 & 0 \\
    0 & 0 & 0 & x_4 - x_5 & x_5 - x_4
\end{pmatrix}$$

$$\det \begin{pmatrix} x_1 - a \\ x_1 - b \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_{11} & x_{12} & 1 \end{pmatrix} = [a, b, x_1]$$

$$\det M_T = [a, b, x_1][x_1, c, x_2][x_1, x_2, x_3][x_1, x_3, x_4][x_2, x_4, x_5].$$
The factors and stresses

The pure condition of $G$ is $[x_1, x_2, x_3][x_1, x_3, x_4][x_2, x_4, x_5]$.

- $[x_1, x_2, x_3] = 0 \iff x_1, x_2$ and $x_3$ are collinear.

- $\iff$ There is a dependence on the rows of

$$
\begin{pmatrix}
 x_1 - x_2 & x_2 - x_1 & 0 & 0 & 0 \\
 x_1 - x_3 & 0 & x_3 - x_1 & 0 & 0 \\
 0 & x_2 - x_3 & x_3 - x_2 & 0 & 0 \\
\end{pmatrix}.
$$

- There is a stress supported on the induced graph on $\{1, 2, 3\}$. 

![Graph diagram]
Factors and stresses

The pure condition of $G$ is $[x_1, x_2, x_3][x_1, x_3, x_4][x_2, x_4, x_5]$.

- $[x_2, x_4, x_5] = 0 \iff x_2, x_4$ and $x_5$ are collinear.
- But there is no dependence relation supported on

$$
\begin{pmatrix}
0 & x_2 - x_5 & 0 & 0 & x_5 - x_2 \\
0 & 0 & 0 & x_4 - x_5 & x_5 - x_4
\end{pmatrix}.
$$

- The induced subgraph on $\{2, 4, 5\}$ cannot support a stress.

**Question:** How can we find a stress associated to this factor?
Let $G = (V, E)$ be a graph with $|V| = n$.

Define $\varphi: (\mathbb{C}^d)^n \to \mathbb{C}^{\binom{n}{2}}$ by

$$\varphi(x) = ((x_i - x_j) \cdot (x_i - x_j)) = \ell_{ij}.$$ 

1. $\ker \varphi = \text{ideal of } \widetilde{CM}^{n,d}$.
2. Every $f \in \ker \varphi$ gives rise to a stress by differentiating $[f(\varphi(x))]' = \nabla f(\varphi(x)) \cdot d\varphi.$

---

Joint work with Rosen, Theran, Vinzant

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Theorem (Rosen-S-Theran-Vinzant)

- $G$ is isostatic
- $\varphi^* : \mathbb{C}[\ell_{ij}] \to \mathbb{C}[x]$  
- $g$ is an irreducible factor of $C_G$

Then

- $P = (\varphi^*)^{-1}(\langle g \rangle)$ is prime
- $\exists f \neq 0 \in P \cap \mathbb{C}[\ell_G]$
- $\nabla f$ is a stress for frameworks in $V(g)$. 
Defining equation of $\varphi_G([x_1, x_2, x_3])$

Let $G$ be

$$
\begin{align*}
\varphi_G(\mathcal{V}(\{x_1, x_2, x_3\})) & \text{ is defined by } \\
& -\ell_{12}^2 + 2\ell_{12}\ell_{13} - \ell_{13}^2 + 2\ell_{12}\ell_{23} + 2\ell_{13}\ell_{23} - \ell_{23}^2 \\
& = \det \begin{pmatrix} 2\ell_{13} & \ell_{13} - \ell_{12} + \ell_{23} \\ \ell_{13} - \ell_{12} + \ell_{23} & 2\ell_{23} \end{pmatrix}
\end{align*}
$$

This is the Cayley condition that a triple of real numbers must satisfy if they are the pairwise distances among 3 collinear points.

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Defining equation of $\varphi_G([x_2, x_4, x_5])$ is defined by an irreducible polynomial of degree 6 supported on every edge of $G$.

**Question:** Can we use algebraic methods to find (true) motions?
Let $G$ be isostatic and $e \notin G$.

**Question:** Are there edge lengths so that $G$ has a motion in which $e$ changes length?

- Add in edge $e = 35$.
- $G + e$ contains a circuit.
- Get circuit polynomial $p$ of degree 4 in each variable.
- $I = \langle$ coefficients of $p(e)\rangle$.
- $I$ has 6 associated primes.

The two that are geometrically significant for motions are

\[
\langle \ell_{13}, \ell_{14} - \ell_{34}, \ell_{12} - \ell_{23} \rangle, \langle \ell_{23} - \ell_{34}, \ell_{12} - \ell_{14}, -\ell_{45} + \ell_{25} \rangle
\]
Summary

- Bar-and-joint rigidity is open in dimension 3.
- Combinatorial rigidity comes in many flavors.
- Polynomial methods may be used to find intuitive, geometric explanations for singular behavior.

For more background on rigidity theory, see the forthcoming:

[Handbook of Geometric Constraint Systems Principles]