Polynomial methods and rigidity theory

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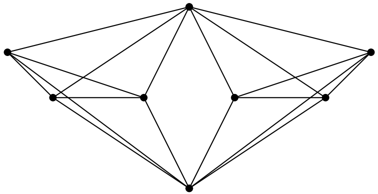
3d bar-and-joint rigidity

Is a generic framework associated $G = (V, E)$ minimally infinitesimally rigid in $\mathbf{R}^3$ if and only if

- $|E| = 3|V| - 6$
- If $V' \subset V$, and $E' \subset E$ is the set of edges induced on V', then $|E'| \leq 3|V'| - 6$?

Figure: A counterexample: the double banana



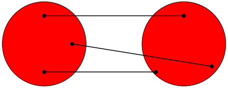
Question

What kinds of frameworks do other constraint systems give?

Bodies and bars

Definition

A d-dimensional body-bar framework in $\mathbf{R}^d \subset \mathbf{P}^d$ is a finite collection of full-dimensional rigid bodies $B_1, \ldots, B_n$ connected at flexible joints by fixed-length bars by m fixed-length bars.



Is a given framework of bodies and bars rigid or flexible?

Combinatorial rigidity theory for body-bar frameworks

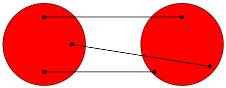


Figure: Two rigid bodies in the plane connected by three bars.

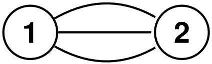


Figure: Multigraph G associated to a framework with 2 bodies joined by 3 bars.

Theorem (Tay [1], White-Whiteley [2])

A generic framework associated to a multigraph $G(V, E)$ is minimally infinitesimally rigid in $\mathbf{P}^d$ if and only if

- $|E| = k|V| - k$
- *If $V' \subset V$, and $E' \subset E$ is the set of edges induced on V', then $|E'| \leq k|V'| - k$,*

where $k = \binom{d+1}{2}$.

[1] Tay, Rigidity of Multi-graphs. I. Linking Rigid Bodies in n-Space, 1984.

[2] White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.

Combinatorial rigidity theory for body-bar frameworks

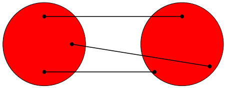


Figure: Two rigid bodies in the plane connected by three bars.

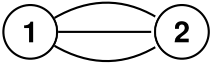


Figure: Multigraph G associated to a framework with 2 bodies joined by 3 bars.

Theorem (Tay [3], White-Whiteley[4])

A generic framework associated to a multigraph $G(V, E)$ is minimally infinitesimally rigid in $\mathbf{P}^d$ if and only if G can be decomposed as a union of k edge-disjoint spanning trees where $k = \binom{d+1}{2}$.

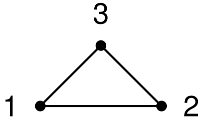
[3] Tay, Rigidity of Multi-graphs. I. Linking Rigid Bodies in n-Space, 1984.

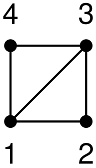
[4] White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.

An aside: bar-joint rigidity theory in terms using spanning trees

Theorem (Laman [5], Pollaczek-Geiringer [6], Haas [7])

A generic framework associated to $G = (V, E)$ is minimally infinitesimally rigid in $\mathbf{R}^2$ if and only if adding any edge e to G (which may give a multigraph) results in a graph that can be decomposed into 2 edge-disjoint spanning trees.





[5] Laman, On graphs and the rigidity of plane skeletal structures, 1970

[6] Pollaczek-Geiringer, Über die gliederung ebener fachwerk, 1927

[7] Haas, Characterizations of Arboricity in Graphs, 2002

The rigidity matrix and the pure condition [8]

Given n bodies in $\mathbf{P}^d$ and m bars, we form an $m \times kn$ rigidity matrix.

- $k = \binom{d+1}{2}$ = dim of group Euclidean motions.
- $\mathbf{x}_{ij} =$ Plücker coordinates of bar from vertex i to vertex j.
- Study the generic minimally rigid case where $m = kn - k$.
- Eliminate trivial motions by tying down a body (1)

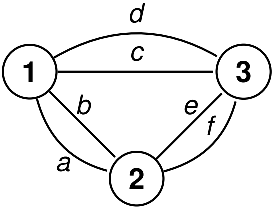
$$C_G := \det \begin{pmatrix} \vdots & & & & & & \vdots \\ 0 & \cdots & \mathbf{x}_{ij} & \cdots & -\mathbf{x}_{ij} & \cdots & 0 \\ \vdots & & & & & & \vdots \\ I_k & 0 & \cdots & & & \cdots & 0 \end{pmatrix}$$

Definition

The polynomial C_G is the pure condition of G .

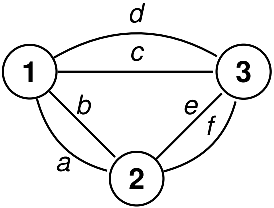
[8] White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.

Key idea of the proof (for 3-frame in $\mathbf{P}^2$)



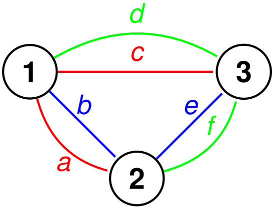
$$\left(\begin{array}{ccccccccc}
a_1 & a_2 & a_3 & -a_1 & -a_2 & -a_3 & 0 & 0 & 0 \\
b_1 & b_2 & b_3 & -b_1 & -b_2 & -b_3 & 0 & 0 & 0 \\
c_1 & c_2 & c_3 & 0 & 0 & 0 & -c_1 & -c_2 & -c_3 \\
d_1 & d_2 & d_3 & 0 & 0 & 0 & -d_1 & -d_2 & -d_3 \\
0 & 0 & 0 & e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 \\
0 & 0 & 0 & f_1 & f_2 & f_3 & -f_1 & -f_2 & -f_3 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Key idea of the proof: rearrange the columns



$$\left(\begin{array}{ccc|ccc|ccc}
a_1 & -a_1 & 0 & a_2 & -a_2 & 0 & a_3 & -a_3 & 0 \\
b_1 & -b_1 & 0 & b_2 & -b_2 & 0 & b_3 & -b_3 & 0 \\
c_1 & 0 & -c_1 & c_2 & 0 & -c_2 & c_3 & 0 & -c_3 \\
d_1 & 0 & -d_1 & d_2 & 0 & -d_2 & d_3 & 0 & -d_3 \\
0 & e_1 & -e_1 & 0 & e_2 & -e_2 & -0 & e_3 & -e_3 \\
0 & f_1 & -f_1 & 0 & f_2 & -f_2 & 0 & f_3 & -f_3 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

Key idea of the proof: expand via trees



$$\left(\begin{array}{ccc|ccc|ccc}
a_1 & -a_1 & 0 & a_2 & -a_2 & 0 & a_3 & -a_3 & 0 \\
b_1 & -b_1 & 0 & b_2 & -b_2 & 0 & b_3 & -b_3 & 0 \\
c_1 & 0 & -c_1 & c_2 & 0 & -c_2 & c_3 & 0 & -c_3 \\
d_1 & 0 & -d_1 & d_2 & 0 & -d_2 & d_3 & 0 & -d_3 \\
0 & e_1 & -e_1 & 0 & e_2 & -e_2 & -0 & e_3 & -e_3 \\
0 & f_1 & -f_1 & 0 & f_2 & -f_2 & 0 & f_3 & -f_3 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)$$

More geometric constraints: body-cad frameworks[9]

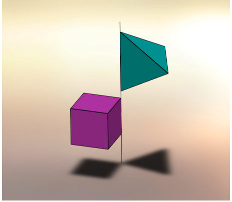


Figure: line-line coincidence constraint

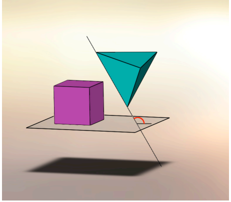


Figure: line-plane angle constraint

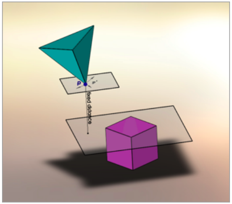


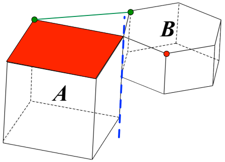
Figure: point-plane distance constraint

Question: Does a given set of constraints specify an infinitesimally rigid structure?

[9] Figures from Haller et al, Body-and-cad geometric constraint systems, 2012.

CAD-frameworks vs body-and-bar frameworks[10]

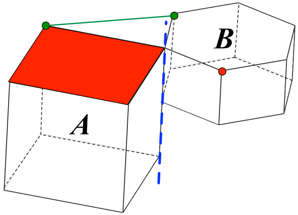
- Constraints may restrict more than one degree of freedom.
- Some constraints only restrict rotational degrees of freedom, so generically there are "extra" zeroes in the rigidity matrix.



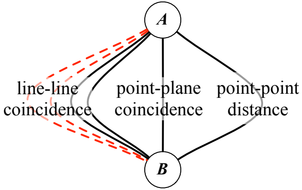
- line-line coincidence
- point-plane coincidence
- point-point distance

[10] Figure from Lee-St. John, S, Combinatorics and the Rigidity of CAD Systems, 2013.

Primitive cad graphs



$H = (V, R \cup B)$[11]



- One vertex per body.
- One edge per row in rigidity matrix.
- Edges partitioned into angular and non-angular constraints.

[11] Figure from Lee-St. John, S, Combinatorics and the Rigidity of CAD Systems, 2013.

A combinatorial characterization of body-cad rigidity[12]

$G = (V, R \cup B)$ is an $[a, b]$-graph if $\exists B' \subset B$ such that

- $(V, R \cup B')$ is the disjoint union of a spanning trees
- $(V, B \backslash B')$ is the disjoint union of b spanning trees

Theorem (Lee-St.John, S.)

A body-and-cad framework is generically minimally infinitesimally rigid iff it is a

- $[1, 2]$*-graph in 2D.*
- $[3, 3]$*-graph in 3D (point-point coincidences are omitted).*

Figure: $[1, 2]$-frame

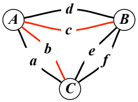
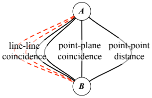


Figure: $[3, 3]$-frame



[12] Lee-St. John, S, Combinatorics and the Rigidity of CAD Systems, 2013.

What about nongeneric frameworks?

- Frameworks designed to be useful may have nongeneric properties.
- Geiss-Schreyer [13]: Realizations of the Stewart-Gough platform via finite fields.

Figure: (Geiss-Schreyer, Figure 2) Motions generate a curve of degree 12 and genus 7 in its canonical embedding.

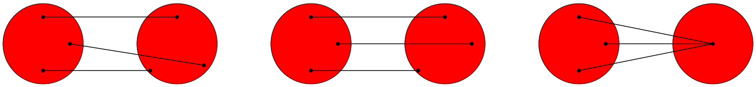
[13] Geiss-Schreyer, A family of exceptional Stewart-Gough mechanisms of genus 7

Nongeneric frameworks and the pure condition

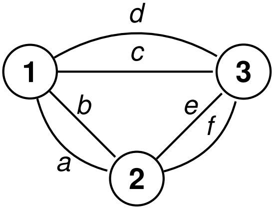
- $C_G(\mathbf{x})$ is a nonzero polynomial if and only if G is generically infinitesimally rigid.
- $V(C_G)$ is the set of frameworks associated to G with infinitesimal motions.

What can we learn from studying C_G?

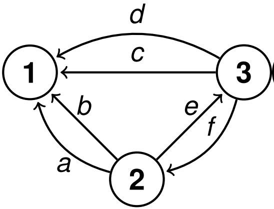
- Rigid substructures?
- Intuitive, geometric descriptions of special positions?

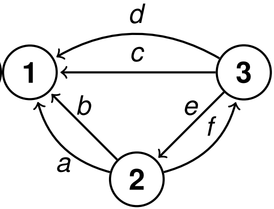


$G =$



$$C_G = \det \begin{pmatrix} a & -a & 0 \\ b & -b & 0 \\ c & 0 & -c \\ d & 0 & -d \\ 0 & e & -e \\ 0 & f & -f \\ I_3 & 0 & 0 \end{pmatrix}$$



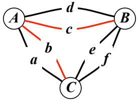


$C_G = [abe][cdf] - [abf][cde] = (a \wedge b) \vee (c \wedge d) \vee (e \wedge f)$
via Grassmann-Cayley factorization

[14] White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.

The rigidity matrix and the pure condition: body-cad[15]

Example



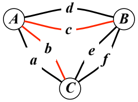
"Tie down" a body by appending $\dim SE(d)$ rows.

The pure condition $= \det M_T$.

$$M_T = \begin{array}{cccccc|ccc}
a_1 & a_2 & a_3 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 \\
b_1 & 0 & 0 & 0 & 0 & 0 & -b_1 & 0 & 0 \\
c_1 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 \\
d_1 & d_2 & d_3 & -d_1 & -d_2 & -d_3 & 0 & 0 & 0 \\
0 & 0 & 0 & e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 \\
0 & 0 & 0 & f_1 & f_2 & f_3 & -f_1 & -f_2 & -f_3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

[15] Farre et al, Algorithms for detecting dependencies and rigid subsystems for CAD, 2016

The structure of the pure condition: body-cad



$$\det M_T = \det \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & -c_1 & 0 & 0 \\ d_1 & d_2 & d_3 & -d_1 & -d_2 & -d_3 \\ 0 & 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & 0 & f_1 & f_2 & f_3 \end{pmatrix}$$

$$= [abc][def] - [abd][cef]$$

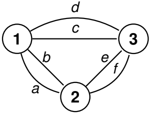
$$= -[abd][cef]$$

$$= b_1(a_2d_3 - a_3d_2)c_1(e_2f_3 - e_3f_2)$$

The pure condition: body-bar vs body-cad

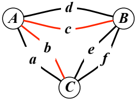
Body-bar [16]

- Each edge is in the support of one irreducible factor.
- Each factor is the pure condition of a graph minor.
- Circuits have the form isostatic plus one edge.



Body-cad[17]

- Each edge is in the support of one irreducible factor
- Irreducible factors may not be pure conditions of $[a, a], [b, b] or [a+b, a+b]-$ graphs.
- Circuits are mysterious!



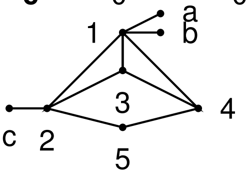
[16] White, Whiteley, The Algebraic Geometry of Motions of Bar-And-Body Frameworks, 1987.

[17] Farre et al, Algorithms for detecting dependencies and rigid subsystems for CAD, 2016

The bar-joint pure condition[18]

The tied-down rigidity matrix

$$\begin{pmatrix} \mathbf{x}_1-\mathbf{x}_2 & \mathbf{x}_2-\mathbf{x}_1 & 0 & 0 & 0 \\ \mathbf{x}_1-\mathbf{x}_3 & 0 & \mathbf{x}_3-\mathbf{x}_1 & 0 & 0 \\ \mathbf{x}_1-\mathbf{x}_4 & 0 & 0 & \mathbf{x}_4-\mathbf{x}_1 & 0 \\ 0 & \mathbf{x}_2-\mathbf{x}_3 & \mathbf{x}_3-\mathbf{x}_2 & 0 & 0 \\ 0 & \mathbf{x}_2-\mathbf{x}_5 & 0 & 0 & \mathbf{x}_5-\mathbf{x}_2 \\ 0 & 0 & \mathbf{x}_3-\mathbf{x}_4 & \mathbf{x}_4-\mathbf{x}_3 & 0 \\ 0 & 0 & 0 & \mathbf{x}_4-\mathbf{x}_5 & \mathbf{x}_5-\mathbf{x}_4 \\ \hline \mathbf{x}_1-\mathbf{a} & 0 & 0 & 0 & 0 \\ \mathbf{x}_1-\mathbf{b} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x}_2-\mathbf{c} & 0 & 0 & 0 \end{pmatrix}$$



[18] White, Whiteley, The algebraic geometry of stresses in frameworks, 1983.

The pure condition of G

$$
M_T = \left(\begin{array}{c|c|ccc}
\mathbf{x}_1 - \mathbf{a} & 0 & 0 & 0 & 0 \\
\mathbf{x}_1 - \mathbf{b} & 0 & 0 & 0 & 0 \\
\hline
\mathbf{x}_1 - \mathbf{x}_2 & \mathbf{x}_2 - \mathbf{x}_1 & 0 & 0 & 0 \\
0 & \mathbf{x}_2 - \mathbf{c} & 0 & 0 & 0 \\
\hline
\mathbf{x}_1 - \mathbf{x}_3 & 0 & \mathbf{x}_3 - \mathbf{x}_1 & 0 & 0 \\
\mathbf{x}_1 - \mathbf{x}_4 & 0 & 0 & \mathbf{x}_4 - \mathbf{x}_1 & 0 \\
0 & \mathbf{x}_2 - \mathbf{x}_3 & \mathbf{x}_3 - \mathbf{x}_2 & 0 & 0 \\
0 & \mathbf{x}_2 - \mathbf{x}_5 & 0 & 0 & \mathbf{x}_5 - \mathbf{x}_2 \\
0 & 0 & \mathbf{x}_3 - \mathbf{x}_4 & \mathbf{x}_4 - \mathbf{x}_3 & 0 \\
0 & 0 & 0 & \mathbf{x}_4 - \mathbf{x}_5 & \mathbf{x}_5 - \mathbf{x}_4
\end{array}\right)
$$

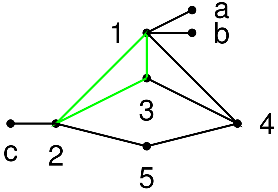
$$
\det \begin{pmatrix} \mathbf{x}_1 - \mathbf{a} \\ \mathbf{x}_1 - \mathbf{b} \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_{11} & x_{12} & 1 \end{pmatrix} = [\mathbf{a}, \mathbf{b}, \mathbf{x}_1]
$$

$$
\det M_T = [\mathbf{a}, \mathbf{b}, \mathbf{x}_1][\mathbf{x}_1, \mathbf{c}, \mathbf{x}_2][\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4][\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5].
$$

The factors and stresses

The pure condition of G is $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4][\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5]$.

- $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 0 \Leftrightarrow \mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are collinear.
- $\Leftrightarrow$ There is a dependence on the rows of
$$\begin{pmatrix} \mathbf{x}_1 - \mathbf{x}_2 & \mathbf{x}_2 - \mathbf{x}_1 & 0 & 0 & 0 \\ \mathbf{x}_1 - \mathbf{x}_3 & 0 & \mathbf{x}_3 - \mathbf{x}_1 & 0 & 0 \\ 0 & \mathbf{x}_2 - \mathbf{x}_3 & \mathbf{x}_3 - \mathbf{x}_2 & 0 & 0 \end{pmatrix}.$$
- There is a stress supported on the induced graph on $\{1, 2, 3\}$.



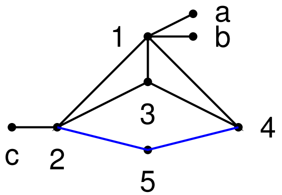
Factors and stresses

The pure condition of G is $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4][\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5]$.

- $[\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5] = 0 \Leftrightarrow \mathbf{x}_2, \mathbf{x}_4$ and $\mathbf{x}_5$ are collinear.
- But there is no dependence relation supported on

$$\begin{pmatrix} 0 & \mathbf{x}_2 - \mathbf{x}_5 & 0 & 0 & \mathbf{x}_5 - \mathbf{x}_2 \\ 0 & 0 & 0 & \mathbf{x}_4 - \mathbf{x}_5 & \mathbf{x}_5 - \mathbf{x}_4 \end{pmatrix}.$$

- The induced subgraph on $\{2, 4, 5\}$ cannot support a stress.



Question: How can we find a stress associated to this factor?

Back to the Cayley-Menger variety [19]

- Let $G = (V, E)$ be a graph with $|V| = n$.
- Define $\varphi : (\mathbf{C}^d)^n \to \mathbf{C}^{\binom{n}{2}}$ by

$$\varphi(\mathbf{x}) = ((\mathbf{x}_i - \mathbf{x}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j)) = \ell_{ij}.$$

$$\begin{tikzcd}
(\mathbf{C}^d)^n \arrow[rr, "\varphi"] \arrow[dr, "\varphi_G"'] & & \mathbf{C}^{\binom{n}{2}} \arrow[dl, "\pi_G"] \\
& \mathbf{C}^{|E|} &
\end{tikzcd}$$

1. $\ker \varphi =$ ideal of $\widehat{CM}^{n,d}$.
2. Every $f \in \ker \varphi$ gives rise to a stress by differentiating $[f(\varphi(\mathbf{x}))]' = \nabla f(\varphi(\mathbf{x})) \cdot d\varphi$.

[19] Joint work with Rosen, Theran, Vinzant

Theorem (Rosen-S-Theran-Vinzant)

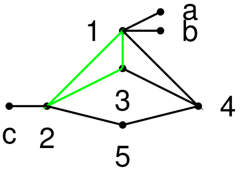
- *G is isostatic*
- $\varphi^* : \mathbf{C}[\ell_{ij}] \to \mathbf{C}[\mathbf{x}]$
- *g is an irreducible factor of C_G*

Then

- $P = (\varphi^*)^{-1}(\langle g \rangle)$ *is prime*
- $\exists f \neq 0 \in P \cap \mathbf{C}[\ell_G]$
- ∇f *is a stress for frameworks in $V(g)$.*

Defining equation of $\varphi_G([\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3])$

Let G be



with pure condition $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4][\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5]$.

- $\varphi_G(V([\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3])$ is defined by

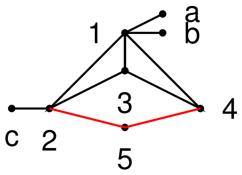
$$-\ell_{12}^2 + 2\ell_{12}\ell_{13} - \ell_{13}^2 + 2\ell_{12}\ell_{23} + 2\ell_{13}\ell_{23} - \ell_{23}^2$$

$$= \det \begin{pmatrix} 2\ell_{13} & \ell_{13} - \ell_{12} + \ell_{23} \\ \ell_{13} - \ell_{12} + \ell_{23} & 2\ell_{23} \end{pmatrix}$$

- This is the Cayley condition that a triple of real numbers must satisfy if they are the pairwise distances among 3 collinear points.

Defining equation of $\varphi_G([\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5]))$

$\varphi_G(V([\mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_5]))$ is defined by an irreducible polynomial of degree 6 supported on every edge of G.

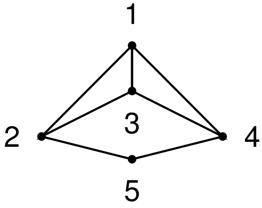


Question: Can we use algebraic methods to find (true) motions?

Finding motions [Rosen-S-Theran-Vinzant]

Let G be isostatic and $e \notin G$..
Question: Are there edge lengths so that G has a motion in which e changes length?

- Add in edge $e = 35$.
- $G + e$ contains a circuit.
- Get circuit polynomial p of degree 4 in each variable.
- $I = \langle$ coefficients of $p(e)\rangle$.
- I has 6 associated primes.



The two that are geometrically significant for motions are

$$\langle \ell_{13}, \ell_{14} - \ell_{34}, \ell_{12} - \ell_{23}\rangle, \langle \ell_{23} - \ell_{34}, \ell_{12} - \ell_{14}, -\ell_{45} + \ell_{25})\rangle$$

Summary

- Bar-and-joint rigidity is open in dimension 3.
- Combinatorial rigidity comes in many flavors.
- Polynomial methods may be used to find intuitive, geometric explanations for singular behavior.

For more background on rigidity theory, see the forthcoming: