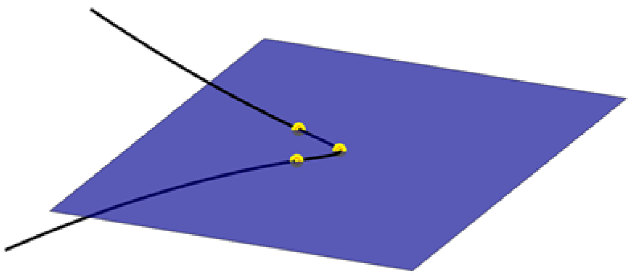
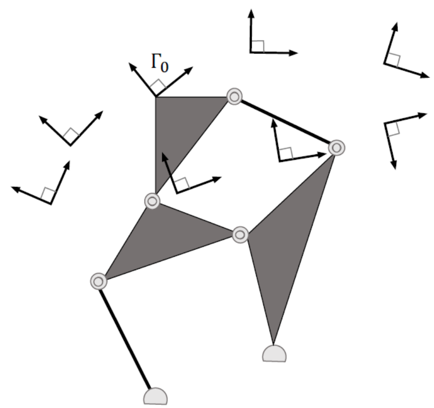
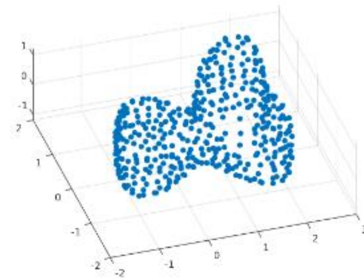
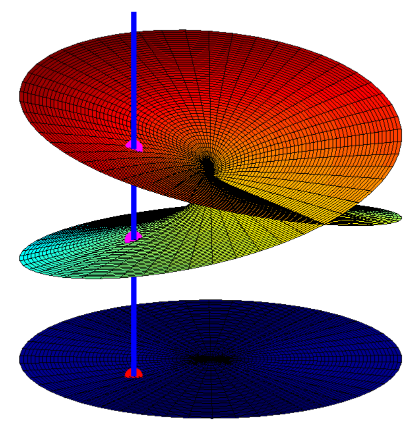
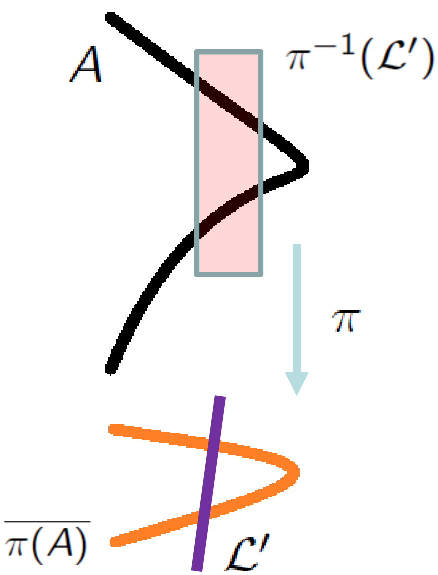
Applications of Sampling in Numerical Algebraic Geometry









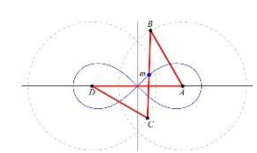


Jonathan Hauenstein

Applications of Polynomial Systems

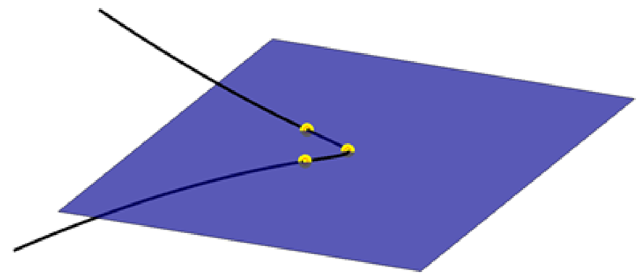
NSF CBMS TCU

June 5, 2018



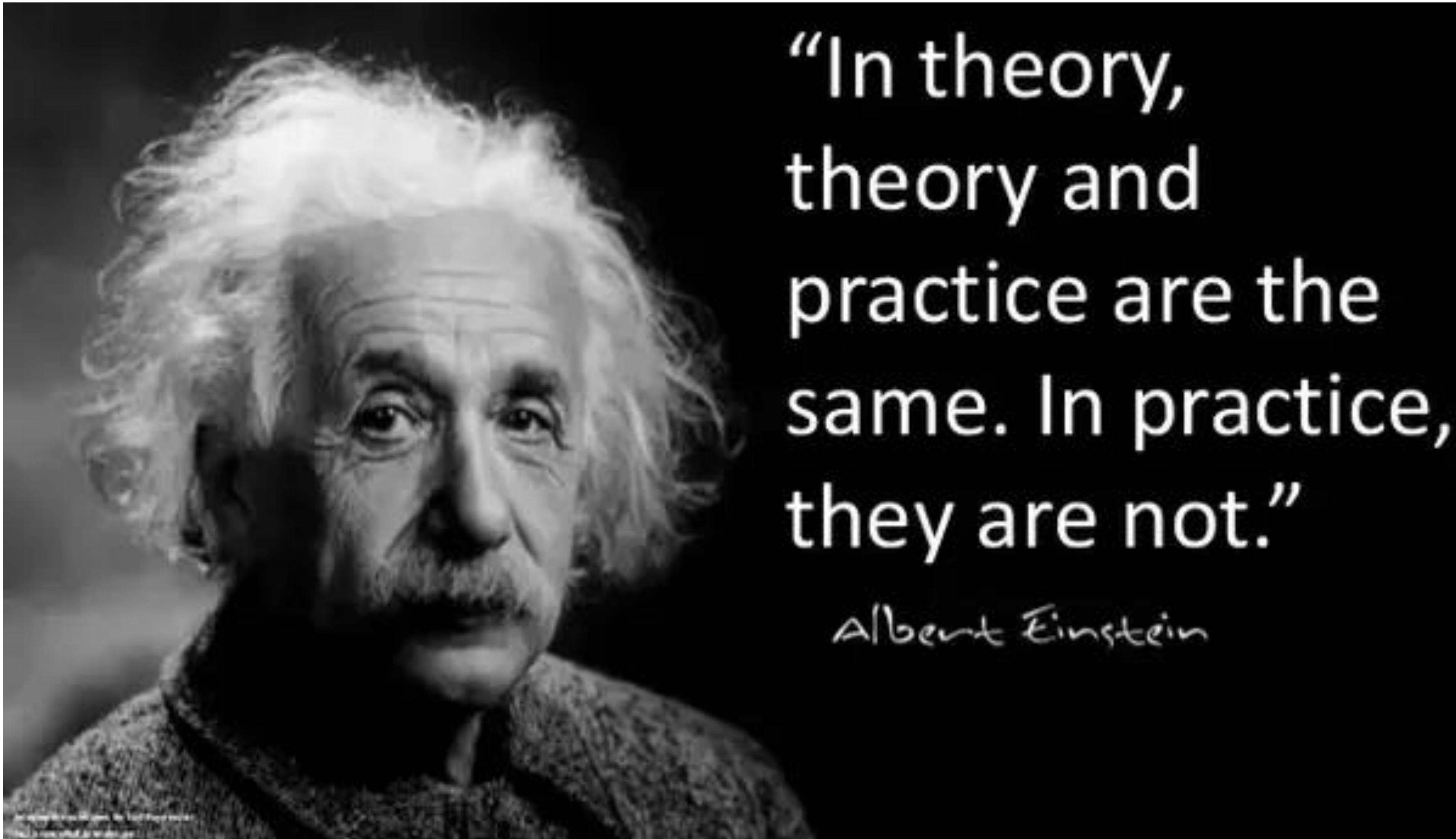
Overview

- Musings on numerical algebraic geometry
- Witness sets



- Applications of sampling

Theory vs. Practice



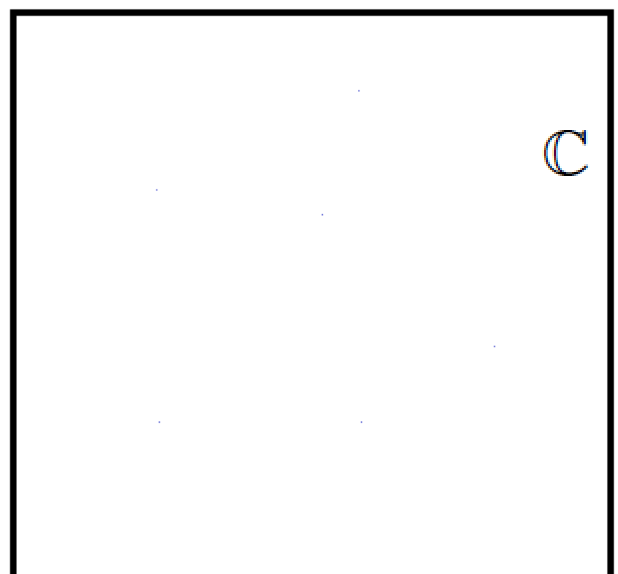
Theory vs. Practice

Let $f \in \mathbb{C}[x]$ be a univariate polynomial.

Theory:

- $f \equiv 0$ if and only if $f(x^*) = 0$ for general $x^* \in \mathbb{C}$.

  If $f \not\equiv 0$, then $\mathcal{V}(f) \subset \mathbb{C}$ has finitely many points. Hence, $\mathbb{C} \setminus \mathcal{V}(f)$ is a Zariski open dense subset of $\mathbb{C}$.



$\#\mathcal{V}(f) = 6$

Theory vs. Practice

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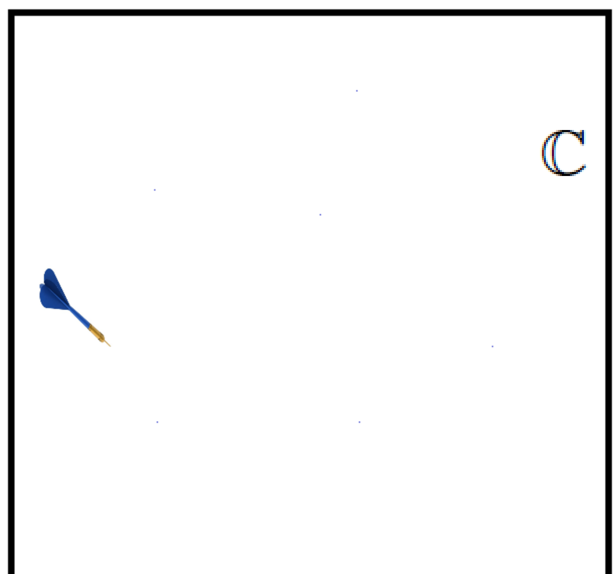
- $f \equiv 0$ if and only if $f(x^*) = 0$ for general $x^* \in \mathbb{C}$.
- $f \equiv 0$ if and only if $f(x^*) = 0$ for random $x^* \in \mathbb{C}$ *with probability* 1.



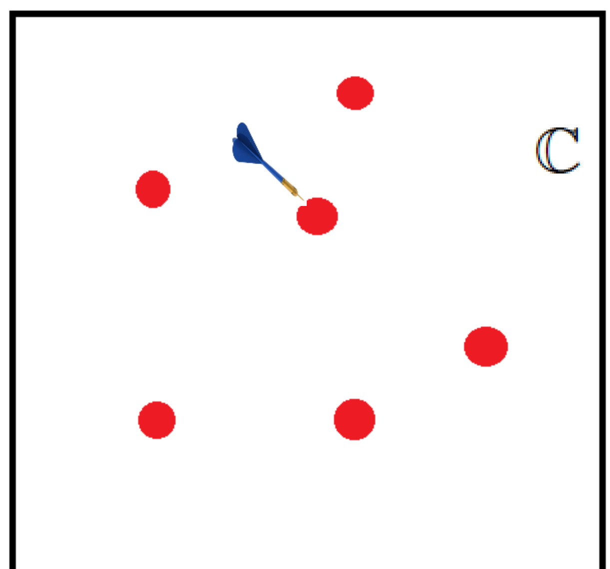
$\#\mathcal{V}(f) = 6$

Theory vs. Practice

Practice:

- Is f known exactly or only approximately?
- What is the scaling of f?
  - $\mathcal{V}(f) = \mathcal{V}(10^{-1000000} \cdot f)$
- How to select a random point in $x^* \in \mathbb{C}$?
- How much error in evaluating $f(x^*)$?
- In the presence of error, what does it mean to be equal to 0?
  - Floating-point arithmetic: select from a finite subset of $\mathbb{C}$

Prob(failure) > 0



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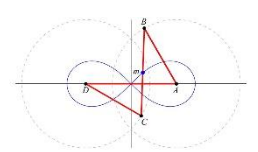
Prob(failure) > 0

Reduce failure rate by:

- using higher precision
- rescale
- reformulate (different geometric description?)
- take advantage of structure
- develop a different numerical approach (Simon Telen's poster)



Theory vs. Practice

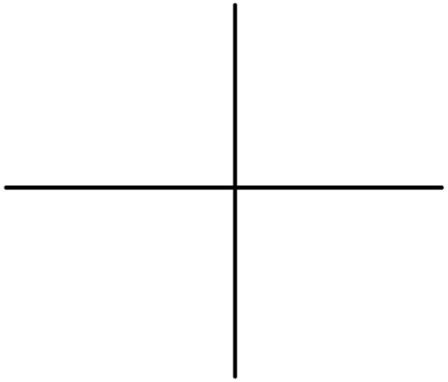
Is $\mathcal{V}(xy - \epsilon)$ reducible or irreducible?

Theory vs. Practice

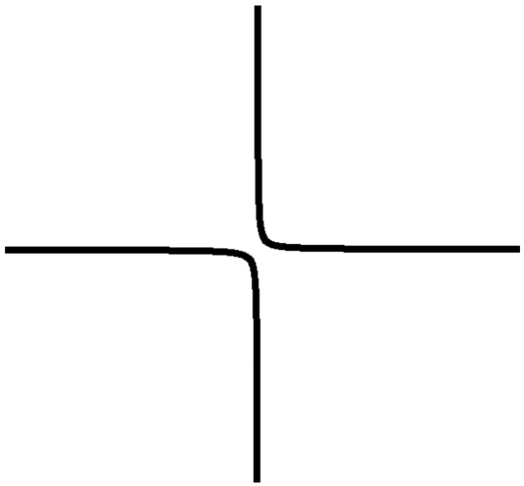
Is $\mathcal{V}(xy-\epsilon)$ reducible or irreducible?

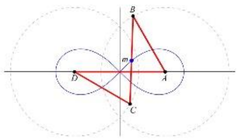
Theory:

- Reducible if $\epsilon = 0$: $\mathcal{V}(xy) = \mathcal{V}(x) \cup \mathcal{V}(y)$



- Irreducible if $\epsilon \neq 0$





Theory vs. Practice

Is $\mathcal{V}(xy - \epsilon)$ reducible or irreducible?

Practice:

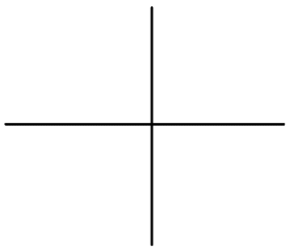
- Problem is *ill-posed*
  - answer does not depend continuously on ϵ

Theory vs. Practice

Is $\mathcal{V}(xy - \epsilon)$ reducible or irreducible?

Practice:

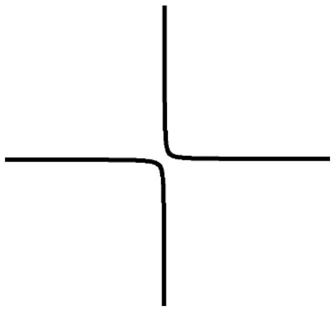
- Is $\epsilon \neq 0$ due to numerical noise or truly nonzero
  - If numerical noise: set to 0 – reducible



  - If nonzero: solve a rescaled version – irreducible

$$f(x, y) = xy - \epsilon$$

$$g(\hat{x}, \hat{y}) = \frac{1}{\epsilon} f(\hat{x}\sqrt{\epsilon}, \hat{y}\sqrt{\epsilon}) = \hat{x}\hat{y} - 1$$



Theory vs. Practice

For numerical methods:

- Solve well-posed, well-conditioned, and num. stable problems!

Theory vs. Practice

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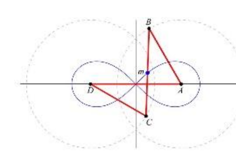
$$
\begin{aligned}
&220701798714766542157344369814603731920649470787977482 09t^6 \\
&+ 55858313927257191953451634705163103627058890428440103 28t^5 \\
&+ 14175569812724447393500233789877848531491265t^4W \\
&- 4477180786035007172164248960407378691578283216077040398 64t^4 \\
&- 865676553865719012232365931516983629620274 40t^3W \\
&+ 571145297696983576247423064 75t^2W^2 \\
&+ 474302309016648096934423520799618219755274954155075926592t^3 \\
&+ 1928563420712290077234813561834612137380576 80t^2W \\
&- 19430270604360445325875295940 0tW^2 - 26371599148125W^3 \\
&+ 2341397816853864817617847981162945070584483528261510775184t^2 \\
&- 1835288562819411262638933768610093443263299 20tW \\
&+ 16496924410592194938861213540 0W^2 \\
&- 539025869397077269511781194383341975448880792033814574656 0t \\
&+ 615504990697001734787240630893876548123084 00W \\
&+ 3193966974265623365398753846860968247266969720956505401600.
\end{aligned}
$$



Theory vs. Practice

What is the difference locally at the origin between

$$f(x,y) = \begin{bmatrix} y - x^2 \\ y^{100} \end{bmatrix} \quad \text{and} \quad g(x,y) = y - x^2?$$

Theory vs. Practice

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Theory:

- f: origin is isolated of multiplicity 200
- g: origin lies on a positive-dimensional component

Theory vs. Practice

What is the difference locally at the origin between

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Practice:

- For $C = \{(x, x^2) \mid |x| < 1/2\}$:
  - $g = 0$ on C
  - $\|f\| \leq 10^{-60}$ on C

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Practice:

- For $C = \{(x, x^2) \mid |x| < 1/2\}$:
  - $g = 0$ on C
  - $\|f\| \leq 10^{-60}$ on C

- Difference is some WD-40

Theory vs. Practice

At 2001 Computational Kinematics Workshop:

- Demonstrated this was a highly accurate machine

Theory: isolated solution of multiplicity 4

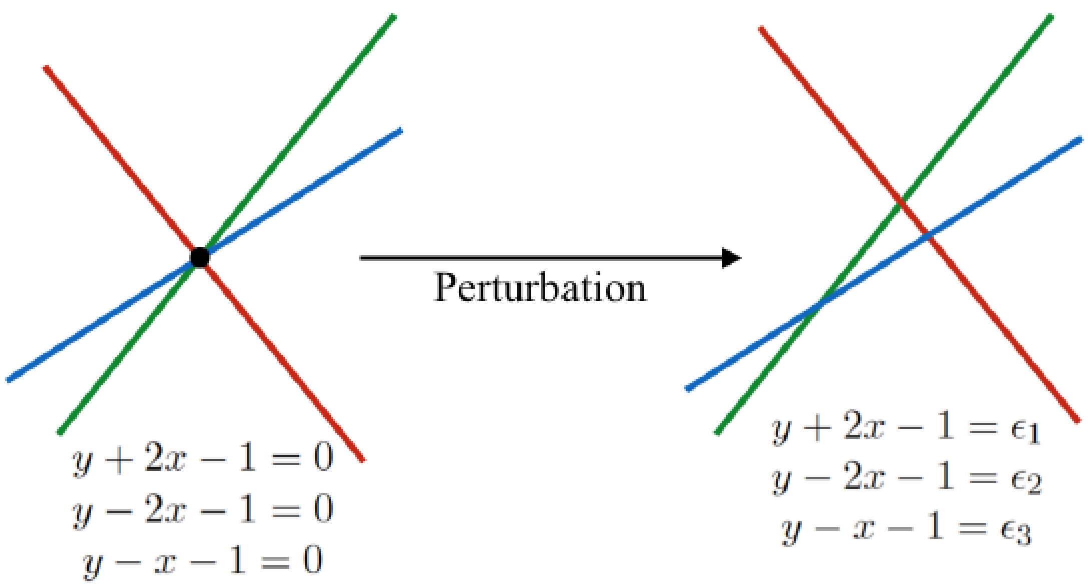
- It should not move but does due to multiplicity, joint tolerances, and link elasticity

F. Park et al.
Seoul National University

Symbolic vs. Numeric

Generally speaking:

- Numerical methods prefer well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension = # equations
  - stable under perturbations



Symbolic vs. Numeric

Generally speaking:

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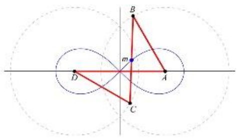
- Gröbner basis methods prefer vastly over-determined systems
  - fewer "new" polynomials to compute
  - Bardet-Faugere-Salvy (2004)

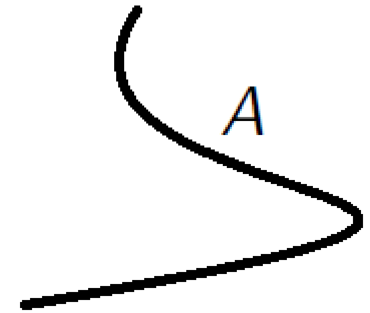
The result of an exact Gröbner basis computation is a proof.

- Num. alg. geom. replaces certainty with "probability 1"



Symbolic vs. Numeric

How to represent an irreducible algebraic variety A on a computer?



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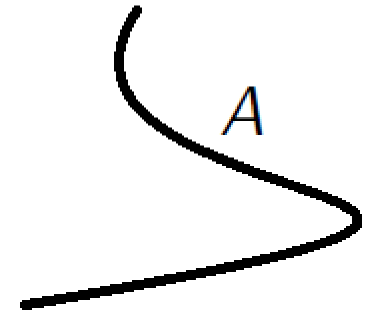


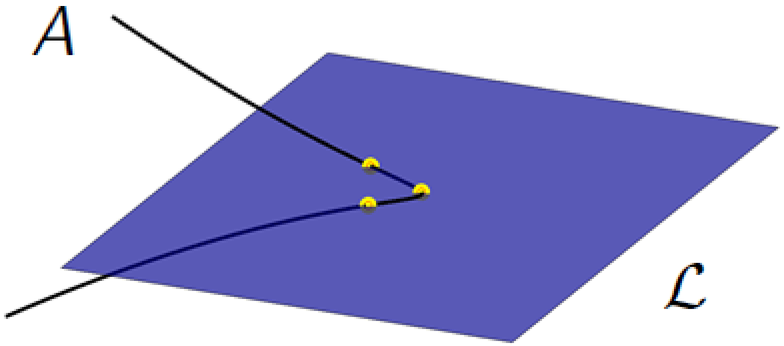
- algebraic: prime ideal $I(A) = \{g \mid g(a) = 0 \text{ for all } a \in A\}$
  - Hilbert Basis Theorem (1890): there exists $f_1, \ldots, f_k$ such that

$$I(A) = \langle f_1, \ldots, f_k \rangle$$

Witness Set

How to represent an irreducible algebraic variety A on a computer?

- geometric: witness set $\{f, \mathcal{L}, W\}$ where
  - f is polynomial system where A is an irred. component of $\mathcal{V}(f)$
  - $\mathcal{L}$ is a linear space with $\mathrm{codim}\ \mathcal{L} = \dim\ A$
  - $W = \mathcal{L} \cap A$ where $\#W = \deg\ A$

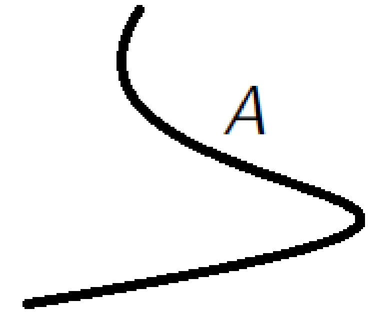


Witness Set

Example

$A = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

- $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$



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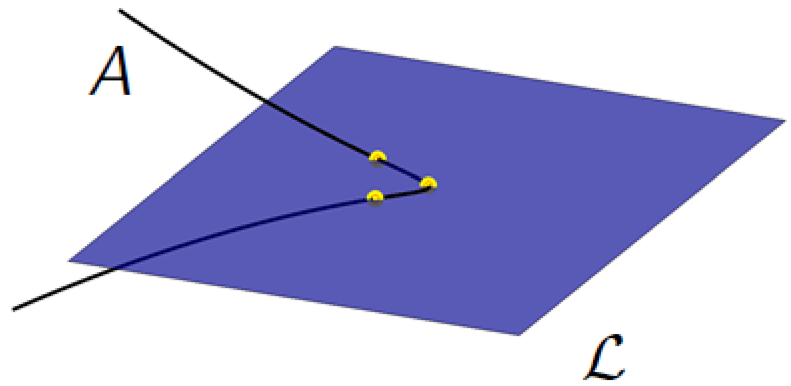
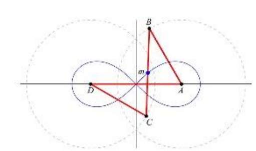


- $\{f, \mathcal{L}, W\}$ where
  - $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$
  - $\mathcal{L} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0\} \subset \mathbb{P}^3$
    - $\mathrm{codim}\ \mathcal{L} = \dim A = 1$
  - $W = \left\{ \begin{array}{l} [1, 3.2731, 10.7130, 35.0644], \\ [1, 0.8596, 0.7389, 0.6351], \\ [1, -2.1326, 4.5481, -9.6995] \end{array} \right\}$
    - $\deg A = 3$



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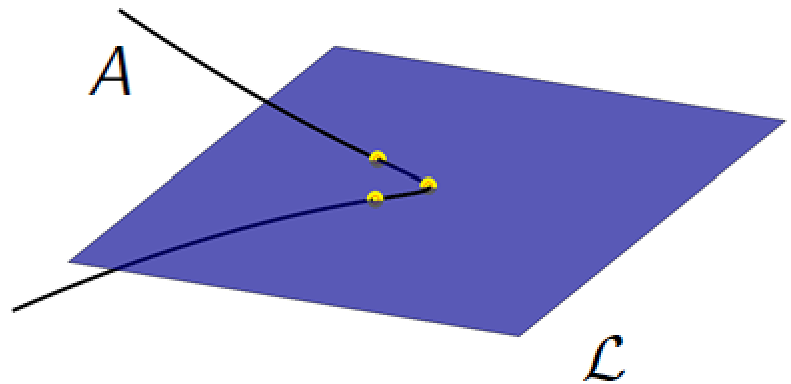
- $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$

- $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$



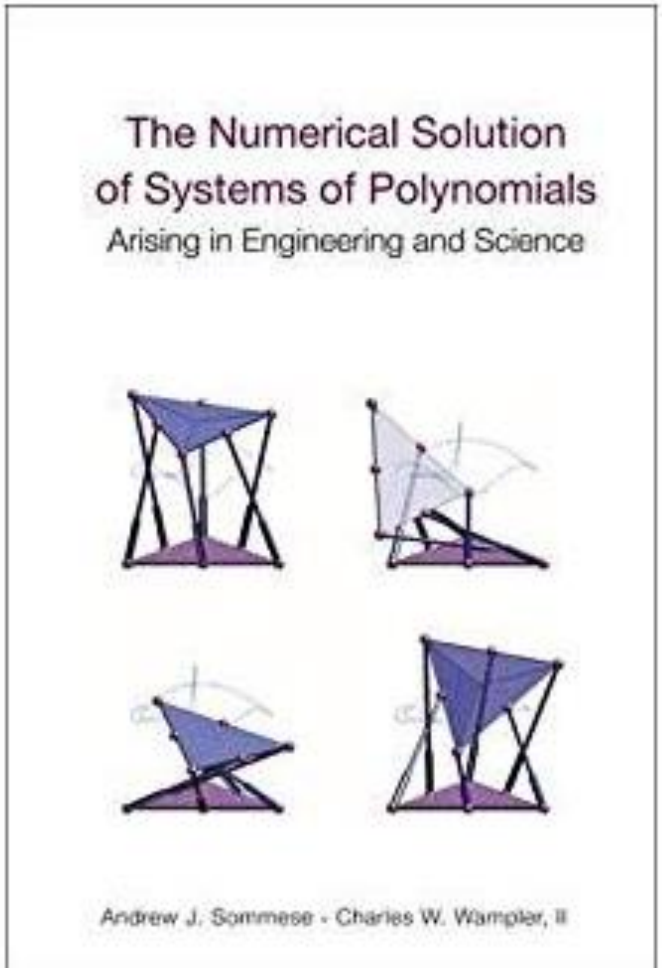
$$\mathcal{V}(f) = A \cup \{x_0 = x_1 = 0\}$$

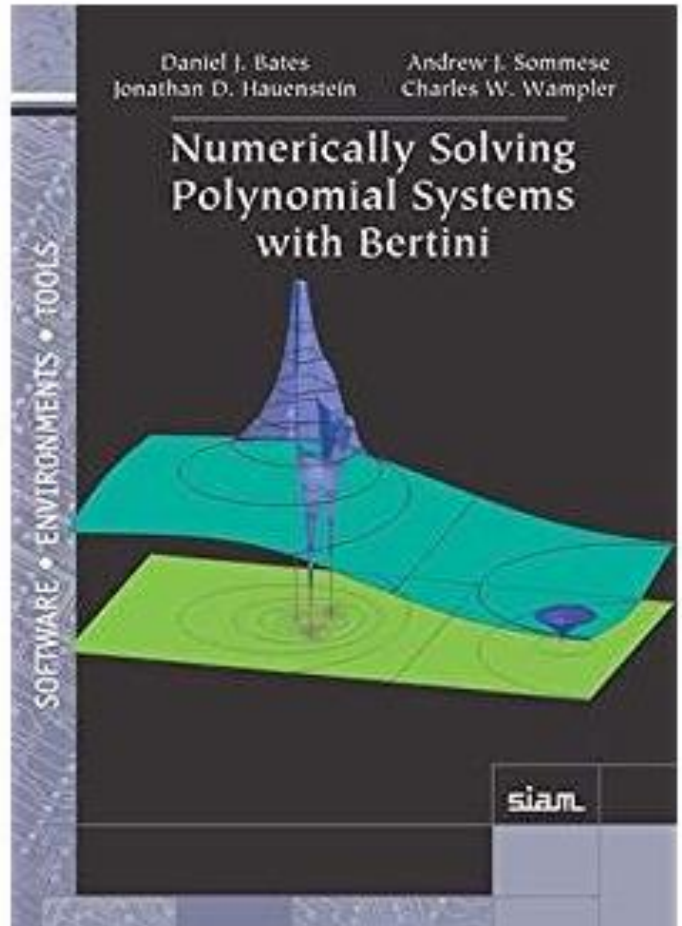
- Witness sets "localize" computations to A effectively ignoring the other irreducible components.

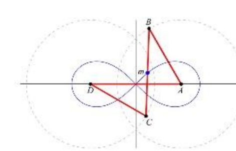
- Sample points from A by moving the linear slice $\mathcal{L}$.

Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:



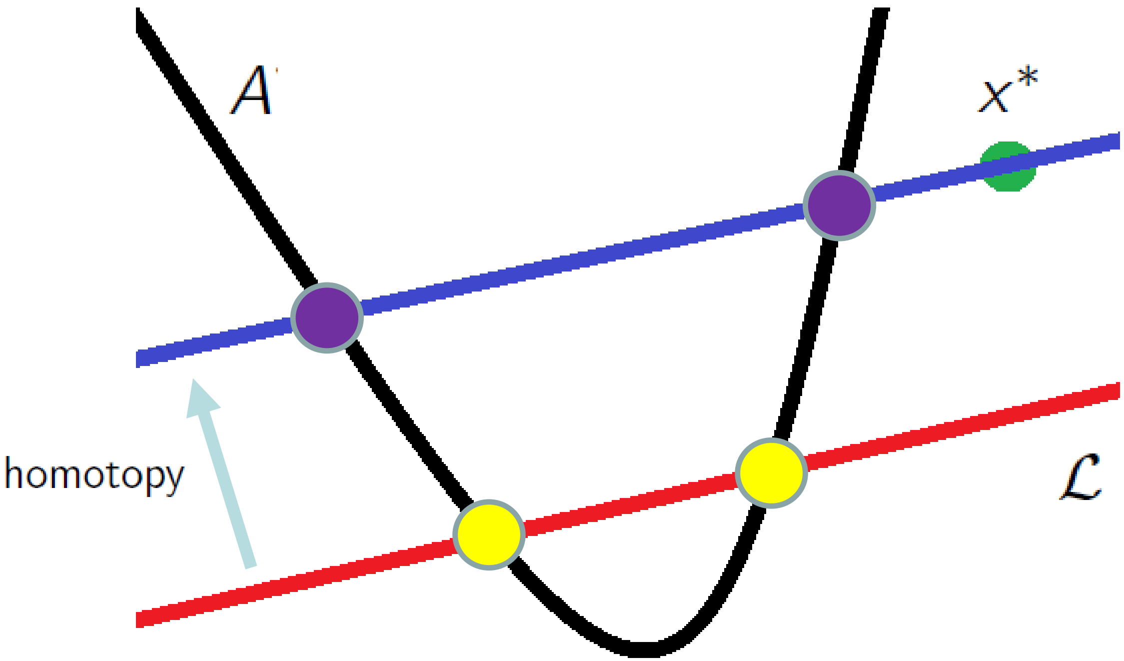




Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

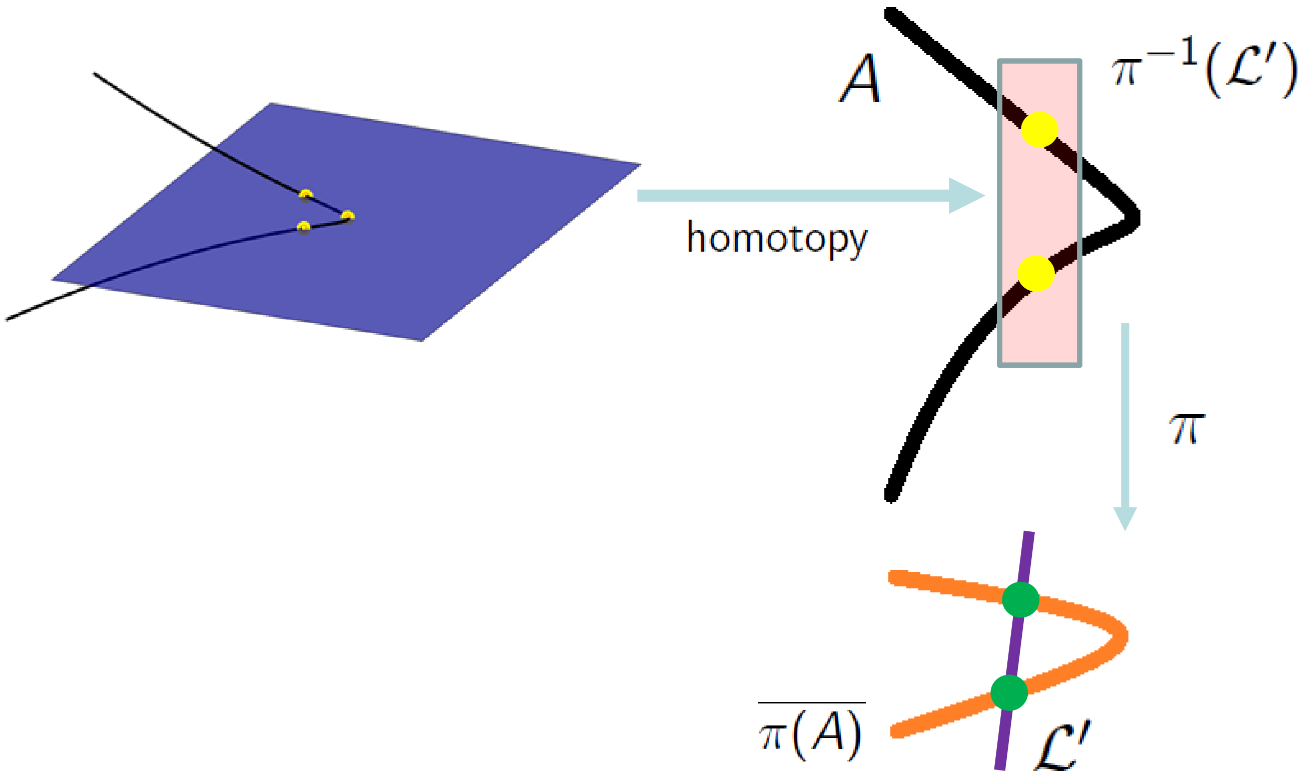
- membership testing: is $x^* \in A$?
  - decide if $g(x^*) = 0$ for every $g \in I(A)$ without knowing $I(A)$



Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

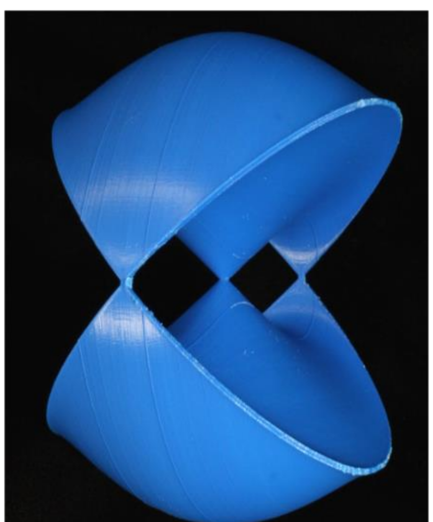
- projection: $\overline{\pi(A)}$
  - perform computations on $\overline{\pi(A)}$ without knowing *any* polynomials that vanish on $\overline{\pi(A)}$

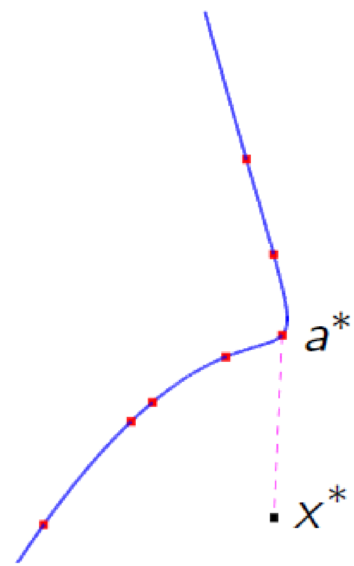


Witness Set

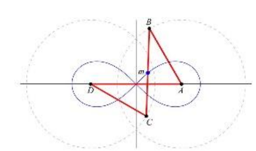
Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- intersection: $A \cap B$
  - special case is *regeneration*
    - $\mathcal{V}(f_1, \ldots, f_k, f_{k+1}) = \mathcal{V}(f_1, \ldots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
  - compute A_{sing}
  - compute critical points of optimization problem





$\min \|x^* - a\|_2$ such that $a \in A \cap \mathbb{R}^n$

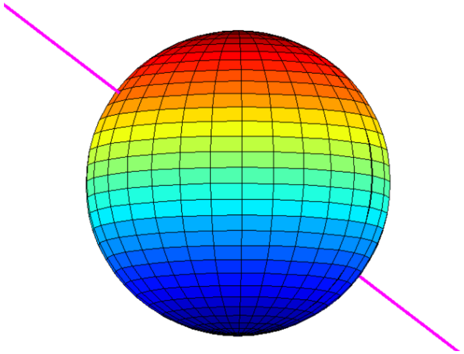


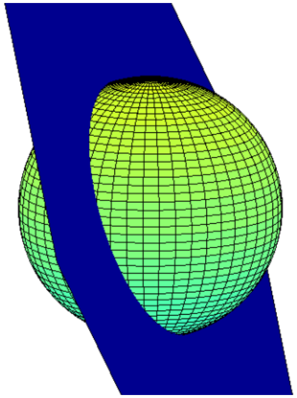
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

Test other algebraic properties of A

- is A arithmetically Cohen Macaulay?
- is A arithmetically Gorenstein?
- is A a complete intersection?





Algebraic properties

Example

$$A = \sigma_4(\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^4) \subset \mathbb{P}^{35}$$

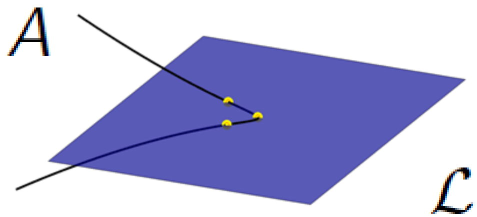
- $\dim A = 31$
- $\deg A = 345$

- $I(A)$ contains 10 poly. of degree 6 and 20 poly. of degree 9
  - Bates-Oeding (2011), Friedland-Gross (2012)
- used sampling to show that A was aCM and that these polynomials generate $I(A)$

N.S. Daleo and J.D. Hauenstein,
Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme.
J. Symb. Comput., 72, 128-146, 2016.

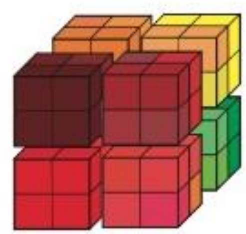
Sampling

Sample points from A by moving the linear slice $\mathcal{L}$.

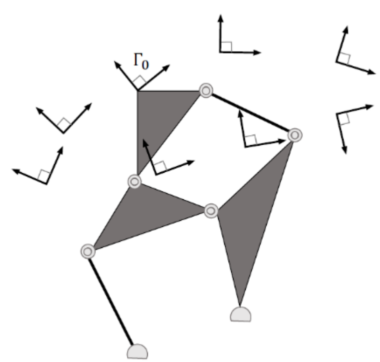


How to utilize sample point(s) to extract data?

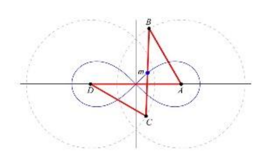
- Vanishing polynomials



- Sampling for solving sum of squares (SOS) programs
- Degree estimation

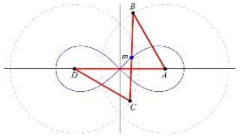


- Topological properties



Vanishing polynomials

For many varieties A, the only known polynomial in $I(A)$ is $f \equiv 0$.



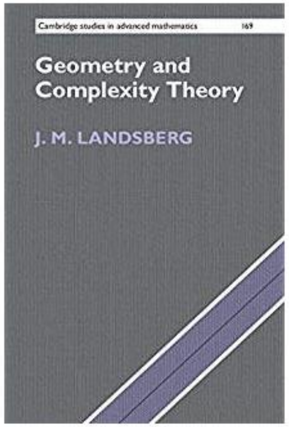
Vanishing polynomials

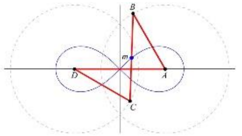
For many varieties A, the only known polynomial in $I(A)$ is $f \equiv 0$.

Problem: Compute the exponent ω of matrix multiplication.

- smallest constant such that two $n \times n$ matrices can be multiplied using $O(n^{\omega+\epsilon})$ arithmetic operations for every $\epsilon > 0$
- Current state of the art: $2 \leq \omega \leq 2.374$

- Could be solved by knowing polynomials that vanish on secant varieties – Landsberg (2017).





Vanishing polynomials

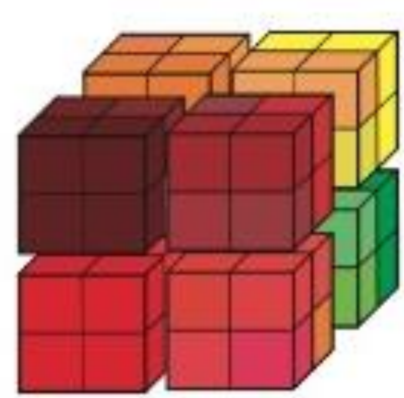
Compute homogeneous polynomials that vanish on

$$\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \overline{\left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \;\middle|\; a_i, b_i, c_i \in \mathbb{C}^4 \right\}} \subset \mathbb{P}^{63}$$

- 6^{th} secant variety of $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$ in $\mathbb{P}^{63}$
- $\dim = 59$

  - If $a, b, c \in \mathbb{C}^4$, then $a \otimes b \otimes c \in \mathbb{C}^{4\times 4\times 4}$ with

$$(a \otimes b \otimes c)_{ijk} = a_i \cdot b_j \cdot c_k$$



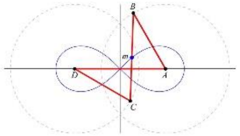
J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Exp. Math., 22(4), 372-383, 2013.

Vanishing polynomials

Cast as a classical elimination problem:

- Eliminate a's, b's, c's from

$$\sum_{\ell=1}^{6} a_{\ell i} \cdot b_{\ell j} \cdot c_{\ell k} - z_{ijk} = 0 \quad \text{where} \quad i, j, k = 1, \ldots, 4.$$



Vanishing polynomials

Cast as a classical elimination problem:

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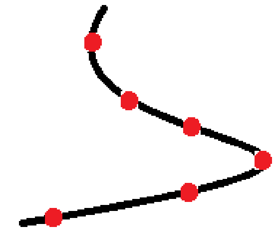
$$\sum_{\ell=1}^{6} a_{\ell i} \cdot b_{\ell j} \cdot c_{\ell k} - z_{ijk} = 0 \quad \text{where} \quad i, j, k = 1, \ldots, 4.$$

Still waiting for Gröbner basis methods to terminate....

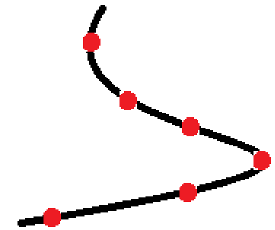
Vanishing polynomials



Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute f where $f(a_i) = 0$.

Vanishing polynomials



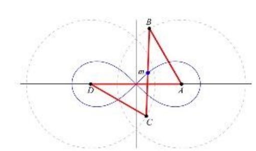
Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute f where $f(a_i) = 0$.

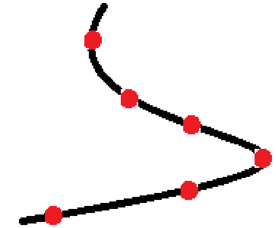
Example

Find homogeneous quadratic polynomials vanishing on:

$$[1,1,1,1], [1,-1,1,-1], [1,2,4,8], [1,-2,4,-8], [1,3,9,27],$$
$$[1,-3,9,-27], [1,4,16,64], [1,-4,16,-64], [1,5,25,125], [1,-5,25,-125]$$



Vanishing polynomials



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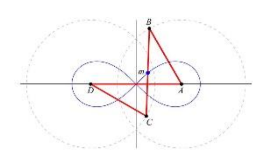
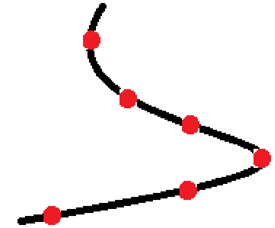
$$[1,-3,9,-27], [1,4,16,64], [1,-4,16,-64], [1,5,25,125], [1,-5,25,-125]$$

x_0^2	x_0x_1	x_0x_2	x_0x_3	x_1^2	x_1x_2	x_1x_3	x_2^2	x_2x_3	x_3^2
1	1	1	1	1	1	1	1	1	1
1	−1	1	−1	1	−1	1	1	−1	1
1	2	4	8	4	8	16	16	32	64
1	−2	4	−8	4	−8	16	16	−32	64
1	3	9	27	9	27	81	81	243	729
1	−3	9	−27	9	−27	81	81	−243	729
1	4	16	64	16	64	256	256	1024	4096
1	−4	16	−64	16	−64	256	256	−1024	4096
1	5	25	125	25	125	625	625	3125	15625
1	−5	25	−125	25	−125	625	625	−3125	15625



Vanishing polynomials

Example



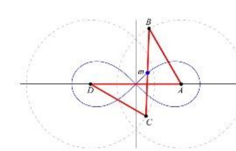
Find homogeneous quadratic polynomials vanishing on:

$[1,1,1,1], [1,-1,1,-1], [1,2,4,8], [1,-2,4,-8], [1,3,9,27],$
$[1,-3,9,-27], [1,4,16,64], [1,-4,16,-64], [1,5,25,125], [1,-5,25,-125]$

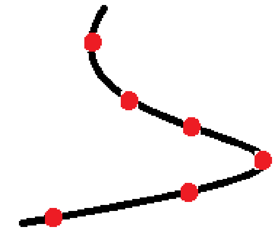
x_0^2	x_0x_1	x_0x_2	x_0x_3	x_1^2	x_1x_2	x_1x_3	x_2^2	x_2x_3	x_3^2
1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	1	1	-1	1
1	2	4	8	4	8	16	16	32	64
1	-2	4	-8	4	-8	16	16	-32	64
1	3	9	27	9	27	81	81	243	729
1	-3	9	-27	9	-27	81	81	-243	729
1	4	16	64	16	64	256	256	1024	4096
1	-4	16	-64	16	-64	256	256	-1024	4096
1	5	25	125	25	125	625	625	3125	15625
1	-5	25	-125	25	-125	625	625	-3125	15625

3-dimensional null space is generated by:

$$x_1^2 - x_0x_2, \quad x_1x_2 - x_0x_3, \quad x_2^2 - x_1x_3$$



Vanishing polynomials



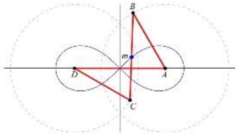
Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute f where $f(a_i) = 0$.

Problem is the number of sample points needed:

- To show no nonconstant polynomials of degree 18 vanish on $\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) \subset \mathbb{P}^{63}$, need

$$N \geq \binom{63+18}{18} \approx 4.567 \cdot 10^{17}$$



Vanishing polynomials

When all else fails, solve a different problem.

- partial information is better than no information

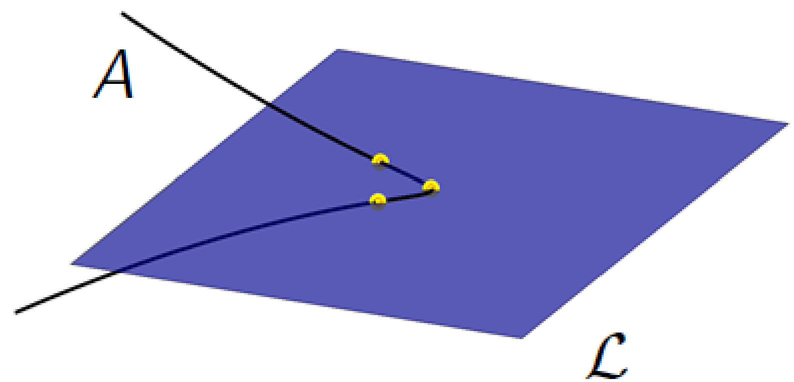
Vanishing polynomials

When all else fails, solve a different problem.

- partial information is better than no information

What polynomials vanish on the set of witness points $A \cap \mathcal{L}$?

- If f vanishes on A, then $f\big|_{\mathcal{L}}$ vanishes on $A \cap \mathcal{L}$.



- Exact correspondence when arithmetically Cohen-Macaulay
- Upper bounds in general

Example

Vanishing polynomials

$$\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \overline{\left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \;\middle|\; a_i, b_i, c_i \in \mathbb{C}^4 \right\}} \subset \mathbb{P}^{63}$$

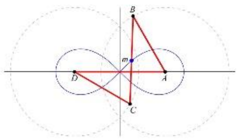
- $\dim = 59$
- $\deg = 15{,}456$

Restricting to $\dim$ 4 linear space $\mathcal{L}$

- To show no nonconstant polynomials of degree 18 vanish:

$$N \geq \binom{4+18}{18} = 7315$$

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Exp. Math., 22(4), 372-383, 2013.



Example

Vanishing polynomials

$$\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \overline{\left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \;\middle|\; a_i, b_i, c_i \in \mathbb{C}^4 \right\}} \subset \mathbb{P}^{63}$$

- $\dim = 59$
- $\deg = 15{,}456$

Interpolating witness point set shows

- No nonconstant polynomials of degree ≤ 18 vanish
- 64 polynomials of degree 19 restricted to $\mathcal{L}$ vanish
  - Go search for polynomials of degree 19!

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Exp. Math., 22(4), 372-383, 2013.

Example

Vanishing polynomials

$$\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \overline{\left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \;\middle|\; a_i, b_i, c_i \in \mathbb{C}^4 \right\}} \subset \mathbb{P}^{63}$$

Representation theory proves existence of 64 polynomials of degree 19 that vanish.

- Used to prove that 2×2 matrix multiplication tensor is not contained in $\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4)$.
- Rank and border rank of 2×2 matrix multiplication tensor is 7

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Exp. Math., 22(4), 372-383, 2013.

SOS programs

It is possible to interpolate over other families of polynomials

- Cifuentes-Parrilo (2017) interpolate sums of squares modulo an ideal without knowing the ideal using sample points

Given polynomial p, compute $g_1, \ldots, g_k$ such that

$$p \equiv \sum_{i=1}^{k} g_i^2 \mod I(A)$$

assuming such a decomposition exists.

- Certificate that $p \geq 0$ on $A \cap \mathbb{R}^n$.

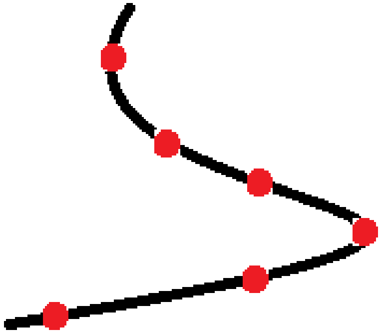
SOS programs

A necessary condition for

$$p \equiv \sum_{i=1}^{k} g_i^2 \mod I(A)$$

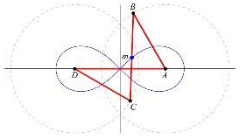
is, for samples $a_1, \ldots, a_N \in A$,

$$p(a_j) = \sum_{i=1}^{k} g_i(a_j)^2$$



- Computation performed using semidefinite program

D. Cifuentes and P.A. Parrilo,
Sampling algebraic varieties for sum of squares programs.
SIAM J. Opt., 27(4), 2381–2404, 2017.



SOS programs

Example (Trace ratio)

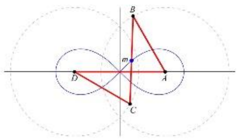
$$A_{n,k} = \{X \in \mathbb{R}^{n\times n} \mid X^T = X, X^2 = X, \mathrm{trace}(X) = k\}$$

Given symmetric matrix $X, Y, Z \in \mathbb{R}^{n\times n}$ where $Y \succ 0$, solve

$$\begin{array}{ll} \max & \gamma \\ \text{s.t.} & \mathrm{trace}(Y\alpha)(\gamma - \mathrm{trace}(Z\alpha)) - \mathrm{trace}(X\alpha) \equiv F(\alpha) \quad \text{mod } I(A_{n,k}), \\ & F \text{ is SOS}, \\ & \deg F = 2. \end{array}$$

n	k	Equations SDP variables	Equations SDP constraints	Equations SDP time(s)	Sampling SDP variables	Sampling SDP constraints	Sampling SDP time(s)	Gröbner bases time(s)
4	2	342	188	0.47	56	45	0.10	0.00
5	3	897	393	0.71	121	105	0.11	0.02
6	4	2062	738	1.34	232	210	0.15	0.20
7	5	4265	1277	3.62	407	378	0.19	6.04
8	6	8106	2073	9.06	667	630	0.34	488.17
9	7	14387	3198	23.83	1036	990	0.61	out of memory
10	8	24142	4733	58.17	1541	1485	1.18	out of memory

D. Cifuentes and P.A. Parrilo,
Sampling algebraic varieties for sum of squares programs.
SIAM J. Opt., 27(4), 2381–2404, 2017.

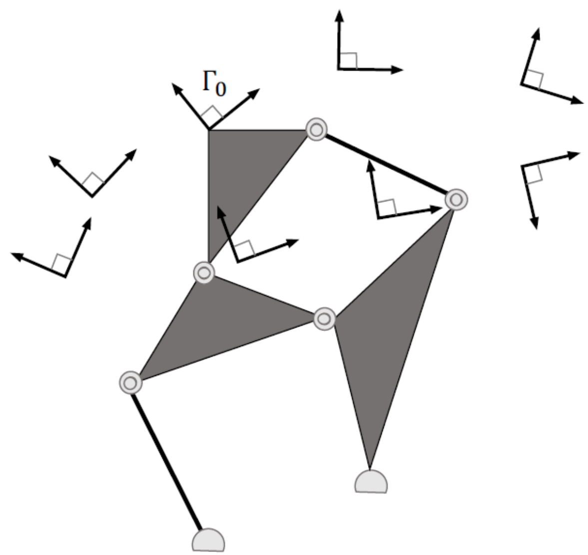


Degree estimation

Can we estimate the degree of a variety by sampling?

Example

How many 6-bar Watt I linkages obtain 8 given poses?



Multihomogeneous Bézout bound: $3.43 \cdot 10^{10}$

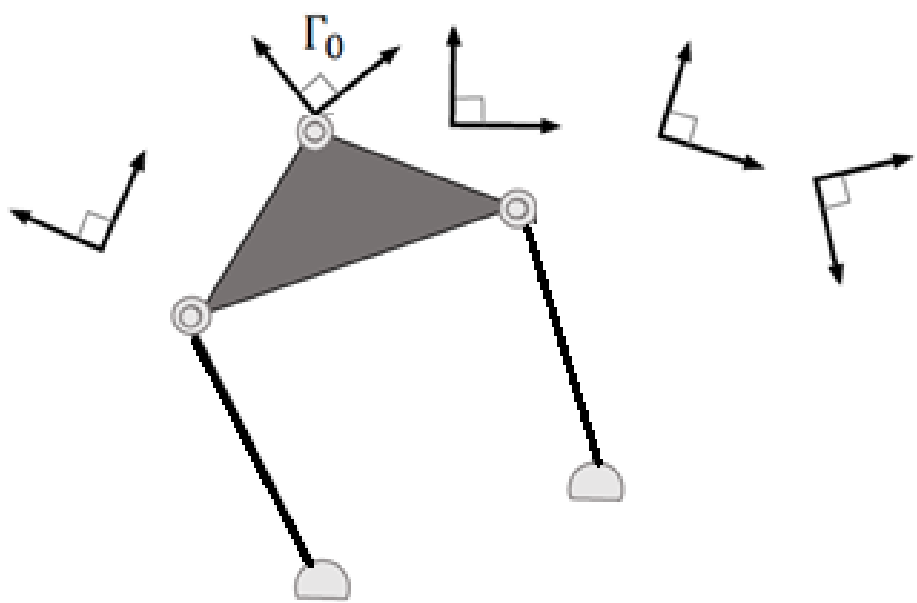
J.D. Hauenstein and S.N. Sherman,
Statistically estimating the number of solutions to motion generation problems.
In preparation.

Degree estimation

How many 6-bar Watt I linkages obtain 8 given poses?

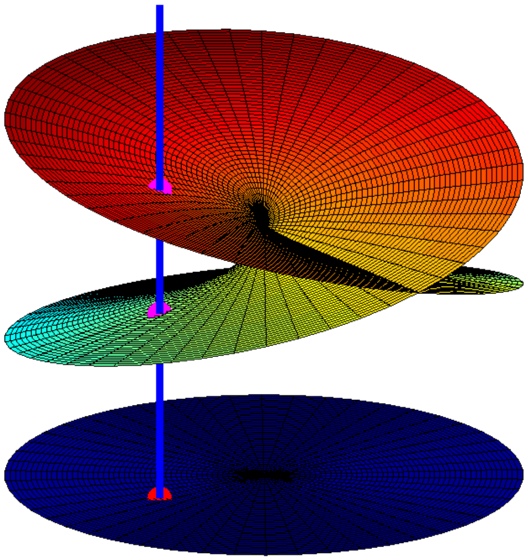
Corresponding problem for 4-bar linkages – Burmester (1886)

- 4 solutions for 5 poses



Degree estimation

Given one point in a witness point set $W = A \cap \mathcal{L}$, generate another point (possibly same point) by using a monodromy loop.

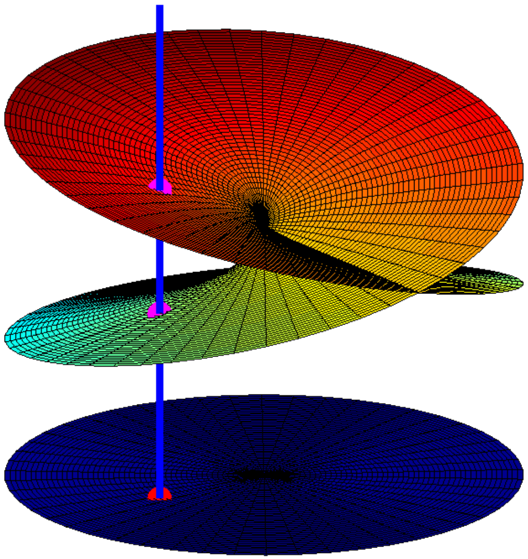


`MonodromySolver`

- Duff-Hill-Jensen-Lee-Leykin-Sommars (2018)
- Bliss-Duff-Leykin-Sommars (2018)

Degree estimation

Given one point in a witness point set $W = A \cap \mathcal{L}$, generate another point (possibly same point) by using a monodromy loop.



- IF we assume that we can generate random subsets of W, we can estimate $\#W$.

Degree estimation

Example (German tank problem)

WWII: Estimate # of tanks Germany was producing per month.

- use serial numbers on parts recovered
  - assume uniformly distributed to generate statistical estimate

Degree estimation

Example (German tank problem)

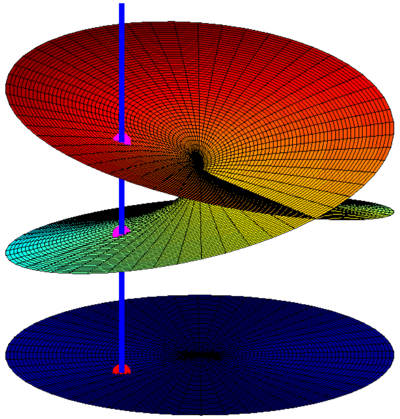
WWII: Estimate # of tanks Germany was producing per month.

- use serial numbers on parts recovered
    - assume uniformly distributed to generate statistical estimate

month	statistical est.	intelligence est.	German records
June 1940	169	1000	122
June 1941	244	1550	271
August 1942	327	1550	342

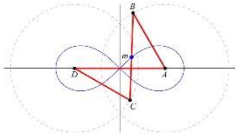
Degree estimation

Hypergeometric estimate of $\deg A = \#W$:

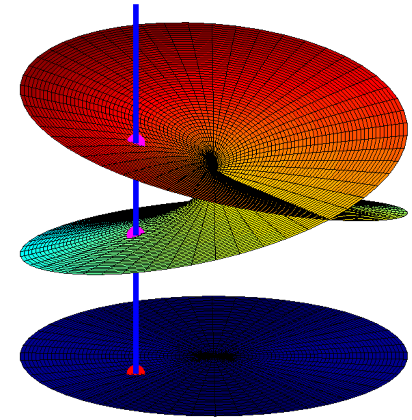


$$\deg A = \#W \approx \frac{n}{p}$$

- $n =$ number of points already known in W
- $p =$ ratio of repeats in sample



Degree estimation



Hypergeometric estimate of $\deg A = \#W$:

$$\deg A = \#W \approx \frac{n}{p}$$

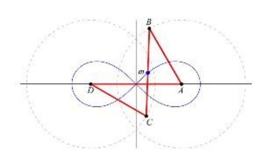
- n = number of points already known in W
- p = ratio of repeats in sample

Example

Assume that $n = 10$ points are already known in W.

- Monodromy loop provides 8 new points and 2 repeats:

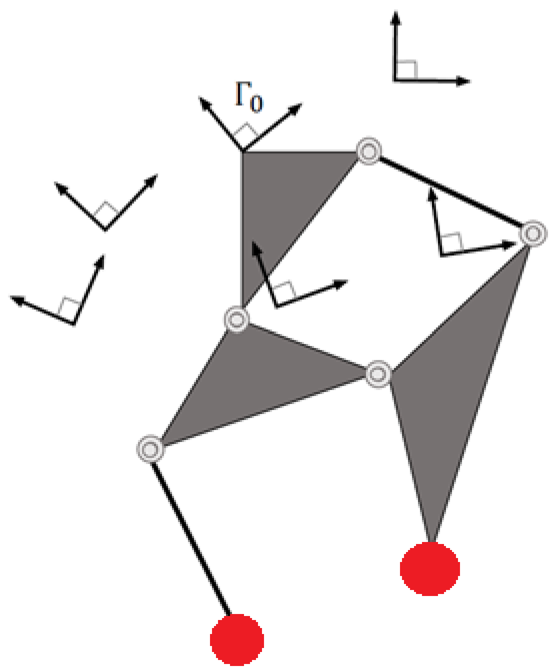
$$\#W \approx \frac{10}{2/10} = 50$$



Degree estimation

Validate statistical model:

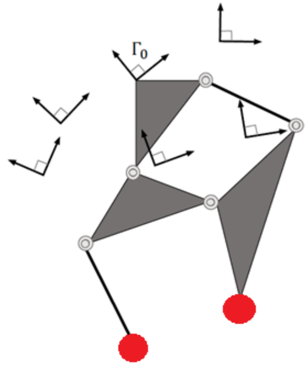
- Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.
  - Problem was studied by Plecnik-McCarthy-Wampler (2014)



Degree estimation

Example

- Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.



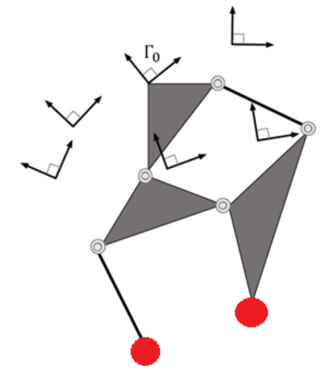
Perform monodromy loops starting from $n = 1000$ known solutions.

- Mean from 10 monodromy loops:
  - Ratio of repeats: 17.51%
  - Estimated number of solutions: 5750.5

Degree estimation

Example

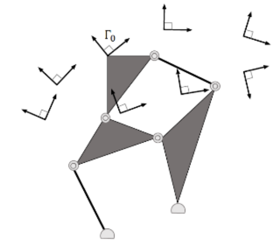
- Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.



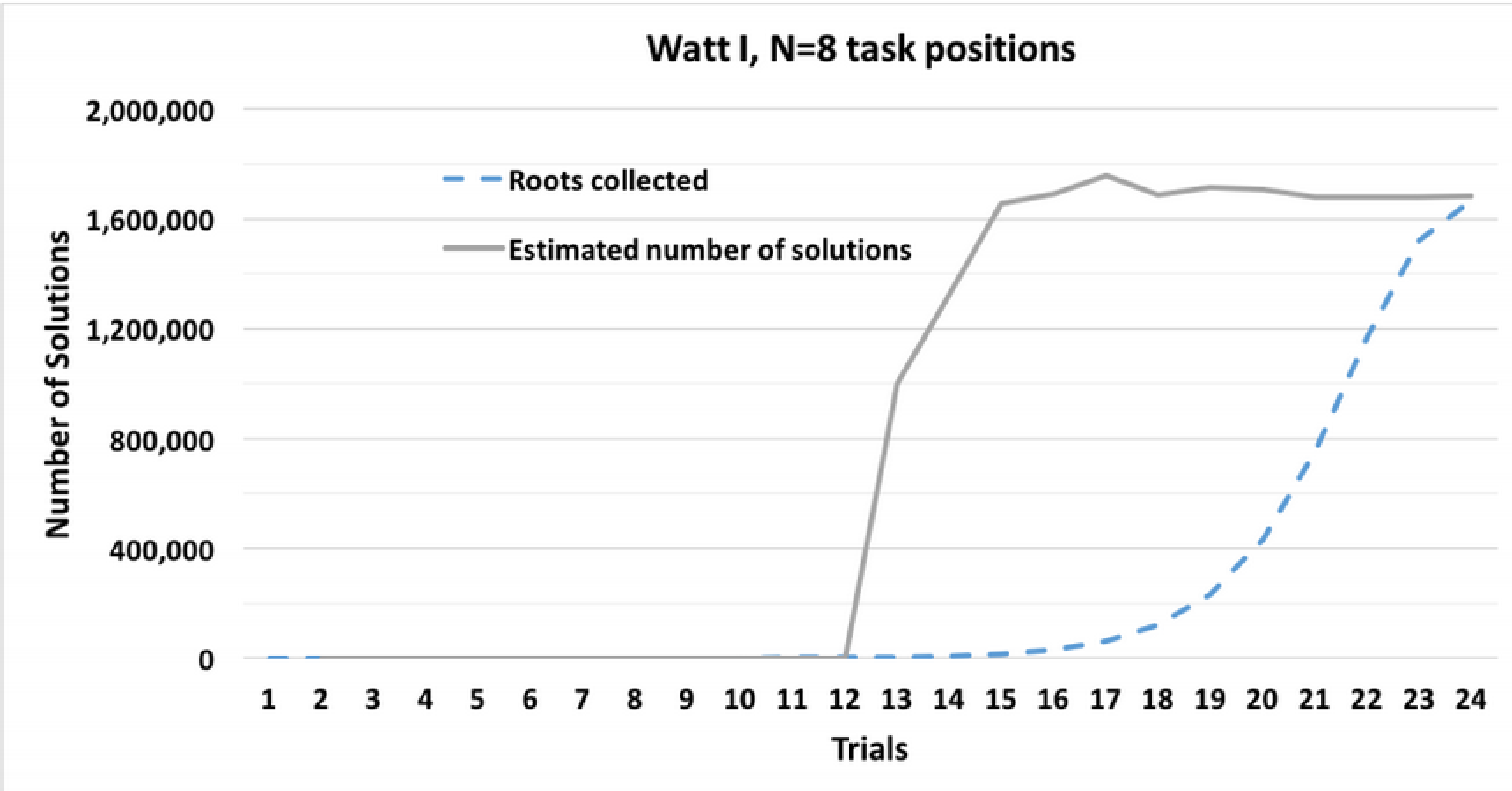
Perform monodromy loops starting from $n = 1000$ known solutions.

- Mean from 10 monodromy loops:
  - Ratio of repeats: 17.51%
  - Estimated number of solutions: 5750.5
- Theoretical values:
  - Ratio of repeats: 17.38%
  - Number of solutions: 5754

Degree estimation



How many 6-bar Watt I linkages obtain 8 given poses?



J.D. Hauenstein and S.N. Sherman,
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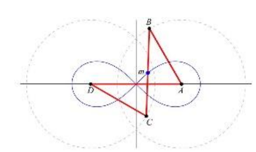
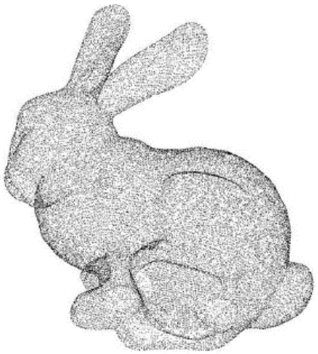
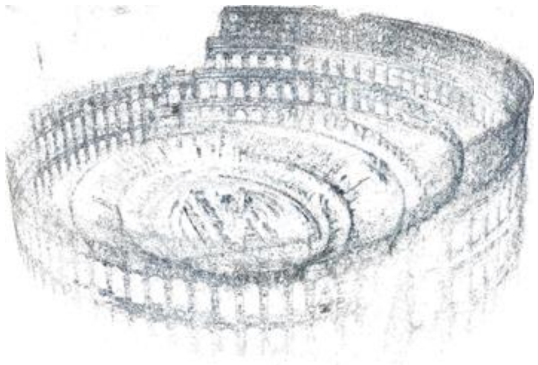
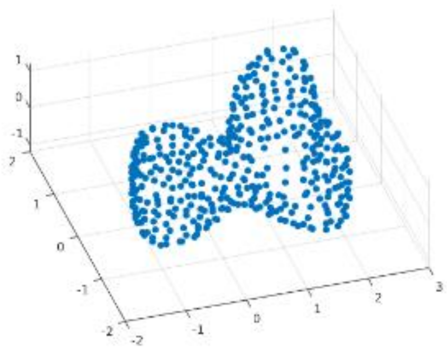


Topological properties

Objects can be identified from point clouds (set of sample points):



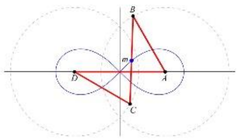




$\beta_0 = 1, \beta_1 = 0, \beta_2 = 1$

Topology of real algebraic surfaces from point clouds:

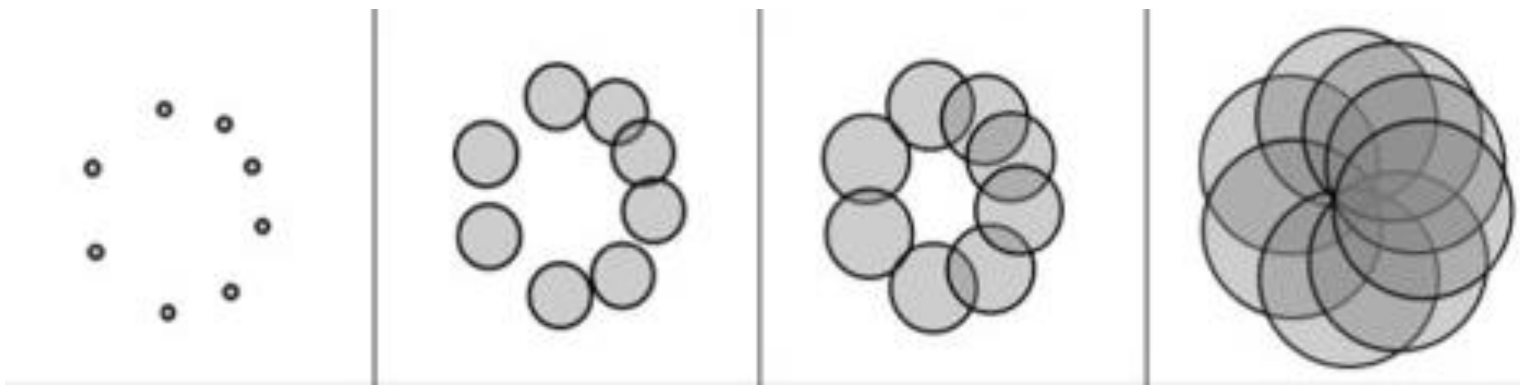
Niyogi-Smale-Weinberger (2008)
Cucker-Krick-Shub (2016)
Dufresne-Edwards-Harrington-H (2018)
Breiding-Kalisnik Verovsek-Sturmfels-Weinstein (2018)
...



Topological properties

Persistent homology

- treat each point as center of a ball where radius changes
- determine features which persist over wide range of radii

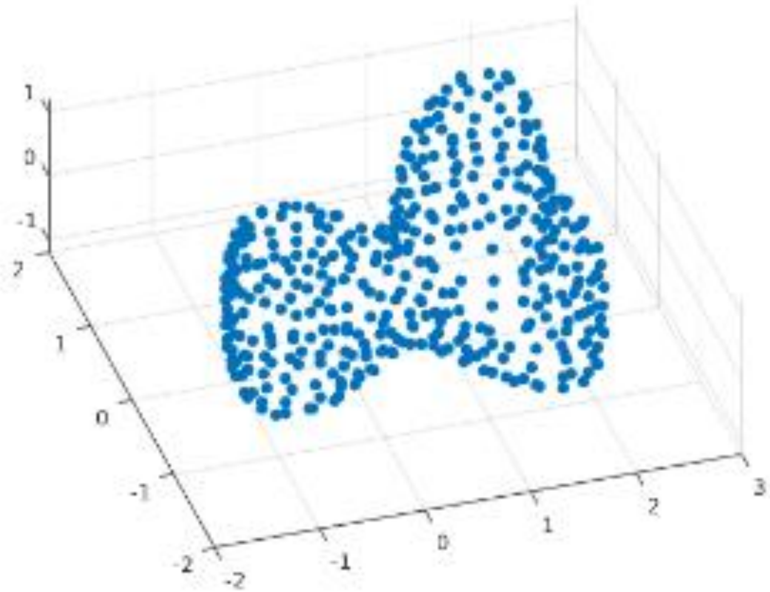


With provably dense samples, we can prove theorems about when topological features must be present in the point cloud.

- Generate provably dense samples using num. alg. geom.

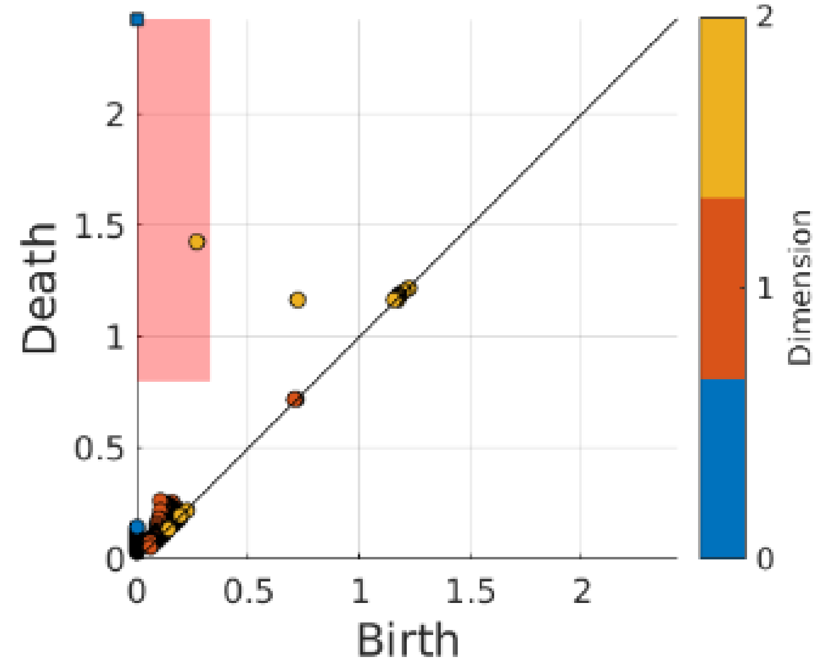
Topological properties

$$4x^4 + 7y^4 + 3z^4 - 3 - 8x^3 + 2x^2y - 4x^2 - 8xy^2 - 5xy + 8x - 6y^3 + 8y^2 + 4y$$



$$\beta_0 = 1, \beta_1 = 0, \beta_2 = 1$$

Persistence diagram

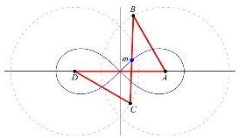


E. Dufresne, P.B. Edwards, H.A. Harrington, and J.D. Hauenstein,
Sampling real algebraic varieties for topological data analysis.
Preprint available at `https://www.nd.edu/~jhauenst/preprints/dehhTDANAG.pdf`

- Shed some light on what makes some computations difficult
  - Solve well-posed, well-conditioned, and num. stable problems!

- Explain witness sets and some applications of sampling
  - With many omissions (sorry)

- See software “in action” during software demonstration with
  - Jose Rodriguez
  - Danielle Brake
  - Maggie Regan



Thank You!