Applications of Sampling in Numerical Algebraic Geometry

Jonathan Hauenstein
Applications of Polynomial Systems
NSF CBMS TCU
June 5, 2018
Overview

- Musings on numerical algebraic geometry
- Witness sets
- Applications of sampling
“In theory, theory and practice are the same. In practice, they are not.”

Albert Einstein
Theory vs. Practice

Let $f \in \mathbb{C}[x]$ be a univariate polynomial.

Theory:

- $f \equiv 0$ if and only if $f(x^*) = 0$ for general $x^* \in \mathbb{C}$.

If $f \neq 0$, then $\mathcal{V}(f) \subset \mathbb{C}$ has finitely many points. Hence, $\mathbb{C} \setminus \mathcal{V}(f)$ is a Zariski open dense subset of $\mathbb{C}$.

\[ \# \mathcal{V}(f) = 6 \]
Theory vs. Practice

Let $f \in \mathbb{C}[x]$ be a univariate polynomial.

Theory:

- $f \equiv 0$ if and only if $f(x^*) = 0$ for general $x^* \in \mathbb{C}$.
- $f \equiv 0$ if and only if $f(x^*) = 0$ for random $x^* \in \mathbb{C}$ with probability 1.

\[ \# \mathcal{V}(f) = 6 \]
Theory vs. Practice

Practice:

- Is $f$ known exactly or only approximately?
- What is the scaling of $f$?
  
    - $\mathcal{V}(f) = \mathcal{V}(10^{-1000000} \cdot f)$
- How to select a random point in $x^* \in \mathbb{C}$?
- How much error in evaluating $f(x^*)$?
- In the presence of error, what does it mean to be equal to 0?
  
    - Floating-point arithmetic: select from a finite subset of $\mathbb{C}$

$$\text{Prob(failure)} > 0$$

$$\#\mathcal{V}(f) = 6$$
Practice:

- Is \( f \) known exactly or only approximately?
- What is the scaling of \( f \)?
  - \( \mathcal{V}(f) = \mathcal{V}(10^{-1000000} \cdot f) \)
- How to select a random point in \( x^* \in \mathbb{C} \)?
- How much error in evaluating \( f(x^*) \)?
- In the presence of error, what does it mean to be equal to 0?
  - Floating-point arithmetic: select from a finite subset of \( \mathbb{C} \)

\[ \text{Prob}(\text{failure}) > 0 \]

Reduce failure rate by:

- using higher precision
- rescale
- reformulate (different geometric description?)
- take advantage of structure
- develop a different numerical approach (Simon Telen’s poster)
Theory vs. Practice

Is $\mathcal{V}(xy - \varepsilon)$ reducible or irreducible?
Theory vs. Practice

Is $\mathcal{N}(xy - \epsilon)$ reducible or irreducible?

Theory:

- Reducible if $\epsilon = 0$: $\mathcal{N}(xy) = \mathcal{N}(x) \cup \mathcal{N}(y)$

- Irreducible if $\epsilon \neq 0$
Theory vs. Practice

Is $\mathcal{V}(xy - \epsilon)$ reducible or irreducible?

Practice:

- Problem is *ill-posed*
  - answer does not depend continuously on $\epsilon$
Theory vs. Practice

Is $\mathcal{V}(xy - \epsilon)$ reducible or irreducible?

Practice:

- Is $\epsilon \neq 0$ due to numerical noise or truly nonzero
  
  - If numerical noise: set to 0 – reducible
  
  - If nonzero: solve a rescaled version – irreducible

\[
f(x, y) = xy - \epsilon
\]

\[
g(\hat{x}, \hat{y}) = \frac{1}{\epsilon} f(\hat{x}\sqrt{\epsilon}, \hat{y}\sqrt{\epsilon}) = \hat{x}\hat{y} - 1
\]
Theory vs. Practice

For numerical methods:

- Solve well-posed, well-conditioned, and num. stable problems!
Theory vs. Practice

For numerical methods:

- Solve well-posed, well-conditioned, and num. stable problems!

\[\begin{align*}
22070179871476654215734436981460373192064947078797748209t^6 \\
+ 5585831392725719195345163470516310362705889042844010328t^5 \\
+ 14175569812724447393500233789877848531491265t^4 W \\
- 447718078603500717216424896040737869157828321607704039864t^4 \\
- 86567655386571901223236593151698362962027440t^3 W \\
+ 57114529769698357624742306475t^2 W^2 \\
+ 474302309016648096934423520799618219755274954155075926592t^3 \\
+ 192856342071229007723481356183461213738057680t^2 W \\
- 194302706043604453258752959400t W^2 - 26371599148125W^3 \\
+ 2341397816853864817617847981162945070584483528261510775184t^2 \\
- 183528856281941126263893376861009344326329920t W \\
+ 164969244105921949388612135400W^2 \\
- 5390258693970772695117811943833419754488807920338145746560t \\
+ 61550499069700173478724063089387654812308400W \\
+ 3193966974265623365398753846860968247266969720956505401600. 
\end{align*}\]
Theory vs. Practice

What is the difference locally at the origin between

\[ f(x, y) = \begin{bmatrix} y - x^2 \\ y^{100} \end{bmatrix} \quad \text{and} \quad g(x, y) = y - x^2 \]
Theory vs. Practice

What is the difference locally at the origin between

\[ f(x, y) = \begin{bmatrix} y - x^2 \\ y^{100} \end{bmatrix} \quad \text{and} \quad g(x, y) = y - x^2? \]

Theory:

- \( f \): origin is isolated of multiplicity 200
- \( g \): origin lies on a positive-dimensional component
Theory vs. Practice

What is the difference locally at the origin between

\[ f(x, y) = \begin{bmatrix} y - x^2 \\ y^{100} \end{bmatrix} \quad \text{and} \quad g(x, y) = y - x^2? \]

Practice:

- For \( C = \{(x, x^2) \mid |x| < 1/2\} \):
  - \( g = 0 \) on \( C \)
  - \( \|f\| \leq 10^{-60} \) on \( C \)
Theory vs. Practice

What is the difference locally at the origin between

\[ f(x, y) = \begin{bmatrix} y - x^2 \\ y^{100} \end{bmatrix} \quad \text{and} \quad g(x, y) = y - x^2? \]

Practice:

- For \( C = \{(x, x^2) \mid |x| < 1/2\}: \)
  - \( g = 0 \) on \( C \)
  - \( \|f\| \leq 10^{-60} \) on \( C \)
- Difference is some WD-40
Theory vs. Practice

At 2001 Computational Kinematics Workshop:

- Demonstrated this was a highly accurate machine

Theory: isolated solution of multiplicity 4

- It should not move but does due to multiplicity, joint tolerances, and link elasticity
Symbolic vs. Numeric

Generally speaking:

- Numerical methods prefer well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension = \# equations
  - stable under perturbations

\[
\begin{align*}
y + 2x - 1 &= 0 \\
y - 2x - 1 &= 0 \\
y - x - 1 &= 0
\end{align*}
\]

\[
\begin{align*}
y + 2x - 1 &= \epsilon_1 \\
y - 2x - 1 &= \epsilon_2 \\
y - x - 1 &= \epsilon_3
\end{align*}
\]
Symbolic vs. Numeric

Generally speaking:

- Numerical methods prefer well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension = \# equations
  - stable under perturbations

- Gröbner basis methods prefer vastly over-determined systems
  - fewer “new” polynomials to compute

The result of an exact Gröbner basis computation is a proof.
- Num. alg. geom. replaces certainty with “probability 1”
Symbolic vs. Numeric

How to represent an irreducible algebraic variety $A$ on a computer?
Symbolic vs. Numeric

How to represent an irreducible algebraic variety $A$ on a computer?

- algebraic: prime ideal $I(A) = \{ g \mid g(a) = 0 \text{ for all } a \in A \}$
  
  - Hilbert Basis Theorem (1890): there exists $f_1, \ldots, f_k$ such that
    
    $$I(A) = \langle f_1, \ldots, f_k \rangle$$
Witness Set

How to represent an irreducible algebraic variety $A$ on a computer?

- geometric: witness set $\{f, \mathcal{L}, W\}$ where
  - $f$ is polynomial system where $A$ is an irreducible component of $\mathcal{V}(f)$
  - $\mathcal{L}$ is a linear space with $\text{codim} \mathcal{L} = \dim A$
  - $W = \mathcal{L} \cap A$ where $\# W = \deg A$
Example

\[
A = \{ [s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1 \} \subset \mathbb{P}^3 - \text{twisted cubic curve}
\]

\[
I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle
\]

A
Example

$A = \{ [s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1 \} \subset \mathbb{P}^3$ – twisted cubic curve

- $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$

- $\{f, \mathcal{L}, W\}$ where
  - $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$
  - $\mathcal{L} = \{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0 \} \subset \mathbb{P}^3$
    - $\text{codim} \mathcal{L} = \text{dim} A = 1$
  - $W = \{ [1, 3.2731, 10.7130, 35.0644], [1, 0.8596, 0.7389, 0.6351], [1, -2.1326, 4.5481, -9.6995] \}$
    - $\text{deg} A = 3$
Example

\[ A = \{ [s^3, s^2 t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1 \} \subset \mathbb{P}^3 \] - twisted cubic curve

- \( I(A) = \langle x_1^2 - x_0 x_2, x_1 x_2 - x_0 x_3, x_2^2 - x_1 x_3 \rangle \)

- \( f = \begin{bmatrix} x_1^2 - x_0 x_2 \\ x_1 x_2 - x_0 x_3 \end{bmatrix} \)

\[ \mathcal{V}(f) = A \cup \{ x_0 = x_1 = 0 \} \]

- Witness sets “localize” computations to \( A \) effectively ignoring the other irreducible components.

- Sample points from \( A \) by moving the linear slice \( \mathcal{L} \).
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- The Numerical Solution of Systems of Polynomials Arising in Engineering and Science
- Numerically Solving Polynomial Systems with Bertini
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- membership testing: is $x^* \in A$?
- decide if $g(x^*) = 0$ for every $g \in l(A)$ without knowing $l(A)$
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- projection: $\overline{\pi(A)}$

- perform computations on $\overline{\pi(A)}$ without knowing any polynomials that vanish on $\pi(A)$
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- intersection: $A \cap B$
  - special case is regeneration
    - $\mathcal{V}(f_1, \ldots, f_k, f_{k+1}) = \mathcal{V}(f_1, \ldots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
  - compute $A_{\text{sing}}$
  - compute critical points of optimization problem

\[
\min \|x^* - a\|_2 \quad \text{such that} \quad a \in A \cap \mathbb{R}^n
\]
Witness Set

Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

Test other algebraic properties of $A$
- is $A$ arithmetically Cohen Macaulay?
- is $A$ arithmetically Gorenstein?
- is $A$ a complete intersection?
Example

\[ A = \sigma_4(\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^4) \subset \mathbb{P}^{35} \]

- \( \text{dim } A = 31 \)
- \( \text{deg } A = 345 \)

- \( I(A) \) contains 10 poly. of degree 6 and 20 poly. of degree 9

- used sampling to show that \( A \) was \( \text{aCM} \) and that these polynomials generate \( I(A) \)

N.S. Daleo and J.D. Hauenstein,
Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme.
Sampling

Sample points from $A$ by moving the linear slice $\mathcal{L}$.

How to utilize sample point(s) to extract data?

- Vanishing polynomials
- Sampling for solving sum of squares (SOS) programs
- Degree estimation
- Topological properties
Vanishing polynomials

For many varieties $A$, the only known polynomial in $I(A)$ is $f \equiv 0$. 
Vanishing polynomials

For many varieties $A$, the only known polynomial in $I(A)$ is $f \equiv 0$.

**Problem:** Compute the exponent $\omega$ of matrix multiplication.

- smallest constant such that two $n \times n$ matrices can be multiplied using $O(n^{\omega+\epsilon})$ arithmetic operations for every $\epsilon > 0$
- Current state of the art: $2 \leq \omega \leq 2.374$

- Could be solved by knowing polynomials that vanish on secant varieties – Landsberg (2017).
Vanishing polynomials

Compute homogeneous polynomials that vanish on

\[ \sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \mid a_i, b_i, c_i \in \mathbb{C}^4 \right\} \subset \mathbb{P}^{63} \]

- 6th secant variety of \( \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4 \) in \( \mathbb{P}^{63} \)
- \( \text{dim} = 59 \)

- If \( a, b, c \in \mathbb{C}^4 \), then \( a \otimes b \otimes c \in \mathbb{C}^{4 \times 4 \times 4} \) with

\[ (a \otimes b \otimes c)_{ijk} = a_i \cdot b_j \cdot c_k \]

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Vanishing polynomials

Cast as a classical elimination problem:

- Eliminate $a$’s, $b$’s, $c$’s from

$$
\sum_{\ell=1}^{6} a_{\ell i} \cdot b_{\ell j} \cdot c_{\ell k} - z_{ijk} = 0 \text{ where } i, j, k = 1, \ldots, 4.
$$
Vanishing polynomials

Cast as a classical elimination problem:

- Eliminate $a$'s, $b$'s, $c$'s from

\[
\sum_{\ell=1}^{6} a_{\ell i} \cdot b_{\ell j} \cdot c_{\ell k} - z_{ijk} = 0 \quad \text{where} \quad i, j, k = 1, \ldots, 4.
\]

Still waiting for Gröbner basis methods to terminate....
Vanishing polynomials

Cast as a classical interpolation problem:

- For sample points \( a_1, \ldots, a_N \in A \), compute \( f \) where \( f(a_i) = 0 \).
Vanishing polynomials

Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute $f$ where $f(a_i) = 0$.

Example

Find homogeneous quadratic polynomials vanishing on:

\[ [1, 1, 1, 1], [1, -1, 1, -1], [1, 2, 4, 8], [1, -2, 4, -8], [1, 3, 9, 27], \\
[1, -3, 9, -27], [1, 4, 16, 64], [1, -4, 16, -64], [1, 5, 25, 125], [1, -5, 25, -125] \]
Vanishing polynomials

Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute $f$ where $f(a_i) = 0$.

Example

Find homogeneous quadratic polynomials vanishing on:

$[1, 1, 1, 1], [1, -1, 1, -1], [1, 2, 4, 8], [1, -2, 4, -8], [1, 3, 9, 27], [1, -3, 9, -27], [1, 4, 16, 64], [1, -4, 16, -64], [1, 5, 25, 125], [1, -5, 25, -125]$

<table>
<thead>
<tr>
<th>$x_0^2$</th>
<th>$x_0x_1$</th>
<th>$x_0x_2$</th>
<th>$x_0x_3$</th>
<th>$x_1^2$</th>
<th>$x_1x_2$</th>
<th>$x_1x_3$</th>
<th>$x_2^2$</th>
<th>$x_2x_3$</th>
<th>$x_3^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>16</td>
<td>32</td>
<td>64</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>4</td>
<td>-8</td>
<td>4</td>
<td>-8</td>
<td>16</td>
<td>16</td>
<td>-32</td>
<td>64</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>81</td>
<td>243</td>
<td>729</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>9</td>
<td>-27</td>
<td>9</td>
<td>-27</td>
<td>81</td>
<td>81</td>
<td>-243</td>
<td>729</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>256</td>
<td>1024</td>
<td>4096</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td>16</td>
<td>-64</td>
<td>16</td>
<td>-64</td>
<td>256</td>
<td>256</td>
<td>-1024</td>
<td>4096</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>25</td>
<td>125</td>
<td>25</td>
<td>125</td>
<td>625</td>
<td>625</td>
<td>3125</td>
<td>15625</td>
</tr>
<tr>
<td>1</td>
<td>-5</td>
<td>25</td>
<td>-125</td>
<td>25</td>
<td>-125</td>
<td>625</td>
<td>625</td>
<td>-3125</td>
<td>15625</td>
</tr>
</tbody>
</table>
Example

Find homogeneous quadratic polynomials vanishing on:

\[ [1, 1, 1, 1], [1, -1, 1, -1], [1, 2, 4, 8], [1, -2, 4, -8], [1, 3, 9, 27], [1, -3, 9, -27], [1, 4, 16, 64], [1, -4, 16, -64], [1, 5, 25, 125], [1, -5, 25, -125] \]

<table>
<thead>
<tr>
<th>( x_0^2 )</th>
<th>( x_0 x_1 )</th>
<th>( x_0 x_2 )</th>
<th>( x_0 x_3 )</th>
<th>( x_1^2 )</th>
<th>( x_1 x_2 )</th>
<th>( x_1 x_3 )</th>
<th>( x_2^2 )</th>
<th>( x_2 x_3 )</th>
<th>( x_3^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>16</td>
<td>32</td>
<td>64</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>4</td>
<td>-8</td>
<td>4</td>
<td>-8</td>
<td>16</td>
<td>16</td>
<td>-32</td>
<td>64</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>81</td>
<td>243</td>
<td>729</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>9</td>
<td>-27</td>
<td>9</td>
<td>-27</td>
<td>81</td>
<td>81</td>
<td>-243</td>
<td>729</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>256</td>
<td>1024</td>
<td>4096</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td>16</td>
<td>-64</td>
<td>16</td>
<td>-64</td>
<td>256</td>
<td>256</td>
<td>-1024</td>
<td>4096</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>25</td>
<td>125</td>
<td>25</td>
<td>125</td>
<td>625</td>
<td>625</td>
<td>3125</td>
<td>15625</td>
</tr>
<tr>
<td>1</td>
<td>-5</td>
<td>25</td>
<td>-125</td>
<td>25</td>
<td>-125</td>
<td>625</td>
<td>625</td>
<td>-3125</td>
<td>15625</td>
</tr>
</tbody>
</table>

3-dimensional null space is generated by:

\[ x_1^2 - x_0 x_2, \quad x_1 x_2 - x_0 x_3, \quad x_2^2 - x_1 x_3 \]
Vanishing polynomials

Cast as a classical interpolation problem:

- For sample points $a_1, \ldots, a_N \in A$, compute $f$ where $f(a_i) = 0$.

Problem is the number of sample points needed:

- To show no nonconstant polynomials of degree 18 vanish on $\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) \subset \mathbb{P}^{63}$, need

$$N \geq \binom{63 + 18}{18} \approx 4.567 \cdot 10^{17}$$
Vanishing polynomials

When all else fails, solve a different problem.
  ▶ partial information is better than no information
Vanishing polynomials

When all else fails, solve a different problem.
  ▶ partial information is better than no information

What polynomials vanish on the set of witness points \( A \cap \mathcal{L} \)?
  ▶ If \( f \) vanishes on \( A \), then \( f \bigg|_{\mathcal{L}} \) vanishes on \( A \cap \mathcal{L} \).

▶ Exact correspondence when arithmetically Cohen-Macaulay
▶ Upper bounds in general
Vanishing polynomials

\[ \sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \mid a_i, b_i, c_i \in \mathbb{C}^4 \right\} \subset \mathbb{P}^{63} \]

- \( \dim = 59 \)
- \( \deg = 15,456 \)

Restricting to \( \dim 4 \) linear space \( \mathcal{L} \)

- To show no nonconstant polynomials of degree 18 vanish:

\[ N \geq \binom{4 + 18}{18} = 7315 \]

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Example

Vanishing polynomials

\[ \sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \mid a_i, b_i, c_i \in \mathbb{C}^4 \right\} \subset \mathbb{P}^{63} \]

- \( \text{dim} = 59 \)
- \( \text{deg} = 15,456 \)

Interpolating witness point set shows

- No nonconstant polynomials of degree \( \leq 18 \) vanish
- 64 polynomials of degree 19 restricted to \( \mathcal{L} \) vanish
- Go search for polynomials of degree 19!

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank.
Example

Vanishing polynomials

\[
\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) = \left\{ \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i \mid a_i, b_i, c_i \in \mathbb{C}^4 \right\} \subset \mathbb{P}^{63}
\]

Representation theory proves existence of 64 polynomials of degree 19 that vanish.

- Used to prove that $2 \times 2$ matrix multiplication tensor is not contained in $\sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4)$.

- Rank and border rank of $2 \times 2$ matrix multiplication tensor is 7

J.D. Hauenstein, C. Ikenmeyer, and J.M. Landsberg,
Equations for lower bounds on border rank. 
SOS programs

It is possible to interpolate over other families of polynomials

- Cifuentes-Parrilo (2017) interpolate sums of squares modulo an ideal without knowing the ideal using sample points

Given polynomial $p$, compute $g_1, \ldots, g_k$ such that

$$p \equiv \sum_{i=1}^{k} g_i^2 \mod I(A)$$

assuming such a decomposition exists.

- Certificate that $p \geq 0$ on $A \cap \mathbb{R}^n$. 
SOS programs

A necessary condition for

\[ p \equiv \sum_{i=1}^{k} g_i^2 \mod \det(A) \]

is, for samples \( a_1, \ldots, a_N \in A \),

\[ p(a_j) = \sum_{i=1}^{k} g_i(a_j)^2 \]

- Computation performed using semidefinite program
SOS programs

Example (Trace ratio)

\[ A_{n,k} = \{ X \in \mathbb{R}^{n \times n} \mid X^T = X, X^2 = X, \text{trace}(X) = k \} \]

Given symmetric matrix \( X, Y, Z \in \mathbb{R}^{n \times n} \) where \( Y \succ 0 \), solve

\[
\begin{align*}
\text{max } & \quad \gamma \\
\text{s.t. } & \quad \text{trace}(Y \alpha)(\gamma - \text{trace}(Z \alpha)) - \text{trace}(X \alpha) \equiv F(\alpha) \mod I(A_{n,k}), \\
& \qquad F \text{ is SOS}, \\
& \qquad \deg F = 2.
\end{align*}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k )</th>
<th>Equations SDP</th>
<th>Sampling SDP</th>
<th>Gröbner bases</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>variables</td>
<td>constraints</td>
<td>time(s)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>342</td>
<td>188</td>
<td>0.47</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>897</td>
<td>393</td>
<td>0.71</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2062</td>
<td>738</td>
<td>1.34</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>4265</td>
<td>1277</td>
<td>3.62</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>8106</td>
<td>2073</td>
<td>9.06</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>14387</td>
<td>3198</td>
<td>23.83</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>24142</td>
<td>4733</td>
<td>58.17</td>
</tr>
</tbody>
</table>

D. Cifuentes and P.A. Parrilo,
Sampling algebraic varieties for sum of squares programs.
Degree estimation

Can we estimate the degree of a variety by sampling?

Example

How many 6-bar Watt I linkages obtain 8 given poses?

Multihomogeneous Bézout bound: $3.43 \cdot 10^{10}$
Degree estimation

How many 6-bar Watt I linkages obtain 8 given poses?

Corresponding problem for 4-bar linkages – Burmester (1886)
- 4 solutions for 5 poses
Degree estimation

Given one point in a witness point set $W = A \cap L$, generate another point (possibly same point) by using a monodromy loop.

MonodromySolver

- Bliss-Duff-Leykin-Sommars (2018)
Degree estimation

Given one point in a witness point set $W = A \cap \mathcal{L}$, generate another point (possibly same point) by using a monodromy loop.

- **IF** we assume that we can generate random subsets of $W$, we can estimate $\# W$. 

Degree estimation

Example (German tank problem)

WWII: Estimate # of tanks Germany was producing per month.

- use serial numbers on parts recovered
  - assume uniformly distributed to generate statistical estimate
Degree estimation

Example (German tank problem)

WWII: Estimate # of tanks Germany was producing per month.

- use serial numbers on parts recovered
  - assume uniformly distributed to generate statistical estimate

<table>
<thead>
<tr>
<th>month</th>
<th>statistical est.</th>
<th>intelligence est.</th>
<th>German records</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 1940</td>
<td>169</td>
<td>1000</td>
<td>122</td>
</tr>
<tr>
<td>June 1941</td>
<td>244</td>
<td>1550</td>
<td>271</td>
</tr>
<tr>
<td>August 1942</td>
<td>327</td>
<td>1550</td>
<td>342</td>
</tr>
</tbody>
</table>
Degree estimation

Hypergeometric estimate of \( \text{deg } A = \# W \):

\[
\text{deg } A = \# W \approx \frac{n}{p}
\]

- \( n = \) number of points already known in \( W \)
- \( p = \) ratio of repeats in sample
Degree estimation

Hypergeometric estimate of $\deg A = \# W$:

$$\deg A = \# W \approx \frac{n}{p}$$

- $n =$ number of points already known in $W$
- $p =$ ratio of repeats in sample

Example

Assume that $n = 10$ points are already known in $W$.

- Monodromy loop provides 8 new points and 2 repeats:

$$\# W \approx \frac{10}{2/10} = 50$$
Example

Degree estimation

Validate statistical model:

- Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.

- Problem was studied by Plecnik-McCarthy-Wampler (2014)
Example

Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.

Perform monodromy loops starting from $n = 1000$ known solutions.

Mean from 10 monodromy loops:
  - Ratio of repeats: 17.51%
  - Estimated number of solutions: 5750.5
Example

- Fix the two ground pivots and find 6-bar Watt I linkages that obtain 6 given poses.

Perform monodromy loops starting from $n = 1000$ known solutions.

- Mean from 10 monodromy loops:
  - Ratio of repeats: 17.51%
  - Estimated number of solutions: 5750.5

- Theoretical values:
  - Ratio of repeats: 17.38%
  - Number of solutions: 5754
Degree estimation

How many 6-bar Watt I linkages obtain 8 given poses?

Watt I, N=8 task positions

![Graph showing the number of solutions vs. trials]

Number of Solutions

0 400,000 800,000 1,200,000 1,600,000 2,000,000

Trials

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

- Roots collected
- Estimated number of solutions

J.D. Hauenstein and S.N. Sherman,
Statistically estimating the number of solutions to motion generation problems.
In preparation.
Topological properties

Objects can be identified from point clouds (set of sample points):

\[ \beta_0 = 1, \beta_1 = 0, \beta_2 = 1 \]

Topology of real algebraic surfaces from point clouds:

Niyogi-Smale-Weinberger (2008)
Cucker-Krick-Shub (2016)
...

---

ND
Topological properties

Persistent homology

- treat each point as center of a ball where radius changes
- determine features which persist over wide range of radii

With provably dense samples, we can prove theorems about when topological features must be present in the point cloud.

- Generate provably dense samples using num. alg. geom.
Topological properties

\[ 4x^4 + 7y^4 + 3z^4 - 3 - 8x^3 + 2x^2y - 4x^2 - 8xy^2 - 5xy + 8x - 6y^3 + 8y^2 + 4y \]

\[ \beta_0 = 1, \beta_1 = 0, \beta_2 = 1 \]

Persistence diagram
- Shed some light on what makes some computations difficult
  - Solve well-posed, well-conditioned, and num. stable problems!

- Explain witness sets and some applications of sampling
  - With many omissions (sorry)

- See software “in action” during software demonstration with
  - Jose Rodriguez
  - Danielle Brake
  - Maggie Regan
Thank You!