Elimination Theory in the 21st century

Carlos D'Andrea

NSF-CBMS Conference on Applications of Polynomial Systems

UNIVERSITAT DE BARCELONA

Before we start...

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Session on Open Problems
Friday 3.30 pm

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- BYOP

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- 5 minutes per presentation

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- to book a time slot, contact Hal or me

The Paradigm of the 21st Century

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Apart from being crazy about applications

The Paradigm of the 21st Century

Apart from being crazy about applications and implementations...

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“Cost” of Elimination

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The number and "size" of the solutions of

$$\begin{cases} F_1(X_1,\ldots,X_n) & = & 0 \\ F_2(X_1,\ldots,X_n) & = & 0 \\ \vdots & \vdots & \vdots \\ F_n(X_1,\ldots,X_n) & = & 0 \end{cases}$$

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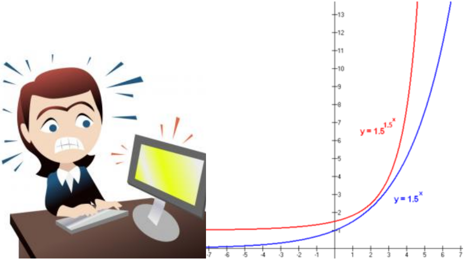
where “size” of $F_i = (d, L)$

is bounded by and generically equal to

$$\boxed{(d^n, nd^{n-1}L)}$$

The output is already exponential!!!



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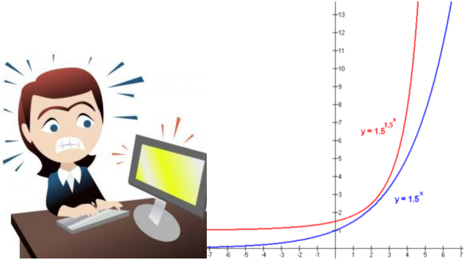

Moreover:

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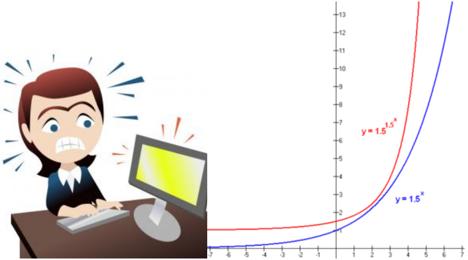

Moreover: the **complexity** of computing **Gröbner bases** is double exponential in (d, L)

What can we do then???

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- "Relax" the input

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- "Relax" the input (and expect the output to be "relaxed" too!)
- Change the Computational Model

Relaxing the input

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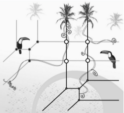
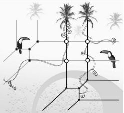


- Tropical Geometry

Relaxing the input



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- "Relaxed" resultants

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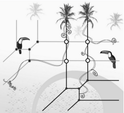

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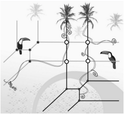

- Tropical Geometry
- "Relaxed" resultants
- Implicitization matrices (Wednesday)

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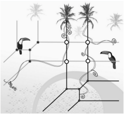

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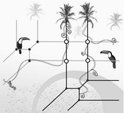

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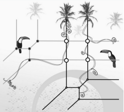

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Changing the model

Changing the model

- Probabilistic algorithms

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- Computations "over the Reals"

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Some "Tapas" into the 21st century

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Tropical: from Exponential to Polynomial

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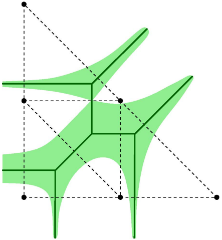

The **logarithmic limit set** of a variety $V \subset \mathbb{C}^n$ is a **fan** in $\mathbb{R}^n$

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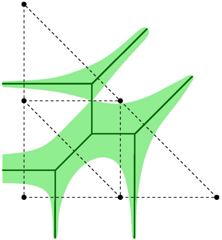

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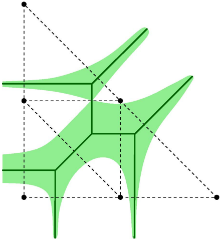

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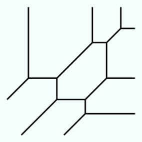
(MSC 2010 **14Txx**)

Computing Tropical Varieties



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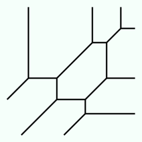

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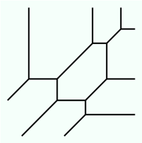
Computing Tropical Varieties



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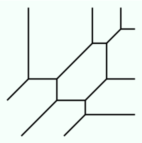
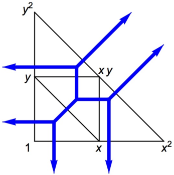


- is "fast"
- encodes a lot of information about the original variety (dimension, degree, singularities...)

First tapa: Tropical Discriminants



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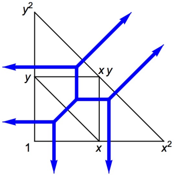

Alicia Dickenstein, Eva Maria Feitchner, Bernd Sturmfels

“Tropical Discriminants”

(JAMS 2007)

Main Result

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Theorem 1.1

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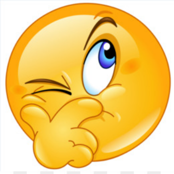
For any $d \times n$ matrix A, the tropical A-discriminant $\tau(X_{A^*})$ equals the **Minkowski sum** of the **co-Bergman fan** $B^*(A)$ and the row space of A.

What is the co-Bergman fan?



What is the co-Bergman fan?

“is” the tropicalization of the kernel of A

Tapa # 2:Tropical Elimination

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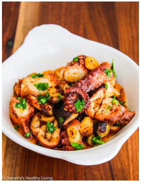

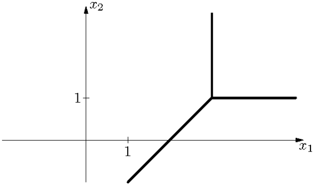

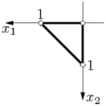

Bernd Sturmfels, Jenia Tevelev

"Elimination Theory for Tropical Varieties"

(MRL 2008)

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What about Tropical Resultants?

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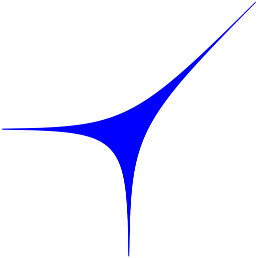

Anders Jensen, Josephine Yu
"Computing Tropical Resultants"
(JA 2013)

From the abstract:

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We fix the supports $A = (A_1, \dots, A_k)$ of a list of **tropical polynomials** and define the tropical resultant $TR(A)$ to be the set of choices of coefficients such that the tropical polynomials have a common solution.

From the abstract II

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We prove that $TR(A)$ is the tropicalization of the algebraic variety of solvable systems and that its dimension can be computed in **polynomial** time...

From the abstract III

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From the abstract III

We also present a new algorithm for recovering a **Newton polytope** from the support of its tropical hypersurface. We use this to compute the Newton polytope of the sparse resultant polynomial in the case when $TR(A)$ is of codimension 1...

“Relaxed” resultants

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In the “real world” systems of equations are neither homogeneous nor all the monomials appear in the expansion

$$\begin{cases} F_0 = a_{01} + a_{02} X_1{}^2 X_2{}^2 + a_{03} X_1 X_2{}^3 \\ F_1 = a_{10} + a_{11} X_1{}^2 + a_{12} X_1 X_2{}^2 \\ F_2 = a_{20} X_1{}^3 + a_{21} X_1 X_2 \end{cases}$$

Sparse Resultants (after GKZ)

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D, Martín Sombra

"A Poisson formula for the sparse resultant"
(PLMS 2015)

$$\begin{array}{ccc} W = \{(\mathbf{c}_{i,\mathbf{a}}, \xi) : F_i(\xi) = 0 \, \forall i\} & \subset & \mathbb{P}^{\mathcal{A}_0} \times \ldots \times \mathbb{P}^{\mathcal{A}_n} \times (\mathbb{C}^\times)^n \\ \downarrow & & \downarrow \pi \\ \pi(W) & \subset & \mathbb{P}^{\mathcal{A}_0} \times \ldots \times \mathbb{P}^{\mathcal{A}_n} \end{array}$$

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$$d^n\mapsto MV(N(\mathcal{A}_1),\ldots,N(\mathcal{A}_n))$$

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The sparse resultant of $F_0, \ldots, F_n$ is the defining equation of the direct image $\pi_* W$

$$d^n \mapsto MV(N(\mathcal{A}_1), \ldots, N(\mathcal{A}_n))$$

$$nd^{n-1}L \mapsto nMV(N(\mathcal{A}_1), \ldots, N(\mathcal{A}_n))L$$

Other “relaxed” resultants

Other "relaxed" resultants

By replacing $(\mathbb{C}^{\times})^n$ with other "efficient" varieties X, we get "efficient" resultants:

$$\begin{array}{ccc} W = \{(\mathbf{c}_{i,\mathbf{a}}, \xi) : F_i(\xi) = 0 \, \forall i\} & \subset & \mathbb{P}^{N_0} \times \ldots \times \mathbb{P}^{N_n} \times X \\ \downarrow & & \downarrow \pi \\ \pi(W) & \subset & \mathbb{P}^{N_0} \times \ldots \times \mathbb{P}^{N_n} \end{array}$$

Examples of "relaxed" resultants

- Reduced Resultants (Zariski, Jouanolou,...)

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- ...

Implicitization matrices

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Recall van der Waerden's matrix introduced this morning:

$$M_s(\lambda)\begin{bmatrix}\vdots\\ \mathbf{x}^{\beta}\\ \vdots\end{bmatrix}=\begin{bmatrix}\vdots\\ \mathbf{x}^{\alpha}F_i\\ \vdots\end{bmatrix}$$

Implicitization matrices

Recall van der Waerden's matrix introduced this morning:

$$M_s(\lambda)\begin{bmatrix}\vdots\\ \mathbf{x}^{\beta}\\ \vdots\end{bmatrix}=\begin{bmatrix}\vdots\\ \mathbf{x}^{\alpha}F_i\\ \vdots\end{bmatrix}$$

For $s \gg 0$, the rank of $M_s(\lambda)$ drops iff there is a common solution

“Rank” vs “Determinant”

"Rank" vs "Determinant"

$$A=\begin{pmatrix} x+2x^3 & -5x^4 & -3+2x \\ -5x & x-2x^2 & 3+4x^3 \\ 2-3x+4x^2 & 4-2x & x^4-x^3 \end{pmatrix}$$

Laurent Busé

"Implicit matrix representations of rational Bézier curves and surfaces"

(CAGD 2014)

Idea

Given a rational parametrization of a spatial curve $(\phi_1(t), \phi_2(t), \phi_3(t))$

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21st century implicitization:

Idea

Given a rational parametrization of a spatial curve $(\phi_1(t), \phi_2(t), \phi_3(t))$

Classic implicitization: Compute the equations $F_i(X, Y, Z)$ of the image

21st century implicitization: Compute a matrix $M(X, Y, Z)$ such that its rank drops on the points of the curve

Properties

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- Entries are linear in X, Y, Z

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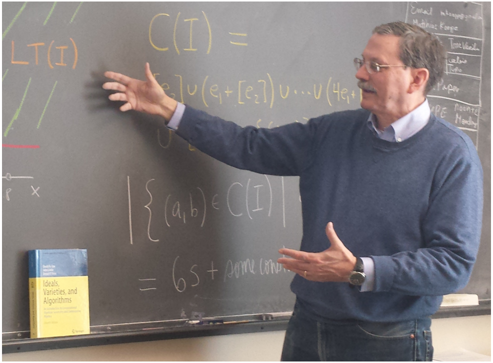
- Entries are linear in X, Y, Z
- Testing rank is fast by using numerical methods
- Over the reals one can use SVD and other numerical methods
- Well suited for a lot of problems in CAGD (properness, inversion,...)

More about this on Wednesday!



Syzygies and Rees Algebras

Warming up tapa

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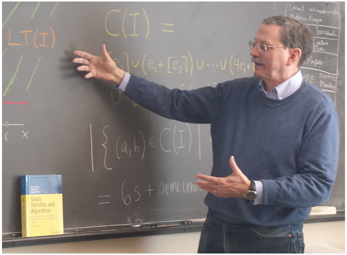

The rest on Wednesday :-)

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The implicit equation of a rational quartic can be computed as a 2×2 determinant.
If the curve has a triple point, then one row is **linear** and the other is **cubic**.
Otherwise, both rows are **quadratic**.

A quartic with a triple point

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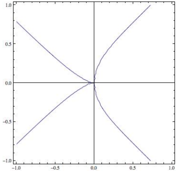
$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$



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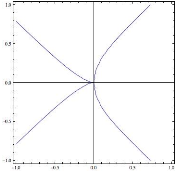
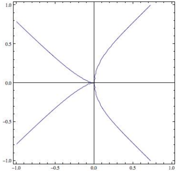


$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

A quartic with a triple point

$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$



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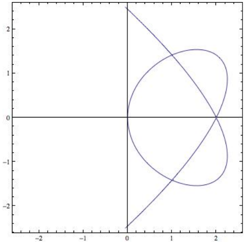
$$\mathcal{L}_{1,1}(\underline{T}, \underline{X}) = T_0 X_2 + T_1 X_1$$

$$\mathcal{L}_{1,3}(\underline{T}, \underline{X}) = T_0(X_1^3 + X_0 X_2^2) + T_1 X_2^3$$

$$\boxed{\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}}$$

A quartic without triple points

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2 t_1^2 - 4t_1^4 : 4t_0^3 t_1 - 4t_0 t_1^3)$$



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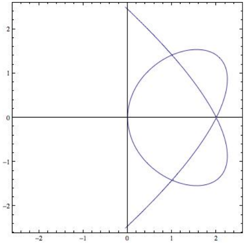

$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

A quartic without triple points

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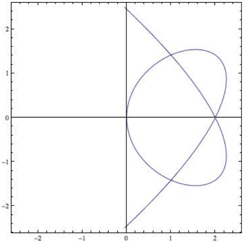

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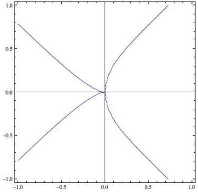
$$\mathcal{L}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$

$$\tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1^2 + \frac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

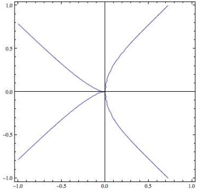
A very compact formula

A very compact formula



If the curve has a point of multiplicity $d-1$

A very compact formula



If the curve has a point of multiplicity $d-1$ the implicit equation is always a 2×2 determinant of a **moving line** and a **moving curve** of degree $d-1$

$$\begin{vmatrix} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{vmatrix}$$

The “method” of moving curves

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The implicit equation of a rational curve should be computed as the determinant of a **small** matrix whose entries are

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The "method" of moving curves

The implicit equation of a rational curve should be computed as the determinant of a **small** matrix whose entries are

some moving lines
some moving conics
some moving cubics
...

the more **singular** the curve, the **simpler** the description of the determinant

In general, we do not know..

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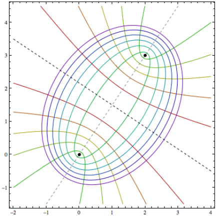
which moving lines?
which moving conics?
which moving cubics?
...



Some order in the 21st century

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David Cox "The moving curve ideal and the Rees algebra" (TCS 2008)

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$\mathcal{K}_\phi :=$ {Moving curves following ϕ} $=$ homogeneous elements in the kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & \phi_0(\underline{T})s \\ X_1 & \mapsto & \phi_1(\underline{T})s \\ X_2 & \mapsto & \phi_2(\underline{T})s \end{array}$$

Some order in the 21st century

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The image of this map is the **Rees Algebra** of ϕ

Method of moving curves revisited

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Method of moving curves revisited

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$$\left| \begin{array}{c} \ldots \\ \text{some } \textbf{minimal generators} \text{ of } \mathcal{K}_\phi \\ \text{and relations among them} \\ \ldots \end{array} \right|$$

The more singular the curve, the "simpler" the description of $\mathcal{K}_\phi$

21st Century Problem

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Compute a minimal system of generators of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

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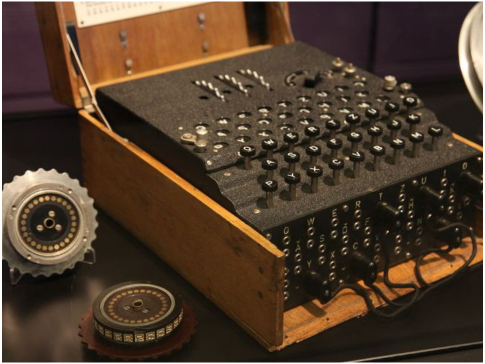
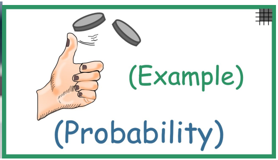
Changing the model of computation



Probabilistic Algorithms: faster they are



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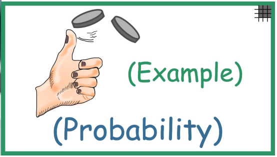

Gabriela Jeronimo, Teresa Krick,
Juan Sabia, Martín Sombra

"The Computational Complexity of the Chow Form"

(JFoCM 2004)

From the abstract:

We present a **bounded probability algorithm** for the computation of the **Chow forms** (resultants) of the equidimensional components of a variety...

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The **expected complexity** of the algorithm is **polynomial** in the size and the geometric degree of the input equation system defining the variety...

As an application, we obtain an algorithm to compute a subclass of **sparse resultants**, whose complexity is **polynomial** in the **dimension** and the **volume** of the input set of exponents...

Main idea

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Given (equations) of a d-dimensional variety

$$V \subset \mathbb{P}^n$$

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- Cut V with "general" (probabilistically) d hyperplanes to obtain finite points: $V \cap \{x_1 = \ldots = x_d = 0\} = \{\xi_1, \ldots, \xi_D\}$
- From (a geometric resolution of) $\{\xi_1, \ldots, \xi_D\}$, **lift** (Newton's Method) to the Chow Form of $V \cap \{x_1 = \ldots = x_{d-1} = 0\}$

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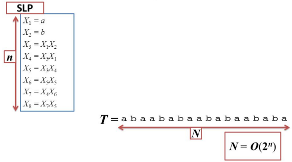
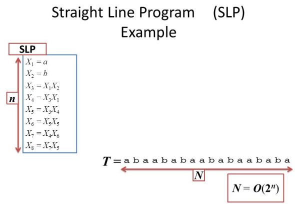
- Do it $(d-1)$ times more...

Star Model: Straight-Line Programs

Straight Line Program (SLP)
Example



Star Model: Straight-Line Programs

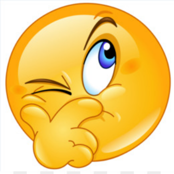


Gabriela Jeronimo, Juan Sabia

"Sparse resultants and straight-line programs"

(JSC 2018)

What is a straight-line program?



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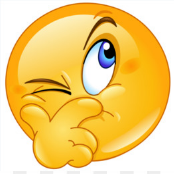

It is a program with **no branches**

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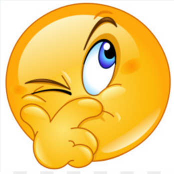

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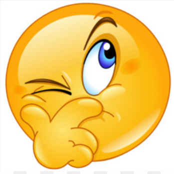

It is a program with **no branches, no loops, no conditional statements**

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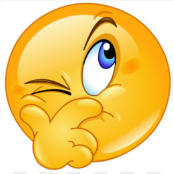

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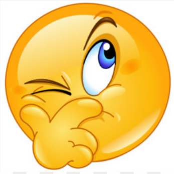
What is a straight-line program?



It is a program with **no branches, no loops, no conditional statements, no comparisons**...just a sequence of basic operations

From the abstract:

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We prove that the **sparse resultant,** redefined by D'Andrea and Sombra and by Esterov as a power of the classical sparse resultant, can be **evaluated** in a number of steps which is **polynomial** in its degree, its number of variables and the size of the exponents ...

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Moreover, we design a **probabilistic algorithm** of this order of complexity to compute a **straight-line program** that evaluates it within this number of steps.

Main ideas

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- SLP's

Main ideas

- SLP's
- Geometric Resolution

Main ideas

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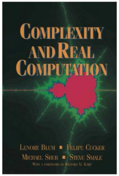
- Newton Hensel Lifting

Main ideas

- SLP's
- Geometric Resolution
- Newton Hensel Lifting
- Padé Approximations...

Anoter "real" model: BSS-machine



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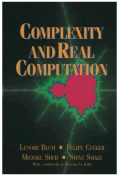
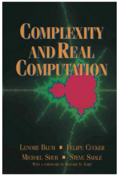


is a **Random Access Machine**

Anoter “real” model: BSS-machine



is a **Random Access Machine**

with registers that can store

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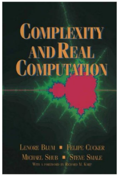
Anoter “real” model: BSS-machine



is a **Random Access Machine** with registers that can store **arbitrary real numbers** and that can compute rational functions over the reals at unit cost

Parallelize: find just one solution

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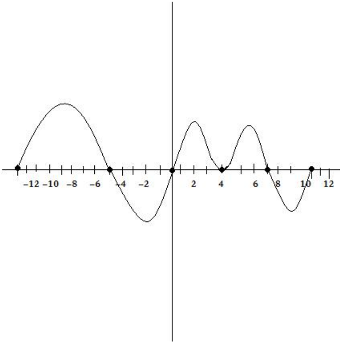
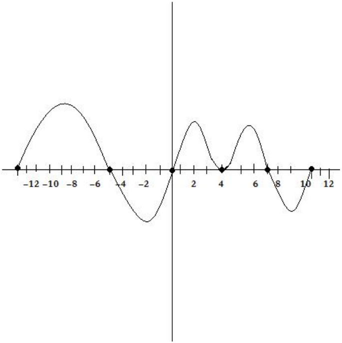


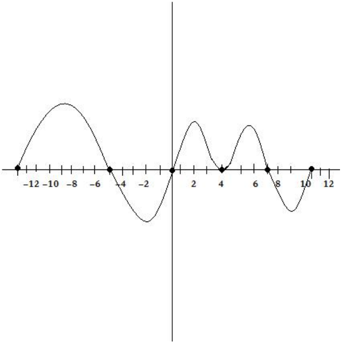
Given $f(x) \in \mathbb{Z}[x]$, compute **a small**

Parallelize: find just one solution



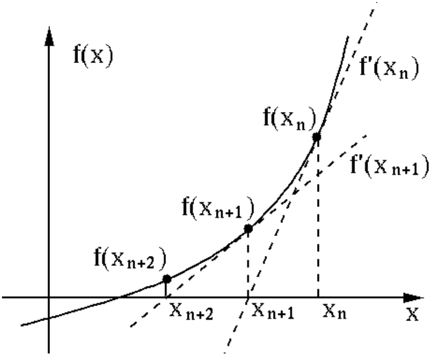
Given $f(x) \in \mathbb{Z}[x]$, compute **a small approximate root**

Parallelize: find just one solution



Given $f(x) \in \mathbb{Z}[x]$, compute **a small approximate root**, **fast**

What is an approximate root?



Steve Smale's 17th's problem (1998)

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Is there an algorithm which computes an **approximate solution** of a system of polynomials in **time polynomial on the average,** in the size of the input ?

Solving Smale's 17th Problem

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- Carlos Beltrán, Luis Miguel Pardo. "On Smale's 17th Problem: A Probabilistic Positive answer" (JFoCM 2008)

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Pierre Lairez (2017)

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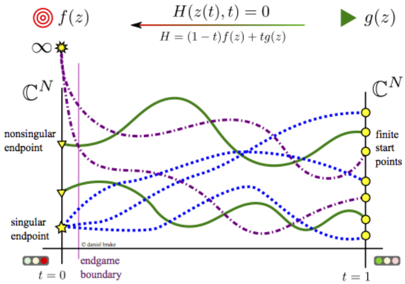
"A deterministic algorithm to compute approximate roots of polynomial systems in polynomial average time"

JFoCM (2017)

Homotopies (In depth tomorrow!)



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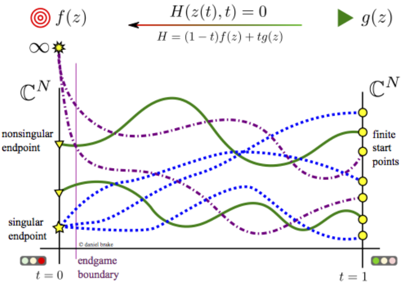
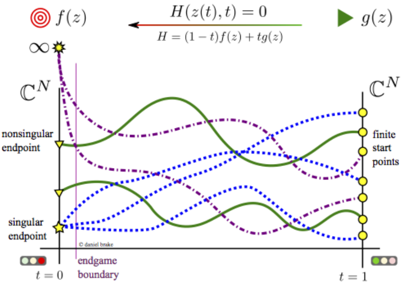


- Start with an "easy" system

Homotopies (In depth tomorrow!)



- Start with an “easy” system
- “Chase” the roots with an homotopy + Newton’s method

Numerical Algebraic Geometry

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By using homotopies, you can compute very fast

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- Components

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Not in the menu today...

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- Differential/Difference Elimination

Not in the menu today...

- Differential/Difference Elimination
- Real Elimination

Not in the menu today...

- Differential/Difference Elimination
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- Applications

Not in the menu today...

- Differential/Difference Elimination
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- **Implementations**

Not in the menu today...

- Differential/Difference Elimination
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Conclusion

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The 21st century is/will be driven by

- Applications

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The 21st century is/will be driven by

- Applications
- Fast & efficient computations

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The 21st century is/will be driven by

- Applications
- Fast & efficient computations

Interesting challenges and problems need your help here!

Thanks!

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http://www.ub.edu/arcades/cdandrea.html