Representation Theory of $SL_2(\mathbb{R})$

- I. $\mathfrak{sl}_2$ irreps
- II. Parabolic induction
- III. Eisenstein series
- IV. Modular forms
- V. Cuspidal auto. forms
- VI. Cohomology.

I) $\mathfrak{sl}_2$ irreps

Lie algebra = vector space $\mathfrak{g}$ w./antisymmetric bilinear form $[\,,\,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$
s.t. $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ (Jacobi).

eg/ End(V)

$$\mathfrak{sl}_{2,\mathbb{R}} = \mathbb{R}\left\langle \overset{Y}{\begin{pmatrix}1 & \\ & -1\end{pmatrix}}, \overset{N_+}{\begin{pmatrix} & 1\\ & \end{pmatrix}}, \overset{N_-}{\begin{pmatrix} & \\ 1 & \end{pmatrix}} \right\rangle \quad \begin{array}{l}[x,y]=xy-yx\\ [Y,N_\pm]=\pm 2N_\pm,\ [N_+,N_-]=Y\end{array}$$ //

Representation (V,ρ) of $\mathfrak{g}$ = linear map $\mathfrak{g}\xrightarrow[\rho]{}\mathrm{End}(V)$ s.t.
($\dim<\infty$)
$$\rho([x,y]) = [\rho(x),y]+[x,\rho(y)].$$

Ex/ Jacobi: $\Rightarrow$ $\mathrm{ad}: \mathfrak{g}\to\mathrm{End}(\mathfrak{g})$ a rep. //
$\qquad x\mapsto [x,\cdot\,]$

eg/ G matrix Lie group, $\mathfrak{g}=\mathrm{Lie}(G)=T_eG$

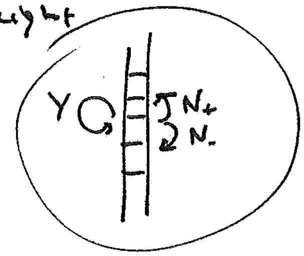
$$e^{\epsilon x}e^{\epsilon y}e^{-\epsilon x}e^{-\epsilon y} = \mathbb{I}+\epsilon^2(xy-yx)+\text{h.o.t.}$$

G connected + simply conn. $\Rightarrow$ 1-1 corr.:

$$\left\{\begin{array}{c}\text{reps. } (C^\infty \text{ homs.})\\ \pi: G\to GL(V)\end{array}\right\} \underset{\exp(x)\mapsto\exp(\rho(x))}{\overset{\text{d at } e}{\rightleftarrows}} \left\{\begin{array}{c}\text{reps.}\\ \rho:\mathfrak{g}\to\mathrm{End}(V)\end{array}\right\}$$ // (V finite dim'l!!)

Given $\underset{\mathfrak{g}}{\mathfrak{sl}_{2,\mathbb{R}}}\xrightarrow{\rho}\mathrm{End}(V)$ irred.:

diagonalize $\rho(Y)$ $\Rightarrow$ $V=\bigoplus_{j\in J}^{\text{finite}} V_j$, $\quad n:=$ highest weight



$v\in V_j \Rightarrow Y(N_+v) = N_+\underbrace{(Yv)}_{jv} + \underbrace{[Y,N_+]}_{2N_+}v = (j+2)N_+v$

$\qquad Y(N_-v) = (j-2)N_-v$

$v\in V_n \Rightarrow N_+v=0$ (off top of ladder)

$\qquad \Rightarrow N_+N_-v = [N_+,N_-]v + 0 = Yv = nv$

$\qquad N_+N_-^2v = \underbrace{[N_+,N_-]}_{Y}N_-v + N_-\underbrace{N_+N_-v}_{nv} = \underbrace{(n+(n-2))}_{2(n-1)}N_-v$

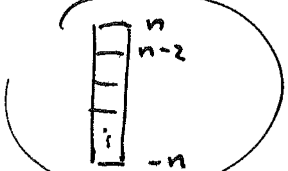
$\vdots$

$$N_+ N_-^k v = k(n-k+1) N_-^k v$$

$\Rightarrow \{N_-^k v\}_{k \geq 0}$ span V

$\Rightarrow \dim V_j = 1 \quad (\forall j \in J)$.

For some m, $\begin{cases} N_-^m v = 0 \\ N_-^{m-1} v \neq 0 \end{cases} \Rightarrow m(n-m+1) = 0 \Rightarrow \begin{cases} n \geq 0 \\ m = n+1 \end{cases} \Rightarrow -n$ is lowest weight.

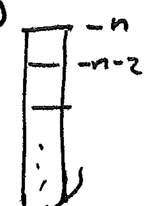


$\Rightarrow J = \{-n, -n+2, \ldots, n\}$

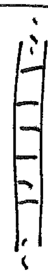
This is the picture for finite-dimensional irreps of $\mathfrak{sl}_2$, $\mathfrak{su}(2)$. (and for $\mathfrak{su}(2)$ that's it.)

For $G = SL_2(\mathbb{R})$, $V = \mathrm{Sym}^n(\underset{\mathbb{C}^2\ \text{dim'l}}{St})$.

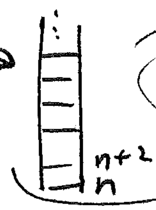
Allow $\dim V = \infty \Rightarrow$ can have $n < 0$, $N_-^k v \neq 0 \quad (\forall k \geq 0)$.



Can also $\begin{cases} \text{flip this up} \\ \text{have bi-infinite ladder} \end{cases}$



(leads to principal series)



(lead to discrete series)

Goal: Construct these in such a way that it is clear that they come from irreps of G.

II) Parabolic induction

$g \in SL_2(\mathbb{R})$

Gram-Schmidt $\Rightarrow g \overset{!}{=} \underset{\text{upper }\Delta,\ \oplus\text{diag}}{\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}} \underset{\text{orthog.}}{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} =: p_\tau k_\theta =: g_{\tau,\theta}$, $\quad \tau = x+iy \in \mathfrak{h}$

Iwasawa: $G = P \cdot K = \underset{\text{unip.}}{N} \overbrace{A\ M}^{\text{Levi}} \cdot K$, where $M \cong \mathbb{Z}/2\mathbb{Z} = P \cap K$.

parabolic: all $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad \pm p_\tau = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}(\pm \mathrm{id})$

Let $\sigma : AM \cong \mathbb{R}^* \longrightarrow \mathbb{C}^*$ be a (1-dim'l) $\mathbb{C}$-rep. of the Levi,

$\tilde{\sigma} : P \to \mathbb{C}^*$ its pullback (trivial on N),

$\pm p_\tau \mapsto \chi^\epsilon(\pm) \cdot y^{\lambda/2} \quad (\lambda \in \mathbb{C})$

$\mathrm{Ad}(\cdot)|_{\mathrm{Lie}(N)} = \Delta_P : P \to \mathbb{C}^*$ the "modular character", and define

$\pm p_\tau \mapsto y$

$$I_{\epsilon,\lambda} := I_P(\sigma) := \mathrm{Ind}_P^G(\underbrace{\Delta_P^{1/2}\sigma}_{\hat\sigma})$$

Remark: This is here in order that σ unitary ($\lambda \in i\mathbb{R}$) $\Rightarrow I_P(\sigma)$ unitary, and so that λ becomes the HC parameter.

(**) $$:= \left\{ f \in C^\infty_{(\mathbb{C})}(G) \;\middle|\; \begin{array}{l} f(pg) = \hat\sigma(p) f(g) \\ f \text{ right } K\text{-finite} \end{array} \right\}$$

(if drop K-finite cond.) G acts by right translation: $(g_0 . f)(g) := f(g g_0)$.

(*) $$\xrightarrow[\text{restrict}]{\cong} \left\{ f \in C^\infty(K) \;\middle|\; \begin{array}{l} f(\pm k) = \chi^\epsilon(\pm)\cdot f(k) \\ f \text{ right } K\text{-finite} \end{array} \right\}.$$

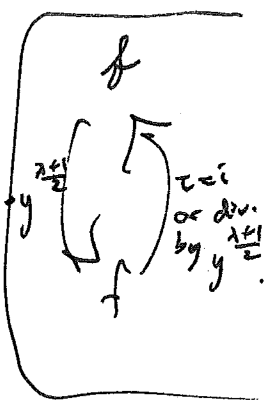
Motivation: (i) Why (**)? Finite groups: $\mathrm{Ind}_H^G(\sigma) = V \otimes_H \mathbb{C}[G] \xrightarrow{\cong}$ V-valued fns. on $H\backslash G$.

(ii) Why induce? Harish-Chandra's subquotient theorem: every admissible (all V_j's finite-dim'l) irrep. is a subquotient of some $I_P(\sigma)$.

Basis of (*): $\boxed{f_n(\theta) := e^{in\theta}}$ with $\begin{cases} n \text{ even}, & \text{if } \epsilon = 0 \\ n \text{ odd}, & \text{if } \epsilon = 1. \end{cases}$

Basis of (**): $p_{x_0+iy_0} \cdot g_{x+iy,\theta} = g_{(xy_0+x_0)+iyy_0,\theta}$ $\Longrightarrow$

$$\boxed{f_n(g_{x+iy,\theta}) = y^{\frac{\lambda+1}{2}} f_n(\theta).}$$



(right) Action of K: $g_{\tau,\theta} k_\phi = g_{\tau,\theta+\phi} \Rightarrow \boxed{k_\phi . f_n = e^{in\theta} f_n}.$

Action of G? Formulas for $g_{\tau,\theta} \cdot g_{\mu,\phi} =: g_{\tilde\tau,\tilde\theta}$ are messy.

Idea for finding them:

$$G \twoheadrightarrow G/K \cong \mathfrak{h}$$

$$g_{\tilde\tau,\tilde\theta} \mapsto g_{\tilde\tau,\tilde\theta}\left[\begin{pmatrix} i \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} \tilde\tau \\ 1 \end{pmatrix}\right] \quad \text{notice } \partial$$

$$G \twoheadrightarrow P\backslash G \cong \mathbb{P}^1(\mathbb{R}) \cong S^1$$

$$g_{\tilde\tau,\tilde\theta} \mapsto [(0,1)] . g_{\tilde\tau,\tilde\theta} = [(-\sin\tilde\theta, \cos\tilde\theta)] \mapsto 2\tilde\theta.$$

Then $(g_{\mu,\phi}\cdot f)(g_{\tau,\theta}) = f(g_{\tilde\tau,\tilde\theta})$, $\quad g_{\mu,\phi}\cdot f := \dfrac{g_{\mu,\phi}\cdot f}{y^{\frac{\lambda+1}{2}}}$

Ex/ (i) $\displaystyle (g_{\mu,\phi}\cdot f)(\theta) = \frac{\mathrm{Im}(\mu)^{\frac{\lambda+1}{2}}}{|\cos\theta - \sigma\sin\theta|^{\lambda+1}} f(\tilde\theta)$

(ii) $\langle f_1, f_2\rangle := \int_{S^1} f_1 \bar f_2 \, d\theta$ is G-invariant if $\lambda \in i\mathbb{R}$.

Replacing K-finite by L^2 condition gives unitary rep of G.

(iii) $dg := \dfrac{dx\wedge dy\wedge d\theta}{2\pi y^2}$ is (right) G-invariant. //

Action of $\mathfrak{g}$? By infinitesimal right-translation:

$$X\in\mathfrak{g} \rightsquigarrow (\mathcal{L}_X f)(g) := \frac{d}{dt} f(g e^{tX})\Big|_{t=0}.$$

Together with the action of K, this gives a (Harish-Chandra) $\underbrace{(\mathfrak{g}, K)\text{-module}}$ structure on $(*)$ resp. $(**)$. In general, for an irrep. of G, this is the underlying space of K-finite vectors.

Eigenvalues of $k = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\right\rangle$ imaginary $\Rightarrow$ convenient to extend $\mathbb{C}$-linearly to action of

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\Big\langle \underset{W}{\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}}, \underset{E_+}{\tfrac12\begin{pmatrix} 1 & i \\ i & -1\end{pmatrix}}, \underset{E_-}{\tfrac12\begin{pmatrix} 1 & -i \\ -i & -1\end{pmatrix}}\Big\rangle \quad \begin{cases} [W, E_\pm] = \pm 2E_\pm \\ [E_+, E_-] = W\end{cases}$$

So $\mathcal{L}_W := -i\,\mathcal{L}_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)}$, $\quad \mathcal{L}_{E_\pm} := \frac12 \mathcal{L}_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)} \pm \frac{i}{2}\mathcal{L}_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)}$.

Writing $\displaystyle g_{\tau,\theta}e^{tX} = g_{\tilde\tau,\tilde\theta} = p_\tau\Big(1 + t\frac{\tilde y'(0)}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} + t\frac{\tilde x'(0)}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \dots\Big)k_\theta\Big(1 + t\tilde\theta'(0)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} + \dots\Big)$

$\overset{\frac{d}{dt}(\cdot)|_{t=0}}{\Longrightarrow}$ $\displaystyle g_{\tau,\theta}X = \frac{\tilde y'(0)}{2y} p_\tau \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} k_\theta + \frac{\tilde x'(0)}{y} p_\tau\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} k_\theta + \tilde\theta'(0) p_\tau k_\theta \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$

$\overset{\text{bottom 2 entries}}{\Longrightarrow}$ $\displaystyle \Big(-\frac{\sin\theta}{\sqrt y}, \frac{\cos\theta}{\sqrt y}\Big)X = \tilde y'(0)\Big(\frac{\sin\theta}{2y^{3/2}}, \frac{-\cos\theta}{2y^{3/2}}\Big) + \tilde\theta'(0)\Big(-\frac{\cos\theta}{\sqrt y}, -\frac{\sin\theta}{\sqrt y}\Big)$.

Putting this together with $\mathbb{C}$-linear extension of $\displaystyle (\mathcal{L}_X f)(g_{\tau,\theta}) = \cancel{\frac{\partial f}{\partial x}}^{=0}\cdot\tilde x'(0) + \frac{\partial f}{\partial y}\cdot\tilde y'(0) + \frac{\partial f}{\partial\theta}\cdot\tilde\theta'(0)$,

have $\displaystyle \mathcal{L}_W f = -i\frac{\partial f}{\partial\theta}$, $\displaystyle \mathcal{L}_{E_\pm} f = e^{\pm 2i\theta}\Big\{y\frac{\partial f}{\partial y} - \frac{i}{2}\frac{\partial f}{\partial\theta}\Big\}$

$\underset{(f = y^{\frac{\lambda+1}{2}} f)}{\Longrightarrow}$ $\displaystyle \mathcal{L}_W f_n = n f_n$, $\displaystyle \mathcal{L}_{E_\pm} f_n = \frac{\lambda \pm n + 1}{2} f_{n\pm 2}$.

Remark: If handed an irrep. of G in $C^\infty(G)$, and wanted to know which one it was, what to do? Need an operator which commutes with all $\mathcal{L}_X$, i.e. $\in \mathcal{Z}(\mathfrak{g})$, regarded as a differential operator. Set

$$\Omega := -\tfrac{1}{4}(\mathcal{L}_W)^2 + \tfrac{1}{2}\mathcal{L}_W - \mathcal{L}_{E_+}\mathcal{L}_{E_-} \qquad (\text{Casimir op.})$$

Ex/ For $f = y^{\frac{\lambda+1}{2}} f$, $\Omega f = \frac{1}{4}(1-\lambda^2) f$. //

eg/ $(\lambda = 0)$ $I_{+,0} = I_P(\mathbb{1})$ $\qquad \cdots \rightleftarrows f_{-2} \underset{1/2}{\overset{-1/2}{\rightleftarrows}} f_0 \underset{-1/2}{\overset{1/2}{\rightleftarrows}} f_2 \underset{-3/2}{\overset{3/2}{\rightleftarrows}} f_4 \rightleftarrows \cdots$

$I_{-,0} = I_P(\chi)$ $\qquad \cdots f_{-3} \rightleftarrows f_{-1} \underset{1}{\overset{-1}{\rightleftarrows}} \big] \underset{0}{\overset{0}{\rightleftarrows}} \big[f_1 \underset{-1}{\overset{1}{\rightleftarrows}} f_3 \rightleftarrows \cdots$ (bi-infinite)

Ω-eigenvalue $\frac{1}{4}$ $\qquad = D_0^- \oplus D_0^+$ (limits of discrete series) //

eg/ $\begin{pmatrix} \lambda = n-1 \\ \pm = \text{parity of } n \end{pmatrix}$ $I_{\pm, n-1}$

$$\underbrace{\cdots \rightleftarrows f_{-n-2} \rightleftarrows f_{-n} \big]}_{D^-_{n-1}} \rightleftarrows \underbrace{f_{-n+2} \rightleftarrows \cdots \rightleftarrows f_{n-2}}_{\substack{\text{finite dim'l} \\ \text{quotient rep.}}} \rightleftarrows \underbrace{\big[f_n \rightleftarrows f_{n+2} \rightleftarrows}_{D^+_{n-1}}$$

//

All other cases are irreducible, and this produces all irreducible admissible $(\mathfrak{g}, K)$-modules & shows that they each have a "globalization" to G-irrep.. This globalization is not unique, though (when one exists) a unitary one is.

Which are unitarizable?

- trivial (1-dim'l quotient of $I_{+,1}$) } obvious
- spherical unitary principal series $I_{+,i\nu}$ $(\nu \in \mathbb{R})$ } induced from unitary character
- non " " " " $I_{-,i\nu}$ $(\nu \in \mathbb{R}^*)$
- LDS D_0^+, D_0^- $(\subset I_{-,0})$ } will arise inside a larger unitary rep. next lecture (& Phillip will exhibit another proof)
- DS D_k^+, D_k^-, $k \in \mathbb{Z}_+$
- complementary series $I_{+,\lambda}$, $0 < |\lambda| < 1$. } won't treat this

III) Eisenstein Series $\qquad \Gamma := SL_2(\mathbb{Z})$

In the next lecture we'll discuss the appearance of the unitary irreps. above as automorphic representations, i.e. as subrepresentations of the automorphic forms

$$\mathcal{A}(G,\Gamma) := \left\{ \Phi \in C^\infty(\Gamma\backslash G) \;\middle|\; \begin{array}{l} |\Phi| \text{ of poly. growth in } y, \\ \Phi \text{ right } K\text{-finite}, \ \Omega\text{-finite} \end{array} \right\}.$$

We can construct some now (of a different flavor). Consider ($\underline{\ell \text{ even}}$) (6)

$$f_{\lambda,\ell}(g_{\tau,\theta}) := y^{\frac{\lambda+1}{2}} e^{i\ell\theta} \in C^\infty(\pm N\backslash G) \subset C^\infty(\pm\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}\backslash G).$$

otherwise would get 0

To get into $C^\infty(\Gamma\backslash G)$, average over

$$\pm\Gamma_N\backslash\Gamma = [(0,1)].\Gamma = \mathbb{P}^1(\mathbb{Q})$$

$$\pm\Gamma_N\overset{\gamma}{\begin{pmatrix}a & b\\ p & q\end{pmatrix}} \longmapsto [(p,q)] \longleftrightarrow \frac{q}{p} = \kappa$$

cosets — rel. prime pairs — rational boundary pts.

$$\gamma g_{\tau,\theta} =: g_{\tilde\tau,\tilde\theta} \Rightarrow \begin{cases} \tilde\tau = \gamma\langle\tau\rangle = \dfrac{a\tau+b}{p\tau+q}, & \tilde y = \operatorname{Im}\gamma\langle\tau\rangle = \dfrac{y}{|p\tau+q|^2} \\ \tilde\theta = \theta - \arctan\left(\dfrac{y}{x+\kappa}\right), & e^{2i\tilde\theta} = e^{2i\theta}\dfrac{p\bar\tau+q}{p\tau+q}. \end{cases}$$

Ex/

Define now

$$E_{\lambda,\ell}(g_{\tau,\theta}) = \sum_{\pm\Gamma_N\gamma\in\pm\Gamma_N\backslash\Gamma} f_{\lambda,\ell}(\gamma g_{\tau,\theta})$$

$$= \sum_{(p,q)=1} \tilde y^{\frac{\lambda+1}{2}} f_\ell(\tilde\theta)$$

$$= y^{\frac{\lambda+1}{2}} e^{i\ell\theta} \sum_{(p,q)=1} \frac{1}{|p\tau+q|^{\lambda+1}} \left(\frac{p\bar\tau+q}{p\tau+q}\right)^{\ell/2}$$

$$= y^{\ell/2} e^{i\ell\theta} \underbrace{\left\{ \frac{1}{\tilde\zeta(\lambda+1)} \sum_{m,n\in\mathbb{Z}^2}{}' \frac{y^{\frac{\lambda-\ell+1}{2}}}{(m\tau+n)^{\frac{\lambda+\ell+1}{2}}(m\bar\tau+n)^{\frac{\lambda-\ell+1}{2}}} \right\}}_{\mathcal{E}_{\lambda,\ell}(\tau) \text{ (for later reference)}}.$$

$$\tilde\zeta(\lambda+1) = \sum_{\substack{k\in\mathbb{Z}\setminus\{0\}\\ (kp,kq)=(m,n)}} \frac{1}{k^{\lambda+1}}$$

eg/ $\mathcal{E}_{2j-1,2j}(\tau) = \dfrac{1}{2\zeta(2j)} \displaystyle\sum_{m,n}{}' \dfrac{1}{(m\tau+n)^{2j}}$ (holo. Eisenstein series) //

__Observations__: (i) the map $f \xrightarrow{\text{avg}} E$ is (right) G-equivariant.

(ii) $\operatorname{Re}(\lambda) > 0 \Rightarrow$ AC.

(iii) (most) other λ: use analytic continuation (fixing f & g) in λ

we conclude the

__Theorem__: $\{E_{\lambda,\ell}\}_{\ell\in 2\mathbb{Z}}$ gives a copy of $I_{\pm,\lambda} \subset \mathcal{A}(G,\Gamma)$ (for "most λ").

Remark. Notice in particular the distinguished role played by the holo. Eisenstein series in the copy of D^+_{2j-1}: it is the lowest-weight vector.

IV) Modular forms (start off a little more generally)

$\Gamma \leq SL_2(\mathbb{Z})$ arithmetic, i.e. commensurable with some (congruence sgp.)
$$\Gamma(N) := \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}.$$

For $f \in \mathcal{O}(\mathfrak{h})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, write
$$f|^k_\gamma(\tau) := \frac{f(\gamma\langle\tau\rangle)}{(c\tau+d)^k}. \qquad \frac{a\tau+b}{c\tau+d}$$

Ex/ $(f|^k_\gamma)|^k_\eta = f|^k_{\gamma\eta}$. //

Definition: $f \in M_k(\Gamma)$ [resp. $S_k(\Gamma)$] $\iff \begin{cases} \text{(i) } f \equiv f|^k_\gamma \quad \forall \gamma \in \Gamma \\ \text{(ii) } \lim_{\tau \to i\infty} (f|^k_\gamma)(\tau) \text{ finite [resp. 0]} \quad \forall \gamma \in SL_2(\mathbb{Z}) \end{cases}$

modular form, cusp form of wt. k & level Γ

Geometric interp.: $\mathcal{E} \supset E_\tau = \mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle$ universal ell. curve

$$\begin{array}{ccc} \mathcal{E} & \supset & E_\tau \\ \downarrow & & \downarrow \\ \mathfrak{h} & \ni & \tau \end{array}, \quad \gamma.(\tau,u) := \left(\gamma\langle\tau\rangle, \frac{u}{c\tau+d}\right)$$

Ex/ well-defined //

Set $\mathcal{E}_\Gamma := {}_\Gamma\backslash\mathcal{E} \xrightarrow{\rho} {}_\Gamma\backslash\mathfrak{h} =: Y_\Gamma$, $\quad \mathcal{E}^k_\Gamma \to Y_\Gamma$ kth fiber product

$$\begin{array}{ccc} \mathcal{E}^k_\Gamma & \to & Y_\Gamma \\ \text{compact.}\cap & & \cap \\ \text{(Shokurov)}\ \overline{\mathcal{E}^k_\Gamma} & \to & \overline{Y}_\Gamma \end{array}$$

$$\gamma^* du = \frac{du}{(c\tau+d)}, \quad \gamma^* d\tau = \frac{(c\tau+d)a - (a\tau+b)c}{(c\tau+d)^2} d\tau = \frac{d\tau}{(c\tau+d)^2}$$

So $f \in M_{k+2}(\Gamma)$ resp. $S_{k+2}(\Gamma)$ $\Rightarrow f(\tau)\, d\tau \wedge du_1 \wedge \cdots \wedge du_k \in \Omega^{k+1}(\overline{\mathcal{E}^k_\Gamma})\langle \log \overline{\mathcal{E}^k_\Gamma} \backslash \mathcal{E}^k_\Gamma \rangle$ resp. $\Omega^{k+1}(\overline{\mathcal{E}^k_\Gamma})$. //

Henceforth, unless stated otherwise,
$\Gamma := SL_2(\mathbb{Z})$: (i) becomes $f(\tau+1) = f(\tau)$ $\left(\Rightarrow f(\tau) = F(\underbrace{e^{2\pi i\tau}}_{q}) = \sum_{n\in\mathbb{Z}} a_n q^n\right)$
$f(-1/\tau) = \tau^k f(\tau)$

(ii) " $a_n = 0$ for $n < 0$ [resp. $n \leq 0$].

$\Gamma \ni -\mathrm{id} \Rightarrow$ no MF's of odd weight

eg/ $E_k(\tau) := \frac{1}{2\zeta(k)} \sum{}' \frac{1}{(m\tau+n)^k} \left(= \mathcal{E}_{k-1,k} \right) \in M_k(\Gamma)$ (Ex/ check directly or use //)

$k \geq 4$ even

$= 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$, where $\sigma_{k-1}(n) := \sum_{\substack{m>0 \\ m|n}} m^{k-1}$.

(Mittag-Leffler)

$\Rightarrow E_4 = 1 + 240q + \cdots \in M_4$

$E_6 = 1 - 504q + \cdots \in M_6$

$\Rightarrow \Delta(\tau) := \frac{E_4^3 - E_6}{1728} = q - 24q^2 + \cdots \in S_{12}(\Gamma)$. //

(modular disc)

eg/ $\eta(\tau) := q^{1/24} \prod_{l \geq 1} (1-q^l) \in \mathcal{O}(\mathfrak{h})$, η^{24} inv. under $\tau \mapsto \tau+1$

(Dedekind eta)

residue theory $\Rightarrow \eta(-1/\tau)^{24} = \tau^{12} \eta(\tau)^{24} \Rightarrow \eta^{24} = q + \cdots \in S_{12}(\Gamma)$ //

On $\mathbb{P}^1 \cong \overline{\Gamma \backslash \mathfrak{h}}$, the local coordinate w at $\begin{cases} 0 & \text{looks like } (\tau - i)^2 \\ 1 & \text{---} \quad (\tau - \zeta_3)^3 \\ \infty & \text{---} \quad q \end{cases}$

$\Rightarrow d\tau$ becomes $\sim \begin{cases} dw/\sqrt{w} \\ dw/w^{2/3} \\ dw/w \end{cases}$. Hence,

$f \in M_{2k}(\Gamma)$ resp. $S_{2k}(\Gamma)$ $\Longleftrightarrow f\, d\tau^{\otimes k} \in \Gamma\left(\mathbb{P}^1, \mathcal{O}\left(-2k + \left\lfloor \frac{k}{2} \right\rfloor [0] + \left\lfloor \frac{2k}{3} \right\rfloor [1] + k[\infty]\right)\right)$ resp. $k-1$

So $\dim M_{2k}(\Gamma) = \underbrace{\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{2k}{3} \right\rfloor - k}_{\dim S_{2k}(\Gamma)} + 1 \quad \forall\, k \geq 2$.

eg/ $\dim S_{12} = 1 \Rightarrow \eta^{24} = \Delta$

$\dim M_8 = 1 \Rightarrow E_4^2 = E_8 \Rightarrow \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120}$ $(\forall n)$. //

Can also show: $\bigoplus M_k(\Gamma) = \mathbb{C}[E_4, E_6]$.

V) Cuspidal automorphic forms G, Γ, dg, etc. as above

Let ${}^\circ L^2(\Gamma\backslash G) := \overline{\left\{ \Phi \in L^2(\Gamma\backslash G) \;\middle|\; \int_0^1 \Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx = 0 \;\; \forall g\in G \right\}}$ ← L^2 closure; ↻ $\pi(g)$ right G-action

$${}^\circ\mathcal{A}(G,\Gamma) := \mathcal{A}(G,\Gamma) \cap {}^\circ L^2(\Gamma\backslash G)$$ = smooth, Ω & K-finite vectors in ${}^\circ L^2$.

(= $(\mathfrak{g}, K)$ rep. by infinitesimal right translation)

Subreps. unitary: which ones appear? By result in rep. thry., they occur discretely: so, only countably many (as a $\oplus$)!

$${}^\circ\mathcal{A}_m(G,\Gamma) := \left\{ \Phi \in {}^\circ\mathcal{A} \cdots \;\middle|\; \pi(k_\theta)\Phi = e^{im\theta}\Phi \right\}$$

(*) $f^{[m]} \downarrow \;\cong\; \uparrow \Phi^{[m]}$

$${}^\circ C^\infty_m(\mathfrak{h},\Gamma) := \left\{ f \in C^\infty(\mathfrak{h}) \;\middle|\; \begin{array}{l} \bullet\; f|^m_\gamma = f \;\; \forall \gamma\in\Gamma \\ \bullet\; \int_{\Gamma\backslash\mathfrak{h}} |f|^2 y^{m-2}\, dx\wedge dy < \infty \\ \bullet\; \int_0^1 f(x+iy)\, dx = 0 \;\; \forall y \in \mathbb{R}_+ \\ \bullet\; f \;\; \omega_m\text{-finite} \end{array} \right\}$$

$\omega_m := -y^2\Delta + 2imy\,\partial_{\bar\tau} + \frac{m}{2}\left(1-\frac{m}{2}\right)$

where $\Phi^{[m]}_f(g_{\tau,\theta}) := e^{im\theta} y^{m/2} f(\tau) \overset{(**)}{=} \left(f|^m_{g_{\tau,\theta}}\right)(i)$

$f^{[m]}_\Phi(\tau) := \Phi(p_\tau)\cdot y^{-m/2}$

are clearly inverse.

(formerly: $E_{\lambda,l}$ $\downarrow\Gamma$ $\mathcal{E}_{\lambda,l}$)

Pf. of (**): $\left(f|^m_{g_{\tau,\theta}}\right)(i) = \left(f|^m_{p_\tau}|^m_{k_\theta}\right)(i) = \dfrac{\left(f|^m_{p_\tau}\right)(k_\theta\langle i\rangle)}{(-i\sin\theta+\cos\theta)^m}$

$$= e^{im\theta}\left(f|^m_{p_\tau}\right)(i) = e^{im\theta}\frac{f(p_\tau\langle i\rangle)}{(0i+1/\sqrt{y})^m} = e^{im\theta} y^{m/2} f(\tau)$$

Sketch of (*): $\bullet\; \left(f^{[m]}_\Phi|^m_\gamma\right)(\tau) = \dfrac{\left(f^{[m]}_\Phi|^m_\gamma|^m_{p_\tau}\right)(i)}{y^{m/2}} \underset{(**)}{=} y^{-m/2}\,\Phi^{[m]}_{f^{[m]}_\Phi}(\gamma p_\tau)$

$$= y^{-m/2}\,\Phi(\gamma p_\tau) = y^{-m/2}\,\Phi(p_\tau) = f^{[m]}_\Phi(\tau).$$

$(\bullet\; \omega_m = f^{[m]}\circ\Omega\circ\Phi^{[m]})$

Remark: ${}^\circ\mathcal{A}_m = 0$ for m odd: $\Phi(g) \underset{\substack{\uparrow \\ -\mathrm{id}\in\Gamma}}{=} \Phi(\overset{g e^{i\pi}}{-g}) = \pi(k_\pi)\,\Phi(g) \overset{m \text{ odd}}{=} -\Phi(g)$.

Now ${}^{\circ}C^{\infty}(\mathfrak{h},\Gamma) := \bigoplus_{m\in\mathbb{Z}} {}^{\circ}C^{\infty}_m \cong {}^{\circ}\mathcal{A}(G,\Gamma) \circlearrowleft \underbrace{(\mathfrak{g},K)\text{-mod}}_{\text{explicitly realize on LHS:}}$

The formulas from before are almost correct (need $\partial/\partial x$ term):

$$\mathcal{L}_W = -i\frac{\partial}{\partial\theta}, \quad \mathcal{L}_{E_+} = e^{2i\theta}\left\{2iy\,\partial_\tau - \frac{i}{2}\partial_\theta\right\}, \quad \mathcal{L}_{E_-} = e^{-2i\theta}\left\{-2iy\,\partial_{\bar\tau} + \frac{i}{2}\partial_\theta\right\}$$

$$L_W := f^{[m]} \circ \mathcal{L}_W \circ \Phi^{[m]} = m.$$

$$L_{E_-} := \quad \ldots \quad = -2iy^2\,\partial_{\bar\tau}$$

$$L_{E_+} := \quad \ldots \quad = 2i\,\partial_\tau + \frac{m}{y}.$$

expand?

Theorem: $\#\left\{\begin{array}{c}\text{(indep.)}\\ \text{Copies of } D^+_{m-1} \subset {}^{\circ}\mathcal{A}(G,\Gamma)\end{array}\right\} = \begin{cases} 0, & m \text{ odd} \\ \left\lfloor \frac{m}{4}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor - \frac{m}{2}, & m \text{ even}\end{cases}$

Pf: $\mathrm{Hom}\left(D^+_{m-1}, {}^{\circ}\mathcal{A}(G,\Gamma)\right) \cong$

lowest wt. vect.

$$\underbrace{\ker L_{E_-}}_{(\ker \partial_{\bar\tau})} \cap {}^{\circ}C^{\infty}_m(\mathfrak{h},\Gamma) = \left\{ f \in \mathcal{O}(\mathfrak{h}) \;\middle|\; \begin{array}{l} \bullet\ f|^m_\gamma = f \quad \forall \gamma\in\Gamma \\ \bullet\ \underbrace{\int_0^1 (a_0 + a_1 q + \ldots)\,dx}_{a_0} = 0 \end{array}\right\}$$

$$= S_m(\Gamma).$$ Take dims. on both sides //

Remark: $\#\{\text{Copies of } D^-_{m-1} \subset {}^{\circ}\mathcal{A}(G,\Gamma)\}$ is same, since $\Phi \to \overline{\Phi}$ inverts weights & $\overline{\mathcal{L}_{E_+}} = \mathcal{L}_{E_-}$

On $\mathfrak{h}$, this is ${}^{\circ}C^{\infty}_m \to {}^{\circ}C^{\infty}_{-m}$
$f \mapsto y^m \bar f$.

Bi-infinite ladders? must pass thru $0\ (=m)$

$${}^{\circ}C^{\infty}_0(\mathfrak{h},\Gamma) = \left\{ f\in C^\infty(\Gamma\backslash\mathfrak{h}) \;\middle|\; \begin{array}{l} \bullet\ \int_{\Gamma\backslash\mathfrak{h}} |f|^2 \frac{dx\wedge dy}{y^2} < \infty \\ \bullet\ \int_0^1 f(x+iy)\,dx = 0 \\ \bullet\ f \quad \omega_0 = \underbrace{-y^2\Delta}_{\text{hyp. Laplacian}} \text{-finite} \end{array}\right\}$$

$$= \bigoplus_{\xi\in\Xi} {}^{\circ}C^{\infty}_0 \cap \ker\{\omega_0 - \xi\}$$

$\uparrow$ indexing set $\subset \mathbb{R}$

$$= \bigoplus M_\xi \quad \text{(Maass cusp forms)} \longleftrightarrow \tfrac{1}{4}(1-\lambda^2) \text{ for the } I_{+,\lambda}\ (\cong I_{+,-\lambda})$$

this lies in

• w_0 positive def. : $\langle -y^2\Delta f, f\rangle = \int_{\Gamma\backslash\mathfrak{h}} (\cancel{-y^2}\Delta f)\,\bar f\,\frac{dx\wedge dy}{\cancel{y^2}}$ (11)

$$\underset{(\int \text{ by parts})}{=} \int_{\Gamma\backslash\mathfrak{h}} \left\{\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2\right\} dx\wedge dy > 0$$

$\Rightarrow \lambda^2 \in \mathbb{R}_{<1} \Rightarrow \lambda \in \underset{\text{comp.}}{\mathbb{D}_1}$ or $\underset{\text{spherical principal}}{i\mathbb{R}}$

• Selberg : $\xi \geq \frac{1}{4} \Rightarrow \lambda = i\nu$. only

<u>Theorem</u> : $\dim \mathcal{M}_{\frac{1+\nu^2}{4}} = \#\{\text{copies of } \underset{\underset{P_+(\nu)}{\shortparallel}}{I_{+,i\nu}} \subset {}^\circ\mathcal{A}(G,\Gamma)\}$

<u>Conclusion</u> : ${}^\circ\mathcal{A}(G,\Gamma) = \bigoplus$ $(\mathfrak{g}, K)$ isotypes of type D^+, D^-, P_+ ← only countably many ν's occur.

<u>Remark</u> : The Eisenstein series of §III are a complement to this in the sense that (a) they lie outside ${}^\circ\mathcal{A}$ (even $\mathcal{A}\cap L^2$ in some cases) and (b) λ varies continuously.

VI) Cohomology

$\Gamma \leq Sl_2(\mathbb{Z})$ congruence sgp. ⇝ LDS & $I_{-,i\nu}$ can now occur in ${}^\circ\mathcal{A}(G,\Gamma)$

$\mathcal{E}_\Gamma \xrightarrow[p]{} Y_\Gamma$, $\omega := p_*\Omega^1_{\mathcal{E}_\Gamma/Y_\Gamma}$ (Hodge bundle)

$K_{Y_\Gamma} \cong \omega^{\otimes 2}$ (from the automorphy factors we computed previously)

ω admits canonical extension to $\overline{Y}_\Gamma$ w./Herm. metric of log growth along $D := \overline{Y}_\Gamma \backslash Y_\Gamma$, $\bar\omega \to \overline{Y}_\Gamma$ (compatible with $\otimes$). We have

$K_{\overline{Y}_\Gamma} = \bar\omega^{\otimes 2}(-D)$, $\Gamma(\overline{Y}_\Gamma, \bar\omega^{\otimes k}) = M_k(\Gamma)$, $\Gamma(\overline{Y}_\Gamma, \bar\omega^{\otimes k}(-D)) = S_k(\Gamma)$.

Write $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{n}\oplus\mathfrak{n}^+ = \mathbb{C}\langle W\rangle \oplus \mathbb{C}\langle E_-\rangle \oplus \mathbb{C}\langle E_+\rangle$.

In general, Lie algebra cohomology of a rep. (V_π, π) of $G (= Sl_2(\mathbb{R}))$ wrt $\mathfrak{n}$ is coh. $H^*(\mathfrak{n}, V_\pi)$ ($\circlearrowleft K$) of a complex

$$0 \to V_\pi \xrightarrow{d} \mathfrak{n}^\vee \otimes V_\pi \xrightarrow{d} \Lambda^2\mathfrak{n}^\vee \otimes V_\pi \xrightarrow{d} \cdots \quad (K \text{ acts by Ad on } \mathfrak{n})$$

which in this case is

$$0 \to V_\pi \xrightarrow{d} \mathrm{Hom}(\mathfrak{n}, V_\pi) \to 0$$
$$\mathrm{ev}_{E_-} \downarrow \cong$$
$$V_\pi \quad (\text{twist } \mathcal{L}_{E_-})$$
$$v \longmapsto (dv)(E_-) = E_- \cdot v$$

So $H^0(\mathfrak{n}, V_\pi) = \ker(E_-) =$ lowest wt. vector
$H^1(\mathfrak{n}, V_\pi) = \mathrm{coker}(E_+) =$ highest wt. vector } only nonzero for DS & LDS!

Letting a subscript k denote the K-eigenspace,

$$H^0(\mathfrak{n}, V_\pi)_k = \begin{cases} \mathbb{C} & \text{if } k \geq 1 \text{ \& } V = D^+_{k-1} \\ 0 & \text{otherwise} \end{cases}$$

$$H^1(\mathfrak{n}, V_\pi)_k = \begin{cases} \mathbb{C} & \text{if } k \leq 1 \text{ \& } V = D^-_{1-k} \\ 0 & \text{otherwise} \end{cases} \qquad (\mathfrak{n}^\vee \text{ contributes } +2 \text{ to } K\text{-types})$$

<u>Definition</u>: The <u>cuspidal automorphic cohomology</u> of Y_Γ is

$$\mathcal{S}^*(Y_\Gamma, \omega^{\otimes k}) := \ker\{H^*(\overline{Y}_\Gamma, \overline{\omega}^{\otimes k}) \to H^*(D, \overline{\omega}^{\otimes k}|_D)\}.$$

This turns out to be equal to the L^2 Dolbeault cohomology $H^*_{(2)}(Y_\Gamma, \omega^{\otimes k})$,

which
$$= H^*\{(\Lambda^\bullet \mathfrak{n}^\vee \otimes \mathcal{A}^2(G,\Gamma) \otimes \mathbb{C}_k)^K\}$$
$$= H^*(\mathfrak{n}, \mathring{\mathcal{A}}(G,\Gamma))_{-k}$$
$$= \bigoplus_{\pi \in \hat{G}_u} H^*(\mathfrak{n}, V_\pi)_{-k}^{\oplus m_\pi(\Gamma)} \longleftarrow (1\text{-dim'l if } \neq 0)$$

corresponds to antiholomorphic differentials in Dolbeault complex

can replace this (L^2 forms) by cuspidal ones — nothing is lost.

So $\mathcal{S}^0(Y_\Gamma, \omega^{\otimes m}) = \mathrm{Hom}(D^+_{k-1}, \mathring{\mathcal{A}}(G,\Gamma)) \cong \mathcal{S}_m(\Gamma)$ (duh)

while $(*)$ $\mathcal{S}^1(Y_\Gamma, K_{Y_\Gamma} \otimes (\omega^{\otimes m})^\vee) = \mathcal{S}^1(Y_\Gamma, \omega^{\otimes(2-m)}) = \mathrm{Hom}(D^-_{m-1}, \mathring{\mathcal{A}}(G,\Gamma)) \cong \mathcal{S}_m(\Gamma)$

$$m-1 = 1-(2-m)$$

Now the +-def. inner product

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\Gamma \backslash G} \Phi_1 \overline{\Phi}_2 \, dg \qquad \text{on } \mathring{\mathcal{A}}(G,\Gamma)$$

$$dg = \frac{dx \wedge dy \wedge d\theta}{2\pi y^2}$$

becomes the Petersson inner product

$$\langle f_1, f_2 \rangle_m = \int_{\Gamma \backslash \mathfrak{H}} f_1 \overline{f_2} \, y^m \frac{dx \wedge dy}{y^2} \quad \text{on } \mathcal{S}_m(\Gamma).$$

Altogether then we get a composite $\cong$:

$$\begin{tikzcd} \mathcal{S}^0(Y_\Gamma, \omega^{\otimes m})^\vee & \mathcal{S}^0(Y_\Gamma, \omega^{\otimes m}) \arrow[l, "\langle \cdot, f\rangle_m \leftarrow f"', "\cong"] \arrow[r, "\cong"'] & \mathcal{S}^1(Y_\Gamma, K_{Y_\Gamma} \otimes \omega^{\vee \otimes m}) \\ \arrow[from=1-1, to=1-3, bend right, "\overset{SD}{\cong}"'] \end{tikzcd}$$

recovering Serre duality in this case. But in fact we have much more: on the level of representatives, the $\cong$ $(*)$ is given by complex conjugation

$$\begin{array}{ccccccc} \mathcal{S}_m(\Gamma) & & D^+_{m-1} & \xrightarrow[\text{c.c.}]{\cong} & D^-_{m-1} & & \in \mathcal{S}^1(Y_\Gamma, K_{Y_\Gamma} \otimes \omega^{\vee \otimes m}) \\ \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; & & \\ f & \longmapsto & \Phi_f & \longmapsto & \overline{\Phi}_f & \longrightarrow & y^m \bar{f} \underbrace{\left(\frac{d\bar\tau}{y^2}\right)}_{\substack{\text{formal} \\ \text{weight 2}}} (\otimes d\tau) \\ & & \| & & \| & & \\ & & e^{im\theta} y^{m/2} f & & e^{-im\theta} y^{-m/2}(y^m \bar{f}) & & \end{array}$$

This does 3 things:

- makes SD in some sense the <u>identity</u> map
- gives us explicit representatives of $\mathcal{S}^1$
- anticipates a special case of the Penrose transforms Phillip will discuss.

<u>Remark</u>: If we had taken instead $\mathfrak{n} = \mathbb{C}\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$, then

Frobenius reciprocity $\Rightarrow$ $H^*(\mathfrak{n}, V_\pi)$ essentially always $\neq 0$ (for admissible), as it has to recover the original rep'n of the Levi we induced from.

While less interesting from our point of view, this observation plays a crucial role in the classification of admissible reps. //

<u>Conclusion</u>: Whether you want to [(a)] use rep'n theory to understand complex geometry of $_\Gamma\backslash D$'s, or [(b)] use arithmetic geometry of $_\Gamma\backslash D$ to attack the Langlands program for automorphic representations, the decomposition of $\mathcal{A}_0(G, \Gamma)$ into tempered irreps, and the computation of their $\mathfrak{n}$-cohomology, are hard problems of great importance; which will be illustrated for more general reductive groups in Phillip's (& Wushi's) lectures.