The Hodge Theorems: sketch, consequences, generalizations

Aroldo Kaplan

◇ Introduction.

◇ Harmonic forms on hermitian manifolds.

◇ Kahler manifolds and the Hodge Decomposition.

◇ Sl_2 action: Lefschetz Decomposition, bilinear relations.

◇ Remarks on algebraicity, generalizations, etc.

$M =$ compact complex manifold of dimension n.

Riemann's Theorem ($\sim$ 1850): *For $n = 1$,*

$$H^1_{dR}(M, \mathbb{C}) \cong \Omega^1(M) \oplus \overline{\Omega^1(M)}.$$

or: the number of independent holomorphic differentials on a compact non-singular Riemann surface is equal to its genus.

Hodge Theorem ($\sim$ 1940): *$M =$ compact Kahler manifold. Then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q} H^{p,q}(M)$$

$$H^{p,q}(M) := \frac{\ker(d) \cap \mathcal{A}^{p,q}(M)}{\mathrm{im}(d) \cap \mathcal{A}^{p,q}(M)} = \overline{H^{q,p}(M)}.$$

Reason for the lapse:

Hodge decomposition cannot not hold for general compact complex M:

$$H^1 = H^{10} \oplus \overline{H^{10}} \qquad \Rightarrow \qquad b_1(M) \;\; \mathit{even}$$

Expected for algebraic M – enough for the study of algebraic integrals.

But requires transcendental proof, so likely to hold for $\{?\} \supset \{\mathit{algebraic}\}$.

Hodge:

$$? = \mathit{Kahler}$$

Will sketch proof of the decomposition theorem.

$\mathcal{A}_M^{p,q}$ = sheaf of germs of forms on M of the form

$$\sum_{I,J} f_{I,J}\ dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}, \qquad f_{I,J} \in C_M^\infty$$

$$d : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q} \oplus \mathcal{A}_M^{p,q+1}$$

$$d = \partial + \bar{\partial}$$

Explicitely, $\bar{\partial}(f\ d\mathbf{z}_I \wedge d\bar{\mathbf{z}}_J) =$

$$\sum_r \frac{\partial f}{\partial \bar{z}_r} d\bar{z}_r \wedge d\mathbf{z}_I \wedge d\bar{\mathbf{z}}_J, \qquad \frac{\partial f}{\partial \bar{z}_r} = \frac{1}{2}\left(\frac{\partial f}{\partial x_r} + i\frac{\partial f}{\partial y_r}\right)$$

Since

$$0 = (\partial + \bar{\partial})^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2$$

$$\text{type:} \qquad (2,0) \qquad (1,1) \qquad (0,2)$$

implies

$$\bar{\partial}^2 = 0 = \partial^2 = \partial\bar{\partial} + \bar{\partial}\partial$$

Dolbeault cohomology

$$H_{\bar{\partial}}^{p,q}(M) = \frac{ker(\bar{\partial} : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M))}{\bar{\partial}(\mathcal{A}^{p,q-1}(M))}$$

No general relation between $H_d^k(M, \mathbb{C})$ and $H_{\bar{\partial}}^{p,q}(M)$ except for (Frolicher) spectral sequence of $(\mathcal{A}^{*,*}(M), \partial, \bar{\partial})$.

Hermitian metrics

Positive hermitian inner product $h(u, v)$ on the holomorphic $T(M)$:

$$h(\lambda u, v) = \lambda h(u, v), \quad h(u, \lambda v) = \bar{\lambda} h(u, v)$$

$$h(u, v) = \overline{h(v, u)}, \quad h(v, v) >> 0$$

J : complex structure on $T^{\mathbb{R}}(M)$:

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

$$T^{\mathbb{C}}(M) =: \mathbb{C} \otimes T^{\mathbb{R}}(M) = T(M) \oplus \overline{T(M)}$$

eigenspaces $\pm i$ of J. As real spaces with a J,

$$(T(M), i) \cong (T^{\mathbb{R}}(M), J)$$

On $T^{\mathbb{R}}(M)$,

$$h = g - i\omega$$

g riemannian metric
ω 2-form of type (1,1).

$$g(Ju, Jv) = g(u, v), \quad \omega(Ju, Jv) = \omega(u, v), \quad g(X, Y) = \omega(X, JY).$$

(M, ω) determines h.
(M, g) too, so hermitian metrics exist.

A hermitian metric on M defines one on $\mathcal{A}^{p,q}(M)$:

$$(\phi, \psi) = \int_M h_x(\phi_x, \psi_x)\Omega_h(x).$$

and corresponding adjoint operators

$$d^*, \partial^*, \overline{\partial}^*.$$

<u>Laplacians</u>:

$$\triangle_d = dd^* + d^*d \qquad \triangle_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

$\triangle_d$ preserves degree and real structure.
$\triangle_{\bar\partial}$ preserves type (p,q).
$(\triangle\phi, \psi) = (\phi, \triangle\psi)$

<u>Harmonic forms</u>:

$$\mathfrak{H}_d^k(M) = \{\phi \in \mathcal{A}^k(M, \mathbb{C}) : \triangle_d\phi = 0\}$$

$$\mathfrak{H}_{\bar\partial}^{p,q}(M) = \{\phi \in \mathcal{A}^{p,q}(M) : \triangle_{\bar\partial}\phi = 0\}$$

Hodge Theorems on harmonic forms

For M compact orientable riemannian,

$$H^k_d(M) \cong \mathfrak{H}^k_d(M)$$

For M compact hermitian

$$H^{p,q}_{\bar\partial}(M) \cong \mathfrak{H}^{p,q}_{\bar\partial}(M)$$

More precisely: every cohomology class contains a unique harmonic representative (= form of smallest norm).

Sketch:

Harmonic $\Rightarrow$ closed and coclosed:

$$0 = (\triangle\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) = |d\alpha|^2 + |d^*\alpha|^2$$

Therefore have map

$$\mathfrak{H}^k \to H^k_{dR}, \qquad \alpha \mapsto [\alpha]$$

Injective: $[\alpha] = 0 \Rightarrow \alpha = dw$. Harmonic $\Rightarrow$ coclosed: $d^* dw = 0$ and therefore

$$(dw, dw) = (d^* dw, w) = 0.$$

Surjective: based on another "Hodge decomposition":

<u>Theorem</u>. *M compact Riemannian $\Rightarrow \mathfrak{H}(M)$ is finite-dimensional and*

$$\mathcal{A}(M) = \mathfrak{H}(M) \oplus \triangle(\mathcal{A}(M))$$

$$\mathcal{A}(M) = \ker\triangle \oplus \mathrm{im}\triangle$$

If $\triangle$ was a symmetric operator on finite-dimensional space, would diagonalize, separate 0-eigenspace, etc.
One does this weakly in the L_2 completion $\widehat{\mathcal{A}(M)}$: for any $\Psi \perp \ker\triangle$ the equation

$$\triangle\phi = \Psi$$

has a unique weak solution $\phi \in \widehat{\mathcal{A}(M)}$, meaning that for all $\alpha \in \mathcal{A}(M)$

$$(\phi, \triangle\alpha) = (\Psi, \alpha).$$

This involves functional analysis.
Next one proves that $\triangle$ is elliptic, which $\Rightarrow$ weak solutions with Ψ smooth, are smooth, QED.

Now

$$\begin{aligned}\mathcal{A}(M) &= \ker\triangle \oplus \mathrm{im}\triangle \\ &= \mathfrak{H}(M) \oplus (dd^* + d^*d)(\mathcal{A}(M)) \\ &= \mathfrak{H}(M) \oplus d(\mathcal{A}(M)) \oplus d^*(\mathcal{A}(M))\end{aligned}$$

because the last 3 terms are $\perp$.
Since

$$d^*(\mathcal{A}(M)) = Closed(M)^{\perp},$$

$$Closed(M) = \mathfrak{H}(M) \oplus d(\mathcal{A}(M))$$

$\Rightarrow$

$$H(M) \cong \mathfrak{H}(M),$$

QED.

Kahler metrics

A hermitian metric $h = g - i\omega$ is *Kahler* if $d\omega = 0$.

M is *Kahler* if it admits a Kahler metric.

◇ If $dim_{\mathbb{C}} M = 1$ any hermitian metric is Kahler.

◇ Projective implies Kahler, because Fubini-Study on $\mathbb{C}P^n$

$$\omega_{FS} = -\frac{1}{2\pi i}\partial\overline{\partial}\log(1 + \sum \mid z_i \mid^2)$$

is, and restricting to complex submanifolds preserves Kahler.

◇ Hopf surface $(\mathbb{C}^2 - 0)/2^{\mathbb{Z}}$ is not Kahler.

Crucial Lemma. *For a Kahler metric,*

$$2\triangle_{\bar\partial} = \triangle_d.$$

The Lemma implies that (p,q)-components of harmonic are harmonic, and therefore

$$\mathfrak{H}^k_d(M) = \bigoplus_{p+q=k} \mathfrak{H}^{p,q}_{\bar\partial}(M).$$

$$H^k_d(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M)$$

s.t. $H^{p,q}(M) = \overline{H^{q,p}(M)}$.

Independent of the metric because

$$H^{p,q}(M) := \frac{\text{global closed (p, q)} - \text{forms}}{\text{exact ones}} \cong \mathfrak{H}^{p,q}(M)).$$

This and the Lemma follow from the Hodge-Kahler identities.

Define $L \in End(A^*)$ by

$$L(\phi) = \omega \wedge \phi$$

$$A^k \to A^{k+2}, \qquad A^{p,q} \to A^{p+1,q+1}$$

and

$$\Lambda = L^*.$$

Hodge-Kahler:

$$[\Lambda, \bar{\partial}] = -i\partial^*, \qquad [\Lambda, \partial] = i\bar{\partial}^*$$

The proof for the euclidean metric

$$\omega = i\Sigma dz_j \wedge d\bar{z}_j$$

is a just a calculation.

In general, one proves that Kahler implies

$$h = \text{Euclidean} + \text{terms of order } \geq 2$$

which implies H-K, these being 1st. order.

One concludes:

Hard Lefschetz Theorem:

(a):

$$L^k : H^{n-k}(M, \mathbb{R}) \to H^{n+k}(M, \mathbb{R})$$

is an isomorphism.
(b): The lowest-weight vectors

$$P^k = H^k(M, \mathbb{R}) \cap (ker\Lambda)$$

generate H:

$$H^k(M, \mathbb{R}) = \bigoplus_j L^j P^{k-2j}$$

(c): (a) and (b) are compatible with the Hodge decomposition (because L is of pure type).

Polarizing form on $H^k(M, \mathbb{R})$:

$$Q([\alpha], [\beta]) = \int_M \alpha \wedge \beta \wedge \omega^{n-k}.$$

Real, symmetric for k even, antisymmetric for odd.

Theorem. *Q is non-degenerate, the Hodge and Lefschetz decompositions are orthogonal under $Q(u, \bar{v})$, and on $P^{p,q}$ and for $\alpha \neq 0$,*

$$i^{p-q}(-1)^{\frac{(n-p-q)(n-p-q-1)}{2}} Q(\alpha, \overline{\alpha}) >> 0$$

Proof: use the Hodge star operator:

$$Q([\alpha], [\beta]) = (-1)(L^{n-k}\alpha, *\beta)$$

and prove

$$*\beta = \frac{(-1)^{\frac{k(k+1)}{2}} i^{p-q}}{(n-k)!} L^{n-k}\beta.$$

Geometrically: Under Poincaré duality

$$H^k(M, \mathbb{R}) \cong H_{2n-k}(M, \mathbb{R})$$

For $M \hookrightarrow \mathbb{C}P^N$,

◇ Primitive (or finite) cycles: don't intersect the hyperplane $\mathbb{C}P^{N-1}$ at ∞.

◇ Hard Lefschetz expresses any cycle in terms of these $+ \bigcap \mathbb{C}P^{N-k}$'s.

◇ $Q =$ intersection form.

Some basic consequences

◇ *The odd Betti numbers of a compact Kahler manifold are even*, because

$$H^{2k+1}(M, \mathbb{C}) = H^{2k+1,0} \oplus ... \oplus H^{k+1,k} \oplus \overline{H^{2k+1,0} \oplus ... \oplus H^{k+1,k}}$$

$\diamond$ Generalization of Riemann's theorem:

$$\Omega^k(M) \cong H^{k,0}(M).$$

Proof: $H^{p,q}(M) \cong H^{p,q}_{\bar\partial}(M) \cong H^q(M, \Omega^p)$ by Dolbeault's Theorem. In particular $H^{p,0}(M) \cong H^0(M, \Omega^p) = \Omega^p(M)$.

$\diamond$ On $\mathbb{C}P^n$ the only holomorphic forms are the constant functions.

Further remarks

◇ Define a "weight filtration" $W_k = H^0 \oplus ... \oplus H^k$. Then

$$(H^*(M, \mathbb{Q}), F^*, W_*)$$

is a *mixed Hodge structure*, polarized by (L_ω, Q) and split $/\mathbb{R}$ by Y.

<u>On Algebraicity</u>.

◇ M Kahler is algebraic iff $[\omega] \in H^k(M, \mathbb{Z})$ (Kodaira).

For M algebraic,

◇ There are algebraic constructions of the filtration

$$F^p(H^k(M, \mathbb{C})) = H^{k,0} \oplus ... \oplus H^{p,k-p}$$

[Deligne-Illusie 1989, etc.].

◇ There cannot be algebraic proofs of

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} F^p \cap \overline{F^q}.$$

Hodge for Characteristic Cohomology

M smooth or analytic manifold
$D \subset T(M)$ distribution on M (e.g. horizontal on a M-T domain)
$E = D^{\perp} \subset T^*(M)$ Exterior Differential System defined by D.
$\mathcal{I} \subset \Omega^*(M)$ ideal generated by the sections of E and their differentials.
Characteristic Cohomology of E: de Rham

$$H^*_E(\Omega^*(M)/\mathcal{I}, d)$$

Assume D bracket-generating.

Then $\dim H^0_E = 1$.

M is connected by piecewise horizontal curves.

Furthermore, let $g =$ subriemannian or subhermitian metric supported on D.
Then d^* exists on $\Omega^*(M)/\mathcal{I}$ and

Sublaplacian on functions:

$$L \sim X_1^2 + ... + X_d^2 + \text{lower order}$$

X_i frame of D.
Regularity goes back to

Hormander 1965: *If X_i generate the tangent sheaf, then $X_1^2 + ... + X_d^2$ is hypoelliptic.*

Theorem (M. Taylor, Griffiths): *Assume M compact. Then $\mathfrak{H}_L(M) = \{\varphi : L\varphi = 0\}$ is finite dimensional,*

$$\Omega(M) = \mathfrak{H}_L(M) \oplus L(\Omega(M))$$

and therefore

$$H_E^*(\Omega^*(M)/\mathcal{I}, d) \cong \mathfrak{H}_L(M).$$

Assume M complex with Kahler *indefinite* metric h, but positive-definite on D (e.g. horizontal distribution).

Hodge decomposition for $H^*_{\mathcal{E}}(\Omega^*(M)/\mathcal{I}, d)$?

"Sub- Hodge decomposition"?

EDS's at the boundary

Let $M =$ a hyperbolic space $\mathbb{K}H^N$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$, with metric g. Fix $o \in M$ and let
$\mathbb{S}_r =$ sphere of hyperbolic radius r around o.
$\mathbb{S}_\infty =$ sphere at infinity.

Then $g|_{\mathbb{S}_r}$ blows up as $r \to \infty$, but $\gamma := \lim_{r\to\infty} e^{-2r} g|_{\mathbb{S}_r}$ defines on $\mathbb{S}_\infty$:

– A standard spherical metric if $\mathbb{K} = \mathbb{R}$ (up to multiples)

– A subriemannian conformal metric supported on a contact distribution $D \subset T(\mathbb{S})$ (and ∞ elsewhere), if $\mathbb{K} = \mathbb{C}$

More precisely, the hyperbolic metric decomposes as

$$g = dr^2 + \sinh^2(2r)\theta^2 + \sinh^2(r)\gamma$$

where $\theta|_{\mathbb{S}} =$ is a "polycontact" ($=$ contact with values in $Im(\mathbb{K})$) 1-form, and $\gamma|_{\mathbb{S}} =$ subriemannian metric supported on $D = ker\theta$.

Changing the origin o changes θ, γ by a multiple, but D and the conformal class of γ are determined by the metric in the interior. Conversely, θ, γ determine g up to multiples.

This is an instance of the AdS-QFT (Maldacena, holographic) correspondence.

Main application: deforming the EDS θ, γ on the boundary via curvatures (in the sense of Cartan Equivalence) leads to Einstein deformations of the hyperbolic metric, which are new.

Generalizes to higher rank, with "parabolic" geometries at the boundary [Biquard-Mazzeo, *"Parabolic Geometries as Conformal Infinities of Einstein Metrics"* (2006)], and other situations, e.g. non-symmetric harmonic spaces $M = RN$ with N of Heisenberg type [K., *"Fundamental solutions for a class of hypoelliptic operators"] (1980)*.

In all cases the limiting structures are real and determine filtrations of $T^{\mathrm{R}}(\partial M)$.

Q: On period domains, how are MT domains reflected on the boundary along horizontal directions?