HODGE THEORY AND REPRESENTATION THEORY

PHILLIP GRIFFITHS

Ten lectures to be given during the NSF/CBMS Regional Conference in the Mathematical Sciences at TCU, June 18–22, 2012.

Table of Contents

These lectures are centered around the subjects of Hodge theory and representation theory and their relationship. A unifying theme is the geometry of homogeneous complex manifolds. One objective is to present, in a general context, some of the recent work of Carayol [C1], [C2], [C3].

Finite dimensional representation theory interacts with Hodge theory through the use of Hodge representations to classify the possible realizations of a reductive, $\mathbb{Q}$-algebraic group as a Mumford-Tate group. The geometry of homogeneous complex manifolds enters through the study of Mumford-Tate domains and Hodge domains.

Infinite dimensional representation theory and the geometry of homogeneous complex manifolds interact through the realization of the Harish-Chandra modules associated to discrete series representations , especially their limits, as cohomology groups associated to homogeneous line bundles (work of Schmid). It also enters through the work of Carayol on automorphic cohomology, the most recent of which involves the Hodge theory associated to boundary components of Mumford-Tate domains.

Throughout these lectures we have kept the "running examples" of SL_2, $S\mathfrak{U}(2,1)$, $\mathrm{Sp}(4)$ and $\mathrm{SO}(4,1)$. Many of the general results whose proofs are not given in the lectures are easily verified in the running examples. They also serve to illustrate and make concrete the general theory.

We have attempted to keep the lecture notes as accessible as possible. Both the subjects of Hodge theory and representation theory are highly developed and extensive areas of current mathematics and we are only able to touch on some aspects where they are related. When more advanced concepts from another area have been used, such as local cohomology and Grothendieck duality from algebraic geometry at the end of Lecture 6, we have illustrated them through the running examples in the hope that at least the flavor of what is being done will come through.

Lectures 1 and 2 are basically elementary, assuming some standard Riemann surface theory. In this setting we will introduce many of the basic concepts that appear in these lectures. At the end of Lecture 2 we have given a more extensive summary of the topics that are covered later in the lectures. Lecture 3 is also essentially self-contatined, although some terminology from Lie theory and algebraic groups will be used. Lecture 4 will draw on the structure and representation theory of complex Lie algebras and their real forms. Lecture 5 will use some of the basic material about infinite dimensional representation theory and the theory of homogeneous complex manifolds. In Lectures 6 and 7 we will draw from complex function theory and, in the last part of Lecture 6, some topics from algebraic geometry. Lectures 8 and 9 will utilize the material that has gone before; they are mainly devoted to specific computations in the framework that has been established. The final Lecture 10 is devoted to issues and questions that arise from the earlier lectures.

After a number of the lectures we have given an appendix whose purposes are to present proofs of results that because of time could not be given in the lecture and to discuss related topics that although perhaps not logically necessary for the lectures present related material that is of interest in its own right.

At the end of the lecture notes we have given a few additional references. These include several expository papers or books where a more complete set of references to the material in these lectures can be found. Lectures 3, 4, 8 and 9 are based on the joint works [GGK1] and [GGK2] with Mark Green and Matt Kerr. A main reference for Lecture 5 is [Sch2] and for Lecture 6 is [FHW]. Lecture 7 is in part drawn from [GG].

It is a pleasure to thank Sarah Warren for doing a marvelous job of typing these lecture notes.

Lecture 1
The classical theory: Part I

The first two lectures will be largely elementary and expository. They will deal with the upper-half-plane $\mathcal{H}$ and Riemann sphere $\mathbb{P}^1$ from the points of view of Hodge theory, representation theory and complex geometry. The topics to be covered will be

(i) compact Riemann surfaces of genus one (= 1-dimensional complex tori) and polarized Hodge structures (PHS) of weight one;
(ii) the space $\mathcal{H}$ of PHS's of weight one and its compact dual $\mathbb{P}^1$ as homogeneous complex manifolds;
(iii) the geometry and representation theory associated to $\mathcal{H}$;
(iv) equivalence classes of PHS's of weight one as $\Gamma\backslash\mathcal{H}$ and automorphic forms;
(v) the geometric representation theory associated to $\mathbb{P}^1$, including the realization of higher cohomology by global, holomorphic data;
(vi) Penrose transforms in genus $g = 1$ and $g \geqq 2$.

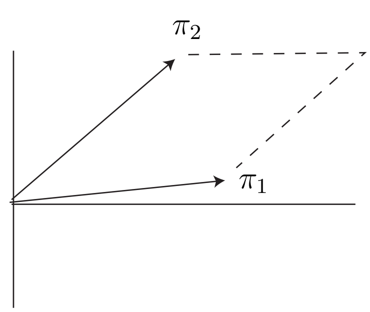
Assumptions:

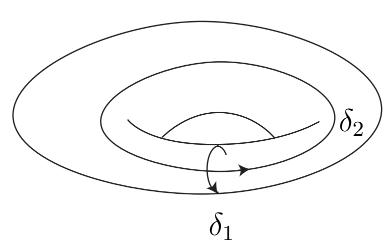
- basic knowledge of complex manifolds (in this lecture mainly Riemann surfaces);
- elementary topology and manifolds, including de Rham's theorem;
- some familiarity with classical modular forms will be helpful but not essential;[1]
- some familiarity with the basic theory of Lie groups and Lie algebras.[2]

Complex tori of dimension one: We let X = compact, connected complex manifold of dimension one and genus one. Then X is a complex torus $\mathbb{C}/\Lambda$ where

$$\Lambda = \{n_1\pi_1 + n_2\pi_2\}_{n_1,n_2\in\mathbb{Z}} \subset \mathbb{C}$$

is a lattice. The pictures are





Here $\delta_1 \leftrightarrow \pi_1$ and $\delta_2 \leftrightarrow \pi_2$ give a basis for $H_1(X, \mathbb{Z})$.

[1]The classical theory will be covered in the lectures by Matt Kerr.
[2]This topic will be covered in Mark Green's lecture.

The complex plane $\mathbb{C} = \{z = x + iy\}$ is oriented by

$$dx \wedge dy = \left(\tfrac{i}{2}\right) dz \wedge d\bar{z} > 0.$$

We choose generators π_1, π_2 for Λ with $\pi_1 \wedge \pi_2 > 0$, and then the intersection number

$$\delta_1 \cdot \delta_2 = +1.$$

We set $V_{\mathbb{Z}} = H^1(X, \mathbb{Z})$, $V = V_{\mathbb{Z}} \otimes \mathbb{Q} = H^1(X, \mathbb{Q})$ and denote by

$$\begin{cases} Q : V \otimes V \to \mathbb{Q} \\ \mathbb{Q}(v, v') = -Q(v', v) \end{cases}$$

the cup-product, which via Poincaré duality $H_1(X, \mathbb{Q}) \cong H^1(X, \mathbb{Q})$ is the intersection form.

We have

$$\begin{array}{ccl} H^1(X, \mathbb{C}) & \cong H^1_{\mathrm{DR}}(X) = & \left\{ \begin{array}{c} \text{closed 1-forms } \psi \\ \text{modulo exact} \\ \text{1-forms } \psi = d\zeta \end{array} \right\} \\ \wr\| & & \\ H^1(X, \mathbb{Z})^* \otimes \mathbb{C} & & \end{array}$$

and it may be shown that

$$H^1_{\mathrm{DR}}(X) \cong \operatorname{span}_{\mathbb{C}} \{dz, d\bar{z}\}.$$

The pairing of cohomology and homology is given by *periods*

$$\pi_i = \int_{\delta_i} dz$$

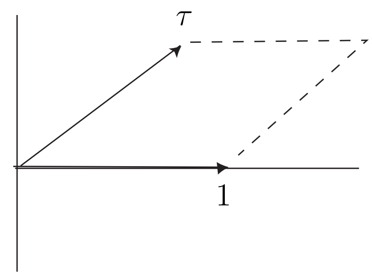
and $\Pi = \binom{\pi_2}{\pi_1}$ is the *period matrix* (note the order of the π_i's).

Using the basis for $H^1(X, \mathbb{C})$ dual to the basis δ_1, δ_2 for $H_1(X, \mathbb{C})$, we have

$$\begin{array}{ccl} H^1(X, \mathbb{C}) & \cong \mathbb{C}^2 & = \text{column vectors} \\ \cup & \cup & \\ dz & = \Pi. & \end{array}$$

We may scale $\mathbb{C}$ by $z \to \lambda z$, and then $\Pi = \lambda \Pi$ so that the period matrix should be thought of as point in $\mathbb{P}^1$ with homogeneous coordinates $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$. By scaling, we may normalize to have $\pi_1 = 1$, so that setting $\tau = \pi_2$ the normalized period matrix is $\left[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right]$

where $\operatorname{Im}\tau > 0$.



Differential forms on an n-dimensional complex manifold Y with local holomorphic coordinates $z_1, \ldots, z_n$ are direct sums of those of *type* (p, q)

$$f\, \underbrace{dz_{i_1} \wedge \cdots \wedge dz_{i_p}}_{p} \times \underbrace{d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}}_{q}.$$

Thus the C^∞ forms of degree r on Y are

$$\begin{cases} A^r(Y) \;\; = \bigoplus\limits_{p+q=r} A^{p,q}(Y) \\ A^{q,p}(Y) = \overline{A^{p,q}(Y)}. \end{cases}$$

Setting

$$H^{1,0}(X) = \operatorname{span}\{dz\}$$
$$H^{0,1}(X) = \operatorname{span}\{d\bar{z}\}$$

we have

$$\begin{cases} H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \\ H^{0,1}(X) \;\; = \overline{H^{1,0}(X)}. \end{cases}$$

This says that the above decomposition of the 1-forms on X induces a similar decomposition in cohomology. This is true in general for a compact Kähler manifold (Hodge's theorem) and is the basic starting point for Hodge theory. This will be discussed in the lectures by Eduardo Cattani.

From $dz \wedge dz = 0$ and $\left(\frac{i}{2}\right) dz \wedge d\bar{z} > 0$, by using that cup-product is given in de Rham cohomology by wedge product and integration over X we have

$$\begin{cases} Q\big(H^{1,0}(X), H^{1,0}(X)\big) = 0 \\ iQ\big(H^{1,0}(X), \overline{H^{1,0}(X)}\big) > 0. \end{cases}$$

Using the above bases the matrix for Q is

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and these relations are

$$\begin{cases} Q(\Pi,\Pi) = {}^t\Pi Q\Pi = 0 \\ iQ(\Pi,\overline{\Pi}) = i{}^t\overline{\Pi}Q\Pi > 0. \end{cases}$$

For $\Pi = \left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$ the second is just $\operatorname{Im}\tau > 0$.

Definitions: *(i) A* **Hodge structure** *of weight one is given by a $\mathbb{Q}$-vector space V with a line $V^{1,0} \subset V_{\mathbb{C}}$ satisfying*

$$\begin{cases} V_{\mathbb{C}} \ = V^{1,0} \oplus V^{0,1} \\ V^{0,1} = \overline{V}^{1,0}. \end{cases}$$

(ii) A **polarized Hodge structure** *of weight one (PHS) is given by the above together with a non-degenerate form*

$$Q : V \otimes V \to \mathbb{Q}, \qquad Q(v, v') = -Q(v', v)$$

satisfying the Hodge-Riemann bilinear relations

$$\begin{cases} Q(V^{1,0}, V^{1,0}) = 0 \\ iQ(V^{1,0}, \overline{V}^{1,0}) > 0. \end{cases}$$

In practice we will usually have $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$. The reason for working with $\mathbb{Q}$ will be explained later.

When $\dim V = 2$, we may always choose a basis so that $V \cong \mathbb{Q}^2 =$ column vectors and Q is given by the matrix above. Then $V^{1,0} \cong \mathbb{C}$ is spanned by a point

$$\left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right] \in \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$$

where $\operatorname{Im}\tau > 0$.

Identification: The space of PHS's of weight one *(period domain)* is given by the upper-half-plane

$$\mathcal{H} = \{\tau : \operatorname{Im}\tau > 0\}.$$

The *compact dual* $\check{\mathcal{H}}$ given by subspaces $V^{1,0} \subset V_{\mathbb{C}}$ satisfying $Q(V^{1,0}, V^{1,0}) = 0$ (this is automatic in this case) is $\check{\mathcal{H}} = \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$ where

$$\mathbb{P}^1 = \{\tau\text{-plane}\} \cup \infty = \text{lines through the origin in } \mathbb{C}^2.$$

It is well known that $\mathcal{H}$ and $\mathbb{P}^1$ are *homogeneous complex manifolds*; i.e., they are acted on transitively by Lie groups. Here are the relevant groups. Writing

$$z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

and using Q to identify $\Lambda^2 V$ with $\mathbb{Q}$ we have

$$Q(z, w) = {}^t w Q z = z \wedge w$$

and the relevant groups are

$$\begin{cases} \operatorname{Aut}(V_{\mathbb{R}}, Q) \cong \mathrm{SL}_2(\mathbb{R}) & \text{for } \mathcal{H} \\ \operatorname{Aut}(V_{\mathbb{C}}, Q) \cong \mathrm{SL}_2(\mathbb{C}) & \text{for } \mathbb{P}^1. \end{cases}$$

In terms of the coordinate τ the action is the familiar one:

$$\tau \to \frac{a\tau + c}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$. This is because $\tau = z_0/z_1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} az_0 + bz_1 \\ cz_0 + dz_1 \end{pmatrix} = z_1 \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}.$$

If we choose for our reference point $i \in \mathcal{H}$ $(= \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1)$, then we have the identifications

$$\begin{cases} \mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \\ \mathbb{P}^1 \cong \mathrm{SL}_2(\mathbb{C})/B \end{cases}$$

where (this is a little exercise)

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : i(a - d) = -b - c \right\}.$$

The Lie algebras are (here $\mathfrak{k} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)

$$\mathrm{sl}_2(\mathfrak{k}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b \in \mathfrak{k} \right\}$$

$$\mathrm{so}(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$\mathfrak{b} = \left\{ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Remark: From a Hodge-theoretic perspective the above identifications of the *period domain* $\mathcal{H}$ and its *compact dual* $\check{\mathcal{H}}$ are the most convenient. From a group-theoretic perspective, it is frequently more convenient to set

$$\zeta = \frac{\tau - i}{\tau + i}, \qquad \operatorname{Im}\tau > 0 \Leftrightarrow |\zeta| < 1$$

and identify $\mathcal{H}$ with the unit disc $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$. When this is done, $\mathrm{SL}_2(\mathbb{R})$ becomes the other real form

$$S\mathcal{U}(1,1)_{\mathbb{R}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : {}^t\bar{g}\mathbb{H}g = \mathbb{H} \right\}$$

of $\mathrm{SL}_2(\mathbb{R})$, where here $\mathbb{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\begin{aligned} \mathcal{H} \ni i &\leftrightarrow 0 \in \Delta \\ \mathrm{SO}(2) &\leftrightarrow \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} \\ B &\leftrightarrow \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \right\}. \end{aligned}$$

Thus, $\mathrm{SO}(2)$ is here a "standard" maximal torus and B is a "standard" Borel subgroup.

We now think of $\mathcal{H}$ as the parameter space for the family of PHS's of weight one and with $\dim V = 2$. Over $\mathcal{H}$ there is the natural *Hodge bundle*

$$\mathbb{V}^{1,0} \to \mathcal{H}$$

with fibres

$$\mathbb{V}^{1,0}_{\tau} =: V^{1,0}_{\tau} = \text{ line in } V_{\mathbb{C}}.$$

Under the inclusion $\mathcal{H} \hookrightarrow \mathbb{P}^1$, the Hodge bundle is the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Both $\mathbb{V}^{1,0}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ are examples of homogeneous vector bundles.

In general, given

- a homogeneous space
$$Y = A/B$$
where A is a Lie group and $B \subset A$ is a closed subgroup, and
- a linear representation $r : B \to \mathrm{Aut}\, E$ where E is a complex vector space,

there is an associated homogeneous vector bundle

$$\begin{array}{ccc} \mathbb{E} & =: & A \times_B E \\ \downarrow & & \downarrow \\ Y & = & A/B \end{array}$$

where $A \times_B E$ is the trivial vector bundle $A \times E$ factored by the equivalence relation

$$(a, e) \sim (ab, r(b^{-1})e)$$

where $a \in A$, $e \in E$, $b \in B$. The group A acts on $\mathbb{E}$ by $a \cdot (a', e) = (aa', e)$ and there is an A-equivariant action on $\mathbb{E} \to Y$. There is an evident notion of a morphism of homogeneous vector bundles; then $\mathbb{E} \to Y$ is trivial *as a homogeneous vector bundle* if, and only if, $r : B \to \mathrm{Aut}(E)$ is the restriction to B of a representation of A.

Example: Let $\tau_0 \in \mathcal{H} \subset \mathbb{P}^1$ be the reference point. For the standard linear representation of $\mathrm{SL}_2(\mathbb{C})$ on $V_{\mathbb{C}}$, the Borel subgroup B is the stability group of the flag

$$(0) \subset V_{\tau_0}^{1,0} \subset V_{\mathbb{C}}.$$

It follows that there is over $\mathbb{P}^1$ an exact sequence of $\mathrm{SL}_2(\mathbb{C})$-homogeneous vector bundles

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{V} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

where $\mathbb{V} = \mathbb{P}^1 \times V_{\mathbb{C}}$ with $g \in \mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{V}$ by $g \cdot ([z], v) = ([gz], gv)$. The restriction to $\mathcal{H}$ of this sequence is an exact sequence of $\mathrm{SL}_2(\mathbb{R})$-homogeneous bundles

$$0 \to \mathbb{V}^{1,0} \to \mathbb{V} \to \mathbb{V}^{0,1} \to 0.$$

The bundle $\mathbb{V}^{1,0}$ is given by the representation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \to e^{i\theta}$$

of $\mathrm{SO}(2)$. Using the form Q the quotient bundle $\mathbb{V}/\mathbb{V}^{1,0} =: \mathbb{V}^{0,1}$ is identified with the dual bundle $\mathbb{V}^{1,0*}$.

The canonical line bundle is

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Thus

$$\omega_{\mathcal{H}} \cong (\mathbb{V}^{1,0})^{\otimes 2}.$$

Convention: We set

$$\omega_{\mathcal{H}}^{1/2} = \mathbb{V}^{1,0}.$$

Proof. For the Grassmanian $Y = G(n, E)$ of n-planes P in a vector space E there is a $\mathrm{GL}(E)$-equivariant isomorphism

$$T_P Y \cong \mathrm{Hom}(P, E/P).$$

In the case above where $E = \mathbb{C}^2$ and $z = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right] \in \mathbb{P}^1$ we have

$$T_z\mathbb{P}^1 \cong V_z^{1,0^*} \otimes V_{\mathbb{C}}/V_z^{1,0}$$

where $V_z^{1,0}$ is the line in $V_{\mathbb{C}}$ corresponding to z. If we use the group $\mathrm{SL}_2(\mathbb{C})$ that preserves Q in place of $\mathrm{GL}_2(\mathbb{C})$, then

$$V_{\mathbb{C}}/V_z^{1,0} \cong V_z^{1,0^*}.$$

Thus the cotangent space

$$T_z^*\mathbb{P}^1 \cong V_z^{2,0}$$

where in general we set $\mathbb{V}^{n,0} = (\mathbb{V}^{1,0})^{\otimes n}$. The above identification $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ is an $\mathrm{SL}_2(\mathbb{C})$, but *not* $\mathrm{GL}_2(\mathbb{C})$, equivalence of homogenous bundles.

The Hodge bundle $\mathbb{V}^{1,0} \to \mathcal{H}$ has an $\mathrm{SL}_2(\mathbb{R})$-invariant metric, *the Hodge metric*, given fibrewise by the 2nd Hodge-Riemann bilinear relation. The basic invariant of a metric is its curvature, and we have the following

General fact: Let $\mathbb{L} \to Y$ be an Hermitian line bundle over a complex manifold Y. Then the **Chern** (or curvature) **form** is

$$c_1(\mathbb{L}) = \frac{i}{2\pi}\overline{\partial}\partial \log \|s\|^2$$

where $s \in \mathcal{O}(\mathbb{L})$ is any local holomorphic section and $\|s\|^2$ is its length squared.

Basic calculation:

$$c_1(\mathbb{V}^{1,0}) = \frac{1}{4\pi}\frac{dx \wedge dy}{y^2} = \frac{i}{2\pi}\frac{d\tau \wedge \overline{d\tau}}{(\operatorname{Im}\tau)^2}.$$

This has the following

Consequence: *The tangent bundle*

$$T\mathcal{H} \cong \mathbb{V}^{0,2}$$

has a metric

$$ds^2_{\mathcal{H}} = \frac{dx^2 + dy^2}{y^2} = \left(\frac{1}{(\operatorname{Im}\tau)^2}\right)\operatorname{Re}(dz\, d\bar{z})$$

of constant negative Gauss curvature.

Before giving the proof we shall make a couple of observations.

Any $\mathrm{SL}_2(\mathbb{R})$ invariant Hermitian metric on $\mathcal{H}$ is conformally equivalent to $dx^2 + dy^2$; hence it is of the form

$$h(x,y)\left(\frac{dx^2 + dy^2}{y^2}\right)$$

for a positive function $h(x,y)$. Invariance under translation $\tau \to \tau + b$, $b \in \mathbb{R}$, corresponding to the subgroup $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, implies that $h(x,y) = h(y)$ depends only on y. Then invariance under $\tau \to a\tau$ corresponding to the subgroup $\begin{pmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{pmatrix}$, $a > 0$, gives that $h(y) =$ constant. A similar argument gives that $c_1(\mathbb{V}^{1,0})$ is a constant multiple of the form above.

The all important sign of the curvature K may be determined geometrically as follows: Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete group such that $Y = \Gamma\backslash\mathcal{H}$ is a compact Riemann surface of genus $g \geqq 2$ with the metric induced from that on $\mathcal{H}$. By the Gauss-Bonnet theorem

$$0 > 2 - 2g = \chi(Y) = \frac{1}{4\pi}\int_Y K dA = K\left(\frac{\mathrm{Area}(Y)}{4\pi}\right).$$

Proof of basic calculation: We define a section $s \in \Gamma(\mathcal{H}, \mathbb{V}^{1,0})$ by

$$s(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathbb{V}^{1,0}_\tau.$$

The length squared is given by

$$\|s(\tau)\|^2 = i\, {}^t\overline{s(\tau)} Q s(\tau) = 2y.$$

Using for $\tau = x + iy$

$$\begin{cases} \partial_\tau = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar\tau} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases}$$

we obtain

$$\frac{i}{2\pi}\bar\partial\partial = -\frac{1}{4\pi}(\partial_x^2 + \partial_y^2) dx \wedge dy.$$

This gives

$$\frac{i}{2\pi}\bar\partial\partial \log \|s(\tau)\|^2 = \frac{1}{4\pi}\frac{dx \wedge dy}{y^2}.$$

Remark: There is also a $S\mathcal{U}(2)$-invariant metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$ induced from the standard metric on $\mathbb{C}^2$. For this metric

$$\|s(\tau)\|_c^2 = 1 + |\tau|^2$$

(the subscript c on $\|\ \|_c^2$ stands for "compact"). Then we have

$$c_1(\mathbb{V}_c^{1,0}) = -\frac{1}{4\pi}\frac{dx \wedge dy}{(1+|\tau|^2)^2}.$$

Thus, $\mathbb{V}^{1,0} \to \mathcal{H}$ is a *positive* line bundle whereas $\mathbb{V}_c^{1,0} \to \mathbb{P}^1$ is a *negative* line bundle

$$\deg \mathcal{O}_{\mathbb{P}^1}(-1) = \int_{\mathbb{P}^1} c_1(\mathbb{V}_c^{1,0}) = -1.$$

This *sign reversal* between the $\mathrm{SL}_2(\mathbb{R})$-invariant curvature on the open domain $\mathcal{H}$ and the $S\mathcal{U}(2)$ (= compact form of $\mathrm{SL}_2(\mathbb{C})$)-invariant metric on the compact dual $\check{\mathcal{H}} = \mathbb{P}^1$ will hold in general and is a fundamental phenomenon in Hodge theory.

Above we have holomorphically trivialized $\mathbb{V}^{1,0} \to \mathcal{H}$ using the section

$$s(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

We have also noted that we have the isomorphism of $\mathrm{SL}_2(\mathbb{R})$-homogeneous line bundles

$$\omega_{\mathcal{H}} \cong \mathbb{V}^{2,0}.$$

Now $\omega_{\mathcal{H}}$ has a section $d\tau$ and a useful fact is that under this isomorphism

$$d\tau = s(\tau)^2.$$

The proof is by tracing through the isomorphism. To see why it should be true we make the following observations: Under the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $s(\tau)$ transforms to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = (c\tau + d)\begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix};$$

i.e., $s(\tau)$ transforms by $(c\tau+d)^{-1}$. On the other hand, using $ad - bc = 1$ we find that

$$d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{d\tau}{(c\tau+d)^2}.$$

Thus $s(\tau)^2$ and $d\tau$ transform the same way under $\mathrm{SL}_2(\mathbb{R})$, and consequently their ratio is a constant function on $\mathcal{H}$.

Beginnings of representation theory

In these lectures we shall be primarily concerned with infinite dimensional representations of real, semi-simple Lie groups and with finite dimensional representations of reductive $\mathbb{Q}$-algebraic groups. Leaving aside some matters of terminology and definitions for the moment we shall briefly describe the basic examples of the former in the present framework.

Denote by $\Gamma(\mathcal{H}, \mathbb{V}^{n,0})$ the space of global holomorphic sections over $\mathcal{H}$ of the n^{th} tensor power of the Hodge bundle, and by $d\mu(\tau)$ the $\mathrm{SL}_2(\mathbb{R})$ invariant area form $dx \wedge dy/y^2$ on $\mathcal{H}$.

Definition: *For $n \geqq 2$ we set*

$$\mathcal{D}_n^+ = \left\{\psi \in \Gamma(\mathcal{H}, \mathbb{V}^{n,0}) : \int_{\mathcal{H}} \|\psi(\tau)\|^2 d\mu(\tau) < \infty\right\}.$$

There is an obvious natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\Gamma(\mathcal{H}, \mathbb{V}^{n,0})$ that preserves the pointwise norms, and it is a basic result [K2] that the map

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathcal{D}_n^+)$$

gives an irreducible, unitary representation of $\mathrm{SL}_2(\mathbb{R})$.

As noted above there is a holomorphic trivialization of $\mathbb{V}^{1,0} \to \mathcal{H}$ given by the non-zero section

$$\sigma(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Then using the definition of the Hodge norm and ignoring the factor of 2,

$$\|\sigma(\tau)\|^2 = y.$$

Writing

$$\psi(\tau) = f_\psi(\tau)\sigma(\tau)$$

we have

$$\int_{\mathcal{H}} \|\psi(\tau)\|^2 d\mu(\tau) = \left(\frac{i}{2}\right) \iint |f_\psi(\tau)|^2 (\operatorname{Im}\tau)^{n-2} d\tau \wedge d\bar{\tau}.$$

Thus we may describe $\mathcal{D}_n^+$ as

$$\left\{ f \in \Gamma(\mathcal{H}, \mathcal{O}_{\mathcal{H}}) : \iint |f_\psi(x+iy)|^2 y^{n-2} dx \wedge dy < \infty \right\}.$$

For $n = 1$ we define the norm by

$$\sup_{y>0} \int_{-\infty}^{\infty} |f_\psi(x+iy)|^2 dx.$$

The spaces $\mathcal{D}_n^-$ are described analogously using the lower half plane.

Fact ([K2])**:** *The $\mathcal{D}_n^\pm$ for $n \geqq 2$ are the discrete series representations of* $\mathrm{SL}_2(\mathbb{R})$. *For $n = 1$, $\mathcal{D}_1^\pm$ are the limits of discrete series.*

The terminology arises from the fact that in the spectral decomposition of $L^2(\mathrm{SL}_2(\mathbb{R}))$ the $\mathcal{D}_n^\pm$ for $n \geqq 2$ occur discretely.

There is an important duality between the orbits of $\mathrm{SL}_2(\mathbb{R})$ and of $\mathrm{SO}(2, \mathbb{C})$ acting on $\mathbb{P}^1$. Anticipating terminology to be used later in these lectures we set

- $\mathbb{P}^1$ = *flag variety* $\mathrm{SL}_2(\mathbb{C})/B$ where B is the Borel subgroup fixing $i = \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$;
- $\mathrm{SL}_2(\mathbb{R})$ = *real form* of $\mathrm{SL}_2(\mathbb{C})$ relative to the conjugation $A \to \overline{A}$;
- $\mathrm{SO}(2)$ = maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ (in this case it is $\mathrm{SL}_2(\mathbb{R}) \cap B$);
- $\mathcal{H}$ = *flag domain* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$;
- $\mathrm{SO}(2, \mathbb{C})$ = complexification of $\mathrm{SO}(2)$.

We note that $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$.

Matsuki duality is a one-to-one correspondence of the sets

$$\{\mathrm{SL}_2(\mathbb{R})\text{-orbits in } \mathbb{P}^1\} \leftrightarrow \{\mathrm{SO}(2, \mathbb{C})\text{-orbits in } \mathbb{P}^1\}$$

that reverses the relation "in the closure of." The orbit structures in this case are

$$\begin{tikzcd}[column sep=small] \mathcal{H} \arrow[dr, dash] & & \overline{\mathcal{H}} \arrow[dl, dash] & \text{open } \mathrm{SL}_2(\mathbb{R}) \text{ orbits} \\ & \mathbb{R}\cup\{\infty\} & & \text{closed } \mathrm{SL}_2(\mathbb{R}) \text{ orbit} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \mathbb{P}^1\backslash\{i,-i\} \arrow[dl, dash] \arrow[dr, dash] & & \text{open } \mathrm{SO}(2,\mathbb{C}) \text{ orbit} \\ i & & -i & \text{closed } \mathrm{SO}(2,\mathbb{C}) \text{ orbit} \end{tikzcd}$$

The lines mean "in the closure of." The correspondence in Matsuki duality is

$$\begin{cases} \mathcal{H} \leftrightarrow i \\ \overline{\mathcal{H}} \leftrightarrow -i \end{cases}$$
$$\mathbb{R}\cup\{0\} \leftrightarrow \mathbb{P}^1\backslash\{i,-i\}.$$

Matsuki duality arises in the context of representation theory as follows: A *Harish-Chandra module* is a representation space W for $\mathrm{sl}_2(\mathbb{C})$ and for $\mathrm{SO}(2,\mathbb{C})$ that satisfies certain conditions (to be explained in Lecture 5). A *Zuckerman module* is, for these lectures, a module obtained by taking finite parts of completed unitary $\mathrm{SL}_2(\mathbb{R})$-modules. For the $\mathcal{D}_n^+$ the modules are formal power series

$$\psi = \sum_{k\geqq 0} a_k(\tau - i)^k d\tau^{\otimes n/2}.$$

We think of these as associated to $G_{\mathbb{R}}$-modules arising from the open orbit $\mathcal{H}$. The Lie algebra $\mathrm{sl}_2(\mathbb{C})$, thought of as vector fields on $\mathbb{P}^1$, operates on ψ above by the Lie derivative, and $\mathrm{SO}(2,\mathbb{C})$ operates by linear fractional transformations.

Associated to the closed $\mathrm{SO}(2,\mathbb{C})$ orbit i are formal Laurent series

$$\gamma = \sum_{l\geqq 1} \frac{b_l}{(\tau - i)^l}\left(\frac{\partial}{\partial\tau}\right)^{\otimes n/2} dz.$$

This is also a $(\mathrm{so}(2,\mathbb{C}), \mathrm{SO}(2,\mathbb{C}))$-module. The pairing between $\mathrm{SO}(2,\mathbb{C})$-finite vectors, i.e., finite power and Laurent series, is

$$\langle\psi,\gamma\rangle = \mathrm{Res}_i(\psi,\gamma).$$

There are also representations associated to the closed $\mathrm{SL}_2(\mathbb{R})$ orbit and open $\mathrm{SO}(2,\mathbb{C})$ orbit that are in duality. We will not have a chance to discuss these in this lecture series (cf. [Sch3]).

There is a similar picture if one takes the other real form $S\mathcal{U}(1,1)_{\mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{C})$. It is a nice exercise to work out the orbit structure and duality in this case.

We shall revisit Matsuki duality in this case, but set in a general context, in Lecture 2.

Why we work over $\mathbb{Q}$: Setting $X_\Lambda = C/\Lambda$ we say that X_Λ and $X_{\Lambda'}$ are *isomorphic* if there is a linear mapping

$$\alpha : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$$

with $\alpha(\Lambda) = \Lambda'$. This is equivalent to X_Λ and $X_{\Lambda'}$ being biholomorphic as compact Riemann surfaces. Normalizing the lattices as above the condition is

$$\tau' = \frac{a\tau + b}{c\tau + c}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Thus the equivalence classes of compact Riemann surfaces of genus one is identified with the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$.

For many purposes a weaker notion of equivalence is more useful. We say that X_Λ and $X_{\Lambda'}$ are *isogeneous* if the condition $\alpha(\Lambda) = \Lambda'$ is replaced by $\alpha(\Lambda) \subseteq \Lambda'$. Then $\Lambda'/\alpha(\Lambda)$ is a finite group and there is an unramified covering map

$$X_\Lambda \to X_{\Lambda'}.$$

More generally, we may say that $X_\Lambda \sim X_{\Lambda'}$ if there is a diagram of isogenies

$$\begin{tikzcd} & X_{\Lambda''} \arrow[dl] \arrow[dr] & \\ X_\Lambda & & X_{\Lambda'}. \end{tikzcd}$$

Identifying each of the universal covers with the same $\mathbb{C}$, we have $\Lambda \subset \Lambda''$, $\Lambda' \subset \Lambda''$ and then

$$\Lambda \otimes \mathbb{Q} = \Lambda'' \otimes \mathbb{Q} = \Lambda' \otimes \mathbb{Q}.$$

The converse is true, which suggests one reason for working over $\mathbb{Q}$.

Remark: Among the important subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are the *congruence subgroups*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Then $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Geometrically the quotient spaces $M_{\Gamma(N)} =: \Gamma(N)\backslash\mathcal{H}$ arise as parameter spaces for complex tori X_τ plus additional "rigidifying" data. In this case the additional data is "marking" the N-torsion points

$$X_\tau(N) =: (1/N)\Lambda/\Lambda \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

When we require that an ismorphism $X_\Lambda(N) \cong X_\Lambda(N)$ take marked points to marked points the the equivalence classes of $X_\Lambda(N)$'s are $\Gamma(N)\backslash\mathcal{H}$.

Later in these talks we will encounter arithmetic groups Γ which have compact quotients.

Lecture 2
The classical theory: Part II

This lecture is a continuation of the first one. In it we will introduce and illustrate a number of the basic concepts and terms that will appear in the later lectures, where also the formal definitions will be given.

Holomorphic automorphic forms: We have seen above that the equivalence classes of PHS's of weight one with $\dim V = 2$ may be identified with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. More generally, for geometric reasons discussed earlier one wishes to consider congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and the quotient spaces

$$M_\Gamma =: \Gamma\backslash\mathcal{H}.$$

We make two important remarks concerning these spaces:

(i) The fixed points of $\gamma \in \Gamma$ acting on $\mathcal{H}$ occur when we have a PHS

$$V_\mathbb{C} = V_\tau^{1,0} \oplus V_\tau^{0,1}$$

left invariant by $\gamma \in \mathrm{Aut}(V_\mathbb{Z}, Q)$. Thus γ is an integral matrix that lies in the compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ which preserves the positive Hermitian form $iQ(V_\tau^{1,0}, \overline{V}_\tau^{1,0})$. It follows that γ is of finite order, so that locally there is a disc Δ around τ with a coordinate t on Δ such that

$$\gamma(t) = \zeta \cdot t, \qquad \zeta^m = 1$$

for some integer m (in fact, $m = 2$ or 3). The map

$$s = t^m$$

then gives a local biholomorphism between Δ modulo the action of the group $\{\gamma^m\}$ and the s-disc. In this way M_Γ is a Riemann surface. We define sections of the bundle $\mathbb{V}^{n,0}$ over the quotient space $\{\gamma^k, k \in \mathbb{Z}\}\backslash\Delta$ of the disc modulo the action of γ to be given by γ-invariant sections of $\mathbb{V}^{n,0} \to \Delta$.

Remark: It will be a general fact, with essentially the same argument as above, that isotropy group of a general polarized Hodge structure that lies in an arithmetic group is finite.

(ii) M_Γ will not be compact but will have *cusps*, which are biholomorphic to the punctured disc Δ^*. The model here is the quotient of the region

$$\mathcal{H}_c = \{\mathrm{Im}\,\tau > c\}, \qquad\qquad c > 0$$

by the subgroup $\Gamma_0 = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ of translations. Setting

$$q = e^{2\pi i\tau}$$

we obtain a biholomorphism

$$\Gamma_0\backslash\mathcal{H}_c \xrightarrow{\sim} \{0 < |q| < e^{-2\pi c}\}$$

of the quotient space with a punctured disc.

Definition: *A* **holomorphic automorphic form of weight** n *is given by a holomorphic section* $\psi \in \Gamma(M_\Gamma, \mathbb{V}^{n,0})$ *that is finite at the cusps.*

These will be referred to simply as *modular forms*.

We recall that $\omega_{\mathcal{H}} \cong \mathbb{V}^{2,0}$, so that $\omega_{\mathcal{H}}^{\otimes n/2} \cong \mathbb{V}^{n,0}$ and the sections of $\omega_{M_\Gamma}^{\otimes n/2}$ around the fixed points of Γ are defined as above. Thus automorphic forms of weight n are given by

$$\psi(\tau) = f_\psi(\tau) d\tau^{n/2}$$

where $f_\psi(\tau)$ is holomorphic on $\mathcal{H}$ and satisfies

$$f_\psi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f_\psi(\tau).$$

Around a cusp as above one sets $q = e^{2\pi i\tau}$ and expands in a Laurent series the resulting well-defined function $F_\psi(q) = f_\psi(\tau)$,

$$F_\psi(q) = \sum_n a_n q^n.$$

By definition, the finiteness condition at the cusp is $a_n = 0$ for $n < 0$.

As will be discussed in the lecture of Cattani, from a Hodge-theoretic perspective there is a *canonical extension* $\mathbb{V}_e^{1,0} \to \Delta$ of the Hodge bundle $\mathbb{V}^{1,0} \to \Delta^*$ given by the condition that the Hodge length of a section have at most *logarithmic growth* in the Hodge norm as one approaches the puncture. Modular forms are then the holomorphic sections of $\mathbb{V}^{n,0} \to \Gamma\backslash\mathcal{H}$ that extend to $\mathbb{V}_e^{n,0} \to \overline{\Gamma\backslash\mathcal{H}}$. In this way they are defined purely Hodge-theoretically.

Among the modular forms are the special class of *cusp forms* ψ, defined by the equivalent conditions

- $\int_{\Gamma\backslash\mathcal{H}} \|\psi\|^2 d\mu < \infty$;[3]
- $a_0 = 0$;
- ψ vanishes at the origin in the canonical extensions at the cusps.

Representation theory associated to $\mathbb{P}^1$: It is convenient to represent $\mathbb{P}^1$ as the compact dual of $\Delta = S\mathcal{U}(1,1)/T$. Thus

$$\mathrm{SL}_2(\mathbb{C}) = S\mathcal{U}(1,1)_{\mathbb{C}}.$$

[3]This is not the usual condition, which involves the integral of f_ψ over a horizontal path in $\mathcal{H}$. We have used it in order to have a purely Hodge-theoretic formulation.

At the Lie algebra level we then have

$$\mathrm{su}(1,1)_{\mathbb{R}} = \left\{ \begin{pmatrix} i\alpha & \beta \\ \overline{\beta} & -i\alpha \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

$$\mathrm{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

where $\mathrm{sl}_2(\mathbb{C}) = \mathrm{su}(1,1)_{\mathbb{R}} + i\,\mathrm{su}(1,1)_{\mathbb{R}}$ via

$$\begin{cases} a = \alpha + i\alpha' \\ b = \beta + i\beta' \\ c = \overline{\beta} + i\overline{\beta}'. \end{cases}$$

As basis for $\mathrm{sl}_2(\mathbb{C})$ we take the standard generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then setting

$$\mathfrak{h} = \mathbb{C}H, \quad \mathfrak{n}^+ = \mathbb{C}X, \quad \mathfrak{n}^- = \mathbb{C}Y$$

$\mathfrak{h}$ is a *Cartan sub-algebra* and the structure equations are

$$\begin{cases} [H,X] = 2X \\ [H,Y] = -2Y \\ [X,Y] = H. \end{cases}$$

The *weight lattice* P are the integral linear forms on $\mathbb{Z}H \subset \mathfrak{h}$. Thus $P \cong \mathbb{Z}$ with $\langle 1, H\rangle = 1$. The *root vectors* are the eigenvectors X, Y of $\mathfrak{h}$ acting on $\mathrm{sl}_2(\mathbb{C})$, and the *roots* are the corresponding eigenvalues $+2, -2$ viewed in the evident way as weights. They generate the *root lattice* $R \subset P$ with $P/R \cong \mathbb{Z}/2\mathbb{Z}$. The *positive root* is $+2$ and

$$\begin{cases} \mathfrak{n}^+ = \text{span of positive root vector } X \\ \mathfrak{n}^- = \text{span of negative root vector } Y. \end{cases}$$

For the *Borel subgroup* $B = \left\{ \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix} \right) \right\}$, which is the stability group of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \in \mathbb{P}^1$ corresponding to the origin $0 \in \Delta$, the Lie algebra

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-.$$

We note that the roots are purely imaginary on the Lie algebra

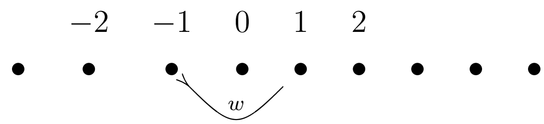
$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

of the maximal torus $T \subset S\mathcal{U}(1,1)_{\mathbb{R}}$.

As is customary notation in representation theory we set

$$\rho = \frac{1}{2}(\Sigma \text{ positive roots}) = 1.$$

The *Weyl group* W acting on $\mathfrak{h}$ is generated by the reflections in the hyperplanes defined by roots; in this case it is just $\pm\,\mathrm{id}$. One usually draws the picture of $i\mathfrak{t} \subset \mathfrak{h}$ with the roots and weights identified. In this case it is $2\pi i\mathfrak{t} = \mathbb{R}$, $P = \mathbb{Z}$, $R = 2\mathbb{Z}$.



where "2" is the positive root and W is generated by the identity and w where $w(x) = -x$.

Given a representation

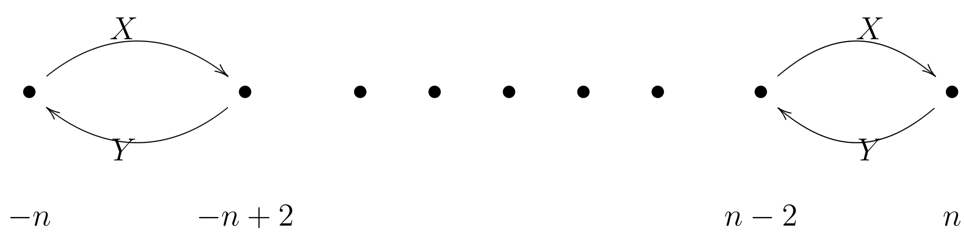
$$r : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}\, E$$

where E is a complex vector space, the *weights* are the simultaneous eigenvalues of $r(\mathfrak{h})$. In this case they are the eigenvalues of $r(H)$. The *standard representation* is given by $E = \mathbb{C}^2$. The *weight vectors* are the eigenvectors for $r(\mathfrak{h})$. For the standard representation they are

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with weights ± 1.

Any irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to $S^n =: \mathrm{Sym}^n E$ for $n = 0, 1, 2, \dots$. The picture of S^n is



where the dots represent the 1-dimensional weight spaces with weights $-n, -n+2, \dots, n-2, n$. The actions on X and Y are as indicated. If we make the identifications

$$\begin{cases} z_0 \leftrightarrow e_+ \\ z_1 \leftrightarrow e_- \end{cases}$$

then

- S^n = homogeneous polynomials $F(z_0, z_1)$ of degree n;
- $X = \partial_{z_1}, Y = \partial_{z_0}$;
- z_0^n is the *highest weight vector*.

As $\mathrm{SL}_2(\mathbb{C})$-modules we have

$$H^0(\mathcal{O}_{\mathbb{P}^1}(n)) \cong S^n.$$

Geometrically, since $\mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathbb{P}^1}(-n)^*$ we see that on each line L in $\mathbb{C}^2$, $F(z_0, z_0)$ restricts to a form that is homogeneous of degree n. Thus

$$F\big|_L \in \mathrm{Sym}^n L^* = \text{fibre of } \mathcal{O}_{\mathbb{P}^1}(n) \text{ at } L.$$

As a homogeneous line bundle

$$\mathcal{O}_{\mathbb{P}^1}(n) = \mathrm{SL}_2(\mathbb{C}) \times_B \mathbb{C}$$

where $\left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \in B$ acts on $\mathbb{C}$ by the character a^n. With our convention above, the differential of this character, viewed as a linear form on $\mathfrak{h}$, is the weight n.

With the notation to be used later we have

$$\mathcal{O}_{\mathbb{P}^1}(n) = L_n$$

where the subscript on L denotes the weight which is the differential of the character that defines the homogeneous line bundle.

By Kodaira-Serre duality

$$H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big)^* \cong H^0(\omega_{\mathbb{P}^1}(k)),$$

and using the isomorphism of $\mathrm{SL}_2(\mathbb{C})$-homogeneous line bundles

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$$
$$H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big)^* \cong H^0\big(\mathcal{O}_{\mathbb{P}^1}(k)\big) \cong S^k.$$

Penrose transform for $\mathbb{P}^1$

One of the main aspects of these lectures will be to use the method of Eastwood-Gindikin-Wong [EGW] to represent higher degree sheaf cohomology by *global, holomorphic* data. We will now illustrate this for $H^1(\mathcal{O}_{\mathbb{P}^1}(-k-2))$.

For this we set (the notation will be explained later in the lectures)

$$\check{\mathcal{W}} = \mathbb{P}^1 \times \mathbb{P}^1 \backslash (\text{diagonal}).$$

Using homogeneous coordinates $z = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$ we have

$$\check{\mathcal{W}} = \{(z, w) \in \mathbb{P}^1 \times \mathbb{P}^1 : z_0 w_1 - z_1 w_0 \neq 0\}.$$

For simplicity of notation we identify $\Lambda^2\mathbb{C}^2 = \mathbb{C}$ and then have $z \wedge w = z_0 w_1 - z_1 w_0$. For calculations it is, as usual, convenient to work upstairs in the open set U in $\mathbb{C}^2 \times \mathbb{C}^2$ lying over $\check{\mathcal{W}}$ and keep track of the bi-homogeneity of a function defined in U.

The *correspondence space* $\check{\mathcal{W}}$ has the properties

(A) $\check{\mathcal{W}}$ is a Stein manifold (it is an affine algebraic variety);

(B) the fibres of the projection $\check{\mathcal{W}} \xrightarrow{\pi} \mathbb{P}^1$ on the first factor are contractible (they are just copies of $\mathbb{C}$).

Under these conditions [EGW] showed that there is a natural isomorphism

$$H^q(\mathcal{O}_{\mathbb{P}^1}(m)) \cong H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(m)); d_\pi\big). \tag{$*$}$$

As we will now briefly explain, the RHS of $(*)$ is a global, holomorphic object. The detailed explanation will be given in Lecture 7. We will explain "in coordinates" what the various terms mean.

- Ω^q_π = sheaf of relative differentials on $\check{\mathcal{W}}$;
- $(\Omega^\bullet_\pi, d_\pi)$ is the complex $\cdots \to \Omega^q_\pi \xrightarrow{d_\pi} \Omega^{q+1}_\pi \to \dots$;
- $\Omega^\bullet_\pi(m) = \Omega^\bullet_\pi \otimes_{\mathcal{O}_{\check{\mathcal{W}}}} \pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ where $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ is the pullback bundle;
- $H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(m)); d_\pi\big)$ is the de Rham cohomology arising from the global sections of the above complex.

The relative forms are defined by

$$\Omega^q_\pi = \Omega^q_{\check{\mathcal{W}}}/\text{image}\left\{\pi^*\Omega^1_{\mathbb{P}^1} \otimes \Omega^{q-1}_{\check{\mathcal{W}}} \to \Omega^q_{\check{\mathcal{W}}}\right\},$$

and d_π is induced by the usual exterior differential d. We think of $\pi^*\mathcal{O}_{\mathbb{P}^1}(m) \to \check{\mathcal{W}}$ as a vector bundle whose transition functions are constant on the fibres of π, and then d_π is well defined on sections of $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$.

The pullback sheaf $\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)$ is the sheaf over $\check{\mathcal{W}}$ whose sections over an open set $Z \subset \check{\mathcal{W}}$ are the sections of $\mathcal{O}_{\mathbb{P}^1}(m)$ over $\pi(Z)$. We have an inclusion

$$\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m) \hookrightarrow \pi^*\mathcal{O}_{\mathbb{P}^1}(m)$$

where the subsheaf $\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)$ is given by the sections of the bundle $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ that are constant on the fibres of $\check{\mathcal{W}} \to \mathbb{P}^1$.

In coordinates $(z, w) = (z_0, z_1; w_0, w_1)$ on U, $\Omega^\bullet_\pi$ means that we mod out by dz_0 and dz_1. Setting

$$\Psi = w_1 dw_0 - w_0 dw_1$$

we have

- $\Gamma\big(\check{\mathcal{W}}, \pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)\big) = \left\{\begin{array}{l} F(z,w) \text{ holomorphic in } U \text{ and homogeneous} \\ \text{of degree } m \text{ in } z \text{ and of degree zero in } w \end{array}\right\}$;
- $d_\pi F(z,w) = F_{w_0}(z,w)dw_0 + F_{w_1}(z,w)dw_1$.[4]

Using Euler's relation

$$w_0 F_{w_0} + w_1 F_{w_1} = 0$$

[4]This equation is true for an $F(z, w)$ with any bi-homogeneity in z, w.

when $F(z,w)$ is homogenous of degree zero in w we obtain

$$d_\pi F(z,w) = \left(\frac{F_{w_0}}{w_1}\right)\Psi = -\left(\frac{F_{w_1}}{w_0}\right)\Psi.$$

For the reasons to be seen below, it is now convenient to set $m = -k-2$. Then

- $\Gamma(\check{\mathcal{W}}, \Omega^1_\pi(-k-2)) = \left\{ \begin{array}{l} \frac{G(z,w)\Psi}{(z\wedge w)^{k+2}} \text{ where } G(z,w) \text{ is homogeneous of} \\ \text{degree zero in } z \text{ and of degree } k \text{ in } w \end{array} \right\}.$

THEOREM: *Every class in* $H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(-k-2))\big)$ *has a unique representative of the form*

$$\frac{H(w)\Psi}{(z\wedge w)^{k+2}}$$

where $H(w)$ *is a homogeneous polynomial of degree* k.

Discussion: Given $\frac{G(z,w)\Psi}{(z\wedge w)^{k+2}}$ as above, we have to show that the equation

$$\frac{G(z,w)\Psi}{(z\wedge w)^{k+2}} = d_\pi\left(\frac{F(z,w)}{(z\wedge w)^{k+2}}\right) + \frac{H(w)\Psi}{(z\wedge w)^{k+2}},$$

where F has degree zero in z and degree $k+2$ in w and $H(w)$ is as above, has a unique solution. Using Euler's relation $w_0F_{w_0} + w_1F_{w_1} = (k+2)F$ we find that

$$d_\pi\left(\frac{F(z,w)}{(z\wedge w)^{k+2}}\right) = \frac{z_0F_{w_0}(z,w) + z_1F_{w_1}(z,w)\Psi}{(z\wedge w)^{k+3}}.$$

Then the equation to be solved is, after a calculation,

$$z_0F_{w_0}(z,w) + z_1F_{w_1}(z,w) = (z_0w_1 - z_1w_0)G(z,w) + (z_0w_1 - z_1w_0)H(w).$$

We shall first show that a solution is unique; i.e.,

$$z_0F_{w_0} + z_1F_{w_1} = (z_0w_1 - z_1w_0)H(w) \Rightarrow H(w) = 0.$$

Taking the forms that are homogeneous of degree one in z_0, z_1 gives

$$\begin{cases} F_{w_0} = w_1H \\ F_{w_1} = -w_0H. \end{cases}$$

Applying ∂_{w_1} to the first and ∂_{w_0} to the second leads to

$$H + w_1H_{w_1} = -H - w_0H_{w_0}.$$

Euler's relation then gives that $H(w)$ is homogeneous of degree -2, which is a contradiction.[5]

[5]One may wonder why the degree -2 appears, when all that is needed is degree -1. The philosophical reason is that $H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = (0)$.

It is an interesting exercise to directly show by a calculation the existence of a solution to be above equation. On general grounds we know that this must be so because the map

$$(**) \qquad H(w) \longrightarrow \frac{H(w)\Psi}{(z\wedge w)^{k+2}}$$

has been shown to be injective and $\dim H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big) = k+1 = \dim S^k$.

The map $(**)$ has the following interpretation: Let $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$ be $\mathbb{P}^1$ with coordinates z and w respectively. Then we have a *correspondence diagram*

$$\begin{tikzcd} & \check{\mathcal{W}} \arrow[dl, "\pi_w"'] \arrow[dr, "\pi_z"] & \\ \mathbb{P}^1_w & & \mathbb{P}^1_z. \end{tikzcd}$$

Setting $\mathcal{O}_{\check{\mathcal{W}}}(a,b) = \pi_z^*\mathcal{O}_{\mathbb{P}^1_z}(a) \boxtimes \pi_w^*\mathcal{O}_{\mathbb{P}^1_w}(b)$ and using the theorem of EGW we obtain a diagram

$$\begin{tikzcd} H^0\big(\mathcal{O}_{\mathbb{P}^1_w}(k)\big) \arrow[r, dashed, "\mathcal{P}"] \arrow[d, phantom, "\wr\|"] & H^1\big(\mathcal{O}_{\mathbb{P}^1_z}(k-2)\big) \arrow[d, phantom, "\wr\|"] \\ H^0_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi_w}(0,k)); d_{\pi_w}\big) \arrow[r, "\frac{\Psi}{(z\wedge w)^{k+2}}"] & H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi_z}(-k-2,0)); d_{\pi_z}\big) \end{tikzcd}$$

where the isomorphism

$$H^0\big(\mathcal{O}_{\mathbb{P}^1_w}(k)\big) \xrightarrow{\mathcal{P}} H^1\big(\mathcal{O}_{\mathbb{P}^1_z}(-k-2)\big)$$

is termed a *Penrose transform*. Letting $\mathrm{SL}_2(\mathbb{C})$ act on $\check{\mathcal{W}} \subset \mathbb{P}^1_w \times \mathbb{P}^1_z$ diagonally in the above correspondence diagram we see that $\mathcal{P}$ is an isomorphism of $\mathrm{SL}_2(\mathbb{C})$-modules.

In fact, it is a geometric way of realizing in this special case the isomorphism in the Borel-Weil-Bott (BWB) theorem. The general discussion of the BWB will be given in the appendices to Lectures 5 and 7, where the special role of the weight ρ and transformation $w(\mu+\rho)-\rho$, where μ is a weight, will be explained.

The line bundle L_{-k-2} has weight $-k-2$, and for $k \geq 0$

$$\underbrace{-k-2} + \rho = \underbrace{-k-1}$$

is *regular* in the sense that its value on every root vector is non-zero. Moreover

$$\#\{\text{positive root vectors } X \text{ with } \langle -k-1, X\rangle < 0\} = 1.$$

For $w \in W$ as above

$$w(-k-1) - \rho = k+1-1 = k.$$

The BWB states that for $k \geqq 0$, $H^q(\mathcal{O}_{\mathbb{P}^1}(-k-2)) \neq 0$ only for $q=1$, and that this group is the irreducible $\mathrm{SL}_2(\mathbb{C})$ module with highest weight $w(-k-2+\rho)-\rho = k$. The Penrose transform $\mathcal{P}$ realizes this identification.

Penrose transform for elliptic curves

The mechanism of the EGW theorem and resulting Penrose transform will be a basic tool in these lectures. We now illustrate it for compact Riemann surfaces of genus $g=1$ and then shall do the same for genus $g>1$.

For reasons to be explained, and in part deriving from the work of Carayol that will be discussed in the last lecture, it is convenient to take our complex torus

$$E' = \mathbb{C}/\mathcal{O}_{\mathbb{F}}$$

where $\mathbb{F}$ is a quadratic imaginary number field and $\mathcal{O}_{\mathbb{F}}$ is the ring of integers in $\mathbb{F}$; e.g., $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$.[6] We set

$$\mathcal{W} = \mathbb{C}\times\mathbb{C} \text{ with coordinates } (z', z'')$$

and consider the diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{O}_{\mathbb{F}}\backslash\mathcal{W} \arrow[dl, "\pi'"'] \arrow[dr, "\pi''"] & \\ E' & & E'' \end{tikzcd}$$

where $\alpha \in \mathcal{O}_{\mathbb{F}}$ acts by $\overline{\alpha}$ in the first factor and by $-\alpha$ in the second. It may be easily checked that $\mathcal{O}_{\mathbb{F}}\backslash\mathcal{W}$ is Stein and the fibres of π', π'' are contractible (they are just $\mathbb{C}$'s). Thus the EGW theorem applies to the above diagram.

We will describe line bundles $L'_r \to E'$ and $L''_r \to E''$, where r is a positive integer, and then shall define the Penrose transform to give an isomorphism

$$H^0(E', L'_r) \xrightarrow{\sim} H^1(E'', L''_{-r}).$$

For this we let β be a complex number with

$$\begin{cases} \beta + \overline{\beta} = |\alpha|^2 \\ \operatorname{Im}\beta = \beta_0 > 0. \end{cases}$$

Sections of $L'_r \to E'$ are given by entire holomorphic functions $\theta'_r(z')$ where

$$\theta'_r(z'+\overline{\alpha}) = \theta'_r(z')\exp\left(\frac{2\pi i r}{\beta_0}\left(\alpha z' + \frac{|\alpha|^2}{2}\right)\right).$$

[6]We apologize for the use of $\mathbb{F}$ to denote a number field rather than the more standard notation for a finite field or its algebraic closure. The traditional symbols for number fields have been taken up by more commonly used notations in these lectures.

These are *theta functions* viewed as sections of $L'_r \to E'$ where

$$L'_r = \mathbb{C} \times_{\mathcal{O}_\mathbb{F}} \mathbb{C}$$

with the equivalence relation

$$(z', \xi) \sim \left(z' + \overline{\alpha}, \exp\left(\frac{2\pi i r}{\beta_0} \left(\alpha z' + \frac{|\alpha|^2}{2} \right) \xi \right) \right).$$

Then

$$p(\theta')(z', z'') =: \theta'(z') \exp\left(\frac{2\pi i r}{\beta_0} z' z'' \right) dz'$$

gives a relative differential for $\pi'' : \mathcal{O}_\mathbb{F} \backslash \mathcal{W} \to E''$, and the functional equation

$$p(\theta')(z' + \overline{\alpha}, z'' - \alpha) = p(\theta')(z', z'') \exp\left(\frac{2\pi i r}{\beta_0} (\alpha z'' + \beta) \right)$$

shows that $p(\theta')$ has values in $\pi''^*(L''_{-r})$. Thus

$$p(\theta') \in H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{O}_\mathbb{F} \backslash \mathcal{W}, \Omega^\bullet_{\pi''}(L''_{-r})); d\pi''\big) \cong H^1(E'', L''_{-r})$$

and defines the Penrose transform alluded to above.

The relative 1-form $\exp\left(\frac{2\pi i r}{\beta_0} z' z''\right) dz'$ plays the role of the form ω in the $\mathbb{P}^1$-case. As suggested above the notion has been chosen to align with Carayol's work which will be discussed in the last lecture.

Penrose transforms for curves of higher genus

We let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a co-compact, discrete group and set

$$X' = \Gamma \backslash \mathcal{H}, \quad X = \Gamma \backslash \overline{\mathcal{H}}.$$

Here we take τ' as coordinate in $\mathcal{H}$ and τ as coordinate in $\overline{\mathcal{H}}$. The perhaps mysterious appearance of $\mathcal{H}$ and $\overline{\mathcal{H}}$ will be "explained" when in Lecture 6 we discuss cycle spaces associated to flag domains $G_\mathbb{R}/T$ where G is of Hermitian type. We set $\mathcal{W} = \mathcal{H} \times \overline{\mathcal{H}}$ and consider the diagram

$$\begin{tikzcd} & \Gamma \backslash \mathcal{W} \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ X' & & X. \end{tikzcd}$$

It is again the case that $\Gamma \backslash \mathcal{W}$ is Stein and the fibres of π, π' are contractible. The Penrose transform will be an isomorphism

$$H^0(X', L'_k) \to H^1(X, L_{k-2}).$$

In order to have $L'_k \to X'$ be a positive line bundle we must have $k = -1, -2, \dots$. Then

$$L_{k-2} = L_k \otimes \omega_X$$

where $L_k \to X$ is *negative* since $X = \Gamma\backslash\overline{\mathcal{H}}$.

We let $f(\tau') \in H^0(X', L'_k)$ be a modular form of weight $-k$, given by a holomorphic function on $\mathcal{H}$ satisfying the usual functional equation under the action of Γ. We then set

$$p(f)(\tau', \tau) = f(\tau') \left(\frac{\tau - \tau'}{2i} \right)^{k-2} d\tau'.$$

This is a relative differential for $\Gamma\backslash\mathcal{W} \to X$, and the transformation formula under $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ given by

$$\gamma^* \left(\left(\frac{\tau - \tau'}{2i} \right)^{k-2} d\tau' \right) = (c\tau + d)^{2-k} (c\tau' + d)^{-k} \left(\frac{\tau - \tau'}{2i} \right)^{k-2} d\tau'$$

shows that we obtain a class

$$p(f) \in H^1_{\mathrm{DR}}\Big(\Gamma\big(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(\pi^* L_{k-2})\big); d\pi\Big) \cong H^1(X, L_{k-2})$$

(apologies for the double appearance of Γ). It is a nice exercise to show that $p(f) \neq 0$, and since

$$\dim H^0(X', L'_k) = \dim H^1(X, L_{k-2})$$

we see that the resulting map $H^0(X', L'_k) \to H^1(X, L_{k-2})$ is an isomorphism.

Orbit structure for $\mathbb{P}^1$ The main groups we shall consider acting on $\mathbb{P}^1$ are

- $G_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$;
- $K = \mathrm{SO}(2)$ and its complexification $K_{\mathbb{C}}$;
- $G_{\mathbb{R}} = \mathrm{SL}_2(R) =$ real form of $G_{\mathbb{C}}$.

The compact real form $G_c = S\mathcal{U}(2)$ also acts on $\mathbb{P}^1$ but in these lectures we shall make only occasional use of it. The complex group $G_{\mathbb{C}}$ acts transitively on $\mathbb{P}^1$, but $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ do not act transitively and their orbit structure will be of interest. The central point is *Matsuki duality*, which is

the orbits of $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ are in a 1-1 correspondence.

We have already mentioned this in Lecture 1; here we formulate it in a manner that suggests the general statement. The correspondence is defined as follows: Let $z \in \mathbb{P}^1$ and $G_{\mathbb{R}} \cdot z, K_{\mathbb{C}} \cdot z$ the corresponding orbits. Then

$G_{\mathbb{R}} \cdot z$ and $K_{\mathbb{C}} \cdot z$ are dual exactly when their intersection consists of one closed K orbit.

The following table illustrates this duality.

$$\begin{array}{llll}
 & G_{\mathbb{R}}\text{-orbits} & K_{\mathbb{C}}\text{-orbits} & \\
\begin{array}{l}\text{open}\\ G_{\mathbb{R}}\text{ orbits}\end{array} & \left\{\begin{array}{l}\mathcal{H}\\ \overline{\mathcal{H}}\end{array}\right. & \left.\begin{array}{r} i\\ -i\end{array}\right\} & \begin{array}{l}\text{closed}\\ K_{\mathbb{C}}\text{ orbits}\end{array}\\
 & & & \\
\begin{array}{l}\text{closed}\\ G_{\mathbb{R}}\text{ orbit}\end{array} & \left\{\mathbb{R}\cup\{0\}\right. & \left.\mathbb{P}^1\backslash\{i,-i\}\right\} & \begin{array}{l}\text{open}\\ K_{\mathbb{C}}\text{ orbit}\end{array}
\end{array}$$

$\langle\circ\rangle$

We will now informally describe the content of the remaining lectures in this series. The general objective is to discuss aspects of the relationship between Hodge theory and representation theory, especially those that may be described using complex geometry. The specific objective is to discuss and prove special cases of recent results of Carayol, and some extensions of his work, that open up new perspectives on this relationship and may have the possiblity to introduce new aspects into arithmetic automorphic representation theory that are thus far inaccessible by the traditional approaches through Shimura varieties. Whether or not this turns out to be successful, Carayol's work is a beautiful story in complex geometry.

Lecture 3 will introduce and illustrate the basic terms and concepts in Hodge theory. We emphasize that we will *not* take up the extensive and central topic of the *Hodge theory of algebraic varieties*. Rather our emphasis is on the Hodge structures as objects of interest in their own right, especially as they relate to representation theory and complex geometry.

The basic symmetry groups of Hodge theory are *Mumford-Tate groups*, and associated to them are basic objects of complex geometry, the *Mumford-Tate domains*, consisting of the set of polarized Hodge structures whose generic member has a given Mumford-Tate group G. In Lecture 4 we will describe which G's can occur as a Mumford-Tate group, and in how many ways this can happen. The fundamental concept here is a *Hodge representation*, consisting roughly of a character and a co-character. As homogeneous complex manifolds the corresponding Mumford-Tate domains depend only on the co-character. This lecture will explain and illustrate this.

Lecture 5 is concerned with *discrete series* (DS) and $\mathfrak{n}$-*cohomology.* The central point is the realization of the DS's via complex geometry, specifically the L^2-cohomology of holomorphic line bundles over flag domains.[7] The latter may be realized, in multiple

[7] Another realization due to Atiyah and Schmid [AS], is via L^2 solutions to the Dirac equation on the associated Riemannian symmetric spaces. This realization has many advantageous aspects, but

ways, as Mumford-Tate domains and this will be seen to be an important aspect in Carayol's work. The realization described above is largely the work of Schmid. An important ingredient is this analysis and the description of the L^2-cohomology groups via Lie algebra cohomology, in this case the so-called $\mathfrak{n}$-cohomology. We will discuss these latter groups in some detail as they will play an important role in the material of the later lectures and the work of Carayol.

Lectures 6 and 7 will take up the basic construction and results in the geometry of homogeneous complex manifolds that will play a central role in the remaining lectures, as well as being a very interesting topic in their own right. The main point is that associated to a flag domain there are complex manifolds that capture aspects of the complex geometry and that provide the basic tools for understanding the cohomology of homogeneous line bundles over flag domains. One of these, the *cycle spaces*, are classical and have been the subject of extensive study over the years, culminating in the recent monograph [FHW]. The other tool, the *correspondence spaces*, are of more recent vintage and in several ways may be the object that best interpolates between flag domains and the various associated spaces. Their basic property of *universality* is closely related to *Matsuki duality* which will be introduced and illustrated in these two lectures.

Lectures 8 and 9 will introduce and study the *Penrose transforms*, which among other things allow one to relate cohomologies on different flag domains and on their quotients by arithmetic groups. The main specific results here are the analysis of Penrose transforms in the case when $G = \mathcal{U}(2,1)$ studied by Carayol in [C1], [C2], [C3] and when $G = \mathrm{Sp}(4)$, which is a new case that is discussed in [GGK2] and in a further work in preparation. Using the Penrose transform to relate classical automorphic forms to non-classical automorphic cohomology, we discuss how the cup-products of the images of Penrose transform reach the automorphic cohomology groups associated to totally degenerate limits of discrete series (TDLDS), which are the central representation-theoretic objects of interest in these lectures. This result for $\mathcal{U}(2,1)$ is due to Carayol and for $\mathrm{Sp}(4)$ will appear in [GGK2] and the sequel to that work.

In the last Lecture 10 we discuss some topics that were not covered earlier and some open issues that arise from the material in the lectures. Particularly noteworthy in the topics not covered is the whole issue of the study of cuspidal automorphic cohomology at boundary components in the Kato-Usui completeion or partial compactifications of quotients of of Mumford-Tate domains by arithmetic groups. This seems to be a very interesting area for further work (cf. [KP]).

since in these lectures our primary interest is in the complex geometric aspects of Hodge theory and representation theory we will not discuss it here.

Lecture 3
Polarized Hodge structures and Mumford-Tate groups and domains

In general we will follow the terminology and notation from [GGK1]. An exception is that the Mumford-Tate groups were denoted by M_φ, whereas here they will be denoted by G_φ.

In this lecture we will introduce and explain the following terms:

- polarized Hodge structures (PHS);
- period domains and their compact duals;[8]
- Hodge bundles;
- Mumford-Tate groups;[9]
- Mumford-Tate domains and their compact duals;
- CM polarized Hodge structures.

We will also introduce three of the basic examples for this lecture series.

We begin with a general linear algebra fact. We define the real Lie group

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_{\mathfrak{m}} \cong \mathbb{C}^* = \mathbb{R}^{>0} \times S^1$$

where $\mathbb{C}^* = \{z = re^{i\theta}\}$ is considered as a real Lie group. If V is a rational vector space with $V_\mathbb{R} = V \otimes_\mathbb{Q} \mathbb{R}$ and we have a representation (a homomorphism of real Lie groups)

$$\widetilde{\varphi} : \mathbb{S} \to \mathrm{Aut}(V_\mathbb{R})$$

satisfying $\widetilde{\varphi} : \mathbb{Q}^* \to \mathrm{Aut}(V)$, then we have

(i) $V = \oplus V^n,\ \widetilde{\varphi}(r) = r^n$ on V^n (*weight decomposition*);

(ii) $V^n_\mathbb{C} = \underset{p+q=n}{\oplus} V^{p,q}, V^{q,p} = \overline{V^{p,q}} \quad \varphi(z) = z^p \bar{z}^q$ on $V^{p,q}$.

The $V^n \subset V$ are subspaces defined over $\mathbb{Q}$, and the $V^{p,q} \subset V^n_\mathbb{C}$ are the eigenspaces for the action of $\widetilde{\varphi}(\mathbb{S})$ on $V^n_\mathbb{C}$. In (i) n is the *weight*, and in (ii) (p,q) is the *type*.

There are three equivalent definitions of a **Hodge structure of weight** n**.**

Definitions: (I) $V_\mathbb{C} \underset{p+q=n}{\oplus} V^{p,q}$, $V^{q,p} = V^{\overline{p,q}}$ *(Hodge decomposition)*;

(II) $(0) \subset F^n \subset \cdots \subset F^{n-1} \subset F^n = V_\mathbb{C}$ *(Hodge filtration) satisfying for each* p

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} \mathbb{C};$$

(III) $\widetilde{\varphi} : \mathbb{S} \to \mathrm{Aut}\, V_\mathbb{R}$ *of weight* n.

---

[8]Jim Carlson and Aroldo Kaplan's lectures will discuss the basic properties of these.

[9]Mark Green's lecture will discuss the basic properties of algebraic groups and Lie groups that will be used. A basic reference for this material is [K1].

The equivalence of the first two definitions is

$$F^p = \mathop{\oplus}_{p' \geqq p} V^{p',q'} \qquad\qquad \mathrm{I} \Rightarrow \mathrm{II}$$

$$V^{p,q} = F^p \cap \overline{F}^q \qquad\qquad \mathrm{II} \Rightarrow \mathrm{I}.$$

We have seen above that the $V^{p,q}$ are the eigenspaces of $\widetilde{\varphi}(\mathbb{S})$ acting on $V_{\mathbb{C}}$, which gives

$$\mathrm{I} \Leftrightarrow \mathrm{III}.$$

We shall primarily use the third definition and shall denote a Hodge structure by $(V, \widetilde{\varphi})$.

In general, without specifying the weight a *Hodge structure* is given by V and $\widetilde{\varphi} : \mathbb{S} \to \mathrm{Aut}(V_{\mathbb{R}})$ as above. The weight summands are then Hodge structures of pure weight n. Unless otherwise stated we shall assume that Hodge structures are of pure weight.

We define the *Weil operator* C on $V_{\mathbb{C}}$ by $C(v) = \widetilde{\varphi}(i)v$.

Hodge structures admit the usual operations

$$\oplus, \quad \otimes, \quad \mathrm{Hom}$$

of linear algebra. A *sub-Hodge structure* is given by a linear subspace $V' \subset V$ with $\widetilde{\varphi}(\mathbb{S})(V'_{\mathbb{R}}) \subseteq V'_{\mathbb{R}}$. An important property is that *morphisms are strict*: Given

$$\psi : V \to V'$$

where V, V' have weights $r, r' = n + r$ (r may be negative) and

$$\psi(F^p) \subseteq F'^{p+r},$$

which is equivalent to

$$\psi(V^{p,q}) \subseteq V'^{p+r,q+r},$$

we have the *strictness property*

$$\psi(V_{\mathbb{C}}) \cap F'^{p+r} = \psi(F^p).$$

That is, anything in the image of ψ that lies in F'^{p+r} already comes from something in F^p. The property of strictness implies that Hodge structures form an abelian category.

Hodge's theorem: *For X a compact Kähler manifold the cohomology group $H^n(X, \mathbb{Q})$ has a Hodge structure of weight n.*[10]

As remarked in the first lecture, the decomposition of the C^∞ differential forms

$$\begin{cases} A^n(X) = \mathop{\oplus}\limits_{p+q=n} A^{p,q}(X) \\ A^{q,p}(X) = \overline{A^{p,q}(X)} \end{cases}$$

[10]This theorem will be discussed in the lectures of Eduardo Cattani and Aroldo Kaplan. It opens the door to the rich, extensive and very active field of the Hodge theory of algebraic varieties. A recent treatment of this subject appears in [ICTP].

where

$$A^{p,q}(X) = \left\{ \sum_{\begin{cases} |I|=p \\ |J|=q \end{cases}} f_{IJ}(z,\bar{z}) dz^I \wedge d\bar{z}^J \right\},$$

and for $I = (i_1, \dots, i_p)$ we have $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, induces via de Rham's theorem the Hodge decomposition on cohomology.

An example of a different sort is given by

Tate Hodge structure $\mathbb{Q}(1)$: Here the $\mathbb{Q}$-vector space is $2\pi i\mathbb{Q}$, the weight $n = -2$ and the Hodge type is $(-1,-1)$.

One sets $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ and $V(n) = V \otimes_{\mathbb{Q}} \mathbb{Q}(n)$ (*Tate twist*). Then

$$H^1(\mathbb{C}^*, \mathbb{Q}) \cong \mathbb{Q}(-1) \text{ with generator } \frac{dz}{z}$$

where for $\gamma = \{|z| = 1\} \in H_1(\mathbb{C}^*, \mathbb{Q})$

$$\gamma \to \int_\gamma \frac{dz}{z}$$

gives an isomorphism $H_1(\mathbb{C}^*, \mathbb{Q}) \cong \mathbb{Q}(1)$. In general, for $Y \subset X$ a smooth hypersurface and

$$H^n(Y, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q})$$

the *Gysin map*, defined to be the Poincaré dual of the map on homology induced by the inclusion and which is dual to the residue map (where the $2\pi i$ comes in), one has a morphism of Hodge structures *of the same weight* $n+2$

$$H^n(Y, \mathbb{Q}(-1)) \to H^{n+2}(X, \mathbb{Q}).$$

This is useful for keeping track of weights in formal Hodge theory.

For these lectures the main definition is the following

Definition: *A* **polarized Hodge structure** (V, Q, φ) *(PHS) is given by a Hodge structure* $\varphi : \mathbb{S} \to \mathrm{Aut}(V_{\mathbb{R}})$ *of weight n together with a non-degenerate form*

$$Q : V \otimes V \to \mathbb{Q}, \quad Q(v, v') = (-1)^n Q(v', v)$$

satisfying the Hodge-Riemann bilinear relations

$$\text{(I)} \qquad Q(F^p, F^{n-p+1}) = 0$$

$$\text{(II)} \qquad Q(v, C\bar{v}) > 0, \qquad 0 \neq v \in V_{\mathbb{C}}.$$

These are equivalent to the more classical versions

$$Q(V^{p,q}, V^{p',q'}) = 0, \qquad p' \neq n - p$$
$$i^{p-q} Q(V^{p,q}, \overline{V}^{p,q}) > 0.$$

A sub-Hodge structure $V' \subset V$ of a polarized Hodge structure is polarized by the restriction

$$Q' = Q\big|_{V'},$$

of Q' to V', and setting $V'' = V'^{\perp}$, $Q'' = Q\big|_{V''}$

$$(V, Q) = (V', Q') \oplus (V'', Q'')$$

is a direct sum of PHS's. As a consequence, *PHS's form a semi-simple abelian category.*

For polarized Hodge structures we set $\varphi = \widetilde{\varphi}\big|_{S^1}$ and have the

Propostion: $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}}, Q)$.

Proof. $Q \in V^* \otimes V^*$ and by Hodge-Riemann (I) it has Hodge type $(-1, -1)$. □

In general for a Hodge structure of even weight $n = 2m$ we define the *Hodge classes* $\mathrm{Hg}_{\varphi}(V)$ to be those of Hodge type (m, m). We will return later to the resulting *algebra of Hodge tensors*

$$\mathrm{Hg}^{\bullet,\bullet}(V) = \underset{k \equiv l(2)}{\oplus} \mathrm{Hg}(V^{\otimes^k} \otimes V^{*\otimes^l}).$$

An important observation is

Given a polarized Hodge structure (V, Q, φ)*,* $\mathrm{Hom}(V, V) = V^* \otimes V$ *has a polarized Hodge structure. Moreover, the Lie algebra*

$$\mathfrak{g} = \mathrm{Hom}_Q(V, V) \subset \mathrm{Hom}(V, V)$$

is a sub-Hodge structure.

For the Hodge decomposition we have

$$\mathfrak{g}_{\mathbb{C}} = \oplus \mathfrak{g}^{i,-i}$$

where

$$\mathfrak{g}^{i,-i} = \left\{ X \in \mathfrak{g}_{\mathbb{C}} : X(V^{p,q}) \subseteq V^{p+i,q-i} \right\}.$$

We note that

$$\left[\mathfrak{g}^{i,-i}, \mathfrak{g}^{j,-j}\right] = \mathfrak{g}^{i+j,-(i+j)}.$$

The case of *Shimura varieties* [Ke], which included PHS's of weight $n = 1$, is when

$$\mathfrak{g}^{i,-i} = 0 \ \text{ unless } \ i = 0, \pm 1.$$

Period domains and their compact duals[11]

For a Hodge structure $(V,\widetilde{\varphi})$ of weight n we set

$$\begin{cases} h^{p,q} = \dim V^{p,q} & (= \textit{Hodge numbers}) \\ f^p = h^{n,0} + \cdots + h^{p,n-p}. \end{cases}$$

Definition: (i) A *period domain* D is the set of PHS's (V,Q,φ) with given Hodge numbers $h^{p,q}$. (ii) The *compact dual* $\check{D}$ is the set of filtrations $F^\bullet$ of $V_\mathbb{C}$ with $\dim F^p = f^p$ and satisfying

$$Q(F^p, F^{n-p+1}) = 0.$$

The group $G_\mathbb{R} =: \mathrm{Aut}(V_\mathbb{R}, Q)$ is a real, simple Lie group that acts transitively on D. The isotropy group H of a reference PHS (V,Q,φ_0) preserves a direct sum of definite Hermitian forms, and therefore it is a compact subgroup of $G_\mathbb{R}$ that contains a *compact maximal torus* T. The following exercises give details.

Exercise: $D = \{\varphi : S^1 \to G_\mathbb{R} : \varphi = g^{-1}\varphi_0 g \text{ for some } g \in G_\mathbb{R}\}$. That is, D is the set of $G_\mathbb{R}$-conjugacy classes of the circle $\varphi_0 : S^1 \to G_\mathbb{R}$.

It follows that $H = Z_{\varphi_0}(G_\mathbb{R})$ is the *centralizer* in $G_\mathbb{R}$ of the circle $\varphi_0(S^1)$. The centralizer of a circle in a real Lie group always contains a *Cartan subgroup*, which is isomorphic to the identity component of a product of $\mathbb{R}^*$'s and S^1's. Since in our case $Z_{\varphi_0}(G_\mathbb{R}) \subset H$ is compact only S^1's occur.

Exercise: For $n = 2m+1$ odd

$$H \cong \mathcal{U}(h^{2m+1,0}) \times \cdots \times \mathcal{U}(h^{m+1,m})$$

is a product of unitary groups, and for $n = 2m$ even with $k = h^{2m,0} + h^{2m-2,2} + \cdots$ and $l = h^{2m-1,1} + h^{2m-3,3} + \cdots +$

$$H \cong \mathcal{U}(h^{2m,0}) \times \cdots \times \mathcal{U}(h^{m+1,m-1}) \times \mathcal{O}(h^{m,m})$$

is a product of unitary groups and over orthogonal group.[12]

The group $G_\mathbb{C} = \mathrm{Aut}(V_\mathbb{C}, Q)$ is a complex, simple Lie group that acts transitively on $\check{D}$. The subgroup P in $G_\mathbb{C}$ that stabilizes a $F_0^\bullet$ is a *parabolic* subgroup with

$$H = G_\mathbb{R} \cap P.$$

Usually we choose $F_0^\bullet$ to be $F_{\varphi_0}^\bullet$ where $\varphi_0 \in D$ is a reference point.

Since the second Hodge-Riemann bilinear relations are strict inequalities, the period domain is an open orbit of $G_\mathbb{R}$ acting on $\check{D}$. The orbit structure of $G_\mathbb{R}$'s acting on $\check{D}$'s will be one theme in Lectures 6 and 7.

[11]These will be discussed more fully in the lectures of Jim Carlson and Aroldo Kaplan.

[12]It is frequently convenient in the even weight case to take V to be oriented, so that $G_\mathbb{R}$ is connected.

Exercise: For $n = 1$ show that

$$D \cong \mathcal{H}_g$$

where $\dim V = 2g$ and $\mathcal{H}_g$, *Siegel's generalized upper half space*, is $= \{Z \in M_{g\times g} : Z = {}^tZ,\ \operatorname{Im} Z > 0\}$. For the PHS associated to $H^1(X, \mathbb{Q})$ where X is a compact Riemann surface of genus, the associated Z is the classical *period matrix* of X. (Here we use $\mathbb{Z}$ instead of $\mathbb{Q}$.)

Exercise: For $n = 2$ and $h^{2,0} = h$, $h^{1,1} = 1$ show that

$$\check{D} = \{E \in Gr(h, \mathbb{C}^{2h+1}) : Q(E, E) = 0\},$$

and that $G_{\mathbb{R}}$ acting on $\check{D}$ has two open orbits, one of which is the period domain. This is the case that arises in the period matrices of the 2nd primitive cohomology of smooth algebraic surfaces.

Hodge bundles: Over $\check{D}$ these are the $G_{\mathbb{C}}$-homogenous vector bundles

$$\mathbb{F}^p \to \check{D}$$

whose fibre at a given point $F^\bullet$ is F^p. Restricting to $D \subset \check{D}$ we have

$$V^{p,q} =: \mathbb{F}^p/\mathbb{F}^{p+1}.$$

These are homogeneous vector bundles for the action of $G_{\mathbb{R}}$. Importantly, they are *Hermitian vector bundles* with $G_{\mathbb{R}}$-invariant Hermitian metrics given in each fibre by the second of the Hodge-Riemann bilinear relations. Their general differential geometric properties will be discussed in the lectures by Jim Carlson. In Lecture 5 we will discuss the special case of homogeneous line bundles.

At a reference point $\varphi \in D$ with the PHS on $\mathfrak{g}$ described above, we have for the Lie algebras $\mathfrak{h}_{\mathbb{C}}$ of $H_{\mathbb{C}}$ and P

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}^{0,0}$$
$$\mathfrak{p} = \bigoplus_{i \geqq 0} \mathfrak{g}^{i,-i}$$

and the holomorphic tangent space

$$T_\varphi D \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{p} \cong \bigoplus_{i<0} \mathfrak{g}^{i,-i}.$$

We shall sometimes write $\mathfrak{g}_\varphi$ and $\mathfrak{g}_\varphi^{i,-i}$ when we wish to emphasize the circle $\varphi : S^1 \to G_{\mathbb{R}}$.

The real tangent space is the $G_{\mathbb{R}}$-homogeneous vector bundle whose fibre of $T_{\varphi,\mathbb{R}}D$ at the reference point φ is

$$\left(\bigoplus_{i\neq 0} \mathfrak{g}_\varphi^{i,-i}\right)_{\mathbb{R}}.$$

Setting $T^{1,0}_\varphi D = T_\varphi D$, we have

$$T_{\mathbb{R},\varphi} D \otimes \mathbb{C} = T^{1,0}_\varphi D \oplus T^{0,1}_\varphi D$$

where $T^{0,1}_\varphi D = \overline{T^{1,0}_\varphi} D$. This gives a $G_\mathbb{R}$-invariant almost complex structure on D, which is integrable by the bracket relations given previously. The Hodge-Riemann bilinear relations for $\mathfrak{g}_\mathbb{R}$ induce a $G_\mathbb{R}$-invariant Hermitian metric on D.

Mumford-Tate groups: These are the basic symmetry groups of Hodge theory, encoding both the $\mathbb{Q}$-structure on V and the complex structure (Hodge decomposition) on $V_\mathbb{C}$.

Definitions: (i) *Given a Hodge structure* $(V, \widetilde{\varphi})$ *the* **Mumford-Tate group** *is the smallest* $\mathbb{Q}$*-algebraic subgroup* $G_{\widetilde{\varphi}} \subset \mathrm{GL}(V)$ *such that*

$$\widetilde{\varphi}(\mathbb{S}) \subset G_{\widetilde{\varphi},\mathbb{R}}.$$

(ii) *Given a PHS* (V, Q, φ) *the* **Mumford-Tate group** *is the smallest* $\mathbb{Q}$ *algebraic subgroup* $G_\varphi \subset \mathrm{Aut}(V, Q)$ *such that*

$$\varphi(S^1) \subset G_{\varphi,\mathbb{R}}.$$

It may be shown, and we will explain why this is so, that

$$G_\varphi = G_{\widetilde{\varphi}} \cap \mathrm{Aut}(V, Q).$$

It is also that case that

$$G_{\widetilde{\varphi}} \textit{ and } G_\varphi \textit{ are reductive, } \mathbb{Q}\textit{-algebraic groups.}$$

For G_φ we may see this as follows: If we have a G_φ-invariant subspace $V' \subset V$, then since $\varphi(S^1) \subset G_{\varphi,\mathbb{R}}$ there is an induced action φ' of $\varphi(S^1)$ on $V'_\mathbb{R}$ and therefore (V', φ') is a sub-Hodge structure. We have observed earlier that it is polarized by $Q' = Q\big|_{V'}$ and that setting $(V'', Q'', \varphi'') = (V', Q', \varphi')^\perp$,

$$(V, Q, \varphi) = (V', Q', \varphi') \oplus (V'', Q'', \varphi'')$$

is a direct sum of PHS's. Then by minimality and since $\varphi(S^1) \subset G_{\varphi',\mathbb{R}} \times G_{\varphi'',\mathbb{R}}$ we have that $G_\varphi \subset G_{\varphi'} \times G_{\varphi''}$.[13] In particular, G_φ preserves the direct sum decomposition $V = V' \oplus V''$.

We note that

$$\mathfrak{g}_\varphi \textit{ is a sub-Hodge structure of } \mathrm{Hom}_Q(V, V).$$

In case G_φ is semi-simple, the polarizing form will, up to scalings, be induced by the Cartan-Killing form of $\mathfrak{g}_\varphi$.

The extreme cases are

[13] This inclusion is in general strict.

- $\varphi \in D$ is a generic point $\Rightarrow G_\varphi = \mathrm{Aut}(V, Q)$;
- $G_\varphi \subset H_\varphi =$ stability group of $(V, Q, \varphi) \Rightarrow G_{\widetilde{\varphi}}$ is a $\mathbb{Q}$-algebraic torus.

The second statement is a result whose proof will be given below just before the next section. When $G_{\widetilde{\varphi}}$ is an algebraic torus, $(V, \widetilde{\varphi})$ is by definition a *complex multiplication* (CM) Hodge structure. If $(V, \widetilde{\varphi})$ is *simple*, i.e., it contains no non-trivial proper sub-Hodge structures, then $\mathrm{Hom}_{\widetilde{\varphi}}(V, V)$ is a division algebra acting on $(V, \widetilde{\varphi})$. We shall discuss more about CM PHS's below.

Example: Let $X_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be as in the first lecture. Then

$$H^1(X_\tau, \mathbb{Q}) \text{ is CM} \Leftrightarrow \tau \text{ is a quadratic imaginary number.}$$

Then $L = \mathbb{Q}(\tau)$ is a number field and $G_{\widetilde{\varphi}} = L^*$ is the group of units with G_φ being those of norm one.

Since G_φ is a $\mathbb{Q}$-algebraic group it is natural to ask:

What are the $\mathbb{Q}$-algebraic equations that define $G_\varphi \subset \mathrm{Aut}(V, Q)$?

This question has a very nice answer as follows. Recall the algebra of Hodge tensors

$$\mathrm{Hg}_\varphi^{\bullet,\bullet} \subset \bigoplus_{k \equiv l(2)} V^{\otimes^k} \otimes V^{*\otimes^l}.$$

We have noted that G_φ fixes $\mathrm{Hg}_\varphi^{\bullet,\bullet}$.

Theorem: *G_φ is equal to the subgroup $\mathrm{Fix}(\mathrm{Hg}_\varphi^{\bullet,\bullet})$ that fixes the algebra of Hodge tensors.*

The reverse inclusion

$$\mathrm{Fix}(\mathrm{Hg}_\varphi^{\bullet,\bullet}) \subseteq G_\varphi$$

is based on a theorem of Chevally:

A linear reductive $\mathbb{Q}$-algebraic group is defined by stabilizing a line $L \subset \bigoplus_{k,l}(V^{\otimes^k} \otimes V^{*\otimes^l})$.

The basic idea is that if $L \subset V^{\otimes^k} \otimes V^{*\otimes^l}$ then since $\varphi(S^1) \subset G_{\varphi,\mathbb{R}}$ we have that $\varphi(S^1)$ acts trivially on $L_\mathbb{C}$. Thus the weight $l - k = 2m$ and $L_\mathbb{C} = L_\mathbb{C}^{m,m}$, which says that $L \subset \mathrm{Hg}_\varphi^{k,l}$.

The above characterization of G_φ holds in a suitably modified form for $G_{\widetilde{\varphi}}$. The modification is that on Hodge classes of weight n, $\widetilde{\varphi}(re)$ acts by r^n. Thus the condition of fixing tensors must be replaced by scaling them, and when this is done the above result extends to general Hodge structures. In particular, given $(V, \widetilde{\varphi})$ and a polarization Q, $\widetilde{\varphi}(re^{i\theta}) \cdot Q = r^{-2}Q$. Thus for Hodge structures that are polarizable the difference between $G_{\widetilde{\varphi}}$ and G_φ is just in the scaling action.

The theorem "explains" why for a direct sum $(V,\varphi)=(V',\varphi')+(V'',\varphi'')$ of Hodge structures, the inclusion

$$G_\varphi \subset G_{\varphi'} \times G_{\varphi''}$$

is in general strict. The inclusion holds because the direct sum has at least as many Hodge tensors as those that come from the two factors. It will be strict if there are additional Hodge tensors that relate (V',φ') and (V'',φ'').

Example: For the PHS $(\mathfrak{g}_\varphi, B, \varphi)$ where B is the Cartan-Killing form, both B and the bracket [,] are Hodge tensors. They essentially generate the algebra of Hodge tensors in a manner to be explained below.

Proof of $G_\varphi \subset H_\varphi \Rightarrow G_{\tilde{\varphi}}$ is an algebraic torus. We first note that $\mathrm{End}(V,\varphi)$, the endomorphisms of V that commute with the action of $\varphi(S^1)$ on $V_{\mathbb{R}}$, is just the space $\mathrm{Hg}^{1,1}$ of Hodge tensors in $V \otimes V^*$. Next, the assumption $G_\varphi \subset H_\varphi$, i.e. that G_φ preserves the Hodge structure (V,φ), implies that

$$G_\varphi \subset \mathrm{End}(V,\varphi).$$

Then $G_\varphi = \mathrm{Fix}(\mathrm{Hg}_\varphi^{\bullet,\bullet})$ says that G_φ is commutative, which is what was to be shown. $\square$

Mumford-Tate domains and their compact duals

Definition: *Given a PHS (V,Q,φ) the associated* **Mumford-Tate domain** *is D_φ, the $G_{\varphi,\mathbb{R}}$-orbit of the corresponding point in the period domain.*

Thus for $H_\varphi \subset G_{\varphi,\mathbb{R}}$ the stability group of (V,Q,φ) the quotient space

$$D_\varphi = G_{\varphi,\mathbb{R}}/H_\varphi$$

is a homogeneous complex manifold. As a set

$$D_\varphi = \{g^{-1}\varphi g : g \in G_{\varphi,\mathbb{R}}\}$$

is the set of $G_{\varphi,\mathbb{R}}$-conjugacy classes of $\varphi : S^1 \to G_{\varphi,\mathbb{R}}$. From this we may infer that

$$H_\varphi = Z_{G_{\varphi,\mathbb{R}}}(\varphi(S^1)) \textit{ is the centralizer of } \varphi(S^1) \textit{ in } G_{\varphi,\mathbb{R}}.$$

Since H_φ is compact we have that

$$H_\varphi \textit{ contains a compact maximal torus } T.$$

From general properties of $\mathbb{Q}$-algebraic groups we obtain the result

A Mumford-Tate group contains an anisotropic, $\mathbb{Q}$-maximal torus.

One may think of a split $\mathbb{Q}$-maximal torus in a reductive $\mathbb{Q}$-algebraic group as a product $(\mathbb{Q}^*)^m \times (\mathbb{S}(\mathbb{Q}))^n$ where

$$\mathbb{S}(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Q} \text{ and } a^2 + b^2 = 1 \right\}.$$

Anisotropic means that $m = 0$.

Among reductive $\mathbb{Q}$-algebraic groups this is a very special property. For example, $\mathrm{GL}_n(\mathbb{Q})$, $\mathrm{SL}_n(\mathbb{Q})$ for $n \geqq 3$ are not Mumford-Tate groups. It is a much more subtle matter to rule out other simple groups as being Mumford-Tate groups.

Example (continued): Given a PHS (V, Q, φ) there is an associated PHS $(\mathfrak{g}_\varphi, B, \varphi)$. It defines a point $\mathrm{Ad}\,\varphi$ in the corresponding period domain D_{Ad}. In case G_φ is simple it may be shown that the Mumford-Tate domain $D_{\mathrm{Ad},\varphi} \subset D_{\mathrm{Ad}}$ is the connected component containing $(\mathfrak{g}_\varphi, B, \varphi)$ of the variety defined by imposing the condition that B and $[\ ,\ ]$ are Hodge tensors. The essential point is the adjoint group

$$G_{\mathbb{C},a} = \mathrm{Aut}^0(\mathfrak{g}_{\mathbb{C}}, [\ ,\ ])$$

is the identity component of the subgroup of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ that preserves $[\ ,\ ]$ (cf. [K1]).

In general, it does not seem to be known in what degrees the algebra of Hodge tensors are effectively generated.

Example: We shall show how to realize the unitary group $\mathcal{U}(2,1)_{\mathbb{R}}$ as the real Lie group associated to a $\mathbb{Q}$-algebraic group $\mathcal{U}(2,1)$, and we will see that $\mathcal{U}(2,1)$ is the Mumford-Tate group of three PHS's, including one of weight $n = 3$ with $h^{3,0} = 1$, $h^{2,1} = 2$. For this we proceed in three steps:

(i) determine Hodge structures of a certain type;
(ii) put a real polarization on them;
(iii) ensure that the polarization is rational.

Let $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ where $d > 0$ is a squarefree positive rational number ($d = 1$ will do), and let V be a 6-dimensional $\mathbb{Q}$-vector space with an $\mathbb{F}$-action; i.e., an embedding

$$\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{Q}}(V)\,.$$

Setting $V_{\mathbb{F}} = V \otimes_{\mathbb{Q}} \mathbb{F}$, we have over $\mathbb{F}$ the eigenspace decomposition

$$V_{\mathbb{F}} = V_+ \oplus V_-$$

where $\overline{V}_+ = V_-$. We will show how to construct polarized Hodge structures of weights $n = 4$, $n = 3$, and $n = 2$ with respective Mumford-Tate groups $\mathcal{U}(2,1)$, $\mathcal{U}(2,1)$, and $S\mathcal{U}(2,1)$. For this we write $V_{\mathbb{C}} = V_{+,\mathbb{C}} \oplus V_{-,\mathbb{C}}$. We shall do the $n = 4$ case first, and for this we consider the following picture:

(4,0)	(3,1)	(2,2)	(1,3)	(0,4)	
*	*	*			$V_{+,\mathbb{C}}$
		*	*	*	$V_{-,\mathbb{C}}$

The notation means this: Choose a decomposition $V_{+,\mathbb{C}} = V_+^{4,0} \oplus V_+^{3,1} \oplus V_+^{2,2}$ into 1-dimensional subspaces. Then define $V_{-,\mathbb{C}} = V_-^{2,2} \oplus V_-^{1,3} \oplus V_-^{0,4}$ where $V_-^{p,q} = \overline{V}_+^{q,p}$. Setting $V^{p,q} = V_+^{p,q} \oplus V_-^{p,q}$ gives a Hodge structure.[14]

Next we define a *real* polarization by requiring $Q(V_+,V_+)=0=Q(V_-,V_-)$, then choosing a non-zero vector $\omega_+^{p,q} \in V_+^{p,q}$ and setting

$$\begin{cases} Q(\omega_+^{4,0}, \overline{\omega}_+^{4,0}) = 1, & \overline{\omega}_+^{4,0} \in V^{(0,4)} \\ Q(\omega_+^{3,1}, \overline{\omega}_+^{3,1}) = -1, & \overline{\omega}_+^{3,1} \in V_-^{(1,3)} \\ Q(\omega_+^{2,2}, \overline{\omega}_+^{2,2}) = 1, & \overline{\omega}_+^{2,2} \in V_-^{(2,2)}\,. \end{cases}$$

All other $Q(*,*) = 0$.

Finally, we may choose the $V_+^{p,q}$ to be defined over $\mathbb{F}$ and $\omega_+^{p,q} \in V_{+,\mathbb{F}}$. Then

$$\begin{cases} \frac{1}{2}(\omega_+^{p,q} + \overline{\omega}_+^{p,q}) = e_{5-p} & p = 4,3,2 \\ \frac{1}{2\sqrt{-d}}(\omega_+^{p,q} - \overline{\omega}_+^{p,q}) = e_{7-p} & p = 3,2,1 \end{cases}$$

gives a basis $e_1, \dots, e_6$ for $V_{\mathbb{R}} \cap V_{\mathbb{F}} = V$. In terms of this basis, the matrix entries of Q are in $\mathbb{R} \cap \mathbb{F} = \mathbb{Q}$.

We observe that, by construction, the action of $\mathbb{F}$ on V preserves the form Q. We set

$$\mathcal{U} = \mathrm{Aut}_{\mathbb{F}}(V, Q)\,.$$

This is an $\mathbb{F}$-algebraic group, and we then set

$$\mathcal{U}(2,1) = \mathrm{Res}_{\mathbb{F}/\mathbb{Q}}\,\mathcal{U}\,.$$

Proposition: (i) $\mathcal{U}(2,1)$ *is a* $\mathbb{Q}$*-algebraic group whose associated real Lie group is* $\mathcal{U}(2,1)_{\mathbb{R}}$. (ii) *If we operate on the reference polarized Hodge structure conjugated by a generic* $g \in \mathrm{Aut}_{\mathbb{F}}(V_{\mathbb{R}}, Q) \cong \mathcal{U}(\mathbb{R})$*, the resulting polarized Hodge structure has Mumford-Tate group* $\mathcal{U}(2,1)$.

Proof. Setting $J = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$, the matrix of Q in the $\mathbb{Q}$-basis $e_1, \dots, e_6$ for V is

$$Q = \begin{pmatrix} J & 0 \\ 0 & \left(\frac{1}{d}\right) J \end{pmatrix}.$$

[14] In general, the number of $*$'s in a box will denote the dimension of the complex vector space.

In terms of this basis, $V_{+,\mathbb{F}}$ is spanned by the columns in the matrix

$$\begin{pmatrix} I \\ \sqrt{-d}I \end{pmatrix}.$$

If $g \in \mathrm{Aut}_{\mathbb{F}}(V)$, then the extension of g to $V_{\mathbb{F}}$ commutes with the projections onto $V_{+,\mathbb{F}}$ and $V_{-,\mathbb{F}}$. A calculation shows that these are equations defined over $\mathbb{Q}$. The conditions that g preserve Q are further equations defined over $\mathbb{Q}$. Thus, $\mathcal{U}$ is a $\mathbb{Q}$-algebraic group. Moreover, g is uniquely determined by its restriction to the induced mapping

$$g_+ : V_{+,\mathbb{F}} \to V_{+,\mathbb{F}}\,.$$

In terms of the basis $\omega_+^{4,0}, \omega_+^{3,1}, \omega_+^{2,2}$ of $V_{+,\mathbb{C}} \cong \mathbb{C}^3$, g_+ preserves the Hermitian form J; i.e.,

$${}^t\overline{g}_+ J g_+ = J\,.$$

This shows that the real points $\mathcal{U}(\mathbb{R})$ have an associated Lie group isomorphic to $\mathcal{U}(2,1)_{\mathbb{R}}$, and therefore proves (i). The proof of (ii) will be omitted (cf. [GGK1]).

The reason that the Mumford-Tate is $\mathcal{U}(2,1)$ and not $S\mathcal{U}(2,1)$ is that the circle $\{z \in \mathbb{C} : |z| = 1\}$ acts on $\omega_+^{p,q}$ by z^{p-q} and $z^4 \cdot z^2 \cdot z^0 = z^6 \neq 1$. $\square$

To obtain a polarized Hodge structure of weight $n = 2$ with Mumford-Tate group $S\mathcal{U}(2,1)$ we do the construction as shown in this figure:

$$\begin{array}{|c|c|c|l} \hline * & * & * & V_{+,\mathbb{C}} \\ \hline * & * & * & V_{-,\mathbb{C}} \\ \hline \multicolumn{1}{c}{h^{2,0}} & \multicolumn{1}{c}{h^{1,1}} & \multicolumn{1}{c}{h^{0,2}} & \end{array}$$

We are in $S\mathcal{U}(2,1)$ because $z^2 \cdot z^0 \cdot z^{-2} = 1$.

To obtain a polarized Hodge structure of weight $n = 3$ with Mumford-Tate group $\mathcal{U}(2,1)$ we do a similar construction

$$\begin{array}{|c|c|c|c|l} \hline * & * & * & & V_{+,\mathbb{C}} \\ \hline & * & * & * & V_{-,\mathbb{C}} \\ \hline \multicolumn{1}{c}{h^{3,0}} & \multicolumn{1}{c}{h^{2,1}} & \multicolumn{1}{c}{h^{1,2}} & \multicolumn{1}{c}{h^{0,3}} & \end{array}$$

A difference is that, in order to have Q alternating, we set

$$iQ(\omega_+^{3,0}, \overline{\omega}_+^{3,0}) = 1\,.$$

All of the above give Mumford-Tate domains that are of the form $G_{\mathbb{R}}/T$ where T is a compact maximal torus. The picture when $n=1$

$$\begin{array}{|c|c|l} \hline ** & * & V_{+,\mathbb{C}} \\ \hline * & ** & V_{-,\mathbb{C}} \\ \hline \multicolumn{1}{c}{h^{1,0}} & \multicolumn{1}{c}{h^{0,1}} & \end{array}$$

gives a Mumford-Tate domain $\mathcal{U}(2,1)/\mathcal{U}(2)\times\mathcal{U}(1)$, which as a complex manifold is $S\mathcal{U}(2,1)/S(\mathcal{U}(2)\times\mathcal{U}(1))$. It is an Hermitian symmetric domain $\mathbb{B}$ parametrizing polarized abelian varieties of dimension 3 with an $\mathbb{F}$-action. The corresponding quotient $G_{\mathbb{R}}/T$, where $T\subset K$ is the unique maximal torus, may be thought of as the set of *Hodge flags* lying over the Mumford-Tate domain $\mathbb{B}$. Here, for $F^1\in\mathbb{B}$ a Hodge flag is given by $0\subset L\subset F^1$ where L is a line in F^1.

Returning to the general discussion, we note that Mumford-Tate domains $D=G_{\varphi,\mathbb{R}}/H_\varphi$ have compact duals

$$\check{D}=G_{\varphi,\mathbb{C}}/P_\varphi$$

where $G_{\varphi,\mathbb{C}}$ is the complex Lie group associated to G_φ and P_φ is the parabolic subgroup of G_φ that stabilizes the Hodge filtration $F^\bullet_\varphi$. The Mumford-Tate domain is an open orbit of $G_{\varphi,\mathbb{R}}$ acting on $\check{D}$.

We will next obtain "pictures" of the D above and of its compact dual. For this we We identify $V_{+,\mathbb{C}}$ with $\mathbb{C}^3$ using the basis $\omega_+^{3,0}$, $\omega_+^{2,1}$, $\omega_+^{1,2}$ above. The Hermitian form has the matrix

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Writing vectors in $\mathbb{C}^3$ as $z=\left(\begin{smallmatrix} z_0\\ z_1\\ z_2\end{smallmatrix}\right)$ with $[z]=\left[\begin{smallmatrix} z_0\\ z_1\\ z_2\end{smallmatrix}\right]\in\mathbb{P}^2$, the condition

$$H(z,\bar{z})<0$$

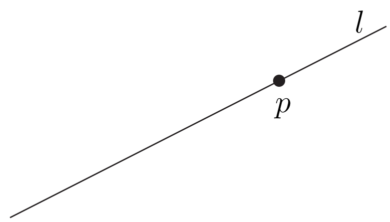
defines the unit ball $\mathbb{B}\subset\mathbb{C}^2\subset\mathbb{P}^2$, where $\mathbb{C}^2$ is given by $z_1=1$.[15]

The compact dual $\check{D}=\mathrm{GL}_3(\mathbb{C})/\mathbb{P}$ where P stabilizes the flag

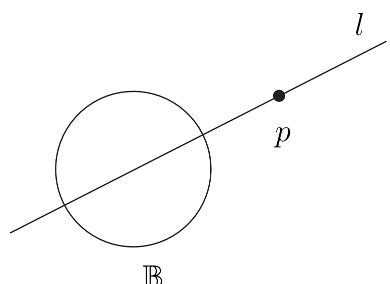
$$\begin{bmatrix} *\\ 0\\ 0\end{bmatrix}\subset\begin{bmatrix} *\\ *\\ 0\end{bmatrix}\subset\begin{bmatrix} *\\ *\\ *\end{bmatrix}$$

[15]This will be one of the "running" examples in the lectures. For computational purposes it will be more convenient to use each of $\left(\begin{smallmatrix} -1 & & \\ & 1 & \\ & & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & & \\ & -1 & \\ & & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & -1\end{smallmatrix}\right)$ for our Hermitian forms in the different lectures where this example appears. We will specify which one is used each time the example is discussed.

in $\mathbb{P}^2$. We may picture $\check{D}$ as the incidence variety in $\mathbb{P}^2 \times \mathbb{P}^{2*}$



where $p \in \mathbb{P}^2$, $l \in \mathbb{P}^{2*}$ is a line and $p \in l$. The Mumford-Tate domain is the open set of all configurations



where, setting $\mathbb{B}^c = \mathbb{P}^2 \backslash (\text{closure of } \mathbb{B})$, we have

$$\begin{cases} p \in \mathbb{B}^c \\ l \cap \mathbb{B} \neq \emptyset. \end{cases}$$

Example: We will describe the period domain D for PHS's of weight $n = 3$ and with all Hodge numbers $h^{p,q} = 1$. This example is of considerable importance in mirror symmetry, as it parametrizes possible PHS's for *mirror quintic* varieties (cf. [GGK0] and the references cited therein).

The construction we now give is an extension of the $S\mathfrak{U}(1,1)$, or unit disc, construction of PHS's of weight $n = 1$ with $h^{1,0} = 1$.

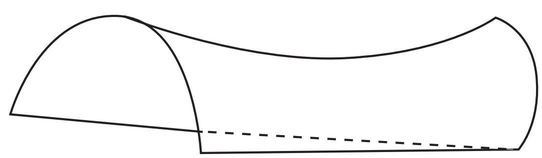
We consider a complex vector space $V_{\mathbb{C}}$ with an alternating form Q where

- there is a basis $v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}$ for $V_{\mathbb{C}}$ such that $Q = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$;
- there is a complex conjugation $\boldsymbol{\sigma} \cdot V_{\mathbb{C}} \to V_{\mathbb{C}}$ where

$$\begin{cases} \boldsymbol{\sigma}(v_{-e_1}) = iv_{e_1} \\ \boldsymbol{\sigma}(v_{-e_2}) = iv_{e_2}, \end{cases}$$

  and then $\boldsymbol{\sigma}(v_{e_1}) = iv_{-e_1}, \boldsymbol{\sigma}(v_{e_2}) = iv_{e_2}$;

- There is a $\mathbb{Q}$-form $V \subset V_{\mathbb{C}}$ given by $V = \operatorname{span}_{\mathbb{Q}}\{w_1, w_2, w_3, w_4\}$ where
$$\begin{cases} w_1 = \frac{1}{\sqrt{2}i}(v_{-e_1} - iv_{e_1}) \\ w_2 = \frac{1}{\sqrt{2}}(v_{-e_1} + iv_{e_1}) \\ w_3 = \frac{1}{\sqrt{2}i}(v_{-e_2} - iv_{e_2}) \\ w_4 = \frac{1}{\sqrt{2}}(v_{-e_2} + iv_{e_2}); \end{cases}$$
The matrix $Q_{\mathbf{w}}$ of Q in this basis is
$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix};$$
- $H : V_{\mathbb{C}} \otimes V_{\mathbb{C}} \to \mathbb{C}$ is the Hermitian form $H(u, v) = iQ(u, \boldsymbol{\sigma} v)$. It has signature $(2, 2)$;
- $H(v, \boldsymbol{\sigma} v) = 0$ defines a real quadratic hypersurface Q_H in $\mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^3$, which we picture as



- $G_{\mathbb{C}} = \operatorname{Aut}(V, Q)$;
- $G_{\mathbb{R}} = \operatorname{Aut}_{\boldsymbol{\sigma}}(V, Q)$. Then $G_{\mathbb{R}}$ is a real form of $G_{\mathbb{C}}$ containing a compact maximal torus T;[16]
- $G_{\mathbb{R}}$ is also the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ that preserves both Q and H.
Proof. For $g \in G_{\mathbb{C}} = \operatorname{Aut}(V_{\mathbb{C}}, Q)$ we have
$$\begin{aligned} H(g(v), g(w)) &= iQ(g(v), \boldsymbol{\sigma}\big(g(w)\big) \\ &= iQ\big(g(v), ((\boldsymbol{\sigma} g)(\boldsymbol{\sigma} w))\big) \end{aligned}$$
where $g \in \mathrm{GL}(V_{\mathbb{C}})$ and $\boldsymbol{\sigma} g$ is the induced conjugation;
- the complexification of the maximal torus $T \subset G_{\mathbb{R}}$ is given by the set of
$$\begin{pmatrix} \lambda_1^{-1} & & & \\ & \lambda_2^{-1} & & \\ & & \lambda_2 & \\ & & & \lambda_1 \end{pmatrix};$$

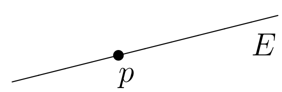
[16] In fact, $G_{\mathbb{R}} = \operatorname{Aut}(V_{\mathbb{R}}, Q_{\mathbf{w}}) \cong \mathrm{Sp}(4)_{\mathbb{R}}$.

- $v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}$ are the eigenvectors for the action of T on $V_{\mathbb{C}}$.

The compact dual $\check{D}$ may be identified with the set of *Lagrange flags*

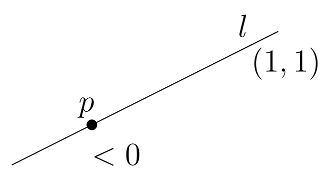
$$(0) \subset F^1 \subset F^2 \subset F^3 = F^{1\perp} \subset V_{\mathbb{C}}$$

where $\dim F^i = i$ and $Q(F^2, F^2) = 0$. In $\mathbb{P}^3 = \mathbb{P}V_{\mathbb{C}}$ such a Lagrange flag is given by a picture



where E $(= \mathbb{P}V^2)$ is a *Lagrange line* in $\mathbb{P}^3$ and p $(= \mathbb{P}V^1)$ is a point on E.

The period domain D may then be pictured as the set of Lagrange lines



where the notation means $H(p) < 0$ and the restriction $H_l =: H\big|_l$ has signature $(1, 1)$. This translates into the condition that the corresponding flag $F^{\bullet}$ satisfy the second Hodge-Riemann bilinear relation.

Example: The "first" non-classical PHS occurs with weight $n = 2$ and Hodge numbers $h^{2,0} = 2$, $h^{1,1} = 1$. Then $\dim V = 5$ and the symmetric bilinear form

$$Q : V \otimes V \to \mathbb{Q}$$

has signature $(4, 1)$. For example, we might take $V = \mathbb{Q}^5$ and Q to have matrix

$$Q = \begin{pmatrix} I_4 & 0 \\ 0 & -1 \end{pmatrix}.$$

For convenience we choose an orientation on V.

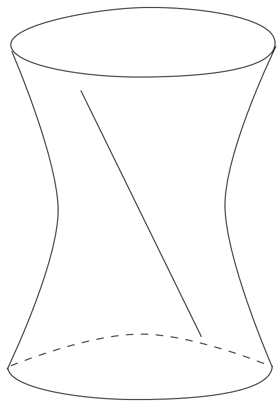
The period domain may be described as

$$D = \{F \in \mathrm{Gr}(2, V_{\mathbb{C}}) : Q(F, F) = 0, Q(F, \overline{F}) > 0\}.$$

Here, $\mathrm{Gr}(2, V_{\mathbb{C}})$ is the Grassmannian of 2-planes in $V_{\mathbb{C}} \cong \mathbb{C}^5$, or equivalently the set $\mathbb{G}(1, 3)$ of lines in $\mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^4$. The compact dual is

$$\check{D} = \{F \in \mathrm{Gr}(2, V_{\mathbb{C}}) : Q(F, F) = 0\}.$$

It is sometimes convenient to denote it by $\mathbb{G}_L(1,3)$, thought of as *Lagrangian lines* in $\mathbb{P}^4$ and pictured something like



As a homogeneous complex manifold

$$D = G_{\mathbb{R}}/H$$

where $H \cong \mathcal{U}(2)_{\mathbb{R}}$ with $A \in \mathcal{U}(2)_{\mathbb{R}}$ mapping to $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SO}(4,1)_{\mathbb{R}}$ using the standard inclusion $\mathcal{U}(2)_{\mathbb{R}} \hookrightarrow \mathrm{SO}(4)_{\mathbb{R}}$ where $\mathcal{U}(2)_{\mathbb{R}}$ is given by the orthogonal transformation on $\mathbb{R}^4$ preserving $J = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$.

Variation of Hodge structure and Mumford-Tate groups

We will only briefly touch on this as it will be discussed in the lectures by Eduardo Cattani and Jim Carlson.

Let D be a period domain for PHS's (V, Q, φ) of weight n and where $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$. We set $\Gamma_{\mathbb{Z}} = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$. In the tangent bundle TD there is a homogeneous sub-bundle $\mathbb{W}$ whose fibre at $\varphi \in D$ is

$$W_\varphi = \mathfrak{g}_\varphi^{-1,1}.$$

In terms of Hodge filtration we may think of the fibre

$$W_\varphi = \{\xi \in T_\varphi D : \xi(F_\varphi^p) \subseteq F_\varphi^{p-1}\}.$$

The condition in the brackets will be called the *infinitesimal period relation* (IPR).

Next, let S be a connected complex manifold. Usually S will be a quasi-projective algebraic variety. A *variation of Hodge structure* (VHS) is given by a locally liftable, holomorphic mapping

$$\Phi : S \to \Gamma_{\mathbb{Z}} \backslash D$$

whose differential satisfies the IPR. Thus, we have

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ S \arrow[r, "\Phi"] & \Gamma_{\mathbb{Z}}\backslash D \end{tikzcd}$$

where $\widetilde{S} \to S$ is the universal cover, and the IPR is expressed by

$$\widetilde{\Phi}_* : T\widetilde{S} \to \mathbb{W}.$$

Choosing a base point $s_0 \in S$, because of the local liftability assumption there is an induced mapping

$$\Phi_* : \pi_1(S, s_0) \to \Gamma_{\mathbb{Z}}.$$

The image $\Phi_*(\pi_1(S, s_0)) =: \Gamma \subset G_{\mathbb{Z}}$ is called the *monodromy group.* It is *the* basic invariant of a global VHS.

Assume now that $s_0 \in S$ is a generic point with $\tilde{s}_0 \in \widetilde{S}$ lying over s_0. Set $\widetilde{\Phi}(\tilde{s}_0) = \varphi_0$ corresponding to a PHS (V, Q, φ_0). Then one may show that outside of a countable union of proper analytic subvarieties of $\widetilde{S}$, the Mumford-Tate groups of $\widetilde{\Phi}(s)$ are the constant subgroup $G_{\varphi_0} =: G_\Phi \subset \operatorname{Aut}(V, Q)$.

Definition: *G_Φ is the* **Mumford-Tate group** *of the VHS.*

The basic facts about G_Φ are:

(i) $\Gamma \subset G_\Phi$.

Thus, the Mumford-Tate group of the VHS contains the $\mathbb{Q}$-Zariski closure $\overline{\Gamma(\mathbb{Q})}$ of the monodromy group.

(ii) If S is a quasi-projective variety, then after passing to a finite covering of S, Γ acts semi-simply on $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$.

If

$$G \sim G_1 \times \cdots \times G_n \times A$$

is the almost product decomposition of the reductive $\mathbb{Q}$-algebraic group G into its $\mathbb{Q}$-simple and abelian parts, then for some $m \leqq n$

(iii) $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ where $\Gamma_i \subset G_i$ for $1 \leqq i \leqq m$ (in fact, $\Gamma_i = \Gamma \cap G_i$).

(iv) If D_i is the $G_{i,\mathbb{R}}$-orbit of $\varphi_0 \in D$, then the VHS splits into a product

$$\Phi : S \to \Gamma_1\backslash D_1 \times \cdots \times \Gamma_m\backslash D_m \times \underbrace{D_{m+1} \times \cdots \times D_n}$$

which is constant in the factor over the brackets.

(v) For $1 \leq i \leq m$, the $\mathbb{Q}$-Zariski closure

$$\overline{\Gamma_i(\mathbb{Q})} = G_i.$$

These statements constitute the *structure theorem for a global VHS.*

It is not the case that Γ_i is commensurable with $\Gamma_{\mathbb{Z}} \cap G_i$; i.e., Γ_i may not be an arithmetic group. But it is the case that it is indistinguishable from one insofar as its tensor invariants are concerned.

Informally, the result says that a global VHS splits into irreducible pieces, each one of which is a quotient of a Mumford-Tate domain in $G_{\mathbb{R}}/H$ where the group G is the $\mathbb{Q}$-Zariski closure of the monodromy group.

A final comment. Given a PHS (V, Q, φ) and an abelian subspace

$$W \subset \mathfrak{g}_{\varphi}^{-1,1},$$

there is an action of $\operatorname{Sym}^{\bullet} W$ on $\oplus V^{p,q}$ where

$$\operatorname{Sym}^k W \otimes V^{p,q} \to V^{p-k,q+k}.$$

The $\operatorname{Sym}^{\bullet} W$-module $\oplus V^{p,q}$ is called the *infinitesimal variation of Hodge structure* (IVHS).

CM polarized Hodge structures

A Hodge structure $(V, \widetilde{\varphi})$ is of *complex multiplication* (CM) *type* if its Mumford-Tate group $G_{\widetilde{\varphi}}$ is an algebraic torus. We shall discuss how to construct PHS's of CM type.

We use the following notations:

- $L =$ a number field $= \mathbb{Q}(\gamma)$ where γ is a primitive element;
- $[L : \mathbb{Q}] = r$;
- $V = L$ as a $\mathbb{Q}$-vector space;
- $A(l) : V \to V =$ multiplication by $l \in L$.

Using $1, \gamma, \dots, \gamma^{r-1}$ as a basis for V

$$A(\gamma) = \begin{pmatrix} 0 & & & & -a_r \\ 1 & 0 & & & -a_{r-1} \\ & 1 & \ddots & & \\ & & & 0 & \cdot \\ & & & 1 & -a_1 \end{pmatrix}$$

where

$$\gamma^r + a_1 \gamma^{r-1} + \cdots + a_r = 0$$

is the minimal equation of γ over $\mathbb{Q}$. If $\gamma_1, \gamma_2, \dots, \gamma_r$ are the roots of this equation, then

$$\eta_i(\gamma) = \gamma_i, \qquad i = 1, \dots, r$$

give the embeddings $L \hookrightarrow \mathbb{C}$. Since

$$\det(\lambda I - A(\gamma)) = \lambda^r + a_1\lambda^{r-1} + \cdots + a_r$$

the eigenvalues of $A(\gamma)$ are $\gamma_1, \dots, \gamma_r$. We let ω_i be an eigenvector associated to γ_i. Since the γ_i are distinct,

$$\{B : V_{\mathbb{C}} \to V_{\mathbb{C}} : [B, A(\gamma)] = 0\} = \{B : B \text{ is diagonal in the basis } \omega_1, \dots, \omega_r\}.$$

Thus

$$T_{\mathbb{Q}} = \{B \in \operatorname{Aut}(V) : [B, A(\gamma)] = 0\}$$

is an algebraic group defined over $\mathbb{Q}$ whose associated complex Lie group $T_{\mathbb{C}}$ is a product of $\mathbb{C}^*$'s.

A short computation gives that a basis of the eigenvectors is

$$\omega_i = \lambda_i \begin{pmatrix} \gamma_i^{r-1} + a_1\gamma_i^{r-2} + \cdots + a_{r-1} \\ \vdots \\ \gamma_i^2 + a_1\gamma_i + a_2 \\ \gamma_i + a_1 \\ 1 \end{pmatrix}$$

where the λ_i are suitable constants of proportionality.

We are looking for a PHS (V, Q, φ) of weight n in which $L \subseteq \operatorname{End}_{\tilde{\varphi}}(V)$. In terms of the basis $\omega_1, \dots, \omega_r$ the circle

$$\varphi : S^1 \to T =: T_{\mathbb{R}}$$

must then be

$$\varphi(t) = \begin{pmatrix} e^{ik_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{ik_r t} \end{pmatrix}.$$

Since the $V^{p,q}$ are eigenspaces of $\varphi(S^1)$ we have

$$\omega_i \in V^{p_i, n-p_i} \text{ where } k_i = 2p_i - n.$$

Because T is defined over $\mathbb{Q}$, by minimality of G_φ we will have

$$G_\varphi \subseteq T_{\mathbb{Q}}.$$

To have a complex multiplication (CM) PMS it remains to find the polarization.[17]

We are thus looking for a non-degenerate pairing

$$Q : L \otimes_{\mathbb{Q}} L \to \mathbb{Q}$$

[17] In general, to construct a PHS the easier part is to construct the HS; finding the polarization is more difficult. We will see this principle operating in generality in Lecture 4.

with $Q(a,b) = (-1)^n Q(b,a)$, and a circle

$$\varphi : S^1 \to \operatorname{Aut}(V_{\mathbb{R}}, Q)$$

with the appropriate signs on the $V^{p,q}$ in order to have the second Hodge-Riemann bilinear relations. Because $V^{p,q} = \overline{V}^{q,p}$, for suitable ω_i and indexing, we must have

$$\begin{cases} \overline{\gamma}_i = \gamma_{r-i+1} \\ \overline{\omega}_i = \omega_{r-i+1}. \end{cases}$$

The cases n odd and n even are somewhat different, and we begin with the easier case.

$n =$ **odd:** The conditions to be satisfied by Q are

$$\begin{cases} Q(\omega_j, \omega_k) = 0, & k \neq r-j+1 \\ i^{2p_j - n} Q(\omega_j, \overline{\omega}_j) > 0 & \text{where } k_i = 2p_i - n. \end{cases}$$

We now use the assumption that L is a *CM field*; i.e., it is a *totally real extension of a purely imaginary quadratic number field* L_0. Specifically, since for n odd we have $\dim V = r = 2s$ is even, and we set

$$L_0 = \mathbb{Q}(|\gamma|^2) \subseteq \mathbb{R}$$

where

$$\begin{cases} [L : L_0] = 2 \\ L_0 = \mathbb{Q}(|\gamma_1|^2, \dots, |\gamma_s|^2). \end{cases}$$

We next use the *trace*

$$\operatorname{Tr}_{L/\mathbb{Q}} : L \to \mathbb{Q}$$

defined by

$$\operatorname{Tr}_{L/\mathbb{Q}}(l) = \sum_{i=1}^{r} \eta_i(l).$$

Then the Galois group $\operatorname{Gal}(L/L_0)$ is generated by $\rho : L \to L$ where

$$\eta_i(\rho(l)) = \overline{\eta_i(l)}, \qquad \rho^2 = \text{identity}.$$

Setting for simplicity of notation $\operatorname{Tr} = \operatorname{Tr}_{L/\mathbb{Q}}$, we observe that since the η_i occur in conjugate pairs we have

$$\operatorname{Tr} \circ \rho = \operatorname{Tr}.$$

We may pick $\xi \in L$ such that $\rho(\xi) = -\xi$. Then we claim that

$$Q(a,b) = \operatorname{Tr}(\xi a \rho(b)) \textit{ is } \mathbb{Q}\textit{-bilinear and alternating.}$$

Proof. We have from $\rho(\xi) = -\xi$ that

$$\operatorname{Tr}(\xi a \rho(b)) = \operatorname{Tr}(\rho(\xi a \rho(b))) = -\operatorname{Tr}(\xi \rho(a) b). \qquad \square$$

Next we define the Q-adjoint B^* of $B \in \mathrm{Aut}(V)$ by

$$Q(B^*a,b) = Q(a,Bb), \qquad\qquad a,b \in V.$$

Then one may verify that

$$\begin{cases} A(l)^* \ = A(\rho(l)) \Rightarrow A(\gamma_i)^* = A(\overline{\gamma}_i) \\ A(\gamma_i)^* = A(\overline{\gamma}_i). \end{cases}$$

These relations imply that $A(\gamma)^*$ has the same eigenspaces as $A(\gamma)$, which noting that $Q(\gamma_i,\gamma_j) = 0$ for $i \neq j$ implies that Hodge-Riemann (I) holds for Q. To have Hodge-Riemann (II) for the Hermitian form

$$iQ(a,\bar{b})$$

it is enough to choose the k_j in the right congruence class mod 4; i.e.,

$$k_j \equiv 2p_j - n \pmod 4.$$

This shows that we can obtain many different PHS's for the same L.

n **even:** In this case we cannot use the symmetric form

$$Q(a,b) = \mathrm{Tr}(a\rho(b))$$

because

$$Q(a,a) = \sum_i \varphi_i(a)\varphi_i(\rho(a)) = \sum_i |\varphi_i(a)|^2 > 0$$

is positive definite. To be able to have an indefinite form Q preserved by L, we observe that for $\xi \in L_0$ if we set

$$Q(a,b) = \mathrm{Tr}(\xi a\rho(b))$$

then since $\rho(\xi) = \xi$ the form Q is symmetric and

$$Q(a,a) = \sum_i \varphi_i(\xi)|\varphi_i(a)|^2.$$

We then have the following result from algebraic number theory, for which we refer to [GGK1] for a proof and discussion.

Lemma: *Let $\psi_1,\dots\psi_m$ be the real embeddings $L_0 \hookrightarrow \mathbb{R}$. Assign to each ψ_i a sign $\epsilon_i = \pm 1$. Then we may choose $\xi \in L_0^*$ so that $\frac{\psi_i(\xi)}{|\psi_i(\xi)|} = \epsilon_i$.*

We now proceed in an analogous manner to the odd case by choosing ω_i for $i = 1,\dots,2m$ and $k_{2m-i} = -k_i$ with

$$k_i \equiv \begin{cases} 0 \bmod 4 & \text{if } \epsilon_i = 1 \\ 2 \bmod 4 & \text{if } \epsilon_i = -1 \end{cases}$$

for $i = 1,\dots,m$. This gives a PHS with

$$V^{p,2m-p} = \mathrm{span}_{\mathbb{C}}\left\{\omega_i : k_i = 2p - m\right\}.$$

Lecture 4
Hodge representations and Hodge domains

A natural question is

> *Which reductive, $\mathbb{Q}$-algebraic groups arise as Mumford-Tate groups of a polarized Hodge structure?*

Classically, this question was addressed by starting with a PHS (V, Q, φ) and asking what the possible Mumford-Tate groups are.

For weight one (abelian varieties), the MT domain $D_\varphi \subset \mathcal{H}_g$ is a complex, homogeneous sub-manifold.[18] Thus, D_φ is an Hermitian symmetric domain (HSD),[19] and a century ago E. Cartan classified the equivariant holomorphic embeddings of an irreducible HSD in $\mathcal{H}_g$. The list is quite short, and does not include any HSD's associated to exceptional groups. In the 1960's this subject was revisited by Satake, Shimura, Kuga, Mumford and others putting in arithmetic aspects arising from the Albert classification of the division algebras which might arise as $\mathrm{End}_{\widetilde{\varphi}}(V)$. In higher weight this approach becomes very complicated, as illustrated by the following is the table of possibilities when $n = 3$ and $h^{3,0} = h^{2,1} = 1$. This table was taken from [GGK1]; we will not attempt to explain it but rather offer it as an illustration of the issue.

[18] This is also the case when the weight $n = 2$ and $h^{2,0} = 1$.

[19] Not every bounded homogeneous domain in $\mathbb{C}^N$ is an HSD. However, a homogeneous sub-domain of an HSD is an HSD.

type	unconstrained? /Herm. symm.?	$\mathrm{ht}(M)$	G	$G(\mathbb{R})^0$
(i)	no/no	2	Sp_4	$\mathrm{Sp}_4(\mathbb{R})$
(ii)	no/yes	2	$\mathrm{Res}_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}\,\mathrm{SL}_{2,\mathbb{Q}(\sqrt{d})}$	$\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$
(iii)	yes/yes	2	$U_{\mathbb{Q}(\sqrt{-d})}(V,\mathfrak{Q})$	$\begin{cases} U(1,1)\cong \\ U(1)\times\mathrm{SL}_2(\mathbb{R})\end{cases}$
(iv)	no/no	2	$U_{\mathbb{Q}(\sqrt{-d})}(V,\mathfrak{Q})$	$U(2)$
(v)	yes/yes	4	SL_2	$\mathrm{SL}_2(\mathbb{R})$
(vi)	yes/yes	2	$\mathrm{Res}_{L_0/\mathbb{Q}}\,U_L$	$U(1)\times U(1)$
(vii)	yes/yes	2	$\mathrm{Res}_{L_0/\mathbb{Q}}\,U_L$	$U(1)\times U(1)$
(viii)	no/yes	2	$\mathrm{SL}_2\times\mathrm{SL}_2$	$\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$
(ix)	no/yes	2	$U_{\mathbb{Q}(\sqrt{-d})}\times\mathrm{SL}_2$	$U(1)\times\mathrm{SL}_2(\mathbb{R})$
(x)	yes/yes	2	$U_{\mathbb{Q}(\sqrt{-d})}\times\mathrm{SL}_2$	$U(1)\times\mathrm{SL}_2(\mathbb{R})$
(xi)	yes/yes	2	$U_{\mathbb{Q}(\sqrt{-d'})}\times U_{\mathbb{Q}(\sqrt{-d''})}$	$U(1)\times U(1)$
(xii)	yes/yes	4	$U_{\mathbb{Q}(\sqrt{-d})}$	$U(1)$

Because of the complexity of this list in the first non-classical case and where $\dim V = 4$ is minimal to have D_φ non-classical, it is natural to invert the above question and ask

In how many ways may a given reductive $\mathbb{Q}$-algebraic group G be realized as a Mumford-Tate group?

This translates into a question in representation theory and leads to the following

Definition: *A* **Hodge representation** (V,ρ,φ) *is given by a representation*

$$\rho : G \to \mathrm{Aut}(V)$$

defined over $\mathbb{Q}$, and a circle

$$\varphi : S^1 \to G_{\mathbb{R}}$$

such that there is a non-degenerate form

$$Q : V\otimes V \to \mathbb{Q}$$

preserved by $\rho(G)$ and where $V(Q,\rho\circ\varphi)$ is a polarized Hodge structure.

We have seen above that if G admits a Hodge representation that is injective on the Lie algebra level, then G contains an *anisotropic* maximal torus $T_{\mathbb{Q}}$. This greatly simplifies the representation theory, and we assume it to be the case and denote by $T \subset G_{\mathbb{R}}$ the compact maximal torus in the corresponding real Lie group.

We write

$$T = \mathfrak{t}/\Lambda$$

where $\Lambda \subset \mathfrak{t}$ is a lattice. We may choose coordinates so that

$$\begin{aligned} &\mathfrak{t} \cong \mathbb{R}^r \text{ where } r \text{ is the } \textit{rank} \text{ of } G; \\ &\Lambda \cong \mathbb{Z}^r \end{aligned}$$

so that $T = \{(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r})\}$. A *character*

$$\chi_\lambda : T \to S^1$$

is given by

$$\chi_\lambda(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r}) = e^{2\pi i(n_1\theta_1 + \cdots + n_r\theta_r)}$$

where $\lambda = (n_1, \dots, n_r) \in \operatorname{Hom}(\Lambda, \mathbb{Z})$. This gives an identification of the character group

$$X(T) \cong \operatorname{Hom}(\Lambda, \mathbb{Z}).$$

We denote by $P \subset i\mathfrak{t}^*$ the *weight lattice*. Elements of P are linear forms that take values in $2\pi i\mathbb{Z}$ on Λ. The differential of the above character is a weight.

The *co-characters* $\check{X}(T)$ are given by homomorphisms

$$\varphi : S^1 \to T,$$

and there is an identification

$$\check{X}(T) \cong \operatorname{Hom}(\mathbb{Z}, \Lambda)$$

where $1 \to (l_1, \dots, l_r) =: l_\varphi$ so that for $t \in S^1 = \mathbb{R}/\mathbb{Z}$

$$\varphi(t) = (e^{2\pi i l_1 t}, \dots, e^{2\pi i l_r t}).$$

In first approximation, a Hodge representation is given by the data (λ, l_φ) of a character λ and co-character l_φ where λ is the highest weight of the induced representation

$$\rho_* : \mathfrak{g}_{\mathbb{C}} \to \operatorname{End}(V_{\mathbb{C}})$$

of the complex Lie algebra. Here we are implicitly assuming that G *is semi-simple*, an assumption that we shall make throughout this lecture.[20] Less essential, as we shall see, is the implicit assumption that the representation is absolutely irreducible. Finally, in general there are several real Lie groups with the same real Lie algebra $\mathfrak{g}_{\mathbb{R}}$, indexed by lattices P' with

$$R \subset P' \subset P$$

[20]The extension to the reductive case has yet to be done, and this is a very important piece of the story that remains to be completed.

where R and P are the respective *root* and *weight* lattices $\mathfrak{t}$. This additional piece of data will enter into the final result.

To explain the main result that leads to an answer to the question posed above we need to introduce some notation. Let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition where $\mathfrak{t} \subset \mathfrak{k}$, $\mathfrak{k}$ being the Lie algebra of the unique maximal compact subgroup K of $G_{\mathbb{R}}$ that contains T. We recall that

$$\begin{cases} [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \end{cases}$$

and the *Cartan involution* θ is defined by $\theta = -\operatorname{id}$ on $\mathfrak{p}$, $\theta = \operatorname{id}$ on $\mathfrak{k}$.

We define a map $\psi : R \to \mathbb{Z}/2\mathbb{Z}$ by

$$\psi(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a compact root} \\ 1 & \text{if } \alpha \text{ is a non-compact root.} \end{cases}$$

Since the Cartan involution is a Lie algebra homomorphism, it follows that ψ is a homomorphism. We next define a homomorphism

$$\Psi : R \to \mathbb{Z}/4\mathbb{Z}$$

by $\Psi =$ "2ψ"; i.e., $\Psi(\alpha) = 0$ for compact roots and $\Psi(\alpha) = 2$ for non-compact roots.

Finally, we shall say that an irreducible representation $\rho : G \to \operatorname{Aut}(V)$ *leads to a Hodge representation* if there is a $\rho : S^1 \to G_{\mathbb{R}}$ such that (V, ρ, φ) is a Hodge representation. This means that there exists at least one $Q : V \otimes V \to \mathbb{Q}$ such that $(V, Q, \rho \circ \varphi)$ is a PHS.

Theorem: *Suppose that $\lambda \in P'$. Let δ be the minimal positive integer such that $\delta\lambda \in R$. Then ρ leads to a Hodge representation if, and only if, there exists an integer m such that*

$$\Psi(\delta\lambda) \equiv \delta m \pmod 4.$$

Here, λ is a weight associated to ρ in a manner to be described now. For this we assume that $\rho : G_{\mathbb{R}} \to \operatorname{Aut}(V_{\mathbb{R}})$ is irreducible. The extension from $\mathbb{Q}$ up to $\mathbb{R}$ is described in [GGK1] and will not be discussed here.

By Schur's lemma $\operatorname{End}_{\mathfrak{g}_{\mathbb{R}}}(V_{\mathbb{R}})$ is a division algebra and there are three cases:

$$\operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{R}}) = \begin{cases} \mathbb{R} & \text{(real case)} \\ \mathbb{C} & \text{(complex case)} \\ \mathbb{H} & \text{(quaternionic case).} \end{cases}$$

Then for $\operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{R}}) \otimes \mathbb{C} \cong \operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{C}})$ where $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, we have

$$\operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{C}}) = \begin{cases} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}), \end{cases}$$

where as usual $\mathbb{H}$ are quaternions and $M_2(\mathbb{C})$ denotes the 2×2 matrices with complex entries. Only in the real case do we get a division algebra over $\mathbb{C}$, so $V_{\mathbb{C}}$ is reducible in the other two cases. The analysis of whether there are invariant forms and whether they are symmetric or alternating will necessitate considering the various cases arising from the three possibilities above.

We denote by $\operatorname{Res}_{\mathbb{C}/\mathbb{R}}$ the operation of *restriction of scalars* that considers a vector space over $\mathbb{C}$ to be one over $\mathbb{R}$, and similarly for $\operatorname{Res}_{\mathbb{H}/\mathbb{R}}, \operatorname{Res}_{\mathbb{H}/\mathbb{C}}$.

We can associate to $V_{\mathbb{R}}$ an irreducible representation U of $\mathfrak{g}_{\mathbb{C}}$ over $\mathbb{C}$ such that as representations of $\mathfrak{g}_{\mathbb{C}}$

$$V_{\mathbb{C}} = \begin{cases} U & \text{(real case)} \\ U \oplus U^*, U \ncong U^* & \text{(complex case)} \\ U \oplus U^*, U \cong U^* & \text{(quaternionic case)} \end{cases}$$

and

$$V_{\mathbb{R}} \cong \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(U) \cong \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(U^*) \quad \text{(complex and quaternionic cases)},$$

while

$$V_{\mathbb{R}} \oplus V_{\mathbb{R}} \cong \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(U) \quad \text{(real case)}.$$

Furthermore, in the quaternionic case there is an irreducible representation $\mathbb{U}$ of $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}$ over $\mathbb{H}$ such that

$$V_{\mathbb{R}} \cong \operatorname{Res}_{\mathbb{H}/\mathbb{R}}(\mathbb{U}) \qquad \text{(quaternionic case)}$$

and then

$$U \cong \operatorname{Res}_{\mathbb{H}/\mathbb{C}}(\mathbb{U}) \qquad \text{(quaternionic case)}.$$

Now

$$\operatorname{End}_{\mathfrak{g}_{\mathbb{R}}}(V_{\mathbb{R}}) \text{ acts on } V_{\mathbb{R}} \text{ as } \begin{cases} \mathbb{R} & & \text{(real case)} \\ \mathbb{C} \text{ acting on } & \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(U) & \text{(complex case)} \\ \mathbb{H} \text{ acting on } & \operatorname{Res}_{\mathbb{H}/\mathbb{R}}(\mathbb{U}) & \text{(quaternionic case)}. \end{cases}$$

In the quaternionic case, if $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}$ acts on the left, then $\operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{R}}) \cong \mathbb{H}$ acts on the right.

Before proceeding we recall the basic notions from the theory of semi-simple complex Lie algebras. The general reference for this is [K1].

- $\mathfrak{g}_{\mathbb{C}}$ is a complex, semi-Lie algebra;
- $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ is a Cartan sub-algebra;
- the common eigenspaces of $\operatorname{ad}\mathfrak{h}$ acting on $\mathfrak{g}_{\mathbb{C}}$ are 1-dimensional *root spaces* $\mathfrak{g}^{\alpha}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}^{\alpha} \right)$$

where $\alpha \in \Phi \subset \mathfrak{h}^*$ is a *root* and $\operatorname{ad}\mathfrak{h}$ acts on $\mathfrak{g}^{\alpha}$ by the linear function α:

$$[H, X] = \langle \alpha, H \rangle X \qquad\qquad H \in \mathfrak{h}, X \in \mathfrak{g}^{\alpha};$$

- since the roots are purely imaginary on $\mathfrak{t}$, we have $\Phi \subset i\mathfrak{t}^*$ and

$$\mathfrak{g}^{-\alpha} = \overline{\mathfrak{g}^{\alpha}}$$

where the conjugation is relative to the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{C}}$;
- we may choose *root vectors* $X_\alpha \in \mathfrak{g}^{\alpha}$ and *co-root vectors* $H_\alpha \in \mathfrak{h}$ such that $\overline{X}_\alpha = \pm X_{-\alpha}$

$$\begin{cases} [H_\alpha, X_\alpha] = 2X_\alpha \\ [H_\alpha, X_{-\alpha}] = -2X_\alpha \\ [X_\alpha, X_{-\alpha}] = H_\alpha. \end{cases}$$

Thus $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ span an $\mathrm{sl}_2(\mathbb{C})$ in $\mathfrak{g}_{\mathbb{C}}$;
- if $\alpha, \beta \in \Phi$ then

$$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$$

where

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} = N_{-\beta,\alpha+\beta} = N_{\alpha+\beta,-\alpha}.$$

If $\alpha + \beta$ is not a root, then $[X_\alpha, X_\beta] = 0$ (where we consider 0 as a root);
- the *Cartan-Killing form*

$$B(x, y) = \operatorname{Trace}(\operatorname{ad} x \operatorname{ad} y), \qquad\qquad x, y \in \mathfrak{g}_{\mathbb{C}},$$

is symmetric, non-singular and positive definite on $i\mathfrak{t}$. Therefore it determines an inner product $(\ ,\ )$ on $i\mathfrak{t}$. The hyperplanes $P_\alpha = \{\lambda \in i\mathfrak{t} : (\lambda, \alpha) = 0) : \alpha \in \Phi\}$ divide it into a finite number of closed, convex cones, the *Weyl chambers*. The reflections s_α in the P_α generate the *Weyl group* W, which leaves Φ invariant and permutes the Weyl chambers simply and transitively;
- a system of *positive roots* Φ^+ is a subset of Φ such that
  (i) $\Phi = \Phi^+ \cup \Phi^-$ (disjoint union) where $\Phi^- = -\Phi$;
  (ii) $\alpha, \beta \in \Phi^+ \Rightarrow \alpha + \beta \in \Phi^+$;

- associated to a positive root system Φ^+ is the *dominant Weyl chamber*
$$C = \{\lambda \in i\mathfrak{t} : (\alpha, \lambda) \geqq 0 \ \text{ for } \ \alpha \in \Phi^+\};$$
- the Weyl group acts simply transitively on the sets of positive roots and establishes a bijection
$$\text{Weyl chambers} \longleftrightarrow \text{positive root systems;}$$
- the Cartan-Killing form has the properties
$$\begin{cases} B(X_\alpha, X_\beta) = \delta_{\alpha,-\beta} \\ B(H_\alpha, H) = \langle \alpha, H \rangle\,, \qquad H \in \mathfrak{h}; \end{cases}$$
- a root is *simple* if it is not a non-trivial sum of roots. Given a set of positive roots there is determined a set $\alpha_1, \dots, \alpha_r$ of simple, positive roots such that the P_{α_i} form the walls of the corresponding Weyl chamber;
- the *weight lattice* P is defined as the set of $\lambda \in i\mathfrak{t}$ such that
$$\langle \lambda, H_\alpha \rangle \in \mathbb{Z}, \qquad\qquad \alpha \in \Phi;$$
- the restriction to $\mathfrak{h}$ of an irreducible representation
$$r : \mathfrak{g}_\mathbb{C} \to \mathrm{End}(V_\mathbb{C})$$
decomposes $V_\mathbb{C}$ into *weight spaces*
$$V_\omega = \{v \in V_\mathbb{C} : r(H)v = \langle \omega, H \rangle\, v \ \text{ for } \ H \in \mathfrak{h}\};$$
- there is a unique *highest weight* λ characterized by
$$r(X_\alpha)V_\lambda = 0 \ \text{ for } \ \alpha \in \Phi^+$$
and $V_\lambda = \mathbb{C}v_\lambda$ where v_λ is a highest weight vector.

Step one: We let λ be the highest weight of U. There is a unique element w_0 of the Weyl group such that $w_0(\Phi^+) = \Phi^-$. It is known that U has an $\mathfrak{g}_\mathbb{C}$-invariant bilinear form if, and only if, $w_0(\lambda) = -\lambda$. By Schur's Lemma, this is non-degenerate, unique up to a constant, and either alternating or symmetric.

If $\alpha_1, \dots \alpha_\mathfrak{r}$ are a choice of simple positive roots for $\mathfrak{g}_\mathbb{C}$ and H_{α_i} are the co-roots, let
$$h^0 = \sum_i H_{\alpha_i}.$$

Then

> *The universal bilinear form is symmetric/alternating depending on whether $\langle \lambda, h^0 \rangle$ is even/odd.*

Further,

$$\langle \alpha_i, h^0 \rangle = 2 \text{ for all } i.$$

Step two: If we write the decomposition into weight spaces

$$U = \bigoplus_{\omega} U_\omega$$

then

$$U^* = \bigoplus_{\omega} U_\omega^*, \qquad U_\omega^* \text{ has weight } -\omega.$$

On $V_{\mathbb{C}} \cong V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ conjugation gives an isomorphism $V_{\mathbb{C}} \xrightarrow{c} V_{\mathbb{C}}$ that gives a natural isomorphism of vector spaces $U_\omega \xrightarrow{c_\omega} U_\omega^*$. In the complex and quaternionic cases, $V_{\mathbb{C}} \cong U \oplus U^*$ and c is c_ω on U_ω and c_ω^{-1} on U_ω^*. Now

$$\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}}(S^2 V_{\mathbb{R}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(S^2 V_{\mathbb{C}}, \mathbb{C})$$

and thus

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}}(S^2 V_{\mathbb{R}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \begin{cases} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(S^2 U, \mathbb{C}) & \text{(real case)} \\ \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(S^2 U, \mathbb{C}) \oplus \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(U \otimes U^*, \mathbb{C}) \oplus \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(S^2 U^*, \mathbb{C}) & \text{(complex and quaternionic cases).} \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{g}_{\mathbb{R}}}(\Lambda^2 V_{\mathbb{R}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \begin{cases} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(\Lambda^2 U, \mathbb{C}) & \text{(real case)} \\ \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(\Lambda^2 U, \mathbb{C}) \oplus \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(U \otimes U^*, \mathbb{C}) \oplus \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(\Lambda^2 U^*, \mathbb{C}) & \text{(complex and quaternionic cases).} \end{cases} \end{aligned}$$

Thus:

Real case: There is a unique (up to a constant) invariant bilinear form on $V_{\mathbb{R}}$, symmetric/alternating depending on the parity of $\langle \lambda, h^0 \rangle$.

Complex case: There are unique (up to constants) symmetric invariant bilinear and alternating invariant bilinear forms on $V_{\mathbb{R}}$.

Quaternionic case: There are unique (up to constants) symmetric invariant bilinear and alternating invariant bilinear forms Q on $V_{\mathbb{R}}$ that pair U and U^* so that $Q(v, \overline{v})$ is non-degenerate on $V_{\mathbb{C}}$.

Step three: We recall our notations from above: R is the root lattice of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$, and the real form $\mathfrak{g}_{\mathbb{R}}$ has a Cartan involution $\mathfrak{g}_{\mathbb{R}} \xrightarrow{\theta} \mathfrak{g}_{\mathbb{R}}$ where

$$\theta = \begin{cases} 1 & \text{on } \mathfrak{k} \\ -1 & \text{on } \mathfrak{p}. \end{cases}$$

We have the map

$$R \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z}$$

where

$$\psi(\alpha) = \begin{cases} 0 & \text{if } X_\alpha \in \mathfrak{k} \\ 1 & \text{if } X_\alpha \in \mathfrak{p}, \end{cases} \qquad \alpha \in \Phi.$$

As noted above, since θ is a Lie algebra homomorphism ψ extends uniquely from Φ to R as a group homomorphism. Then

$$R \xrightarrow{\Psi} \mathbb{Z}/4\mathbb{Z}$$

is defined by

$$\begin{cases} \Psi(x) = 2 & \text{if, and only if, } \psi(x) = 1 \\ \Psi(x) = 0 & \text{if, and only if, } \psi(x) = 0. \end{cases}$$

Step four: Associated to $\mathfrak{g}_{\mathbb{R}}$ are connected Lie groups $G_{P'}$ for each lattice P' with

$$P \supseteq P' \supseteq R, \qquad R = \text{root lattice}, \ P = \text{weight lattice}$$

where

$$\pi_1(G_{P'}) \cong P/P', \quad Z(G_{P'}) \cong P'/R.$$

Note that

$$\begin{aligned} &G_R = G_a \ \text{adjoint form}, \ G_a \overset{\mathrm{Ad}}{\hookrightarrow} \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}) \\ &G_P = G_s \ \text{simply connected form}, \ \pi_1(G_s) = 0. \end{aligned}$$

The maximal torus T of $G_{P'}$ is

$$T = \mathfrak{t}/\Lambda, \quad \Lambda \cong \mathrm{Hom}(P', \mathbb{Z}).$$

In order to have U defined on $G_{P'}$, we need $\lambda \in P'$.

Step five: The weights that occur for U belong to $\lambda + R$, and

$$\mathrm{span}_{\mathbb{Z}}(\text{weights of } U) = \mathbb{Z}\lambda + R.$$

Note that $\lambda \in R \otimes_{\mathbb{Z}} \mathbb{Q}$, so this is not a direct sum. Let

$$P' = \mathbb{Z}\lambda + R, \quad \Lambda = \mathrm{Hom}(P', \mathbb{Z})$$

and $\mathfrak{l}_\varphi \in \Lambda$ be the lattice point such that the line $\mathbb{R}\mathfrak{l}_\varphi$ projects in $T \subset G_{\mathbb{R}}$ to give the circle $\varphi(S^1)$.

The key computation that must be done is:

Let $\mathbb{Z}\lambda + R \xrightarrow{\mathfrak{l}_\varphi} \mathbb{Z}$ *project to* $\mathbb{Z}\lambda + R \xrightarrow{\tilde{\mathfrak{l}}_\varphi} \mathbb{Z}/4\mathbb{Z}$. *Then* $\mathfrak{l}_\varphi$ *gives a polarized Hodge structure for* Q *or* $-Q$ *if, and only if,*

$$\begin{cases} \tilde{\mathfrak{l}}_\varphi|_R = \Psi \\ \tilde{\mathfrak{l}}_\varphi(\lambda) \ \text{even/odd} \quad \text{if, and only if, } Q \text{ symmetric/alternating.} \end{cases}$$

Step six: In the complex and quaternionic cases, there exist both symmetric and alternating Q's, so the parity of $\tilde{\mathfrak{l}}_\varphi(\lambda)$ can always be matched.

To deal with the real case, one needs an additional result. In the real case, $w_0(\lambda) = -\lambda$. Since for any element w of the Weyl group and any $\lambda \in P$, we have $w(\lambda) \equiv \lambda \bmod R$, it follows that $\lambda - w_0(\lambda) \in R$, and consequently $2\lambda \in R$. Write

$$2\lambda = \sum_{i=1}^{r} m_i \alpha_i, \qquad m_i \in \mathbb{Z}$$

where $\alpha_1, \dots, \alpha_r$ are the simple positive roots. Then it may be shown, and this is the crucial step to which we refer to [GGK1] for the proof, that we are in the

- real case if, and only if, $\sum_{\psi(\alpha_i)=0} m_i$ is even
- quaternionic case if, and only if, $\sum_{\psi(\alpha_i)=0} m_i$ is odd.

Now

$$\tilde{\mathfrak{l}}_\varphi(\lambda) = \frac{1}{2}\Sigma m_i \Psi(\alpha_i) = \sum_{\psi(\alpha_i)=1} m_i$$

$$\langle \lambda, h^0 \rangle = \left\langle \tfrac{1}{2} \sum_i m_i \alpha_i, h^0 \right\rangle = \sum_i m_i = \tilde{\mathfrak{l}}_\varphi(\lambda) + \sum_{\psi(\alpha_i)=0} m_i.$$

In the real case, this implies

$$\langle \lambda, h^0 \rangle \equiv \tilde{\mathfrak{l}}_\varphi(\lambda) \ (\mathrm{mod}\, 2),$$

and thus in the real case

$$Q \ \text{ is symmetric if, and only if, } \tilde{\mathfrak{l}}_\varphi(\lambda) \text{ is even}$$
$$Q \ \text{ is alternating if, and only if, } \tilde{\mathfrak{l}}_\varphi(\lambda) \text{ is odd.}$$

We then have:

In all cases — real, complex, quaternionic — for an appropriate choice of invariant Q,

$\mathfrak{l}_\varphi$ *gives a polarized Hodge structure if, and only if,* $\tilde{\mathfrak{l}}_\varphi|_R = \Psi$.

Step seven: Since the weights of $V_{\mathbb{C}}$ belong to the $\lambda + R$, we have that $\varphi(z)$ acts on V_ω as $z^{\mathfrak{l}_\varphi(\omega)}$.[21] Thus

$$V_\omega \subset V^{p,q}, \ \text{ where } \ p-q = \mathfrak{l}_\varphi(\omega).$$

The weight n of the PHS must satisfy

$$\begin{cases} n \geq \max \mathfrak{l}_\varphi(\omega), & \omega \text{ a weight of } V_{\mathbb{C}} \\ n \equiv \bar{\mathfrak{l}}(\lambda) \text{ mod } 2.^{22} \end{cases}$$

Once such a weight n is chosen,

$$V_\omega \subset V^{p,q} \ \text{ where } \ p = \frac{n + \mathfrak{l}_\varphi(\omega)}{2}, \quad q = \frac{n - \mathfrak{l}_\varphi(\omega)}{2}.$$

At this stage the analysis proceeds by considering the action on V_ω of an sl_2 generated by $H_\alpha, X_\alpha, X_{-\alpha}$.

Step eight: It is possible to compute ψ, and hence Ψ, using the *Vogan diagram.*[23] For the compact form, $\psi = 0$. For other real forms, using $\alpha_1, \dots, \alpha_r$ to denote the simple positive roots corresponding to the Dynkin diagram,

$$\psi(\alpha_i) = \begin{cases} 1 & \text{if node } i \text{ is "painted" in the Vogan diagram} \\ 0 & \text{if node } i \text{ is "unpainted" in the Vogan diagram.} \end{cases}$$

The existence of a compact maximal torus is equivalent to the Vogan diagram being "non-folded."

Example: We shall illustrate the above in the simplest case when $V = \mathbb{Q}^2$ thought of as column vectors with the bilinear form $Q(u,v) = {}^t vQu$ where $Q = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In this case the group is SL_2 with maximal torus $T = \mathrm{SO}(2)$ given by $\left\{\left(\begin{smallmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{smallmatrix}\right)\right\}$. Thus, identifying $\mathfrak{t} \cong \mathbb{R}$ with coordinate θ, we have $T \cong \mathbb{R}/\mathbb{Z}$. We set $H = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $H_l = lH$ and will check by linear algebra that

$\exp(-i \log z H_l)$ *gives a polarized Hodge structure if, and only if,*

$$l \equiv 1 \ (\mathrm{mod}\, 4).$$

Here we are thinking of $z = e^{2\pi i \xi} \in S^1 = \mathbb{R}/\mathbb{Z}$ so that for $l = 1$, $z \to \exp(-i \log zH)$ gives the circle S^1 in $\mathrm{SL}_2(\mathbb{R})$.

The eigenvectors and eigenvalues of H are given by setting

$$v_+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad v_- = \begin{pmatrix} 1 \\ i \end{pmatrix} = \overline{v}_+,$$

[21] We use the notation $\mathfrak{l}_\varphi(\omega)$ for the pairing $\langle \omega, \mathfrak{l}_\varphi \rangle$ between $\mathfrak{h}^*$ and $\mathfrak{h}$.

[22] We are here assuming that the HS is $V_{\mathbb{C}} = V^{n,0} \oplus V^{n-1,1} \oplus \cdots \oplus V^{0,n}$; i.e., for all non-zero $V^{p,q}$ we have $p \geqq 0, q \geqq 0$.

[23] This will be discussed in Mark Green's lecture.

and then

$$Hv_\pm = \pm i v_\pm.$$

We note that

$$Q(v_+, \overline{v}_+) = -2i$$

so that

$$\begin{cases} iQ(v_+, \overline{v}_+) > 0 \\ i^3 Q(v_-, \overline{v}_-) > 0. \end{cases}$$

Since Q is alternating, the weight n must be odd. The only possible Hodge decompositions are

$$V_{\mathbb{C}} = V^{n,0} \oplus V^{0,n}$$

where $V^{n,0} = V_\pm$. Thus $n = l$ and the bilinear relation

$$i^l Q(v, \overline{v}) > 0 \qquad v \in V^{n,0}$$

gives

$$\begin{cases} l \equiv 1 \pmod 4 & V^{n,0} = V_+ \\ -l \equiv 3 \pmod 4 & V^{n,0} = V_-. \end{cases}$$

The second is redundant, so that we have confirmed the italicized statement above.

For the root-weight approach to the computation, since the roots are purely imaginary it is more convenient notationally to set

$$h = -iH = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The root spaces are then the spans of

$$\begin{cases} X = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right) \\ Y = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) = \overline{X}. \end{cases}$$ [24]

Then

$$\begin{cases} [h, X] = 2X \\ [h, Y] = -2Y \\ [X, Y] = h \end{cases}$$

and

$$\begin{cases} h \cdot v_+ = v_+ \\ X \cdot v_+ = 0. \end{cases}$$

---

[24]For the polarized Hodge structure on $\mathrm{sl}_{2,\varphi}$ where $\varphi = i \in \mathcal{H}$, we have $\mathrm{sl}_{2,\varphi}^{(0,0)} = \mathbb{C}H, \mathrm{sl}_{2,\varphi}^{-1,1} = \mathbb{C}X$ and $\mathrm{sl}_{2,\varphi}^{1,-1} = \mathbb{C}Y$.

This gives us that, identifying $i\mathfrak{t}$ with $\mathbb{R}$ where $h \leftrightarrow 1$, the weight and root lattices are

$$\begin{aligned} &P \cong \mathbb{Z} \\ &\cup \\ &R \cong 2\mathbb{Z}. \end{aligned}$$

Moreoever, the standard representation of SL_2 on $\mathbb{Q}^2$ has highest weight 1. Thus, in the above notations we have

- $U = \mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-$
- $\langle \lambda, h \rangle = 1$ and v_+ is the highest weight vector
- $\langle \alpha, h \rangle = 2$ where $[h, X] = 2X$
- $\psi(\alpha) = 1$, $\Psi(\alpha) = 2$.

Setting $l_\varphi = lh$, $l_\varphi(\alpha) = 2l$. Thus the condition $\tilde{l}_\varphi|_R = \Psi$ on the map $\mathbb{Z}\lambda + R \to \mathbb{Z}/4\mathbb{Z}$ is

$$2l = \langle \alpha, l_\varphi \rangle \equiv 2 \ (\mathrm{mod}\, 4).$$

This is exactly the condition that l_φ give a polarized Hodge structure for $\pm Q$ ($+Q$ when $l \equiv 1 \ (\mathrm{mod}\, 4)$, $-Q$ when $l \equiv 3 \ (\mathrm{mod}\, 4)$).

- The list of non-compact real forms that admit Hodge representations is

$$\begin{array}{ll} A_r & \mathrm{su}(p,q),\ p+q = r+1,\ \mathrm{sl}(2,\mathbb{R}) \\ B_r & \mathrm{so}(2p, 2q+1),\ p+q = r \\ C_r & \mathrm{sp}(p,q),\ p+q = r \\ D_r & \mathrm{so}(2p,2q),\ p+q = r,\ \mathrm{so}^*(2r) \\ E_6 & EII, EIII \\ E_7 & EV, EVI, EVII \\ E_8 & EVIII, EIX \\ F_4 & FI, FII \\ G_2 & G. \end{array}$$

Missing are $\mathrm{sl}(m,\mathbb{R})$, $m \geqq 3$, $\mathrm{sl}(m,\mathbb{H}), EI, EIV$. Those with the more rare odd weight Hodge representations are

$$\begin{array}{l} \mathrm{su}(2p,2q),\ p+q \equiv 0(2) \\ \mathrm{su}(2k+1, 2l+1) \\ \mathrm{so}(4p+2, 2q+1),\ \mathrm{so}^*(4k) \\ \mathrm{sp}(2n,\mathbb{R}) \\ EV \ \text{ and } \ EVII. \end{array}$$

- The passage from real forms to $\mathbb{Q}$-forms is greatly simplified by the assumption that $M \supset T$, which implies that the roots are purely imaginary on $\mathfrak{t}$. It does require the assumption that M be absolutely simple. We refer to [GGK1] for details.
- There is also a classification of which M have *faithful* Hodge representations. There are a few simple groups that have Hodge representations but none that are faithful. We again refer to [GGK1] for details.

The adjoint representation

We recall our notation of a maximal compact subgroup $K \subset G_{\mathbb{R}}$ with $T \subset K$. Then we have the Cartan decomposition

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{t} \subset \mathfrak{k}$, and the standard bracket relations

$$\begin{cases} [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \end{cases}$$

hold. We will denote by $\alpha_1, \dots, \alpha_d$ the roots of T belonging to $\mathfrak{k}$ (the *compact roots*), by $\beta_1, \dots, \beta_e$ the roots of T belonging to $\mathfrak{p}$ (the *non-compact roots*). B denotes the Cartan-Killing form. The basic observations are

(i) the representation $\mathrm{Ad} : M \to \mathrm{Aut}(\mathfrak{g}, B)$ preserves the symmetric form B;

(ii) B is negative on the compact root spaces $\mathfrak{g}^{\alpha_j}$ and is positive on the non-compact root spaces $\mathfrak{g}^{\beta_k}$.

This means that $B < 0$ on $(\mathfrak{g}^{\alpha_j} \oplus \mathfrak{g}^{-\alpha_j}) \cap \mathfrak{k} = (\mathfrak{g}^{\alpha_j} \oplus \mathfrak{g}^{-\alpha_j})_{\mathbb{R}}$, and $B > 0$ on $(\mathfrak{g}^{\beta_k} \oplus \mathfrak{g}^{-\beta_k}) \cap \mathfrak{p} = (\mathfrak{g}^{\beta_k} \oplus \mathfrak{g}^{-\beta_k})_{\mathbb{R}}$. Thus, *both the issue of an invariant form and the signs of the form on eigenspaces are determined in this case.*

We consider a co-character

$$\varphi : S^1 \to T$$

given by

$$\varphi(z) = \left(z^{l_1}, \dots, z^{l_r}\right)$$

where $\mathfrak{l}_{\varphi} = (l_1, \dots, l_r) \in \mathrm{Hom}(\mathbb{Z}, \Lambda)$. As before, we identify $\mathfrak{l}_{\varphi}$ with $\mathfrak{l}_{\varphi}(1) \in \Lambda$.

Proposition: *φ gives a polarized Hodge structure on $(\mathfrak{g}, B)$ if, and only if*

$$\begin{cases} \langle \alpha_j, \mathfrak{l}_{\varphi} \rangle \equiv 0 \pmod 4 \\ \langle \beta_k, \mathfrak{l}_{\varphi} \rangle \equiv 2 \pmod 4. \end{cases}$$

Proof. Since B is symmetric, the weight $n = 2n'$ must be even. In fact, by tensoring with a Tate twist $\mathbb{Q}(-n')$ we may assume that $n = 0$. Because, as previously noted

$$\begin{aligned}\mathfrak{k}_{\mathbb{C}} &= \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_j \mathfrak{g}^{\alpha_j}\Big) = \bigoplus_i \mathfrak{g}^{-2i,2i}\\ \mathfrak{p}_{\mathbb{C}} &= \bigoplus_j \mathfrak{g}^{\beta_j} = \bigoplus_i \mathfrak{g}^{-2i-1,2i+1},\end{aligned}$$

the conditions in the proposition exactly mean that the form $Q = -B$ satisfies the second Hodge-Riemann bilinear relations. $\square$

Remarks: (i) The Lie algebra $\mathfrak{h}_\varphi$ of the isotropy group is given by

$$\mathfrak{h}_\varphi = \mathfrak{t} \oplus \bigoplus_{\left\{\substack{\langle\alpha_j,\mathfrak{l}_\varphi\rangle=0\\ \alpha_j\in\Phi_c^+}\right.} (\mathfrak{g}^{\alpha_j} \oplus \mathfrak{g}^{-\alpha_j})_{\mathbb{R}}.$$

We note the inclusion $\mathfrak{h}_\varphi \subset \mathfrak{k}$, consistent with the fact that H_φ is compact.

(ii) We have a map

$$\Lambda/4\Lambda \xrightarrow{(\alpha_1,\dots,\alpha_d,\beta_1,\dots,\beta_e)} \Big(\bigoplus_{\frac{1}{2}(\dim\mathfrak{k}-r)} \mathbb{Z}/4\mathbb{Z}\Big) \oplus \Big(\bigoplus_{\frac{1}{2}(\dim\mathfrak{p})} \mathbb{Z}/4\mathbb{Z}\Big)$$

where $r = \dim T$ is the rank, and the conditions in the proposition are conditions on this map.

The reason that all the congruences are "mod 4" is of course that $i^4 = 1$; more specifically

- the 2^{nd} bilinear relations are $i^{p-q}Q(v,\bar{v}) > 0$ for $0 \neq v \in V^{p,q}$;
- the $V^{p,q}$ are eigenspaces V_m with eigenvalues mi for the action of the differential $\mathfrak{l}_\varphi = (l_1,\dots,l_r)$ of φ;
- thus on the one hand $p - q = m$, so that i^{p-q} depends only on $m \pmod 4$, while on the other hand for the adjoint representation the V_m are direct sums of root spaces $\mathfrak{g}_{\alpha_j}, \mathfrak{g}_{\beta_k}$ so that the m's above are given by $m = \langle\alpha_j, \mathfrak{l}_\varphi\rangle$, $m = \langle\beta_k, \mathfrak{l}_\varphi\rangle$.

$\mathbf{G_2}$**:**

We will determine the Hodge representations for the exceptional Lie group G_2. In $V = \mathbb{Q}^7$ with basis $e_1,\dots,e_7$ we set

$$\omega = (e_1\wedge e_4 + e_2\wedge e_3 + e_3\wedge e_6)\wedge e_7 - 2e_1\wedge e_2\wedge e_3 + 2e_4\wedge e_5\wedge e_6.$$

Then one characterization of G_2 is

$$G_2 = \operatorname{Aut}(V,\omega).$$

We will proceed in several steps.

Step 1: Make a change of basis

$$\begin{aligned} u_1 &= e_1 - e_4, & v_1 &= e_1 + e_4 \\ u_2 &= e_2 - e_5, & v_2 &= e_2 + e_5 \\ u_3 &= e_3 - e_6, & v_3 &= e_3 + e_6 \\ & & v_4 &= e_7. \end{aligned}$$

Then

$$\begin{aligned} 4\omega = &- u_1 \wedge u_2 \wedge u_3 + u_1 \wedge (v_1 \wedge v_4 - v_2 \wedge v_3) + u_2 \wedge (v_2 \wedge v_4 - v_3 \wedge v_1) \\ &+ u_3 \wedge (v_3 \wedge v_4 - v_1 \wedge v_2). \end{aligned}$$

Step 2: Define $Q(X, Y) = (X \rfloor \beta) \wedge (Y \rfloor \beta) \wedge \beta$ where $\beta = -4\omega$.

In terms of the basis $u_1, u_2, u_3, v_1, \ldots, v_4$,

$$Q = \begin{pmatrix} -I_3 & 0 \\ 0 & I_4 \end{pmatrix}.$$

Step 3: $\mathfrak{g}_{2,\mathbb{R}} \subset \mathrm{so}(4,3)$ is defined by infinitesimally preserving β. If

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{}^{3} & \overbrace{}^{4} \end{array} \\ \begin{array}{c} 3\{ \\ 4\{ \end{array} & \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix} \end{array}, \qquad \text{where } A = -{}^tA, C = -{}^tC \text{ is an element of } \mathrm{so}(4,3),$$

then the equations to preserve β are

$$\begin{aligned} a_{12} &= c_{12} + c_{43} & b_{14} &= b_{32} - b_{23} \\ a_{23} &= c_{23} + c_{41} & b_{24} &= b_{13} - b_{31} \\ a_{31} &= c_{31} + c_{42} & b_{34} &= b_{21} - b_{12} \\ & & b_{11} &+ b_{22} + b_{33} = 0. \end{aligned}$$

Step 4: Note that if $E = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$H_1 = \left(\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad H_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{array}\right)$$

satisfy the equations of Step 3, mutually commute, and $\exp(tH_1)$, $\exp(tH_2)$ are circles in $G_2(\mathbb{R})$ with period $2\pi i$. They commute and span a maximal torus T. The exponentials of $2\pi i$ times their real linear combinations give a torus T in $G_2(\mathbb{R})$, which must then be a maximal torus since G_2 has rank two.

Proposition: *The co-character φ whose differential is $\mathfrak{l}_\varphi = l_1H_1+l_2H_2$ gives a polarized Hodge structure for* **every** *representation of G_2 if, and only if, the conditions*

$$\begin{cases} l_1 \equiv 0 \pmod 4 \\ l_2 \equiv 2 \pmod 4 \end{cases}$$

are satisfied.

Proof. We first show that the standard representation of G_2 on $V \cong \mathbb{Q}^7$ with Q as in Step 2, has a polarized Hodge structure. For this we think of $V_{\mathbb{R}}$ as column vectors and let

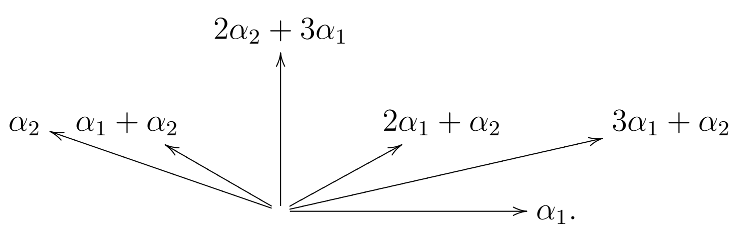
$$\begin{cases} V^- = \text{column vectors } \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ where } Q < 0 \\ \\ V^+ = \text{column vectors } \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \end{pmatrix} \text{ where } Q > 0. \end{cases}$$

Then $l_1H_1 + l_2H_2$ has

$$\begin{cases} \text{eigenvalues } \pm l_1 i, 0 & \text{on } V^- \\ \text{eigenvalues } \pm (l_1+l_2)i, \pm l_2 i & \text{on } V^+. \end{cases}$$

This gives a polarized Hodge structure if, and only if, $l_1 \equiv 0 \pmod 4$, $l_2 \equiv 2 \pmod 4$ and $l_1 + l_2 \equiv 2 \pmod 4$. The third condition is a consequence of the first two, which are just the conditions in the proposition.

At this point we recall the root diagram of $\mathfrak{g}_2$ with positive roots



For this choice the co-roots are

$$\begin{cases} H_{\alpha_1} = H_1 \\ H_{\alpha_2} = H_2 - H_1. \end{cases}$$

Then the dominant weights of the irreducible $\mathfrak{g}_{2,\mathbb{C}}$-modules are linear combinations

$$\lambda = m_1\lambda_1 + m_2\lambda_2,$$

where m_1, m_2 are non-negative integers, and where

$$\begin{cases} \lambda_1 = 2\alpha_1 + \alpha_2 \\ \lambda_2 = 3\alpha_1 + 2\alpha_2. \end{cases}$$

The standard representation has highest weight λ_1, corresponding to the co-weight $H_1 + H_2$. The adjoint representation has highest weight $\lambda_2 = 3\alpha_1 + 2\alpha_2$. It follows that the representation with highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2$ occurs in $S^{m_1}V \otimes S^{m_2}\mathfrak{g}_2$, and hence has a polarized Hodge structure when the conditions in the proposition are satisfied. $\square$

Hodge domains

In this section G will be a reductive $\mathbb{Q}$-algebraic group, not necessarily semi-simple (e.g., $\mathcal{U}(m,n)$). We assume that $G_{\mathbb{R}}$ contains a compact maximal torus T, meaning that the Lie algebra

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_{a,\mathbb{R}} \oplus \mathfrak{A}$$

where $\mathfrak{g}_a$ is the Lie algebra of the adjoint group and where

$$\mathfrak{t} = \mathfrak{t} \cap \mathfrak{g}_{a,\mathbb{R}} \oplus \mathfrak{A}.$$

Writing $T = \mathfrak{t}/\Lambda$, for a given circle

$$\varphi : S^1 \to T$$

given by $l_\varphi \in \Lambda$, we have seen that there may be many representations

$$\rho : G \to \mathrm{Aut}(V, Q)$$

such that $(V, Q, \rho \circ \varphi)$ is a PHS. Setting

$$H = Z_{G_{\mathbb{R}}}(\varphi(S^1)),$$

the same homogeneous complex manifold $D = G_{\mathbb{R}}/H$ therefore appears in many different ways as a Mumford-Tate domain.

Definition: *A* **Hodge domain** *is a homogeneous complex manifold*

$$D = G_{\mathbb{R}}/H$$

where $H = Z_{G_{\mathbb{R}}}(\varphi(S^1))$ and where G admits a Hodge representation $(V, Q, \rho \circ \varphi)$.

We emphasize that the data $\varphi : S^1 \to T$ is part of the definition of a Hodge domain — there will be many such circles in T with the same centralizer. A more precise but less agreeable notation would be (G, φ) consisting of a reductive $\mathbb{Q}$-algebraic group G and a co-character $\varphi : S^1 \to T$ for the maximal torus of $G_{\mathbb{R}}$ containing $\varphi(S^1)$.

Example: We have seen in Lecture 3 that the homogeneous complex manifold

$$D = \mathcal{U}(2,1)_{\mathbb{R}}/T$$

is a Mumford-Tate for PHS's of weights $n = 4, 3$. There we also saw that the homogeneous complex manifold

$$\widetilde{D} = S\mathcal{U}(2,1)_{\mathbb{R}}/T_S,$$

where $T_S = T \cap S\mathcal{U}(2,1)_{\mathbb{R}}$ is a Mumford-Tate domain for PHS's of weight $n = 2$. We note that D and $\widetilde{D}$ are the same as complex manifolds but are not the same as homogeneous complex manifolds. In this case the groups $\mathrm{Pic}_h(D), \mathrm{Pic}_h(\widetilde{D})$ of equivalence classes of homogeneous line bundles are quite different (cf. [GGK2] for details).

We have noted above that a Hodge domain $D = G_{\mathbb{R}}/H$ is associated to the data (G, φ). Since by definition there is at least one PHS $(V, Q, \rho \circ \varphi)$ for a representation $\rho : G \to \mathrm{Aut}(V)$, it follows that there is a PHS on $\mathfrak{g}$ where

$$\mathrm{Ad}\,\varphi : S^1 \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$$

gives the circle. We thus have

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_i \mathfrak{g}^{-i,i}$$

and the infinitesimal period relation (IPR) is given by the $G_{\mathbb{R}}$-invariant distribution

$$W \subset TD$$

where $W = G_{\mathbb{R}} \times_H \mathfrak{g}^{-1,1}$. The IPR is independent of the representation ρ, and thus depends only on the data (G, φ); i.e.,

The IPR is an invariant of the Hodge domain.

Examples (cf. [GGK1] for details)**:** We consider two examples for G_2:
(A) $l_1 = 4, l_2 = -2$.

Then the standard representation gives a PHS of weight $n = 4$ and with Hodge numbers

$$h^{4,0} = 1,\ h^{3,1} = 2,\ h^{2,2} = 1.$$

For the adjoint representation we may see that

$$\mathfrak{g}_2^{-1,1} = \mathrm{span}\left\{X_{\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}\right\}.$$

The corresponding Hodge domain D_a has dimension five and

$$W \subset TD_a$$

is a field of 2-planes. We claim that

W is bracket generating.

Proof. From
$$[X_{\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}] = aX_{-\alpha_1}, \qquad a \neq 0,$$
we see that the bracket is non-trivial and
$$W + [W, W] = \operatorname{span}\{X_{\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}, X_{-\alpha_1}\}.$$
Then from
$$\begin{cases} [X_{-\alpha_1}, X_{\alpha_1+\alpha_2}] = bX_{\alpha_2}, & b \neq 0 \\ [X_{-\alpha_1}, X_{-2\alpha_1-\alpha_2}] = cX_{-3\alpha_1-\alpha_2}, & c \neq 0 \end{cases}$$
we see that $W + [W, W] + [W[W, W]] = \underset{i>0}{\oplus} \mathfrak{g}_2^{-i,i}$. □

(B) $l_1 = 0, l_2 = 2$**.**

For the standard representation we obtain a PHS of weight $n = 2$ and Hodge numbers
$$h^{2,0} = 2,\ h^{1,1} = 3.$$
For the adjoint representation one finds that
$$\mathfrak{g}_2^{-1,1} = \operatorname{span}\{X_{-3\alpha_1-\alpha_2}, X_{-2\alpha_1-\alpha_2}, X_{-\alpha_1-\alpha_2}, X_{-\alpha_2}\}.$$
The matrix of brackets is
$$\begin{matrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{matrix}$$
where each $*$ is an $aX_{-3\alpha_1-2\alpha_2}$, $a \neq 0$. This says that

W defines a contact structure.

But there is much more geometry here. *The infinitesimal variation of Hodge structure*[25] gives a map
$$\operatorname{Sym}^2 W \to \operatorname{Sym}^2 V^{0,2},$$
the dual of which defines three quadrics in $\mathbb{P}W^*$. The common zeroes of these quadrics give a twisted cubic curve
$$C \subset \mathbb{P}W.$$

Historical remark: In his famous 1905 "Five variables" paper Elie Cartan gave two realizations of G_2 as the group of symmetries of a 5-manifold M in which there was a "Cartan geometry" in TM. These examples were (A) and (B), the Cartan geometry being the bracket generating field of 2-planes in (A), and the contact structure with a field of twisted cubic curves in the contact planes in (B).

[25] This will be discussed in the lectures by Jim Carlson.

(C)

Again we take the standard representation and $l_1 = 4$, $l_2 = 2$. This gives a PHS of weight $n = 6$ and with all Hodge numbers

$$h^{p,6-p} = 1.$$

Recently it has been shown ([KP]) that some points of D_C are “motivic” in the sense that they arise from part of the Hodge structure on the cohomology of a projective algebraic variety. A consequence of their work is that

G_2 is a motivic Mumford-Tate group.

Lecture 5
Discrete series and $\mathfrak{n}$-cohomology

Introduction

In this section

- $G_{\mathbb{R}}$ will be a real, semi-simple Lie group containing a compact maximal torus T.

Essentially everything we will discuss will hold in case $G_{\mathbb{R}}$ is reductive, and in fact we will use these results in one of two running examples when $G_{\mathbb{R}} = \mathcal{U}(2,1)_{\mathbb{R}}$. Our main interest will be in the case when $G_{\mathbb{R}}$ is the real Lie group associated to a $\mathbb{Q}$-algebraic group G. Throughout we assumed fixed a maximal compact subgroup K with $T \subset K \subset G_{\mathbb{R}}$. We also assume that $G_{\mathbb{R}}$ is connected as a real Lie group. Thus for every weight PHS's (V, Q, φ) we assume given an orientation of $V_{\mathbb{R}}$.

- $\Gamma \subset G_{\mathbb{R}}$ will be a discrete subgroup.

Unless mentioned otherwise we shall assume that Γ is *co-compact* and *neat*. Although the main eventual interest is the case where $\Gamma \subset G$ is an arithmetic subgroup which may not be co-compact, it will simplify the exposition to assume co-compactness. Neat means that Γ contains no non-trivial elements of finite order. This is a convenient but inessential technical assumption that may always be achieved by passing to a finite index subgroup of Γ.

The representations we will be interested in are

- The discrete summands in $L^2(G_{\mathbb{R}})$, the *discrete series* (DS), and the related *limits of discrete series* (LDS).

Here, $G_{\mathbb{R}}$ acts unitarily on both the left and right. The unitary dual $\hat{G}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is defined to be the set of equivalence classes of irreducible unitary representations

$$\pi : G_{\mathbb{R}} \to \operatorname{Aut}(V_\pi)$$

of $G_{\mathbb{R}}$ on a Hilbert space V_π. One then has the *Plancherel formula*

$$L^2(G_{\mathbb{R}}) = \int_{\hat{G}_{\mathbb{R}}} \operatorname{End}_{\mathrm{HS}}(V_\pi) d\pi$$

and the DS's are those for which the Plancherel measure $d\pi$ assigns a strictly positive point mass. This is equivalent to the matrix coefficients $(\pi(g)u, v)$ being in $L^2(G_{\mathbb{R}})$.

The DS's are parametrized by weights μ belonging to the weight lattice P and such that $\mu + \rho$ is regular, which in particular implies that $\mu + \rho$ belongs to a unique Weyl chamber C.[26] In these lectures we will be especially interested in LDS's, which are parametrized by pairs (μ, C) where $\mu + \rho \in \overline{C}$ but is singular and therefore is orthogonal

[26] We recall that a weight λ is *regular* if $(\lambda, \alpha) \neq 0$ for all non-zero roots α. Otherwise, λ is *singular*.

to some root of $G_{\mathbb{R}}$ but is not orthogonal to any C-simple compact root.[27] The *infinitesimal character* (defined below) associated to a DS or LDS will be denoted by $\chi_{\mu+\rho}$. Of very particular interest will be the *totally degenerate limits of discrete series* (TDLDS) $(0, C)$ where $\mu = -\rho$ and which have infinitesimal character χ_0.

- The unitary $G_{\mathbb{R}}$-module $L^2(\Gamma\backslash G_{\mathbb{R}})$.

In both of the cases of $G_{\mathbb{R}}$ and of $\Gamma\backslash G_{\mathbb{R}}$ the objective of this lecture is to relate the representation theory to complex geometry. In the first case this will involve the cohomology groups

$$H^q(D, L_\mu)$$

where D is a *flag domain* as defined below and $L_\mu \to D$ is a $G_{\mathbb{R}}$-homogeneous line bundle associated to the weight μ. In the second case the relevant cohomology groups are

$$H^q(\Gamma\backslash D, L_\mu).$$

Representation theory enters via the formula

$$H^q(\Gamma\backslash D, L_\mu) = \bigoplus_{\pi\in\hat{G}_{\mathbb{R}}} H^q(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)}$$

where the RHS is a finite sum, $m_\pi(\Gamma)$ is the multiplicity of V_π^* in $L^2(\Gamma\backslash G_{\mathbb{R}})$, and the notation for the $\mathfrak{n}$-*cohomology groups* $H^q(\mathfrak{n}, V_\pi^*)_{-\mu}$ will be explained below.

In summary, the theme of this lecture is to begin to develop the *relationship between representation theory and the geometry of locally homogeneous complex manifolds.* References are [GS], [Sch1], [Sch2], [Sch3] and the references cited therein, [W1], [FHW], and [GGK2] and the references cited there.

We remark that the most classical relation between representation theory and the geometry of homogeneous complex manifolds is the *Borel-Weil-Bott* (BWB) *theorem.* This deals with the $G_{\mathbb{C}}$-modules $H^q(G_{\mathbb{C}}/B, L_\mu)$ where B is a Borel subgroup of $G_{\mathbb{C}}$ and μ is a holomorphic character of B. In the appendix to this lecture we have given a discussion of the BWB theorem in the framework of the overall perspective of these lectures.

Harish-Chandra modules and their infinitesimal character

In these lectures it will frequently be more convenient to work with the *Harish-Chandra module* (HC module) associated to one of the types of unitary representation mentioned above, and also with the corresponding *infinitesimal character*. It will also be convenient to work with flag domains rather than general homogeneous complex manifolds. We now explain these terms.

[27]The singular parameters that are not orthogonal to any compact root K are the *non-degenerate* LDS's.

We recall our notations

- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is a complex, semi-simple Lie algebra;
- $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ is a Cartan sub-algebra;
- $K_{\mathbb{C}}$ is the complex Lie group corresponding to the unique maximal compact subgroup $K \subset G_{\mathbb{R}}$ that contains T;
- $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is the *universal enveloping algebra* of $\mathfrak{g}_{\mathbb{C}}$ with *center* $Z(\mathfrak{g}_{\mathbb{C}})$.

A $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-*module* is a complex vector space M that is a $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$-module and is a linear $K_{\mathbb{C}}$-module, and where the conditions

- The action of $K_{\mathbb{C}}$ is locally finite; i.e., every $m \in M$ lies in a finite dimensional $K_{\mathbb{C}}$-invariant subspace on which $K_{\mathbb{C}}$-acts holomorphically; and
- The differentiated $K_{\mathbb{C}}$-action agrees with the action of the subspace $\mathfrak{k}_{\mathbb{C}}$ of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$

are satisfied.

Definition: *A* **Harish-Chandra module** *is a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module that is finitely generated as a $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$-module and is admissible in the sense that every irreducible $K_{\mathbb{C}}$-module occurs in M with finite multiplicity.*

Examples: (i) The subspace $V_{\pi,K\text{-finite}} \subset V_\pi$ of K-finite vectors in a unitary representation, in particular in a DS or LDS gives an HC-module.

(ii) With notations to be explained below, the $G_{\mathbb{R}}$-module given by a non-zero cohomology group $H^d(D, L_\mu)$ where $\mu + \rho \in \overline{C}$, the closure of the anti-dominant Weyl chamber and $d = \dim K/T$, gives an HC-module.

We now turn to the definition of the infinitesimal character associated to a weight μ. For this we set

- $\mathcal{H} = \mathcal{U}(\mathfrak{h})$, the universal enveloping algebra for $\mathfrak{h}$; and
- $\mathcal{P} = \sum_{\alpha \in \Phi^+} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}^\alpha$ where $\mathfrak{g}^\alpha \subset \mathfrak{g}_{\mathbb{C}}$ is the α-weight space.

By the Poincaré-Birkhoff-Witt theorem, $\mathcal{H} \cap \mathcal{P} = (0)$ and

$$Z(\mathfrak{g}_{\mathbb{C}}) \subset \mathcal{H} \oplus \mathcal{P}.$$

Explicitly, elements of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ are $\sum X_{-\beta_1} \cdots X_{-\beta_j} H_{i_1} \cdots H_{i_k} X_{\alpha_1} \cdots X_{\alpha_l}$ where the β_i, α_i are positive roots and the H_i are a basis for $\mathfrak{h}$. Then $Z(\mathfrak{g}_{\mathbb{C}})$ is contained in the sums of such terms with no $X_{-\beta_i}$'s.

We define

$$\sigma : \mathfrak{h} \to \mathcal{H}$$

by

$$\sigma(H) = H - \rho(H)1$$

where, as usual, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha \in \mathfrak{h}^*$. We next define

$$\gamma' : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{H}$$

to be the projection and set

$$\gamma = \sigma \circ \gamma'$$

where $\sigma(H_{i_1}\cdots H_{i_k}) = \sigma(H_{i_1})\cdots\sigma(H_{i_k})$. It is a theorem of Harish-Chandra that this gives an algebra isomorphism

$$\gamma : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} \mathcal{H}^W$$

where the RHS is the sub-algebra of elements of $\mathcal{H}$ invariant under the Weyl group W of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Note that we may identify $\mathcal{H}^W$ with the algebra $\mathbb{C}[\mathfrak{h}]$ of polynomial functions on the dual $\mathfrak{h}^*$ of $\mathfrak{h}$.

Definition: *For $\zeta \in \mathfrak{h}^*$, we define the* **infinitesimal character** *as the homomorphism*

$$\chi_\zeta : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$

given for $z \in Z(\mathfrak{g}_{\mathbb{C}})$

$$\chi_\zeta(z) = \gamma(z)(\zeta).$$

The RHS is the value of the polynomial $\gamma(z) \in \mathbb{C}[\mathfrak{h}]$ on $\zeta \in \mathfrak{h}^*$.

It is another result of Harish-Chandra that

> *Every character of $Z(\mathfrak{g}_{\mathbb{C}})$ is an infinitesimal character χ_ζ, and $\chi_\zeta = \chi_{\zeta'}$ if, and only if, $\zeta' = w(\zeta)$ for some $w \in W$.*

Below we shall give a geometric interpretation of χ_λ.

Flag varieties and flag domains

Definitions: (i) *A* **flag variety** *is a homogeneous complex manifold*

$$\check{D} = G_{\mathbb{C}}/B$$

where $B \subset G_{\mathbb{C}}$ is a Borel subgroup.[28] (ii) *A* **flag domain** *is an open orbit*

$$D = G_{\mathbb{R}} \cdot x_0$$

of $G_{\mathbb{R}}$ acting on $\check{D}$ where the isotropy group is a compact maximal torus T.

We have noted above that every flag domain arises as a Mumford-Tate domain. In fact, if $\varphi \cdot S^1 \to G_{\mathbb{R}}$ is a circle with $\varphi(S^1) \subset T$ and corresponding to $l_\varphi \in \Lambda$ where $T = \mathfrak{t}/\Lambda$, the condition $Z_{G_{\mathbb{R}}}(\varphi(S^1)) = T$ is equivalent to

$$\langle \alpha, l_\varphi \rangle \neq 0, \qquad \alpha \in \Phi.$$

[28] In the literature a common notation for flag varieties is X. Here our emphasis is on the $D = G_{\mathbb{R}}/T$'s and we are thinking of $\check{D} = G_{\mathbb{C}}/B$ as the compact dual of D. We will generally use the notation for X as a quotient $X = \Gamma\backslash D$.

Since every such pair (G, φ) leads to a Hodge representation it follows that $D = G_{\mathbb{R}}/T$ is a Mumford-Tate domain.

If $D = G_{\mathbb{R}}/H$ is a general Mumford-Tate domain as discussed in Lecture 3 and with compact dual $\check{D} = G_{\mathbb{C}}/P$, there is a unique Borel subgroup $B \subset G_{\mathbb{C}}$ with $B \subseteq P$, and we have a diagram

$$\begin{array}{ccc} G_{\mathbb{R}}/T & \subset & G_{\mathbb{C}}/B \\ \downarrow & & \downarrow \\ D = G_{\mathbb{R}}/H & \subset & G_{\mathbb{C}}/P = \check{D}. \end{array}$$

Then the flag domain $G_{\mathbb{R}}/T$ may be interpreted as the set of *Hodge flags* associated to D. In fact, given a point $F^{\bullet} \in \check{D}$, a point in $G_{\mathbb{C}}/B$ lying over $F^{\bullet}$ may be interpreted as a *full flag* on $V_{\mathbb{C}}$ and the points of $G_{\mathbb{R}}/T$ are the Hodge flags. We shall not give the formal definitions as they are not needed in these lectures, although they will be illustrated in several examples below. The point is that from the point of view of representation theory flag varieties and flag domains are especially convenient, and when we have a Mumford-Tate domain the points in the corresponding flag domain may be thought of as PHS's with the additional data given by full flags satisfying certain conditions in each $V^{p,q}$.

A more intrinsic description of the flag variety as a set is

$$\check{D} = \{\text{set of Borel sub-algebras } \mathfrak{b}_x \subset \mathfrak{g}_{\mathbb{C}}\}.$$

Since any two Borel sub-algebras are conjugate in $G_{\mathbb{C}}$, upon choice of a reference $\mathfrak{b}_{x_0}$ with B_{x_0} the corresponding Borel subgroup we have an identification with the RHS above with $G_{\mathbb{C}}/B_{x_0}$.

Root theoretic descriptions

These are especially useful for computational purposes. Given the choice of B with Lie algebra $\mathfrak{b}$ containing $\mathfrak{h}$ there is a unique choice of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ such that

$$\begin{cases} \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \\ \mathfrak{n} = \bigoplus\limits_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}. \end{cases}$$

We set $\mathfrak{n}^+ = \bigoplus\limits_{\alpha \in \Phi^+} \mathfrak{g}^{\alpha}$. The reason for writing just $\mathfrak{n}$ instead of $\mathfrak{n}^-$ is that it makes the notation for $\mathfrak{n}$-cohomology more convenient; we shall occasionally use $\mathfrak{n}^-$ where warranted by the circumstances.

At the reference point $x_0 = eB \in \check{D}$ we have for the holomorphic tangent space

$$T_{x_0}\check{D} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{b} \cong \mathfrak{n}^+.$$

Choosing $x_0 = eT \in D$ there are the identifications

$$T_{x_0,\mathbb{R}}D \cong \mathfrak{g}_{\mathbb{R}}/\mathfrak{t}$$
$$T_{x_0,\mathbb{C}}D \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{h} \cong \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

of the real tangent space and of its complexification. Setting $T^{1,0}_{x_0,\mathbb{C}}D = \mathfrak{n}^+$ with $T^{0,1}_{x_0,\mathbb{C}} = \overline{T^{1,0}_{x_0,\mathbb{C}}} = \mathfrak{n}^-$ gives a $G_{\mathbb{R}}$-invariant almost complex structure on $G_{\mathbb{R}}/T$, one that due to $[\mathfrak{n}^+, \mathfrak{n}^+] \subseteq \mathfrak{n}^+$ is integrable.

Conversely, a choice Φ^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ determines an integrable almost complex structure on $G_{\mathbb{R}}/T$ as well as a Borel sub-algebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$. An important observation is

> *Two such choices Φ^+, Φ'^+ give equivalent homogeneous complex structures on $G_{\mathbb{R}}/T$ if, and only if, $\Phi'^+ = w(\Phi^+)$ for some $w \in N_{G_{\mathbb{R}}}(T)/T = N_K(T)/T =: W_K$, the Weyl group for K.*

If G_c is a compact real form of $G_{\mathbb{C}}$ with $T \subset G_c$, then the above discussion applies to G_c/T, and one has that homogeneous complex structues on G_c/T given by Φ^+ and Φ'^+ are equivalent since the Weyl group $N_{G_c}(T)/T = N_{G_{\mathbb{C}}}(H)/H$ acts transitively on the set of choices of positive roots. We note that G_c/T and $G_{\mathbb{C}}/B$ are the same complex manifolds, although they are of course different as *homogeneous* complex manifolds.

A $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$-module W is said to have an infinitesimal character if $Z(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$ acts on it by scalars. The resulting homomorphism

$$Z(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$$

is then χ_μ for some $\mu \in \mathfrak{h}^*$. An element $w \in W$ is said to be a *highest weight vector with weight* $\mu \in \mathfrak{h}^*$ if

$$\begin{cases} \mathfrak{n}^+ \cdot w = 0 \\ \mathfrak{h} \text{ acts on } w \text{ by } h(w) = \langle \mu, h \rangle\, w. \end{cases}$$

Then

> *if W is finite dimensional, irreducible $\mathfrak{g}_{\mathbb{C}}$-module with highest weight μ in the usual sense, W has infinitesimal character $\chi_{\mu+\rho}$.*

Homogeneous line bundles and their curvature forms

For $T = \mathfrak{t}/\Lambda$ we recall the character group

$$X(T) \cong \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset i\mathfrak{t}^* \subset \mathfrak{h}^*.$$

Here, $T = \{(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r})\}$ where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_r) \in \mathbb{R}^r/\mathbb{Z}^r$ and $\mu \in \mathrm{Hom}(\Lambda, \mathbb{Z})$ is given by $(\mu_1, \dots, \mu_r)$ where

$$\mu(\boldsymbol{\theta}) = \mu_1\theta_1 + \cdots + \mu_r\theta_r.$$

We identify $\mathrm{Hom}(\Lambda, \mathbb{Z})$ with the character group $X(T)$ by

$$\chi_\mu(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r}) = e^{2\pi i\mu(\boldsymbol{\theta})}.$$

Up to a factor of $2\pi i$, μ is the differential of χ_μ. Identifying the Cartan subgroup H with $(\mathbb{C}^*)^r$, χ_μ extends to a holomorphic character $\chi_\mu : H \to \mathbb{C}^*$ where $\chi_\mu(z_1, \dots, z_r) = z_1^{\mu_1} \cdots z_r^{\mu_r}$. Finally using $H = B/[B, B]$ we obtain a holomorphic homomorphism

$$\chi_\mu : B \to \mathbb{C}^*.$$

Notation: *The holomorphic homogeneous line bundle $L_\mu \to \check{D}$ is given by*

$$L_\mu = G_{\mathbb{C}} \times_B \mathbb{C}$$

where B is represented in $\mathrm{Aut}(\mathbb{C}) = \mathbb{C}^*$ *by χ_μ.*

Denoting by $\mathcal{L}_\mu$ the sheaf $\mathcal{O}_{\check{D}}(L_\mu)$ of holomorphic sections of $L_\mu \to \check{D}$, the action of $G_{\mathbb{C}}$ on $L_\mu \to \check{D}$ induces an action of $\mathfrak{g}_{\mathbb{C}}$, and hence of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, on $\mathcal{L}_\mu$. It is a nice exercise to show that

Any $z \in Z(\mathfrak{g}_{\mathbb{C}})$ acts on $\mathcal{L}_\mu$ by the scalar $\gamma(z)(\mu + \rho)$

where $\gamma : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{H}^W$ is the Harish-Chandra isomorphism introduced above. Recall that $\mu + \rho \in \mathfrak{h}^*$ while $\gamma(z) \in \mathbb{C}[\mathfrak{h}]$, the polynomial functions on $\mathfrak{h}^*$.

For notational simplicity, we shall identify locally free coherent sheaves with the corresponding holomorphic vector bundles, and shall therefore just set $\mathcal{L}_\mu = L_\mu$.

We shall also denote by $L_\mu \to D$ the restriction to D of the homogeneous C^∞ line bundle

$$G_{\mathbb{R}} \times_T \mathbb{C}$$

with the holomorphic structure given above. Since $\chi_\mu\big|_T$ is unitary, the line bundle $L_\mu \to D$ has an invariant Hermitian structure. Denoting by $X_\alpha \in \mathfrak{g}^\alpha$ the standard root vector with dual ω^α we have

$$\begin{cases} T^{1,0}_{x_0} D = \mathrm{span}\{X_\alpha : \alpha \in \Phi^+\} \\ \quad\overline{\omega}^\alpha = \pm\omega^{-\alpha}. \end{cases}$$

We will determine the $\pm$ sign below.

Basic calculation: The *Chern form*, expressed in terms of the curvature, is given by

$$c_1(L_\mu) = \frac{i}{2\pi}\left\{ \sum_{\alpha \in \Phi_c^+} (\mu, \alpha)\omega^\alpha \wedge \overline{\omega}^\alpha - \sum_{\beta \in \Phi_{nc}^+} (\mu, \beta)\omega^\beta \wedge \overline{\omega}^\beta \right\}.$$

Before deriving the formula we comment that there is a G_c-invariant metric in $L_\mu \to \check{D}$ whose Chern form is given by a similar expression but with a $+(\mu, \beta)$ instead of a

$-(\mu,\beta)$ coefficient of $\omega^\beta \wedge \overline{\omega}^\beta$ for $\beta \in \Phi^+_{nc}$. This sign reversal was noted in Lecture 1 in the simplest case of SL_2.

The calculation is based on rather general principles and will be given in a sequence of steps.

Step one: Let A, B be the connected Lie groups with $B \subset A$ a closed, *reductive* subgroup. Here, reductive means that the real Lie algebra $\mathfrak{A}$ of A has an $\operatorname{Ad} B$-invariant splitting

$$\begin{cases} \mathfrak{A} = \mathfrak{b} \oplus \mathfrak{h} \text{ [29]} \\ [\mathfrak{b}, \mathfrak{h}] \subseteq \mathfrak{h}. \end{cases}$$

The homogeneous vector bundle $\mathbb{H} = A \times_B \mathfrak{h}$ then gives an A-invariant connection

$$TA = \pi^* TB \oplus \mathbb{H}$$

in the principal bundle $A \xrightarrow{\pi} A/B$. The basic observation is that the curvature form Ω of this connection is given at the identity coset eB by

$$\Omega(u,v) = -1/2[u,v]_{\mathfrak{b}}$$

where $u, v \in \mathfrak{h} \cong T_{eB}(A/B)$ and $[\ ,\ ]_{\mathfrak{b}}$ is the $\mathfrak{b}$-component of the bracket. The $-1/2$ comes from the *Maurer-Cartan equation*

$$d\omega(u,v) = -\frac{1}{2}[u,v]$$

where ω is a left-invariant 1-form on A and $u, v \in \mathfrak{A}$ are left invariant vector fields.

As a check on signs and constants, let X_i be a basis for the Lie algebra of left-invariant vector fields with dual basis ω^i. Setting

$$[X_j, X_k] = \sum_k c^i_{jk} X_i$$

we claim that

$$d\omega^i = \left(\frac{-1}{2}\right) \sum_{j,k} c^i_{jk} \omega^j \wedge \omega^k.$$

Let

$$d\omega^i = \sum_{j,k} a^i_{jk} \omega^j \wedge \omega^k, \qquad a^i_{jk} + a^i_{kj} = 0.$$

Using the standard formula

$$\left\langle d\omega^i, (X_j, X_k)\right\rangle = \cancel{X_j \cdot \omega^i(X_k)} - \cancel{X_k \cdot \omega^i(X_j)} - \left\langle \omega^i, [X_j, X_k]\right\rangle$$

[29]Here, $\mathfrak{h}$ stands for "horizontal" and is not related to the Cartan sub-algebra, which is the $\mathfrak{h}$ everywhere else in these lectures. The same applies to $\mathfrak{b}$ and to the $\mathbb{H}$ defined below.

the crossed out terms are zero by left-invariance. The RHS is $-c^i_{jk}$, while the LHS is $a^i_{jk} - a^i_{kj}$. Thus

$$2a^i_{jk} = -c^i_{jk}$$

as was to be shown.

For GL_n the *Maurer-Cartan matrix* is

$$\omega = g^{-1}dg, \qquad g = \|g^i_j\| \in \mathrm{GL}_n\,.$$

Then $dg^{-1} = -g^{-1}dgg^{-1}$ gives

$$d\omega = -\omega \wedge \omega,$$

which again serves to check the sign.

Step two: Next let $r : B \to \mathrm{Aut}(E)$ be a linear representation and

$$\begin{array}{c} \mathbb{E} \; = A \times_B E \\ \downarrow \\ A/B \end{array}$$

the corresponding homogeneous vector bundle. Then this bundle has an induced connection whose curvature form $\Omega_{\mathbb{E}}$ is given by

$$\Omega_{\mathbb{E}}(u, v) = r_*\Omega(u, v).$$

Step three: We now apply this when $A = G_{\mathbb{R}}$, $B = T$ and r is given by χ_μ as above. Writing

$$\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

from $[\mathfrak{n}^\pm, \mathfrak{n}^\pm] \subseteq \mathfrak{n}^\pm$ we see that the curvature form is of type $(1,1)$ whose only non-zero terms are

$$\Omega(X_\alpha, \overline{X}_\alpha) = -1/2[X_\alpha, X_{-\alpha}] = \left(-\frac{1}{2}\right) h_\alpha.$$

We now use that $\mathfrak{g}_{\mathbb{C}}$ has two conjugations corresponding to the two real forms $\mathfrak{g}_c$ and $\mathfrak{g}_{\mathbb{R}}$

$$\begin{cases} \tau \leftrightarrow \;\text{ compact form } \mathfrak{g}_c \\ \sigma \leftrightarrow \;\text{ non-compact form } \mathfrak{g}_{\mathbb{R}} \end{cases}$$

and

$$\begin{cases} \tau(X_\alpha) = -X_{-\alpha} \\ \sigma(X_\alpha) = \begin{cases} -X_{-\alpha} \text{ if } \alpha \text{ is compact} \\ X_\alpha \text{ if } \alpha \text{ is non-compact.} \end{cases} \end{cases}$$

The conjugation signs on $\overline{X}_\alpha$ above and on $\overline{\omega}^\alpha$ below are relative to σ. Then

$$\Omega = \sum_{\alpha\in\Phi_c^+} h_\alpha \omega^\alpha \wedge \overline{\omega}^\alpha - \sum_{\beta\in\Phi_{nc}^+} h_\beta \omega^\beta \wedge \overline{\omega}^\beta.$$

Here, the $-1/2$ has gone away using $\overline{\omega}^\alpha \wedge \omega^\alpha = -\omega^\alpha \wedge \overline{\omega}^\alpha$. Denoting by Ω_μ the curvature form Ω_{L_μ} and using $\langle \mu, h_\alpha\rangle = (\mu,\alpha)$ we obtain

$$\Omega_\mu = \sum_{\alpha\in\Phi_{nc}^+} (\mu,\alpha)\omega^\alpha \wedge \overline{\omega}^\alpha - \sum_{\beta\in\Phi_{nc}^+} (\mu,\beta)\omega^\beta \wedge \overline{\omega}^\beta.$$

This concludes the proof of the basic calculation.

For later use we introduce the notation

$$\boxed{q(\mu) = \#\left\{\alpha\in\Phi_c^+ : (\mu,\alpha) > 0\right\} + \#\left\{\beta\in\Phi_{nc}^+(\mu,\beta) < 0\right\}.}$$

Then we note that

- the curvature form Ω_μ is non-degenerate (non-singular) if, and only if, μ is regular;
- in this case, Ω_μ has *signature* $(q(\mu), n - q(\mu))$ where $n = \#\Phi^+ = \dim_{\mathbb{C}} D$.

The classical and non-classical cases

Among the homogeneous complex manifolds we have been considering a very special and important class are the *Hermitian symmetric domains* (HSD's)

$$D = G_{\mathbb{R}}/K.$$

Here, K is the maximal compact subgroup. Relative to the Cartan decomposition

$$\begin{cases} \mathfrak{g}_{\mathbb{R}} = \mathfrak{k}\oplus\mathfrak{p} \quad \text{where} \\ [\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{p},\ [\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{k}\end{cases}$$

identifying at $x_0 = eK \subset D$ the complexified tangent space

$$T_{e,\mathbb{R}} D\otimes\mathbb{C} = \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+\oplus\mathfrak{p}^-$$

where $\mathfrak{p}^+ = T^{1,0}_{x_0} D$, $\mathfrak{p}^- = T^{0,1}_{x_0} D = \overline{\mathfrak{p}^+}$ we have

$$\mathfrak{p}^\pm = \bigoplus_{\beta\in\Phi_{nc}^\pm} \mathfrak{g}^\beta.$$

Since $[\mathfrak{p}^+,\mathfrak{p}^+]\subseteq\mathfrak{k}_{\mathbb{C}}$ this implies (and is equivalent to)

$$\mathfrak{p}^\pm \textit{ are abelian Lie algebras.}$$

This is equivalent to the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$ decomposing into conjugate K-submodules.

Definition: *A homogeneous complex manifold is* **classical** *if it fibres holomorphically or anti-holomorphically over an HSD. Otherwise it is* **non-classical**.

From the above we see that

$$D \text{ is classical} \Leftrightarrow \mathfrak{p}^+ \text{ is an abelian Lie algebra.}$$

We will now show that

There exists a μ such that $q(\mu) = 0$ if, and only if, D is classical.

Proof. If $q(\mu) = 0$ then we have

$$\begin{cases} (\mu, \alpha) > 0 & \text{for } \alpha \in \Phi_c^+ \\ (\mu, \beta) < 0 & \text{for } \beta \in \Phi_{nc}^+. \end{cases}$$

For $\mathfrak{p}^\pm = \bigoplus_{\beta \in \Phi^\pm} \mathfrak{g}^\beta$ we have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Writing

$$\mathfrak{k}_\mathbb{C} = \mathfrak{h} \oplus \mathfrak{n}_c^+ \oplus \mathfrak{n}_c^-$$

where $\mathfrak{n}_c^\pm = \mathfrak{n}^\pm \cap \mathfrak{k}_\mathbb{C}$, we have from

$$[\mathfrak{k}, \mathfrak{p}_\mathbb{C}] \subseteq \mathfrak{p}_\mathbb{C}$$

that, since the sum of two negative roots is negative,

$$[\mathfrak{n}_c^-, \mathfrak{p}^-] \subseteq \mathfrak{p}^-.$$

We must show that

$$[\mathfrak{n}_c^-, \mathfrak{p}^+] \subseteq \mathfrak{p}^+.$$

This is the same as

for $\alpha \in \Phi_c^+, \beta \in \Phi_{nc}^+$, either $-\alpha+\beta$ is not a root or we have $-\alpha+\beta \in \Phi_{nc}^+$.

Now, if $-\alpha + \beta$ is a root it must be a non-compact root, and if $-\alpha + \beta \in \Phi_{nc}^-$ then

$$(\mu, -\alpha + \beta) > 0.$$

This gives

$$(\mu, \beta) > (\mu, \alpha) > 0$$

which is a contradiction. □

As an application, if D is non-classical we have

$$H^0(\Gamma \backslash D, L_\mu) = 0$$

for any $\mu \neq 0$. The proof is by noting that for a section $s \in H^0(\Gamma \backslash D, L_\mu)$ we have

$$\frac{i}{2\pi} \overline{\partial} \partial \log \|s\|^2 = \Omega_\mu.$$

At a maximum point of $\|s\|^2$ we must have $\Omega_\mu \geqq 0$. But in the non-classical case Ω_μ has a negative eigenvalue.

Root diagrams of the complex structures

The homogeneous complex structures on flag domains are given by choices of positive root systems, or equivalently of Weyl chambers. Two such are equivalent as homogeneous complex manifolds if, and only if, the two Weyl chambers are congruent under the action of the *compact Weyl group* $W_K = N_K(T)/Z_K(T)$. In examples it is convenient to use the root diagram to picture things.

$\underline{S\mathfrak{U}(2,1)}$: This is the subgroup of $\mathrm{SL}(3,\mathbb{C})$ that preserves the Hermitian form with matrix

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

The maximal torus

$$T_S = \left\{ \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & e^{2\pi i\theta_2} & \\ & & e^{2\pi i\theta_3} \end{pmatrix} \right\}$$

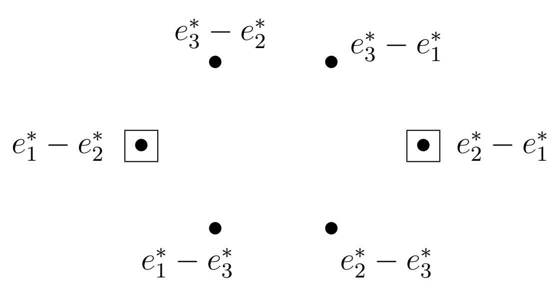
where $\theta_1 + \theta_2 + \theta_3 \equiv 0 \bmod \mathbb{Z}$. We let $\mathbb{R}^3$ have standard basis e_1, e_2, e_3 viewed as column vectors, and the dual space will be row vectors with dual basis $e_1^* = (1,0,0)$, $e_2^* = (0,1,0)$, $e_3^* = (0,0,1)$. Then the Lie algebra

$$\mathfrak{t}_S \subset \mathbb{R}^3$$

is defined by the relation

$$e_1^* + e_2^* + e_3^* = 0.$$

The root diagram is then

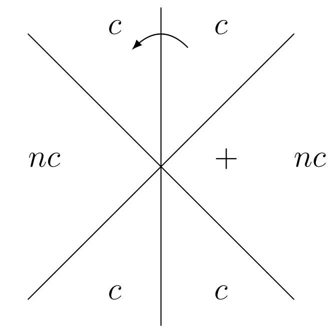


The maximal compact subgroup $K \cong \mathfrak{U}(2)$ is

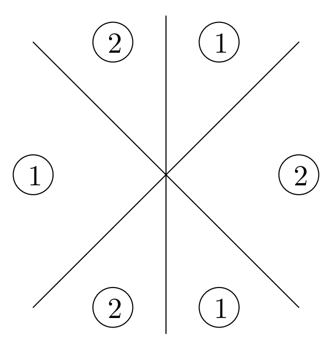
$$\left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in \mathfrak{U}(2), a = \det A^{-1} \right\},$$

and from this we see that the compact roots are those within a box.

The Weyl chambers are

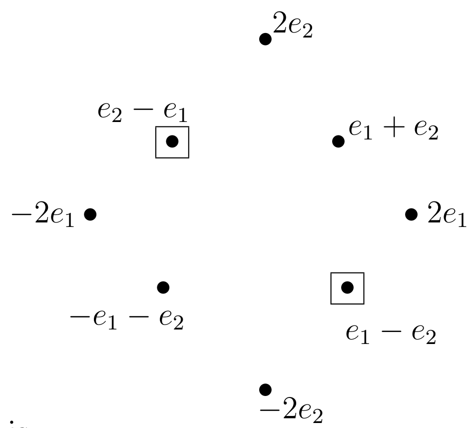


where "c" means a classical complex structure and "nc" a non-classical one. The action of the compact Weyl group is pictured by the arrow. The Mumford-Tate domains for the $\mathcal{U}(2,1)$ example corresponding to PHS's of weight $n = 3$ with $h^{3,0} = 1, h^{2,1} = 2$ in Lecture 3 correspond to the Weyl chamber marked with a $+$. We note that for this choice the compact root $e_2^* - e_1^*$ is *positive*. For the non-classical complex structure given by the Weyl chamber with the $+$ the values of $q(\mu + \rho)$ are

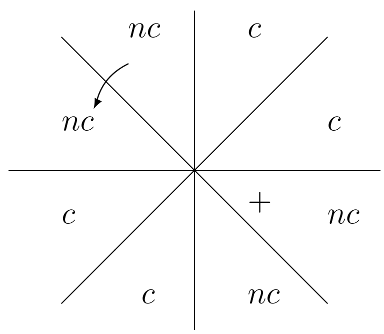


We note that $q(\mu) \neq 0, 3$. We also note the duality that the $q(\mu + \rho)$'s in opposite Weyl chambers add up to $\dim D$, a general phenomenon.

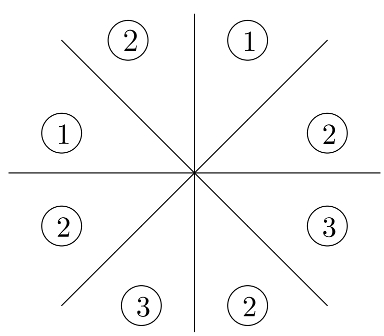
<u>Sp(4)</u>: In this case we will not need to distinguish between $\mathbb{R}^4$ and its dual and can use the more standard notation for the roots. The root diagram is



The Weyl chamber picture is

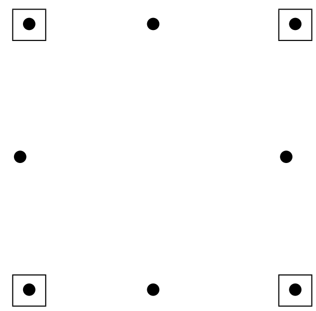


The values of $q(\mu+\rho)$ for the homogeneous complex structure corresponding to the Weyl Chamber marked $+$ are

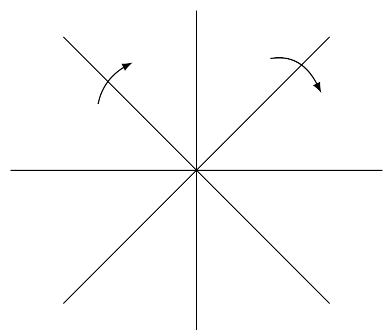


We note that for this Weyl chamber the compact root $e_1 - e_2$ is *positive*. This is the opposite convention to the $\mathcal{U}(2,1)$ example, the two conventions being chosen for Hodge-theoretic reasons. The difference will need to be kept in mind when we do the calculations in Lecture 9. We note again that $q(\mu+\rho) \neq 0, 4$ and the symmetry.

<u>SO(4, 1)</u>: The root diagram is



The Weyl chamber picture is



There are two equivalence classes of non-classical homogeneous complex structures. There are no classical ones.

Realization of the DS

The basic story here is due to Schmid, beginning with his thesis [Sch1] and continuing through a series of papers appearing in the Annals of Mathematics.[30] Here we shall largely follow the expository lecture [Sch2] which contains an extensive bibliography including references to his papers on the subject.

The basic results we shall discuss are

(A) *Let μ be a character giving a homogeneous holomorphic line bundle $L_\mu \to D$. Then*
  (i) *the L^2-cohomology group $H^q_{(2)}(D, L_\mu)$ is zero unless $\mu + \rho$ is regular and $q = q(\mu + \rho)$;*
  (ii) *if $\mu + \rho$ is regular, then $H^{q(\mu+\rho)}(D, L_\mu)$ is a unitary $G_\mathbb{R}$-module that realizes the discrete series representation whose Harish-Chandra parameter is $\mu + \rho$. In particular it has infinitesimal character $\chi_{\mu+\rho}$.*

[30] We note again the paper [AS], which gives a different way of realizing the DS's. The approach taken here is one that uses the geometry of homogeneous complex manifolds.

Schmid's proof of this result uses the realization of L^2-cohomology as $\mathfrak{n}$-cohomology. Recalling the Plancherel decomposition

$$L^2(G_{\mathbb{R}}) = \int_{\hat{G}_{\mathbb{R}}} V_\pi^* \hat{\otimes} V_\pi d\pi$$

the result is

$$H^q_{(2)}(D, L_\mu) = \int_{\hat{G}_{\mathbb{R}}} V_\pi^* \hat{\otimes} H^q(\mathfrak{n}, V_\pi)_{-\mu}.$$

Here the terms in the integrand on the RHS are $\mathfrak{n}$-cohomology groups that will be discussed below.

(B) *Let V_π be in the DS, and μ a weight with $\mu+\rho$ regular, and $H^q(\mathfrak{n}, V_\pi)_{-\mu} \neq 0$. Then*
  (i) $q = q(\mu+\rho)$*;*
  (ii) *for this q,* $\dim H^q(\mathfrak{n}, V_\pi)_{-\mu} = 1$*;*
  (iii) *V_π^* has infinitesimal character $\chi_{\mu+\rho}$;*
  (iv) $H^*(\mathfrak{n}, V_\pi)_{-\mu'} \neq 0 \Rightarrow \mu' + \rho = w(\mu+\rho)$ *for some $w \in W_K$.*

In the section below on the Hochschild-Serre spectral sequence we will use curvature considerations to sketch a proof of (B) for μ sufficiently non-singular.

We remark that in [Sch2] there are three $G_{\mathbb{R}}$-modules that are used:

- V_π = unitary $G_{\mathbb{R}}$-module;
- $V_\pi^\infty = C^\infty$ vectors in V_π;
- $V_{\pi, K\text{-finite}}$ = K-finite vectors in V_π.

The arguments given there show that the $\mathfrak{n}$-cohomology is the same in all three cases. We will give a general discussion of $\mathfrak{n}$-cohomology later in this lecture.

Next we let $Z_0 = K \cdot x_0 \subset D$ be the K-orbit of the reference point $x_0 = eT \in D$. Then $Z_0 = K/T$ is a maximal compact, complex analytic subvariety of D. There will be a general discussion of these in Lecture 6.

(C) *Let μ be a weight such that $\mu+\rho \in -\overline{C}$, the closure of the anti-dominant Weyl chamber. Then for* $d = \dim K/T = \dim Z_0$*,*
  (i) *$H^d(D, L_\mu)$ is a Harish-Chandra module with infinitesimal character $\chi_{\mu+\rho}$;*
  (ii) *the K-type of $H^d(D, L_\mu)$ may be obtained by expanding the cohomology about the maximal compact subvariety Z_0.*

The above results, stated somewhat more precisely, are in [Sch2] in the case when $\mu+\rho$ is regular. The extension to the case when $\mu+\rho$ is on a wall of the anti-dominant Weyl chamber is by a personal communication by Schmid. This latter includes the cases of a LDS and a TDLDS that are of particular importance in these lectures.

We will now explain the other terms used above. For L^2-cohomology we used the $G_{\mathbb{R}}$-invariant metrics in $L_\mu \to D$ and in the tangent bundle TD to define an inner product (pre-Hilbert space structure) on the space $A_c^{0,q}(D, L_\mu)$ of compactly supported, smooth, L_μ-valued $(0,q)$ forms on D. Using this one then defines the adjoint $\overline{\partial}^*$ of the $\overline{\partial}$-operator and Laplace-Beltrani operator $\square$. After completing $A_c^{0,q}(D, L_\mu)$ in L^2, one may regard $\overline{\partial}, \overline{\partial}^*$ as densely defined unbounded operators and setting $\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ define the *L^2-cohomology groups*

$$\begin{aligned} H^q_{(2)}(D, L_\mu) &= \ker \square \\ &= (\ker \overline{\partial}) \cap (\ker \overline{\partial}^*). \end{aligned}$$

The proof of the vanishing statement (i) in (A) for $\mu+\rho$ sufficiently far from the walls of the Weyl chamber was proved in [GS] using the classical method of Bochner-Yano, the same method used by Kodaira in the proof of his vanishing theorem.. That regularity alone is insufficient follows by observing that this same method gives the vanishing of

$$H^q(\Gamma\backslash D, L_\mu), \quad q \neq q(\mu+\rho)$$

for Γ co-compact and neat, while for D non-classical and $n = \dim D$, which implies that $q(\mu+\rho) \neq 0$ for all μ,

$$H^n(\Gamma\backslash D, L_{-2\rho}) = H^n(\Gamma\backslash D, \omega_{\Gamma\backslash D}) \neq 0.$$

The two ingredients used in removing the restriction of sufficient regularity are Zuckerman's translation principle and the lemma of Casselman-Osborne, both of which will be discussed in Lecture 9.

$\mathfrak{n}$-cohomology

We will assume the basic definition and elementary properties of Lie algebra cohomology, and will now explain how it arises in the study of the cohomology groups $H^q(D, L_\mu)$. The basic idea is the following. In the fibration $G_\mathbb{R} \xrightarrow{\pi} D$ the space $\Gamma(D, C^\infty(\mathbb{E}))$ of sections of any homogeneous bundle

$$\begin{array}{c} \mathbb{E} = G_\mathbb{R} \times_T E \\ \downarrow \\ D \end{array}$$

are given by E-valued smooth functions $f : G_\mathbb{R} \to E$ such that under the right action of T

$$f(gt) = r(t^{-1}) f(g)$$

where $r : T \to \operatorname{Aut}(E)$ is the given representation of T. Taking

$$\mathbb{E} = \Lambda^q T^{0,1^*} D \otimes L_\mu$$

to be the bundle of L_μ-valued $(0,q)$ forms where $\chi_\mu : T \to \mathrm{Aut}(\mathbb{C}_\mu)$ is the character that gives L_μ, we have the identification

$$A^{0,q}(D, L_\mu) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^* \otimes \mathbb{C}_\mu)^T.$$

Here we have used the identification

$$T^{0,1}_{x_0} D = \mathfrak{n},$$

and the notation $(\quad)^T$ means "T-invariants" where T acts on $C^\infty(G_\mathbb{R})$ by right translation on $G_\mathbb{R}$, on $\mathfrak{n}$ by the adjoint action Ad and on $\mathbb{C}_\mu$ by χ_μ. We may abbreviate this by letting

$$\big(C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*\big)_{-\mu}$$

be the elements that transform under the Lie algebra $\mathfrak{t}$ of T by the weight $-\mu$. Here the action of $\mathfrak{t}$ on $C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*$ is given for $H \in \mathfrak{t}$ by $R_{\exp H} \otimes 1 + 1 \otimes \mathrm{Ad}\, H$ where R_g denotes the action of right translation by $g \in G_\mathbb{R}$. Equivalently, we consider H as a left-invariant vector field on $G_\mathbb{R}$, and then the infinitesimal action is

$$H \cdot (f \otimes \omega) = (\mathcal{L}_H f) \otimes \omega + f \otimes \mathrm{ad}^* H(\omega)$$

where $\mathcal{L}_H$ is the Lie derivative and ad^* is the dual of the adjoint action ad of $\mathfrak{t}$ on $\mathfrak{n}$. Then $(C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*)_{-\mu}$ are sums of terms $f \otimes \omega$ where

$$H \cdot (f \otimes \omega) = -\langle \mu, H \rangle\, f \otimes \omega.$$

The final step is to note that under the identification

$$A^{0,q}(D, L_\mu) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*)_{-\mu}$$

the $\overline{\partial}$-operator on the LHS becomes the Lie algebra coboundary operator δ on the RHS (cf. [GS]). Here, $\mathfrak{n}$ acts on $C^\infty(G_\mathbb{R})$ by considering $X \in \mathfrak{n}$ as a left-invariant vector field, so that for $g \in G_\mathbb{R}$

$$\exp(tX)(g) = \frac{d}{dt}(g \cdot \exp(tX))_{t=0}$$

where the LHS is the action of the 1-parameter group on $G_\mathbb{R}$ and the RHS is multiplication in the group. Briefly we say that "left-invariant vector fields act by infinitesimal right translation." We note that the group $G_\mathbb{R}$ acts on both sides of the identification above; on the RHS it acts by left translation on $C^\infty(G_\mathbb{R})$ and acts trivially on $\mathfrak{n}^*$.

Summarizing we have the identifications of complexes of $G_\mathbb{R}$-modules

$$A^{0,\bullet}(D, L_\mu), \overline{\partial}) \cong \big((C^\infty(G_\mathbb{R}) \otimes \Lambda^\bullet \mathfrak{n}^*)_{-\mu}, \delta\big)$$

which gives the isomorphism of $G_\mathbb{R}$-modules

$$H^q(D, L_\mu) \cong H^q(\mathfrak{n}, C^\infty(G_\mathbb{R}))_{-\mu}.$$

Replacing $C^\infty(G_\mathbb{R})$ by $L^2(G_\mathbb{R})$ or other subspaces of $L^2(G_\mathbb{R})$, involves analytic issues that are treated in [Sch2]. The end result is the identification

$$H^q_{(2)}(D, L_\mu) = \int_{\hat{G}_\mathbb{R}} V_\pi^* \hat{\otimes} H^q(\mathfrak{n}, V_\pi)_{-\mu} d\pi$$

mentioned above.

The K-type

The *K-type of a Harish-Chandra module M* is the decomposition of the $K_\mathbb{C}$-module M into irreducible $K_\mathbb{C}$-modules with finite multiplicities

$$M\big|_{K_\mathbb{C}} = \underset{\lambda \in \hat{K}_\mathbb{C}}{\oplus} m_\lambda W^\lambda$$

where W^λ is the $K_\mathbb{C}$-module corresponding to $\lambda \in \hat{K}_\mathbb{C}$. A subtlety that we shall encounter in examples is that K will in general be reductive but not semi-simple; e.g., $G_\mathbb{R} = S\mathcal{U}(2,1)_\mathbb{R}$ in which case $K = \mathcal{U}(2)$. Thus $\hat{K}_\mathbb{C}$ will not just be the set of highest weights of the derived group of K, but will have additional parameters arising from the characters of K itself.

Let $Z_0 = K/T = K_\mathbb{C}/B_K$ where $B_K = K_\mathbb{C} \cap B$ be the maximal compact, complex submanifold of $D = G_\mathbb{R}/T$ given by the K-orbit of $x_0 = eT$. The general properties of the space of compact, complex submanifolds of D will be discussed in the next lecture. Here we want to explain the statement (C) above.

For simplicity of notation, we set $Z = Z_0$ and use the notations

- $\mathcal{I}_Z \subset \mathcal{O}_D$ is the ideal sheaf of Z;
- $N_{Z/D} \to Z$ is the normal bundle of $Z \subset D$;
- $N^*_{Z/D}$ is the dual and $S^k N^*_{Z/D}$ is the k^{th} symmetric product.

Then we have

$$S^k N^*_{Z/D} \cong \mathcal{I}^k_Z / \mathcal{I}^{k+1}_Z.$$

Proof. Locally there are holomorphic coordinates $x^1, \dots, x^d$, $y^1, \dots, y^{n-d}$ on D such that

$$Z = \left\{ y^1 = \cdots = y^{n-d} = 0 \right\}.$$

Then, identifying locally free sheaves and vector bundles

- $N_{Z/D} \cong$ span over $\mathcal{O}_Z$ of $\partial/\partial y^1, \dots, \partial/\partial y^{n-d}$;
- $N^*_{Z/D} \cong$ span over $\mathcal{O}_Z$ of $dy^1, \dots, d^{n-d}$;
- $\mathcal{I}_Z \cong$ span over $\mathcal{O}_D$ of $y^1, \dots, y^{n-d}$.

The map

$$\begin{array}{ccc} \mathcal{I}_Z & \to & N^*_{L/D} \\ \cup & & \cup \\ \sum_i f_i(x,y)y^i & \to & \sum_i f_i(x,0)dy^i \end{array}$$

is well defined, the point being that a change of coordinates is of the form

$$\tilde{y}^i = \sum_j F^i_j(x,y)y^j.$$

The kernel is $\mathcal{I}_Z^2$ and the resulting map $\mathcal{I}_Z/\mathcal{I}_Z^2 \to N^*_{Z/D}$ is readily seen to be an isomorphism. A similar argument works for $\mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to \operatorname{Sym}^k N^*_{Z/D}$.

From the above, setting $\mathcal{I}_Z^k(L_\mu) = \mathcal{I}_Z^k \otimes_{\mathcal{O}_D} L_\mu$ and $\mathcal{O}_Z(L_\mu) = L_\mu/\mathcal{I}_Z(L_\mu)$ we obtain exact sequences

$$\begin{aligned} &0 \to \mathcal{I}_Z(L_\mu) \to L_\mu \to \mathcal{O}_Z(L_\mu) \to 0 \\ &0 \to \mathcal{I}_Z^2(L_\mu) \to \mathcal{I}_Z(L_\mu) \to N^*_{Z/D}(L_\mu) \to 0 \\ &0 \to \mathcal{I}_Z^3(L_\mu) \to \mathcal{I}_Z^2(L_\mu) \to S^2N^*_{Z/D}(L_\mu) \to 0 \\ &\qquad\qquad\qquad\qquad \vdots \end{aligned}$$

The induced maps on cohomology

$$\begin{array}{c} \to H^d(D,\mathcal{I}_Z(L_\mu)) \to H^d(D,L_\mu) \to H^d(Z,L_\mu) \to H^{d+1}(D,\mathcal{I}_Z(L_\mu)) \\ H^d(D,\mathcal{I}_Z^2(L_\mu)) \to H^d(D,\mathcal{I}_Z(L_\mu)) \to H^d(Z,N^*_{Z/D}(L_\mu)) \to H^{d+1}(D,\mathcal{I}_L^2(L_\mu)) \\ H^d(D,\mathcal{I}_Z^3(L_\mu)) \to H^d(D,\mathcal{I}_Z^2(L_\mu)) \to H^d(Z,S^2N^*_{Z/D}(L_\mu)) \to H^{d+1}(D,\mathcal{I}_Z^3(L_\mu)) \\ \vdots \end{array}$$

are what is meant by the phrase *expanding the cohomology group* $H^d(D,L_\mu)$ *about* Z. As will be seen in the next lecture, for any coherent sheaf $\mathcal{F} \to D$

$$H^q(D,\mathcal{F}) = 0 \ \text{ for } \ q > d.$$

Thus all the above maps on cohomology

$$H^d(D,\mathcal{I}_Z^k(L_\mu)) \to H^d(Z,S^kN_{Z/D}(L_\mu))$$

are *surjective*. More formally: Define the filtration

$$F^kH^d(D,L_\mu) = \text{image}\left\{H^d(\mathcal{I}_Z^k(L_\mu)) \to H^d(D,L_\mu)\right\}.$$

We will note below that $\cap^k F^kH^d(D,L_\mu) = 0$, which gives

The filtration $F^\bullet H^d(D, L_\mu)$ on the Harish-Chandra module $H^d(D, L_\mu)$ is $K_{\mathbb{C}}$-invariant and has associated graded

$$\bigoplus_{k \geqq 0} H^d(Z, S^k N^*_{Z/D}(L_\mu)).$$

This $K_{\mathbb{C}}$-module is the K-type of $H^d(D, L_\mu)$.

For the TDLDS, which is the case of particular interest in these lectures, the Harish-Chandra parameter is zero so that the TDLDS is specified by a choice C of positive Weyl chamber for which no simple root is compact. Then C determines a set Φ^+ of positive roots and the Harish-Chandra module is $H^d(D, L_{-\rho})$.

In general, the K-type does not determine the HC-module. Here one may think of the principle series which has continuous parameters all with the same K-type such as the TDLDS for $S\mathcal{U}(2,1)_{\mathbb{R}}$. The TDLDS occur for special values of the parameters (cf. [CK]). However, as will be explained below and in Lecture 6 this geometric realization of the K-type gives more: we will see that the cup-product mappings

$$H^0(Z, N_{Z/D}) \otimes H^d(Z, S^k N^*_{Z/D}(L_\mu)) \to H^d(Z, S^{k-1} N^*_{Z/D}(L_\mu))$$

will enable us to reconstruct the $\mathfrak{g}_{\mathbb{C}}$-module $H^d(D, L_\mu)$ from its K-type.

Remark: Suppose that $\mu + \rho$ is regular but may not be anti-dominant. Then there is a Weyl chamber C' such that $\mu + \rho \in -C'$. With $\rho' = \left(\frac{1}{2}\right)$ (sum of the positive roots corresponding to C') we define the weight μ' by

$$\mu' + \rho' = \mu + \rho.$$

Then $H^d(D', L_{\mu'})$ is a Harish-Chandra module with infinitesimal character $\chi_{\mu'+\rho'} = \chi_{\mu+\rho}$ and we may determine the K type by expanding about $Z' = K/T \subset D'$ as above. We note that D' will in general have a different complex structure than D.

If we want to keep an equivalent complex structure we may choose $w \in W_K$ such that $w(\mu + \rho)$ is only K-anti-dominant and write $w(\mu + \rho) = \mu' + \rho$ and proceed as above ([Sch2]).

In Lecture 8 we will use a modification of this method. There we will have $\mu' + \rho' = \mu + \rho$ as above but where for the Weyl chamber C' we will have

$$q(\mu' + \rho') = 0.$$

Then the corresponding homogeneous complex manifold D' will be classical and the Harish-Chandra module will be $H^0(D', L_{\mu'})$. In fact, μ' will be orthogonal to all the compact roots and $H^0(D', L_{\mu'})$ will correspond to a holomorphic discrete series arising from L^2 holomorphic sections of a line bundle over an HSD.

The Hochschild-Serre spectral sequence (HSSS)

Let V be an $\mathfrak{n}$-module. Identifying

$$\mathfrak{p}^+ = \mathfrak{p}_{\mathbb{C}}/\mathfrak{n}$$

we have from $[\mathfrak{b},\mathfrak{n}] \subseteq \mathfrak{n}$ that $\mathfrak{p}^+$ is a $\mathfrak{b}$-module, and hence also an $\mathfrak{n}$-module.[31] Using the Cartan-Killing form we also have the identification of $\mathfrak{b}$-modules

$$\mathfrak{n}^* \cong \mathfrak{p}^+.$$

The HSSS is a spectral sequence abutting to $H^*(\mathfrak{n}, V)$ and with E_1-term

$$E_1^{p,q} = H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes V).$$

The differentials in the spectral sequence commute with the action of $\mathfrak{h}$, and therefore for any weight μ we have a spectral sequence abutting to $H^*(\mathfrak{n}, V)_{-\mu}$ and with E_1 term

$$E_1^{p,q} = H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes V)_{-\mu}.$$

In practice we will assume that V is an admissible Harish-Chandra module and decompose it into K-types (the reason for using W^{λ^*} will appear below)

$$V = \underset{\lambda\in\hat{K}_{\mathbb{C}}}{\oplus} m_\lambda W^{\lambda^*}.$$

Then

$$E_1^{p,q} = \underset{\lambda\in\hat{K}_{\mathbb{C}}}{\oplus} H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes W^{\lambda^*})_{-\mu}^{\oplus m_\lambda}.$$

For V the space of K-finite vectors in $H^d_{(2)}(D, L_\mu)$, the K-type is $V = \oplus V_n$ where

$$V_n = \mathrm{Gr}^n V = H^d\big(Z, \mathrm{Sym}^n N^*_{Z/D}(L_\mu)\big).$$

As noted previously, we have an inclusion

$$\mathfrak{p}_{\mathbb{C}} \hookrightarrow H^0(Z, N_{Z/D}),$$

and then the cup-products on cohomology induce

$$\mathfrak{p}_{\mathbb{C}} \otimes V_n \to V_{n-1}.$$

Using that V is unitarizable with the V_n being unitary summands and that $\mathfrak{p} \cong \mathfrak{p}^*$ as unitary $\mathfrak{n}$-modules, we have dually

$$\mathfrak{p}_{\mathbb{C}} \otimes V_n \to V_{n+1}$$

It is these maps that enable one to compute the differentials in the HSSS. In particular, setting

$$E_{r,n}^{p,q} = \ker d_1 \cap \cdots \cap \ker d_{r-1} \ \text{ on } \ H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes V_n)$$

[31] There is an important subtlety here in that except in the classical case we do *not* have $[\mathfrak{b},\mathfrak{n}^+] \subseteq \mathfrak{n}^+$. The $\mathfrak{b}$-module structure is that on the quotient $\mathfrak{p}_{\mathbb{C}}/\mathfrak{n}$.

we see that

$$d_r \text{ involves only } V_n, V_{n+1}, \dots, V_{n+r};$$

i.e., the action of $\underset{l \leqq r}{\oplus} \operatorname{Sym}^l \mathfrak{p}$ on V_n.

In the appendix to Lecture 7, for E any $\mathfrak{b}_K$-module with corresponding homogeneous vector bundle $\mathbb{E} \to Z$, we will see that

$$H^q(Z, \mathbb{E}(\mu)) = \underset{\lambda \in \hat{K}}{\oplus} W^\lambda \otimes H^q(\mathfrak{n}_K, E \otimes W^{\lambda^*})_{-\mu}.$$

Using this and the HSSS we will give a sketch of how one my prove Schmid's result (B) for μ sufficiently regular, denoted here by $|\mu| \gg 0$ where $|\ |$ is the minimum distance to the a wall of a Weyl chamber.

We first assume that $\mu + \rho$ is anti-dominant. Then since $L_\mu \to Z$ is a negative line bundle, by the Kodaira vanishing theorem for $|\mu| \gg 0$ we will have

$$H^q(Z, \wedge^p N_{Z/D}(L_\mu)) = 0, \qquad 0 \leqq q \leqq d-1 \text{ and all } p.$$

Using the above and taking for E the $\wedge^p \mathfrak{p}^+$'s this gives

$$H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes W^{\lambda^*})_{-\mu} = 0, \qquad 0 \leqq q \leqq d-1 \text{ and all } p$$

for any finite dimensional irreducible $K_{\mathbb{C}}$-module W^{λ^*}. In particular, for $|\mu| \gg 0$

$$E_1^{p,q} = \underset{n}{\oplus} H^q(\mathfrak{n}_K, \wedge^p \mathfrak{p}^+ \otimes V_n)_{-\mu} = 0, \qquad 0 \leqq q \leqq d-1 \text{ and all } p.$$

Thus the E_1 term of the HSSS for $H^*(\mathfrak{n}, V)_{-\mu}$ looks like

$$\begin{array}{|c|c|c|c|}
\hline
* & * & \cdot\ \cdot\ \cdot & * \\
\hline
0 & 0 & \cdot\ \cdot\ \cdot & 0 \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
0 & 0 & \cdot\ \cdot\ \cdot & 0 \\
\hline
\end{array}$$

and then $E_2 = E_\infty$; i.e.,

- $H^q(\mathfrak{n}, V)_{-\mu} = 0, \qquad 0 \leqq q \leqq d-1;$
- $H^d(\mathfrak{n}, V)_{-\mu} \cong \ker\{d_1 \cdot E_1^{0,1} \to E_1^{1,d}\}.$

On the other hand, again for $|\mu| \gg 0$ we have

$$H^q_{(2)}(D, L_\mu) = 0, \qquad q \neq d.$$

This is using the same curvature argument for vanishing of cohomology that we have mentioned above. Since

$$H^q_{(2)}(D, L_\mu) = \int_{\hat{G}_{\mathbb{R}}} V_\pi^* \widehat{\otimes} H^q(\mathfrak{n}, V_\pi)_{-\mu} d\pi,$$

for V the Harish-Chandra module associated to the DS $H^d_{(2)}(D, L_\mu)$ we may infer that

$$\begin{aligned} &E_2^{0,d} = E_\infty^{0,d} \cong H^d(\mathfrak{n}, V)_{-\mu} \\ &E_2^{p,d} = 0 \ \text{ for } \ 2 \leqq p \leqq n-d \ \text{ where } \ n = \dim D. \end{aligned}$$

Moreover, as will be seen in the appendix to the next lecture, $E_2^{0,d} = \ker d_1 : E_1^{0,d} \to E_1^{1,d}$ and

> $E_2^{0,q} \subset E_1^{0,q}$ *is the* 1*-dimensional with generator the Kostant class* κ_μ *of the lowest* K*-type* $V_0 = H^d(Z, L_\mu)$.

This establishes Schmid's result for $V_\pi^* = V$ and $|\mu| \gg 0$. Proof analysis shows that we have really only used that μ is K-anti-dominant. Then

$$q(\mu) = d + e$$

where

$$e = \#\{\beta \in \Phi^+_{nc} : (\mu, \beta) > 0\}.$$

Then vector

$$\bigwedge_{\substack{\beta \in \Phi^+_{nc} \\ (\mu,\beta)>0}} X_\beta =: J_\mu \in \wedge^p \mathfrak{p}^+$$

defines a line in $\wedge^p \mathfrak{p}^+$. The E_2-term of the HSSS looks like

$$\begin{array}{|c|cccc|c|} \hline * & * & \cdot & \cdot & * & * \\ \hline 0 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ \hline \end{array}$$

and using $H^q_{(2)}(D, L_\mu) \neq 0$ only for $q = q(\mu) = d + e$ the E_2 is

$$\begin{array}{cccccccc} & & & & e & & & \\ \hline \multicolumn{1}{|c}{0} & \cdot & \cdot & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{\mathbb{C}} & 0 & \cdot & \multicolumn{1}{c|}{\cdot\ \ 0} \\ \hline \multicolumn{1}{|c}{0} & \cdot & \cdot & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & 0 & \cdot & \multicolumn{1}{c|}{\cdot\ \ 0} \\ \multicolumn{1}{|c}{\cdot} & & & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{\cdot} & \cdot & & \multicolumn{1}{c|}{\cdot} \\ \multicolumn{1}{|c}{0} & \cdot & \cdot & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & 0 & \cdot & \multicolumn{1}{c|}{\cdot\ \ 0} \\ \hline \end{array}$$

where the non-zero term is

$$J_\mu \otimes \left(\text{Kostant class of the lowest } K\text{-type in } H^{q(\mu)}_{(2)}(D, L_\mu)\right).$$

Finally, the condition $|\mu| \gg 0$ may be removed, as in [Sch2], using Zuckerman translation and Casselman-Osborne.

The above is of course not meant to give a proof of Schmid's results in (B), but rather to indicate why they might hold.

Appendix to Lecture 5: The Borel-Weil-Bott (BWB) theorem

The most classical relation between representation theory and complex geometry is the BWB theorem. For reference and for use in Lecture 7 we shall briefly discuss a special case of it here.

The special case deals with a flag variety $\check{D} = G_{\mathbb{C}}/B$. The general case is that of a homogeneous projective variety $G_{\mathbb{C}}/P$, and it may be reduced to the special case using the Leray spectral sequence for the fibration $G_{\mathbb{C}}/B \to G_{\mathbb{C}}/P$.

We consider a weight μ giving rise to a $G_{\mathbb{C}}$-homogeneous line bundle $L_\mu \to \check{D}$. Let

$$q_c(\mu+\rho) = \#\left\{\alpha \in \Phi^+ : (\mu+\rho,\alpha) < 0\right\}.$$

This is the same $q(\mu+\rho)$ as defined earlier, but where we take for our real form of $G_{\mathbb{C}}$ the compact real form G_c, so that then $\check{D} = G_c/T$. The statement of the BWB theorem has two parts, the first of which is

(i) *if $\mu+\rho$ is singular, then all the cohomology groups $H^q(\check{D}, L_\mu) = 0$.*

If $\mu+\rho$ is regular, then there is a unique element $w \in W$ in the Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ such that $w(\mu+\rho) \in C$, the interior of the positive Weyl chamber for Φ^+.

(ii) *$H^{q_c(\mu+\rho)}(\check{D}, L_\mu)$ is the irreducible $G_{\mathbb{C}}$-module with highest weight $w(\mu+\rho)-\rho$.*

Thus the same $G_{\mathbb{C}}$-module may appear in different ways as cohomology groups. In the appendix to Lecture 7 we shall show that these different realizations are all related geometrically via Penrose transformations (which in fact leads to yet another proof of the BWB theorem in this case).

We want to make a couple of observations about the BWB theorem. The first is an explanation of the pervasive appearance of the expression

$$w(\mu+\rho)-\rho$$

in the subject: *it is forced by Kodaira-Serre duality.* In more detail, the original Borel-Weil theorem was the case when $\mu \in \overline{C}$, in which case $\mu+\rho \in C$, and it states that:

$H^0(\check{D}, L_\mu)$ is *the irreducible $G_{\mathbb{C}}$-module with highest weight μ.*

This result may be proved rather directly (cf. [Sch2]).

Keeping $\mu \in \overline{C}$, setting $\dim \check{D} = n$ and noting that $\omega_{\check{D}} = L_{-2\rho}$, Kodaira-Serre duality gives that

$$H^n(\check{D}, L_{-\mu-2\rho}) \text{ is the dual } G_{\mathbb{C}}\text{-module to } H^0(\check{D}, L_\mu).$$

Define $f(\nu)$ by "$H^n(\check{D}, L_\nu)$ has highest weight $f(\nu)$." Then $H^n(D, L_{-\mu-2\rho})$ has highest weight $f(-\mu-2\rho)$ and since $H^0(\check{D}, L_\mu)^*$ has lowest weight $-\mu$,

$$f(-\mu-2\rho) = w(-\mu)$$

where $w(\Phi^-) = \Phi^+$. Replacing μ by $-\lambda$ gives

$$f(\lambda - 2\rho) = w(\lambda).$$

Then formally replacing $\lambda - 2\rho$ by μ and using that $w(\rho) = -\rho$ (see below) gives

$$f(\mu) = w(\mu + 2\rho) = w(\mu + \rho) + w(\rho) = w(\mu + \rho) - \rho,$$

which was what we wanted to show.

Next, following the classical paper [Ko], we want to give the $\mathfrak{n}$-cohomology interpretation of the BWB theorem. For this we use here the following notations, which with apologies are not the same as those in the lecture:

- $\mathfrak{n}_c = \bigoplus\limits_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}$.

The subscript "c" here refers to the compact real form G_c of $G_{\mathbb{C}}$, where $\Phi_c^+ = \Phi^+$ is the set of all positive roots.

- For $w \in W$ we set $\Psi_w = w\Phi^- \cap \Phi^+$.

This is the set of negative roots that change sign under w.

- For $\Psi \in \{\psi_1, \dots, \psi_q\} \subset \Phi^+$ we set

$$\begin{cases} \langle \Psi \rangle = \psi_1 + \cdots + \psi_q \\ \omega^{-\Psi} = \omega^{-\psi_1} \wedge \cdots \wedge \omega^{-\psi_q}. \end{cases}$$

Here, for $\alpha \in \Phi^+$ we are denoting by $\omega^{-\alpha} \in \mathfrak{n}_c^*$ the dual to the negative root vector $X_{-\alpha}$. In the appendix to Lecture 7 we will interpret the $\omega^{-\alpha}$ geometrically in the context of the EGW-theorem.

- If $\alpha_1, \dots, \alpha_r$ are the positive roots, then
  - (i) $\rho - \langle \Psi \rangle = \frac{1}{2}(\pm\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_r)$ for some choices of signs, and as Ψ runs through all subsets of Φ^+ all choices of signs are possible;
  - (ii) Ψ_w and $\Psi_w^c =: \Phi^+ \backslash \Psi_w$ are both closed under addition;
  - (iii) if $\Psi \subset \Phi^+$ has this property, then $\Psi = \Psi_w$ for a unique $w \in W$;
  - (iv) $w(\rho) = \rho - \langle \Psi_w \rangle$; and
  - (v) $\langle \Psi \rangle = \langle \Psi_w \rangle \Rightarrow \Psi = \Psi_w$.

Let V^λ be the irreducible $G_{\mathbb{C}}$-module with highest weight λ. The dual $G_{\mathbb{C}}$-module V^{λ^*} has lowest weight $-\lambda$ and we let $v_{-\lambda}^*$ be a lowest weight vector. Then for any $w \in W$, $w(-\lambda)$ is an extremal weight for V^{λ^*} and we let $v_{w(-\lambda)}^*$ be the corresponding weight vector. Finally we set

$$\begin{aligned} \mu &= w^{-1}(\lambda + \rho) - \rho \Rightarrow \lambda = w(\mu + \rho) - \rho, \\ \kappa_\mu &= v_{w(-\lambda)}^* \otimes \omega^{-\langle \Psi_m \rangle}. \end{aligned}$$

Theorem: (i) $H^q(\mathfrak{n}_c, V^{\nu^*})_{-\mu} = 0$ *for* $\nu \neq w(\mu+\rho)-\rho$ *and* $q \neq q_c(\mu+\rho)$.

(ii) $\dim H^{q_c(\mu+\rho)}(\mathfrak{n}_c, V^{w(\mu+\rho)-\rho})_{-\mu} = 1$ *with generator* κ_μ.

We shall refer to κ_μ as the *Kostant class*. It is a harmonic form in the sense of [Ko], and also in the sense of the EGW-theorem to be discussed in the appendix to Lecture 7.

It is instructive to see why the Lie algebra coboundary $\kappa_\mu = 0$. This follows from property (ii) above and

$$X_{-\beta} v^*_{w(-\lambda)} = 0 \text{ for } \beta \in \Psi^c_w.$$

We will give the calculation in the appendix to Lecture 7 when we discuss the BWB in the context of the EGW-theorem and Penrose transforms there.

Finally we remark that we shall need the BWB when G_c is only reductive. Here we are considering G_c/T as a G_c-homogeneous complex manifold. As a complex manifold this is the same as $G_c^{\mathrm{ad}}/T^{\mathrm{ad}}$ where $T^{\mathrm{ad}} = G_c^{\mathrm{ad}} \cap T$. But as homogeneous complex manifolds G_c/T and $G_c^{\mathrm{ad}}/T^{\mathrm{ad}}$ are quite different. One may think here of $\mathbb{P}^1 = \mathcal{U}(2)/T$. The characters of T are a semi-direct product of those on T^{ad} and on G_c itself, and the action of the latter on cohomology must be added to the usual statement of the BWB theorem. We see this already when the line bundle L is associated to a character of G_c. It is trivial as a holomorphic line bundle but non-trivial as a homogeneous one; the action of G_c on $H^0(G_c/T, L)$ is non-trivial.

Lecture 6
Geometry of flag domains: Part I

In these two lectures we will introduce and explain the relationships among and major properties of three constructions associated to a flag domain $D = G_{\mathbb{R}}/T$:

- cycle space $\mathcal{U}$;
- incidence variety $\mathcal{I}$;
- correspondence space $\mathcal{W}$.

Both $\mathcal{U}$ and $\mathcal{W}$ will be shown to be Stein manifolds and there will be a basic diagram on which $G_{\mathbb{R}}$ acts equivariantly:

$$\begin{tikzcd}
& \mathcal{W} \arrow[d, "\pi"'] \arrow[ddl, "\pi_{\mathcal{I}}"'] \arrow[ddr, "\pi'"] & \\
& \mathcal{I} \arrow[dl, "\pi_D"] \arrow[dr, "\pi_{\mathcal{U}}"'] & \\
D & & \mathcal{U}
\end{tikzcd}$$

The fibres of $\mathcal{W} \xrightarrow{\pi} D$ and of $\mathcal{W} \xrightarrow{\pi_{\mathcal{I}}} \mathcal{I}$ will be seen to be contractible, so that the basic theorem of Eastwood-Gindikin-Wong [EGW], discussed in the next lecture, will apply to this situation. In particular the cohomology groups $H^q(D, L_\mu)$ will be represented by global, holomorphic data on $\mathcal{W}$.

As a consequence of *Matsuki duality*, which is explained below, we will see that $\mathcal{U}$ and $\mathcal{W}$ have the property of universality. One implication is that $\mathcal{U}$ and $\mathcal{W}$ depend only on the flag variety $\check{D} = G_{\mathbb{C}}/B$ and not on the particular flag domain D. A consequence of this is that if we index the open $G_{\mathbb{R}}$-orbits in $\check{D}$ as

$$D_w \subset \check{D}, \qquad\qquad w \in W/W_K$$

then there are diagrams

$$\begin{tikzcd}
& \mathcal{W} \arrow[dl, "\pi_w"'] \arrow[dr, "\pi_{w'}"] & \\
D_w & & D_{w'}
\end{tikzcd}$$

and applying the [EGW] theorem from Lecture 7 enables us to relate the cohomology groups on D_w to those on $D_{w'}$. It is this property that suggests the name *correspondence space* for $\mathcal{W}$.

Pseudo-convexity of D

We shall use the notations

$$\begin{aligned} n &= \dim D \\ d &= \dim K/T. \end{aligned}$$

A basic result, dating to [Sch1], [GS] and discussed in [FHW] and the references cited there, is

THEOREM: *There exists an exhaustion function*

$$f : D \to \mathbb{R}$$

whose Levi form $\mathcal{L}(f)$ has everywhere at least $n - d$ positive eigenvalues.

The argument will proceed in several steps.

Step one: For a holomorphic line bundle over a complex manifold the ratio of the lengths of any section relative to two Hermitian metrics is a well-defined positive function. We let h, h_c be the length function for the $G_{\mathbb{R}}$, respectively G_c invariant metrics in $\omega_D, \omega_{\check{D}}$. Then

$$f = -\log(h/h_c)$$

is a well-defined function from D to $\mathbb{R}$. Moreover, the *Levi form*

$$\mathcal{L}(f) = \frac{i}{2\pi}\partial\overline{\partial} f = c_1(\omega_D) - c_1(\omega_{\check{D}}),$$

and we have seen in Lecture 5 that

- $c_1(\omega_{\check{D}}) < 0$;
- $c_1(\omega_D)$ has everywhere $\geqq n - d$ positive eigenvalues.

It follows that $\mathcal{L}(f)$ has the property in the theorem. Therefore, it remains to show that f is an exhaustion function; i.e.,

$$f(x) \to \infty \ \text{ as } \ x \to \partial D.$$

Since the volumes of D relative to h and h_c satisfy

$$\begin{cases} \operatorname{vol}(D, h) = \infty \\ \operatorname{vol}(D, h_c) < \infty \end{cases}$$

this is at least plausible. For $D = \Delta$ the unit disc in $\check{D} = \mathbb{P}^1$ we have

$$\begin{aligned} h_c &= \frac{dz d\bar{z}}{(1 + |z|^2)^2} \\ h &= \frac{dz d\bar{z}}{(1 - |z|^2)^2} \end{aligned}$$

which gives

$$\begin{aligned} f &= -\log\left(\frac{1-|z|^2}{1+|z|^2}\right) \\ &= \log\frac{1}{1-|z|} + O(1). \end{aligned}$$

Step two: There are G_c, respectively $G_{\mathbb{R}}$ invariant metrics $(\ ,\ )_{\check{D}}, (\ ,\ )_D$ in the tangent bundles $T\check{D}, TD$, and h_c, h are the induced metrics in the dual top exterior powers. Thus we need to compare $(\ ,\ )_c$ and $(\ ,\ )$ in D as we approach ∂D. Specifically, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of h relative to h_c, it will suffice to show

- the λ_i extend to continuous functions on $\overline{D} = D \cup \partial D$;
- at least one $\lambda_i(x) \to 0$ as $x \to \partial D$.

We recall our notations

- $B_x =$ Borel subgroup corresponding to $x \in \check{D}$;
- $B = B_{x_0}$ where $x_0 \in D$ is the reference point.

Then B determines a set Φ^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ where $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ and where

- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$;
- $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}^{\alpha}$;
- $T_{x_0} D \cong \mathfrak{n}^+$.

Each of $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{g}_c$ is a real form of $\mathfrak{g}_{\mathbb{C}}$ and we let

$$\begin{cases} \sigma : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} \\ \tau : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} \end{cases}$$

be the respective conjugation. Then denoting by $\mathbf{B}$ the Cartan-Killing form, for $u, v \in \mathfrak{n}^+$ we have[32]

$$\begin{cases} (u,v)_D = \mathbf{B}(u, \sigma(v)) \\ (u,v)_{\check{D}} = -\mathbf{B}(u, \tau(v)). \end{cases}$$

We let

- $\mathcal{O} = G_{\mathbb{C}}$-orbit of $T_{x_0}\check{D}$ in $T\check{D}$;
- $\mathcal{O}_{\mathbb{R}} = G_{\mathbb{R}}$-orbit of $T_{x_0} D$ in $T\check{D}$;
- $\mathcal{O}_c = G_c$-orbit of $T_{x_0}\check{D}$ in $T\check{D}$.

[32]We here use $\mathbf{B}$ to denote the Cartan-Killing form, since the customary notation for it denotes in this lecture the Borel subgroup.

Then we note that each of $(\ ,\ )_D$ and $(\ ,\ )_{\check{D}}$ give continuous functions on $\mathcal{O}$ which restrict to the respective $G_{\mathbb{R}}$, G_c invariant metrics on TD and $T\check{D}$. Here we are identifying $T_x\check{D} \cong \operatorname{Ad} g_x \cdot \mathfrak{n}^+$ where $g_x \cdot x_0 = x$ with $g_x \in G_{\mathbb{C}}$. We have to show

If $x \in \partial D$, then $(\ ,\ )_D(x)$ is degenerate.

It is positive semi-definite by continuity.

Step three: The crucial step is

Bruhat's Lemma: *Any two Borel sub-algebras of $\mathfrak{g}_{\mathbb{C}}$ contain a common σ-stable Cartan sub-algebra.*

We apply this to $\mathfrak{b}_x$ and $\sigma(\mathfrak{b}_x)$ and denote by $\mathfrak{h}_x$ a common Cartan sub-algebra of $\mathfrak{g}_{\mathbb{C}}$ with Φ_x denoting the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_x)$. Then

$$T_x\check{D} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{b}_x$$

singles out a set Φ_x^+ of positive roots.

So far this discussion applies to any $x \in \check{D}$. We need to use the assumption that $x \in \partial D$, which implies for the $G_{\mathbb{R}}$-orbit $G_{\mathbb{R}} \cdot x$ of x that the real codimension

$$\operatorname{codim}_{\check{D}} G_{\mathbb{R}} \cdot x > 0.$$

Let $V_x \subset G_{\mathbb{R}}$ be the stability group of x with Lie algebra $\mathfrak{v}_x$. Then

$$\mathfrak{v}_x = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{b}_x.$$

We note that

- $\mathfrak{v}_x$ is a real form of $\mathfrak{b}_x \cap \sigma(\mathfrak{b}_x)$, and
- $\mathfrak{h}_x = \mathfrak{h}_{x,\mathbb{R}} \otimes \mathbb{C}$

where $\mathfrak{h}_{x,\mathbb{R}} = \mathfrak{h}_x \cap \sigma(\mathfrak{h}_x)$ is a real Cartan-sub-algebra of $\mathfrak{g}_{\mathbb{R}}$ with

$$\mathfrak{h}_{x,\mathbb{R}} \subset \mathfrak{v}_x.$$

We have

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \mathfrak{h}_x \oplus \left(\bigoplus_{\alpha \in \Phi_x} \mathfrak{g}_x^{\alpha} \right) \\ \mathfrak{b}_x &= \mathfrak{h}_x \oplus \left(\bigoplus_{\alpha \in \Phi_x^+} \mathfrak{g}_x^{-\alpha} \right) =: \mathfrak{h}_x \oplus \mathfrak{n}_x \end{aligned}$$

where $\mathfrak{g}_x^{\alpha} \subset \mathfrak{g}_{\mathbb{C}}$ is the α-root space for $\alpha \in \Phi_x$. Also,

$$\mathfrak{v}_{x,\mathbb{C}} = \mathfrak{h}_{x,\mathbb{C}} \oplus (\oplus \text{ root spaces}).$$

Since $\mathfrak{v}_x$ is a real form of $\mathfrak{b}_x \cap \sigma(\mathfrak{b}_x)$,

$$\mathfrak{v}_x = \mathfrak{h}_{x,\mathbb{R}} \oplus \left(\bigoplus_{\alpha \in \Phi_x^+ \cap \sigma\Phi_x^+} \mathfrak{g}_x^{-\alpha} \right)_{\mathbb{R}}$$

where $\left(\quad \right)_{\mathbb{R}}$ is the root space. This gives for the *real codimension*

$$\operatorname{codim}_{\check{D}} G_{\mathbb{R}} \cdot x = \#\left(\Phi_x^+ \cap \sigma\Phi_x^+\right).$$

Thus

$$\operatorname{codim}_{\check{D}} G_{\mathbb{R},x} > 0 \Leftrightarrow \Phi_x^+ \cap \sigma\Phi_x^+ \neq \emptyset.$$

We are now done. Namely, let $0 \neq v \in \left(\bigoplus_{\alpha \in \Phi_x^+ \cap \sigma\Phi_x^+} \mathfrak{g}^{\alpha} \right)_{\mathbb{R}}$. Then $v \in \mathfrak{n}_x$ and

$$(v, v)_D = \mathbf{B}(v, \sigma v) = \mathbf{B}(v, v) = 0$$

since $\mathbf{B}(\mathfrak{n}_x, \mathfrak{n}_x) = 0$. $\square$

Remark: Intuitively, the $G_{\mathbb{R}}$-invariant metric in TD is induced from the metrics in the Hodge bundles using the inclusion

$$TD \subset \oplus \operatorname{Hom}(\mathbb{F}^p, \mathbb{V}_{\mathbb{C}}/\mathbb{F}^p).$$

The metrics in the $\mathbb{F}^p$ are non-singular in D, but at least one becomes singular on the boundary $\partial D = \overline{D} \backslash D$. This means that the second Hodge-Riemann bilinear relations become only positive semi-definite. This heuristic may help to explain what is behind the above argument.

Remark: For the complexified tangent space we have

$$(TG_{\mathbb{R}} \cdot x)_{x,\mathbb{C}} = \left\{ \left(\bigoplus_{\alpha \in \Phi_x^+ \cap \sigma\Phi_x^+} \mathfrak{g}^{\alpha} \right)_{\mathbb{R}} \otimes \mathbb{C} \right\} \oplus \left(\bigoplus_{\alpha \notin \Phi_x^+ \cap \sigma\Phi_x^+} \mathfrak{g}^{\alpha} \right).$$

The first factor is the complexification of the "real" part of the tangent space and the second factor is the complexification of the *Cauchy-Riemann* or complex part of the tangent space

$$T^{\mathrm{CR}} G_{\mathbb{R} \cdot x} = (TG_{\mathbb{R},x})_x \cap J_x (TG_{\mathbb{R} \cdot x})_x$$

where J_x is the almost complex structure in the real tangent space $T_{x,\mathbb{R}} \check{D}$ to $\check{D}$ at x.

A simple example will illustrate the mechanism in the argument. For $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$ and $D = \mathcal{H}$, $\check{D} = \mathbb{P}^1$ we let

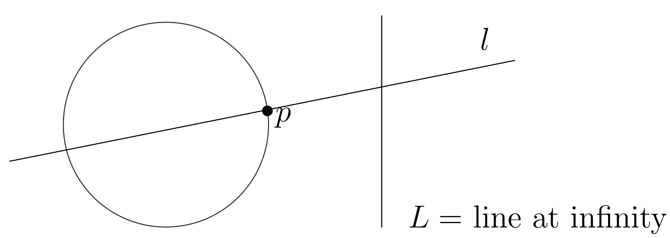
$$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \partial D$$

be the origin. Then

- $$\mathfrak{b}_x = \left\{ \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\},$$
- $$\sigma \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} = \begin{pmatrix} \bar{a} & 0 \\ \bar{b} & -\bar{a} \end{pmatrix},$$
- $$\mathfrak{h}_x = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{C} \right\},$$
- $$\mathfrak{v}_x = \left\{ \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$
- $$v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ is a } \sigma\text{-real root vector relative to } \mathfrak{h}_{x,\mathbb{R}}.$$

A more substantive example is this.

Example: For $G_{\mathbb{R}} = S\mathcal{U}(2,1)_{\mathbb{R}}$ with non-classical domain $D \subset \check{D}$ as discussed before, we consider the point $x = (p, l) \in \partial D$ given by



Here

$$p = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad L = [0,0,1], \quad l = [0,1,0].$$

We have

- $$\mathfrak{g}_{\mathbb{C}} = \mathrm{sl}(3,\mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & a_{21} & a \\ a_{12} & a_{22} & b \\ c & d & e \end{pmatrix} : a_{11} + a_{22} + e = 0 \right\};$$

- $\sigma \begin{pmatrix} a_{11} & a_{21} & a \\ a_{12} & a_{22} & b \\ c & d & e \end{pmatrix} = \begin{pmatrix} -\bar{a}_{11} & -\bar{a}_{12} & \bar{c} \\ -\bar{a}_{21} & -\bar{a}_{22} & \bar{d} \\ \bar{a} & \bar{b} & -\bar{e} \end{pmatrix}$;

- $\mathfrak{b}_x = \left\{ \begin{pmatrix} a_{11} & a_{21} & a \\ 0 & a_{22} & 0 \\ c & d & e \end{pmatrix} : \begin{array}{l} a_{11} + a = c + e \\ a_{11} + a_{22} + e = 0 \end{array} \right\}$.

Note that $\dim_{\mathbb{C}} \mathfrak{b}_x = 5 = \dim \mathfrak{g}_{\mathbb{C}} - \dim \check{D}$.

- $\mathfrak{b}_x \cap \sigma(\mathfrak{b}_x) = \left\{ \begin{pmatrix} a_{11} & a_{21} & a \\ 0 & a_{22} & 0 \\ c & 0 & e \end{pmatrix} : \begin{array}{l} a_{11} + a_{22} + e = 0 \\ a_{11} + a = c + e \end{array} \right\}$;

- $\mathfrak{v}_x = \left\{ \begin{pmatrix} i\alpha & 0 & \gamma - i(2\alpha + \beta) \\ 0 & i\beta & 0 \\ \gamma + i(2\alpha + \beta) & 0 & -i(\alpha + \beta) \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$.

Note that for the real dimensions we have $\dim_{\mathbb{R}} \mathfrak{v}_x = 3 = \dim_{\mathbb{R}} G_{\mathbb{R}} - \dim_{\mathbb{R}}(G_{\mathbb{R}} \cdot x)$.

We take

$$H_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} \in \mathfrak{v}_x$$

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{v}_x$$

$$X = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} \in \mathfrak{v}_x$$

Then

- $\mathfrak{v}_x = \operatorname{span}_{\mathbb{R}} \{H_1, H_2, X\}$;

- $\mathfrak{h}_{x,\mathbb{R}} = \operatorname{span}_{\mathbb{R}} \{H_1, H_2\}$;

- $[H_1, X] = 2X$, $[H_2, X] = 0$;

- $V_x \sim S^1 \times N$ where $N \cong \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}, u, v \in \mathbb{R} \right\}$, where $\sim$ denotes is "isogeneous to";
- $\Phi_x^+ \cap \sigma\Phi_x^+ =$ "2".

In particular, $\Phi_x^+ = \sigma(\Phi_x^+)$ and $\operatorname{codim}_{\mathbb{R}} G_{\mathbb{R}} \cdot x = 1$.

Remark: In this example the real part of the tangent space maps onto the real part to the tangent space to the sphere $S^3 = \partial\mathbb{B}$. The Cauchy-Riemann part has dimension 2 and may be described as the direct sum of two pieces

(i) p varies in the CR-part of T_pS^3;
(ii) l varies in the $\mathbb{P}^1$ of lines in $\mathbb{P}^2$ passing through p.

The Levi form is positive on the first part and zero on the second part, which is a holomorphic curve in $G_{\mathbb{R}} \cdot x$.

Using a standard result in complex analysis (cf. the references in [GS], [Sch1] and [FHW]) the above theorem has the following

Corollary: *For any coherent analytic sheaf $\mathcal{F} \to D$,*

$$H^q(D, \mathcal{F}) = 0 \text{ for } d > n - d.$$

The cycle space

Let $D = G_{\mathbb{R}}/T$ be a homogeneous complex manifold as above. Then $Z_0 =: K/T$ is a compact, complex submanifold of D.

Definition: *The* **cycle space**

$$\mathcal{U} = \{gZ_0 : g \in G_{\mathbb{C}} \text{ and } gZ_0 \subset D\}.$$

That is, $\mathcal{U}$ is the set of translates by elements in $G_{\mathbb{C}}$ of the compact, complex submanifold Z_0 that remain in D.

A basic fact is given by the

Theorem: *$\mathcal{U}$ is a Stein manifold.*

We shall give one argument, following [W3], and shall then discuss another argument from [BHH] that gives additional information that will be used later.

Proof. It will suffice to produce a strictly plurisubharmonic exhaustion function

$$F : \mathcal{U} \to \mathbb{R}.$$

We denote points of $\mathcal{U}$ by u and let $Z_u \subset D$ be the corresponding compact, complex submanifold of D. Set

$$F(u) = \sup_{x \in Z_u} f(x).$$

Then F is an exhaustion function. There are some technical issues regarding the smoothness of F for which we refer to [W3]. We want to show that the Levi form

$$\mathcal{L}(F) > 0.$$

Let $x_u \in Z_u$ be a point at which $f\big|_{Z_u}$ has a maximum. Then by the maximum principle

- $df\big|_{T_{x_u} Z_u} = 0$
- $\mathcal{L}\left(f\big|_{Z_u}\right) \leqq 0$

We want to show that for $\xi \in T_u\mathcal{U}$

$$\mathcal{L}(F)(\xi) > 0.$$

For this we identify ξ with a normal vector field to Z_u in D; i.e.,

$$\xi \in H^0(Z_u, N_{Z_u/D}).$$

Then

$$\mathcal{L}(F)(\xi) = \mathcal{L}(f)(\xi(x_u)).$$

Since $\mathcal{L}(F)$ has everywhere at least $\dim D - \dim Z_u$ positive eigenvalues and $\mathcal{L}(F) \leqq 0$ in $T_{x_u} Z_u$, we may infer that $\mathcal{L}(F)(\xi) > 0$ as desired. $\square$

The cycle space in the non-classical case

The structure of the cycle space $\mathcal{U}$ defined above is quite different in the classical and non-classical cases. If D is a classical flag domain, then there is a fibration

$$D \to D_{\mathrm{HSD}} = G_{\mathbb{R}}/K$$

over an Hermitian symmetric domain that is either holomorphic or anti-holomorphic. It follows that the image of any compact, connected complex analytic submanifold of D is a point. Thus $\mathcal{U} \cong G_{\mathbb{R}}/K$ with one of the two homogeneous complex structures. In these lectures we are primarily interested in the non-classical case, and therefore in the remainder of this lecture we shall assume that

D is non-classical.

We then have the

PROPOSITION: $\mathcal{U} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$.

Proof. Let $u_0 \in \mathcal{U}$ be the reference point corresponding to $Z_0 = K/T \subset D$. We will show that there is a natural identification

$$T_{u_0}\mathcal{U} = \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}.$$

This will establish the proposition at the tangent space level, and we refer to [FHW] for the proof of the full statement. We will also assume that $\mathfrak{g}_{\mathbb{C}}$ is simple, as the general case may be reduced to this one.

We may think of $\mathfrak{g}_{\mathbb{C}}$ as a Lie algebra of holomorphic vector fields on $\check{D}$. Restricting these vector fields to Z_0 gives a map

$$\mathfrak{g}_{\mathbb{C}} \to H^0(Z_0, N_{Z_0/D}),$$

where the normal vector fields are thought of as infinitesimal deformations of Z_0 in D. With this interpretation there is an inclusion

$$T_{u_0}\mathcal{U} \hookrightarrow H^0(Z_0, N_{Z_0/D}),$$

and by the definition of $\mathcal{U}$ we have

$$\mathfrak{g}_{\mathbb{C}} \to T_{u_0}\mathcal{U} \hookrightarrow H^0(Z_0, N_{Z_0/D}).$$

Since the complexification $K_{\mathbb{C}}$ of K acts on the compact, homogenous complex manifold K/T, we see that the vector fields corresponding to $\mathfrak{k}_{\mathbb{C}}$ are tangent to Z_0, so that we have the natural surjective mapping

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}} \twoheadrightarrow T_{u_0}\mathcal{U}$$

that we want to show is injective. Thinking of $\mathfrak{g}_{\mathbb{C}}$ as normal vector fields along Z_0, the subspace of those that are tangent to Z_0 is a *sub-algebra*. Thus we have to show

> *Let $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ be a sub-algebra with $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ and where both inclusions are proper. Then there is a choice of positive roots such that*
>
> $$\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$$
>
> *where* $\mathfrak{p}^- = \underset{\beta\in\Phi^-_{nc}}{\oplus} \mathfrak{g}^\beta$.

Since $G_{\mathbb{R}}$ is assumed to be simple it is known [K1] that in the Cartan decomposition

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}.$$

$\operatorname{Ad} K$ acts irreducibly, and it acts absolutely irreducibly if, and only if, $G_{\mathbb{R}}$ is not of Hermitian type. If $G_{\mathbb{R}}$ is of Hermitian type, then $K = Z(S^1)$ where the circle $S^1 \subset K$

is the center. Moreover, $\operatorname{Ad} S^1$ acting on $\mathfrak{p}_{\mathbb{C}}$ decomposes into conjugate eigenspaces

- $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$
- $\mathfrak{p}^- = \overline{\mathfrak{p}^+}$

where the conjugation is relative to the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{C}}$; i.e., the σ above. It then follows that since the inclusions $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ are proper

$$\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^{\pm}.$$

There is then a choice of positive roots such that $\mathfrak{q}$ is as in the italicized statement above. $\square$

A natural question that arises from the above argument is:

Are all the deformations of Z_0 in D obtained from the cycle space?

Here one should be a little fussy and phrase the question more precisely as follows:

(i) Does $\mathfrak{U}$ contain a (topological) neighborhood of u_0 in the Hilbert scheme of $\check{D}$?
(ii) Is the Hilbert scheme reduced at u_0?

The answer to (ii) is "yes" (cf. [FHW]), and the answer to (i) is "no" in general. We shall see below that

> For $D = S\mathfrak{U}(2,1)_{\mathbb{R}}/T$ with a non-classical complex structure we have $\mathfrak{g}_{\mathbb{C}} \cong H^0\big(Z_0, N_{Z_0/0}\big)$. As we shall see in the appendix to Lecture 9, for $D = \mathrm{Sp}(4)_{\mathbb{R}}/T$ with a non-classical complex structure, $\dim H^0(Z_0, N_{Z_0/D}) = \dim \mathfrak{g}_{\mathbb{C}} + 1$.

The definition of $\mathfrak{U}$ depended on a particular choice of flag domains $D \subset \check{D}$. It was proved in [AkG] that $\mathfrak{U}$ has the following *universality property.*

THEOREM: *$\mathfrak{U} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ is the same for any D.*[33]

In fact they prove more. Let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$$

be a Cartan decomposition and $\mathfrak{A} \subset \mathfrak{p}$ a maximal abelian sub-algebra. It is known that any two such are conjugate under $\operatorname{Ad} K$, and that $\operatorname{Ad} K(\mathfrak{A}) = \mathfrak{p}$. Let $\Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})$ be the *restricted root system.* It is also known that $\mathfrak{g}_{\mathbb{R}}$ is an orthogonal direct sum of the restricted root spaces, which are the common eigenspaces of the $\operatorname{ad} H$ for $H \in \mathfrak{A}$, all of the eigenvalues being real. Following [AkG] one defines

$$\omega_0 = \{H \in \mathfrak{A} : |\alpha(H)| < \pi/2 \text{ for } \alpha \in \Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})\}.$$

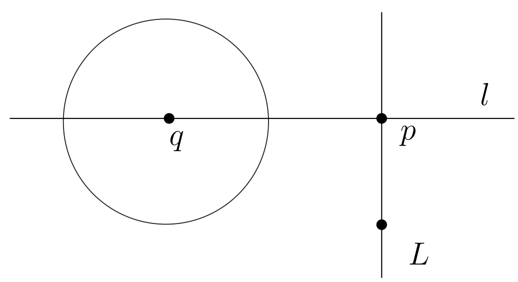
[33]This will be formulated somewhat more precisely below.

Recalling that $\mathcal{U} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, [AkG] prove that

$$\boxed{\mathcal{U} = G_{\mathbb{R}} \exp(i\omega_0) \cdot u_0.}$$

In the appendix to this lecture there are more details about the root space decomposition of $\mathfrak{g}_{\mathbb{R}}$ under the action of $\mathfrak{A}$ with application to the computation of the tangent spaces to $\mathcal{U}$ along $\exp(i\omega_0) \cdot u_0$.

Example: Referring to the $S\mathcal{U}(2,1)$ example, for the open $G_{\mathbb{R}}$-orbit D we have the picture



where

- $q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ =origin in $\mathbb{B}$;
- $L = [0,0,1]$ = line at infinity;
- $p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = point on L

and $l = \overline{qp}$. Then u_0 corresponds to the maximal compact subvariety $Z(q,L)$. Taking

$$\mathfrak{A} = \left\{ H = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\} \cong \mathbb{R}$$

the action of $\exp itA$ on u_0 may be described geometrically as follows: As t increases the point p and line L move at equal speed to where at $t = t_0$ we have

$$p_{t_0} \in \partial\mathbb{B}, \;\; L_{t_0} \text{ tangent to } \partial\mathbb{B} \text{ at } p_{t_0}.$$

In coordinates we take $a = 1$ in H above. Then

$$\exp(itH) = \begin{pmatrix} \cos t & 0 & i\sin t \\ 0 & 1 & 0 \\ i\sin t & 0 & \cos t \end{pmatrix}$$

$$\exp(itH) = \begin{pmatrix} i\sin t \\ 0 \\ \cos t \end{pmatrix}$$

and the condition $\exp(itH)q \in \mathbb{B}$ is $|t| < \pi/4$, which is consistent with the boxed result and in this case the root being "2."

Example: $\mathrm{SO}(4,1)$. We may first identify

$$\check{\mathcal{W}} = G_{\mathbb{C}}/K_{\mathbb{C}} = \left\{ E \in \mathrm{Gr}(4, V_{\mathbb{C}}) : Q\big|_E \text{ non-singular} \right\}.$$

Then we let $u = E^{\perp}$ where "$\perp$" means "Q orthogonal complement." Then one may show that

$$E \in \mathcal{U} \Leftrightarrow \left\{ \begin{array}{c} \mathrm{span}(u, \bar{u}) \text{ has dimension one and } H\big|_{\mathrm{span}(u,\bar{u})} < 0, \\ \text{or } H\big|_{\mathrm{span}(u,\bar{u})} \text{ has signature } (1,1) \end{array} \right\}$$

where $H(u, v) = Q(u, \bar{v})$. The first condition is equivalent to $E = \overline{E}$ and $H\big|_E > 0$. The set of such E's is just the real symmetric space $\mathrm{SO}(4,1)_{\mathbb{R}}/\mathrm{SO}(4)_{\mathbb{R}} \subset \mathcal{U}$. Either of the two conditions is equivalent to

$$E \in \mathcal{U} \Leftrightarrow \text{ for } 0 \neq v \in E, \text{ if } Q(v,v) = 0 \text{ then } H(v, \bar{v}) > 0.$$

We will interpret this result Hodge-theoretically. For this we let $D_0 \cong \mathrm{SO}(4,1)_{\mathbb{R}}/\mathcal{U}(2)$ be the period domain for PHS's of weight $n = 2$ and with $h^{2,0} = 2$, $h^{1,1} = 1$. A point of D_0 is $F^2 \in \mathrm{Gr}(2, \mathbb{C}^5)$ with

$$\begin{cases} Q(F^2, F^2) = 0, \quad \text{i.e., } F^2 \in \mathrm{Gr}_L(2, \mathbb{C}^5) \\ Q(F^2, \overline{F}^2) > 0. \end{cases}$$

The flag domain D is the set of Hodge flags given by $J \subset F^2$, $\dim J = 1$. Thus $D \to D_0$ is a $\mathbb{P}^1$-bundle. We note that given $J \subset F^2$ there is a full flag

$$0 \subset J \subset F^2 \subset F^{2\perp} \subset J^{\perp} \subset \mathbb{C}^5.$$

The maximal compact subvarieties of D and D_0 are in one-to-one correspondence under the map $D \to D_0$. For D and E as above, the maximal compact subvariety is

$$Z(E) = \left\{ F^2 \subset E : Q(F^2, F^2) = 0 \right\} = \mathrm{Gr}_L(2, E).$$

It is interesting to interpret the [AkG] result in this case. Taking for Q the standard form

$$Q = \begin{pmatrix} I_4 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\mathfrak{g}_{\mathbb{R}} = \left\{ \begin{pmatrix} 0 & {}^t b & a \\ -b & A & c \\ \underbrace{a}_{1} & \underbrace{{}^t c}_{3} & \underbrace{0}_{1} \end{pmatrix} : a \in \mathbb{R} \text{ and } b, c \in \mathbb{R}^3, A \in \mathrm{so}(3)_{\mathbb{R}} \right\}.$$

Taking

$$\mathfrak{A} = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \right\}$$

and letting $\xi \in \mathfrak{A}$ be given by the above matrix when $a = t$ we have

$$\exp(i\xi) = \begin{pmatrix} \cot t & 0 & i \sin t \\ 0 & I_3 & 0 \\ i \sin t & 0 & \cos t \end{pmatrix}.$$

For our reference point $u_0 \in \mathfrak{U}$ we take the point $u_0 = (0, \dots, 0, 1)$ with corresponding $E_0 = \mathbb{C}^4 \subset \mathbb{C}^5$. Then for $u_t = \exp(iH) u_0$ we have

$$u_t = (i \sin t, 0, 0, 0 \cos t).$$

Then for t not an integral multiple of $\pi/2$, $\dim \operatorname{span}(u_t, \bar{u}_t) = 2$ and on this span

$$H = \begin{pmatrix} \sin^2 t & 0 \\ 0 & -\cos^2 t \end{pmatrix},$$

which has signature $(1,1)$ for $0 < t < \pi/2$.

Interlude on Grauert domains

The result of Akheizer-Gindikin [AkG], and the use we shall make of it below following [BHH], is part of a very nice story in complex geometry that we want to briefly outline.

We let M be a Riemannian manifold with metric g and set

- $TM \cong T^*M$ (identification using g);
- $\rho : TM \to \mathbb{R}$ the Riemannian distance;
- $\alpha =$ canonical 1-form on T^*M and $\omega = d\alpha$.

Grauert's idea was that there is a Stein complex structure in a neighborhood N of $M \subset TM$; in this way N has lots of real analytic functions. The basic result is this:

> *there exists a unique complex structure on a sufficiently small N such that (i) ρ^2 is strictly plurisubharmonic and the corresponding Kähler metric restricts to g on M; and (ii) $(\overline{\partial}\partial\rho)^n = 0$ on $N\backslash M$.*

The complex structure is the unique one such that for every geodesic $\gamma : [0, \epsilon] \to M$

$$s + it \to (\gamma(s), t\gamma'(s))$$

is a holomorphic curve in N. We shall refer to N as a *Grauert domain.*

A natural question is: *What is the maximal Grauert domain?* It is known (cf. [BHH] and the references cited there) that negative curvature of M implies that any N must have finite radius. This is because of the following result:

> *The almost complex structure tensor J is a solution of the Jacobi equation along holomorhpic curves as above.*

In this way the curvature enters the picture, and negative curvature turns out to imply that J develops a singularity in finite time.

In the case when $M = G_{\mathbb{R}}/K$ and

$$TM = G_{\mathbb{R}} \times_K \mathfrak{p}$$

the Jacobi operator is

$$Y \to R(Y, X)X = -(\operatorname{ad} X)^2 Y$$

where $X, Y \in \mathfrak{p}$ and R is the curvature. Then the Jacobi equation for J may be explicitly analyzed in terms of the eigenvalues of the operator $\operatorname{ad} X$ and it follows that the maximal Grauert domain is

$$\mathcal{G} = G_{\mathbb{R}} \times (\operatorname{Ad} K(\omega_0)) \subset TM.$$

The basic result in [AkG], with another proof given in [BHH], is

> *The map*
>
> $$\mathcal{G} \to \mathcal{U}$$
>
> *given by $(g, \operatorname{Ad} k(H)) \to gk \exp(iH)u_0$ is a $G_{\mathbb{R}}$-equivariant biholomorphism.*

If we identify the tangent space

$$T_{u_0}(G_{\mathbb{C}}/K_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \oplus i\mathfrak{p}$$

then the differential at the identity of the above map is the identity.

The above result leads to another proof, again following [BHH], that $\mathcal{U}$ is Stein. For this we first note that

the action of $G_{\mathbb{R}}$ on $\mathcal{U}$ is proper.

Proof. Let $u_0 \in \mathcal{U}$ be fixed and $\{g_n\} \in G_{\mathbb{R}}$ a sequence with $u_n = g_n u_0$. Assuming that $\{u_n\}$ is a bounded sequence in $\mathcal{U}$, we have to show that a subsequence converges to a point in $G_{\mathbb{R}}$. The maximal compact subvarieties $Z_{u_n} = g_n Z_{u_0}$ lie in a bounded subset in D. Then from the fact that $G_{\mathbb{R}}$ acts property on D, we may infer that $\{g_n\}$ is a bounded sequence in $G_{\mathbb{R}}$, and hence has a convergent sequence. $\square$

Another proof follows from the [AkG] result in the box above. Namely, one may observe that

> *For $u = \exp(iH) \cdot u_0$ where $H \in \omega_0$, the isotropy group $G_{\mathbb{R},u}$ is the centralizer of H in K.*

In particular, $G_{\mathbb{R},u} = Z_K(H)$ is compact.

A consequence is that the orbits are closed and the quotient space $G_{\mathbb{R}} \backslash \mathcal{U}$ is Hausdorff. A $G_{\mathbb{R}}$-invariant function

$$f : \mathcal{U} \to \mathbb{R}$$

is said to be an *exhaustion function modulo $G_{\mathbb{R}}$* if for a sequence $u_n \in \mathcal{U}$ that is divergent in $G_{\mathbb{R}} \backslash \mathcal{U}$ we have $f(u_n) \to \infty$. As shown in [BHH] such a function is uniquely determined by the restriction f_{ω_0} to a function on ω_0 that is invariant under the Weyl group $N_K(\mathfrak{A})/Z_K(\mathfrak{A})$, and any such function f_{ω_0} extends to a $G_{\mathbb{R}}$-invariant function f on $\mathcal{U}$. Moreover, f has Levi form $\mathcal{L}(f) > 0$ exactly when f_{ω_0} is strictly convex. It follows that there exist strictly plurisubharmonic functions f that are exhaustion functions modulo $G_{\mathbb{R}}$. In the appendix to this lecture we will further discuss this result.

Let now $\Gamma \subset G_{\mathbb{R}}$ be a co-compact, neat discrete group. Then the projection $\Gamma \backslash \mathcal{U} \to G_{\mathbb{R}} \backslash \mathcal{U}$ is proper. This implies that

$$f : \Gamma \backslash \mathcal{U} \to \mathbb{R}$$

is an exhaustion function in the usual sense, so that

$$\Gamma \backslash \mathcal{U} \ \textit{is Stein,}$$

as then is also its covering space $\mathcal{U}$.

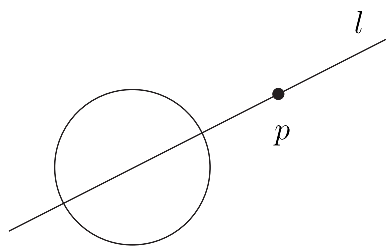
The result about $\Gamma \backslash \mathcal{U}$ will be used below.

Although we shall not need it in these lectures there is an interesting result describing the cycle space $\mathcal{U}$ in case *G is of Hermitian type*, meaning that the quotient $G_{\mathbb{R}}/K$ has the structure of an Hermitian symmetric domain $\mathcal{B}$.

Theorem ([BHH] and [FHW]): *If G is of Hermitian type and D is non-classical, then there is a biholomorphism*

$$\mathcal{U} \cong \mathcal{B} \times \overline{\mathcal{B}}.$$

Example: $S\mathfrak{U}(2,1)$. We recall that $\check{D}$ is the flag manifold for $\mathbb{P}^2$ consisting of pairs (p,l) where $p \in \mathbb{P}^2$, $l \in \mathbb{P}^{2^*}$ is a line in $\mathbb{P}^2$, and $p \in l$. We also recall that $\mathbb{B} \subset \mathbb{C}^2$ denotes the unit ball with $\mathbb{B}^c$ the complement of the closure. Points of the non-classical D are then given by the following sets of points $(p,l) \in \check{D}$

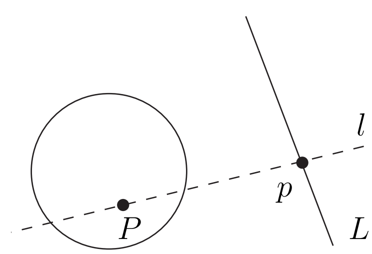


$$p \in \mathbb{B}^c,\ l \cap \mathbb{B} \neq \emptyset.$$

We next note that any pair (P,L) where

$$\begin{cases} P \in \mathbb{B} \\ L \cap \mathbb{B}^c = \emptyset \end{cases}$$

gives a compact subvariety $Z(P,L) \cong \mathbb{P}^1$ in D as described by the picture



That is, $Z(P,L) = \{(p,l) : l$ is a line through P and $p = l \cap L\}$. The cycle space $\mathfrak{U}$ is the set of all such $Z(P,L)'s$. We note that the set

$$\{L \in \mathbb{P}^{2^*}, L \subset \mathbb{B}^c\} \cong \overline{\mathbb{B}}.$$

Indeed, the LHS is just the set of points $L \in \mathbb{P}^{2^*}$ on which the Hermitian form $\mathbb{H}$ is negative, and $\mathbb{H}$ gives a conjugate linear isomorphism $\mathbb{C}^3 \xrightarrow{\sim} \mathbb{C}^{3^*}$. From this we see that

$$\mathfrak{U} \cong \mathbb{B} \times \overline{\mathbb{B}}.$$

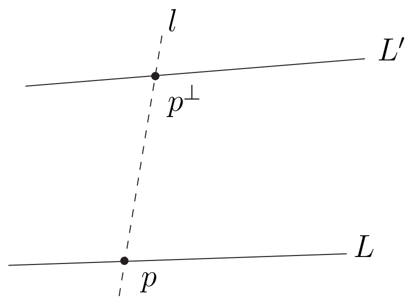
Example: Sp(4). We recall that $\check{D}$ is the set of Lagrange flags (p,l) in $\mathbb{P}^3$, where $l \in \mathbb{P}^3$ is a line that is Lagrangian for the alternating form Q and $p \in l$. One of the two non-classical flag domains $D \subset \check{D}$ is given by

$$D = \{(p,l) : \mathbb{H}(p) < 0,\ \mathbb{H}_l \text{ has signature } (1,1)\}$$

where $\mathbb{H}$ is the Hermitian form described in Lecture 3 and $\mathbb{H}_l$ is the restrction of $\mathbb{H}$ to l. For each pair L, L' of Lagrangian lines with

$$\mathbb{H}_L < 0,\ \mathbb{H}_{L'} > 0$$

we have a compact subvariety $Z(L, L') \cong \mathbb{P}^1$ in D described by the pictures



Here, for $p \in L$ the point $p^\perp \in L'$ is the unique point on L with $\mathbb{H}(p, p^\perp) = 0$, and the line $l = \overline{pp^\perp}$. It follows that

$$\mathcal{U} \cong \mathcal{H}_3 \times \overline{\mathcal{H}}_3$$

where $\mathcal{H}_3$ is Siegel's generalized upper-half-space.

Hyperbolicity of $\mathcal{U}$

Another nice result, not required for these lectures but of interest, is

For D non-classical, $\mathcal{U}$ is Kobayashi hyperbolic.

This follows from the fact mentioned above that there is a *bounded* strictly plurisubharmonic function ρ^2 on $\mathcal{U}$. Any complex manifold with this property is Kobayashi hyperbolic. In case G is of Hermitian type the result also follows from the above identification $\mathcal{U} \cong \mathcal{B} \times \overline{\mathcal{B}}$.

Although for G not of Hermitian type $\mathcal{U}$ is far from being homogeneous, the two properties of being Stein and hyperbolic mean that, from the point of view of complex function theory, $\mathcal{U}$ has many of the function-theoretic characteristics of an HSD.

Matsuki duality

Let $\mathcal{O} = (\mathcal{O}_{K_\mathbb{C}}, \mathcal{O}_{\mathbb{G}_\mathbb{R})}$ be a pair of orbits

$$\begin{cases} \mathcal{O}_{K_\mathbb{C}} = K_\mathbb{C} \cdot x_\mathcal{O} \\ \mathcal{O}_{G_\mathbb{R}} = G_\mathbb{R} \cdot x_\mathcal{O}. \end{cases}$$

Definition: *We say that* $\mathcal{O}$ *is a* **dual pair** *if the intersection*

$$\mathcal{O}_{K_{\mathbb{C}}} \cap \mathcal{O}_{G_{\mathbb{R}}} = K \cdot x_{\mathcal{O}}$$

is a K*-orbit.*

We note that the orbit $K \cdot x_{\mathcal{O}}$ is unique.

The relation "contained in the closure of" partially orders the sets of $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ orbits. Matsuki's result [Ma] is that the notion of duality between pairs of orbits induces a bijection

$$\{G_{\mathbb{R}}\text{-orbits in } \check{D}\} \leftrightarrow \{K_{\mathbb{C}}\text{-orbits in } \check{D}\}$$

that reverses the partial ordering.

We set

$$\mathcal{U}_{\mathcal{O}} = \{g \in G_{\mathbb{C}} \cdot (g\mathcal{O}_{K_{\mathbb{C}}}) \cap \mathcal{O}_{G_{\mathbb{R}}} \text{ is closed and non-empty}\}^o / K_{\mathbb{C}}$$

where $\{\ \}^o$ is the connected component of e. The precise universality statement is

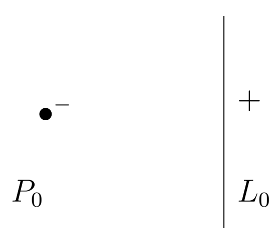
$$\mathcal{U}_{\mathcal{O}} \textit{ is independent of } \mathcal{O}.$$

We shall illustrate this in our running examples. In these lectures we shall mainly use it for open $G_{\mathbb{R}}$-orbits, which by Matsuki duality correspond to closed $K_{\mathbb{C}}$-orbits.

Example: $S\mathcal{U}(2,1)$. We shall illustrate the $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$-orbits with pictures. For this we denote by

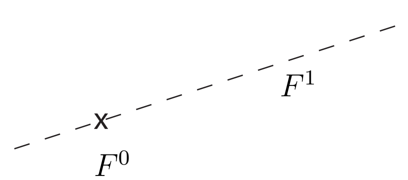
$$P_0, L_0 \in \mathbb{P}^2 \times \mathbb{P}^{2^*}$$

the standard pair on which the Hermitian form has the indicated signatures

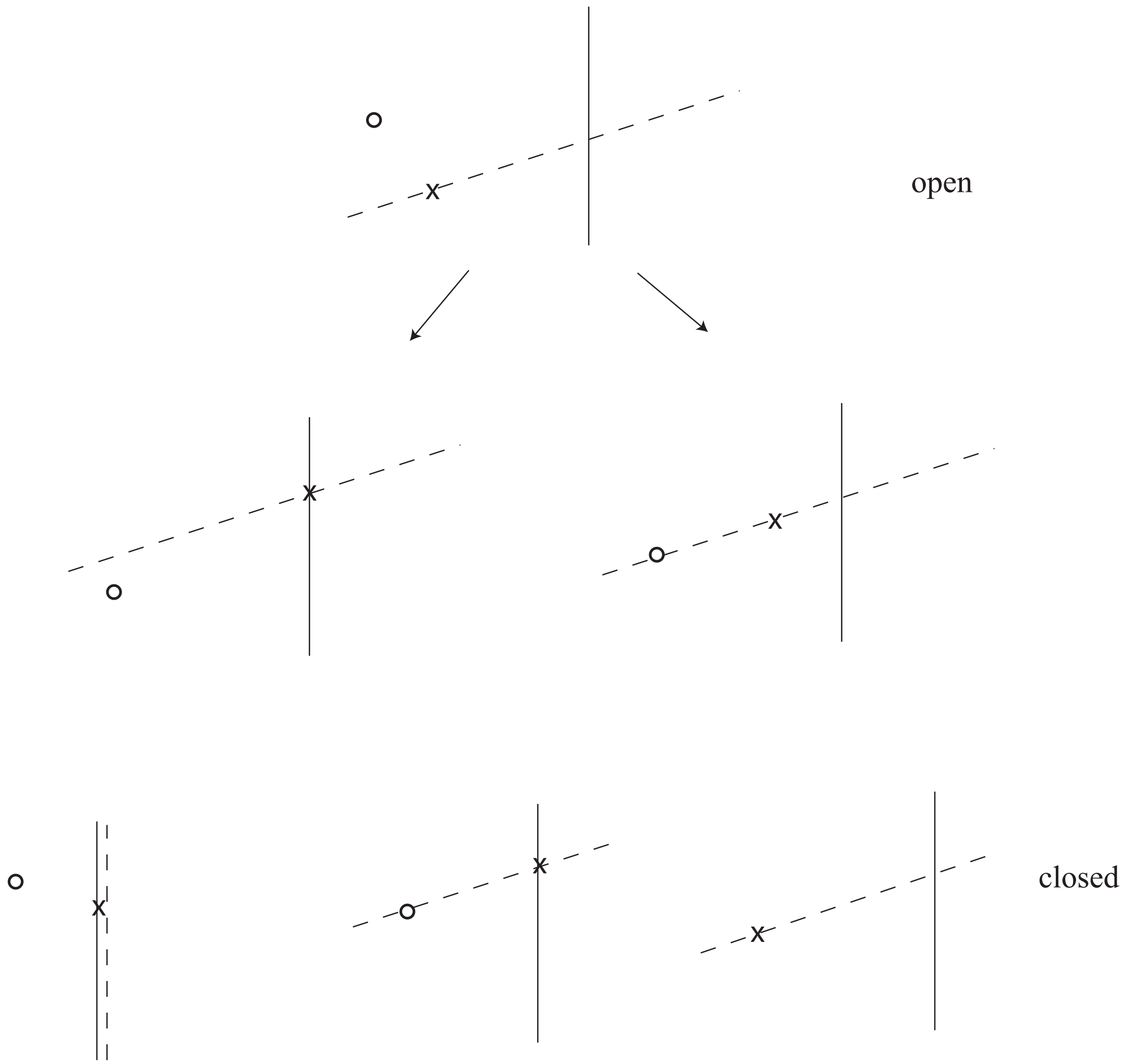


Here P_0 is the origin in $\mathbb{C}^2 \subset \mathbb{P}^2$ and L_0 is the line at infinity.

We will here denote points of $\check{D}$ by a flag $F^0 \subset F^1$



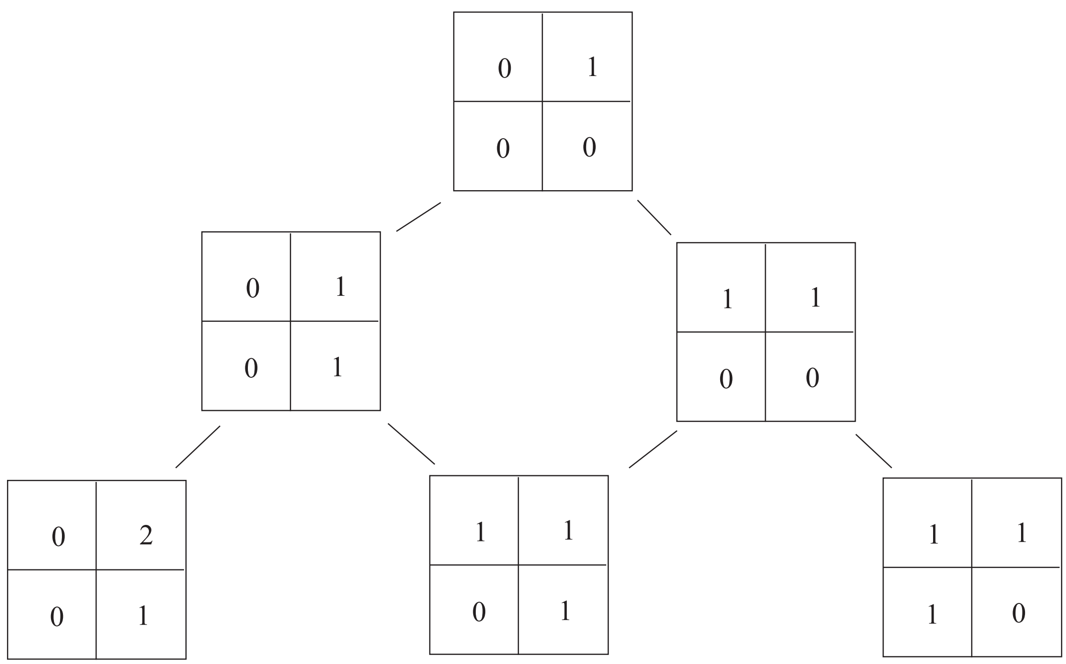
Denoting by $\searrow$ the relation "contained in the closure" there are six $K_{\mathbb{C}}$-orbits



We may denote each of these by a table

	P_0	L_0
F^1		
F^0		

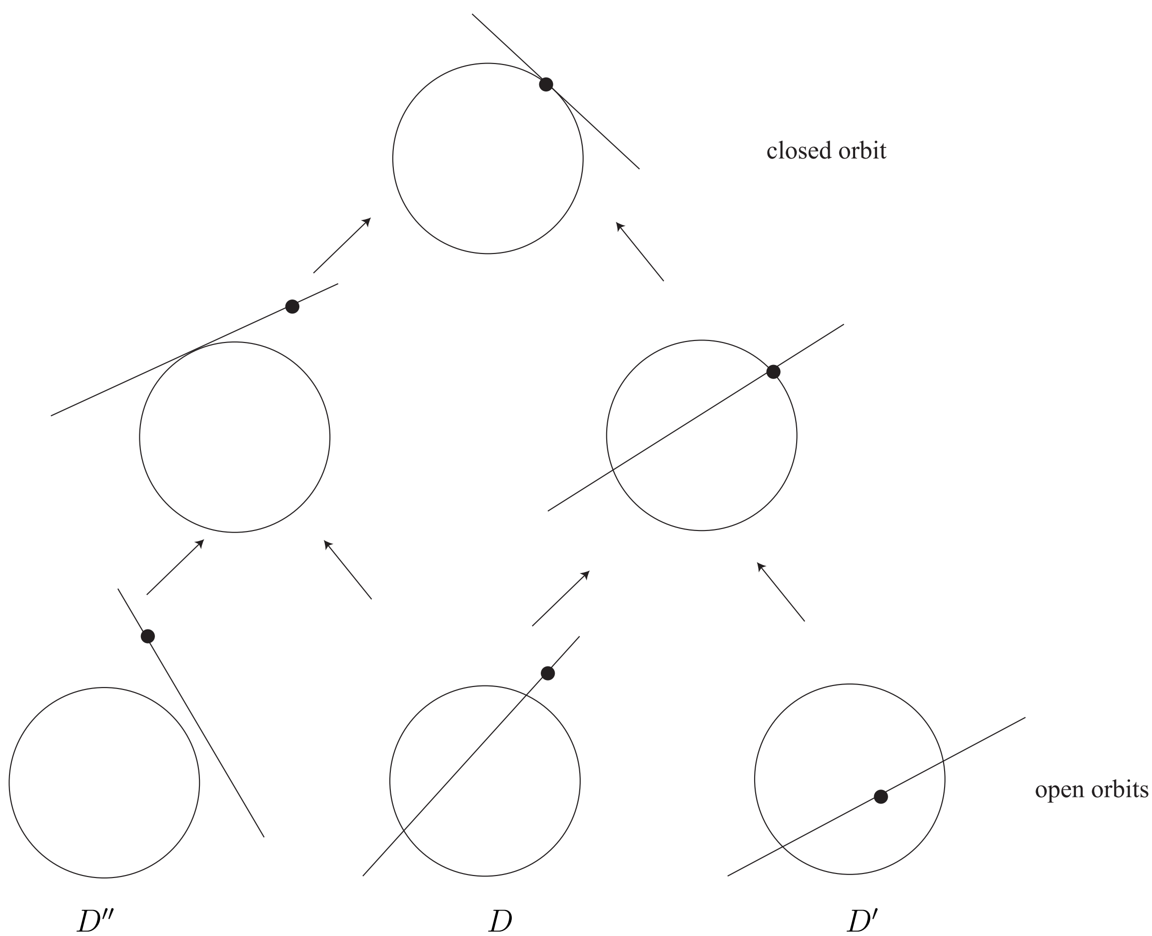
where the entries are $\dim F^i \cap P_0$ and $\dim F^i \cap L_0$. Then the above picture is



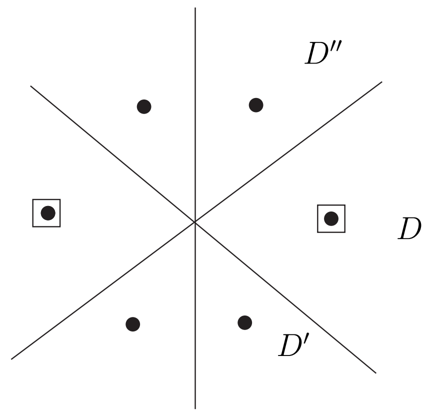
The correspondence to $G_{\mathbb{R}}$ orbits is

# negative eigenvalues on F^1	# positive eigenvalues on F^1
# negative eigenvalues	# positive eigenvalues

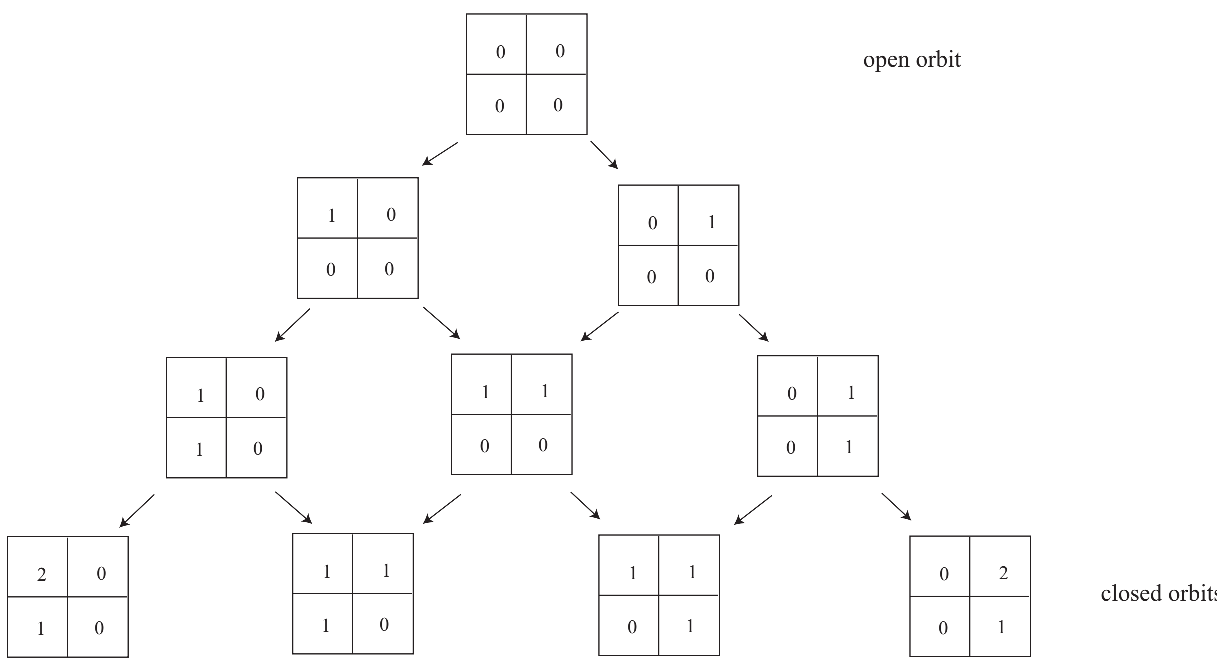
The pictures of the $G_{\mathbb{R}}$-orbits are



Here, D is the non-classical flag domain for $S\mathcal{U}(2,1)$ and D', D'' are the two classical ones. The root diagram with the positive Weyl chambers labelled is



Example: Sp(4)**.** In $\mathbb{P}^3$ we have the two standard reference Lagrange planes $L^{\pm}$, represented by lines in P^3, on which the Hermitian form H is positive, respectively negative definite. There are ten $K_{\mathbb{C}}$ orbits, which may be pictured as



The picture of the dual $G_{\mathbb{R}}$-orbits it given below. We set $H_{F^j} = H\big|_{F^j}$ and the notation $H_{F^j}(a,b)$ means that $H\big|_{F^j}$ has signature (a,b).

$\boxed{H_{F^1} = 0}$ closed orbit

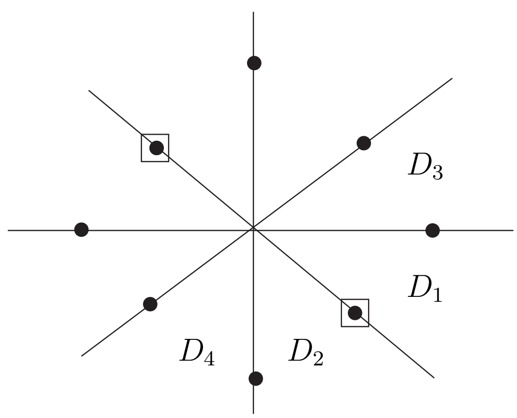
$\boxed{\begin{matrix} H_{F^1}(1,0) \\ H_{F^0} = 0 \end{matrix}}$ $\boxed{\begin{matrix} H_{F^1}(0,1) \\ H_{F^0} = 0 \end{matrix}}$

$\boxed{\begin{matrix} H_{F^1}(1,0) \\ H_{F^0} > 0 \end{matrix}}$ $\boxed{\begin{matrix} H_{F^1}(1,1) \\ H_{F^0} = 0 \end{matrix}}$ $\boxed{\begin{matrix} H_{F^1}(0,1) \\ H_{F^0} < 0 \end{matrix}}$

$\boxed{H_{F^1} > 0}$ $\boxed{\begin{matrix} H_{F^1}(1,1) \\ H_{F^0} > 0 \end{matrix}}$ $\boxed{\begin{matrix} H_{F^1}(1,1) \\ H_{F^0} < 0 \end{matrix}}$ $\boxed{H_{F^1} < 0}$ open orbits

$\underbrace{D_3}_{\text{classical}}$ $\underbrace{D_1 \qquad\qquad D_2}_{\text{non-classical}}$ $\underbrace{D_4}_{\text{classical}}$

The root diagram is



Example: SO(4, 1).[34] With $Q = \left(\begin{smallmatrix} I_4 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $H(u,v) = Q(u,\bar{v})$ as above, we set

$$P = \{z_5 = 0\} \subset \mathbb{C}^5.$$

[34]This description is due to Mark Green.

We shall also denote by $\mathbf{Q} \subset \mathbb{P}^4$ the corresponding quadric and $\mathbf{P} \cong \mathbb{P}^3$ the projectivization of P above. Then setting $P_{\mathbb{R}} = P \cap \mathbb{R}^5$ the maximal compact subgroup

$$K = \{g \in \mathrm{SO}(4,1)_{\mathbb{R}} : gP_{\mathbb{R}} = P_{\mathbb{R}}\} \cong O(4)_{\mathbb{R}}.$$

We denote by $\mathfrak{p}_\infty = [0,0,0,0,1]$ and by

$$\pi : \mathbb{P}^3 \backslash \mathfrak{p}_\infty \to \mathbf{P}$$

the projection.

We shall first describe the orbit structure for the period domain D, and then say how this lifts to the orbit structure of the flag domain $\widetilde{D}$ lying over D. Recall that the compact dual

$$\check{D} = \text{ space of lines lying in } \mathbf{Q}.$$

For the $G_{\mathbb{R}}$-orbits, we let $F_2 \in \check{D}$ and set $H_{F_2} = H\big|_{F_2}$. Then we have

(i) $\dim(F_2 \cap \overline{F}_2) = 0$ or 1;
(ii) $\dim(F_2 \cap \overline{F}_2) = 0 \Rightarrow H_{F_2}$ has signature (2,0) or (1,0);
(iii) $\dim(F_2 \cap \overline{F}_2) = 0 \Rightarrow H_{F_2 \cap \overline{F}_2} = 0$ and H_{F_2} has signature (1,0).

These describe the $G_{\mathbb{R}}$-orbits in $\check{D}$.

Turning to the $K_{\mathbb{C}}$-orbits, we have

(i) $\pi(F_2)$ is tangent to $\mathbf{P} \cap \mathbf{Q}$ or lies in $\mathbf{P} \cap \mathbf{Q}$;
(ii) $\mathrm{span}\{\mathfrak{p}_\infty, F_2\} \cap \mathbf{Q} = F_2 \cup$ line. This line may be F_2 or distinct from F_2.

These describe the $K_{\mathbb{C}}$-orbits in $\check{D}$. The duality between them and the $G_{\mathbb{R}}$-orbits is

$$\left\{ \begin{array}{l} \dim F_2 \cap \overline{F}_2 = 0 \\ \text{signature } H_{F_2} = (2,0) \end{array} \right\} \longleftrightarrow \pi(F_2) \subset \mathbf{P} \cap \mathbf{Q} \Leftrightarrow F_2 \subset \mathbf{P} \cap \mathbf{Q}.$$

These are two open $G_{\mathbb{R}}$-orbits corresponding to the two components of $O(4)_{\mathbb{R}}$ and two closed $K_{\mathbb{C}}$-orbits corresponding to the two rulings of $\mathbf{P} \cap \mathbf{Q}$

$$\left\{ \begin{array}{l} \dim F_2 \cap \overline{F}_2 = 0 \\ \text{signature } H_{F_2} = (1,0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathrm{span}\{\mathfrak{p}_\infty, F_2\} \cap \mathbf{Q} \text{ is a} \\ \text{double line, } F_2 \not\subset \mathbf{P} \cap \mathbf{Q} \end{array} \right\}$$

$$\dim F_2 \cap \overline{F}_2 = 1 \longleftrightarrow \left\{ \begin{array}{l} \mathrm{span}\{\mathfrak{p}_\infty, F_2\} \cap \mathbf{Q} \text{ is two} \\ \text{distinct lines, } F_2 \not\subset \mathbf{P} \cap \mathbf{Q} \end{array} \right\}.$$

For the flag domain whose points are $0 \subset F_1 \subset F_2$ with $F_2 \in \check{D}$ and $\dim F_i = i$, we will break the three cases for $\check{D}$ down into sub-cases.

Case 1: There are no sub-cases;

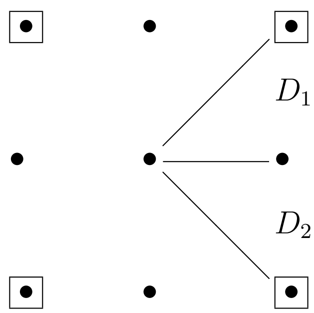
Case 2:

$G_{\mathbb{R}}$-side	$K_{\mathbb{C}}$-side
signature $H\big\|_{F_1}$ can be $(1,0)$ or $(0,0)$	$\dim F_1 \cap P$ can be 0 or 1

Case 3:

$G_{\mathbb{R}}$-side	$K_{\mathbb{C}}$-side
$\dim F_1 \cap \overline{F}_1$ can be 0 or 1	$\dim F_1 \cap P$ can be 0 or 1

As previously noted, the root diagram is



and the two inequivalent complex structures on $G_{\mathbb{R}}/T$ are given by the two marked Weyl chambers.

Relationship between Matsuki duality and representation theory

There is an extension of Matsuki duality to sheaves [MUV]. Because of the realization of certain Harish-Chandra modules as cohomology groups of homogeneous line bundles over open $G_{\mathbb{R}}$-orbits in a flag manifold it is reasonable to surmise that some sort of dual objects can be realized as cohomology groups associated to line bundles over closed $K_{\mathbb{C}}$-orbits in the same flag manifold. This is in fact the case; the basic reference is [HMSW] with an exposition given in [Sch3]. Referring to these works for precise statements and the definitions of *Beilinson-Bernstein localization* and *Zuckerman modules* which will be used below, we may very informally express a special case of the duality as follows.

> *Between the open $G_{\mathbb{R}}$-orbit D and the closed $K_{\mathbb{C}}$-orbit Z there is a duality between the Harish-Chandra modules associated to $H^d(D, L_\mu)$ and to $H_Z^{n-d}(\check{D}, L_\mu \otimes \omega_{\check{D}})$.*

Here, $H_Z^*(\check{D}, \mathcal{F})$ denotes the local cohomology of the coherent sheaf $\mathcal{F}$ along the closed subvariety Z. The general duality result involves $\mathcal{D}$-modules, but since Z is closed and smooth the cohomology of $\mathcal{D}$-modules may be replaced by local cohomology. The result holds under an assumption of regularity; the condition that $\mu+\rho$ is regular and anti-dominant is sufficient. We will illustrate it in two examples where one may verify that it also holds when $\mu+\rho$ is in the closure of the anti-dominant Weyl chamber, the case of particular interest in these lectures.

SL_2 example

In this case, $D=\mathcal{H}$ and $L_\mu=\omega_{\mathcal{H}}^{\otimes n/2}$, $n\geq 1$ as explained in Lecture 1. The *Zuckerman module* associated to $H^0(\mathcal{H},\omega_{\mathcal{H}}^{\otimes n/2})$ consists of finite germs f of sections about the closed $K_{\mathbb{C}}$-orbit i

$$f=\sum_{k=0}^{m} a_k(\tau-i)^k d\tau^{\otimes n/2}.$$

The local cohomology group is

$$\begin{aligned} H_Z^1(\check{D},\omega_{\check{D}}^{\otimes n/2}\otimes\omega_{\check{D}}) &\cong H^0\big(\check{D},\mathcal{H}_Z^1(\omega_{\check{D}}^{*\otimes n/2}\otimes\omega_{\check{D}})\big)\\ &\cong \mathcal{H}_{Z,i}^1(\omega_{\check{D}}^{*\otimes n/2}\otimes\omega_{\check{D}}) \end{aligned}$$

where $\mathcal{H}_Z(*)$ denotes the local cohomology sheaf and $\mathcal{H}_{Z,i}(*)$ is the stalk of that sheaf at i. Elements of this are

$$f=\sum_{l=0}^{m} b_l(\tau-i)^{-l-1}(d\tau)^{-n/2}\otimes d\tau,$$

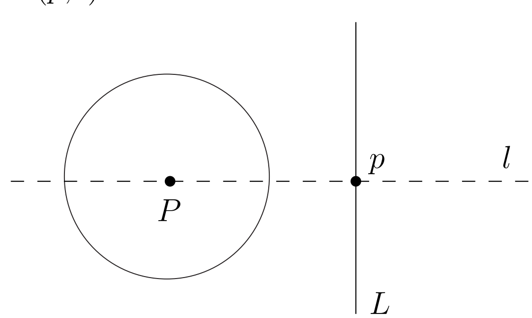
and the duality pairing is

$$\psi\otimes f\to \operatorname{Res}_i(f\psi)=\sum_k a_k b_k.$$

We note that each of $H^0(\mathcal{H},\omega_{\mathcal{H}}^{\otimes n/2})$ and $H_Z^1(\check{D},\omega_{\check{D}}^{*\otimes n/2})$ are $(\mathfrak{g}_{\mathbb{C}},K_{\mathbb{C}})$ modules, where $\mathfrak{g}_{\mathbb{C}}\cong \mathrm{sl}_2(\mathbb{C})$ acts as holomorphic vector fields on $\mathbb{P}^1$.

$S\mathcal{U}(2,1)$ example

We take for D the non-classical complex structure on $S\mathcal{U}(2,1)/T_S$ and for $Z=Z(P,L)$ the $K_{\mathbb{C}}$-orbit of the point (p,l)



where

- $P = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right] \in \mathbb{B} \subset \mathbb{P}^2$;
- $L = [0,\ 0,\ 1] \in \check{\mathbb{P}}^2$.

We label the above picture of the $K_{\mathbb{C}}$-orbits as

$$\begin{tikzcd}[column sep=small] & Q \arrow[dl, dash] \arrow[dr, dash] & \\ E_1 \arrow[dr, dash] & & E_2 \arrow[dl, dash] \\ & Z & \end{tikzcd}$$

where Q is the open $K_{\mathbb{C}}$-orbit, E_1 and E_2 are the two codimension-one $K_{\mathbb{C}}$-orbits and

$$Z = \overline{E}_1 \cap \overline{E}_2.$$

We observe from the picture that

> *Each of E_1, E_2 is a bundle over $\mathbb{P}^1$ with fibres $\mathbb{C}$, and $\overline{E}_1, \overline{E}_2$ are smooth and meet transversely along Z.*

We may also see from the picture that the normal bundle

$$N_{Z/D} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)$$

which has the geometric meaning

$$= \left\{ \begin{matrix} \text{hold } l \\ \text{fixed and} \\ \text{vary } p \end{matrix} \right\} \oplus \left\{ \begin{matrix} \text{hold } p \\ \text{fixed and} \\ \text{vary } l \end{matrix} \right\}.$$ [35]

Set

$$r = -\deg(L_\mu\big|_Z) - 2 \geqq 0$$

where the inequality follows from our assumptions on $\mu + \rho$ (cf. the appendix to Lecture 9). The Zuckerman module associated to $H^1(D, L_\mu)$ are finite sums of the K-type

[35] $K = \mathcal{U}(2)$ and $N_{Z/D}$ is a K-homogeneous vector bundle. The above isomorphism is a splitting as a $S\mathcal{U}(2)$-homogeneous bundle. The structure as a $\mathcal{U}(2)$-homogeneous bundle is more subtle and will be presented in the appendix to Lecture 9. For present purposes this is not needed.

with elements

$$f \in \bigoplus_{m=0}^{n} H^1\big(Z, \operatorname{Sym}^m N^*_{Z/D}(-r-2)\big)$$
$$\wr\|$$
$$\bigoplus_{m=0}^{n} H^0\big(Z, \operatorname{Sym}^m N_{Z/D}(r)\big).$$

We want to see what this means in local coordinates. For this we choose a point $p \in Z$ and local coordinates x, y, z such that p is the origin and Z is given by $x = y = 0$. For any locally free coherent sheaf $\mathcal{F} \to Z$ an element $g \in H^1(Z, \mathcal{F})$ may be written relative to the Čech covering

$$\begin{aligned} \mathcal{U}_0 &= \{p \neq 0\} \\ \mathcal{U}_1 &= \{\text{neighborhood of } p\} \end{aligned}$$

of Z as $g = \delta G$ where $G \in \Gamma(\mathcal{U}_0 \cap \mathcal{U}_1, \mathcal{F}) \cong \Gamma(\Delta^*, \mathcal{F})$ has a pole at p.[36] With this notation, the f above is given by

$$f = \sum_{\left\{\substack{m \\ a+b=m}\right.} \delta \tilde{f}_{a,b}(z) dx^a dy^b$$

where, after locally trivializing $\mathcal{O}_Z(1)$,

$$\tilde{f}_{a,b}(z) = \sum_{c>0} \tilde{f}_{a,b,c} z^{-c}$$

is a finite Laurent series.

On the other hand, a standard result in duality theory gives in this case that

$$H^2_Z(\check{D}, \mathcal{O}_Z(r) \otimes \omega_{\check{D}}) = H^0\left(\check{D}, \mathcal{H}^2_Z(\mathcal{O}_Z(r) \otimes \omega_{\check{D}})\right).^{37}$$

Sections on the RHS are locally

$$\psi = \sum_{\left\{\substack{m \\ a+b=m}\right.} \psi_{a,b}(z) \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial y}\right)^b dx \wedge dy \wedge dz$$

where the $\psi_{a,b}(z)$ is a holomorphic function. We set

$$F = \sum \tilde{f}_{a,b}(z) x^a y^b;$$

[36] This is because $H^1(Z, \mathcal{F}(kp)) = 0$ for $k \gg 0$ where $\mathcal{F}(kp)$ are the sections of $\mathcal{F}$ with a pole of order k at p.

[37] In general there is a spectral sequence abutting to $H^*_Z(\check{D}, \mathcal{F})$ and with E_2-term $H^q(\check{D}, \mathcal{H}^p_Z(\mathcal{F}))$. In the case at hand this spectral sequence collapses to give the stated result.

i.e., replace dx^a by x^a and dy^b by y^b, and then set

$$\Psi = \sum_{\left\{ \substack{m \\ a+b=m} \right.} \psi_{a,b}(z) x^{-a} y^{-b} dx \wedge dy \wedge dz,$$

i.e., replace $\left(\frac{\partial}{\partial x}\right)^a$ by x^{-a} and $\left(\frac{\partial}{\partial y}\right)^b$ by y^{-b}. When this is done the pairing is

$$\psi \otimes f \to \operatorname{Res}(F\Psi)$$

where the RHS is the *Grothendieck residue symbol*, which in this case is just the iterated 1-variable residue.

Appendix to Lecture 6: The Iwasawa decomposition and applications

Many of the results about the cycle space $\mathcal{U}$, especially its $G_{\mathbb{R}}$-orbit structure, may be best interpreted using the Iwasawa decomposition. In this appendix we shall recall this decomposition and shall illustrate its application in our running examples.

We shall work at the level of Lie algebras. For this we let

- $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition with Cartan involution θ;
- $\mathfrak{A} \subset \mathfrak{p}$ a maximal abelian sub-algebra;
- $B =$ Cartan-Killing form.

We recall that for $X, Y \in \mathfrak{g}_{\mathbb{R}}$, $X \neq 0$

- $B(X,Y) = B(\theta X, \theta Y)$ and $B(X, \theta X) < 0$.

Setting $B_\theta(X,Y) = B(X, \theta Y)$, for $H \in \mathfrak{A}$ the transformations $\operatorname{Ad} H$ are a commuting family of self-adjoint transformations on $\mathfrak{g}_{\mathbb{R}}$, and hence they may be simultaneously diagonalized with real eigenvalues. Setting

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g}_{\mathbb{R}} : (\operatorname{ad} H)X = \lambda X \ \text{ for all } \ H \in \mathfrak{A}\},$$

the non-zero λ's give the *restricted root system* $\Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})$ with the properties

(i) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^0 \oplus \left(\underset{\lambda \in \Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})}{\oplus} \mathfrak{g}^\lambda \right)$ where $\mathfrak{g}^0 = Z_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{A})$;
(ii) $\theta \mathfrak{g}^\lambda = \mathfrak{g}^{-\lambda}$;
(iii) $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{A}$ orthogonally, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{A})$ is the centralizer of $\mathfrak{A}$ in $\mathfrak{k}$;
(iv) $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subseteq \mathfrak{g}^{\lambda+\mu}$.

$\Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})$ contributes an abstract root system and we choose a set $\Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})$ of positive roots (e.g., by using a lexicographic ordering on $\mathfrak{A}^*$).

Definition: *We set* $\mathfrak{n} = \underset{\lambda \in \Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})}{\oplus} \mathfrak{g}^\lambda$.

In this appendix the notation $\mathfrak{n}$ replaces the notation $\mathfrak{n} = \oplus$ (negative root spaces for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$) used elsewhere in these talks.

We have from (i), (ii) above

$$(*) \qquad \mathfrak{g}_{\mathbb{R}} = \mathfrak{A} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \theta\mathfrak{n}.$$

From this we may infer the *Iwasawa decomposition*

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{A} \oplus \mathfrak{n}.$$

Proof. We first check that the sum is direct. If $X \in \mathfrak{k} \cap (\mathfrak{A} \oplus \mathfrak{n})$, then $\theta X = X$ while $\theta = -\operatorname{id}$ on $\mathfrak{A}$ and $\theta\mathfrak{n} \cap \mathfrak{n} = (0)$. To see that the sum spans $\mathfrak{g}_{\mathbb{R}}$, we use (i) above and

$\mathfrak{g}^0 = \mathfrak{A} \oplus \mathfrak{m}$ to write $X \in \mathfrak{g}_{\mathbb{R}}$ as

$$(**) \qquad X = \underbrace{H}_{\mathfrak{A}} + \underbrace{X^0 + \sum_{\lambda \in \Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})} (X^{-\lambda} + \theta X^{-\lambda})}_{\mathfrak{k}} + \underbrace{\sum_{\lambda \in \Phi^-(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})} (X^{\lambda} - \theta X^{-\lambda})}_{\mathfrak{n}}$$

where the terms above the brackets belong to the indicated sub-spaces. Here, H is the component of X in $\mathfrak{A}$, $X^0 \in \mathfrak{m}$ and X^{λ} is the component of X in $\mathfrak{g}^{\lambda}$ for $\lambda \in \Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})$. This establishes the Iwasawa decomposition. $\square$

To relate the expression $(**)$ for X to the decomposition $(*)$ we write more simply

$$X = \underbrace{H}_{\mathfrak{A}} + \underbrace{X^0}_{\mathfrak{m}} + \underbrace{\sum_{\lambda \in \Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})} X^{\lambda}}_{\mathfrak{n}} + \underbrace{\sum_{\lambda \in \Phi^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{A})} X^{-\lambda}}_{\theta\mathfrak{n}}.$$

Example: $S\mathfrak{U}(2,1)$**.** We take $\mathfrak{A}$ to be spanned by $H = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. Then using the notation in the lecture, the elements of the orthogonal $H^{\perp}$ under B are

$$A = \begin{pmatrix} a_{11} & a_{21} & i\alpha \\ a_{12} & a_{22} & c \\ -i\alpha & \bar{c} & e \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

From

$$[H, A] = \begin{pmatrix} -2i\alpha & \bar{c} & e - a_{11} \\ -c & 0 & -a_{12} \\ -(e - a_{11}) & a_{21} & 2i\alpha \end{pmatrix}$$

we see that non-zero restricted roots have $a_{22} = 0$. The equation $[H, A] = \lambda A$ then gives

$$\begin{cases} -2i\alpha = \lambda a_{11}, \quad 2i\alpha = \lambda e \Rightarrow 4\lambda^2 \alpha = \alpha \\ -c = \lambda a_{12}, \qquad -a_{12} = \lambda c \Rightarrow \lambda^2 c = c. \end{cases}$$

The possible eigenvalues λ are then given by

$$\begin{cases} \alpha \neq 0 \Rightarrow c = a_{12} = 0 \text{ and } \lambda^2 = 4 \\ \alpha = 0 \Rightarrow a_{11} = e = 0 \text{ and } \lambda^2 = 1. \end{cases}$$

We take as positive restricted roots $\lambda = 2$ and $\lambda = 1$. As root vectors we may then take

$$\lambda = 2 \qquad \begin{pmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{pmatrix}$$

$$\lambda = 1 \qquad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and then

$$\mathfrak{n} = \left\{ \begin{pmatrix} i\alpha & \beta - i\gamma & i\alpha \\ -\beta + i\gamma & 0 & \beta + i\gamma \\ -i\alpha & \beta - i\gamma & -i\alpha \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

Finally

$$\mathfrak{m} = \left\{ \begin{pmatrix} i\alpha & 0 & 0 \\ 0 & -2i\alpha & 0 \\ 0 & 0 & i\alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

SO$(4,1)$**:** We have

$$\mathfrak{g}_{\mathbb{R}} = \left\{ \begin{pmatrix} 0 & {}^t b & a \\ -b & A & c \\ \underbrace{a}_{1} & \underbrace{{}^t c}_{3} & \underbrace{0}_{1} \end{pmatrix} : a \in \mathbb{R} \text{ and } b, c \in \mathbb{R}^3, A \in \mathrm{so}(3)_{\mathbb{R}} \right\}.$$

By calculations similar to the above we find that $\lambda^2 = 1$, taking $\lambda = 1$ to correspond to the positive root we have

$$\mathfrak{A} = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & {}^t b & 0 \\ -b & A & 0 \\ 0 & 0 & 0 \end{pmatrix} : A + {}^t A = 0 \right\}$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & -{}^t b & 0 \\ b & 0 & b \\ 0 & {}^t b & 0 \end{pmatrix} : b \in \mathbb{R}^3 \right\}$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} : A + {}^t A = 0 \right\}.$$

First application: The tangent space to $G_{\mathbb{R}}$-orbits in $\mathcal{U}$ (based on [FHW])

Letting $u_0 \in \mathcal{U}$ be our reference point corresponding to the identity coset in $G_{\mathbb{R}}/K \subset \mathcal{U}$, for $H \in \mathfrak{A}$ we set

$$u = \exp(iH) \cdot u_0 \in \mathcal{U}.$$

It will be convenient to use the notations

- $\mathcal{O}_u =$ the orbit $G_{\mathbb{R}} \cdot u \subset \mathcal{U}$;
- $\mathfrak{g}_{\mathbb{R},u} =$ Lie algebra of the isotropy subgroup $G_{\mathbb{R},u} \subset G_{\mathbb{R}}$ of u.

Then the real tangent space

$$T_{u,\mathbb{R}}\mathcal{O}_u = \mathfrak{g}_{\mathbb{R}}/\mathfrak{g}_{\mathbb{R},u}.$$

Letting J_u be the almost-complex structure acting on $T_{u,\mathbb{R}}\mathcal{U}$ we will determine $\mathfrak{g}_{\mathbb{R},u}$ and J_u. We note that since the vector fields given by the action of $\mathfrak{g}_{\mathbb{C}}$ span $T_{u,\mathbb{C}}\mathcal{U}$ we have

$$T_{u,\mathbb{R}} = T_{u,\mathbb{R}}\mathcal{O}_u \oplus J_u T_{u,\mathbb{R}}\mathcal{O}_u.$$

The intersection

$$T_{u,\mathbb{R}}\mathcal{O}_u \cap J_u T_{u,\mathbb{R}}\mathcal{O}_u =: T_u^{\mathrm{CR}}\mathcal{O}_u$$

is by definition the *Cauchy-Riemann tangent space*. See the note at the end of this subsection for the definition of the *intrinsic Levi form* and an argument that it is non-degenerate on this space.

The basic geometric observation is that

The $G_{\mathbb{R}}$-orbits are transverse in $\mathcal{U}$ to the submanifold $K \cdot \exp(i\omega_0) \cdot u_0$.

Here in the tangent space to $\mathcal{U}$ at u_0 we have the picture

$$\begin{aligned} i\mathfrak{p} &= T_{u_0\mathbb{R}} K \cdot \exp(i\omega_0) \cdot u_0 \\ \mathfrak{p} &= T_{u_{0,\mathbb{R}}} \mathcal{O}_{u_0}, \end{aligned}$$

which using $\mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{p}$ and $\operatorname{Ad} K \cdot \mathfrak{A} = \mathfrak{p}$ gives that the tangent space to $K \cdot \exp(i\omega_0) \cdot u_0$ is $i\mathfrak{p}$. It follows from this observation that, as previously noted, the Lie algebra of the isotropy subgroup $G_{\mathbb{R},u}$ at $\exp(iH) \cdot u_0$ depends only on the centralizer of H in $\mathfrak{k}$, with the extremes being

- $\mathfrak{g}_{\mathbb{R},u_0} = \mathfrak{k} \qquad (H = 0)$;
- $\mathfrak{g}_{\mathbb{R},u} = \mathfrak{m} \qquad (H$ generic).

We now shall make this precise.

For $X \in \mathfrak{g}_{\mathbb{R}}$ we let $\widehat{X}$ denote the corresponding vector field on $\mathcal{U}$ with value $X(u) \in T_{u,\mathbb{R}}\mathcal{U}$. Then the basic formula, which results from $\operatorname{Ad} \cdot \exp = e^{\operatorname{Ad}}$ and $\theta H = -H$ is

$$(***) \qquad X \in \mathfrak{g}^{\lambda} \Rightarrow \begin{cases} \widehat{X}(u) = e^{-i\langle \lambda, H\rangle} \exp(iH)_* \widehat{X}(u_0) \\ \widehat{\theta X}(u) = e^{i\langle \lambda, H\rangle} \exp(iH)_* \widehat{\theta X}(u_0). \end{cases}$$

It follows also that

- $\langle \lambda, H\rangle \neq 0 \Rightarrow \widehat{X}(u)$ and $\widehat{\theta X}(u)$ span a complex line in $T_{u,\mathbb{R}}\mathcal{O}_u$;
- $\langle \lambda, H\rangle = 0 \Rightarrow \widehat{X}(u) = -\widehat{\theta X}(u)$.

The latter equation follows by adding the equations in $(***)$ and using that $X + \theta X \in \mathfrak{k}$, so that $(\widehat{X} + \widehat{\theta X})(u_0) = 0$.

Definition: *We set*

$$\begin{aligned} \mathfrak{n}_H^0 &= \bigoplus_{\left\{\substack{\lambda > 0 \\ \langle \lambda, H\rangle = 0}\right.} \mathfrak{g}^{\lambda} \\ \mathfrak{n}_H^1 &= \bigoplus_{\left\{\substack{\lambda > 0 \\ \langle \lambda, H\rangle \neq 0}\right.} \mathfrak{g}^{\lambda}. \end{aligned}$$

Then we observe the properties

- $\mathfrak{n}_H^0$ and $\mathfrak{n}_H^1$ are sub-algebras;
- $\mathfrak{n} = \mathfrak{n}_H^0 + \mathfrak{n}_H^1$ is a semi-direct sum and $\mathfrak{n}_H^1$ is an ideal in $\mathfrak{n}$;
- $\mathfrak{A} + \mathfrak{n}_H^0 + \theta(\mathfrak{n}_H^0)$ is the normalizer of H in $\mathfrak{g}_{\mathbb{R}}$;
- $\mathfrak{g}_{\mathbb{R},u} = Z_{\mathfrak{k}}(H) = \mathfrak{m} + \mathfrak{m}_H$, where $\mathfrak{m}_H \subset \mathfrak{k}$ is the span of the $X + \theta X$ for $X \in \mathfrak{n}_H^0$.

In particular, for the real codimension of $\mathcal{O}_u$ in $\mathcal{U}$

$$\operatorname{codim}_{\mathcal{H}} \mathcal{O}_u = \dim \mathfrak{A} + \dim \mathfrak{n}_H^0.$$

Proposition: *For the real tangent space $\mathcal{O}_u$ we have*

$$\boxed{T_{u,\mathbb{R}}\mathcal{O}_u \cong \mathfrak{A} \oplus \mathfrak{n}_H^0 \oplus \left(\mathfrak{n}_H^1 \oplus \theta\mathfrak{n}_H^1\right).}$$

The Cauchy-Riemann part of the tangent space is the right-hand term in the parenthesis.

Here the second term in the direct sum is equal to $\{X - \theta X : X \in \mathfrak{n}_H^0\}$. In one extreme case when $H = 0$, this term is isomorphic to $\mathfrak{n}$, and then

$$T_{u,\mathbb{R}}\mathcal{O}_{u_0} \cong \mathfrak{A} \oplus \mathfrak{n}$$

as should be the case from the Iwasawa decomposition in the form $G_{\mathbb{R}} = NAK$. In the other extreme case when H is regular

$$T_{u,\mathbb{R}}\mathcal{O}_u \cong \mathfrak{A} \oplus (\mathfrak{n} \oplus \theta\mathfrak{n}).$$

Examples: In the two above examples we have only the two respective cases for the real codimension

$$\begin{aligned} &H = 0 \Rightarrow \operatorname{codim}_{\mathfrak{U}} \mathcal{O}_{u_0} = \dim i\mathfrak{p} = \begin{cases} 4 \\ 6 \end{cases} \\ &H \neq 0 \Rightarrow \operatorname{codim}_{\mathfrak{U}} \mathcal{O}_u = 1. \end{aligned}$$

Note on Levi geometry: Let M be a complex manifold and $S \subset M$ a real submanifold. At a point $x \in S$, the Cauchy-Riemann part of the tangent space is

$$T_x^{\mathrm{CR}} S = T_x S \cap J_x T_x S,$$

where J_x is the complex structure. The *intrinsic Levi form* $\mathcal{L}_{S,x}$

$$\mathcal{L}_{S,x} : T_x^{\mathrm{CR}} S \otimes T_x^{\mathrm{CR}} S \to T_x S / T_x^{\mathrm{CR}} S$$

is defined by

$$\mathcal{L}_{S,x}(u, v) = [\tilde{u}, J\tilde{v}](x) \bmod T_x^{\mathrm{CR}} S.$$

Here, $\tilde{u}, \tilde{v}$ are local vector field extensions of u, v; they are bracketed and then the bracket is evaluated at x and projected to $T_x S / T_x^{\mathrm{CR}} S$. Three general properties are

(i) iff S is a hypersurface defined by $g = 0$, then up to a scale factor

$$\mathcal{L}_{S,x} = \pm i\bar{\partial}\partial g\big|_{T_x^{\mathrm{CR}} S};$$

(ii) if $S \subset S'$, then

$$T^{\mathrm{CR}} S = TS \cap T^{\mathrm{CR}} S',$$

and there is an obvious functoriality property relating $\mathcal{L}_S$ and $\mathcal{L}_{S'}$;

(iii) in particular, if S is continued in a hypersurface $S' = \{g = 0\}$ in which $i\bar{\partial}\partial g\big|_{T_{S'}^{\mathrm{CR}}}$ is positive definite, then $\mathcal{L}_S$ is non-degenerate.

This is the case for the $G_{\mathbb{R}}$-orbits $\mathcal{O}_u$, since by [AkG] there is a biholomorphism between the maximal Grauert tube of $G_{\mathbb{R}}/K$ in $TG_{\mathbb{R}}/K$ and $\mathcal{U}$, and the norm function ρ^2 on $TG_{\mathbb{R}}/K$ is $G_{\mathbb{R}}$-invariant and strictly plurisubharmonic.

Second application: The tangent space to $G_{\mathbb{R}}$-invariant hypersurfaces in $\mathcal{U}$ (result from [BHH])

The Stein property of $\Gamma\backslash\mathcal{U}$, for Γ co-compact and discrete in $G_{\mathbb{R}}$, is central to the study of automorphic cohomology using Penrose transforms. As discussed in the lecture, its proof is based on constructing strictly plurisubharmonic exhaustion functions f of $\mathcal{U}$ modulo $G_{\mathbb{R}}$. Such a function f will have as level sets

$$\mathcal{U}_c = \{u \in \mathcal{U} : f(u) = c\}$$

which are $G_{\mathbb{R}}$-invariant hypersurfaces in $\mathcal{U}$. As discussed in the lecture, if f_{ω_0} is a strictly convex function defined in ω_0 and invariant under the Weyl group $N_K(\mathfrak{A})/Z_K(\mathfrak{A})$, then f_{ω_0} uniquely determines a $G_{\mathbb{R}}$-invariant function on $\mathcal{U} = G_{\mathbb{R}} \exp(i\omega_0) \cdot u_0$.

One wants to show that

such a function f is strictly plurisubharmonic.

The details of this are given in [FHW], and we shall only comment on the main points. For this we let N_H^1 be the complex Lie group with Lie algebra $\mathfrak{n}_H^1 + i\mathfrak{n}_H^1$. Then from the proposition one may infer that

$$T_u^{\mathrm{CR}}\mathcal{O}_u = T_u(N_H^1 \cdot u).$$

This does not mean that $N_H^1 \cdot u$ is contained in $\mathcal{O}_u$, which is impossible since $\mathcal{L}_{\mathcal{O}_u}$ is non-degenerate.

Next, and this is the key point where the convexity of the function f_{ω_0} enters, using the identification of $\mathcal{U}$ with the maximal Grauert tube of $G_{\mathbb{R}}/K$ in $TG_{\mathbb{R}}/K$, along the 0-section the almost complex structure is given by the tautological identification of $T_xG_{\mathbb{R}}/K$ with the fibre of the Grauert tube at x. When this is done, using the $G_{\mathbb{R}}$-invariance of f the complex Hessian of f at u_0 is identified with the real Hessian of the restriction of f to that fibre $T_{u_0}G_{\mathbb{R}}/K \cong \mathfrak{p}$. This restriction is strongly convex on $f\big|_{\omega_0} = f_{\omega_0}$, and using the K-invariance and $\operatorname{Ad} K \cdot \mathfrak{A} = \mathfrak{p}$ it follows that f is strictly convex on $T_{u_0}G_{\mathbb{R}}/K$.

The above is only intended to indicate the ingredients in the argument. A final remark is that in the classical case when $\mathcal{U} \cong \mathcal{B} \times \overline{\mathcal{B}}$ where $\mathcal{B}$ is an HSD, then $\Gamma\backslash\mathcal{U}$ is a holomorphic fibre space over the projective algebraic variety $\Gamma\backslash\mathcal{B}$ with the HSD $\overline{\mathcal{B}}$ as fibre. It may be that a proof that $\Gamma\backslash\mathcal{U}$ is Stein may be given in this setting.

Addendum to Lecture 6: On the structure of the $G_{\mathbb{R}}$ and $K_{\mathbb{C}}$-orbits

From correspondence with Mark it seems possible to give a more detailed description than in the notes. Here we will discuss only the $G_{\mathbb{R}}$-orbits. Let $x \in \check{D} = G_{\mathbb{C}}/B$. Then by Bruhat's lemma we may choose a σ-stable Cartan sub-algebra

$$\mathfrak{h}_x = \mathfrak{b}_x \cap \sigma \mathfrak{b}_x.$$

From [K1], pages 386 and 458 we may conjugate $\mathfrak{h}_x = \mathfrak{h}_x \cap \sigma \mathfrak{h}_x$ in $G_{\mathbb{R}}$ to be θ-stable (this may move x in the $G_{\mathbb{R}}$-orbit). We shall refer to this $\mathfrak{h}_x$ as a *(σ, θ)-stable Cartan subalgebra.*

We denote by Φ_x the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_x)$ and by Φ_x^+ the positive roots determined by $\mathfrak{b}_x$. By (σ, θ) stability we have

$$\begin{cases} \mathfrak{h}_{x,\mathbb{R}} = \mathfrak{t}_x \oplus \mathfrak{A}_x \\ \mathfrak{t}_x = \mathfrak{h}_{x,\mathbb{R}} \cap \mathfrak{k},\ \mathfrak{A}_X = \mathfrak{h}_{x,\mathbb{R}} \cap \mathfrak{p}. \end{cases}$$

Dropping now the subscripts x, using this decomposition for $\alpha \in \Phi$ we have

$$\alpha = i\alpha_1 + \alpha_2$$

where $\alpha_1 \in \mathfrak{t}^*$, $\alpha_2 \in \mathfrak{A}^*$ are both real. Let $X \in \mathfrak{g}^\alpha$ and write

$$\begin{cases} X = X_1 + X_2 \\ X_1 \in \mathfrak{k}_{\mathbb{C}}, X_2 \in \mathfrak{p}_{\mathbb{C}}. \end{cases}$$

Then setting $\theta\alpha = i\alpha_1 - \alpha_2 = -\overline{\alpha}$

$$X_1 - X_2 \in \mathfrak{g}^{\theta\alpha}.$$

We note that

$$\alpha_2 \neq 0 \Rightarrow X_1 \neq 0,$$

since if $v_1 = 0$ then $X \in \mathfrak{g}^\alpha \cap \mathfrak{g}^{\theta(\alpha)} = 0$. From this we may group the roots as follows:

$$\begin{array}{rl} \textit{complex roots} & \alpha_1 \neq 0, \alpha_2 \neq 0 \\ \textit{real roots} & \alpha_1 = 0 \\ \textit{imaginary roots} & a_2 = 0 \begin{cases} \text{compact} \\ \text{non-compact} \end{cases} \end{array}$$

where the compact, non-compact means that the root vector is in k, $\mathfrak{A}$ respectively.[38] We note that

[38] If $\mathfrak{h}_{\mathbb{R}}$ is maximally non-compact, then any imaginary root is compact.

- $\alpha \in \Phi \Rightarrow \pm i\alpha_1 \pm \alpha_2 \in \Phi$; i.e., Φ is stable under σ, θ;
- thus the complex roots come in *quartets*, and the real an imaginary roots come in pairs;
- for any choice of Φ^- there are exactly 2 roots from each quartet and exactly 1 root from each pair in Φ^-. Moreover,
- for any quartet, there is either an $\alpha, \sigma(\alpha)$ or and $\alpha, \theta(\alpha)$ pair.

For the isotropy algebra we have $\alpha, \sigma(\alpha) \in \Phi^- \Leftrightarrow \alpha \in \Phi^- \cap \sigma\Phi^-$ and

$$\mathfrak{v}_{x,\mathbb{R}} = \mathfrak{b}_x \cap \mathfrak{g}_{\mathbb{R}} = \mathfrak{h}_{x,\mathbb{R}} + \underset{\alpha \in \Phi^- \cap \sigma\Phi^-}{\oplus} (\mathfrak{g}^\alpha + \mathfrak{g}^{\sigma(\alpha)})_{\mathbb{R}}$$

where

$$\dim_{\mathbb{R}} (\mathfrak{g}^\alpha + \mathfrak{g}^{\sigma(\alpha)})_{\mathbb{R}} = \begin{cases} 1 & \alpha = \sigma(\alpha) \\ 2 & \alpha \neq \sigma(\alpha). \end{cases}$$

This again gives for the real codimension

$$\operatorname{codim} G_{\mathbb{R}} \cdot x = \# \left(\Phi^- \cap \sigma\Phi^- \right).$$

A couple of examples will illustrate the above. We will restrict to non-open $G_{\mathbb{R}}$-orbits.

$\underline{S\mathcal{U}(2,1)}$: The only possibility is

$$\mathfrak{h}_{x,\mathbb{R}} = \mathfrak{t}_x \oplus \mathfrak{A}_x$$

where each summand has dimension 1. We may take as above

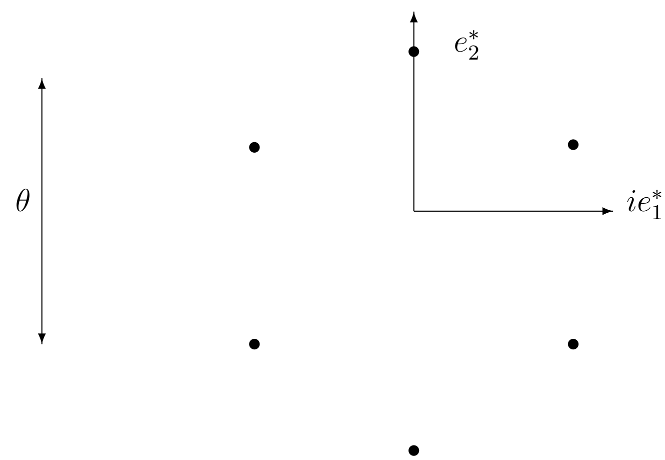
$$\mathfrak{t}_x = \operatorname{span} \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} =: e_1$$

$$\mathfrak{A}_x = \operatorname{span} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =: e_2.$$

The roots are

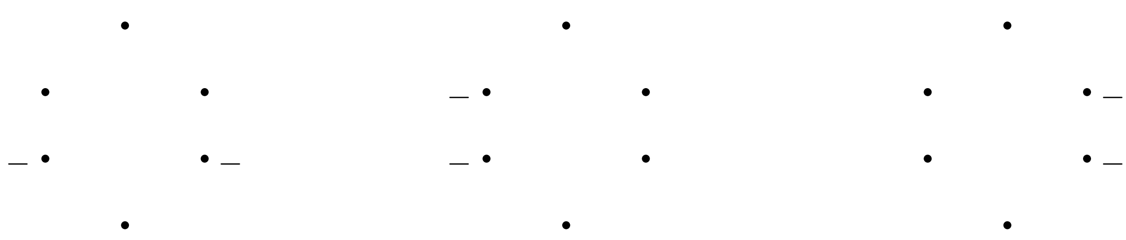
$$\pm 2e_2^*, \ \pm 3ie_1^* + e_2^*$$

with the picture (not the same as the usual picture when the roots are all imaginary — the open orbit case)



The choices of Φ^- are

(a) (b) (c)



Here conjugation is reflection in the e_2^* axis and the action of the Weyl group for $\mathfrak{h}$ is reflection in the ie_1^* axis. Then

$$\begin{aligned} \text{for (a),} \qquad & \Phi^- = \sigma\Phi^- \leftrightarrow \text{codimension-3 orbit} \\ \text{for (b), (c),} \qquad & \#\left(\Phi^- \cap \sigma\Phi^-\right) = 1 \leftrightarrow \text{codimension-1 orbits.} \end{aligned}$$

<u>SO(4, 1)</u> : Here all possibilities occur. As before in the non-open orbit case we have

$$\mathfrak{h}_{x,\mathbb{R}} = \mathfrak{t}_x \oplus \mathfrak{A}_x$$

where each summand has dimension 1. Taking now

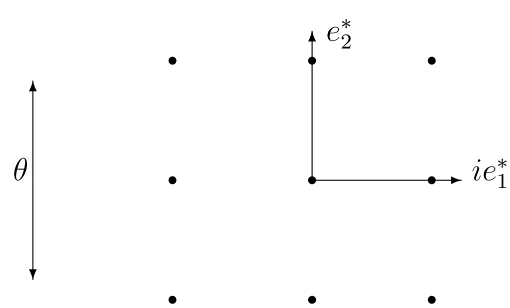
$$e_1 = \left\{\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right\}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the roots are

$$\begin{array}{rl} \pm 2ie_1^* & \text{imaginary} \\ \pm 2e_2^* & \text{real} \\ \pm 2ie_1^* + 2e_2^* & \text{complex.} \end{array}$$

With the root picture



the possibilities for Φ^- are

$$
\begin{array}{ccc}
 & \text{(a)} & \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet\, - \\
\underline{\odot} & \underline{\odot} & \underline{\odot}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \text{(b)} & \\
\bullet & \bullet & \bullet\, - \\
\bullet & \bullet & \bullet\, - \\
\bullet & \underline{\odot} & \underline{\bullet}
\end{array}
$$

where the roots in $\Phi^- \cap \sigma\Phi^-$ are circled. For the non-open $G_{\mathbb{R}}$-orbits, there is one each of codimension 1,3.

There is a similar story for the $K_{\mathbb{C}}$-orbits that we will not be able to give here. An interesting consequence is the "conservation law"

$$\tfrac{1}{2}\operatorname{codim}_{\mathbb{R}} G_{\mathbb{R}} \cdot x + \operatorname{codim}_{\mathbb{C}} K_{\mathbb{C}} \cdot x \textit{ is independent of } \Phi^-.$$

In more detail, if

$$
\begin{aligned}
q &= \#\ \text{quartets} \\
r &= \#\ \text{of real pairs} \\
i_c &= \#\ \text{of imaginary compact roots} \\
l_c &= \dim_{\mathbb{R}} \mathfrak{t}_x, l_{nc} = \dim_{\mathbb{R}} \mathfrak{A}_x,
\end{aligned}
$$

and if we break up

$$q = q_\sigma(\Phi^-) + q_\theta(\Phi^-)$$

where

$$
\begin{aligned}
q_\sigma(\Phi^-) &= \#\ \text{ of } \ \alpha, \sigma(\alpha) \ \text{ pairs in } \ \Phi^- \\
q_\theta(\Phi^-) &= \#\ \text{ of } \ \alpha, \theta(\alpha) \ \text{ pairs in } \ \Phi^-
\end{aligned}
$$

then

$$
\begin{aligned}
\operatorname{codim}_{\mathbb{R}} G_{\mathbb{R}} \cdot x &= r + 2q_\sigma(\Phi^-) \\
\operatorname{codim}_{\mathbb{C}} K_{\mathbb{C}} \cdot x &= |\Phi^-| - \dim \mathfrak{k}_{\mathbb{C}} + l_c + i_c + q_\theta(\Phi^-)
\end{aligned}
$$

and the sum is independent of the choice of Φ^- ($|\Phi^-| = \dim \check{D}$). We also note that

$$\dim_{\mathbb{R}} T_x^{CR} G_{\mathbb{R}} \cdot x = 2q_\sigma(\Phi^-).$$

The conservation law may be written more succinctly as

$$\operatorname{codim}_{\mathbb{R}} G_{\mathbb{R}} \cdot x + \operatorname{codim}_{\mathbb{R}} K_{\mathbb{C}} \cdot x = \dim_{\mathbb{R}} \check{D} + \dim_{\mathbb{R}}(\mathfrak{k}/\mathfrak{t}_x).$$

Lecture 7
Geometry of flag domains: Part II

Correspondence spaces: The basic diagram

In this lecture we shall first define the correspondence space $\mathcal{W}$ associated to a real, semi-simple Lie group $G_{\mathbb{R}}$ containing a compact maximal torus T. This will then lead to the basic diagram given below. Next we shall give the result relating the cohomology groups associated to the spaces in the basic diagram. The main point here is to give different ways of realizing higher degree sheaf cohomology of holomorphic line bundles over D by global, holomorphic data on the associated spaces.

As a first step towards the definition of $\mathcal{W}$ we shall define its dual $\check{\mathcal{W}}$, and before doing this we shall make the following change of notation from Lecture 6:

$$\check{\mathfrak{U}} = G_{\mathbb{C}}/K_{\mathbb{C}}.$$

Then $\check{\mathfrak{U}}$ is an affine algebraic variety. For a non-classical flag domain $D \subset \check{D}$ with distinguished maximal compact subvariety $Z_0 \subset D$ where $Z_0 = K_{\mathbb{C}} \cdot x_0$ with $x_0 \in D$ and Z_0 the distinguished K-orbit under Matsuki duality, $\check{\mathfrak{U}}$ is a smooth subvariety in the component of the Hilbert scheme of $\check{D}$ containing the point corresponding to Z_0.

Example: For $G = \mathrm{SO}(4,1)$ with $Q = \begin{pmatrix} I_4 & \\ & -1 \end{pmatrix}$ we have

$$\check{\mathfrak{U}} = \{E \in \mathrm{Gr}(4, \mathbb{C}^5) : Q_E \text{ non-singular}\}$$

where $Q_E = Q\big|_E$. Identifying $\mathbb{C}^5$ with $\mathbb{C}^{5^*}$ using Q, there is an equivalent identification

$$\check{\mathfrak{U}} = \{[u] \in \mathbb{P}^{4^*} : Q(u,u) \neq 0\},$$

where we still denote by Q the corresponding quadratic form on $\mathbb{C}^{5^*}$. Here $E = [u]^{\perp}$, and in the second description we have

$$\check{\mathfrak{U}} = \mathbb{P}^4 \backslash (\text{non-singular quadric}),$$

and the distinguished point $u_0 = [0, \ldots, 0, 1]$.

Returning to the general discussion, we have $\check{D} = G_{\mathbb{C}}/B$ where B is a Borel subgroup containing the Cartan subgroup $H = T_{\mathbb{C}}$ whose Lie algebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$. Finally we have the Borel subgroup $B_K =: B \cap K_{\mathbb{C}}$ of $K_{\mathbb{C}}$ whose Lie algebra is

$$\mathfrak{b}_K = \mathfrak{h} \oplus \mathfrak{n}_K^-$$

where $\mathfrak{n}_K^- = \bigoplus_{\alpha \in \Phi_c^+} \mathfrak{g}^{-\alpha}$ is the direct sum of the negative compact root spaces. We set

- $\check{\mathcal{W}} = G_{\mathbb{C}}/H =$ *enhanced flag variety*
- $\check{\mathfrak{I}} = G_{\mathbb{C}}/B_K$.

The dual of the basic diagram to be defined below is

$$\begin{tikzcd} & \check{\mathcal{W}} \arrow[d] & \\ & \check{\mathcal{I}} \arrow[dl] \arrow[dr] & \\ \check{D} & & \check{\mathcal{U}}. \end{tikzcd}$$

It has the properties:

(i) the fibres of $\check{\mathcal{I}} \to \check{D}$ and of $\check{\mathcal{W}} \to \check{\mathcal{I}}$ are contractible affine algebraic varieties;
(ii) the fibres of $\check{\mathcal{I}} \to \check{\mathcal{U}}$ are projective algebraic varieties; and
(iii) the fibres of $\check{\mathcal{W}} \to \check{\mathcal{U}}$ are affine algebraic varieties.

Here "dual" means that this is a diagram relating the compact dual $\check{D}$ with $\check{\mathcal{U}}$. The basic diagram as defined below will be the restriction of the above diagram to the part lying above D.

Discussion (see [FHW] and [GG] for detailed proofs): From

$$\mathfrak{b} = \mathfrak{b}_K \oplus \mathfrak{p}^-$$

where $\mathfrak{b}_K = \mathfrak{b} \cap \mathfrak{k}_{\mathbb{C}}$ as above we have that

$$\exp : \mathfrak{p}^- \xrightarrow{\sim} B/B_K$$

where B/B_K, a typical fibre of $\check{\mathcal{I}} \to \check{D}$, is affine and contractible. A similar observation holds for the typical fibre B_K/H of $\check{\mathcal{W}} \to \check{\mathcal{I}}$. A typical fibre of $\check{\mathcal{I}} \to \check{\mathcal{U}}$ is $Z_0 = K_{\mathbb{C}}/B_K$. Finally, a typical fibre of $\check{\mathcal{W}} \to \check{\mathcal{U}}$ is $K_{\mathbb{C}}/H$, the enhanced flag variety for $K_{\mathbb{C}}/B_K$.

Definition: *The* **correspondence space** $\mathcal{W}$ *is the inverse image of* $\mathcal{U}$ *in the diagram*

$$\begin{tikzcd} \mathcal{W} \arrow[d] & \subset & \check{\mathcal{W}} \arrow[d] \\ \mathcal{U} & \subset & \check{\mathcal{U}}. \end{tikzcd}$$

The term correspondence space derives from the universality property of $\mathcal{U}$: Given open $G_{\mathbb{R}}$-orbits $D_w, D_{w'}$ in $\check{D}$ we have

$$\begin{tikzcd} & \mathcal{W} \arrow[dl] \arrow[dr] & \\ D_w & & D_{w'}. \end{tikzcd}$$

Moreover, from (i) above *the fibres of each map are contractible affine algebraic varieties.* This diagram will be used to relate the cohomologies $H^q(D_w, L_\mu)$ and $D^{q'}(D_{w'}, L'_{\mu'})$ via Penrose transforms which will be defined below.

From Lecture 6 and (iii) above we have

$$\mathcal{W} \textit{ is a Stein manifold.}$$

In fact, $\mathcal{W}$ fibres over the Stein manifold $\mathcal{U}$ with affine algebraic varieties as fibres. Since $\mathcal{U}$ has the function-theoretic characteristics of a bounded domain of holomorphy, we see that $\mathcal{W}$ has a mixed algebro-geometric/complex function-theoretic character.

Definition: *For D a non-classical flag domain the* **basic diagram** *is the open subset of the above diagram containing the* $\widehat{\quad}$*'s on the terms*

$$\begin{tikzcd}
& \mathcal{W} \arrow[ddl, "\pi"'] \arrow[d, "\pi_{\mathcal{J}}"] \arrow[ddr, "\pi'"] & \\
& \mathcal{J} \arrow[dl, "\pi_D"] \arrow[dr, "\pi_{\mathcal{U}}"'] & \\
D & & \mathcal{U}.
\end{tikzcd}$$

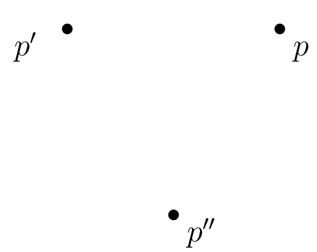
The intermediate space

$$\mathcal{J} = \{(x, u) : x \in Z_u\} \subset D \times \mathcal{U}$$

is, for evident reasons, called the *incidence variety*. From ([FHW], (6.23)), in case G is of Hermitian type the fibres of $\mathcal{J} \to D$ are contractible. This covers the main examples discussed in these lectures. The result seems to be true in general, but is more complicated and will be discussed elsewhere.

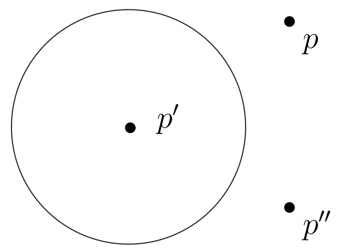
Because of the universality of $\mathcal{W}$ there are basic diagrams as above for all open $G_{\mathbb{R}}$-orbits in $\check{D}$.

Example: $S\mathcal{U}(2,1)$**.** We may picture $\check{\mathcal{W}}$ as the set of *projective frames*



These are sets of triples of independent points in $\mathbb{P}^2$.

Then $\mathcal{W}$ is the sub-set of projective frames

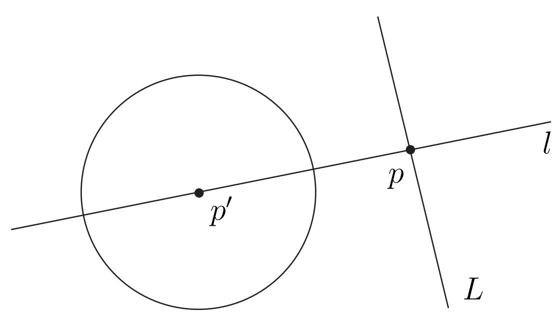


$$p' \in \mathbb{B},\ \overline{pp''} \subset \mathbb{B}^c$$

The maps of $\mathcal{W}$ to D, D', D'' are

$$p, p', p'' \to \begin{cases} (p, \overline{pp'}) \in D \\ (p', \overline{p'p}) \in D' \\ (p, \overline{pp''}) \in D''. \end{cases}$$

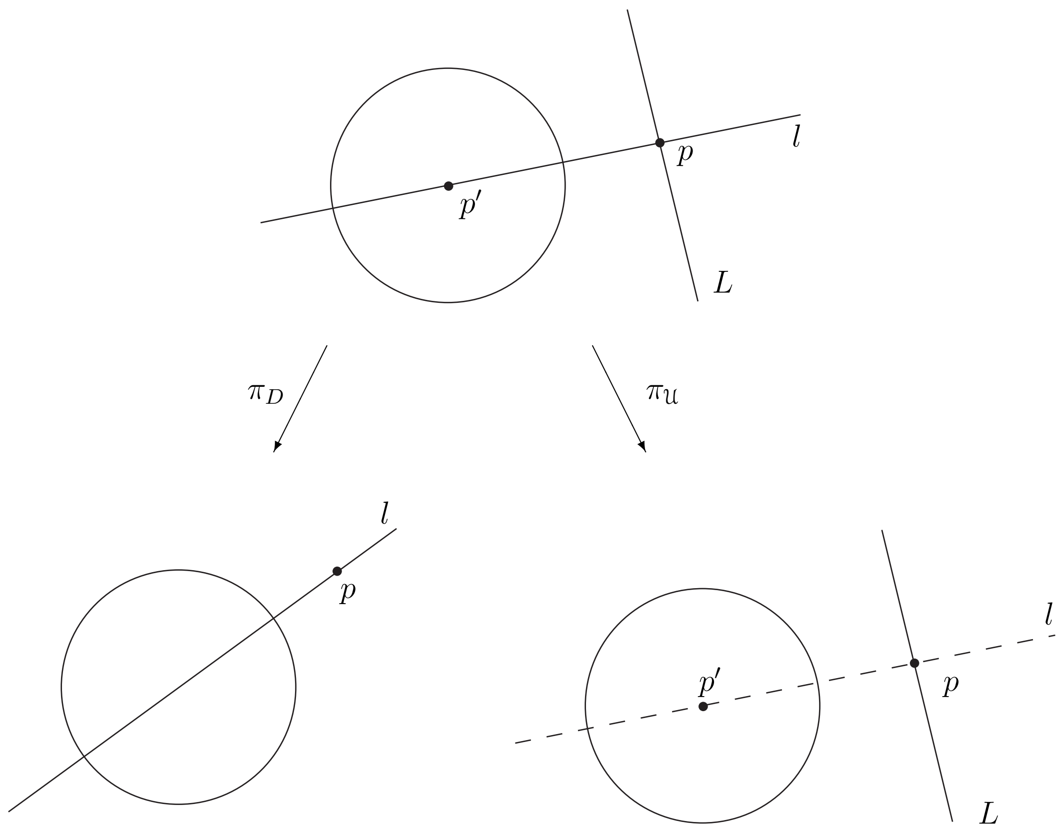
The incidence variety $\mathcal{I}$ is the subset of $\{(p, l), (p', L)\} \in D \times \mathcal{U}$ given by the configurations



The map $\mathcal{W} \to \mathcal{I}$ is given by

$$p, p', p'' \to \text{above figure when } L = \overline{pp''}.$$

The maps $\mathcal{I} \to D$, $\mathcal{I} \to \mathcal{U}$ are

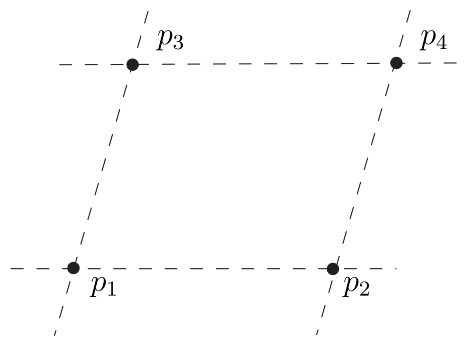


Here, we recall the compact subvariety $Z(p', L) \subset D$ given by the set of all points $\{(p, l)\}$ in the lower right hand figure.

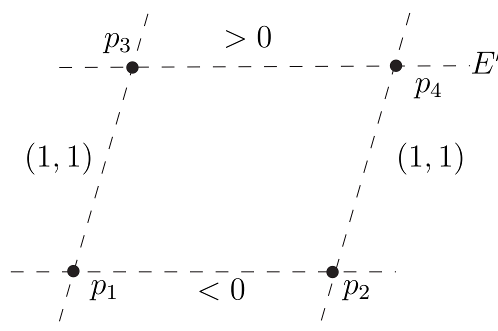
All of the stated properties of the basic diagram may be readily verified from the above pictures. For example, the fibre of $\mathcal{I} \to D$ is given by holding p, l fixed. Then

$$\begin{cases} p' \in l \cap \mathbb{B} \cong \Delta \\ L = \text{lines through } p \text{ in } \mathbb{B}^c \cong \overline{\Delta}. \end{cases}$$

Example: Sp(4)**.** For $\check{\mathcal{W}}$ we have the description as the set of *Lagrange quadrilaterals*



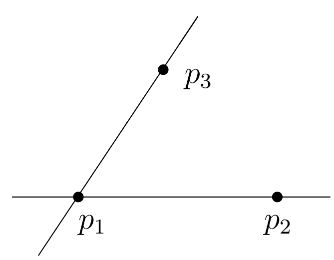
given by projective frames p_1, p_2, p_3, p_4 in $\mathbb{P}^3$ and where the dashed lines are all Lagrangian. The diagonal lines are *not* Lagrangian. The correspondence space $\mathcal{W}$ is the open subset of $\check{\mathcal{W}}$ of all Lagrange quadrilaterals



where the Hermitian form has the indicated signature on the Lagrangian lines.

The maps of $\mathcal{W}$ to one of the two non-classical domains D and to one of the two classical domains D' are given by

$$p_1, p_2, p_3, p_4 \longrightarrow$$



where $(p_1, \overline{p_1 p_3}) \in D$ and $(p_1, \overline{p_1 p_2}) \in D'$. The map of $\mathcal{W}$ to $\mathcal{U}$ is given by

$$p_1, p_2, p_3, p_4 \to E, E'$$

where E, E' are the pair of Lagrangian lines in the above figure where the Hermitian form is negative, respectively positive definite.

Again all of the stated properties in the basic diagram may be directly verified. We note that it is frequently easier to verify a property of $\mathcal{W} \to D$ by factoring it into $\mathcal{W} \to \mathcal{I}$ followed by $\mathcal{I} \to D$.

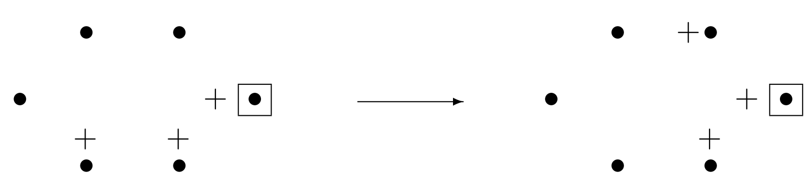
In the study of Penrose transforms we shall use the diagram

$$\begin{array}{ccc} & \mathcal{W} & \\ & \downarrow & \\ & \mathcal{J} & \\ \swarrow & & \searrow \\ D_w & & D_{w'} \end{array}$$

where $\mathcal{J} \subset \check{\mathcal{J}} =: G_{\mathbb{C}}/P$ where $P \subset G_{\mathbb{C}}$ is a parabolic subgroup containing B and B'. The reason that such diagrams arise is that the basic operation in passing from the complex structure $D_{w'}$ to the complex structure D_w will be

one non-compact simple β root changes sign.

Thus, $w^{-1}w' = s_\beta$ is the reflection in the root plane corresponding to β. For example, in the $S\mathcal{U}(2,1)$ example passing from the classical complex structure D' given above to the non-classical complex structure given by



This process is reminiscent of Bott's original proof of the BWB (cf. [Sch2]); as we shall see in the appendix to Lecture 8 this is not accidental.

In general, the Levi component of P will correspond to a subset $\Psi \subset \Phi^+$ such that both Ψ and the complement $\Phi^+\backslash\Psi$ are closed under addition. The nilpotent radial of the Lie algebra of P will be $\bigoplus_{\alpha\in\Phi^+\backslash\Psi} \mathfrak{g}^{-\alpha}$.

The theorem of [EGW]

Although fairly simple to state and prove, this result will for us have multiple applications. Let M, N be complex manifolds and

$$\pi : M \to N$$

a holomorphic submersion. We identify holomorphic vector bundles and their sheaves of sections. For $F \to N$ a holomorphic vector bundle we let

- $\pi^{-1}F$ be the pullback to M of the sheaf F;
- $\pi^* F$ be the pullback to M of the bundle F.

We may think of $\pi^{-1}F \subset \pi^* F$ as the sections of $\pi^* F$ that are constant along the fibres of $M \to N$.

Next we let Ω^q_π be the sheaf over M of relative holomorphic q-forms. We have

$$0 \to \pi^*\Omega^1_N \to \Omega^1_M \to \Omega^1_\pi \to 0,$$

and this defines a filtration $F^m\Omega^q_M$ with

$$\Omega^q_\pi \cong \Omega^q_M / F^q\Omega^q_M.$$

In local coordinates (x^i, y^α) on M such that $\pi(x^i, y^\alpha) = (y^\alpha)$, $F^m\Omega^q_M$ are the holomorphic differentials generated over Ω^{q-m}_M by terms $dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_m}$. Thus $F^m\Omega^q_M =$ image$\left\{\pi^*\Omega^m_N \otimes \Omega^{q-m}_M \to \Omega^q_M\right\}$. From this description we see that we have

$$d : F^m\Omega^q_M \to F^m\Omega^{q+1}_M,$$

and consequently there is an induced *relative differential*

$$d_\pi : \Omega^q_\pi \to \Omega^{q+1}_\pi.$$

Setting $\Omega^q_\pi(F) = \Omega^q_\pi \otimes_{\mathcal{O}_M} \pi^* F$, since the transition functions of $\pi^* F$ may be taken to involve only the y^α's, we may define

$$d_\pi : \Omega^q_\pi(F) \to \Omega^{q+1}_\pi(F)$$

to obtain the complex $(\Omega^\bullet_\pi(F); d_\pi)$. Using the holomorphic Poincaré lemma with holomorphic dependence on parameters one has the resolution

$$0 \to \pi^{-1}F \to \Omega^0_\pi(F) \xrightarrow{d_\pi} \Omega^1_\pi(F) \xrightarrow{d_\pi} \Omega^2_\pi(F) \to \cdots .$$

Denoting by $\mathbb{H}^*(M, \Omega^\bullet_\pi(F))$ the hypercohomology of the complex $(\Omega^\bullet_\pi(F), d_\pi)$, from this resolution we have

$$H^*(M, \pi^{-1}F) \cong \mathbb{H}^*(M, \Omega^\bullet_\pi(F)).$$

We denote by

$$H^*_{\mathrm{DR}}\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big)$$

the de Rham cohomology groups arising by taking the global holomorphic sections of the complex $(\Omega^\bullet_\pi(F); d_\pi)$.

Theorem ([EGW]): *Assume that M is Stein and the the fibres of $M \to N$ are contractible. Then*

$$H^*(N, F) \cong H^*_{\mathrm{DR}}\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big).$$

Discussion: Using the standard spectral sequence associated to the above resolution of $\pi^{-1}F$

$$E_2^{p,q} = H^q\big(H^p(M, \Omega^\bullet_\pi(F)); d_\pi\big) \Rightarrow \mathbb{H}^{p+q}\big(M, \Omega^\bullet_\pi(F)\big)$$

and the assumption that M is Stein to have $H^p(M, \Omega^\bullet_\pi(F)) = 0$ for $p > 0$ gives

$$H^*(M, \pi^{-1}F) \cong H^*_{\mathrm{DR}}\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big).$$

Next, in the situations with which we shall be concerned, the submersion $M \to N$ will be locally over N a topological product. Then by the contractibility of the fibres the direct image sheaves

$$R^q_\pi(\pi^{-1}F) = 0 \ \text{ for } \ q > 0.$$

The Leray spectral sequence thus gives

$$H^q(N, F) \cong H^q(M, \pi^{-1}F);$$

here the LHS is $H^q(N, R^0_\pi(\pi^{-1}F)) = H^q(N, F)$. Combining the above isomorphisms gives the theorem. □

From what we have seen above the [EGW] theorem applies to $\mathcal{W} \to D$ and to give

$$H^q(D, L_\mu) \cong H^q_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big).$$

In this way the coherent cohomology $H^q(D, L_\mu)$ is realized by global, holomorphic data. In our examples there will be canonical, or "harmonic," representatives of the de Rham cohomology groups.

Quotienting by a discrete group

Let $\Gamma \subset G_{\mathbb{R}}$ be a discrete, co-compact and neat subgroup. A principal motivation for [GGK2] was to understand some of the geometric and arithmetic properties of the *automorphic cohomology groups* $H^q(\Gamma\backslash D, L_\mu)$, objects that had arisen many years ago [GS], [WW1], [WW2], [Wi1] but whose above mentioned properties had to us remained largely mysterious until the works [C1], [C2], and [C3]. In studying the automorphic cohomology groups it is important to be able to take the quotient of the basic diagram by Γ, the quotient being

$$\begin{tikzcd} & \Gamma\backslash\mathcal{W} \arrow[dl, "\pi"'] \arrow[d, "\pi_{\mathcal{J}}"] \arrow[dr, "\pi'"] & \\ & \Gamma\backslash\mathcal{J} \arrow[dl, "\pi_D"] \arrow[dr, "\pi_{\mathcal{U}}"'] & \\ \Gamma\backslash D & & \Gamma\backslash\mathcal{U}. \end{tikzcd}$$

Here we note that the group $G_{\mathbb{R}}$ acts equivariantly on the diagram, and so the quotient diagram is well-defined. The basic result concerning it is

THEOREM: $\Gamma\backslash\mathcal{W}$ *is Stein, and the fibres of* π, π_D *and* $\pi_{\mathcal{J}}$ *are contractible.*

Proof. We first note that no $\gamma \in \Gamma$, $\gamma \neq e$, has a fixed point acting on D or on $\mathcal{U}$. For D this is because the isotropy subgroup of $G_{\mathbb{R}}$ fixing any point $x \in D$ is compact. For $u \in \mathcal{U}$, if γ fixes u then it maps the compact subvariety $Z_u \subset D$ to itself, so again γ is of finite order. It follows that the above fibres are biholomorphic to those in the basic diagram before quotienting by Γ.

The next, and crucial, step is the result that there exists strictly plurisubharmonic functions on $\mathcal{U}$ that are exhaustion functions modulo $G_{\mathbb{R}}$. As in the second proof that $\mathcal{U}$ is Stein discussed in Lecture 6, this induces a strictly plurisubharmonic exhaustion function of $\Gamma\backslash\mathcal{U}$, which is therefore a Stein manifold. Then $\Gamma\backslash\mathcal{W} \to \Gamma\backslash\mathcal{U}$ is a fibration over a Stein manifold with affine algebraic varieties as fibres, which implies that $\Gamma\backslash\mathcal{W}$ is itself Stein. $\square$

The proof of the result on $H^*(D, L_\mu)$ then applies verbatim to give

$$H^*(\Gamma\backslash D, L_\mu) \cong H^*_{\mathrm{DR}}\big(\Gamma(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big).$$

The double appearance of the notation Γ in the RHS is unfortunate, but we hope that the meaning is clear.

Relating cohomologies on $\mathcal{W}$ and $\mathcal{U}$

To state the main result we first define bundles

$$F^{p,q}_\mu \to \mathcal{U}$$

as follows: For $u \in \mathcal{U}$ let $Z_u \subset D$ be the corresponding maximal compact subvariety. Let $F^{p,q}_\mu = R^q_{\pi_{\mathcal{U}}}(\Omega^p_{\pi_D}(L_\mu))$. Then the fibre

$$F^{p,q}_{\mu,u} = H^q(Z_u, \Lambda^p N_{Z_u\backslash D}(L_\mu)).$$

THEOREM ([GG]): *There exists a spectral sequence with*

$$\begin{cases} E^{p,q}_1 = H^0(\mathcal{U}, F^{p,q}_\mu), \quad \textit{and} \\ E^{p,q}_\infty = \mathrm{Gr}^p H^{p+q}_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \end{cases}$$

Using our earlier result on the relation between $H^q(D, L_\mu)$ and $H^q_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$ we have the following result:

COROLLARY: *There exists a spectral sequence with*

$$\begin{cases} E^{p,q}_1 = H^0(\mathcal{U}, F^{p,q}_\mu) \\ E^{p,q}_\infty = F^p H^{p+q}(D, L_\mu). \end{cases}$$

If $H^0(Z, \Lambda^{q+1} N_{Z/D}(L_\mu)) = \cdots = H^{q-1}(Z, \Lambda^2 N_{Z/D}(L_\mu)) = 0$, *then*

$$H^q(D, L_\mu) \cong \ker \left\{ H^0(\mathcal{U}, F_\mu^{0,q}) \xrightarrow{d_1} H^0(\mathcal{U}, F_\mu^{1,q}) \right\}.$$

The latter is related to section 14.3 in [FHW]. Under the vanishing condition in the corollary, the coherent cohomology $H^q(D, L_\mu)$ is, in a different way from above using the EGW theorem, realized as a global, holomorphic object. The vanishing condition is satisfied for μ anti-dominant and sufficiently far from the walls of the Weyl chamber.

The differentials d_r are linear, first order differential operators. Below we will comment further on d_1.

Proof. Referring to the basic diagram we have on $\mathcal{W}$ the exact sequence of relative differentials

$$0 \to \pi_{\mathcal{J}}^* \Omega^1_{\pi_D} \to \Omega^1_\pi \to \Omega^1_{\pi_{\mathcal{J}}} \to 0.$$

This induces a filtration on $\Omega^\bullet_\pi$, and hence one on the complex

$$\Gamma\big(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big).$$

This filtration then leads to a spectral sequence abutting to

$$H^*_{\mathrm{DR}}(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi).$$

We will identify the E_1-term with that given in the statement of the theorem.

The first observation is that in this spectral sequence we have

$$\begin{cases} E_0^{p,q} \cong \Gamma\big(\mathcal{W}, \Omega^p_{\pi_{\mathcal{J}}} \otimes \pi_{\mathcal{J}}^* \Omega^q_{\pi_D}(L_\mu)\big) \\ d_0 \quad = d_{\pi_{\mathcal{J}}}. \end{cases}$$

Thus

$$\begin{cases} E_1^{p,q} \cong H^q_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi_{\mathcal{J}}} \otimes \pi_{\mathcal{J}}^* \Omega^p_{\pi_D}(L_\mu)); d_{\pi_{\mathcal{J}}}\big) \\ d_1 \text{ is induced by } d_{\pi_D}. \end{cases}$$

By [EGW] applied to $\mathcal{W} \xrightarrow{\pi_{\mathcal{J}}} \mathcal{J}$ we have

$$\begin{cases} E_1^{p,q} \cong H^q\big(\mathcal{J}, \Omega^p_{\pi_D}(L_\mu)\big) \\ d_1 \text{ is induced by } d_{\pi_D}. \end{cases}$$

Since $\mathcal{U}$ is Stein, the Leray spectral sequence applied to $\mathcal{J} \xrightarrow{\pi_{\mathcal{U}}} \mathcal{U}$ and $\Omega^q_{\pi_D}(L_\mu)$ gives

$$\begin{cases} E_1^{p,q} \cong H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big) \\ d_1 \text{ is induced by } d_{\pi_D}. \end{cases}$$

It remains to establish the identification

$$R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu) \cong F_\mu^{p,q}.$$

This will be done by identifying the various tangent spaces at the reference point $(x_0, u_0) \in \mathcal{I}$. For this we continue to identify locally free sheaves F with vector bundles and denote by F_p the fibre at the point p. We then have the identifications

- $T_{x_0}D = \mathfrak{n}^+$;
- $T_{x_0}Z = \mathfrak{n}^+_c$;
- $N_{Z/D,x_0} = \mathfrak{p}^+$;
- $T_{u_0}\mathcal{U} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$;
- $T_{(x_0,u_0)}\mathcal{I} \subset \mathfrak{n}^+ \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

In this last identification we write $\mathfrak{n}^+ = \mathfrak{n}^+_c \oplus \mathfrak{p}^+$ and then we have

- $T_{(x_0,u_0)}\mathcal{I} = \mathfrak{n}^+_c \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$

where the inclusion $\mathfrak{n}^+_c \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \subset \mathfrak{n}^+_c \oplus \underbrace{\mathfrak{p}^+ \oplus \mathfrak{p}^+} \oplus \mathfrak{p}^-$ is given by the diagonal mapping in the term over the bracket. It follows that

- $\Omega^1_{\pi_D,(x_0,u_0)} = \mathfrak{p}^{-*} \cong \mathfrak{p}^+ = N_{Z/D,x_0}$

where the isomorphism is via the Cartan-Killing form. $\square$

The proof also allows us to identify the symbol $\sigma(d_1)$ of the differential operator d_1, as follows: Recall that

$$\sigma(d_1) : F^{0,d}_{\mu,u_0} \otimes T^*_{u_0}\mathcal{U} \to F^{1,q}_{\mu,u_0},$$

or using the definition of the $F^{p,q}_\mu$

$$\sigma(d_1) : H^q(Z, L_\mu) \otimes T^*_{u_0}\mathcal{U} \to H^q(Z, N_{Z/D}(L_\mu)).$$

Using the identification $T^*_{u_0}\mathcal{U} \cong \mathfrak{p}^* \cong \mathfrak{p}$ we have the mapping

$$\mathfrak{p} \to H^0(Z, N_{Z/D})$$

given geometrically by considering $X \in \mathfrak{p} \subset \mathfrak{g}$ as a holomorphic vector field along Z and then taking the normal part of X. Combining this with the evident map

$$H^q(Z, L_\mu) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, N_{Z/D}(L_\mu))$$

gives the symbol map. We will give a proof of this below.

Remark on the above corollary: Under the assumptions in the corollary, the E_1 term of the spectral sequence looks like

$$\begin{array}{|c|c|c|c|}
\hline
* & * & & * \\
\hline
0 & 0 & \cdots & 0 \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
0 & 0 & \cdots & 0 \\
\hline
\end{array}$$

and we have an exact sequence of $G_{\mathbb{R}}$-modules

$$0 \to H^d(D, L_\mu) \to H^0(\mathcal{U}, F_\mu^{0,d}) \xrightarrow{d_1} H^0(\mathcal{U}, F_\mu^{1,d}) \to \cdots \to H^0(\mathcal{U}, F_\mu^{n-d,d}) \to 0$$

where $\dim d = n$. As noted above the symbol of the first d_1 is a bundle map whose value at u_o is

$$F^{0,1}_{\mu,u_0} \to F^d_{\mu,u_0} \otimes (\mathfrak{p}^+ \oplus \mathfrak{p}^-),$$

which looks very much like the complexification to the Grauert tube $\mathcal{U} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ of the Dirac operator over $G_{\mathbb{R}}/K \subset \mathcal{U}$ used by Atiyah-Schmid [AS]. We have not checked whether or not this is so.

By localizing the above exact sequence and using that $\mathcal{U}$ is Stein, one obtains over $\mathcal{U}$ exact sheaf sequence

$$F_\mu^{0,d} \xrightarrow{d_1} F_\mu^{1,d} \xrightarrow{d_1} \cdots \xrightarrow{d_1} F_\mu^{n-d,d} \to 0.$$

This is reminiscent of the Spencer sequence giving an involutive resolution of the sheaf whose sections are the localizations of $H^d(D, L_\mu)$ along the maximal compact subvarieties in the cycle space. Again we have not checked this.

Finally, the assumption that μ is “sufficiently regular” is a common one in the theory. As previously noted, it is necessary when vanishing theorems are used, since the curvature calculations that are used apply also to quotients by co-compact discrete subgroups $\Gamma \subset G_{\mathbb{R}}$. Many results in the theory are proved first in the sufficiently regular case, and then extended to the general case using Zuckerman translation and Casselman-Osborne ([Sch2]). We will comment further on this.

$\mathfrak{n}$-cohomology interpretation

A familiar theme in the study of cohomology of homogeneous spaces and their quotients is to represent that cohomology by Lie algebra cohomology. As we have noted in an earlier lecture, for flag domains one considers $\mathfrak{n}$-cohomology where $\mathfrak{n}$ is the direct sum of the negative root spaces. Even though $\mathcal{W}$ is not a homogeneous space for $G_{\mathbb{R}}$, we will show that the global de Rham cohomology groups $H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$ can be realized as $\mathfrak{n}$-cohomology for a certain $G_{\mathbb{R}}$-module $\mathcal{O}G_{\mathcal{W}}$. Using this interpretation we will then observe that our spectral sequence is just the Hochschild-Serre spectral sequence.

The definition of $\mathcal{O}G_{\mathcal{W}}$ is as follows: From the earlier basic diagrams we obtain

$$\begin{tikzcd}
G_{\mathcal{W}} \arrow[d] & \subset & G_{\mathbb{C}} \arrow[d, "f"] \\
\mathcal{W} \arrow[d, "\pi_{\mathcal{J}}"] \arrow[ddd, bend right, "\pi"'] & \subset & G_{\mathbb{C}}/H \arrow[d] \\
\mathcal{J} \arrow[d, "\pi_D"] & \subset & G_{\mathbb{C}}/B_K \arrow[d] \\
D & \subset & G_{\mathbb{C}}/B.
\end{tikzcd}$$

Definition: *$G_{\mathcal{W}} = f^{-1}(\mathcal{W})$ is the open subset of $G_{\mathbb{C}}$ lying over $\mathcal{W}$ in the above diagram, and*

$$\mathcal{O}G_{\mathcal{W}} = \Gamma(G_{\mathcal{W}}, \mathcal{O}_{G_{\mathcal{W}}})$$

is the algebra of holomorphic functions on $G_{\mathcal{W}}$.

Now $\mathcal{O}G_{\mathcal{W}}$ is a somewhat strange object, but it is not as intractable as the definition might suggest. Since $G_{\mathcal{W}} \subset G$ is $G_{\mathbb{R}}$-invariant, $\mathcal{O}G_{\mathcal{W}}$ is a $G_{\mathbb{R}}$-module and therefore $\mathfrak{n}$-cohomology with coefficients in $\mathcal{O}G_{\mathcal{W}}$ is well-defined.

In fact, since

$$D = G_{\mathbb{R}} \cdot x_0 \subset G/B$$

and

$$\mathcal{W} = \{g \in G_{\mathbb{C}} : gK \cdot x_0 \subseteq D\}/H$$

we have

$$G_{\mathbb{R}}\mathcal{W} \subseteq \mathcal{W}, \qquad \mathcal{W}K \subseteq \mathcal{W}.$$

Thus, $G_{\mathbb{R}}$ and K act on $\mathcal{O}G_{\mathcal{W}}$ by

$$\begin{cases} (gf)(h) = f(gh) & g \in G_0, f \in \mathcal{O}G_{\mathcal{W}}, h \in G_{\mathcal{W}} \\ (fk)(h) = f(hk^{-1}) & k \in K. \end{cases}$$

Because $G_{\mathcal{W}} \subset G_{\mathbb{C}}$ is an open set, the Lie algebra $\mathfrak{g}$, viewed as right invariant vector fields on $G_{\mathbb{C}}$, acts on $\mathcal{O}G_{\mathcal{W}}$ on the left. When $\mathfrak{g}$ is viewed as left invariant vector fields it acts on $\mathcal{O}G_{\mathcal{W}}$ on the right. These two actions commute, and we will use the right action of $\mathfrak{n}$ to define $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$. These groups then have an action on $G_{\mathbb{R}}$ on the left and an action of the Cartan subgroup H on the right.

Theorem: (i) *There is a natural isomorphism*

$$H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^{\bullet}_{\pi}(L_\mu)); d_\pi\big) \cong H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}.$$

(ii) *The Hochschild-Serre spectral sequence associated to the sub-algebra* $\mathfrak{n}_K \subset \mathfrak{n}$ *coincides with the spectral sequence given in the earlier theorem.*

Proof. The notation $(\ )_{-\mu}$ on the RHS of the above isomorphism means the following: The Cartan subgroup H acts on the right on $G_{\mathcal{W}}$ and therefore acts on the complex $(\Lambda^{\bullet}\mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}, \delta)$ that computes Lie algebra cohomology. Then $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}$ is that part of $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$ that transforms by the character χ_μ^{-1} of H corresponding to the weight $-\mu$. This enters the picture because holomorphic sections of $\pi^* L_\mu \to \mathcal{W}$ are given by holomorphic functions on $G_{\mathcal{W}}$ that transform by χ_μ^{-1} under the right action of H.

The proof of (i) in the above theorem is essentially the observation from the proof of the earlier theorem, and using the identifications there, that we have the natural identification complexes

$$\Gamma\big(\mathcal{W}, \Omega^{\bullet}_{\pi}(L_\mu); d_\pi\big) \cong (\Lambda^{\bullet}\mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}; \delta)_{-\mu}.$$

Here "natural" means that the action of $G_{\mathbb{R}}$ on the LHS is given by the $G_{\mathbb{R}}$-module structure of $\mathcal{O}G_{\mathcal{W}}$.

Turning to (ii) in the theorem, here the basic observation is that when pulled back to $G_{\mathcal{W}}$, the exact sequence used in the earlier argument is the dual to the restriction to $G_{\mathcal{W}} \subset G_{\mathbb{C}}$ of the exact sequence of homogeneous vector bundles over G/H given by the exact sequence of H-modules

$$0 \to \mathfrak{n}_K \to \mathfrak{n} \to \mathfrak{p}^- \to 0.$$

From this we may infer (ii) in the theorem. $\square$

We note that using the above identifications and $\mathfrak{p}^{-*} \cong \mathfrak{p}^+$ via the Cartan-Killing form,

$$E_1^{p,q} = H^q(\mathfrak{n}_K, \Lambda^p \mathfrak{p}^+ \otimes \mathcal{O}G_{\mathcal{W}})_{-\mu}.$$

Using this interpretation we shall now compute the symbol $\sigma(d_1)$ of

$$d_1 : H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big) \to H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^{p+1}_{\pi_D}(L_\mu)\big).$$

Following the identification there of the fibre of the vector bundle $F^{p,q}_{\mu,u_0} \to \mathcal{U}$ and tangent space $T_{u_0}\mathcal{U}$ at the reference point, and identifying Z_{u_0} with Z to simplify the notation, the symbol $\sigma(d_1)$ of the 1st-order linear differential operator is a map

$$\sigma(d_1) : H^q\big(Z, \Lambda^p N_{Z/D}(L_\mu)\big) \otimes \mathfrak{p}^* \to H^q\big(Z, \Lambda^{p+1} N_{Z/D}(L_\mu)\big).$$

Theorem: *With the identifications* $\mathfrak{p}^* \cong \mathfrak{p}$ *given by the Cartan-Killing form and inclusion* $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ *the symbol is given by*

$$\sigma(d_1)\varphi \otimes X = \varphi \wedge X.$$

Here, on the LHS we have $X \in \mathfrak{p}$ and $\varphi \in H^q(Z, \Lambda^p N_{Z/D}(L_\mu))$, and on the RHS X is the corresponding normal vector field in $H^0(Z, N_{Z/D})$. The map is $H^q(Z, \Lambda^p N_{Z/D}(L_\mu)) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, \Lambda^{p+1} N_{Z/D}(L_\mu))$ induced by $\Lambda^p N_{Z/D} \otimes N_{Z/D} \to \Lambda^{p+1} N_{Z/D}$.

Proof. To compute the symbol on $\varphi \otimes X$, we take a section f of $F^{p,q}$ defined near u_0 with $f(u_0) = 0$ and whose linear part is $\varphi \otimes X$. Then by definition

$$\sigma(d_1)\varphi \otimes X = (d_1 f)(u_0).$$

We shall give the computation when $p = 0$, $q = 1$ as this will indicate how the general case goes. Pulled back to $G_{\mathcal{W}}$ we may write

$$f = \sum_{\alpha \in \Phi_c^+} f_\alpha \omega^{-\alpha}$$

where the f_α are holomorphic functions that vanish along the inverse image of Z_{u_0}. Then

$$d_1 f = \sum_{\left\{\substack{\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+}\right.} (f_\alpha X_{-\beta}) \omega^{-\beta} \wedge \omega^{-\alpha} + \sum_{\alpha \in \Phi_c^+} f_\alpha d_\pi \omega^{-\alpha}.$$

The second term vanishes along the inverse image of Z_{u_0}. As for the first term, under the pairing

$$\begin{pmatrix} \text{normal vector fields} \\ \text{to } Z_{u_0} \end{pmatrix} \otimes \begin{pmatrix} \text{holomorphic functions} \\ \text{vanishing along } Z_{u_0} \end{pmatrix} \to \mathcal{O}_{Z_0}$$

when evaluated along Z_{μ_0} the first term is the value along Z_{u_0} of

$$\sum_{\left\{\substack{\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+}\right.} (f_\alpha X_{-\beta}) X_\beta \otimes \omega^{-\alpha}$$

where $X_\beta \otimes \omega^{-\alpha} \in \mathfrak{p}^+ \otimes \mathfrak{n}^*$ and $\sum f_\alpha X_{-\beta}\big|_{Z_0} \in \mathcal{O}_{Z_0}$. $\square$

Discussion: The $G_\mathbb{R}$-module $\mathcal{O}G_{\mathcal{W}}$ is certainly not a Harish-Chandra module, but it does have an interesting structure, reflecting the fact that $\mathcal{W}$ is a mixed algebro-geometric/complex analytic object, as we now explain. The fibres of

$$\begin{array}{ccc} G_{\mathcal{W}} & \subset & G_\mathbb{C}/H \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow \\ \mathcal{U} & \subset & G_\mathbb{C}/K_\mathbb{C} \end{array}$$

are affine algebraic varieties isomorphic to the enhanced flag variety $K_\mathbb{C}/H$. We may smoothly and equivariantly compactify $G_\mathbb{C}/H$ so that each fibre $g^{-1}(u)$, $u \in \mathcal{U}$, is the complement of a divisor with normal crossings. Then we may consider the $G_\mathbb{R}$-invariant sub-algebra $\mathcal{O}G_{\mathcal{W}}^{\text{alg}} \subset \mathcal{O}G_{\mathcal{W}}$ of functions that are rational along each fibre, and by truncating Laurent series we may write $\mathcal{O}G_{\mathcal{W}}^{\mathrm{alg}}$ as the union of $G_{\mathbb{R}}$-submodules that are fibrewise K-finite acting on the right. Thus as a $G_{\mathbb{R}}$-module over the $G_{\mathbb{R}}$-module $\mathcal{O}(\mathcal{U}) = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ we see that $\mathcal{O}G_{\mathcal{W}}$ has a reasonable structure.

As for the $G_{\mathbb{R}}$-module $\mathcal{O}(\mathcal{U})$, we have noted above that $\mathcal{U}$ has the function-theoretic characteristics of a bounded domain of holomorphy (contractible, Stein, Kobayashi hyperbolic). In fact, for $G_{\mathbb{R}}$ of Hermitian type, $\mathcal{U} \cong \mathbb{B} \times \overline{\mathbb{B}}$ where $\mathbb{B}$ is an Hermitian symmetric domain and where $G_{\mathbb{R}}$ acts diagonally. Again, $\mathcal{O}(\mathcal{U})$ is not a HC-module but it seems to be a reasonable object. Here we shall illustrate it in the case of $S\mathcal{U}(2,1)$.

Examples: We represent elements of $G_{\mathbb{C}} = \mathrm{SL}(3,\mathbb{C})$ as

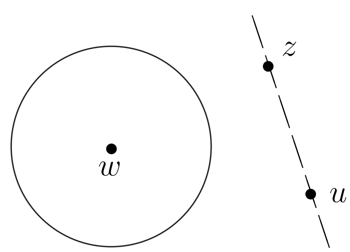
$$g = \begin{pmatrix} z_1 & w_1 & u_1 \\ z_2 & w_2 & u_2 \\ z_3 & w_3 & u_3 \end{pmatrix} = (z, w, u).$$

Taking as Hermitian form $\mathbb{H} = \mathrm{diag}(1,-1,1)$, $G_{\mathcal{W}} \subset G_{\mathbb{C}}$ is defined by the conditions

$$\begin{cases} \mathbb{H}(w) < 0 \\ \mathbb{H}(z \wedge u) > 0. \end{cases}$$

The map $G_{\mathcal{W}} \to \mathcal{W}$ is given by

$(z, w, u) \to$



the dashed line indicating that the line $\overline{zu}$ lies in $\mathbb{B}^c$. The space $\mathcal{O}G_{\mathcal{W}}$ is spanned by the functions

$$w_1^i w_2^j w_3^k (z_2u_3 - z_3u_2)^l (z_3u_1 - z_1u_3)^m (z_1u_2 - z_2u_1)^n z_1^p z_2^q z_3^r u_1^a u_2^b u_3^c$$

where

$$i, j, i+j+k, l, m, l+m+n, p, q, r, a, b, c \geq 0.$$

There are relations among the generators, such as

$$\left(\frac{z_2u_3 - z_3u_2}{z_1u_2 - z_2u_1}\right)(z_1u_2 - z_2u_1) = z_2u_3 - z_3u_2.$$

Appendix to Lecture 7: The BWB theorem revisited

We shall interpret the BWB theorem in the context of the EGW theorem. Using this interpretation we shall introduce the Penrose transforms in this situation; in fact, this construction leads to yet another proof of the BWB theorem. The bottom line of the discussion will be this:

> *For a flag domain $\check{D} = G_{\mathbb{C}}/B$, the various manifestations of an irreducible finite dimension $G_{\mathbb{C}}$-module as cohomology groups $H^q(\check{D}, L_\mu)$ are realized geometrically by Penrose transforms between these groups.*

We shall use the piece

$$\begin{tikzcd} G_{\mathbb{C}} \arrow[d] \\ \check{\mathcal{W}} = G_{\mathbb{C}}/H \arrow[d, "\pi"] \\ \check{D} = G_{\mathbb{C}}/B \end{tikzcd}$$

of the diagram at the beginning of the lecture. By the EGW theorem we have

$$H^q\big(\check{D}, L_\mu\big) = H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big).$$

Here, as in the discussion of $\mathfrak{n}$-cohomology given in Lecture 5 and above, we may pull everything up to $G_{\mathbb{C}}$ to obtain an isomorphism of $G_{\mathbb{C}}$-modules

$$H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big) \cong H^q(\mathfrak{n}, \mathcal{O}G_{\mathbb{C}})_{-\mu}.$$

On the RHS $\mathcal{O}G_{\mathbb{C}}$ is the algebra of holomorphic functions on $G_{\mathbb{C}}$. Denoting by $\mathcal{O}^{\mathrm{alg}}_{G_{\mathbb{C}}}$ the algebra of holomorphic, rational, functions it seems reasonable that using GAGA type arguments the inclusion $\mathcal{O}^{\mathrm{alg}}_{G_{\mathbb{C}}} \hookrightarrow \mathcal{O}G_{\mathbb{C}}$ induces an isomorphism on $\mathfrak{n}$-cohomology, and we shall assume this. Then the algebraic version of the Peter-Weyl theorem gives

$$\mathcal{O}^{\mathrm{alg}}_{G_{\mathbb{C}}} = \bigoplus_{\lambda \in \hat{G}_{\mathbb{C}}} V^\lambda \otimes V^{\lambda^*},$$

where the RHS are the finite direct sums of the $G_{\mathbb{C}}$-modules $\mathrm{Hom}(V^\lambda, V^\lambda)$ ranging over the equivalence classes of irreducibles V^λ indexed by their highest weights. Putting things together yields

$$H^q(\check{D}, L_\mu) \cong \bigoplus_{\lambda \in \hat{G}_{\mathbb{C}}} V^\lambda \otimes H^q(\mathfrak{n}_c, V^{\lambda^*})_{-\mu}.$$

Here we are conforming to the notation $\mathfrak{n}_c$ for the direct sum of all of the negative root spaces. This is the same $\mathfrak{n}$ as above — the subscript "c" is used to signify that we are working with the compact real form of $G_{\mathbb{C}}$. By Kostant's theorem, the only non-zero term on the RHS occurs when $\mu+\rho$ is non-singular, $q=q_c(\mu+\rho)$ and $\lambda=w(\mu+\rho)-\rho$. Thus for this λ

$$H^{q_c(\mu+\rho)}(\check{D},L_\mu)\cong V^\lambda\otimes H^{q_c(\mu+\rho)}(\mathfrak{n},V^{\lambda^*})_{-\mu}.$$

We also have for this λ

$$H^0(\check{D},L_\lambda)\cong V^\lambda\otimes H^0(\mathfrak{n},V^{\lambda^*})\cong V^\lambda\otimes\mathbb{C}v_{-\lambda}$$

where $v_{-\lambda}$ is a non-zero lowest weight vector for V^{λ^*}. This leads to a diagram of $G_{\mathbb{C}}$-modules

$$\begin{tikzcd} H^0_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}},\Omega^\bullet_\pi(L_\lambda));d_\pi\big) \arrow[r, "\Theta_\mu"] \arrow[d, phantom, "\wr\|"] & H^{q_c(\mu+\rho)}_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}},\Omega^\bullet_\pi(L_\mu));d_\pi\big) \arrow[d, phantom, "\wr\|"] \\ H^0(\check{D},L_\lambda) \arrow[r, dashed, "\mathcal{P}"] & H^{q_c(\mu+\rho)}(\check{D},L_\mu) \end{tikzcd}$$

where the top row is multiplication by

$$\Theta_\mu=:v_{-\lambda}\otimes\kappa_\mu$$

where κ_μ is the Kostant form from the appendix to Lecture 5 and where $v_\lambda\in V^\lambda$ is a highest weight vector and $\langle v_\lambda,v_{-\lambda}\rangle=1$. The vertical isomorphism are given by the EGW theorem, and the bottom dotted arrow is by definition a *Penrose transform.* This diagram and interpretation is what is meant in the italicized statement at the beginning of this appendix.

We note that when we are using $\mathfrak{n}$-cohomology to represent $\overline{\partial}$-cohomology as was done in Lecture 5, the $\omega^{-\alpha}$ for $\alpha\in\Phi^+$ are the pullbacks to $G_{\mathbb{R}}$ of $(0,1)$ forms on D; i.e.,

$$\omega^{-\alpha}=\pm\overline{\omega}^\alpha \textit{ where } \omega^\alpha \textit{ is dual to } X_\alpha\in T^{1,0}_eD.$$

However, here the $\omega^{-\alpha}$ for $\alpha\in\Phi^+$ are the pullbacks to $G_{\mathbb{C}}$ of holomorphic relative differentials; i.e.,

$$\omega^{-\alpha}\in\Omega^1_\pi.$$

The Lie algebra cohomology calculations are formally the same; the interpretation is different.

Using this we shall now give the proof, promised in the appendix to Lecture 5, that the Kostant form is closed. In the present notation we have to show that

$$d_\pi\kappa_\mu=0.$$

We first show that

$$d_\pi\omega^{-\langle\Psi_w\rangle}=0.$$

For this we use the Maurer-Cartan equation, which gives

$$d\omega^{-\alpha} \equiv \left(\frac{-1}{2}\right) \sum_{\beta,\gamma} c^{\alpha}_{\beta\gamma} \omega^{-\beta} \wedge \omega^{-\gamma} \operatorname{mod} \mathfrak{h}^* \wedge \mathfrak{g}^*_{\mathbb{C}}.$$

Here,

$$c^{\alpha}_{\beta\gamma} \neq 0 \Rightarrow \alpha = \beta + \gamma.$$

Passing to relative differentials means that we set

$$\omega^{\beta} \equiv 0 \text{ if } \beta \in \Phi^+.$$

Then

$$d_\pi \omega^{-\alpha} = \left(\frac{-1}{2}\right) \sum_{\beta,\gamma \in \Phi^+} c^{\alpha}_{\beta\gamma} \omega^{-\beta} \wedge \omega^{-\gamma}.$$

If $\Psi_w = \{\psi_1, \dots, \psi_q\} \subset \Phi^+$, this gives

$$d_\pi \omega^{-\langle \Psi_w \rangle} = \left(\frac{-1}{2}\right) \sum_j (-1)^j c^{\psi_j}_{\beta\gamma} \omega^{-\beta} \wedge \omega^{-\gamma} \wedge \omega^{-\psi_1} \wedge \cdots \wedge \widehat{\omega^{-\psi_j}} \wedge \cdots \wedge \omega^{-\psi_q}$$

where the sum is over $\beta, \gamma \in \Psi^c_w = \Phi^+ \backslash \Psi_w$. Since by (ii) in the properties of Ψ_w in Lecture 5, Ψ^c_w is closed under addition, we have $c^{\psi_j}_{\beta\gamma} = 0$, as desired.

We next compute $d_\pi v^*_{w(-\lambda)}$:

$$d_\pi v^*_{w(-\lambda)} = \sum_{\beta \in \Phi^+} X_{-\beta} \cdot v^*_{w(-\lambda)} \otimes \omega^{-\beta}.$$

The only terms that will contribute to $d_\pi \kappa_\mu = d_\pi (v^*_{w(-\lambda)} \otimes \omega^{-\langle \Psi_w \rangle})$ are the

$$X_{-\beta} v^*_{w(-\lambda)}, \quad \beta \in \Psi^c_w.$$

To see this, we have $\Phi^+ = \Psi_w \cup \Psi^c_w$ (disjoint union). For every $\alpha \in \Phi^+$, since $v^*_{-\lambda}$ is a lowest weight vector

$$X_{-\alpha} v^*_{-\lambda} = 0.$$

It follows that for every $\beta \in w\Phi^+$

$$X_{-\beta} \cdot v^*_{w(-\lambda)} = 0.$$

But $\beta \in \Psi^c_w \Rightarrow \beta \in w\Phi^+$ and we are done.

Finally, using the above diagram a proof of BWB may be given as follows:

- the first statement (i) in the theorem follows from our earlier curvature calculations and the Kodaira vanishing theorem, as was the case in the original proof by Bott;
- the Kostant form κ_μ is harmonic in the sense of EGW, from which it follows that the Penrose transform is injective;

- finally, by the Hirzebruch-Riemann-Roch theorem the bottom two groups have the same dimension.

We mention this argument here because a similar one will be used later in a more involved setting.

Lecture 8
Penrose transforms in the two main examples

In this and the next lecture we shall study automorphic cohomology defined on quotients by a co-compact, neat arithmetic subgroup $\Gamma \subset G$ for the flag domains associated to $\mathcal{U}(2,1)_{\mathbb{R}}/T$ and $\mathrm{Sp}(4)_{\mathbb{R}}/T$. Specifically, with the notation

- $X = \Gamma\backslash D$ where D is non-classical,
- $X' = \Gamma\backslash D'$ where D' is classical,
- $\mathcal{W}_\Gamma = \Gamma\backslash\mathcal{W}$

we will use the correspondence diagram

$$\begin{tikzcd} & \mathcal{W}_\Gamma \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X & & X' \end{tikzcd}$$

and the EGW theorem to construct Penrose transforms

$$H^0(X', L'_{\mu'}) \xrightarrow{\mathcal{P}} H^1(X, L_\mu)$$

where μ', μ are certain weights related by

$$\mu' + \rho' = \mu + \rho,$$

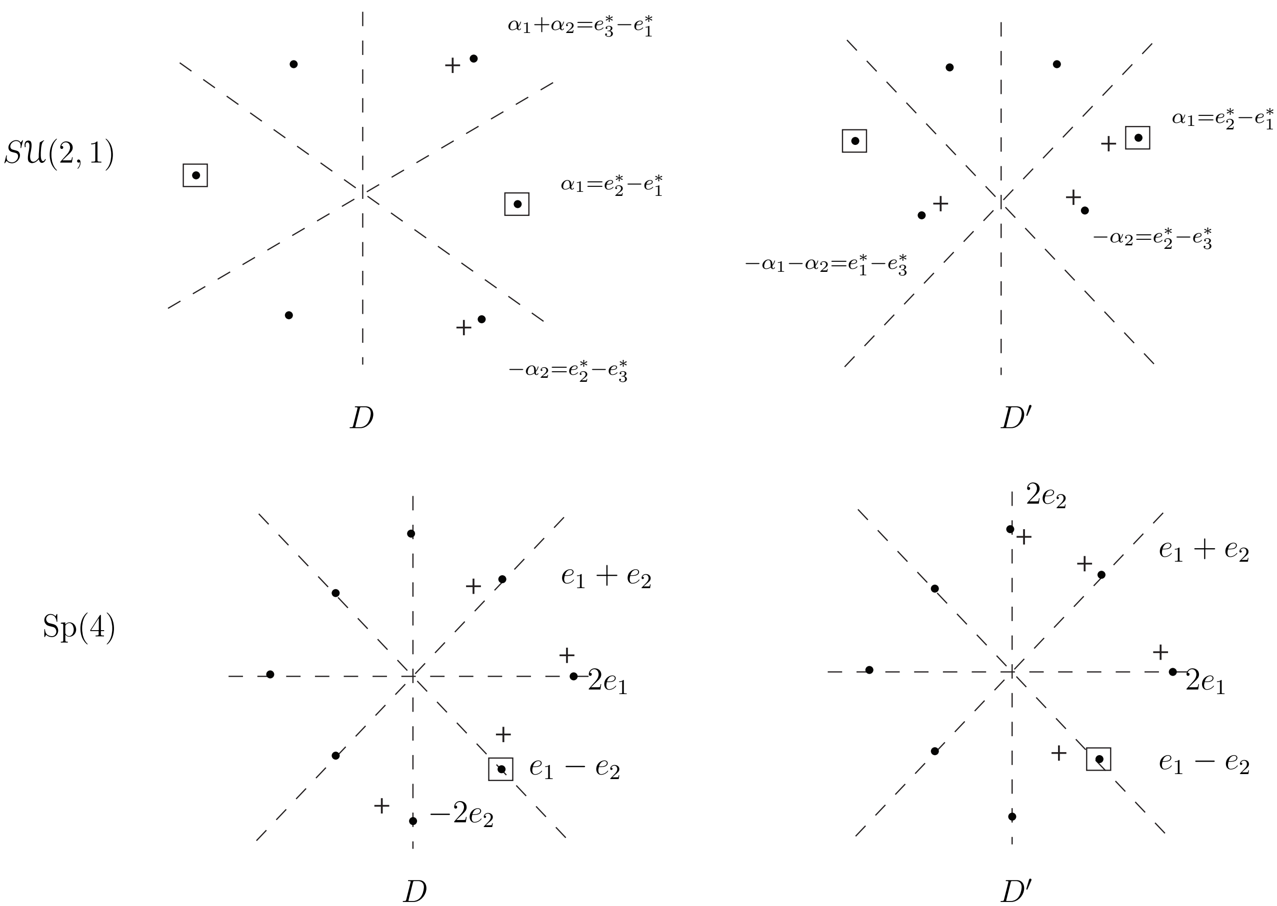
and where $\mathcal{P}$ is an isomorphism taking Picard modular forms, respectively Siegel modular forms to the *non-classical automorphic cohomology group* $H^1(X, L_\mu)$.[39] This will be done by constructing an injective Penrose transform map

$$H^0(D', L'_{\mu'}) \to H^1(D, L_\mu)$$

and then passing to the quotient by Γ. There the Penrose will still be injective and then by equality of dimensions it will be an isomorphism. In this lecture we shall discuss the proof of the result for D' and D and in the next we shall treat it for X' and X.

[39]Here, non-classical means that D is non-classical. It seems plausible, but has not yet been proved in generality that X is *not* an algebraic variety (cf. Lecture 10). Nevertheless, certain of its coherent cohomology groups are naturally isomorphic to those of a projective algebraic variety.

For easy reference the homogeneous complex structures we shall use are illustrated by the following root diagrams:[40]



In the above the positive roots are labelled with a + and the compact roots are denoted $\boxed{\bullet}$. Note that

> *we are in the non-classical case if, and only if, the positive compact root is* **not** *simple.*

This is the case when the cohomology group $H^1(D, L_{-\rho})$ is a Harish-Chandra module corresponding to a TDLDS with infinitesimal character χ_0 and given by the data $(0, C)$ where C is the positive Weyl chamber, the case of particular interest in these lectures.

[40]In these lectures we will fairly consistently use the notations $e_i^* - e_j^*$ for the weights in the $S\mathfrak{U}(2,1)$ case. We have included here, and have used in Lecture 4, the alternative α_1, α_2 notation as this is used in [GGK2] where detailed proofs of several of results discussed below are given.

Line bundles for $S\mathcal{U}(2,1)$

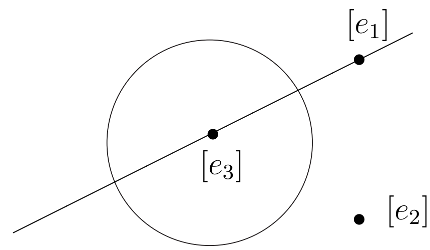
We use the following notations:

- $\mathbb{C}^3$ = column vectors with standard basis $e_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$;
- H is the Hermitian form with matrix $\left(\begin{smallmatrix}1&&\\&1&\\&&-1\end{smallmatrix}\right)$ and where
$$H(u,v) = {}^t\bar{v}Hu;$$
- setting $H(u) = H(u,u)$, the unit ball $\mathbb{B} \subset \mathbb{P}^2$ is defined
$$H(u) < 0.$$
- e_1^*, e_2^*, e_3^* is the dual basis to e_1, e_2, e_3, considered as row vectors;
- the maximal torus T of $S\mathcal{U}(2,1)_{\mathbb{R}}$ is
$$\left\{ g = \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & e^{2\pi i\theta_2} & \\ & & e^{2\pi i\theta_3} \end{pmatrix} \right\};$$
- the isomorphism between T and $\mathfrak{t}/\Lambda$ is given by
$$g \to \boldsymbol{\theta} = \begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3;$$
Here, $\mathfrak{t} \cong \mathbb{R}^3$ and $\Lambda \cong \mathbb{Z}^3$;
- the inclusion $S\mathcal{U}(2,1)_{\mathbb{R}} \hookrightarrow \mathcal{U}(2,1)_{\mathbb{R}}$ induces
$$\mathfrak{t}_S \hookrightarrow \mathfrak{t}$$
where $T_S =: T \cap S\mathcal{U}(2,1)_{\mathbb{R}} = \mathfrak{t}_S/\Lambda_S$;
- $\mathfrak{t}_S = \operatorname{span}_{\mathbb{R}}\{u_1, u_2\}$ where
$$\begin{cases} u_1 = e_1 - e_2 \\ u_2 = e_2 - e_3; \end{cases}$$
- $\mathfrak{t}_S \subset \mathfrak{t}$ is defined by the equation
$$e_1^* + e_2^* + e_3^* = 0.$$

In the above root diagram for su$(2,1)$ we are thinking of the e_i^* as linear functions on $\mathfrak{t}_S$. In the literature the roots of su$(2,1)$ are frequently denoted by $e_i - e_j$, but for reasons that will appear below in this case we feel it is better to use $e_i^* - e_j^*$ in order to notationally distinguish between $\mathfrak{t}_S$ and $\mathfrak{t}_S^*$.

Homogeneous line bundles for D

The *reference flag* for D is $[e_1] \subset [e_1, e_3] \subset \mathbb{C}^3$ (note the ordering), where $[\ ]$ denotes linear span. Then $[e_1, e_3]$ is given by the line $[e_2]^\perp \subset \mathbb{C}^3$. The picture is



We have chosen the indexing this way so as to have for the maximal compact subgroup $K \subset S\mathcal{U}(2,1)$

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in \mathcal{U}(2), a = \det A^{-1} \right\}.$$

We shall consider three types of $S\mathcal{U}(2,1)_{\mathbb{R}}$-homogeneous line bundles over D:

(a) $F_{(a,b)}$ = restriction to D of the line bundle $\mathcal{O}_{\mathbb{P}^2}(a) \boxtimes \mathcal{O}_{\mathbb{P}^{2*}}(b)$ over $\mathbb{P}^2 \times \mathbb{P}^{2*}$;

(b) $L_{\mathfrak{k}}$ obtained from the character corresponding to the weight $\mathfrak{k} = (k_1, k_2, k_3) \in \mathrm{Hom}(\Lambda, \mathbb{Z})$;

(c) the Hodge bundles $\mathbb{V}^{p,q}$ for the PHS of weight $n = 3$ with Hodge numbers $h^{3,0} = 1$, $h^{2,1} = 2$ described in Lecture 3.

We note that

$$L_{\mathfrak{k}} \cong L_{\mathfrak{k}+} \ \textit{as homogeneous lines bundles} \ \Leftrightarrow \mathfrak{k} = \mathfrak{k}' + m(1,1,1) \ \text{for} \ m \in \left(\frac{1}{3}\right)\mathbb{Z}.$$

The $1/3$ appears because the root lattice R and weight lattice P are related by

$$P/R \cong \mathbb{Z}/3\mathbb{Z}.$$

We say that $\mathfrak{k}$ *is normalized* if $k_1 + k_2 + k_3 = 0$. Given any $\mathfrak{k}'$ we may uniquely chose m as above so that $\mathfrak{k}$ is normalized. The relation between (a) and (b) is

$$F_{(a,b)} = L_{\left(\frac{2a+b}{3}\right)(e_2^* - e_1^*) + \left(\frac{a-b}{3}\right)(e_3^* - e_2^*)} = L_{\left(\frac{2a+b}{3}\right)\alpha_1 + \left(\frac{a-b}{3}\right)\alpha_2}.$$

Proof. The fibre $F_{(-1,0)}$ at the reference flag is the line $[e_1]$ on which T acts by the character $e^{2\pi i \theta_1}$ corresponding to the weight $e_1^* = (1,0,0)$. Thus $F_{(-1,0)} = L_{e_1^*}$. Similarly, the fibre of $F_{(0,-1)}$ is the line $[e_2]^\perp \subset \mathbb{C}^3$ on which T acts by the character whose corresponding weight is $-e_2^*$. Thus

$$F_{(a,b)} = L_{-ae_1^* + be_2^*}.$$

For $\mathfrak{k}' = (-a, b, 0)$, $m = 1/3(a-b)$ and the normalized weight is

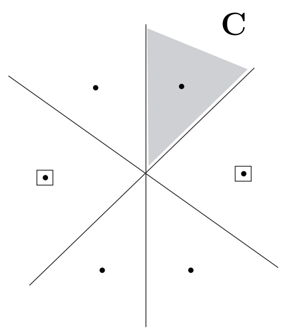
$$\begin{aligned}\mathfrak{k} &= \frac{1}{3}(-2a-b, 2b+a, a-b)\\ &= \frac{1}{3}(-2a-b, 2a+b, 0) + \frac{1}{3}(0, -a+b, a-b),\end{aligned}$$

which gives the result. $\square$

A similar argument gives for the Hodge bundles

$$\begin{cases}\mathbb{V}_{+}^{3,0} = L_{(1,0,0)}\\ V_{+}^{2,1} = L_{(0,0,1)}\\ V_{+}^{1,2} = L_{0,1,0)}.\end{cases}$$

We picture Weyl chambers in the usual way



We note that for μ with $\mu + \rho \in \mathbf{C}$ we have $q(\mu+\rho) = 1$, and hence

$$H^1_{(2)}(D, L_\mu) \neq 0.$$

Although this is not the anti-dominant Weyl chamber it is the one that will play a central role in the Penrose transform discussed below.

Homogeneous line bundles for D'

Here the reference flag is $[e_3] \subset [e_3, e_1] \subset \mathbb{C}^3$. The picture is the same as above but where now the pair $(p, l) \in \check{D}$ has $p = [e_3] \in \mathbb{B}$. A similar argument to the one above gives

$$F'_{(a',b')} = L'_{\left(\frac{b'-a'}{3}\right)(e_2^*-e_1^*)+\left(\frac{-2a'-b'}{3}\right)(e_3^*-e_2^*)} = L'_{\left(\frac{b'-a'}{3}\right)\alpha_1+\left(\frac{-2a'-b'}{3}\right)\alpha_2}.$$

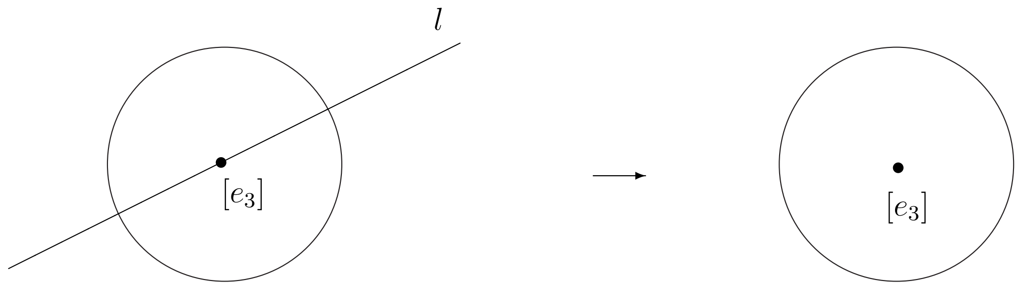
For the PHS of weight $n = 3$ with $h^{1,0} = 3$ we have

$$\mathbb{V}_{+}^{1,0} = F'_{(-1,0)}.$$

We recall the holomorphic fibration

$$D' \to \mathbb{B}$$

of D' over the unit ball given by the picture



The homogeneous line bundles on $\mathbb{B}$ are the $L'_{\mathfrak{k}'}$ for which $\mathfrak{k}'$ is orthogonal to the compact root $e_2^* - e_1^*$. By the above these are the line bundles

$$F'_{(a',-a')}$$

when $b' = a'$. Of particular importance is the pullback $\omega'_{\mathbb{B}}$ to D' of the canonical bundle $\omega_{\mathbb{B}}$. We have

$$\omega'_{\mathbb{B}} = L'_{2e_3^* - e_1^* - e_2^*} = \mathbb{V}_+^{1,0\,\otimes\,3}$$

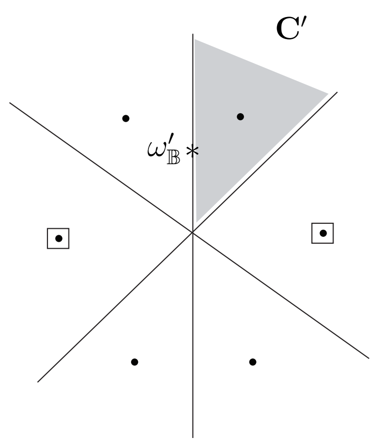
which, setting ${\omega'}_{\mathbb{B}}^{1/3} = \mathbb{V}_+^{1,0}$, gives

$${\omega'}_{\mathbb{B}}^{\otimes k/3} = \otimes^k \mathbb{V}_+^{1,0} = F'_{(-k,0)}.$$

For D' we have

$$\rho' = e_2^* - e_3^*.$$

We picture a Weyl chamber as follows:



This $\mathbf{C}'$ is the same Weyl chamber as that labelled $\mathbf{C}$ for D. This Weyl chamber is *not* the dominant one for the complex structures on either D or D'. The roots are marked with $\bullet$ and $\boxed{\bullet}$ and the weight corresponding to $\omega'_{\mathbb{B}}$ with a $*$. Note that $*$ is perpendicular to the compact root since $\omega'_{\mathbb{B}}$ is the pullback of a line bundle over $G_{\mathbb{R}}/K$. We also note that

- *for* $\mu \in \mathbf{C}'$, $q(\mu) = 0$.
- *if* $\omega'^{\otimes k/3}_{\mathbb{B}} \otimes L_{\rho'} = L'_{\mu'_k}$, *then* $\mu'_k \in \mathbf{C}'$ *for* $k \geqq 3$.

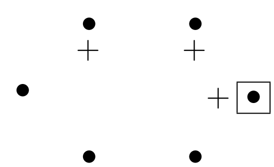
By Schmid's theorems this gives

$$H^0_{(2)}(D', \omega'^{\otimes k/2}_{\mathbb{B}}) \neq 0 \ \textit{for} \ k \geqq 3.$$

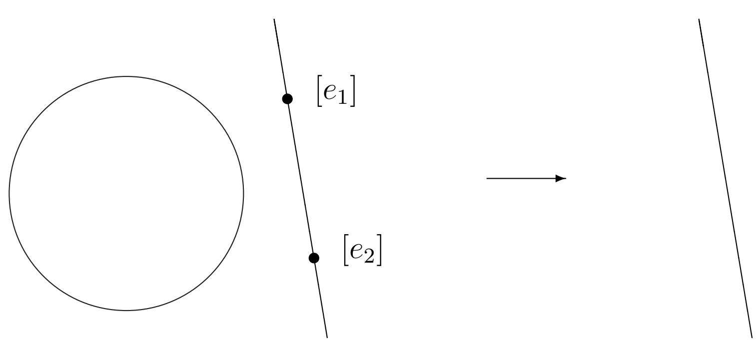
These are among the *holomorphic discrete series* (HDS) for $S\mathcal{U}(2,1)_{\mathbb{R}}$, and may be thought of as an analogue of the $\mathcal{D}^+_n$, $n \geqq 2$, in Lecture 1. We note that this is a Weyl chamber where for $\mu + \rho$, $\mu' + \rho'$ in it we have $H^1_{(2)}(D, L_\mu) \neq 0$, $H^0_{(2)}(D', L_{\mu'}) \neq 0$.

Holomorphic line bundles for D''

Here the classical complex structure is given by



The map



gives a holomorphic fibration $D'' \to \mathbb{B}^c$. The above discussion for D' may be repeated for D'', and the results will be used below.

Line bundles for $\mathrm{Sp}(4)$

The discussion is similar to, but simpler (no 1/3's), than that for $S\mathcal{U}(2,1)$, so we will just summarize what comes out.

We recall our notations:

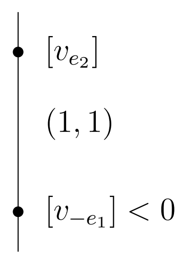
- $\check{D}$ consists of all Lagrangian flags $F^\bullet = \{F^1 \subset F^2 \subset F^3 \subset \mathbb{C}^4\}$ where $\dim F^i = i$ and $F^{1^\perp} = F^3, F^{2\perp} = F^2$;
- $F_{(a,b)} \to \check{D}$ is defined to be the homogeneous line bundle whose corresponding weight is $ae_1 + be_2$;
- our reference flag is

$$[v_{-e_1}] \subset [v_{-e_1}, v_{-e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}].$$

At the reference flag the fibre

$$\begin{aligned} F_{(1,0)} &= [v^*_{-e_1}] \leftrightarrow e_1 \\ F_{(0,1)} &= [v_{e_2}] \quad \leftrightarrow e_2. \end{aligned}$$

Our reference point in D is $[v_{-e_1}], [v_{-e_1}, v_{e_2}]$



where the < 0 and $(1,1)$ denote the sign of the Hermitian form on the point $[v_{-e_1}]$ and line $[v_{-e_1}, v_{e_2}]$ respectively. Thus for the Hodge bundles over D

$$\begin{cases} \mathbb{V}^{3,0} = F_{(-1,0)} \\ \mathbb{V}^{2,1} = F_{(0,1)}. \end{cases}$$

Turning to D', keeping the same reference flag as above we have at the reference point of D' the flag $[v_{-e_1}], [v_{-e_1}, v_{-e_2}]$

which gives

$$\begin{cases} F'_{(1,0)} = [v^*_{-e_1}] \leftrightarrow e_1 \\ F'_{(0,1)} = [v^*_{-e_2}] \leftrightarrow e_2. \end{cases}$$

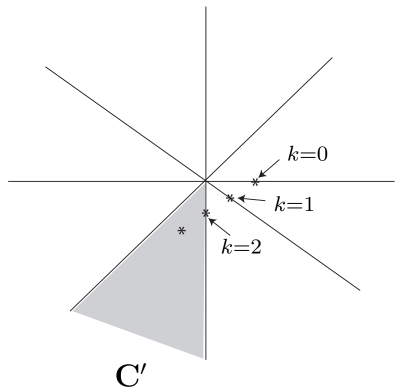
The Hodge-theoretic interpretation of D' we shall use is:

- $\mathcal{H}$ is the space of PHS's of weight $n = 1$ given by Lagrangian 2-planes $F^2 \subset \mathbb{C}^4$ with $H < 0$ on F^2;
- D' is the set of Hodge flags $F^1 \subset F^2$ lying over points of $\mathcal{H}$.

Thus D' is a $\mathbb{P}^1$-bundle over an HSD. Denoting by $\omega'_{\mathcal{H}}$ the pullback to D' of the canonical bundle $\omega_{\mathcal{H}}$, arguing in a similar way to the $S\mathcal{U}(2,1)$ case we find that

$$\omega'_{\mathcal{H}} = F_{(-3,-3)}.$$

In the Weyl chamber diagram



the shaded one is where for $\mu+\rho\in\mathbf{C}$, $\mu'+\rho'\in\mathbf{C}'$ we have

$$H^1_{(2)}(D,L_\mu)\neq 0,\quad H^0_{(2)}(D',L_{\mu'})\neq 0.^{41}$$

Note that for $k\geq 3$ we have

$$\omega'^{\otimes k/3}_{\mathcal{H}}\otimes\mathbb{L}_{\rho'}\in\mathbf{C}'.$$

The picture of the corresponding weights are the $*$'s above.

Penrose transforms for $S\mathcal{U}(2,1)$

We now come to one of the main results in this lecture series. In the diagram from Lecture 7, and where D, D' are the non-classical, respectively classical complex structures on $S\mathcal{U}(2,1)/T_S$,

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D & & D' \end{tikzcd}$$

we will first show that over $\mathcal{W}$

$$\begin{cases} \pi^* F_{(1,-1)} \cong \pi'^* F'_{(-1,0)} \\ \pi^* F_{(0,-1)} \cong \pi'^* F'_{(0,-1)}. \end{cases}$$

[41]Here, as in the $S\mathcal{U}(2,1)$ case, we are using the notation $\mathbf{C}$ and $\mathbf{C}'$ for the same Weyl chamber, the point being to indicate whether we have D or D' in mind.

Then taking $\omega = \omega^{-\alpha}$ where $\alpha = e_3^* - e_1^*$ we will have a commutative diagram

$$\begin{array}{ccc} H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'}(F'_{(a',b')})); d_\pi\big) & \xrightarrow{\omega} & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(F_{(a,b)})); d_\pi\big) \\ \wr\| & & \wr\| \\ H^0(D', F'_{(a',b')}) & \xrightarrow{\mathcal{P}} & H^1(D, F_{(a,b)}) \end{array}$$

where

$$\begin{cases} a = -a' - 2 \\ b = a' + b' + 1. \end{cases}$$

This defines the Penrose transform and the main result is the

THEOREM: *The Penrose transform*

$$H^0(D', F'_{(-3-l,0)}) \to H^1(D, F_{(l+1,-2-l)})$$

is injective for $l \geq 0$.

We have seen above that

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}$$

so the LHS above is

$$H^0\big(D', {\omega'}_{\mathbb{B}}^{\otimes(l/3+1)}\big) \cong H^0\big(\mathbb{B}, \omega_{\mathbb{B}}^{\otimes(l/3+1)}\big).$$

The Γ-invariant sections will be *Picard modular forms* of weight $l/3+1$, a classical object. We will see that the quotient by Γ of the above diagram and maps gives an isomorphism

$$H^0(X, F'_{(-3-l,0)}) \xrightarrow{\sim} H^1(X, L_{(l-1,-2-l)})$$

relating the classical object on the left to the non-classical one on the right.

We will not have time to give the details of the proof in the lecture. These appear in the appendix to the lecture; here we will comment on the essential ideas behind the argument.

Relation of the line bundles on D and D' pulled back to $\mathcal{W}$

This is given by

$$\begin{cases} \pi'^* F'_{(-1,0)} \cong \pi^* F_{(1,1)} \\ \pi'^* F'_{(0,-1)} \cong \pi^* F_{(0,-1)}. \end{cases}$$

Here the isomorphisms are as homogeneous line bundles over $\mathcal{W}$. These follow from

$$\pi^* F_{(-1,0)} \cong L_{e_1^*}, \quad \pi^* F_{(0,-1)} = L_{-e_2^*}$$
$$\pi'^* F'_{(-1,0)} \cong L_{e_3^*}, \quad \pi'^* F'_{(0,1)} = L_{-e_2^*}.$$

A consequence is

$$\boxed{\pi'^* F'_{(a',b')} \cong \pi^* F_{(-a',a'+b')}.}$$

Geometric interpretation of the form ω

This contains the essential geometric idea in the construction. We will identify $S\mathcal{U}(2,1)_{\mathbb{C}} \cong \mathrm{SL}(3,\mathbb{C})$ with the set of frames in $\mathbb{C}^3$. For f_1, f_2, f_3 independent column vectors we set them side by side to form a matrix

$$(f_1\, f_2\, f_3) = g \in \mathrm{SL}(3,\mathbb{C}).$$

The equations of a moving frame

$$df_i = \sum_j \omega_i^j f_j$$

have as coefficients the entries in the Maurer-Cartan matrix

$$\begin{pmatrix} \omega_1^1 & \omega_2^1 & \omega_3^1 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 \end{pmatrix} = g^{-1}dg.$$

Here the f_i are viewed as vector-valued maps $f_i : \mathrm{SL}(3,\mathbb{C}) \to \mathbb{C}^3$. The forms ω_i^j are linearly independent subject to the relation $\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$. Geometrically the ω_i^i each reflect the scaling action of the corresponding weight as we move in the fibres of $\mathrm{SL}(3,\mathbb{C}) \to \mathcal{W}$. For $\alpha = e_3^* - e_1^*$

$$\omega^{-\alpha} = \omega_3^1.$$

The root $e_3^* - e_1^*$ is the one that changes sign when we pass from D' to D. Geometrically

ω_3^1 *measures how* e_3 *moves along the line* $\overline{e_3 e_1}$.

The passage from D' to D is given symbolically by

$$([e_3], [e_3, e_1]) \to ([e_1], [e_3, e_1]).$$

We hope that this gives some intuitive indication of the geometry behind the Penrose transform.

Next we note that

ω *is a holomorphic section of* $\Omega_\pi^1 \otimes \pi^* F_{(-2,1)} \to \mathcal{W}$.

Assuming this and combining it with the boxed isomorphism above we see why the Penrose transform takes

$$\pi'^* F'_{(a',b')} \to \pi^* F_{(-a'-2,a'+b'+1)}.$$

For the proof of the italicized statement we use

- the *Maurer-Cartan equations*

$$d\omega_i^k = \sum_j \omega_i^j \wedge \omega_j^k$$

which result from $d^2 f_i = 0$;

- Ω^1_π means that we mod out by $\omega^2_1, \omega^3_1, \omega^2_3$, which we write as $\equiv_\pi 0$.

More precisely, working in the open set $G_{\mathcal{W}} \subset \mathrm{SL}(3,\mathbb{C})$ lying over $\mathcal{W}$, the 1-forms $\omega^2_1, \omega^3_1, \omega^2_3$ are semi-basic for the projection $G_{\mathcal{W}} \to D$. Note that

$$\begin{cases} \omega^2_1 = \omega^3_1 = 0 \Rightarrow \text{the point } [e_1] \text{ doesn't move} \\ \omega^2_3 = 0 \Rightarrow \text{the line } [e_3, e_1] \text{ doesn't move.} \end{cases}$$

Thus the integral manifolds of this (integrable) Pfaffian system define the fibres of $\mathcal{W} \xrightarrow{\pi} D$. This is the meaning of the last bullet above.

Remark: In general, for a submersion $f : M \to N$, we recall that differential forms ψ on M are *semi-basic differential form* if the contraction $X \rfloor \psi = 0$ for any vertical tangent vector field X (i.e., $f_* X = 0$). The sub-bundle of T^*M given by semi-basic 1-forms satisfy the Frobenius integrability condition, and the leaves of the foliation of M they define are the fibres of the above submersion.

For the proof of the italicized statement we have from the Maurer-Cartan equation

$$\begin{aligned} d\omega^1_3 &\equiv_\pi \left(\omega^3_3 - \omega^1_1\right) \wedge \omega^1_3 \\ &\equiv_\pi \left(-2\omega^1_1 - \omega^2_2\right) \wedge \omega^1_3 \end{aligned}$$

using $\omega^1_1 + \omega^2_2 + \omega^3_3 = 0$. This says that ω^1_3 scales by the character with weight $-2e^*_1 - e^*_2$ as we move in the fibres of $G_{\mathcal{W}} \to \mathcal{W}$, which was to be shown.

We next let F be a holomorphic function on $G_{\mathcal{W}}$ that is the pullback of a holomorphic section of $L'_{\mu'} \to D'$. We claim that

$$dF \equiv 0 \operatorname{mod}\left\{\omega^1_1, \omega^2_2, \omega^3_3, \omega^2_1, \omega^1_3, \omega^2_3\right\}.$$

The reason is that first the coefficients of the ω^i_i give the scaling action corresponding to the weight. Next, the 1-forms $\omega^2_1, \omega^1_3, \omega^2_3$ are semi-basic for $\mathcal{W} \xrightarrow{\pi'} D$, which implies the claim.

Since $\omega^2_1, \omega^2_3 \equiv_\pi 0$, we infer that

$$d_\pi F \omega^1_3 = 0.$$

This proves that the map

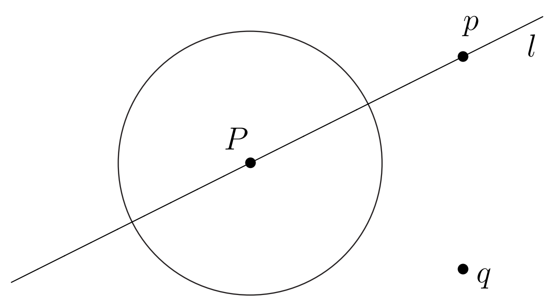
$$H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'}(F'_{\mu'}))d_{\pi'}\big) \xrightarrow{\omega} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(F_\mu)); d_\pi\big)$$

is well defined, where the weight μ is determined by the scaling action of $F\omega^1_3$. The difficult part of the proof of the result stated above is to show that for the range of indices stated in the theorem *the Penrose transform is injective.* That is

$$F\omega = d_\pi G \Rightarrow G = 0$$

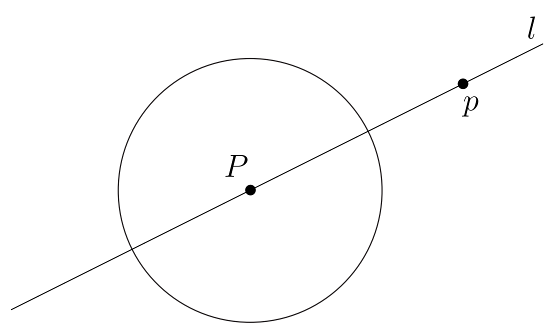
where $G \in \Gamma(\mathcal{W}, \pi^{-1}F_\mu)$.

The geometric idea behind the proof of this statement is the following: Recall that $\mathcal{W}$ consists of all configurations $\{p, P, q\}$ in the figure



, $\overline{pq} \subset \mathbb{B}^c$

The transformation from D' to D does not involve q; intuitively, P moves along the fixed line l to p. This suggests that we consider the quotient space $\mathcal{J}$ of all configurations



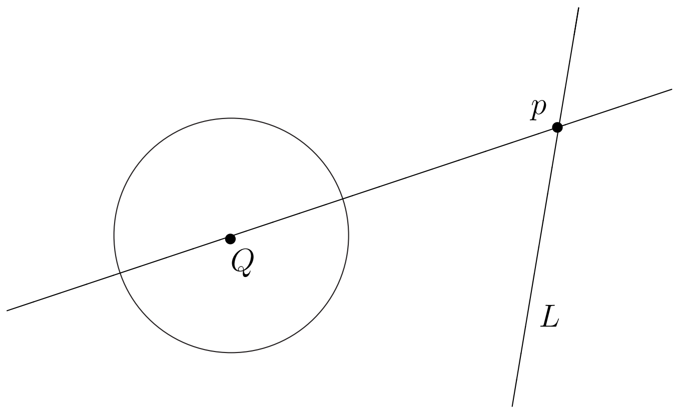
There is an evident diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[ddl, "\pi"'] \arrow[d, "\tau"] \arrow[ddr, "\pi'"] & \\ & \mathcal{J} \arrow[dl, "\sigma"] \arrow[dr, "\sigma'"'] & \\ D & & D' \end{tikzcd}$$

where

$$\begin{cases} \sigma(P, p, l) = (p, l) \\ \sigma'(P, p, l) = (P, l). \end{cases}$$

The variety $\mathcal{J}$ is not Stein — it contains the compact subvarieties $Z_L \cong \mathbb{P}^1$ given by fixing $Q \in \mathbb{B}$, taking a line $L \subset \mathbb{B}^c$ and looking at all the configurations $(Q, p, \overline{Qp}) \in \mathcal{J}$



However, the fibres of $\mathcal{W} \xrightarrow{\tau} \mathcal{J}$ and $\mathcal{J} \xrightarrow{\sigma} D$ are contractible, so at least part of the proof of the EGW theorem applies. When one works out what this means one finds that

- $F\omega$ lives on $\mathcal{J}$; i.e., on $\mathcal{W}$ it is the pullback under τ of a form on $\mathcal{J}$;
- $d_\pi G = F\omega \Rightarrow G$ lives on $\mathcal{J}$.

Then from the second statement one may restrict G to be section of line bundles over the space of all $Z_L \cong \mathbb{P}^1$ described above. Under the conditions in the statement of the theorem these line bundles turn out to have negative degree; hence $G = 0$.

We shall not give the details here (cf. the appendix to this lecture) but will also use the following result:

$$H^0(D, F_{(k-2,1-k)}) = 0 \text{ for all } k \in \mathbb{Z}.$$

Penrose transforms for $\mathrm{Sp}(4)$

The discussion largely parallels that for $S\mathcal{U}(2,1)$, the end result being

THEOREM: *The Penrose transform*

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})$$

is defined as in the $S\mathcal{U}(2,1)$ *case where* $a = a'$, $b = b' + 2$. *It is injective when* $a < b$.

We recall that the line bundles $F(a,b)$ and $F'_{(a',b')}$ were defined by the respective weights $ae_1 + be_2$, $a'e_1 + b'e_2$, from which it follows that in the diagram

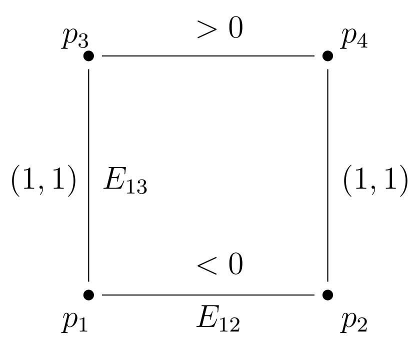
$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D & & D' \end{tikzcd}$$

we have

$$\pi^* F_{(a,b)} \cong \pi'^* F'_{(a',b')}.$$

The Sp(4) case is in this way notationally simpler than the $S\mathcal{U}(2,1)$ case.

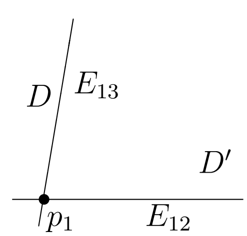
Next we recall that $\mathcal{W}$ may be pictured as Lagrange quadrilaterals



where the symbols $> 0, < 0, (1,1)$ indicate the signature of the Hermitian form on the Lagrange lines. The maps in the above diagram are given by

$$\begin{cases} \pi(p_1, p_2, p_3, p_4) = (p_1, E_{13}) \\ \pi'(p_1, p_2, p_3, p_4) = (p_1, E_{12}) \end{cases}$$

or pictorially



The passage from D' to D is given by

$$p_2 \to p_3.$$

Thus the component of the Maurer-Cartan matrix of a moving frame that reflects this transformation is

$$\omega_2^3.$$

The space $\mathcal{J}$ that encodes the passage from D' to D is the set of configurations in the diagram just above, and the maps in

$$\begin{tikzcd} & \mathcal{W} \arrow[d] & \\ & \mathcal{J} \arrow[dl] \arrow[dr] & \\ D & & D' \end{tikzcd}$$

are the obvious ones.

The Maurer-Cartan equations give

$$d\omega_2^3 \equiv_\pi (\omega_2^2 - \omega_3^3) \wedge \omega_2^3$$

which when we interpret the ω_j^i in terms of the indexing of weights in this case says that

$$\omega_2^3 \text{ transforms as a section of } \pi^* F_{(0,2)}.$$

This is why the Penrose transform takes $F'_{(a',b')} \to F_{(a',b'+2)}$.

For the case

$$\omega'^{\otimes k/3}_{\mathcal{H}} = F'_{(-k,-k)}$$

whose Γ-invariant sections correspond to *Siegel modular forms* of weight k for the Penrose transform will be injective for $k \geqq 1$.

Summary: We have been referring to D and its quotient $X = \Gamma\backslash D$ as non-classical, and D' and its quotient $X' = \Gamma\backslash D'$ as classical. The Penrose transform gives a mechanism for relating the cohomology of line bundles L_μ in the non-classical case to that of $L'_{\mu'}$ in the classical case. The condition for this is the relation

$$\chi_{\mu+\rho} = \chi_{\mu'+\rho'}$$

between the infinitesimal characters.[42] In Lecture 10 we discuss the open (so far as I know) question of whether this is sufficient.

In the classical case the groups $H^{q'}(X, L'_{\mu'})$ have an arithmetic structure. One may ask if a Penrose transform between two classical cases $H^{q'}(X', L'_{\mu'})$ and $H^{q''}(X'', L''_{\mu''})$ preserves the arithmetic structures. This is plausible but, so far as I know, has not been established.

[42] To be more precise, $\mu + \rho$ and $\mu' + \rho'$ should be related by W_K to give the necessary condition.

Appendix to Lecture 8: Proofs of the results on Penrose transforms for D and D'

This appendix is largely reproduced from notes for a seminar at the IAS and from [GGK2]. There is some repetition with the material in Lecture 8.

Step one: With the notations from Lecture 7, we consider the diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[d, "\tau"] \arrow[dr, "\pi'"] & \\ & \mathcal{J} \arrow[dl, "\sigma"] \arrow[dr, "\sigma'"'] & \\ D & & D' \end{tikzcd}$$

We will denote by ω_i^j the restriction to the open subset lying over $\mathcal{W}$ in $G_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$ of the Maurer-Cartan forms and we set

$$\omega = \omega_3^1 \, .$$

Proposition: *ω is a holomorphic section of*

$$\Omega_\pi^1 \otimes \pi^* F_{(-2,1)} \, .$$

Proof. Denoting congruence modulo $\Omega_\pi^\bullet$ by $\equiv_\pi$, by the Maurer-Cartan equation we have

$$d\omega_3^1 \equiv_\pi (\omega_3^3 - \omega_1^1) \wedge \omega_3^1 \, .$$

From $\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$ we obtain

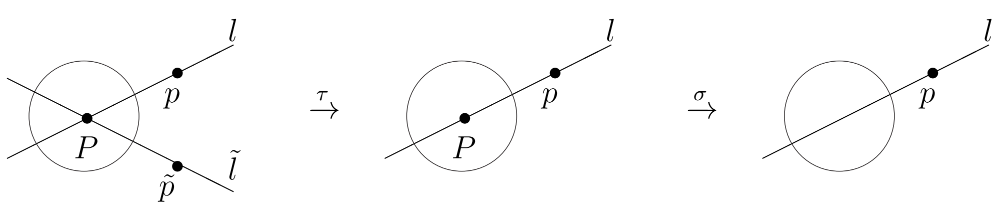
$$d\omega_3^1 \equiv_\pi (-2\omega_1^1 - \omega_2^2) \wedge \omega_3^1 \, .$$

From Lecture 8, we obtain that over D

$$F_{(a,b)} = L_{-ae_1^* + be_2^*} \, ,$$

from which the result follows. $\square$

Remark: The maps are



The fibres are

- $$\tau^{-1}(p,l,P) = \left\{ \begin{array}{l} \text{set of lines } \tilde{l} \text{ through } P,\ \tilde{l} \neq l, \\ \text{and points } \tilde{p} \in \tilde{l} \text{ such that } \overline{p\tilde{p}} \subset \mathbb{B}^c \end{array} \right\}$$
  $$= \text{disc bundle over } \mathbb{C}$$
- $$\sigma^{-1}(p,l) = \{P \in l \cap \mathbb{B}\} \cong \Delta\,.$$

These are contractible Stein manifolds, so that at least one half of the proof of the EGW theorem applies to each map. However,

$$\mathcal{J} \cong \{(p,P) : P \in \mathbb{B} \text{ and } p \in \mathbb{B}^c\}$$

is not Stein. Thus even though the diagram

$$\begin{tikzcd} & \mathcal{J} \arrow[dl] \arrow[dr] & \\ D & & D' \end{tikzcd}$$

is the most natural one to interpolate between D and D', we need to go up to the correspondence space $\mathcal{W}$ to be able to apply the EGW theorem to holomorphically realize the cohomologies of D and D' and then to relate them via the Penrose transform. This situation is the general one when B and B' are not "opposite" Borel subgroups. In this case for the group $A = B \cap B'$ we may expect to have

$$\begin{tikzcd} & \mathcal{J} \subset G_{\mathbb{C}}/A \arrow[dl] \arrow[dr] & \\ D & & D' \end{tikzcd}$$

as the natural space to connect D and D'.

Even though $\mathcal{J}$ is not Stein the geometry is reflected in the exact sequence

$$0 \to \tau^*\Omega^1_\sigma \to \Omega^1_\pi \to \Omega^1_\tau \to 0\,,$$

where the geometric meanings are

- $\tau^*\Omega^1_\sigma$ means dP moves along l,
- Ω^1_π means $d\tilde{p}, d\tilde{l}$ move subject to $d\left\langle \tilde{l}, \tilde{p} \right\rangle = 0$,
- Ω^1_τ means that $d\tilde{p}$ moves, where $\tilde{l} = \overline{P\tilde{p}}$ is determined by $\tilde{p}$.

The above exact sequence gives a filtration of $\Omega^\bullet_\pi$. For any line bundle $L \to D$ we may tensor it with

$$\pi^*L \cong \tau^*(\sigma^*L)$$

to obtain a spectral sequence

$$E_0^{p,q} = \Gamma\big(\mathcal{W}, \Omega_\tau^q \otimes \tau^*\Omega_\sigma^p(\pi^* L)\big) \Rightarrow H_{\mathrm{DR}}^{p+q}\big(\Gamma(\mathcal{W}, \Omega_\pi^\bullet(\pi^* L)); d_\pi\big)\,.$$

For fixed p, the relative differentials are d_τ and since the fibres of τ are contractible and Stein we may apply the proof of EGW to infer that

$$E_1^{p,q} = H^1\big(\mathcal{J}, \Omega_\sigma^p(\sigma^* L)\big)\,.$$

One may then identify the canonical form ω as representing a class in the image of the natural mapping

$$\begin{array}{ccc} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma(\sigma^*\mathcal{O}_D(-2,1)))\big) & \xrightarrow{\tau^*} & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(\pi^*\mathcal{O}_D(-2,1)))\big) \\ \wr\!\| & & \\ E_2^{1,0}\,. & & \end{array}$$

Step two: We want to relate the following

- over D we have the line bundles $F_{(a,b)}$;
- over D' we have the line bundles $F'_{(a',b')}$;
- over $\mathcal{W}$ we have the homogeneous line bundles

  $$\mathcal{L}_{e_i^*} \to \mathcal{W}$$

  given by the identification $\mathcal{W} \cong G_{\mathbb{C}}/T_{\mathbb{C}}$ and the characters of $T_{\mathbb{C}}$ corresponding to the e_i^*.

Proposition: *Over $\mathcal{W}$ we have*

$$\begin{cases} \pi'^* F'_{(-1,0)} \cong \pi^* F_{(1,-1)} \\ \pi'^* F'_{(0,-1)} \cong \pi^* F_{(0,-1)}\,. \end{cases}$$

Corollary: *Over $\mathcal{W}$ we have*

$$\begin{cases} \pi'^* F'_{(a',b')} \cong \pi^* F_{(-a',a'+b')} \\ \pi'^* F'_{(a',b')} \otimes \pi^* F_{(-2,1)} \cong \pi^* F_{(-a'-2,a'+b'+1)}\,. \end{cases}$$

Proof. The result follows from

$$\begin{cases} \pi^* F_{(-1,0)} & \cong \mathcal{L}_{e_1^*}, \pi^* F_{(0,-1)} \cong \mathcal{L}_{-e_2^*} \\ \pi'^* F'_{(-1,0)} & \cong \mathcal{L}_{e_3^*}, \pi'^* F'_{(0,-1)} \cong \mathcal{L}_{-e_2^*}\,. \end{cases} \qquad \square$$

Definition: *The* **Penrose transform**

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})\,,$$

where $a = -a' - 2$ and $b = a' + b' + 1$, is defined by the commutative diagram

$$\begin{array}{ccc} H^0_{DR}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes \pi'^* L'_{(a',b')})\big) & \xrightarrow{\omega} & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^* L_{(a,b)})\big) \\ \wr\Vert & & \wr\Vert \\ H^1(D', L'_{(a',b')}) & \xrightarrow{\quad\mathcal{P}\quad} & H^1(D, L_{(a,b)})\,. \end{array}$$

Remark: We have noted that ω corresponds to the simple root α that changes sign when we pass from D' to D. Geometrically, ω is the EGW representative of the fundamental class (a divisor in this case) of the *Bruhat cell* corresponding to the parabolic subgroup associated to B' and α; i.e., the one whose Lie algebra is $\mathfrak{b}' \oplus \mathbb{C}X_\alpha$. We do not know what, if any, generality this method has.

Step three: We begin with the

Observation: *For $F \in H^0(D', L'_{(a',b')}) \cong H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes \pi'^* L'_{(a',b')})\big)$,*

$$F\omega \in \Gamma(\mathcal{W}, \Omega^1_\pi \otimes \pi^* L_{(a,b)})$$

is harmonic.

Proof. Lifted up to the open set in $G_\mathbb{C}$ lying over $\mathcal{W}$, F is a function of f_1, f_2, f_3 of the form

$$F = F(f_3, f_1 \wedge f_3)\,.$$

If $\alpha = e_3^* - e_1^*$ is the root with

$$\omega_3^1 = \omega^{-\alpha}\,,$$

then the harmonic condition from [EGW] is

$$X_\alpha \cdot (X_{-\alpha} \rfloor F\omega) = X_\alpha \cdot F = 0\,.$$

This is equivalent to

$$F_3^1 = 0 \iff \text{the coefficient of } \omega_1^3 \text{ in } dF \text{ is zero.}$$

By the chain rule, dF will be a linear combination of the forms in df_3 and in $d(f_1 \wedge f_3)$. The former are the ω_3^j, and for the latter we have

$$\begin{aligned} d(f_1 \wedge f_3) &\equiv (df_1) \wedge f_3 \bmod\{\omega_3^3, \omega_3^2, \omega_3^1\} \\ &\equiv 0 \bmod\{\omega_1^1, \omega_1^2, \omega_3^3, \omega_3^2 \wedge \omega_3^1\} \end{aligned}$$

since $f_3 \wedge f_3 = 0$.

Since ω_1^3 does not appear in the bracket term we have $F_3^1 = 0$. $\square$

Theorem 2.13 in [EGW] gives conditions on (a,b) such that a de Rham class in $H^1_{\mathrm{DR}}\big(\mathcal{W}, \Omega^1_\pi(L_{(a,b)}); d_\pi\big)$ has a unique harmonic representative. Unfortunately, this result does not apply in our situation. Geometrically, one may say that the reason for this is that the [EGW] proof uses the diagram

$$\begin{tikzcd} \mathcal{W} \arrow[rr] \arrow[dr] & & \mathcal{I} \arrow[dl] \\ & D & \end{tikzcd}$$

rather than the above diagram which more closely captures the geometric relationship between D and D'. This brings us to the

Proposition: *(i) If $H^0(D', L'_{(a',b')}) \neq 0$, then $b' \geqq 0$.*

(ii) The Penrose transform is injective if $b' \leqq 0$. The common solutions to (i) and (ii) are $b' = 0$.

Remarks: (i) In terms of (a,b) these conditions are

$$\begin{cases} a+b+1 \geqq 0 \\ a+b+1 \leqq 0\,. \end{cases}$$

(ii) The Weyl chamber where $H^0_{(2)}(D', L'_{(a',b')})$ is non-zero is given by

$$\begin{cases} \quad b'+1 > 0 \\ a'+b'+2 < 0\,. \end{cases}$$

If $b' = 0$ these reduce to

$$a' \leqq -3\,.$$

As we have seen, the pullback $\omega'_{\mathbb{B}}$ to D' of the canonical bundle on $\mathbb{B}$ is given by

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}\,.$$

Also, we have noted that the pullback $\mathbb{V}'^{1,0}_{+}$ to D' of the Hodge bundle $\mathbb{V}^{1,0}_{+}$ over $\mathbb{B}$ is given by

$$\mathbb{V}'^{1,0}_{+} = F'_{(-1,0)}\,.$$

Thus

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}\,.$$

We set $\omega'^{\otimes k/3}_{\mathbb{B}} = F'_{(-k,0)} = \otimes^k V'^{1,0}_{+}$ and have defined Picard automorphic forms of weight k to be Γ-invariant sections of $\omega'^{\otimes k/3}_{\mathbb{B}}$. Picard automorphic forms of weight $k \geqq 1$ then give sections of

$$F'_{(-k,0)} \to D'\,.$$

From the above we have the

Corollary: *The Penrose transform*

$$\mathcal{P} : H^0(D', F'_{(-3-l,0)}) \to H^1(D, F_{(l+1,-2-l)})$$

is injective for $l \geqq 0$.

In particular, $\mathcal{P}$ will be seen to be injective on Picard modular forms of weight $k \geqq 3$.

Remark: In Carayol (cf. Proposition (3.1) in [C2]) it is proved that

$$\mathcal{P} \text{ is injective for } b' \geqq 0, a' + b' + 2 \leqq 0\,.$$

The common solutions to the two sets of conditions are are

$$b' = 0, \qquad a' \leqq -2\,.$$

The solutions when $b' = 0$ are

$$a' < -2$$

which is exactly the range in Carayol's condition.

Proof of (ii). The first step is to use the above diagram and the spectral sequence arising from the above exact sequence of relative differentials to reduce the question to one on $\mathcal{J}$. The spectral sequence leads to the maps

$$\begin{aligned} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma \otimes \sigma^* L_{(a,b)}); d_\sigma\big) &\xrightarrow{\tau^*} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \tau^*\Omega^\bullet_\sigma \otimes \pi^* L_{(a,b)}); d_\tau\big) \\ &\longrightarrow H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^* L_{(a,b)}); d_\pi\big). \end{aligned}$$

We will show that

(a) $F\omega \in \Gamma(\mathcal{J}, \Omega^1_\sigma \otimes \sigma^* L_{(a,b)})$;

(b) the image of $F\omega$ under the natural map

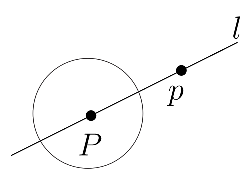
$$H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma \otimes \sigma^* F_{(a,b)}); d_\sigma\big) \to H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^* F_{(a,b)}; d_\pi)\big)$$

is non-zero in $H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^* L_{(a,b)}); d_\pi\big)$ for (a,b) in the range stated in the Proposition.

Proof of (a): We let $G_{\mathbb{C}}(\mathcal{J})$ be the inverse image of $\mathcal{J}$ under the mapping

$$(f_1, f_2, f_3) \to ([f_1] \in \mathbb{B}^c, [f_3] \in \mathbb{B} \cdot [f_1 \wedge f_3])$$

where the RHS is the point



of $\mathcal{J}$ given by $p = [f_1]$, $P = [f_3]$ and $l = \overline{Pp} = f_1 \wedge f_3$. The 1-forms $\omega_1^2, \omega_1^3, \omega_3^2, \omega_3^1$ are semi-basic for $G_{\mathbb{C}}(\mathcal{J}) \to \mathcal{J}$, and $\omega_1^2, \omega_1^3, \omega_3^2$ are semi-basic for $G_{\mathbb{C}}(\mathcal{J}) \to D$. Then $\omega = \omega_3^1$, $F = F(f_3, f_1 \wedge f_3)$ and

$$d(F\omega) \equiv 0 \operatorname{mod}\{\omega_1^1, \omega_2^2, \omega_3^3, \omega_1^2, \omega_1^3, \omega_3^2\}$$

implies that $F\omega \in \Gamma(\mathcal{J}, \Omega_\sigma^1 \otimes \sigma^* L_{(a,b)})$.

Suppose now that

$$F\omega = d_\pi G$$

where $G \in \Gamma(\mathcal{W}, \pi^* L_{(a,b)})$. Pulling G back to the open subset $G_{\mathbb{C}}(\mathcal{J})$ of $G_{\mathbb{C}}$ we have that $G = G(f_1, f_2, f_3, f_1 \wedge f_3, f_2 \wedge f_3)$. Then $F\omega = d_\pi G$ implies that dG has no ω_2^1, ω_2^3 term, which then gives that $G = G(f_1, f_3, f_1 \wedge f_3)$, and when the scaling is taken into account

$$G \in \Gamma(\mathcal{J}, \sigma^* L_{(a,b)})\,.$$

This reduces the question to one on $\mathcal{J}$; we have to show that the equation on $\mathcal{J}$

$$F\omega = d_\sigma G$$

implies that $F = 0$. We will prove the stronger result

> *For (a', b') in the range stated in the above proposition, this equation implies that $G = 0$.*

The idea is to show that (i) the maximal compact subvarieties $Z \subset D$ have natural lifts to compact subvarieties $\widetilde{Z} \subset \mathcal{J}$, and the $\widetilde{Z}$ cover $\mathcal{J}$; (ii) the restrictions $G\big|_{\widetilde{Z}}$ are zero. In fact, we have that $Z \cong \mathbb{P}^1$ and under the projection $\sigma\widetilde{Z} \to Z$ we will show that

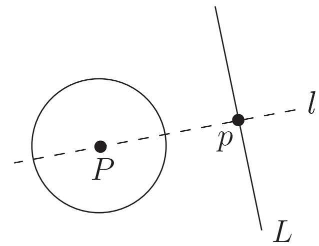
$$\sigma^* F_{(a,b)}\big|_{\widetilde{Z}} \cong \mathcal{O}_{\mathbb{P}^1}(a+b)\,.$$

Thus

$$G\big|_{\widetilde{Z}} \in H^0(\mathcal{O}_{\mathbb{P}^1}(a+b))\,,$$

and we see that the range of (a', b') in the proposition is exactly $a + b < 0$.

For the details, we identify $\mathcal{J}$ with pairs $(P, p) \in \mathbb{B} \times \mathbb{B}^c$ and $\overline{\mathbb{B}}$ with lines $L \subset \mathbb{B}^c$. Then $\mathbb{B} \times \overline{\mathbb{B}} = \mathcal{U}$ is the cycle space, and we have seen that each point $(P, L) \in \mathcal{U}$ gives a maximal compact subvariety $Z(P, L) \subset D$ as in the picture



where

$$Z(P,L) = \{(p,l) \in D\} \cong \mathbb{P}^1 .$$

The lift $\widetilde{Z}(P,L) \subset \mathcal{J}$ of $Z(P,L)$ is then given by

$$\widetilde{Z}(P,L) = \{(p,l,P) \in \mathcal{J}\}$$

where P is constant. We have

$$\begin{tikzcd} & \widetilde{Z}(P,L) \arrow[dl, "f"'] \arrow[dr, "\check{f}"] & \\ \mathbb{P}^2 & & \mathbb{P}^{2*} \end{tikzcd}$$

where

$$\begin{cases} f(p,l,P) = p \\ \check{f}(p,l,P) = l\,. \end{cases}$$

From this we may infer the formula for $\sigma^* F_{(a,b)}\big|_{\widetilde{Z}}$ where $\widetilde{Z} = \widetilde{Z}(P,L)$. This completes the proof of (ii) in the proposition.

The proof of (i) is similar. Given $(P,L) \in \mathcal{U}$ we define $Z'(P,L) \subset D'$ by

$$Z'(P,L) = \{(P,L) \in D'\}$$

in the above figure. Then we have

$$\begin{tikzcd} & Z'(P,L) \arrow[dl, "f'"'] \arrow[dr, "\check{f}'"] & \\ \mathbb{P}^2 & & \mathbb{P}^{2*} \end{tikzcd}$$

where

$$\begin{cases} f'(P,l) = P \\ \check{f}'(P,l) = l\,. \end{cases}$$

Since P is fixed we have that

$$F'_{(a',b')}\big|_{Z'(P,L)} \cong \mathcal{O}_{\mathbb{P}^1}(b')\,.$$

If follows that

$$b' < 0 \Rightarrow \Gamma(\mathcal{J}, \sigma'^* F'_{(a',b')}) = 0 \Rightarrow \Gamma(D', F'_{(a',b')}) = 0$$

where $\sigma'(P,p) = P$. $\square$

Discussion: The argument in [C2] is rather different in that Carayol uses the pseudo-concavity of $\mathcal{J}$ rather than the compact subvarieties. It goes as follows.

Since f_1 and f_3 obviously determine $f_1 \wedge f_3$, we may write

$$G(f_1, f_3, f_1 \wedge f_3) = H(f_1, f_3)\,.$$

Then for each fixed f_3 the LHS is bi-homogeneous of degree (a, b) in f_1 and $f_1 \wedge f_3$. The RHS is then bi-homogeneous of degree $a + b = b' - 1$ in f_1 and $b = a' + b' + 1$ in f_3. Now as noted above

$$\mathcal{J} \cong \mathbb{B} \times \mathbb{B}^c$$

where $[f_3] \in \mathbb{B}$ and $[f_1] \in \mathbb{B}^c$. For fixed f_3, $H(f_1, f_3)$ is a holomorphic function defined for $f_1 \in (\mathbb{C}^3 \backslash \{0\}) \backslash \widetilde{\mathbb{B}}^c$, where $\widetilde{\ }$ denotes the inverse image in $\mathbb{C}^3 \backslash \{0\}$ of $\mathbb{B}^c \subset \mathbb{P}^2$. By Hartogs' theorem, $H(f_1, f_3)$ extends to a holomorphic function of f_1 to all of $\mathbb{C}^3$ where it is homogeneous of degree $b' - 1$. Then if $b' \leqq 1$, the case we shall be primarily interested in, it follows that $G = 0$.

As noted in [C2], the above argument gives the following

Observation: *Every section $s \in \Gamma(D, F_{(a,b)})$ is the restriction to D of a section $\hat{s} \in \Gamma(\check{D}, \check{F}_{(a,b)})$.*

Proof. The section s lifts to a function $(f_1, f_1 \wedge f_3)$ defined on an open set of $G_{\mathbb{C}}$ and homogeneous of degree (a, b) in $(f_1, f_1 \wedge f_3)$. We then define

$$S(f_1, f_3) = s(f_1, f_1 \wedge f_3)$$

and apply Hartogs' theorem to S to give the result (cf. [C2] for the details). $\square$

Corollary:

$$H^0(D, F_{(k-2, 1-k)}) = (0) \;\; \textit{for all} \;\; k \in \mathbb{Z}\,.$$

Proof. We must show $H^0(\check{D}, F_{(k-2,1-k)}) = (0)$. For $k \in \mathbb{Z}$ and $\mu_k = \frac{k-3}{3}\alpha_1 + \frac{2k-3}{3}\alpha_2$ we have

$$\begin{cases} \mu_k + \rho \;\text{ singular } \;(k = 1, 2) \\ \text{or} \\ q(\mu_k + \rho) = 1 \end{cases}$$

which gives the result. $\square$

The Penrose transform in the second example

The objectives of this section are

(i) to define the Penrose transform

$$\mathcal{P} : H^0(D', L'_{\mu'}) \to H^1(D, L_\mu)$$

in the second example, where D and D' are $\mathrm{Sp}(4,\mathbb{R})/T$ with the non-classical and classical complex structures described in Lecture 3;

(ii) to show that $\mathcal{P}$ is injective for certain μ and μ'.

The discussion will be carried out in several steps.

Step one: We first will carry out for Sp(4) the calculations that were given for $S\mathcal{U}(2,1)$ just below the statement of the theorem in that case. As was done there, we first discuss the compact case where we have

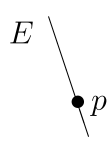
$$\begin{cases} M = G_{\mathbb{C}}/B \\ M' = G_{\mathbb{C}}/B' \end{cases}$$

where B, B' are the Borel subgroups where $D = G_{\mathbb{R}}/T$, $T = G_{\mathbb{R}} \cap B$ and $D' = G_{\mathbb{R}}/T'$, $T' = G_{\mathbb{R}} \cap B'$. Of course, $M = \check{D}$ and $M' = \check{D}'$ are isomorphic as homogeneous complex manifolds, but after making this identification D and D' will be different $G_{\mathbb{R}}$ orbits.

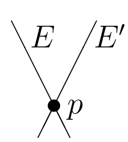
The first step is to describe in the compact case the diagram

$$\begin{tikzcd}
 & G_{\mathbb{C}} \arrow[d] & & \\
 & \check{\mathcal{W}} \arrow[d] \arrow[ddl, "\pi"'] \arrow[ddr, "\pi'"] & & = G_{\mathbb{C}}/T_{\mathbb{C}} \\
 & \check{\mathcal{J}} \arrow[dl] \arrow[dr] & & = G_{\mathbb{C}}/A \\
G_{\mathbb{C}}/B = M & & M' = G_{\mathbb{C}}/B'. &
\end{tikzcd}$$

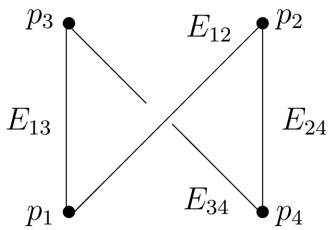
Here the pictures are

- $G_{\mathbb{C}}/B \longleftrightarrow$ 

$=$ Lagrange flag

- $G_{\mathbb{C}}/A \longleftrightarrow$ 

$= \left\{ \begin{array}{c} \text{pairs of} \\ \text{Lagrange flags} \\ \text{meeting in} \\ \text{a point} \end{array} \right\}$

- $G_{\mathbb{C}}/T_{\mathbb{C}} \longleftrightarrow$ 

= Lagrange quadrilaterals

- $G_{\mathbb{C}} =$ frames (f_1, f_2, f_3, f_4).

The maps are

$$\begin{cases} (f_1, f_2, f_3, f_4) \longrightarrow (p_1, p_2, p_3, p_4), & p_i = [f_i] \\ (p_1, p_2, p_3, p_4) \longrightarrow (p_1, E_{13}, E_{12}), & \\ (p, E, E') \longrightarrow (p, E) \text{ and } (p, E, E') \to (p, E'), & p = p_1 \text{ and} \\ & E = E_{13}, E' = E_{12}\,. \end{cases}$$

Step two: We have

$$\begin{array}{ccc} H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi \otimes \pi^* L_\mu); d_\pi\big) & \overset{\omega}{\dashleftarrow} & H^0_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi'} \otimes \pi'^* L'_{\mu'}); d_\pi\big) \\ \wr\| & & \wr\| \\ H^1(M, L_\mu) & & H^0(M', L'_{\mu'}). \end{array}$$

We shall show that

The form ω^3_2 gives the pullback to $G_{\mathbb{C}}$ of a canonical form

$$\omega \in \Gamma\big(\mathcal{W}, \Omega^1_\pi \otimes \pi^* L_\mu \otimes \pi'^* \check{L}'_{\mu'}\big)$$

that gives the map indicated by the dotted line above.

Here, μ and μ' are characters of T that give homogeneous line bundles $L_\mu, L'_{\mu'}$ over M, M', where $\mu + \rho = \mu' + \rho'$ (see below). The calculations are parallel to those given below.

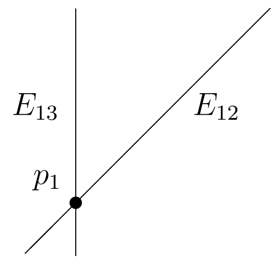
Proof. The method is similar to that used below. The fibres of the map $G_{\mathbb{C}} \to M$ are given by

$$\begin{cases} \omega^2_1 = 0,\ \omega^3_1 = 0,\ \omega^4_1 = 0 \\ \omega^2_3 = 0 \end{cases} .$$

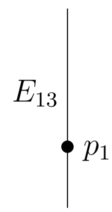
where we have used $\omega^2_1 + \omega^4_3 = 0$ and $\omega^3_1 + \omega^4_2 = 0$. The fibres of $G_{\mathbb{C}} \to \check{\mathcal{J}}$ are given by the above Pfaffian equations together with

$$\omega^3_2 = 0\,.$$

Geometrically the above means that along the fibres of $G_{\mathbb{C}} \to \check{\mathcal{J}}$ the configuration



is constant, while along the fibres of $G_{\mathbb{C}} \to M$ the configuration



is constant.

We next observe that

$$\omega_2^3 \textit{ spans an integrable sub-bundle } J \subset \Omega_\pi^1 .$$

Indeed, using $\omega_2^4 + \omega_1^3 = 0$, the Maurer-Cartan equation

$$\begin{aligned} d\omega_2^3 &= \omega_2^1 \wedge \omega_1^3 + \omega_2^2 \wedge \omega_2^3 + \omega_2^3 \wedge \omega_3^3 + \omega_2^4 \wedge \omega_4^3 \\ &= \omega_2^1 \wedge \omega_1^3 + (\omega_2^2 - \omega_3^3) \wedge \omega_2^3 + \omega_4^3 \wedge \omega_1^3 \end{aligned}$$

gives

$$d\omega_2^3 \equiv_\pi (\omega_2^2 - \omega_3^3) \wedge \omega_2^3 .$$

This implies first that J is a sub-bundle and secondly that it is integrable.

Step three: We next have the observation

> *Let F be a holomorphic function, defined in an open set in $G_{\mathbb{C}}$ that is the pullback of a holomorphic section of $L'_{\mu'} \to M'$. Then*
>
> $$dF \equiv 0 \bmod \left\{\omega_j^j, \omega_1^2, \omega_1^3, \omega_1^4, \omega_2^3\right\} .$$

Here, $1 \leqq j \leqq 4$. It follows that, where again $1 \leqq j \leqq 4$,

$$d_\pi F \equiv 0 \bmod \left\{\omega_j^j, \omega_2^3\right\} .$$

From the preceeding we conclude that

$$d_\pi(F\omega_2^3) \equiv 0 .$$

We now let ω be the form on $\check{\mathcal{J}}$ that pulls back to ω_2^3 on $G_{\mathbb{C}}$. More precisely, there is a line bundle $L \to \check{\mathcal{J}}$ that will be identified below and then

$$\omega \;\; \textit{is a section of} \quad J \otimes L \subset \Omega^1_\pi \otimes L\,.$$

The above calculations then give the main step:

PROPOSITION: *For $F \in H^0(M, L'_{\mu'})$, the map*

$$F \to F\omega$$

induces a map given by the dotted arrow above.

Finally it remains to identify the relation among the line bundles $L'_{\mu'}, L_\mu$ and L. Let

$$\begin{cases} L'_{\mu'} = F'_{(a',b')} \\ L_\mu \;= F_{(a,b)}\,. \end{cases}$$

Then it follows that

$$L = \pi^* F_{(0,2)}\,.$$

Using this and $\pi^* F_{(a,b)} = \pi'^* F'_{(a,b)}$ on $\mathcal{J}$, the identifications give for the Penrose transformation

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})$$
$$\begin{cases} a = a' \\ b = b' + 2\,. \end{cases}$$

This is the same as

$$\mu + \rho = \mu' + \rho'\,.$$

Step four: For $F \in H^0(D', F'_{(a',b')})$, we may pull F back to an open set in $G_{\mathbb{C}}$ where it is a holomorphic function

$$F(f_1, f_1 \wedge f_2)\,.$$

It follows that

$$F\omega \in \operatorname{Image}\left\{H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma \otimes \sigma^* F_{(a,b)})\big) \to H^1\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^* F_{(a,b)})\big)\right\}.$$

Suppose that

$$F\omega = d_\pi G$$

where $G \in H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\sigma \otimes \pi^* F_{(a,b)})\big)$. We will show that

The pullback of G to an open set in $G_{\mathbb{C}}$ is
a function of the form $G(f_1, f_1 \wedge f_2, f_1 \wedge f_3)$.

Proof: As in the first example, we shall work modulo the differential scaling coefficients ω_j^j, which will take care of themselves at the end. We recall that

$$\Omega^1_\pi = \operatorname{span}\left\{\omega_1^2 = -\omega_3^4,\ \omega_1^3 = -\omega_2^4, \omega_2^4, \omega_3^2\right\}.$$

Then we have

$$dG \ \textit{does not involve}\ \omega_2^1 = -\omega_4^3,\ \omega_3^1 = -\omega_4^2, \omega_4^1\,.$$

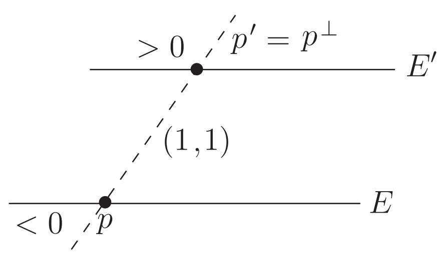
It follows first that $G = G(f_1, f_2, f_3)$. Next, since ω_2^1 and ω_3^1 do not appear in dG, we infer that

$$G = G(f_1, f_1 \wedge f_2, f_1 \wedge f_3)\,.$$

This gives the

Conclusion: *If* $F_\omega = d_\pi G$, *then* $G \in \Gamma(\mathcal{J}, \sigma^* F_{(a,b)})$.

Step five: The space $\mathcal{J}$ has maximal compact subvarieties $Z = Z(E, E')$ given by the picture



That is, the locus

$$\{p, \overline{pp'}, E\}, \qquad E \text{ fixed}$$

gives a $\mathbb{P}^1$ in $\mathcal{J}$. The line $\overline{pp'}$ is Lagrangian since $p' = p^\perp$, and H has signature $(1,1)$ on $\overline{pp'}$ since $H(p) < 0$ and $H(p') > 0$. Since $\mathcal{J}$ is covered by such $Z(E, E')$, to show that the equation

$$d_\pi G \equiv F\omega, \qquad G \in \Gamma(\mathcal{J}, \sigma^{-1} F_{(a,b)})$$

cannot hold non-trivially it will suffice to establish the stronger result that all

$$G\big|_{Z(E,E')} \equiv 0\,.$$

But we have seen that

$$F_{(a,b)}\big|_{Z(E,E')} = \mathcal{O}_{\mathbb{P}^1}(a-b)\,.$$

This gives the

Theorem: *The Penrose transform*

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})$$

is injective for $a < b$, *or equivalently for*

$$a' + b' + 1 < 0\,.$$

Corollary: *The Penrose transform*

$$\mathcal{P} : H^0(D, \omega_{\mathcal{H}}^{\prime \otimes k/3}) \to H^1(D, F_{(-k,-k+2)})$$

is injective for $k \geqq 1$.

Remark: As a check on the signs we recall that the distinguished Weyl chamber $\mathbf{C}$ is the unique one where

$$\begin{cases} \mu' + \rho' \in \mathbf{C} \Rightarrow H^0_{(2)}(D', L'_{\mu'}) \neq 0 \\ \mu + \rho \in \mathbf{C} \Rightarrow H^1_{(2)}(D, L_\mu) \neq 0\,. \end{cases}$$

Then for $\mu' = a'e_1 + b'e_2$

$$\mu' + \rho' \in \mathbf{C} \Longleftrightarrow \begin{cases} a' < -2 \\ b' < a' + 1\,, \end{cases}$$

and for $\mu = ae_1 + be_2$

$$\mu + \rho \in \mathbf{C} \Longleftrightarrow \begin{cases} a < -2 \\ b < a + 3\,. \end{cases}$$

The Penrose transform

$$\begin{cases} \mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)}) \\ \qquad a = a', b = b' + 2 \end{cases}$$

exactly takes the μ' satisfying the above to the μ satisfying its conditions.

Lecture 9
Automorphic cohomology

The purposes of this lecture are

- to complete the proof of the injectivity of the Penrose transform for Picard and Siegel modular forms;
- to present the calculation of cup-products, which allows one to reach the groups
$$H^q(X, L_{-\rho}), \qquad q = 1, 2,$$
corresponding to the TDLDS by cup-products of Penrose transforms of Picard/Sigel modular forms.

The intricate calculations here involve computations in $\mathfrak{n}$-cohomology, which are particularly subtle for TDLDS's. In Appendix I to this lecture we have given the analysis of the $\mathfrak{U}(2)$-modules and $\mathfrak{n}_K$-cohomologies that will form the basis for some of these calculations. In Appendix II we have given the proofs, due to Schmid, of the degeneration of the Hochschild-Serre spectral sequences in the cases of TDLDS's that are of particular interest in these lectures.

Before getting into the specifics I would like to make one remark from the perspective of an algebraic geometer. Namely, from
$$\omega_X = L_{-2\rho}$$
we see that the canonical bundle ω_X has a natural square root
$$\omega_X^{1/2} = L_{-\rho},$$
which by Kodaira-Serre duality gives
$$H^q(X, L_{-\rho}) \cong H^{n-q}(X, L_{-\rho})^*$$
where $\dim X = n$. Thus *the groups $H^q(X, L_{-\rho})$ come in dual pairs.*

In general, if $\mu + \rho$ is singular then the sheaf cohomology Euler characteristic
$$\chi(X, L_\mu) = \sum_q (-1)^q \dim H^q(X, L_\mu) = 0.$$

Proof. For $\check{D}$ it follows from the BWB theorem that all the groups $H^q(\check{D}, L_\mu) = 0$. Next, by the Hirzebruch-Riemann-Roch theorem, for any weight μ
$$\chi(\check{D}, L_\mu) = \int_{\check{D}} P(\Omega_{\hat{\mu}}, \Omega_{T\check{D}})$$

where $P(\Omega_{\hat{\mu}}, \Omega_{T\check{D}})$ is a G_c-invariant polynomial in the curvature forms $\Omega_{\hat{\mu}}$ for L_μ and $\Omega_{T\check{D}}$ for the tangent bundle $T\check{D}$. When $\mu+\rho$ is singular, the BWB theorem and G_c-invariance give that $P(\Omega_{\hat{\mu}}, \Omega_{T\check{D}}) = 0$. For D, by the curvature considerations from Lecture 5 at the identity coset we have

$$P(\Omega_\mu, \Omega_{TD}) = \pm P(\Omega_{\hat{\mu}}, \Omega_{T\check{D}}).$$

When $\mu+\rho$ is singular the RHS is zero, while by the Atiyah-Singer version of the Hirzebruch-Riemann-Roch theorem for X

$$\chi(X, L_\mu) = \int_X P(\Omega_\mu, \Omega_{TD}) = 0. \qquad \square$$

We see from this that the line bundles $L_{-\rho} \to X$ have much the flavor of *special divisors* on algebraic curves, especially those corresponding to line bundles of degree $g-1$. In fact, from $L_{-\rho} = \omega_X^{1/2}$ they resemble *theta characteristics*. For $G = \mathrm{SL}_2$ the line bundle $L_{-1} \to X$ is a distinguished theta characteristic, meaning here a particular square root of the canonical bundle arising from the uniformization $\mathcal{H} \to X$.

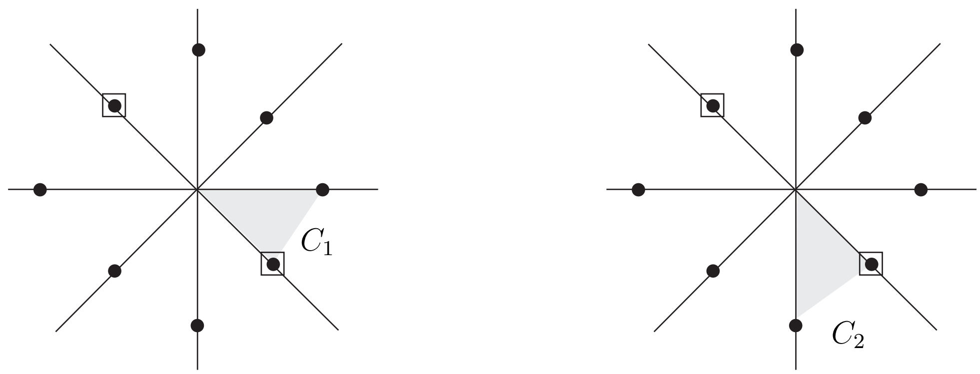
Returning to the main topic of this lecture, the calculation in $\mathfrak{n}$-cohomology will yield the results stated in examples just below. One observes that the Euler characteristic phenomenon is already evident. One notes also the difference with the Schmid results in Lecture 5 on $\mathfrak{n}$-cohomology for DS, where there is only one non-zero group and consequently non-zero Euler characteristic as well.

Examples

$\underline{S\mathcal{U}(2,1)}$: There is one equivalence class of a TDLDS $(0, C)$ where C is the positive Weyl chamber for the non-classical complex structure. For the corresponding Harish-Chandra module V_0 we will see that

$$h^1(\mathfrak{n}, V_0)_\rho = h^2(\mathfrak{n}, V_0)_\rho = 1.$$

$\underline{\mathrm{Sp}(4)}$: There are then two equivalence classes of TDLDS's $(0, C_1)$ and $(0, C_2)$ corresponding to the two non-classical complex structures $D = D_1$ and D_2. The pictures are



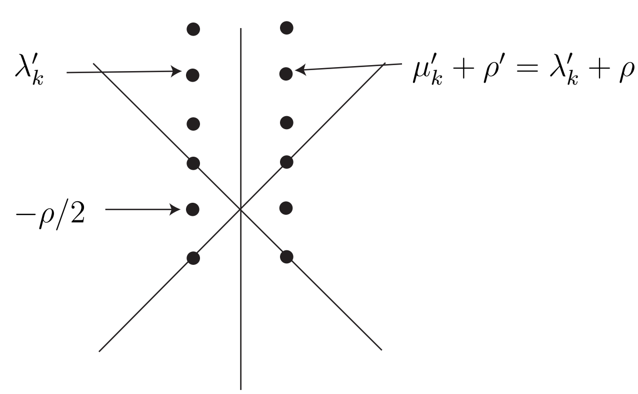
We will then see that

$$\begin{cases} h^1(\mathfrak{n}, V_1)_\rho = h^2(\mathfrak{n}, V_2)_\rho = 1 \\ h^2(\mathfrak{n}, V_2)_\rho = h^3(\mathfrak{n}, V_2)_\rho = 1. \end{cases}$$

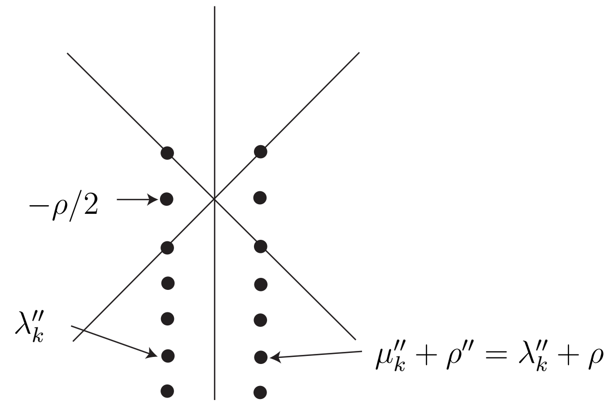
Before turning to specifics we want to give an approximate statement of the main results for $S\mathcal{U}(2,1)$. The Sp(4) case will be discussed later. Recall our notations from Lecture 8

- ${\omega'}_{\mathbb{B}}^{\otimes k/3} = L'_{\mu'_k}$ = line bundles over D' whose sections over X are Picard modular forms of weight k;
- $L_{\lambda'_k} \to D$ where $\lambda'_k + \rho = \mu'_k + \rho'$;
- $H^0(X', L'_{\mu'_k}) \xrightarrow{\sim} H^1(X, L_{\lambda'_k})$ for $k \geqq 4$ via the Penrose transform.

The picture is



For the other classical complex structure D'' with anti-holomorphic fibration $D'' \to \overline{\mathbb{B}}$ over the ball, there are similar results with the picture



From this we see that

$$\lambda'_k + \lambda''_k = -\rho$$

so that cup-products in cohomology give a map

$$\boxed{H^1(X, L_{\lambda'_k}) \otimes H^1(X, L_{\lambda''_k}) \to H^2(X, L_{-\rho}).}$$

The very approximate statement is that *this cup-product is surjective for* $k \geqq 5$.[43] Since as noted above

$$H^1(X, L_{-\rho}) = H^2(X, L_{-\rho})^*,$$

and since as we have seen from the curvature considerations that $H^0(X, L_{-\rho}) = H^3(X, L_{-\rho}) = 0$, this means that the non-classical groups $H^q(X, L_{-\rho})$ can be reached by classical groups.

Other than the very rich connection between complex geometry and representation theory that is involved, one may ask *why is this of interest to arithmetic algebraic geometers?* One answer is that the group $H^1(X, L_{-\rho})$ appears as the infinite component of an automorphic representation that is not associated to the cohomology, either l-adic or coherent, of a Shimura variety.[44] Thus defining an arithmetic structure on this vector space is not possible by classical methods. In the above boxed map, the vector spaces on the LHS have an arithmetic structure, and if one could show that the kernel of the cup is defined over $\overline{\mathbb{Q}}$, this would give an arithmetic structure to the RHS.

[43] The boxed statement is what one might initially try to prove from the above weight considerations. The precise result, discussed below, involves a duality and taking a limit over the discrete groups Γ.

[44] This topic will be discussed in the lecture by Wushi Goldring.

Williams lemma and application

We first recall the Casselman-Osborne lemma in its original form and then shall dualize it to the form we shall use. Let V be the Harish-Chandra module associated to an irreducible unitary representation $\widetilde{V}$ of $G_{\mathbb{R}}$ and suppose that V has a highest weight vector v of weight μ. Then

- $v \in H^0(\mathfrak{n}^+, V)_\mu$ where $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+} = \mathbb{C}v$;
- V has infinitesimal character $\chi_{\mu+\rho}$.

The Casselman-Osborne lemma states that in general

$$H^q(\mathfrak{n}^+, V)_\mu \neq 0 \Rightarrow \chi_V = \chi_{\mu+\rho}$$

(we don't asume V has a highest weight vector of weight μ). A consequence is

If $H^q(\mathfrak{n}, V)_{-\mu} \neq 0$, then V has infinitesimal character $\chi_{-(\mu+\rho)}$.

In order to ensure that $\widetilde{V}$ is in the discrete series we need an extra hypothesis, given by

Williams lemma: *Given an irreducible unitary representation V and a weight μ satisfying*

(i) *$\mu + \rho$ is regular;*

(ii) *Property **P**: For each $\beta \in \Phi_{nc}$ with $(\mu + \rho, \beta) > 0$*

$$(\mu + \rho - \frac{1}{2} \sum_{\begin{cases} \alpha\in\Phi \\ (\mu+\rho,\alpha)>0 \end{cases}} \alpha, \beta) > 0.$$

Then

$$H^q(\mathfrak{n}, V)_{-\mu} \neq 0 \Rightarrow \begin{cases} \bullet \quad q = q(\mu+\rho) \\ \bullet \quad \dim H^q(\mathfrak{n}, V)_{-\mu} = 1 \\ \bullet \quad \widetilde{V} = \widetilde{V}_{-(\mu+\rho)} \end{cases}$$

where $\widetilde{V}_{-(\mu+\rho)}$ is a discrete series representation with infinitesimal character $\chi_{-(\mu+\rho)}$.

For both the $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$ examples we may define the Penrose transform

$$H^0(X', L'_{\mu'_k}) \xrightarrow{\mathcal{P}} H^1(X, L_{\lambda'_k})$$

where μ'_k are the weights giving Picard, respectively Seigel modular forms. Indeed, from the diagram

$$\begin{tikzcd} & \Gamma\backslash\mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \Gamma\backslash D = X & & X' = \Gamma\backslash D' \end{tikzcd}$$

we have seen in Lecture 7 that $\Gamma\backslash\mathcal{W}$ is Stein and the fibres of π, π' are contractible. Moreover, the form ω is $G_{\mathbb{R}}$, and hence Γ, invariant. Thus the constructions in the previous lecture apply here to define the mapping $\mathcal{P}$ above. We note that

- $H^0(X', L'_{\mu'_k}) = H^0(D, L'_{\mu'_k})^{\Gamma}$;[45]
- $\mathcal{P}$ is an isomorphism for $k \geqq 4$.

For the proof of the second statement the same argument as in the previous lecture shows that $\mathcal{P}$ is injective. For the surjectivity we will give below a proof using $\mathfrak{n}$-cohomology. That it should be true, at least for $k \gg 0$, may be seen as follows:

We first have, for $k \gg 0$, from the vanishing of cohomology arising from the sign properties of the curvature forms

$$H^{q'}(X', L'_{\mu'_k}) = 0 \text{ for } q' \neq 0,\ H^q(X, L_{\lambda'_k}) = 0 \text{ for } q \neq 1.$$

It first follows by using the Leray spectral sequence and noting that $\omega_{\mathbb{B}} \to \mathbb{B}$ is a positive line bundle (in fact, $k \geqq 4$ works here if we use duality and Kodaira vanishing) that

$$H^{q'}(X', L'_{\mu'_k}) \cong H^{q'}(\Gamma\backslash\mathbb{B}, \omega_{\mathbb{B}}^{\otimes k/3}).$$

For the $H^q(X, L_{\lambda'_k})$ we use that the curvature form $\Omega_{\lambda'_k}$ has one positive and all the rest negative eigenvalues. Standard vanishing theorems then give the result.

Once we have injectivity and the vanishing result, it will suffice to show that the sheaf Euler characteristics are the same. Noting that $\mathrm{vol}(X') = \mathrm{vol}(X)$, this follows from the proportionality property of the curvature forms at the identity coset and the Atiyah-Singer version of the Hirzebruch-Riemann-Roch theorem.

The application of Williams lemma is this:

> *For both $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$ and for $k \geqq 4$ the condition $\mathbf{P}$ is satisfied for the μ'_k giving Picard, respectively Siegel automorphic forms.*

Proof for $S\mathcal{U}(2,1)$. The Penrose transform is given symbolically by

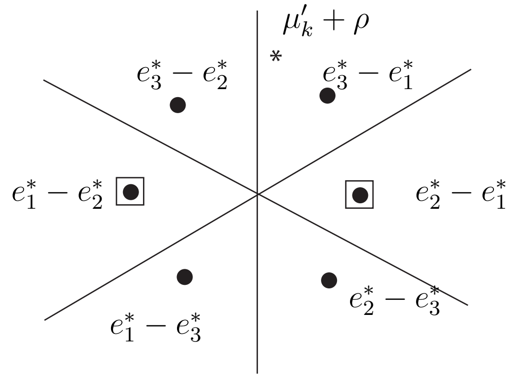
$$F'_{(-k,0)} \to F_{(k-2,1-k)}.$$

Then referring to the formulas for line bundles in the $S\mathcal{U}(2,1)$ case in Lecture 8 we find that

$$\mu'_k + \rho' = \lambda'_k + \rho = \left(\frac{k}{3}\right)(e_2^* - e_1^*) + \left(\frac{2k-3}{3}\right)(e_3^* - e_2^*).$$

[45]In general there is a spectral sequence abutting to $H^*(X, L_\mu)$ and with $E_1^{p,q} = H^q(\Gamma, H^p(D, L_\mu))$. Similarly for X' and X''. Somewhat miraculously for the groups that appear in the main results in this lecture we will always have $H^q(X, L_\mu) = H^q(D, L_\mu)^{\Gamma}$ (cf. [GGK2]).

Using the picture



the non-compact roots β with $(\lambda'_k + \rho, \beta) > 0$ are $e_3^* - e_1^*$ and $e_3^* - e_2^*$, where we have assumed $k \geq 3$ and used $(e_2^* - e_1^*, e_2^* - e_1^*) = (e_3^* - e_2^*, e_3^* - e_2^*) = 2$ and $(e_2^* - e_1^*, e_3^* - e_2^*) = -1$. Then

$$\left(\frac{1}{2}\right) \sum_{\substack{\alpha \in \Phi \\ ({\mu'}_k + \rho, \alpha) > 0}} \alpha = e_3^* - e_1^*$$

and

$$\begin{cases} (\lambda'_k + \rho + e_1^* - e_3^*, e_3^* - e_1^*) = \left(\frac{2}{3}\right)(k-3) \\ (\lambda'_k + \rho + e_1^* - e_3^*, e_3^* - e_2^*) = k - 3. \end{cases}$$

Thus condition **P** holds for $k \geqq 4$.

The argument for Sp(4) is similar. □

Note: For $S\mathcal{U}(2,1)$ and for $k = 1, 2$ the weight $\lambda'_k + \rho$ is irregular and, even though it is regular for $k = 3$ condition **P** fails in this case.

Automorphic cohomology in terms of $\mathfrak{n}$-cohomology We first recall the general formula

$$\boxed{H^q(X, L_\mu) = \bigoplus_{\pi \in \hat{G}_\mathbb{R}} H^q(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)}}$$

where $m_\pi(\Gamma)$ is the multiplicity of V_π in $L^2(\Gamma \backslash G_\mathbb{R})$. Assuming that μ satisfies condition **P** with $q(\mu + \rho) = 1$, and denoting by $V_{-(\mu+\rho)}$ the Harish-Chandra module corresponding to a DS representation with infinitesimal character $\chi_{-(\mu+\rho)}$, using Casselman-Osborne the above becomes

$$H^1(X, L_\mu) \cong H^1\left(\mathfrak{n}, V_{-(\mu+\rho)}\right)^{m_{-(\mu+\rho)}(\Gamma)}$$

where the $\mathfrak{n}$-cohomology group is 1-dimensional and $H^q(X, L_\mu) = 0$ for $q \neq 1$.

There is a similar result

$$H^0(X', L'_{\mu'}) \cong H^0\left(\mathfrak{n}', V_{-(\mu'+\rho')}\right)^{m_{-(\mu'+\rho')}(\Gamma)}$$

for X'. When $\mu+\rho=\mu'+\rho'$, which is the case for Penrose transforms, we see that $h^1(X', L'_{\mu'}) = h^0(X, L_\mu)$ which gives a proof of the claim above that $\mathcal{P}$ is an isomorphism for Picard modular forms of weight $k \geqq 4$. A similar argument works for Siegel modular forms.

Remark: As one might expect on general geometric grounds, since the Penrose transforms are $G_{\mathbb{R}}$-equivariant they may be defined at the level of $\mathfrak{n}$-cohomology, and then when sheaf cohomology is expressed in terms of $\mathfrak{n}$-cohomology the two ways of defining Penrose transforms agree. In the $S\mathcal{U}(2,1)$ case this goes as follows:

We want to construct

$$H^0\left(\mathfrak{n}', V_{-(\mu'+\rho')}\right)_{-\mu'} \xrightarrow{\omega} H^1\left(\mathfrak{n}, V_{-(\mu+\rho)}\right)_{-\mu}.$$

Now

$$\begin{cases} \mathfrak{n}' = \operatorname{span}\left\{X_{e_3^*-e_1^*}, X_{e_3^*-e_2^*}, X_{e_1^*-e_2^*}\right\} \\ \mathfrak{n} = \operatorname{span}\left\{X_{-(e_3^*-e_1^*)}, X_{e_3^*-e_2^*}, X_{e_1^*-e_2^*}\right\} \end{cases}$$

and $\omega = \omega^{-(e_3^*-e_1^*)}$, which since $\mu-\mu' = e_1^*-e_3^*$ transforms exactly the right way to give the desired map.

$\mathfrak{n}$-cohomology for the TDLDS: The $S\mathcal{U}(2,1)$ case

Let $V_0 = H^1(D, L_{-\rho})$ be the Harish-Chandra module associated to the TDLDS for $S\mathcal{U}(2,1)$. We will show that

$$\boxed{\begin{aligned} &H^0(\mathfrak{n}, V_0)_\rho = H^3(\mathfrak{n}, V_0)_\rho = 0 \\ &H^1(\mathfrak{n}, V_0)_\rho \cong W_0^{(0)} \\ &H^2(\mathfrak{n}, V_0)_\rho \cong W_0^{(0)}. \end{aligned}}$$

Here we are using the notation from Appendix I to this lecture:

- W = standard $\mathcal{U}(2)$-module;
- $W_k^{(n)} = \operatorname{Sym}^n W \otimes (\det W)^k$ as a $\mathcal{U}(2)$-module.

The proof of the boxed statement will actually produce generators for these groups; these will be used in the computation of cup-products.

The calculation uses what is arguably *the* basic tool; namely, the Hochschild-Serre spectral sequence (HSSS). This is a spectral sequence that abuts to $H^*(\mathfrak{n}, V_0)_\rho$ and has E_1-term

$$E_1^{p,q} = H^q\left(\mathfrak{n}_K, V_0 \otimes \wedge^p \mathfrak{p}^+\right)_\rho.$$

Here we use the notation $X_{ij} = X_{e_i^* - e_j^*}$ for the root vector corresponding to a root $e_i^* - e_j^*$, and have

- $\mathfrak{n}_K = \mathfrak{n} \cap \mathfrak{k}_{\mathbb{C}} = \mathbb{C}X_{12}$;
- $\mathfrak{p}^+ \cong \mathfrak{p}/\mathfrak{p}^-$

with the isomorphism being given by the Cartan-Killing form. The HSSS is usually stated without the ρ, but the terms in it are T-modules and the differentials are T-morphisms so that it makes sense to take the part that transforms by the weight ρ.

The idea of computing the E_1-term is to expand V_0 into its K-type and then use Kostant's theorem from the appendix to Lecture 7. In general there is a significant subtlety in that $\mathfrak{p}^+$ is generally not a trivial $\mathfrak{n}_K$-module, but rather has a composition series whose successive quotients are 1-dimensional trivial $\mathfrak{n}_K$-modules to which Kostant's theorem applies. For $S\mathfrak{U}(2,1)$ the situation is simpler in that as a $\mathfrak{b}_K = \mathfrak{h} \oplus \mathfrak{n}_K$-module

$$\mathfrak{p}^+ \cong \mathbb{C}X_{31} \oplus \mathbb{C}X_{23},$$

reflecting the geometric fact that, as a $\mathfrak{U}(2)$-homogeneous vector bundle, the normal bundle $N_{Z/D}$ of the maximal compact subvariety $Z \subset D$ is a direct sum of line bundles. From Appendix I the K-type of V_0 is

$$\begin{aligned} \mathrm{Gr}^\bullet V_0 &= \bigoplus_{n=0}^{\infty} V_{0,n} \\ V_{0,n} &= \bigoplus_{0 \leqq k \leqq \left[\frac{n}{3}\right]} W_{n-3k}^{(n)} \end{aligned}$$

From the calculations in that appendix

$$\begin{cases} H^0(\mathfrak{n}_K, W_l^{(n)})_{-\mu} \neq 0 \iff \mu = (l, n+l) \\ H^1(\mathfrak{n}_K, W_l^{(n)})_{-\mu} \neq 0 \iff \mu = (n+l+1, l-1) \end{cases}$$

we may readily fill in the E_1-term in the HSSS. The notation (a,b) means that $\mu = ae_1^* + be_2^*$.

The lowest K-type in V_0 is the trivial K-module $W_0^{(0)}$ with generator v_0. With the notation

$$\omega_{ij} = X_{ij}^*$$

the generator of $E_1^{0,1}$ is $v_0\omega_{12}$. We will show that

> *The HSSS degenerates at E_1. Since we will see trivially that $d_1(v_0\omega_{12}) = 0$, this is equivalent to $d_2(v_0\omega_{12}) = 0$.*

We will present two proofs of this result. The first is by direct computation and it will yield an expression for the generator of $H^1(\mathfrak{n}, V_0)_\rho$. The second proof, due to Wilfried

Schmid, uses his results recalled in Lecture 5 and Zuckerman translation and Casselman-Osborne. It will be given in Appendix II to this lecture, where a similar argument for the degeneracy of the HSSS in the Sp(4) case, also due to Schmid, will also be presented.

For the computation we shall let d_π denote the Lie algebra coboundary. Then

> *$d_\pi \omega_{ij}$ means take the usual $d\omega_{ij}$ given by the Maurer-Cartan equation and mod out any terms with an $(\mathfrak{h} \oplus \mathfrak{n}^+)^*$ factor.*

Then

$$\begin{cases} d_\pi \omega_{12} = -\omega_{13} \wedge \omega_{32} \\ d_\pi \omega_{13} = 0, \; d_\pi \omega_{32} = 0. \end{cases}$$

The reason for the notation d_π is that it agrees with that used in the EGW formalism.

Step one: $d_1(v_2\omega_{12}) = 0$ means to solve the equation

$$d_\pi(v_0\omega_{12}) \equiv 0 \operatorname{mod}(\text{terms with an } \mathfrak{n}_K^* \text{ entry}),$$

which using the above and $\mathfrak{n}_K^* = \mathbb{C}\omega_{12}$ is to determine A, B so that

$$d_\pi(v_0\omega_{12} + A\omega_{13} + B\omega_{32}) \equiv 0 \operatorname{mod} \omega_{12}.$$

Using

$$\begin{aligned} d_\pi(v_0\omega_{12} + A\omega_{13} + B\omega_{32}) &= (-v_0 + X_{13}B - X_{32}A)\omega_{13} \wedge \omega_{32} \\ &\quad + (X_{13}v_0 - X_{12}A)\omega_{13} \wedge \omega_{12} + (X_{32}v_0 - X_{12})\omega_{32} \wedge \omega_{12} \end{aligned}$$

we have

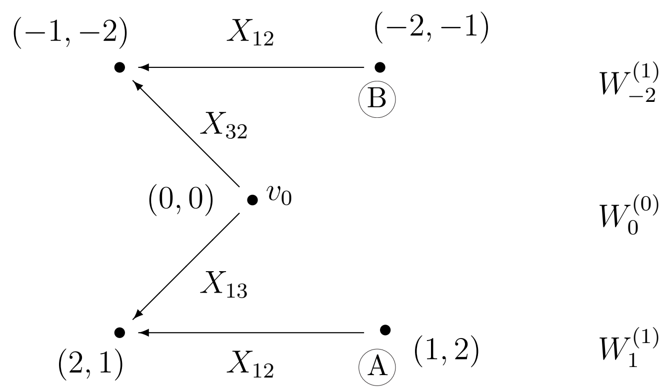
$$d_1(v_0\omega_{12}) = 0 \Longleftrightarrow v_0 = X_{13}B - X_{32}A.$$

This gives

(∗) *For A, B satisfying $v_0 = X_{13}B - X_{32}A$, to have $d_\pi(v_0\omega_{12} + A\omega_{13} + B\omega_{32}) = 0$ we must have*

$$\begin{cases} X_{12}A = X_{13}v_0 \\ X_{12}B = X_{32}v_0. \end{cases}$$

Step two: We will proceed to analyze these equations. As will be seen below, this analysis will involve only the first two graded pieces in the K-type. For book-keeping purposes it is convenient to draw these with the 1-dimensional weight spaces labelled



This diagram will be amplified and further explained in the first appendix to this lecture. The horizontal arrows are the action of the compact root vector X_{12}. The actions of the non-compact root vectors are described by

$$\begin{aligned} X_{13} &\longleftrightarrow (2,1) \quad , \quad X_{31} \longleftrightarrow (-2,-1) \\ X_{23} &\longleftrightarrow (1,2) \quad , \quad X_{32} \longleftrightarrow (-1,-2) \end{aligned}$$

where the notation means: "the action of X_{ij} takes an (a,b) weight space to an $(a+i,b+j)$ weight space." Above, we have drawn in the actions of X_{13} and X_{32} on the $(0,0)$ weight space and have indicated the weight spaces where A and B are situated.

Using $X_{21}X_{12}A = [X_{21}, X_{12}]A = A$ and $X_{21}X_{13} - X_{13}X_{21} = X_{23}$ we have

$$X_{12}A = X_{13}v_0 \Longleftrightarrow A = X_{21}X_{13}v_0 = X_{23}v_0 + \cancel{X_{13}X_{21}v_0}$$

where the crossed out term is zero. Similarly

$$X_{12}B = X_{32}v_0 \Longleftrightarrow B = X_{21}X_{32}v_0 = -X_{31}v_0 + \cancel{X_{32}X_{21}v_0}.$$

From these two equations we obtain

$$X_{32}A - X_{13}B = -v_0 \Longleftrightarrow (X_{32}X_{23} + X_{13}X_{31})v_0 = -v_0.$$

Thus we must show that

$$\boxed{(X_{32}X_{23} + X_{13}X_{31})v_0 = -v_0.}$$

Step three: We have yet to use that among the Harish-Chandra modules with the same K-type we are considering the TDLDS V_0. The LHS of the boxed equation suggests the *Casimir operator* $\Omega \in Z(\mathfrak{g}_{\mathbb{C}})$. For this we have the

Lemma: *For any TDLDS*

$$\chi_0(\Omega) = -\|\rho\|^2.$$

Assuming the lemma we will complete the proof of the equation in the box above.

For $S\mathfrak{U}(2,1)$, $\rho = e_2^* - e_1^*$ so that $\|\rho\|^2 = 2$. Since $z \in Z(\mathfrak{g}_{\mathbb{C}})$ acts on V_0 by the scalar $\chi_0(z)$ from the lemma we have

$$\left(\frac{1}{2}\right)\Omega v_0 = -v_0.$$

Thus we need to show that the LHS of the boxed equation is $\left(\frac{1}{2}\right)\Omega v_0$. Certainly the LHS of that equation looks like at least part of the Casimir operator. We shall show that the remaining part acts trivially on v_0. The reasons this is so are

- $\mathfrak{t}$ acts by zero on $v_0 \in W_0^{(0)}$;
- $\mathfrak{n}_K$ acts also by zero.

The calculation is

$$\begin{aligned}
\Omega &= X_{32}X_{23} + X_{23}X_{32} + X_{13}X_{31} + X_{31}X_{13} + X_{12}X_{21} + X_{21}X_{12} \\
&= 2(X_{32}X_{23} + X_{13}X_{31}) + [X_{23}, X_{32}] + [X_{31}, X_{13}] + [X_{21}, X_{12}] + X_{12}X_{21} \\
&= 2(X_{32}X_{23} + X_{13}X_{31}) + e_2^* - e_3^* + e_3^* - e_1^* + e_2^* - e_1^* + X_{12}X_{21} \\
&= 2(X_{32}X_{23} + X_{13}X_{31}) + \underbrace{2(e_2^* - e_1^*) + X_{12}X_{12}}
\end{aligned}$$

and the terms over the bracket act trivially on v_0. $\square$

Proof of the lemma. From [K] we want to show that the constant term of $\gamma(\Omega)$ is $-\|\rho\|^2$. From loc. cit., page 295,

$$\Omega = \sum_{i=1}^{l} H_i^2 + 2H_\rho + 2\sum_{\alpha\in\Phi^+} X_{-\alpha}X_\alpha$$

where $H_1, \dots, H_l$ are an orthonormal basis of $\mathfrak{h}$ with respect to the Cartan-Killing form and H_ρ is the co-weight corresponding to ρ. Following the notations in Lecture 5

$$\begin{aligned}
\gamma'(\Omega) &= \sum H_i^2 + 2H\rho \\
\gamma(\Omega) = \sigma\big(\gamma'(\Omega)\big) &= \sum_i \big(H_i - \rho(H_i)\big)^2 + 2\big(H_\rho - \rho(H_\rho)\big).
\end{aligned}$$

The constant term here is

$$\sum \rho(H_i)^2 - 2\rho(H_\rho) = \|\rho\|^2 - 2\|\rho\|^2. \qquad \square$$

For later use, if we denote by $v_{(a,b)}$ a highest weight for $W_b^{(a)}$ then the generator of $H^1(\mathfrak{n}, V_0)_\rho$ is

$$\omega_0 =: v_0\omega_{12} - v_{(1,1)}\omega_{13} - v_{(1,-2)}\omega_{32}.$$

The cup-product result

We let V_k' and V_k'' be the DS representations with infinitesimal characters $\lambda_k' + \rho = \mu_k'$ and $\lambda_k'' + \rho = \mu_k''$. Then

$$q(\mu_k') = q(\mu_k'') = 1,$$

and so by Schmid's results we have

$$\dim H^1(\mathfrak{n}, V_k')_{-\lambda_k'} = \dim H^1(\mathfrak{n}, V_k'')_{-\lambda_k''} = 1$$

and all the other $\mathfrak{n}$-cohomology groups are zero. We will very roughly show that

- *V_0 occurs as a direct summand in $V_k' \hat{\otimes} V_k''$;*
- *the cup-product followed by the projection $V_k' \hat{\otimes} V_k'' \to V_0$ induces an isomorphism for $k \geqq 5$*

$$H^1(\mathfrak{n}, V_k')_{-\lambda_k'} \otimes H^1(\mathfrak{n}, V_k'')_{-\lambda_k''} \xrightarrow{\sim} H^2(\mathfrak{n}, V_0)_\rho.$$

Here "very roughly" means that what is actually proved in Carayol is that V_k' is a direct summand of $V_0 \hat{\otimes} V_k'$, and then the cup-product statement follows by applying the $\mathfrak{n}$-cohomology version of Serre duality.

Remark: Before presenting some details of the argument we will give an heuristic for the result. From Appendix I to this lecture we have using the HSSS

$$H^2(\mathfrak{n}, V_0)_\rho \cong H^0(\mathfrak{n}_K, \wedge^2 \mathfrak{p}^+)_\rho \cong W_0^{(0)}.$$

Taking (here we change notation slightly to make the formulas come out more transparent)

$$\begin{cases} \lambda' = \left(\frac{c}{2}\right)(e_1^* + e_2^*) - \rho/2 \\ \lambda'' = -\left(\frac{c}{2}\right)(e_1^* + e_2^*) - \rho/2 \end{cases}$$

and letting V', V'' be the Harish-Chandra modules corresponding to DS representations with infinitesimal characters $\chi_{\lambda'+\rho}, \chi_{\lambda''+\rho}$, from the degeneracy of the HSSS we have that

$$\begin{cases} H^1(\mathfrak{n}, V')_{-\lambda'} \cong H^0(\mathfrak{n}_K, \mathfrak{p}^+)_{-\lambda'} \\ H^1(\mathfrak{n}, V'')_{-\lambda''} \cong H^0(\mathfrak{n}_K, \mathfrak{p}^+)_{-\lambda''}. \end{cases}$$

Now denoting by $\mathbb{C}_\gamma$ the $\mathfrak{t}_\mathbb{C}$-module with weight γ, we have

$$\mathfrak{p}^+ = \mathbb{C}_{e_3^* - e_1^*} \oplus \mathbb{C}_{e_2^* - e_3^*}.$$

Let's suppose that the $'$ group comes from $\mathbb{C}_{e_3^* - e_1^*}$ and the $''$ group from $C_{e_2^* - e_3^*}$. A little computation gives

$$\begin{cases} -(\lambda' + e_3^* - e_1^*) = -\left(\frac{c-3}{2}\right)(e_1^* + e_2^*) \\ -(\lambda'' + e_2^* - e_3^*) = \left(\frac{c-3}{2}\right)(e_1^* + e_2^*). \end{cases}$$

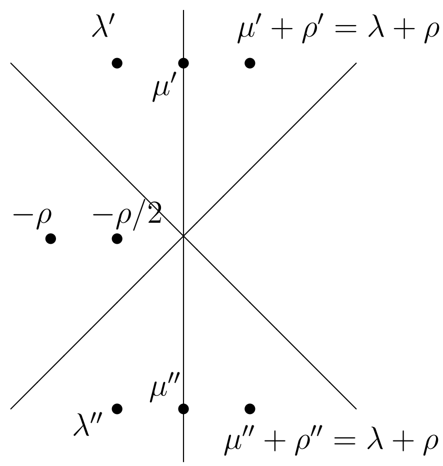
Then from Appendix I we have

$$\begin{cases} H^0(\mathfrak{n}_K, \mathbb{C}_{e_3^*-e_1^*})_{-\lambda'} \cong W^{(0)}_{-\left(\frac{c-3}{2}\right)} \\ H^0(\mathfrak{n}_K, \mathbb{C}_{e_2^*-e_3^*})_{-\lambda''} \cong W^{(0)}_{\left(\frac{c-3}{2}\right)}. \end{cases}$$

Thus heuristically the cup-product should be

$$W^{(0)}_{-\left(\frac{c-3}{2}\right)} \otimes W^{(0)}_{\left(\frac{c-3}{2}\right)} \xrightarrow{\sim} W^{(0)}_0.$$

Turning to the statement and proof of the cup-product result, the idea is this: Recall the root diagram where we omit the subscripts "k." Here the μ' and μ'' are the Blattner parameters and $\mu' + \rho'$, $\mu'' + \rho''$ the Harish-Chandra parameters of the holomorphic and anti-holomorphic DS representations[46]



We have Penrose transforms

$$\begin{cases} H^0(X', L'_{\mu'}) \xrightarrow{\sim} H^1(X, L_{\lambda'}) \\ H^0(X'', L'_{\mu''}) \xrightarrow{\sim} H^1(X, L_{\lambda''}). \end{cases}$$

[46]We recall that given a nonsingular weight ξ we may define a positive root system

$$\Phi^+(\xi) = \{\alpha \in \Phi : (\xi, \alpha) > 0\}.$$

If then $\xi + \rho(\xi)$ is integral there exists a unique DS representation π_ξ with infinitesimal character χ_ξ. Moreover, $\pi_\xi\big|_K$ contains with multiplicity one the K-type with highest weight

$$\Xi = \xi + \rho(\xi) - 2\rho_c(\xi).$$

Finally, if Ξ' is the highest weight of a K-type in $\pi_\xi\big|_K$, then

$$\Xi' = \Xi + \sum_{\alpha \in \Phi^+(\xi)} n_\alpha \alpha, \qquad n_\alpha \in \mathbb{Z}^{\geqq 0}.$$

Two such representations are equivalent if, and only if, their parameters are equivalent under W_K. Then ξ is the *Harish-Chandra parameter* and Ξ is the *Blattner parameter*.

The LHS's are Picard authomorphic forms and their conjugates, and hence are "classical." We have

$$\boxed{\lambda' + \lambda'' = -\rho}\,,$$

and because of this the cup-product is a mapping

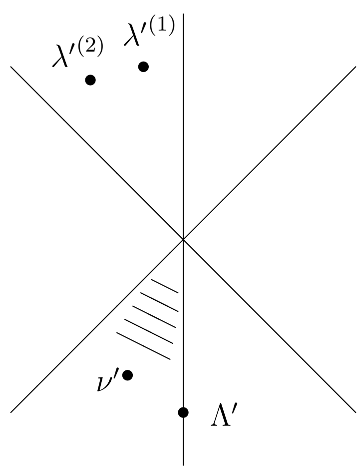
$$H^1(X, L_{\lambda'}) \otimes H^1(X, L_{\lambda''}) \to H^2(X, L_{-\rho})$$

where the RHS is non-classical. We want to show that

the above cup-product is surjective

so that in this way we can reach a non-classical object with a classical one.[47] By using the expressions for the above cohomology groups in terms of $\mathfrak{n}$-cohomology we are reduced to proving a result about cup-products in $\mathfrak{n}$-cohomology. As previously noted we shall actually prove a dual form of the desired result.

We are going to work with each of D', D'' separately and then combine the results. For D' we use the picture



$\lambda'^{(1)} = \lambda'$ in the earlier picture

$\nu = \lambda''$ in the earlier picture

We shall denote by V' the unique DS representation of $S\mathfrak{U}(2,1)_{\mathbb{R}}$ with *Harish-Chandra parameter* ν'; Λ' will denote the *Blattner parameter*, which is the "lowest highest weight" in the K-type. Explicitly,

$$\begin{cases} \Lambda' = \left(\frac{k}{3}\right)(e_1^* + e_2^*) \\ \nu' = \Lambda' + e_3^* - e_2^* \end{cases}$$

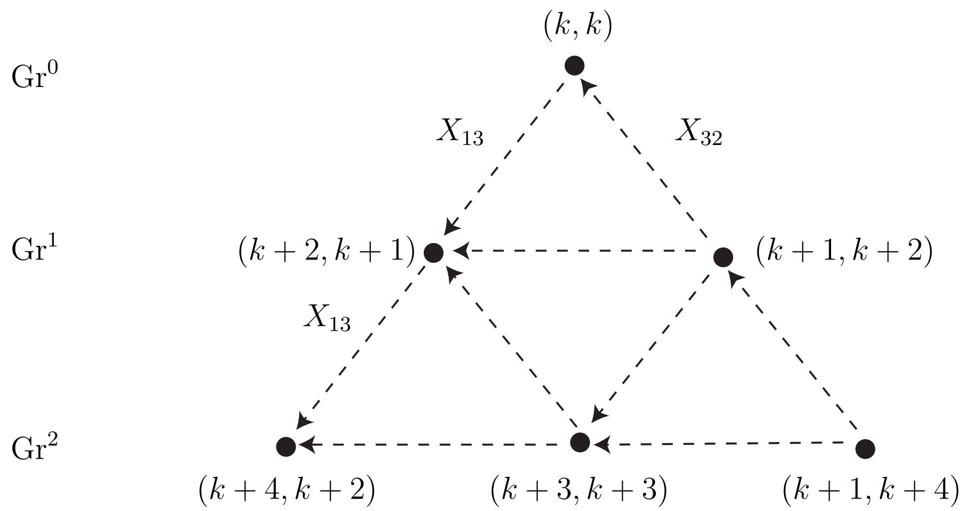
[47]The actual result will be a little weaker in that it will involve the limit over Γ's. But the essential idea is the above.

$$\lambda'^{(1)} = -\left(\tfrac{k}{3}\right)(e_1^* + e_2^*) + 2e_1^* + e_2^*$$
$$\lambda'^{(2)} = -\left(\tfrac{k}{3}\right)(e_1^* + e_2^*) + 3e_1^*.$$

Then the K-type of V' is

$$\bigoplus_{n \geqq 0} W_{k+3n}^{(n)}.$$

The picture of the K-type is



From the picture we see that there is a non-zero vector $v' \in (k, k)$ such that

$$\begin{cases} X_{32} \cdot v' = 0 \\ (e_1^* + e_2^*)v' = kv'; \end{cases}$$

i.e., v' transforms like $\det^k$.

The reason for using V' will appear below. From the results of Schmid we have

$$H^q(\mathfrak{n}, V')_{-\mu} \neq \{0\} \Leftrightarrow \begin{cases} q = 1 \ \text{ and } \ \mu = \lambda'^{(1)} \\ q = 2 \ \text{ and } \ \mu = \lambda'^{(2)}. \end{cases}$$

Moreover, $H^1(\mathfrak{n}, V')_{-\lambda'^{(1)}}$ is generated by $v'\omega_{31}$ and $H^2(\mathfrak{n}, V')_{-\lambda'^{(2)}}$ by $v'\omega_{12} \wedge \omega_{13}$. For V'' we have a similar picture flipped about the horizontal axis and with

$$H^q(\mathfrak{n}, V'') \neq \{0\} \Leftrightarrow \begin{cases} q = 1 \ \text{ and } \ \mu = \lambda''^{(1)} \\ q = 2 \ \text{ and } \ \mu = \lambda''^{(2)}. \end{cases}$$

From a result in representation theory (cf. [C1]), there is a unique 1-dimensional subspace of $V' \otimes V_0$ that is killed by $X_{32} = X_{e_3^* - e_2^*}$ and on which K acts by $\det^k$.

Intuitively, from the formal expansion into K-types of $V' \otimes V_0$ we need to have a summand

$$W^{(n)}_{k+3n} \otimes W^{(l)}_{-3n}$$

in the tensor product such that there is a weight vector killed by $X_{e_3^* - e_2^*}$. By inspection of the pictures of the K-types we see that $W^{(0)}_k \otimes W^{(0)}_0$ is the unique such subspace.

The projection of Harish-Chandra modules

$$V' \otimes V_0 \to V'$$

then induces

$$H^1(\mathfrak{n}, V')_{-\lambda'^{(1)}} \otimes H^1(\mathfrak{n}, V_0)_\rho \to H^2(\mathfrak{n}, V')_{-\lambda'^{(2)}}$$

which using the above notation for the generator of $H^1(\mathfrak{n}, V_0)_\rho$ and the fact that $v' \otimes v_{(1,-2)}$ projects to zero, is given by a generator

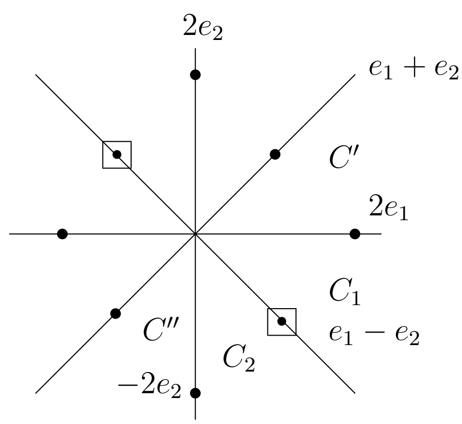
$$v'\omega_{13} \otimes (v_0\omega_{12} - v_{(1,1)}\omega_{13} - v_{(1,-2)}\omega_{32}) \to cv'\omega_{13} \wedge \omega_{12}$$

for some non-zero constant c. Dualizing gives the desired surjectivity of

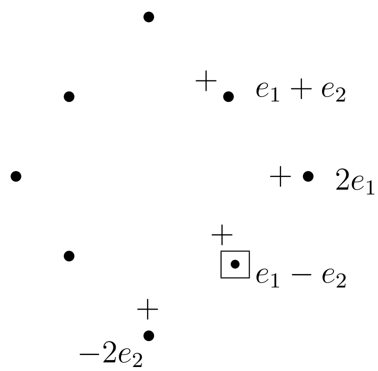
$$H^1(\mathfrak{n}, V')_{-\lambda'^{(1)}} \otimes H^1(\mathfrak{n}, V'')_{-\lambda''^{(1)}} \to H^2(\mathfrak{n}, V_0)_\rho.$$

The case of $\mathrm{Sp}(4)$

We first recall the root diagram



Up to equivalence under the Weyl group there are two TDLDS's V_1 and V_2 corresponding to $(0, C_1)$ and $(0, C_2)$. We will focus on the complex structure on $D = \mathrm{Sp}(4)_\mathbb{R}/T$ given by the Weyl chamber C_1, for which we have the picture



, $\rho_1 = 2e_1 - e_2.$

From the argument given by Schmid in appendix II to this lecture we have that for the $\mathfrak{n} = \oplus$ (negative root spaces in this picture)

> *The Hochschild-Serre spectral sequence for each of $H^*(\mathfrak{n}, V_i)_{-\rho}$, $i = 1, 2$, degenerates at E_1*

This will enable us to compute the $\mathfrak{n}$-cohomology from the E_1-term, and using the results in appendix I to this lecture this can be done once we know the K-types of V_1 and V_2. We will now give this computation for

$$V_1 = H^1(D, L_{-\rho}).$$

The argument for V_2 is similar using the Weyl chamber C_2.

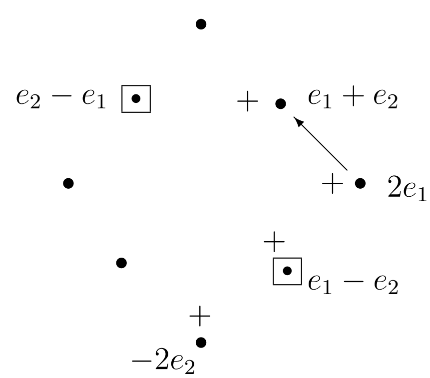
K-type of $H^1(D, L_{-\rho})$

We will first show that

> *As a holomorphic line bundle*

$$N_{Z/D} \cong L_{-2e_2} \oplus L_{e_2} \oplus L_{2e_1+e_2}.$$

Proof. We shall use the picture



The normal bundle is the $\mathcal{U}(2)$-homogeneous bundle given by the action of the negative compact root vector $X_{e_2-e_1}$ on $\mathfrak{p}^+ = \mathfrak{p}_{\mathbb{C}}/\mathfrak{p}^-$. For

$$\mathfrak{p}^+ = \operatorname{span}\{X_{-2e_2}, X_{2e_1}, X_{e_1+e_2}\}$$

we see that as a $\mathcal{U}(2)$ module

$$\mathfrak{p}^+ = \mathbb{C}X_{-2e_2} \oplus \mathbb{C}\{X_{2e_1} \xrightarrow{X_{21}} X_{e_1+e_2}\},$$

where the term in the brackets denotes the 2-dimensional vector space $\operatorname{span}\{X_{2e_1}, X_{e_1+e_2}\}$ with the indicated action of X_{21}. Thus as $\mathcal{U}(2)$ homogeneous bundle

$$N_{Z/D} \cong L_{-2e_2} \oplus N'$$

where

$$0 \to L_{e_1+e_2} \to N' \to L_{2e_1} \to 0.$$

As *holomorphic* vector bundles this is a Koszul sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Z \arrow[r, "{(z_1,z_2)}"] \arrow[d, phantom, "\wr\|"] & \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1) \arrow[r, "{\binom{-z_2}{z_1}}"] \arrow[d, phantom, "\wr\|"] & \mathcal{O}_Z(2) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
0 \arrow[r] & L_{e_1+e_2} \arrow[r] & N' \arrow[r] & L_{2e_1} \arrow[r] & 0
\end{tikzcd}$$

where $[z_1, z_2]$ are homogeneous cooredinates on $\mathbb{P}^1$. This gives for the dimension

$$h^0(Z, N_{Z/D}) = 7 = 6 + 1 = \dim \operatorname{sp}(4)_{\mathbb{C}} + 1.$$

Since $h^1(Z, N_{Z/D}) = 0$ the deformations of Z in D are unobstructed. Thus

$$H^0(Z, N_{Z/D}) = \operatorname{image}\{\operatorname{sp}(4)_{\mathbb{C}} \to H^0(Z, N_{Z/D})\} \oplus \mathbb{C}$$

and we see that

> *the deformations of Z in D consist of the cycle space plus one "extra" deformation.*

For the K-type we have as holomorphic vector bundles

$$S^m N^*_{Z/D} \cong \underset{i+j+k=n}{\oplus} L_{(-j-2k)e_1+(2i-k)e_2}$$
$$S^m N^*_{Z/D}(L_{-\rho}) \cong \underset{i+j+k=m}{\oplus} L_{(-j-2k-2)e_1+(2i-k+1)e_2}.$$

Of course we need these as $\mathcal{U}(2)$-homogeneous vector bundles, which involves the $\mathrm{Sym}^n\, N'^*$'s where N' is as above. But from the above we see that $H^0(Z,*) = (0)$ for all the line bundles $*$ on the RHS. It follows that $H^1(Z, \mathrm{Sym}^m\, N^*_{Z/D}(L_{-\rho}))$ is, as a $\mathcal{U}(2)$-module, the same as $\oplus H^1(Z,*)$. The point is that a filtered $\mathcal{U}(2)$-module is, as a $\mathcal{U}(2)$-module, the same as the associated graded

$$H^1\big(Z, \mathrm{Sym}^m\, N^*_{Z/D}(L_{-\rho})\big) = \overset{m}{\underset{k=0}{\oplus}}\ \overset{k}{\underset{i=0}{\oplus}} W^{(2m+2i-2m+1)}_{-i-m-1}.$$

We may now fill in the following table of the E_1-term for the HSSS for $H^*(\mathfrak{n}, V_1)_\rho$

	$\wedge^0\mathfrak{p}^*$	$\wedge^1\mathfrak{p}^*$	$\wedge^2\mathfrak{p}^*$	$\wedge^3 p^*$
H^1	$n=1$ $k=-1$	$n=1$ $k=0$	0	0
H^0	0	0	$n=1$ $k=-1$	$n=1$ $k=0$

The notation means that in the non-zero blocks only the $W_k^{(n)^*}$ in the K-type occurs for the given n and k. We may abbreviate this by the table

$W_1^{(1)^*}$	$W_0^{(1)^*}$	0	0
0	0	$W_{-1}^{(1)^*}$	$W_0^{(1)^*}$

Since the d_1's are maps of $\mathfrak{b}_K$ modules we see that they are zero. This is true for any Harish-Chandra module with the above K-type. For the particular $V_1 = H^1(D, L_\rho)$ Schmid has given a proof, reproduced in appendix II, that $d_2 = 0$.

Remark: We recall the corresponding picture for $S\mathcal{U}(2,1)$ is

$W_0^{(0)}$	0	0
0	0	$W_0^{(0)}$

The symmetry is due to the special feature

$$L_{-\rho}\big|_Z = \omega_Z$$

in this case. In general, for $X = \Gamma\backslash D$ where Γ is co-compact we have from $\omega_X = L_{-2\rho}$ and Kodaira-Serre duality

$$H^q(X, L_{-\rho})^* \cong H^{d-q}(X, L_{-\rho}).$$

Here, $d = 3$ for $S\mathfrak{U}(2,1)$ and $d = 4$ for $\mathrm{Sp}(4)$; both sides are zero for $q = 0, d$. In the computation of the automorphic cohomology in terms of $\mathfrak{n}$-cohomology the $\mathfrak{n}_K$-cohomology groups for groups

$$H^q(Z, L_{-\rho}) \ \textit{and its dual}\ H^{d-q}(Z, L_\rho \otimes \omega_X)$$

appear. For $S\mathfrak{U}(2,1)$

$$L_\rho \otimes \omega_X = L_{-\rho}.$$

But for $\mathrm{Sp}(4)$

$$\begin{aligned} L_\rho \otimes \omega_X &= L_{2e_1-e_2} \otimes L_{-2\rho_c} \\ &= L_{2e_1-e_2} \otimes L_{e_2-e_1} \\ &= L_{e_1} = L_{-\rho} \otimes L_{3e_1-e_2} \end{aligned}$$

and this reflects the dualities

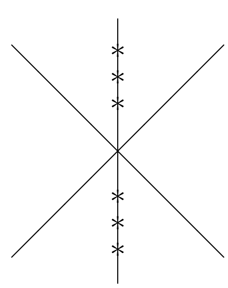
$$\begin{cases} W_{-1}^{(1)^*} \cong W_0^{(1)} \\ W_0^{(1)^*} \cong W_{-1}^{(1)} \end{cases}$$

between the $E_1^{0,1}$ and $E_1^{3,0}$ terms and $E_1^{0,2}$ and $E_1^{4,0}$ terms in the table for $\mathrm{Sp}(4)$.

If we try to mimic for $\mathrm{Sp}(4)$ the cup-product story given above for $S\mathfrak{U}(2,1)$ we find that the asymmetry

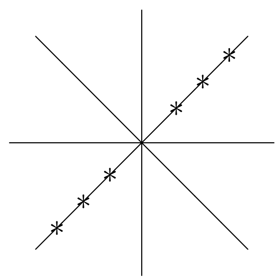
$$\rho \neq 2\rho_c$$

does not allow a direct analogy. For $S\mathfrak{U}(2,1)$ the Picard automorphic forms and their conjugates occurred as in the picture

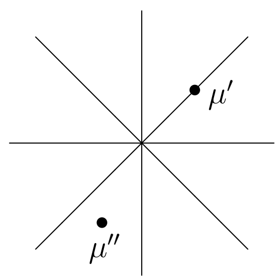


The symmetry $\rho = 2\rho_c$ in this case led to the picture given at the beginning of a previous section giving the cup-product result in this case.

For Sp(4) the Siegel modular forms and their conjugates are pictured as



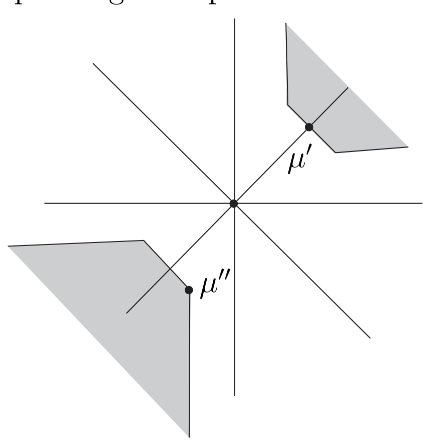
Because of the aforementioned asymmetry the $S\mathcal{U}(2,1)$ picture must be replaced by



Thus we let

- V' = holomorphic DS with Blattner parameter $\mu' = k(e_1 + e_2)$;
- V'' = almost-holomorphic DS with Blattner parameter $\mu'' = -k(e_1 + e_2) - e_2$.

The weights of the corresponding DS representations are contained in the shaded regions



An analysis simlar to, but in several ways more intricate than that given for the $S\mathcal{U}(2,1)$ case, leads to the "surjectivity" of

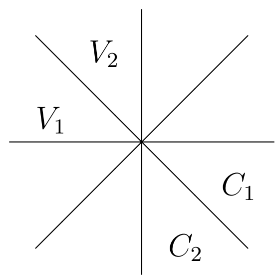
$$H^1(D, L_{\mu'(1)}) \otimes H^2(D, L_{\mu''(2)}) \to H^3(D, L_{-\rho}).$$

The quotation marks mean that as in the $S\mathcal{U}(2,1)$ case one only has surjectivity in the limit over Γ's.

The $\mathfrak{n}$-cohomology result is the isomorphism

$$H^1(\mathfrak{n}, V')_{-\mu'^{(1)}} \otimes H^2(\mathfrak{n}, V'')_{-\mu''^{(2)}} \xrightarrow{\sim} H^3(\mathfrak{n}, V_2)_{-\rho}$$

where V_2 is the "other" TDLDS



The proof of this result will be given in the sequel to [GGK2].

Appendix I to Lecture 9: The K-types of the TDLDS for $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$

In this appendix we establish the notation for the relevant representation theory of $\mathcal{U}(2)$ and determine the above K-types and subsequent structure of the above mentioned $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules.

We shall first deal with the representation theory of $\mathcal{U}(2)$ and recall the notations:

- $K = \mathcal{U}(2)$ with maximal torus $T = \left\{ \left(\begin{smallmatrix} e^{2\pi i\theta_1} & 0 \\ 0 & e^{2\pi i\theta_2} \end{smallmatrix} \right) \right\}$;
- $\mathfrak{t}$ has coordinates $\boldsymbol{\theta} = \left(\begin{smallmatrix} \theta_1 \\ \theta_2 \end{smallmatrix} \right)$ so that $T \cong \mathbb{R}^2/\mathbb{Z}^2$;
- $e_1^*, e_2^* \in \mathfrak{t}^*$ are the weights giving a $\mathbb{Z}$-basis for the character lattice $\mathrm{Hom}(\Lambda, \mathbb{Z})$ of $T = \mathfrak{t}/\Lambda$ and where
$$\langle e_1^*, \boldsymbol{\theta} \rangle = \theta_1,\ \langle e_2^*, \boldsymbol{\theta} \rangle = \theta_2;$$
- $e_2^* - e_1^* = \alpha$ is the *positive* root for $\mathcal{U}(2)$;
- $Z = \mathcal{U}(2)_{\mathbb{R}}/T = \mathcal{U}(2)_{\mathbb{C}}/B$ where B is the Borel subgroup with Lie algebra
$$\mathfrak{b}_K = \mathfrak{t}_{\mathbb{C}} \otimes \mathfrak{n}_K$$
where
$$\begin{aligned} \mathfrak{n}_K &= \mathbb{C}X_{12} \ \text{ and} \\ X_{12} &= X_{e_1^* - e_2^*} \end{aligned}$$
is the negative root vector;
- for a weight $\mu = ae_1^* + be_2^*$, $L_\mu \to Z$, or $L_{(a,b)} \to Z$, is the corresponding $\mathcal{U}(2)$-homogeneous, holomorphic line bundle;
- $W = \mathbb{C}^2$ is the standard $\mathcal{U}(2)$-module with highest weight e_2^*, and where we set
$$\begin{cases} w_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = \text{ lowest weight vector} \\ w_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) = \text{ highest weight vector;} \end{cases}$$
- $\Delta = \mathcal{U}(2)$-module $\Lambda^2 W$ with $\mathcal{U}(2)$ acting by the character det with weight $e_1^* + e_2^*$; we set
$$\delta = w_2 \wedge w_1;$$
- $W_k^{(n)}$ is the $\mathcal{U}(2)$-module $\mathrm{Sym}^n W \otimes \Delta^k$; it has weight vectors
$$w_1^n \delta^k, w_1^{n-1} w_2 \delta^k, \ldots, w_2^n \delta^k,$$
where the weights increase from left to right;
- we shall sometimes abuse notation and write the above weight vector as $w_1^{n+k} w_2^k, \ldots, w_1^k w_2^{n+k}$;

- as $\mathcal{U}(2)$-modules

$$W_k^{(n)^*} \cong W_{-n-k}^{(n)}$$
$$W_k^{(n)} \otimes W_l^{(m)} = \bigoplus_{i=0}^{m} W_{i+k+l}^{(n+m-2i)}, \qquad m \leqq n,$$

- we have

$$\mathcal{O}_Z(1) = L_{e_2^*}$$

so that

$$Z = \mathbb{P}W^*;$$

- from this we have as $\mathcal{U}(2)$-modules

$$\begin{cases} H^0(Z, L_{ae_1^*+be_2^*}) & = W_a^{(b-a)} \\ H^1(Z, L_{ae_1^*+be_2^*}) & = W_{b+1}^{(a-b-2)}. \end{cases}$$

Note: The BWB theorem is usually stated for semi-simple groups. Suitably interpreted it also holds for reductive groups. Thus we write

$$ae_1^* + be_2^* = (b-a)e_2^* + a(e_1^* + e_2^*)$$

and think of $W_a^{(b-a)}$ as the $\mathcal{U}(2)$-module with highest weight $(b-a)e_2^*$ and determinant weight $a(e_2^* + e_2^*)$, so that the statement is

> $H^0(Z, L_{ae_1^*+be_2^*})$ *is the* $\mathcal{U}(2)$*-module with highest weight* $(b-a)e_2^*$ *and determinant weight* $a(e_1^* + e_2^*)$.

As for $H^1(Z, L_{ae_1^*+be_2^*})$ we have

$$\rho_c = \left(\frac{1}{2}\right)(-e_1^* + e_2^*)$$

so that $ae_1^* + be_2^* + \rho_c$ is singular if, and only if,

$$a = b + 1.$$

The linear form ρ_c is not integral on Λ where $T = t/\Lambda$, but since $(\rho_c, e_2^* - e_1^*) = 1$ it is integral on the root $e_2^* - e_1^*$ and is therefore a "weight" in this sense. The geometric point is that ω_Z does have an $S\mathcal{U}(2)$-invariant square root, but it does not have a $\mathcal{U}(2)$-invariant one.

As a check we will verify that the above formulas for the $\mathcal{U}(2)$-modules $H^q(Z, L_{ae_1^*+be_2^*})$ are consistent with the formula

$$H^q(Z, L_\mu) = \bigoplus_{k,n} W_k^{(n)} \otimes H^q\left(\mathfrak{n}_K, W_k^{*(n)}\right)_{-\mu}.$$

Using $W_k^{*(n)} = W_{-k-n}^{(n)}$ we have

$$H^0\left(\mathfrak{n}_K, W_k^{*(n)}\right) = \mathbb{C} \text{ (lowest weight vector in } W_{-k-n}^{(n)}).$$

This lowest weight vector is $w_1^{-k} w_2^{-n-k}$ and transforms by $-\mu$ exactly when

$$\mu = ke_1^* + (n+k)e_2^* = ne_2^* + k(e_1^* + e_2^*),$$

which was to be proved.

Next

$$H^1\left(\mathfrak{n}_K, W_{-k-n}^{*(n)}\right) \cong \mathbb{C} \text{ (highest weight vector in } W_{-k-n}^{(n)} \otimes X_{12}^*).$$

Using $X_{12}^* = w_1^{-1} w_2$ the term in parenthesis is $w_1^{n-k-1} w_2^{-k+1}$ so that

$$\mu = (n+k+1)e_1^* + (k-1)e_2^*$$

as desired.

- Finally, the notation
$$(a,b) = ae_1^* + be_2^*$$
will make the book-keeping easier. Thus
$$\begin{cases} L_{ae_1^*+be_2^*} = L_{(a,b)} \\ \omega_Z = L_{(1,-1)}. \end{cases}$$

The K-type for the TDLSD V_0 for $S\mathcal{U}(2,1)$

We have $V_0 = H^1(D, L_{-\rho})$ and the K-type is the $\mathcal{U}(2)$-module

$$\bigoplus_{n \geqq 0} H^1\left(Z, \operatorname{Sym}^n N_{Z/D}^*(L_{-\rho})\right).$$

Recalling that

$$\mathfrak{p}^+ = \operatorname{span}\{X_{31}, X_{23}\} = \operatorname{span}\{X_{e_3^*-e_1^*}, X_{e_2^*-e_3^*}\}$$

and identifying $\mathfrak{p}^+ \cong \mathfrak{p}_{\mathbb{C}}/\mathfrak{p}_-$ as $\mathfrak{b}_K$-modules using the Cartan-Killing form the normal bundle is the $\mathcal{U}(2)$-homogeneous vector bundle

$$\begin{array}{ccc} N_{Z/D} & = & \mathcal{U}(2) \times_T \mathfrak{p}^+ \\ \big\downarrow & & \big\downarrow \\ Z & = & \mathcal{U}(2)/T. \end{array}$$

Since $\mathfrak{n}_K = \mathbb{C}X_{12}$ acts trivially on $\mathfrak{p}^+$, as $\mathfrak{b}_K$-modules we have

$$\mathfrak{p}^+ = \mathbb{C}X_{31} \oplus \mathbb{C}X_{23}.$$

Now $e_3^* = -e_1^* - e_2^*$ so that

$$\begin{aligned} &X_{31} \text{ has weight } e_3^* - e_1^* = -2e_1^* - e_2^* = -2(e_1^* + e_2^*) + e_2^* \\ &X_{23} \text{ has weight } e_2^* - e_3^* = e_1^* + 2e_2^* = (e_1^* + e_2^*) + e_2^*. \end{aligned}$$

This gives the conclusion

$$N_{Z/D} = \Delta^{-2}(1) \oplus \Delta(1) = L_{(-2,1)} \oplus L_{(1,2)}$$

where $\Delta^k(1) = \Delta \otimes \mathcal{O}_Z(k)$. Then

- $N_{Z/D}^* = L_{(2,-1)} \oplus L_{(-1,-2)}$;
- $\operatorname{Sym}^n N_{Z/D}^* = \bigoplus_k L_{(2n-3k,n-3k)}$;
- $\operatorname{Sym}^n N_{Z/D}^*(L_{-\rho}) = \oplus L_{(2n-3k+1,n-3k-1)}$

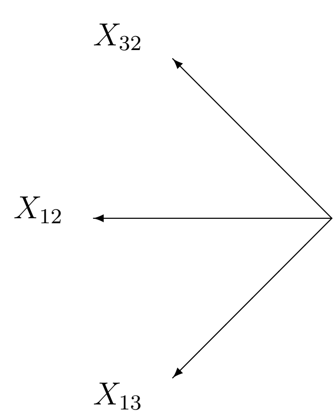
where the last step uses $-\rho = (1,-1)$. Using the formula for $H^1(Z, L_{(a,b)})$ above, for the K-type of V_0 we find that

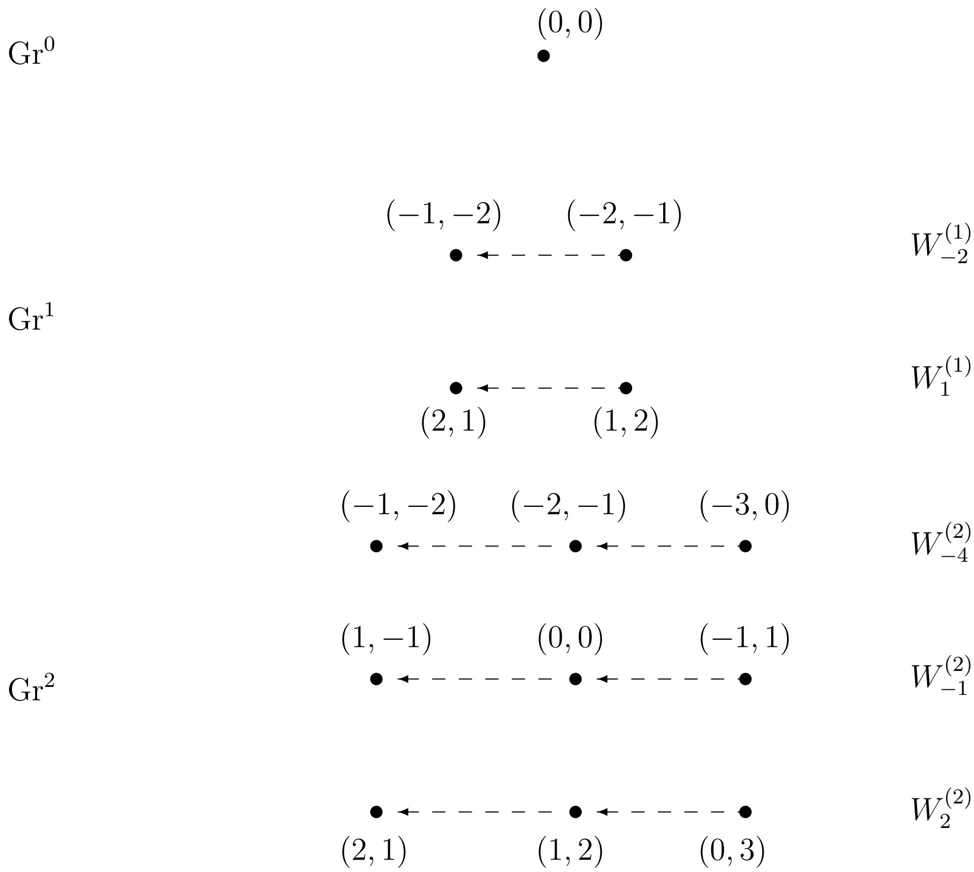
$$\operatorname{Gr}^n \cdot V_0 = \bigoplus_k W_{n-3k}^{(n)}.$$

The first few terms are

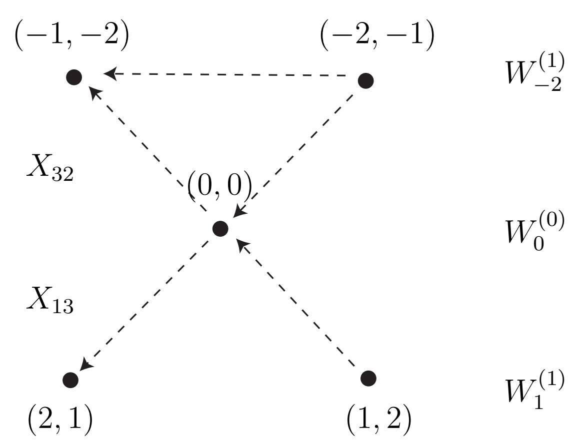
$$\begin{array}{lc} \operatorname{Gr}^0 & W_0^{(0)} \\ \operatorname{Gr}^1 & W_1^{(1)} \oplus W_{-2}^{(1)} \\ \operatorname{Gr}^3 & W_2^{(2)} \oplus W_{-1}^{(2)} \oplus W_{-4}^{(2)} \\ & \vdots \end{array}$$

Remark: For later use we give the following picture of the K-type with action of $\mathfrak{n}$ as depicted by





Here the dots represent 1-dimensional weight spaces where (a,b) corresponds to the weight $ae_1^* + be_2^*$. The dashed arrows give the action of X_{12} as depicted above. In contrast to the DS we see that weights such as $(0,0)$ can appear infinitely after in the K-type. To get the action of $\mathfrak{n}$ we have to overlay these diagrams. For example, overlaying Gr^0 and Gr^1 gives the picture

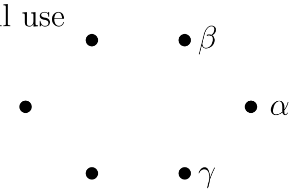


This diagram was used in the computation showing that $d_1 = d_2 = 0$ in the HSSS given above.

Appendix II to Lecture 9: Schmid's proof of the degeneracy of the HSSS for TDLDS in the $\mathcal{SU}(2,1)$ and $\mathrm{Sp}(4)$ cases

The $\mathcal{SU}(2,1)$ case

For notational simplicity we shall use



for the positive root system. Then

$$\begin{cases} \rho = \alpha \\ \rho_c = \alpha/2 \Rightarrow \omega_Z = \mathcal{O}_Z(L_{-\rho}) \\ \rho_{nc} = \alpha/2. \end{cases}$$

We let V_ρ be the Harish-Chandra module associated to the DS realized as the L^2-cohomology group $H^1_{(2)}(D, L_{-2\rho})$. Since

$$\mathcal{O}_Z(L_{-2\rho}) = \mathcal{O}_Z(L_{-\rho}) \otimes \omega_Z,$$

we have $H^1(Z, L_{-2\rho}) \cong H^0(Z, L_\rho)^*$ so that V_ρ has lowest K-type the irreducible $\mathcal{SU}(2)$-module with highest weight ρ.[48] We also denote by V_0 the Harish-Chandra module $H^1(D, L_{-\rho})$ associated to the TDLDS.

We denote by M^ρ the irreducible finite dimensional representation of $\mathcal{SU}(2,1)_{\mathbb{C}}$ with highest weight ρ. It is the adjoint representation and has weights

$$\pm\alpha, \pm\beta, \pm\gamma, \text{ and } 0 \text{ (twice)}.$$

The argument uses the basic operation of *Zuckerman tensoring*, which consists of taking an infinite dimensional representation and tensoring it with a finite dimensional one to obtain a representation that is not irreducible but has an infinitesimal character which is a sum containing the one in which we are interested. Thus we consider $V_\rho \otimes M^\rho$, which involves composition factors with infinitesimal characters $\chi_{\rho+\nu}$ where ν is a weight of M^ρ. Moreover, it is a basic general fact that

> if the weight ν has multiplicity one, if $\rho+\nu$ is dominant and if $\rho+\nu$ is not Weyl equivalent to $\rho+\nu'$ for any other weight ν' of W^ρ, then the DS or LDS Harish-Chandra module with infinitesimal character $\chi_{\rho+\nu}$ occurs once in the tensor product as a subrepresentation and no other composition factors have infinitesimal character $\chi_{\rho+\nu}$.

For our TDLDS V_0 this gives

---

[48] We note that the degeneracy argument will only involve the weights of $\mathcal{SU}(2)$-modules.

V_0 occurs as a summand in $V_\rho \otimes M^\rho$ and no other composition factor has infinitesimal character χ_0.

For the $\mathfrak{n}$-cohomology of V_ρ, we have from Lecture 5 that it is the Harish-Chandra module associated to each of

- $H^1_{(2)}(D, L_{-2\rho})$ $\qquad (q(-2\rho+\rho)=1)$
- $H^2_{(2)}(D, L_{2\rho})$ $\qquad (q(2\rho+\rho)=2)$

Here the compact Weyl group $W_K = \{\mathrm{id}, s_\alpha\}$ where s_α is reflection in the compact root line, and since

$$-2\rho+\rho = s_\alpha(2\rho)+\rho$$

the above two $S\mathcal{U}(2,1)$-modules are equivalent realizations of the DS with Harish-Chandra module V_ρ. From Schmid's results on the $\mathfrak{n}$-cohomology of V_ρ in Lecture 5 we have

$$H^q(\mathfrak{n}, V_\rho) = \left\{ \begin{array}{l} \text{one dimensional of weight } 2\alpha \text{ for } q=1, \\ \text{one dimensional of weight } 0 \text{ for } q=2, \\ 0 \text{ for } q \neq 1,2 \end{array} \right\}.$$

The generator of $H^1(\mathfrak{n}, V_\rho)$ is the lift of the Kostant class $\kappa_\mu \in H^1(\mathfrak{n}_K, H^1(Z, L_{-2\rho}))$ which was discussed in the appendix to Lecture 5.

Since V_0 is a summand of $V_\rho \otimes M^\rho$ we have an inclusion

$$H^*(\mathfrak{n}, V_0) \hookrightarrow H^*(\mathfrak{n}, V_\rho \otimes M_\rho).$$

By Casselman-Osborne the cohomology of V_0 occurs in weight ρ,[49] and no other composition factors can contribute cohomology in weight ρ. Thus

$$H^*(\mathfrak{n}, V_0) = \rho\text{-weight space in } H^*(\mathfrak{n}, V_\rho \otimes M^\rho).$$

For a weight ν let $\mathbb{C}_\nu$ denote the 1-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts via ν. As a $\mathfrak{b}$-module, M^ρ has a composition series with composition factors $\mathbb{C}_\nu$ as ν runs over the weights of M^ρ. Specifically,

- $\mathbb{C}_{-\rho}$ occurs as a $\mathfrak{b}$-submodule of M^ρ;
- $\mathbb{C}_\rho$ occurs as a $\mathfrak{b}$-quotient module of M^ρ.

Thus we obtain morphisms

- $H^*(\mathfrak{n}, V_\rho) \otimes \mathbb{C}_{-\rho} \to H^*(\mathfrak{n}, V_\rho \otimes M^\rho)$;
- $H^*(\mathfrak{n}, V_\rho \otimes M^\rho) \to H^*(\mathfrak{n}, V_\rho) \otimes \mathbb{C}_\rho$.

Specializing this to the ρ-weight components and using the above description of $H^*(\mathfrak{n}, V_\rho)$ we find

[49] As we saw in Lecture 9, the E_1-term of the HSSS has generators $v_0\omega^{-\alpha}$ in $E_1^{0,1}$ and $v_0\omega^{-\beta} \wedge \omega^{-\gamma}$ in $E_1^{2,0}$.

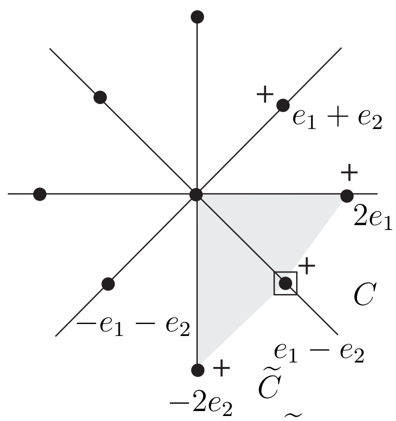
- $H^1(\mathfrak{n}, V_\rho) \otimes \mathbb{C}_{-\rho} \to H^1(\mathfrak{n}, V_\rho \otimes M^\rho)$;
- $H^2(\mathfrak{n}, V_\rho \otimes M^\rho) \to H^2(\mathfrak{n}, V_\rho) \otimes \mathbb{C}_\rho$.

The composition series for the $\mathfrak{b}$-module M^ρ gives a spectral sequence abutting to $H^*(\mathfrak{n}, V_\rho \otimes M^\rho)$, and since cohomology in the lower degree maps in and cohomology in the higher degree maps out, there cannot be cancellation in the ρ-weight space in $H^*(\mathfrak{n}, V_\rho \otimes M^\rho)$. Thus

$$h^1(\mathfrak{n}, V_0) = h^2(\mathfrak{n}, V_0) = 1$$

which implies that the HSSS degenerates at E_1.

The $\mathrm{Sp}(4)$ **case:**
We recall the root diagram



In this case there are two TDLDS's V_0 and $\widetilde{V}_0$ corresponding to the Weyl chambers C, $\widetilde{C}$.[50] Taking

$$\Phi^+ = \{e_1 + e_2, 2e_2, e_1 - e_2, -2e_2\}$$

we want to show that the HSSS degenerates at E_1 for each of $H^*(\mathfrak{n}, V_0)$ and $H^*(\mathfrak{n}, \widetilde{V}_0)$. We first take the case of $H^*(\mathfrak{n}, V_0)$. We have

$$\rho = 2e_1 - e_2$$
$$\rho_c = \frac{1}{2}(e_1 - e_2)$$
$$\rho_{nc} = \frac{1}{2}(3e_1 - e_2).$$

Since V_0 is the Harish-Chandra module associated to $H^1(D, L_{-\rho})$ and

$$\mathcal{O}_Z(L_{-\rho}) = \mathcal{O}_Z\left(L_{-\rho_{nc}+\rho_c}\right) \otimes \omega_Z,$$

[50]In the lecture these were denoted by C_1 and C_2. Here it is more convenient to use the above notation.

using Kodaira-Serre duality we see that V_0 has lowest K-type the K-module with highest weight

$$\rho_{nc} - \rho_c = e_1.$$

We shall also consider the Harish-Chandra module $V_\rho =: H^1(D, L_{-\rho-2\rho_c})$ which has lowest K-type the K-module with highest weight

$$\rho + \rho_{nc} - \rho_c = 3e_1 - e_2.$$

Again by Casselman-Osborne, the $\mathfrak{n}$-cohomology of V_0 occurs in weight ρ and that of V_ρ in weights $w\rho + \rho$ where $w \in W$

- $H^*(\mathfrak{n}, V_0) = H^*(\mathfrak{n}, V_0)_\rho$;
- $H^*(\mathfrak{n}, V_\rho) = \bigoplus\limits_{w \in W} H^*(\mathfrak{n}, V_\rho)_{w\rho+\rho}$.

From Schmid's results in Lecture 5

$$H^q(\mathfrak{n}, V_\rho) = \begin{cases} \mathbb{C}_{4e_1-2e_2} & \text{if } q = 1 \\ \mathbb{C}_{e_1+e_2} & \text{if } q = 2 \\ 0 & \text{if } q \neq 1, 2. \end{cases}$$

Let $W_{-\rho}$ be the irreducible Sp(4)-module of *lowest* weight $-\rho$. Then as before

$$V_0 \text{ is a direct summand of } V_\rho \otimes W_{-\rho}$$

and no other composition factor of $V_\rho \otimes W_{-\rho}$ has infinitesimal character χ_0. Hence

$$H^*(\mathfrak{n}, V_0) = H^*(\mathfrak{n}, V_\rho \otimes W_{-\rho})_\rho.$$

Note that $4e_1 - 2e_2 = 2\rho$, $e_1 + e_2 = \rho + s_{e_1-e_2}\rho$. Filtering $W_{-\rho}$ by $\mathfrak{b}$-submodules we obtain a spectral sequence with E_2-term

$$\big(H^*(\mathfrak{n}, V_\rho) \otimes W_{-\rho}\big)_\rho \Rightarrow H^*(\mathfrak{n}, V_0).$$

The reason that the E_2-term is as given is because the action of $\mathfrak{n}$ shifts the filtration down by *two*, so that $d_1 = 0$. The notation "$\Rightarrow$" means that the spectral sequence abuts to $H^*(\mathfrak{n}, V_0) = H^*(\mathfrak{n}, V_\rho \otimes W_{-\rho})_\rho$. We then have

$$(H^q(\mathfrak{n}, V_\rho) \otimes W_{-\rho})_\rho = \begin{cases} H^1(\mathfrak{n}, V_\rho) \otimes (W_{-\rho})_{-\rho} & q = 1 \\ H^2(\mathfrak{n}, V_\rho) \otimes (W_{-\rho})_{-s_{e_1-e_2}\rho} & q = 2 \\ 0 & \text{otherwise}. \end{cases}$$

But $-s_{e_1-e_2}\rho > -\rho$, so the non-zero term in H^2 occurs at a higher level in the filtration than the non-zero term in H^1. This implies that the spectral sequence degenerates at

E_2 and

$$H^1(\mathfrak{n}, V_0) = \begin{cases} \mathbb{C}_\rho & q = 1 \\ \mathbb{C}_\rho & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

which was to be shown.

For $H^*(\mathfrak{n}, \widetilde{V}_0)$, it is convenient to equivalently compute $H^*(\tilde{\mathfrak{n}}, V_0)$ where $\tilde{\mathfrak{n}}$ corresponds to the direct sum of the negative root spaces for the system of positive roots

$$\widetilde{\Phi}^+ = \{-e_1 - e_2, e_1 - e_2, 2e_1, -2e_2\}.$$

The corresponding quantities are

$$\tilde{\rho} = e_1 - 2e_2, \tilde{\rho}_c = \frac{1}{2}(e_1 - e_2), \tilde{\rho}_{nc} = \frac{1}{2}(e_1 - 2e_2).$$

In this situation

$$H^q(\mathfrak{n}, V_\rho) = \begin{cases} \mathbb{C}_{3(e_1-e_2)} & \text{if } q = 2 \\ \mathbb{C}_0 & \text{if } q = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as before we find that

$$H^*(\tilde{\mathfrak{n}}, V_0) = H^*(\tilde{\mathfrak{n}}, V_\rho \otimes W_{-\rho})_{\tilde{\rho}}.$$

We have $3(e_1 - e_2) = \tilde{\rho} + s_{e_1+e_2}\tilde{\rho}$, $0 = \tilde{\rho} - \tilde{\rho}$. Then as before we obtain a spectral sequence with E_2-term

$$(H^*(\tilde{\mathfrak{n}}, V_\rho) \otimes W_{-\rho})_{\tilde{\rho}} \Rightarrow H^*(\tilde{\mathfrak{n}}, V_0),$$

and for the LHS

$$(H^q(\tilde{\mathfrak{n}}, V_\rho) \otimes W_{-\rho})_{\tilde{\rho}} = \begin{cases} H^2(\tilde{\mathfrak{n}}, V_\rho) \otimes (W_{-\rho})_{-s_{e_1+e_2}\tilde{\rho}} & \text{if } q = 2 \\ H^3(\tilde{\mathfrak{n}}, V_\rho) \otimes (W_{-\rho})_{\tilde{\rho}} & \text{if } q = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Again, the non-zero term in the higher q occur at a higher level of the filtration than the non-zero term for the lower q, because $\tilde{\rho} > -s_{e_1+e_2}\tilde{\rho}$ relative to the ordering given by $\widetilde{\Phi}^+$. In conclusion

- $H^2(\tilde{\mathfrak{n}}, V_0) = \mathbb{C}_{\tilde{\rho}}$;
- $H^3(\tilde{\mathfrak{n}}, V_0) \cong \mathbb{C}_{\tilde{\rho}}$

and $H^q(\tilde{\mathfrak{n}}, V_0)$ is zero otherwise.

Remark: We are aware of three methods of computing the $\mathfrak{n}$-cohomology for a TDLDS.

(i) by direct computation knowing the explicit form of the representation ([C1] and [C2]);

(ii) by direct computation using the HSSS, where both the K-type and the action of $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ are known geometrically (as was done for $S\mathfrak{U}(2,1)$ above);

(iii) by Schmid's method, using his results for the $\mathfrak{n}$-cohomology of DS's and Zuckerman tensoring and Casselman-Osborne as in this appendix.

Lecture 10
Selected topics and potential areas for research

We begin by giving a brief preview of the items to be covered in this lecture. Remark that there is an extended appendix on boundary components and degenerations of PHS's with emphasis on the examples $S\mathfrak{U}(2,1)$ (Carayol) and $\mathrm{SO}(4,1)$, both cases where there is an arithmetic structure on the boundary components but not on the Mumford-Tate domain itself (cf. [KP] for recent work in this direction).

Hermitian symmetric sub-domains of non-classical Mumford-Tate domains

It may be shown that a Hodge domain with trivial IPR is an Hermitian symmetric domain; the argument will be given below. It is beginning to appear that of particular interest are equivariantly embedded Hodge domains

$$D_H \subset D$$

where D is non-classical and where D_H is an integral manifold of the IPR. As just noted, D_H is then an HSD. We will discuss two particular cases of this.

- The recent work of Freidman-Laza [FL]. In first approximation they show that if Γ is an arithmetic group and $S \subset \Gamma\backslash D$ is a closed integral manifold of the IPR where S is quasi-projective and where the inverse image $\widetilde{S} \subset D$ is the intersection of D with an algebraic subvariety in the compact dual $\check{D}$, then $\widetilde{S}$ is an HSD. They then use this and other methods to analyze the VHS's of Calabi-Yau type having this property.
- An extremely interesting issue to arithmetic algebraic geometers is to

  put a "natural" arithmetic structure on $H^q_o(X, L_\mu)$.

  Here, Γ is an arithmetic group and $H^q_o(X, L_\mu)$ is the *cuspidal automorphic cohomology*, which we have not yet defined (cf. [C3], [GGK2] and the appendix to this lecture, where $H^q_o(X, L_\mu)$ will be denoted $H^q_e(X, L_\mu)$). An *arithmetic structure* means a "natural" subspace $H^q_o(X, L_\mu)_{\mathbb{F}} \subset H^q_o(X, L_\mu)$ that is defined over a number field $\mathbb{F} \subset \mathbb{C}$ and with $H^q_o(X, L_\mu)_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{C} = H^q_o(X, L_\mu)$. For those cuspidal cohomology groups that are Penrose transforms of classical cuspidal automorphic groups $H^{q'}_o(X', L'_{\mu'})$, there is an arithmetic structure arising from the fact that $L'_{\mu'} \to X'$ is an algebraic line bundle defined over a number field. Not obtainable in this way are the cuspidal automorphic cohomology groups $H^q_o(X, L_{-\rho})$ corresponding to TDLDS's.

  Classically one criterion for arithmeticity of modular forms is by taking arithmetic values at CM points. In fact, this was the central topic in Shimura's CBMS lecture series here some years ago [Shi]. For $X = \Gamma\backslash D$ where $D = \mathcal{U}(2,1)_{\mathbb{R}}/T$ as in the earlier lectures, an analogue of this would be to evaluate classes in $H^1_o(X, L_{-\rho})$ on *Shimura curves*, which are 1-dimensional quotients of HSD's $D_H \subset D$.[51] There are in fact three types of Shimura curves, but the program of evaluating classes in $H^1_o(X, L_{-\rho})$ on them to give a criterion for arithmeticity has yet to be carried out. We will briefly discuss this below.

Boundary components of Mumford-Tate domains

For period domains D, Kato-Usui [KU] have defined extensions $D_\Sigma \supset D$ leading to *completions* of VHS's[52]

$$\begin{array}{rcl} \Phi : S & \longrightarrow & \Gamma\backslash D \\ \cap & & \cap \\ \overline{\Phi} : \overline{S} & \longrightarrow & \Gamma\backslash D_\Sigma. \end{array}$$

Here S is a smooth, quasi-projective variety having a smooth completion $\overline{S}$ where $\overline{S}\backslash S$ is a normal crossing divisor around which the VHS Φ has unipotent monodromies (an inessential assumption) and $\overline{\Phi}$ is an extension of Φ to $\overline{S}$. As a set we may think of $\Gamma\backslash D_\Sigma$ as certain Γ-equivalence classes, specified by the *fan* Σ, of limiting mixed Hodge structures (LMHS's). The boundary components $D_\sigma \subset D_\Sigma\backslash D$ correspond to *nilpotent cones* $\sigma \subset \mathfrak{g}$ in the fan Σ (see below for discussion of the terms).

Although it has only been carried out in detail in a few cases, it is reasonable to assume that the Kato-Usui theory can be extended to general Mumford-Tate domains D. Both the extent to which the extension D_Σ depends only on the underlying Hodge domain and not on the particular Mumford-Tate domain, and the relation of the boundary components to the orbit structures under Matsuki duality, are interesting issues that remain to be clarified.

- One classical definition of arithmeticity of modular forms defined on $\Gamma(N)\backslash\mathcal{H}$ is in terms of the arithmeticity of the coefficients in the Fourier expansions about a cusp. In [C3] Carayol has given a similar definition for the cohomology group $H^1_o(X, L_{-\rho})$ in the $S\mathcal{U}(2,1)$ case. In this he takes $\Gamma = \mathcal{U}(2,1)_{\mathcal{O}}$ where $\mathcal{O}$ is the ring of integers in the number field $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ as was used in the Mumford-Tate domain with generic Mumford-Tate group $\mathcal{U}(2,1)$ discussed in Lecture 3.

[51] In [GGK2] there is a result that shows how, using EGW, one may in some cases "evaluate" cohomology classes at points of $\Gamma\backslash\mathcal{W}$, and that when this is done the Penrose transform of arithmetic classes in $H^0_o(X', L'_{\mu'_k})$ take arithmetic values at CM points. However, for evident dimension reasons it is more natural to evaluate classes in $H^1_o(X, L_{-\rho})$ on algebraic curves in X.

[52] Here the term "completion of Φ" refers to extending Φ to the completion $\overline{S}$ of S. The term "partial compactification" is also sometimes used for $\Gamma\backslash D_\Sigma$.

Carayol's method uses that two of the boundary components of X are $\mathbb{C}^*$-bundles over CM elliptic curves E', E'' which are "arithmetic objects." He then extends his Penrose transform method to the Kato-Usui completions to define a Fourier expansion of an automorphic cohomology class in $H^1_o(X, L_\mu)$ about an arithmetically defined boundary component. The details of the argument over the boundary lead to the Penrose transforms between pairs of CM elliptic curves that was presented in Lecture 2. We shall briefly discuss this below; in the appendix to this lecture we have included notes from a seminar talk given at the IAS that gives a more comprehensive treatment of the story, including an informal introduction to the Kato-Usui theory.

Existence of Penrose transforms

We have seen that in the case of flag varieties $G_{\mathbb{C}}/B$ the different ways of realizing a given irreducible $G_{\mathbb{C}}$-module as a cohomology group may all be achieved through Penrose transforms among them, which in fact leads to yet another proof of the BWB theorem. The analogous issue in the non-compact case seems to be an open question, one that we will briefly discuss.

Lifting the Kostant class

As we have seen in Lecture 5, the $\mathfrak{n}$-cohomology of the Harish-Chandra module $H^d(D, L_\mu)$, where $\mu + \rho$ is in the closure $-\overline{C}$ of the anti-dominant Weyl chamber, is a topic of interest. In the case where $\mu + \rho \in -C$ is in the interior, this group is known by the work of Schmid as presented in Lecture 5. In the case where $\mu + \rho$ is on the boundary and corresponds to a non-degenerate LDS there is the result by Williams [Wi2] extending that of Schmid. For the case $\mu = -\rho$ of a TDLDS the general result seems not to be known. The standard techniques include the use of the HSSS, and below we give an heuristic geometric argument that the differentials in the spectral sequence all vanish on the important Kostant class.

Three other topics that will be mentioned are

On the Stein property of quotients $\Gamma\backslash\mathcal{W}$ by a generally non-co-compact arithmetic group.

Relations between the Kato-Usui boundary components and the $G_{\mathbb{R}}$ and $K_{\mathbb{C}}$ orbit structures.

On the presumed non-algebraicity of quotients $\Gamma\backslash D$ when D is non-classical.

We now turn to a discussion of the above topics, beginning with the

Work of Friedman-Laza

We begin with the

Observation: *A Mumford-Tate domain $D = G_{\mathbb{R}}/H$ with trivial infinitesimal period relation is an Hermitian symmetric domain.*

Proof. We have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus (\underset{i\neq 0}{\oplus}\, \mathfrak{g}^{-i,i})$$

where at the reference point φ_0 of D the holomorphic tangent space

$$T_{\varphi_0} D \cong \underset{i>0}{\oplus}\, \mathfrak{g}^{-i,i}.$$

The assumption that the IPR is trivial is

$$\mathfrak{g}^{-i,i} = (0), \qquad\qquad i \geqq 2,$$

i.e.

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}.$$

From $[\mathfrak{g}^{-i,i}, \mathfrak{g}^{-j,j}] \subset \mathfrak{g}^{-(i+j),i+j}$ we infer that

$$\mathfrak{g}^{-1,1} \text{ and } \mathfrak{g}^{1,-1} = \overline{\mathfrak{g}^{-1,1}} \text{ are abelian sub-algebras of } \mathfrak{g}_{\mathbb{C}}.$$

We also have that $\mathfrak{g}^{-1,1}$ and $\mathfrak{g}^{1,-1}$ are direct sums of non-compact root spaces, while $\mathfrak{h}_0 \subseteq \mathfrak{k}_{\mathbb{C}}$. It follows that $\mathfrak{h}_0 = \mathfrak{k}_{\mathbb{C}}$, and in the Cartan decomposition

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$$

we have the $\mathfrak{k}$-invariant decomposition

$$\begin{aligned} \mathfrak{p}_{\mathbb{C}} &= \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} \\ &=: \mathfrak{p}^{+} \oplus \mathfrak{p}^{-} \end{aligned}$$

where $\mathfrak{p}^{\pm}$ are abelian sub-algebras. In particular, $G_{\mathbb{R}}/K$ has an $G_{\mathbb{R}}$-invariant, integrable almost complex structure. □

Next we let

$$\Phi : S \to \Gamma\backslash D$$

be a variation of Hodge structure where the global monodromy group $\Gamma \subset G_{\mathbb{Z}}$ is irreducible over $\mathbb{Q}$. By the structure theorem in Lecture 3 one may always reduce to this case. Also, without loss of generality we may assume that $\Phi(S)$ is closed in $\Gamma\backslash D$.[53]

[53]This is a standard result in VHS: period mappings extend across divisors around which the local monodromy group is finite.

THEOREM ([FL]): *Assume that the inverse image $\widetilde{S}$ of S in D is*

$$\widetilde{S} = \widehat{S} \cap D$$

where $\widehat{S} \subset \check{D}$ is an algebraic variety. Then $\widetilde{S} = D_H$ where D_H is an HSD equivariantly embedded in D.

Proof (sketch — details in [FL]): By the structure theorem from Lecture 3 we may assume that $D = G_{\mathbb{R}}/H$ is a Mumford-Tate domain where G is the Mumford-Tate group of a generic point of S. Then we have

- $\overline{\Gamma(\mathbb{Q})} = G$ (from Lecture 3);
- $\Gamma\widetilde{S} \subseteq \widetilde{S} \Rightarrow \Gamma(\mathbb{C})$ stabilizes $\widehat{S}$.

This is because $\widetilde{S} \subset \check{D}$ is defined by algebraic equations together with inequalities; in particular, $\widetilde{S}$ is Zariski dense in $\widehat{S}$.

- $\overline{\Gamma}(\mathbb{C}) = G_{\mathbb{C}}$ acts transitively on $\check{D}$ and stabilizes $\widehat{S} \subset \widehat{D} \Rightarrow \widehat{S} = \widehat{D}$.

Here one must take some care with connected components (loc. cit.).

- Then $\widetilde{S} = D$;
- Finally, D is a Mumford-Tate domain with trivial IPR; hence is an HSD. □

Lifting of the Kostant class:

This discussion is speculative and some of the issues raised are probably well known to experts.

Let μ be a weight such that $\mu + \rho \in -\overline{C}$, the closure of the anti-dominant Weyl chamber. Recall that $\mu + \rho \in -C$ is the situation when the L^2-cohomology and ordinary cohomology "line up" in the sense that the natural map (cf. [Sch2])

$$H^d_{(2)}(D, L_\mu) \to H^d(D, L_\mu)$$

is injective with dense image. It is also the situation where $H^d(D, L_\mu)$ is an irreducible Harish-Chandra module $V_{\mu+\rho}$ with infinitesimal character $\chi_{\mu+\rho}$. The issue we will discuss is the

Question: *Is there an $\mathfrak{n}$-cohomology interpretation of the surjectivity of the mapping*

$$H^d(D, L_\mu) \twoheadrightarrow H^d(Z, L_\mu)?$$

To explain this, we recall from the discussion of Kostant's theorem in the appendix to Lecture 7 that

$$H^d(Z, L_\mu) \cong \bigoplus_{\lambda \in \widehat{K}} W^{\lambda^*} \otimes H^d(\mathfrak{n}_K, W^\lambda)_{-\mu}.$$

By Kodaira-Serre duality and using $\omega_Z = L_{-2\rho_c}$

$$\begin{aligned} H^d(Z, L_\mu) &\cong H^0(Z, L_{-\mu} \otimes L_{-2\rho_c})^* \\ &\cong W^{-\mu-2\rho_c *} \end{aligned}$$

Thus

$$H^d(Z, L_\mu) \cong W^{-\mu-2\rho_c *} \otimes \kappa_\mu$$

where

$$\kappa_\mu \in H^d(\mathfrak{n}_K, W^{-\mu-2\rho_c})_{-\mu} \cong \mathbb{C}$$

is the *Kostant class*

$$\kappa_\mu = v_{\mu+2\rho_c} \otimes \bigwedge_{\alpha \in \Phi_c} \omega^{-\alpha}.$$

Note that the Kostant class determines the irreducible K-module $H^d(Z, L_\mu)$, since from it we know its highest weight $-\mu - 2\rho_c$.

We note that $H^d(Z, L_\mu)$ *is the lowest K-type of* $V_{\mu+\rho}$. That is, all the other irreducible K-summands W^λ in $V_{\mu+\rho}$ have highest weight $\lambda > -\mu - 2\rho_c$. When $\mu + \rho$ is non-singular this implies that the K-module $H^d(Z, L_\mu)$ determines the discrete series $H^d_{(2)}(D, L_\mu)$ and its associated Harish-Chandra module $V_{\mu+\rho}$.

By the results of Schmid from Lecture 5, in case $\mu + \rho$ is non-singular we have

$$\dim H^d(\mathfrak{n}, V^*_{\mu+\rho})_{-\mu} = 1.$$

We denote by $\sigma_\mu \in H^d(\mathfrak{n}, V^*_{\mu+\rho})$ a generator and refer to it as the *Schmid class.*

We next consider the diagram

$$\begin{array}{ccc} H^d(\mathfrak{n}_K, W^{-\mu-2\rho_c})_{-\mu} & \longrightarrow & H^d(\mathfrak{n}_K, V^*_{\mu+\rho})_{-\mu} \\ & & \uparrow \\ & & H^d(\mathfrak{n}, V^*_{\mu+\rho})_{-\mu} \end{array}$$

where the top arrow results from the inclusion $W^{-\mu-2\rho_c} \subset V^*_{\mu+\rho}$ and the vertical arrow from the inclusion $\mathfrak{n}_K \hookrightarrow \mathfrak{n}$. It seems quite plausible, but we do not have a proof, that *in the above diagram the Schmid class maps to the image of the Kostant class.* If so we have the following

Conclusion: *In the Hochschild-Serre spectral sequence for* $H^*(\mathfrak{n}, V^*_{\mu+\rho})_{-\mu}$ *we have*

$$\kappa_\mu \in H^d(\mathfrak{n}_K, V^*_{\mu+\rho})_{-\mu} = E_1^{0,d}.$$

Moreover, the differentials

$$d_1\kappa_\mu = d_2\kappa_\mu = d_3\kappa_\mu = \cdots = 0$$

and the Schmid class $\sigma_\mu \in H^d(\mathfrak{n}, V^*_{\mu+\rho})_{-\mu}$ *maps to the Kostant class* $\kappa_\mu \in E_\infty^{0,d}$.

We would like to have the same result when $\mu+\rho$ is singular; e.g., when $\mu=-\rho$ corresponds to a TDLDS V_0. For $S\mathcal{U}(2,1)$ by explicit calculations we have seen that the Kostant class

$$\kappa_{-\rho}=v_0\omega_{12}$$

can be lifted, where the explicit lifting to the Schmid class in $H^1(\mathfrak{n},V_0)_\rho$ is given by

$$\sigma_{-\rho}=v_0\omega_{12}+A\omega_{13}+B\omega_{32}$$

with

$$\begin{cases} A=X_{21}X_{13}v_0 \\ B=X_{21}X_{32}v_0. \end{cases}$$

For both $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$ it follows from Schmid's arguments in Appendix I to Lecture 9 that $\kappa_{-\rho}$ can be lifted.

In general, an heuristic could be this:

> *The Kostant class determines, and is determined by, the lowest K-type of V_0. The lowest K-type lifts naturally — i.e., geometrically — to V_0. Hence the Kostant class should lift, in fact to a Schmid class $\sigma_{-\rho}\in H^1(\mathfrak{n},V_0)_\rho$ that determines V_0.*

Shimura curves: This discussion pertains to the non-classical Mumford-Tate domains $D=G_{\mathbb{R}}/T$ when $G_{\mathbb{R}}=\mathcal{U}(2,1)_{\mathbb{R}}$ or $\mathrm{Sp}(4)_{\mathbb{R}}$.

A classical criterion for arithmeticity of modular forms f defined on $\Gamma(N)\backslash\mathcal{H}$ is that f should assume arithmetic values (suitably defined — cf. [Shi]) at CM points. For D as above, say in the $S\mathcal{U}(2,1)$ case, the interesting automorphic cohomology for $X=\Gamma\backslash D$ is $H^1(X,L_\mu)$ where $\mu+\rho$ is in the closure of the anti-dominant Weyl chamber. As noted above, in [GGK2] there is a discussion of how to "evaluate" classes $\eta\in H^1(X,L_\mu)$ at compatible pairs of CM points in $\Gamma\backslash\mathcal{W}$ where $\mathcal{W}\subset D\times D'$, and there an arithmeticity result is proved when $\eta=\mathcal{P}(\eta')$ where $\eta'\in H^0(X',L'_{\mu'})$ is an arithmetic Picard modular form.

However, it is more natural to evaluate η on algebraic *curves* $C\subset X$. We will briefly explain some of the issues involved in the $G=S\mathcal{U}(2,1)$ case. Let $\widetilde{G}\subset G$ be a $\mathbb{Q}$-algebraic subgroup such that $\widetilde{G}_{\mathbb{R}}=S\mathcal{U}(1,1)_{\mathbb{R}}$. For the Hermitian form $\mathrm{diag}(1,1,-1)$ on $\mathbb{Q}^3$ there are two evident such

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

A third is by taking the Hermitian form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on $\mathbb{C}^2$ and identifying $\mathrm{Sym}^2\,\mathbb{Q}^2=\mathbb{Q}^3$; this gives an embedding $S\mathcal{U}(1,1)\hookrightarrow S\mathcal{U}(2,1)$. The group $\widetilde{\Gamma}=\Gamma\cap\widetilde{G}$ is an arithmetic subgroup of Γ, and setting $\widetilde{T} = T_S \cap \widetilde{G}_{\mathbb{R}}$ where $T_S \subset S\mathcal{U}(2,1)_{\mathbb{R}}$ is the evident maximal torus, we have

$$\begin{array}{ccc} \widetilde{G}_{\mathbb{R}}/\widetilde{T} & \subset & G_{\mathbb{R}}/T_S \\ \downarrow & & \downarrow \\ \widetilde{\Gamma}\backslash\widetilde{G}_{\mathbb{R}}/\widetilde{T} & \subset & \Gamma\backslash G_{\mathbb{R}}/T_S. \end{array}$$

For the non-classical and classical complex structures D, D' on $G_{\mathbb{R}}/T_S$, and for $X = \Gamma\backslash D$, $X' = \Gamma\backslash D'$ we let

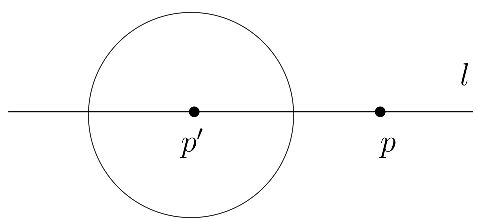
$$C \subset X,\ C' \subset X'$$

be the corresponding algebraic curves given by the two appearances of $\widetilde{\Gamma}\backslash\widetilde{G}_{\mathbb{R}}/\widetilde{T}$.

Definition: *We shall call C, C'* **Shimura curves***.*

Each of C, C' is a quotient of the unit discs Δ, Δ' equivariantly embedded in D, D' respectively.

Example: For a suitable choice of a fixed line l, we have



and

$$\begin{cases} \Delta' = l \cap \mathbb{B} \\ \Delta = l \cap \mathbb{B}^c. \end{cases}$$

An evident issue is the

Question: *Describe the maps $H^1(X, L_\mu) \to H^1(C, L_\mu)$.*

Since C is an algebraic curve defined over a number field, $H^1(C, L_\mu)$ has an arithmetic structure. As pointed out by Carayol, this could be a step towards defining an arithmetic structure on the automorphic cohomology group $H^1(X, L_\mu)$.

In order for this question to make sense one is led to the following consideration: Let μ, μ' be weights giving line bundles $L_\mu \to D$, $L'_{\mu'} \to D'$ whose cohomology groups are related by a Penrose transform $\mathcal{P}$ as in Lecture 9. Although we have not checked the details, it seems reasonable that there should be a commutative diagram

$$\begin{tikzcd} H^0(X', L'_{\mu'}) \arrow[r, "\mathcal{P}"] \arrow[d] & H^1(X, L_\mu) \arrow[d] \\ H^0(C', L'_{\mu'}) \arrow[r, dashed, "\widetilde{\mathcal{P}}"] & H^1(C, L_\mu) \end{tikzcd}$$

where the vertical arrows are restriction maps and the dashed horizontal arrow is a Penrose transform between algebraic curves as discussed in Lecture 2. The left vertical arrow is an arithmetic map; i.e., a map preserving arithmetic structures on the vector spaces. In order to use $\mathcal{P}$ to define an arithmetic structure on $H^1(X, L_\mu)$ one would need to know that

$$\widetilde{\mathcal{P}} \textit{ preserves arithmetic structures.}$$

More precisely, there should be a complex number δ such that

$$\widetilde{\mathcal{P}}\big(H^0(C', L'_{\mu'})_{\overline{\mathbb{Q}}}\big) \subseteq \delta H^1(C, L_\mu)_{\overline{\mathbb{Q}}}.$$

This seems plausible since on the correspondence space, $\widetilde{\mathcal{P}}$ is given by multiplication by a fixed cohomology class.

The map $H^1(X, L_{-\rho}) \to H^1(C, L_{-\rho})$ is of course particularly interesting. We note that on X, $L_{-\rho} = \omega_X^{1/2}$ and $\mathcal{O}_C(L_{-\rho}) = \omega_C^{1/2}$.

Realization by Penrose transforms:

We shall not attempt to formulate a precise question here. The issue is this: For flag manifolds $G_{\mathbb{C}}/B$ we have seen in the appendix to Lecture 7 that the different realizations of an irreducible $G_{\mathbb{C}}$-module as the cohomology groups of a homogenous line bundle over $G_{\mathbb{C}}/B$ are all related by Penrose transforms. This provides a geometric way of realizing the identifications provided by the BWB theorem.

There is a similar issue for Harish-Chandra modules constructed from the cohomology of homogeneous line bundles over flag domains. Here new subtleties arise. First, as we have seen in Lectures 8 and 9 the Penrose transform may be between flag domains with inequivalent complex structures. A second is that, whereas in the BWB case the cohomology groups will vanish in case $\mu + \rho$ is singular, this will not be so in the flag domain case. In fact, the case when $\mu + \rho$ is singular is a particularly interesting one. Finally, the topology on the cohomology groups may enter; e.g., the ordinary cohomology vanishes in degree bigger than $d = \dim_{\mathbb{C}} K/T$. To avoid this issue, it may be more convenient to ignore it and consider Penrose transforms on quotients by arithmetic groups where at least in the co-compact case, the topology does not matter.

This is also the case that is particularly interesting in representation theory. The first hypothesis to test might be:

$H^q(X, L_\mu)$ and $H^q(X', L'_{\mu'})$ are Penrose related if, and only if, the infinitesimal characters satisfy $\chi_{\mu+\rho} = \chi_{\mu'+\rho'}$.

Boundary components of Mumford-Tate domains:

As indicated above we shall briefly discuss Carayol's result about defining arithmeticity of automorphic cohomology in terms of "Fourier expansions" about an arithmetic boundary component. In the appendix to this lecture we have reproduced the relevant part of the unpublished IAS lecture notes that goes into the proof of Carayol's theorem.

Notational remark: In the work of Kato-Usui they use Σ, corresponding to a fan consisting of a family of nilpotent cones $\sigma \subset \mathfrak{g}$ satisfying certain conditions, to denote the particular extension $D \subset D_\Sigma$. Here we shall sometimes simply use the subscript "e," as in $D_e, X_e, H^1_e(X, L_\mu)$ to denote extensions of the corresponding object $D, X, H^1(X, L_\mu)$ to a larger object.

We recall notations from Lecture 3:

- $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$
- $V = \mathbb{Q}$-vector space of dimension 6
- $Q : V \otimes V \to \mathbb{Q}$ an alternating non-degenerate form
- $\mu : \mathbb{F} \to \operatorname{End}_{\mathbb{Q}}(V)$
- $V_{\mathbb{F}} = V_+ \oplus V_-$
- $\mathbb{H}(u, v) = -iQ(u, v), \quad u, v \in V_{+,\mathbb{C}}$
- $D = \begin{cases} \text{Mumford-Tate domain for PHS's of} \\ \text{weight } n = 3 \text{ with } h^{3,0} = 1,\ h^{2,1} = 2 \\ \text{and with generic Mumford-Tate group} \\ \widetilde{G} = \operatorname{Sp}(V, Q) \cap \operatorname{Res}_{\mathbb{F}/\mathbb{Q}}(\operatorname{GL}(V_+)). \end{cases}$

Then $\widetilde{G} \cong U(2,1)$ and $D = G_{\mathbb{R}}/T$ where $G_{\mathbb{R}} \cong \operatorname{SU}(2,1)$.

- $\Gamma \subset G$ an arithmetic subgroup; we shall eventually take $\Gamma = \mathcal{U}_{\mathbb{H}}(\mathcal{O}_{\mathbb{F}})$ where $\mathcal{O}_{\mathbb{F}}$ are the integers $\mathbb{F}$ and $\mathcal{U}_{\mathbb{H}}(\mathcal{O}_{\mathbb{F}}) =: \widetilde{G}(\mathcal{O}_{\mathbb{F}})$;
- $X = \Gamma\backslash D$.

For $L = L_{-k,0}$ or $L = L_{0,-k}$ we will define

- a "parabolic" subspace $H^1_e(X, L) \subset H^1(X, L)$.

Following Carayol, we will then define what it means for a class $\alpha \in H^1_e(X, L)$ to be arithmetic (in this case, this means defined over $\mathbb{F}^{ab}$ = maximal abelian extension of $\mathbb{F}$).

Theorem (Carayol): *$H^1_e(X, L)$ is generated by arithmetic classes.*

This is an analogue of the classical result that cuspidal modular forms are generated by ones whose Fourier expansions at the cusps are arithmetic; i.e., whose coefficients lie in a fixed number field.

Step one. In Lecture 9, we had diagrams

$$\begin{tikzcd} & \mathcal{W} \arrow[dl] \arrow[dr] & \\ D & & D' \arrow[d] \\ & & \Delta \end{tikzcd}$$

and the quotient by Γ (co-compact in Lecture 9 but definitely not assumed to be so here)

$$\begin{tikzcd} & \Gamma\backslash\mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X & & X' \arrow[d, "p"] \\ & & Y'. \end{tikzcd}$$

Using the [EGW] Penrose-type transform, this allowed one to give isomorphisms

$$\begin{aligned} H^1(X, L_\lambda) &\cong H^0(X', L_{\lambda'}) \\ &\cong H^0(Y', \widetilde{L}') \end{aligned}$$

where, for suitable λ and λ', $\widetilde{L}'$ is an explicit line bundle over the Picard modular surface Y'.

Step two. As noted above Kato-Usui ([KU]) have developed a theory of extensions, or partial compactifications,

$$D \subset D_\Sigma$$

of period domains with the property that period mappings

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \Gamma\backslash D \\ \cap & & \cap \\ \overline{S} & \xrightarrow{\overline{\Phi}} & \Gamma\backslash D_\Sigma \end{array}$$

extend as indicated. Here, S is a smooth quasi-projective variety, $\overline{S}$ is a smooth completion with $\overline{S}\backslash S := Z$ a normal crossing divisor with unipotent monodromies around the branches of Z (this may always be assumed), and Φ is a "period mapping" arising from a global variation of Hodge structure over S. In the classical case when D is an Hermitian symmetric domain, $\Gamma\backslash D_\Sigma$ is a toroidal compactification constructed by Mumford and his collaborators.

As remarked above, it is plausibly the case that the KU theory can be extended to Mumford-Tate sub-domains

$$D_M \subset D$$

of period domains. The issue is that the KU theory is based on the limiting mixed Hodge structures (LMHS's) constructed by Cattani-Kaplan-Schmid using the several variable nilpotent and SL_2-orbit theorems that give precise approximations to the period mapping in punctured polycylinders around points of Z (cf. [CKS]). The nilpotent orbits may be chosen to lie in D_M but this is, at least to me, not clear for the SL_2-orbits. It is OK, however, for the case $D = \mathrm{SU}(2,1)/T$ of interest here, and the analysis of the boundary

$$\partial D_\Sigma = D_\Sigma \backslash D$$

and of the quotients by Γ

$$\partial X_\Sigma = X_\Sigma \backslash X$$

is step two. It turns out that there is one "principal" boundary component $E' \subset \partial X_\Sigma$, which is a $\mathbb{C}^*$-bundle over a CM elliptic curve

$$E' \cong \mathbb{C}/\mathcal{O}_{\mathbb{F}}.$$

Step three. The correspondence space picture extends to

$$\begin{tikzcd}
 & \Gamma\backslash \mathcal{W}_\Sigma \arrow[dl] \arrow[dr] & \\
X_\Sigma & & X'_\Sigma \arrow[d] \\
 & & \widetilde{Y}'_\Sigma
\end{tikzcd}$$

where $Y'_\Sigma = (\Gamma\backslash\Delta) \cup \{\text{points}\}$ is compact.[54] Moreover, the [EGW] Penrose-type transforms extend to this situation to relate $H^1(X_\Sigma, L_\Sigma)$ and $H^0(\widetilde{Y}'_\Sigma, \widetilde{L}'_\Sigma)$. We define

$$H^1_e(X, L) = \ker\left\{H^1(X_\Sigma, L_\Sigma) \to H^1(\partial X_\Sigma, L_\Sigma)\right\}$$

and similarly for $H^0_e(\widetilde{Y}', \widetilde{L})$. Then there is a map

$$H^0_e(\widetilde{Y}', \widetilde{L}') \to H^1_e(X, L).$$

[54]This diagram is only a first approximation to the actual picture, in which Carayol uses a desingularization $\widetilde{Y}'_\Sigma \to Y'_\Sigma$ of Y_Σ with the elliptic curve E' appearing over the cusp in Y'_Σ.

The group $H^0_e(\widetilde{Y}', \widetilde{L}')$ may be identified as "parabolic Picard modular forms of weight k". Using results of Siegel and Shimura, this vector space has an arithmetic structure. In fact, after suitably trivializing the canonical bundle ω_Δ, and therefore also $\widetilde{L}'$, with coordinates $(x,y) \in \mathbb{C}^2$ the sections in $H^0_e(Y', \widetilde{L}')$ are given by

$$f(x,y) = \sum_{r \in \mathbb{N}^*} g_r(y) \exp\left(\left(\frac{-2\pi i r}{\beta_0}\right) x\right),$$

where $\beta \in \mathbb{F}$ is related to the logarithm N of monodromy, $\beta_0 = \operatorname{Im}\beta$ and the $g_r|y|$ are theta functions on E'. These theta functions are sections of an arithmetic line bundle over E', and thus it makes sense to say that $g_r(y)$ is "arithmetic". The arithmetic $f(x,y)$'s are shown to generate $H^0_e(Y', \widetilde{L}')$, and their images under the above map are then shown to generate $H^1_e(X, L)$.

On the Stein property of quotients $\Gamma\backslash\mathcal{W}$ where Γ is an arithmetic group

In Lecture 9 we have used the Penrose transform method applied to the diagram

$$\begin{array}{ccc} & \Gamma\backslash\mathcal{W} & \\ \swarrow & & \searrow \\ \Gamma\backslash D & & \Gamma\backslash D' \end{array}$$

where $\Gamma \subset G_\mathbb{R}$ is a co-compact discrete group. Just above we have discussed Carayol's use of it in the $S\mathcal{U}(2,1)$ case when Γ is an arithmetic group. In this special case he showed by a direct argument that $\Gamma\backslash\mathcal{W}$ is Stein ([C3]). In general one would like to show that

$$\Gamma\backslash\mathcal{W} \textit{ is Stein.}$$

In Lecture 6, and in the appendix to that lecture, we have discussed the construction of $G_\mathbb{R}$-invariant, strictly plurisubharmonic functions

$$f : \mathcal{U} \to \mathbb{R}.$$

For co-compact Γ these give an exhaustion function

$$f : \Gamma\backslash\mathcal{U} \to \mathbb{R},$$

proving that $\Gamma\backslash\mathcal{U}$ is Stein. From this and the fact that the fibres of

$$\Gamma\backslash\mathcal{W} \to \Gamma\backslash\mathcal{U}$$

are affine algebraic varieties one concludes that $\Gamma\backslash\mathcal{W}$ is Stein. This leads naturally to the question:

> *For f, constructed as explained in Lecture 6 from a strictly convex function on ω_0 where $\mathcal{U} = G_{\mathbb{R}} \exp(i\omega_0) \cdot u_0$, and for Γ an arithmetic group is the induced function $f : \Gamma\backslash\mathcal{U} \to \mathbb{R}$ an exhaustion function?*

For co-compact Γ the argument is of a general topological nature based on the observation that the projection

$$\Gamma\backslash\mathcal{U} \to G_{\mathbb{R}}\backslash\mathcal{U}$$

is a proper map. In general, to answer the above question it may be necessary to consider the geometry of a fundamental domain (Siegel set) for the action of Γ on D.

Relation between Kato-Usui boundary components and the $G_{\mathbb{R}}$ and $K_{\mathbb{C}}$ orbit structure[55]

For D a classical period domain there is in [KU] an extensive theory of extensions of D given by adding to the "boundary" families of limiting mixed Hodge structures D_σ associated to nilpotent cones $\sigma \subset \mathfrak{g}$. Here, the word "boundary" is in quotation marks as the D_σ's are not defined as subsets of the topological boundary $\partial D = \overline{D}\backslash D$ of D in the compact dual $\check{D}$. In fact, it is not even clear to me that there are natural maps $D_\sigma \to \partial D$.

As noted the [KU] theory has been worked out in case D is a period domain. It is reasonable to anticipate that the theory can be extended to general Mumford-Tate domains. We will assume this, and will in fact take for D a flag domain embedded in its compact dual $\check{D}$, which is a flag manifold. As has been discussed in Lecture 6, there is a rich orbit structure for the action of $G_{\mathbb{R}}$ on ∂D, and these orbits are in duality to $K_{\mathbb{C}}$ orbits in $\check{D}$. The general question, not precisely formulated here, is

> *Is there a relation between the KU boundary components D_σ and the $G_{\mathbb{R}}$-orbit structure of ∂D? If so, what is the relation between the D_σ's and the dual $K_{\mathbb{C}}$ orbits? In particular what is the Hodge theoretic interpretation of this relation?*

For a nilpotent cone σ one has that

$$\sigma \otimes \mathbb{R} \subset \mathfrak{A}$$

for a suitable abelian sub-algebra $\mathfrak{A} \subset \mathfrak{p}_{\mathbb{C}}$. In fact, for a suitable reference point in D we will have

$$\sigma \otimes \mathbb{R} \subset \mathfrak{A} \subset \mathfrak{g}_{\varphi}^{-1,1}.$$

[55] Some of what follows was suggested by Mark Green.

According to the theorem of Cattani-Kaplan [CKn], one has the very strong property that *the monodromy weight filtration associated to* N *in the interior of* $\sigma \otimes \mathbb{R}$ *is independent* of N. This is easily proved in case $\sigma \otimes \mathbb{R}$ is spanned by a strongly orthogonal set of root vectors X_{α_i}; this means that

$$\pm\alpha_i \pm \alpha_j \text{ is not a root for } i \neq j.$$

We do not know how general this is; i.e., does the Cattani-Kaplan property imply that $\sigma \otimes \mathbb{R}$ is spanned by the X_{α_i} for a strongly orthogonal set of roots? This observation suggests that there may be a relation between the KU theory and the group theoretic structure of the orbits of $G_{\mathbb{R}}$ acting on ∂D, and then perhaps also to the dual $K_{\mathbb{C}}$-orbits.

On the non-algebraicity of quotients $\Gamma\backslash D$ when D is non-classical

A theorem of Carlson-Toledo states that

> *For D a period domain for PHS's of even weight, excluding the case when $n = 2$ and $h^{2,0} = 1$ when D is an HSD, and for $\Gamma \subset G_{\mathbb{R}}$ discrete and co-compact, the quotient $\Gamma\backslash D$ does not have the homotopy type of a compact Kähler manifold.*

In particular, X is not a projective algebraic variety. An obvious question is

> *For D a non-classical Mumford-Tate domain and Γ an arithmetic group, can one prove that $\Gamma\backslash D$ is not a quasi-projective algebraic variety?*

Of course, in case G is of Hermitian type and Γ is co-compact the quotient will have the homotopy type of a compact Kähler manifold. So even in this case addressing the question above will necessitate new methods and seems to me to be an interesting issue in complex geometry, one that so far as I am aware has not been addressed in the literature.

Appendix to Lecture 10: Boundary components and Carayol's result

The contents of this appendix is reproduced from notes for an IAS seminar in March, 2011. In addition to discussing [C3], it gives an introduction, illustrated by examples, to the Kato-Usui theory.

1. Limiting mixed Hodge structures (LMHS's)

Recall that a **mixed Hodge structure** is given by the data $(V, W_\bullet, F^\bullet)$ where

- V is a $\mathbb{Q}$-vector space
- $\{0\} \subset W_0 \subset \cdots \subset W_m = V$ is the *weight filtration*
- $F^n \supset F^{n-1} \supset \dots$ is the *Hodge filtration* of $V_{\mathbb{C}}$

and where

- $F^p \cap W_{k,\mathbb{C}}/W_{k-1,\mathbb{C}} := F^p_k$ is a pure Hodge structure of weight k, meaning that
$$F^p_k \oplus \overline{F}{}^{k-p+1}_k \xrightarrow{\sim} \mathrm{Gr}_{k,\mathbb{C}}, \qquad 0 \leqq p \leqq k.$$

Thus, for $H^{p,q}_k := F^p_k \cap \overline{F}{}^q_k$ we have
$$\begin{cases} \mathrm{Gr}_{k,\mathbb{C}} = \underset{p+q=k}{\oplus} H^{p,q}_k \\ \overline{H}{}^{p,q}_k = H^{q,p}_k. \end{cases}$$

There is also the notion of a polarized mixed Hodge structure, whose formal definition in the case we need it will be given below.

The definitions of a pure Hodge structure and a mixed Hodge structure are of course motivated by geometry. If X is a smooth, complete complex algebraic variety, then $H^n(X,\mathbb{Q})$ has a (canonical) pure Hodge structure of weight n. If X is an arbitrary complex algebraic variety, then $H^n(X,\mathbb{Q})$ has a (canonical) mixed Hodge structure. If X is complete then the $m = n$ in the weight filtration. If X is projective, then the pure Hodge structure in the smooth case and mixed Hodge structure in the general case are polarized.

If we have a family X_t of smooth varieties degenerating to a generally singular variety X_0, then one might suspect that the pure Hodge structures $H^n(X_t,\mathbb{Q})$ have as limit a mixed Hodge structure related to $H^n(X_0,\mathbb{Q})$. This is in fact the case as we shall now briefly summarize.

First, we recall that if $N : V \to V$ is a nilpotent endomorphism with $N^{n+1} = 0$, there is a unique weight filtration $W_\bullet(N)$ such that
$$\begin{cases} N : W_k(N) \to W_{k-2}(N), \text{ and the induced maps} \\ N^k : \mathrm{Gr}_{n+k}(N) \xrightarrow{\sim} \mathrm{Gr}_{n-k}(N) \text{ are isomorphisms.} \end{cases}$$

One defines

$$\begin{cases} W_0(N) &= N^n(V) \\ W_{2n-1}(N) &= \ker N^n, \end{cases}$$

and then for $\overline{V} = V/W_0$ and $\overline{N} : \overline{V} \to \overline{V}$ induced by N and with $\overline{N}^n = 0$, we have

$$W_0(\overline{N}), \qquad W_{2n-3}(\overline{N})$$

and for $V \xrightarrow{\pi} \overline{V}$ we set

$$\begin{cases} W_1 &= \pi^{-1}(W_0(\overline{N})) \\ W_{2n-2} &= \pi^{-1}(W_{2n-3}(\overline{N})) \end{cases}$$

and proceed inductively.

Suppose now we are given (V, Q) and a nilpotent endomorphism $N \in \mathrm{End}_Q(V)$.

Definition: *A* **limiting mixed Hodge structure** *(LMHS) is given by* $(V, Q, W_\bullet(N), F^\bullet)$ *where*

(i) $(V, W_\bullet(N), F^\bullet)$ *is a mixed Hodge structure,*

(ii) N *is a morphism of mixed Hodge structures of type* $(-1,-1)$*; in particular*

$$N(F^p) \subset F^{p-1}, \text{ [56]}$$

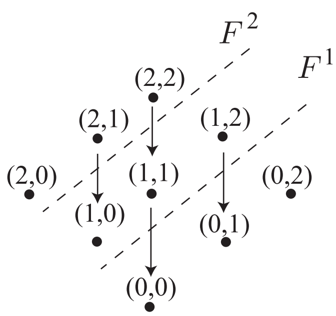
and

(iii) *on* $\mathrm{Gr}_{n+k,\mathrm{prim}} := \ker N^{k+1}$*, the bilinear forms*

$$Q_k(u,v) = Q(N^k u, v) \qquad u, v \in \mathrm{Gr}_{n+k,\mathrm{prim}}$$

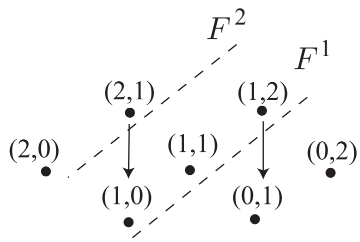
define a polarized Hodge structure of weight $n+k$ *(here there may be a sign).*

One picture of a LMHS is given by a Hodge diamond which looks like the cohomology of a compact Kähler manifold with N playing the role of the Lefschetz operator but going in the opposite direction. For $n = 2$ we have the possibilities



$$\begin{cases} N^3 = 0, \quad N^2 \neq 0 \\ (1,1) = N(2,2) \oplus (1,1)_{\mathrm{prim}} \end{cases}$$

[56] In the classical case of weight $n = 1$ PHS's (families of abelian varieties), the condition $N(F^p) \subset F^{p-1}$ is automatically satisfied, but this is *not* the case in the non-classical case, and in fact is the controlling feature in the non-classical theory.



$$\begin{cases} N^2 = 0, \quad N \neq 0 \\ \mathrm{Gr}_2 = \mathrm{Gr}_{2,\mathrm{prim}}. \end{cases}$$

Example 1: When $\dim V = 5$, $\dim F^2 = 2$, as in the case of PHS's of weight two with $h^{2,0} = 2$, $h^{1,1} = 1$, the only possibility is when $N^3 = 0$ and we have the picture

$$\begin{array}{ccc} & \overset{(2,2)}{\bullet} & \\ & \downarrow & \\ \overset{(2,0)}{\bullet} & \overset{(1,1)}{\bullet} & \overset{(0,2)}{\bullet} \\ & \downarrow & \\ & \overset{(0,0)}{\bullet} & \end{array} \qquad \begin{cases} N^3 = 0, \quad N^2 \neq 0 \\ \mathrm{Gr}_2 = \mathrm{Gr}_{2,\mathrm{prim}}. \end{cases}$$

Then $\mathrm{Gr}_2 \cong \mathbb{Q}(-1) \oplus H$ where H is a PHS of weight two with $h^{2,0} = 1$. We will return to this example below.

Another way of displaying this is

$$\begin{array}{ccccc} \mathbb{Q}(-2) & & \mathbb{Q}(-1) & & \mathbb{Q} \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & \bullet & & \\ & & H & & \end{array}$$

where $H_{\mathbb{C}} = H^{2,0} \oplus H^{0,2}$ is a weight 2 PHS.

Example 2: In this case we shall take $N \in \mathfrak{g}$ where $G = U(2,1)$ viewed as a $\mathbb{Q}$-algebraic subgroup of $\mathrm{Aut}(V, Q)$ where V is a 6-dimensional $\mathbb{Q}$-vector space with an action of $\mathbb{F}$ as above. Then because of the picture of the PHS's, $N^3 = 0$ and the possibilities are

$$\text{(A)} \qquad \begin{array}{cc} (3,2) & (2,3) \\ \bullet & \bullet \\ \downarrow & \downarrow \\ (2,1) & (1,2) \\ \bullet & \bullet \\ \downarrow & \downarrow \\ (1,0) & (0,1) \\ \bullet & \bullet \end{array} \qquad N^3 = 0, N^2 \neq 0$$

(B)

$$\begin{tikzcd}[column sep=small]
(3,1)\ \bullet \arrow[dd] & & & (1,3)\ \bullet \arrow[dd] \\
& (2,1)\ \bullet & (1,2)\ \bullet & \\
(2,0)\ \bullet & & & (0,2)\ \bullet
\end{tikzcd}
\qquad
\begin{array}{l}
N^2=0, \quad N \neq 0 \\
\mathrm{Gr}_3 = \mathrm{Gr}_{3,\mathrm{prim}}
\end{array}$$

(C)

$$\begin{tikzcd}[column sep=small]
& (2,2)^{\oplus 2}\ \bullet \arrow[dd] & \\
(3,0)\ \bullet & & (0,3)\ \bullet \\
& (1,1)^{\oplus 2}\ \bullet &
\end{tikzcd}
\qquad
\begin{array}{l}
N^2=0, \quad N \neq 0 \\
\mathrm{Gr}_2 = \mathrm{Gr}_{3,\mathrm{prim}}.
\end{array}$$

The alternative diagrams for these given above are

(A)

$$\underset{\bullet}{H(-2)} \longrightarrow \underset{\bullet}{H(-1)} \longrightarrow \underset{\bullet}{H}$$

where $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$

(B)

$$\begin{array}{ccc}
\underset{\bullet}{J(-1)} & \longrightarrow & \underset{\bullet}{J} \\
& \underset{\bullet}{K} &
\end{array}
\qquad
\begin{cases}
J = J^{2,0} \oplus J^{0,2} \\
K = K^{3,0} \oplus K^{0,3}
\end{cases}$$

(C)

$$\begin{array}{ccc}
Q(-2)^{\oplus 2} & \longrightarrow & \mathbb{Q}(-1)^{\oplus 2} \\
& \underset{\bullet}{K} &
\end{array}$$

We shall also return to this example below. Without the requirement that $N \in \mathfrak{g}$ as above, there are additional possibilities for LMHS's; e.g. with $N^3 \neq 0$.

Remark. In Lecture 9 we discussed the third example of the non-classical $\mathrm{Sp}(4)/T$. The LMHS's are worked out in detail in [GGK0]; they have a rich arithmetic structure.

2. Kato-Usui boundary components (nilpotent orbits)

We asume given a $\mathbb{Q}$-vector space V, weight n and a non-degenerate bilinear form $Q : V \otimes V \to \mathbb{Q}$ with $Q(u,v) = (-1)^n Q(v,u)$, and a set of Hodge numbers $h^{p,q} = h^{q,p}$ for $p+q=n$ and with $\sum_{p,q} h^{p,q} = \dim V$. There is then a period domain $\widetilde{D}$ and we let $D \subset \widetilde{D}$ be a Mumford-Tate domain consisting of PHS's whose generic member has Mumford-Tate group G where $D = G_{\mathbb{R}}/H$. Finally, we let $w \in \mathbb{C}$ and for $\operatorname{Im} w > 0$ we set

$$q = e^{2\pi i w}.$$

Definition: *A* **nilpotent orbit** $(V, Q, N, F^\bullet)$ *is given by a nilpotent element* $N \in \mathfrak{g}$ *and a point* $F^\bullet \in \check{D}$ *satisfying*

$$\begin{cases} \text{(i)} \qquad \exp(wN)F^\bullet \in D \ \text{ for } \ \operatorname{Im} w \gg 0 \\ \text{(ii)} \qquad N(F^p) \subset F^{p-1}. \end{cases}$$

We set $T = \exp N \in G$. In practice there will frequently, but not always, be a lattice $V_{\mathbb{Z}} \subset V$ such that $T \in G_{\mathbb{Z}} := \{g \in G : g(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}}\}$. Since

$$\exp((w+1)N)F^\bullet = T \exp(wN) F^\bullet$$

and $T \cdot D = D$, condition (i) depends only on w with $|\operatorname{Re} w| \leqq 1/2$. Recalling that

$$T_{F^\bullet} \check{D} \subset \oplus \operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p)$$

we set

$$T^h_{F^\bullet} \check{D} = \left\{ \xi \in T_{F^\bullet} \check{D} : \xi(F^p) \subseteq F^{p-1}/F^p \right\}.$$

This gives a $G_{\mathbb{C}}$-invariant sub-bundle

$$T^h \check{D} \subset T \check{D}$$

and condition (ii) means exactly that the tangents to the orbit $\exp(wN)F^\bullet$ are in $T^h\check{D}$. Rescaling so that (i) is satisfied for $\operatorname{Im} w > 0$ and setting $\Gamma_N = \{T^k\}_{k \in \mathbb{Z}} \subset G$, we have a map from the disc $\Delta^*_d = \{q : 0 < |q| < 1\}$

$$F^\bullet : \Delta^*_d \to \Gamma_N \backslash D$$

where $F^\bullet(q) = \exp(wN) \cdot F^\bullet$. Condition (ii) says exactly that *the map* $(*)$ *is a variation of Hodge structure* (VHS). Results of Schmid give that any VHS over Δ_d^* may be approximated by a nilpotent orbit (*nilpotent orbit theorem*).

Example: It is well-known that a degenerating family of elliptic curves over Δ_d has a period point in the upper-half-plane $\mathcal{H}$ given by

$$m\,\frac{\log q}{2\pi i} + f(q)$$

where $m \in \mathbb{Z}^{\geqq 0}$ and $f(q)$ is holomorphic in Δ_d. The approximating nilpotent orbit is given by taking $f(0)$ to be constant in the above expression.

In fact, a much deeper and more precise result than the nilpotent orbit theorem was proved by Schmid. Namely, for $T_0 \in \mathrm{SL}_2(\mathbb{Z})$ given by $T_0(w) = w+1$, $\Gamma_0 = \{T_0^k\}_{k\in\mathbb{Z}}$ and identifying

$$\Gamma_0\backslash\mathcal{H} = \Delta_d^*,$$

the nilpotent orbit is itself approximated by an *equivariant VHS*

$$(**) \qquad F^\bullet : \Delta_d^* \to \Gamma_N\backslash\widetilde{D}$$

induced by a representation

$$\begin{cases} r : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(V_{\mathbb{R}}, Q) \\ r(T_0) = T. \end{cases}$$

Note that $(**)$ maps to $\Gamma_N\backslash\widetilde{D}$; it seems not to be known if we can keep the image in $\Gamma_N\backslash D$.

Using Schmid's result it follows that

> *Associated to a nilpotent orbit* $(V, Q, N, F^\bullet)$ *is a limiting polarized mixed Hodge structure* $(V, Q, W_\bullet(N), F^\bullet)$*. Conversely, associated to a polarized LMHS* $(V, Q, W_\bullet(N), F^\bullet)$*, there is a nilpotent orbit whose associated LMHS is the given one.*

There is a several variable version of the above, due to Cattani-Kaplan-Schmid [CKS], where Δ_d^* is replaced by $(\Delta_d^*)^k$ and T by commuting nilpotent monodromies $T_1, \ldots, T_k$ with logarithms $N_i = \log T_i$. Set

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{N_1, \ldots, N_k\} \subset \mathfrak{g}_{\mathbb{R}}$$

and for $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$ set

$$N_\lambda = \sum_i \lambda_i N_i.$$

Then each N_λ is nilpotent, and a basic result (conjectured by Deligne and proved by Cattani-Kaplan) is

the weight filtration $W_\bullet(N_\lambda)$ is independent of λ.

This result paved the way for the several variable SL_2-orbit theorem in [CKS], which in turn provided the foundation for the Kato-Usui theory.

In general, we let $\sigma \subset \mathfrak{g}_\mathbb{R}$ be a nilpotent cone as above where

$$\begin{cases} N_i^{n+1} &= 0 \\ [N_i, N_j] &= 0. \end{cases}$$

Setting $\sigma_\mathbb{C} = \mathrm{span}_\mathbb{C}\{N_1, \dots, N_k\} = \sigma \otimes_\mathbb{R} \mathbb{C} \subset \mathfrak{g}_\mathbb{C}$, a *$\sigma$-nilpotent orbit $Z \in \check{D}$* is given by

$$Z = \exp(\sigma_\mathbb{C}) \cdot F^\bullet, \qquad F^\bullet \in \check{D}$$

where the conditions corresponding to (i), (ii) above are satisfied.

Now let Σ be a *fan* (not defined here) of nilpotent cones in $\mathfrak{g}_\mathbb{R}$ and set

$$D_\Sigma = \{(\sigma, Z) : \sigma \in \Sigma \text{ and } Z \text{ is a } \sigma\text{-nilpotent orbit}\}/(\text{rescalings}).$$

Then $D \subset D_\Sigma$ by $F^\bullet \to (0, F^\bullet)$ (trivial nilpotent orbit). Next let $\Gamma \subset G$ be an arithmetic group. There are natural conditions, also not spelled out here, that Σ be *compatible* and *strongly compatible* with Γ (cf. [KU]). In this case we may form

$$X_\Sigma := \Gamma \backslash D_\Sigma.$$

Kato-Usui prove that X_Σ *has the structure of a Hausdorff log-analytic variety with slits*, and that any VHS

$$(\Delta_d^*)^k \to \Gamma_\sigma \backslash D,$$

extends to a morphism of log analytic varieties

$$(\Delta_d)^k \to X_\Sigma.$$

In fact, for each $\sigma \in \Sigma$ there is a boundary component that we denote here by ∂D_σ and which is constructed as follows: First, we let

$$D_\sigma^\# = D_{\{\text{faces of } \sigma\}}.$$

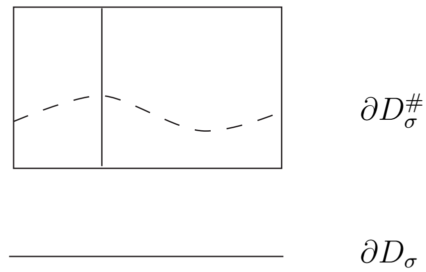
We remark that $\{\text{faces of } \sigma\}$ is a fan. Here the subset of $D_\sigma^\#$ consisting of (σ, Z) gives the orbit as a subset of $\check{D}$

$$Z = \exp\left(\sum_j w_j N_j\right) \cdot F^\bullet;$$

we obtain the same orbit by rescaling $w_j \to w_j + c_j$. We set

$$D_\sigma = D_\sigma^\# / \{\text{rescalings}\}.$$

The picture over $\partial D_\sigma^\# =: D_\sigma^\# \backslash D$ is something like



where the fibres are $\mathbb{C}^k$'s, and where in practice we describe D_σ by taking a slice as pictured (somtimes referred to as *normalizing* the nilpotent orbit).

Next, we let

$$\Gamma_\sigma = \{\text{normalizer of } \sigma \text{ in } \Gamma\}\,.$$

Then Γ_σ acts on $D_\sigma^\#$, and this action preserves the fibres in the above picture but in general will *not* preserve the slice. We let $D_\sigma = D_\sigma^\#/(\text{rescalings})$. As a set $D_\sigma = \partial D_\sigma \sqcup D$ (disjoint union). In the cases of interest here the topology on D_σ will be discussed below. The quotient

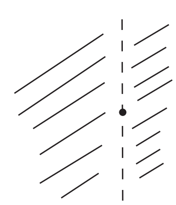
$$\Gamma_\sigma \backslash \partial D_\sigma =: \partial X_\sigma$$

is the boundary component corresponding to the nilpotent cone σ. As a set we have

$$\partial X_\Sigma = \bigcup_{\sigma \in \Sigma} \partial X_\sigma.$$

We remark that X_Σ is a log analytic variety with slits. A log-analytic variety with slits looks locally something like

$$\mathbb{C}^2 \backslash (\{0\} \times \mathbb{C}^*)$$



This condition is forced by the condition $N_i(F^p) \subseteq F^{p-1}$ if one wants a *separated* extension of $X = \Gamma\backslash D$ to which VHS's extend. It is *not* present in the classical case when D is an Hermitian symmetric domain and where the Kato-Usui construction reduces to the toroidal compactification. But it is present in the situation studied by Carayol.

Toy example: Before turning to the determination of the nilpotent orbits in our two "running" examples, we consider the case when

$$\begin{aligned}
&D = \mathcal{H} \text{ is the upper-half-plane} \\
&\cap \\
&\check{D} = \mathbb{P}^1 \ \text{ with points } \ F = \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}$$

where

$$\mathcal{H} = \{F : i(\bar{x}y - x\bar{y}) > 0\}\,.$$

As in Lecture 1 we normalize $F \in \mathcal{H}$ by taking $F = \left[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right]$, $\operatorname{Im}\tau > 0$.

We take $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$\exp(wN)F = \begin{bmatrix} x + wy \\ y \end{bmatrix}.$$

If $\exp(wN)F \in \mathcal{H}$, then $y \neq 0$ and we may take $y = 1$. The nilpotent orbit is then

$$Z = \left\{ \begin{bmatrix} x + w \\ 1 \end{bmatrix} : w \in \mathbb{C} \right\} = \mathbb{P}^1 \Big\backslash \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which we may think of as $\mathbb{P}^1 \backslash \infty$. There is thus one nilpotent orbit. By rescaling we may take $F = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = 0 \in \mathbb{C} \subset \mathbb{P}^1$. The normalized nilpotent orbit is then

$$w \to \begin{bmatrix} w \\ 1 \end{bmatrix}$$

so that $\{w : \operatorname{Im} w > 0\} \to \mathcal{H} \subset \mathbb{P}^1$.

Example 1: This is the case of $D = \mathrm{SO}(4,1)/U(2)$, the period domain for weight 2 polarized Hodge structures with $h^{2,0} = 2$, $h^{1,1} = 1$. We will analyze the nilpotent orbits in this case. To get a sense of what to expect we recall that the LMHS's are of the form

$$(*) \qquad \mathbb{Q}(-2) \xrightarrow{N} \mathbb{Q}(-1) \xrightarrow{N} \mathbb{Q}$$

$$H$$

where H is a PHS of weight 2 with $H_{\mathbb{C}} = H^{2,0} \oplus \overline{H}^{2,0}$ where $\dim H^{2,0} = 1$. A picture is

$$\begin{tikzcd}[row sep=large]
\mathrm{Gr}_4^W = & \mathbb{Q}(-2) \arrow[d, leftrightarrow, bend right, "\alpha"'] \arrow[dr, leftrightarrow, bend left, "\beta"] \arrow[dd, leftrightarrow, bend left=80, "\gamma"] & \\
\mathrm{Gr}_2^W = & \mathbb{Q}(-1) + & H \\
\mathrm{Gr}_0^W = & \mathbb{Q}(0) \arrow[u, leftrightarrow, bend left, "\alpha^*"] \arrow[ur, leftrightarrow, bend right, "\beta^*"'] &
\end{tikzcd}$$

One may replace the $\mathbb{Q}$'s by $\mathbb{Z}$'s by working carefully. The extension classes corresponding to α and its dual α^* may be normalized out by choosing the right point $F^\bullet$ in the nilpotent orbit. One of the remaining extension classes β is in

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2), H) \cong H^{0,2}/H.$$

If we have a lattice $V_\mathbb{Z}$ with $V = \mathbb{Q} \otimes_\mathbb{Z} V_\mathbb{Z}$, the extension class will be in $H^{0,2}/H_\mathbb{Z}$ where $H_\mathbb{Z}$ is the image of $V_\mathbb{Z} \cap W_2$ in H. If we have $H_\mathbb{Z} \cong \mathbb{Z}^2$ and $H^{0,2} = \mathbb{C}v_\tau$ where $\tau = (\tau, 1)$, then $\operatorname{Im}\tau \neq 0$. If

$$Q_H : H \otimes H \to \mathbb{Q}$$

is the polarizing form, then $Q(v_\tau, v_\tau) = 0$ gives a quadratic equation for τ with rational coefficients. Thus

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), H) \ \textit{is a CM elliptic curve.}$$

The other extension class is $\gamma \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), \mathbb{Z}) \cong \mathbb{C}^*$. Thus the boundary component is 2-dimensional.[57]

To carry out the calculations we will use the following notations:

- $$Q = \left(\begin{array}{c|cc} -I_3 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right).$$

- $$N_a = \begin{pmatrix} 0 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & a_3 \\ a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

- $N_a \in \mathrm{so}(4,1)$ and $N_a \in \mathfrak{g} = \mathrm{so}(4,1)$ is defined over $\mathbb{Q}$ if the $a_j \in \mathbb{Q}$.
- $[N_a, N_b] = 0$.

[57]There are actually two CM elliptic curve boundary components, depending on a choice of $\pm$ sign below.

- $$N_a^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a^2/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a^2 = a \cdot a.$$

- $N_a^3 = 0$.

- $$\exp N_a = \begin{pmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ a_1 & a_2 & a_3 & 1 & -a^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The standard basis for $\mathbb{Q}^5$, written as column vectors, is $e_1, \dots, e_5$. Then

$$\begin{cases} Q(e_i, e_j) = -\delta_{ij} & 1 \leqq i,\ j \leqq 3 \\ Q(e_4, e_5) = 1 \\ \text{all other } Q(e_\alpha, e_\beta) = 0. \end{cases}$$

LEMMA: (i) *Any nilpotent cone can be conjugated into the above.* (ii) *If σ gives a nilpotent orbit then* $\dim \sigma = 1$.

For simplicity of calculation we shall take $a_1 = 1$, $a_2 = a_3 = 0$. Then

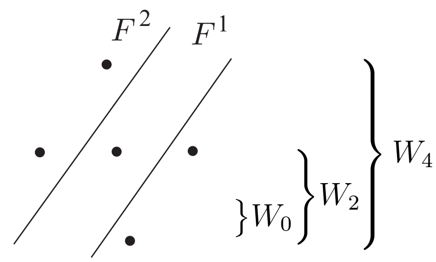
$$\begin{cases} Ne_1 = e_4 \\ Ne_5 = e_1 \Rightarrow N^2 e_5 = e_4 \\ \text{all other } Ne_\alpha = 0. \end{cases}$$

The weight filtration is then

- $$\begin{array}{l} W_0 = \{e_4\} \\ \cap \\ W_2 = \{e_1, e_2, e_3, e_4\} \end{array}$$

where here $\{\ \}$ denotes span over $\mathbb{Q}$.

We now determine the conditions on $F = F^2 = \operatorname{span}\{f_1, f_2\}$ over $\mathbb{C}$ so that $(V, Q, W_\bullet(N), F)$ gives a LMHS. We know the picture of the LMHS must be



Since $W_2 = \operatorname{span}\{e_1, e_2, e_3, e_4\}$, if $f_1 \in F^2$ projects to a non-zero element in W_4/W_2 its e_5-component must be non-zero. This is the topmost dot above. Thus we may take

$$f_1 = \begin{pmatrix} v_1 \\ a \\ 1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} \in \mathbb{C}^3.$$

Adding a multiple of f_1 to f_2, we may take

$$f_2 = \begin{pmatrix} v_2 \\ b \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} \in \mathbb{C}^3.$$

The bilinear relations $Q(F, F) = 0$ are

$$\begin{cases} 0 = Q(f_1, f_1) = -v_1^2 + 2a \\ 0 = Q(f_2, f_2) = -v_2^2 \\ 0 = Q(f_1, f_2) = -v_1 \cdot v_2 + b \end{cases}$$

which give

$$\begin{cases} a = v_1^2/2 \\ b = v_1 \cdot v_2 \\ v_2^2 = 0. \end{cases}$$

Since $\dim F = 2$ these imply that $v_2 \neq 0$.

Next, we determine the conditions that

$$N(F^2) \subseteq F^1 \iff Q(N(F), F) = 0.$$

Using

$$\begin{cases} Nf_1 = v_{12}e_4 + e_1 \\ Nf_2 = v_{21}e_4, \end{cases}$$

the equations $Q(N(F),F)=0$ give $v_{21}=0$. From $v_2^2=0$ and $v_2\neq 0$, we may normalize to have $v_{22}=1$, and then by the middle bilinear relation above $v_{23}=\pm i$. We take the $+$ sign and set $v_1=z$ to have

$$f_1=\begin{pmatrix} z\\ z^2/2\\ 1\end{pmatrix},\qquad f_2=\begin{pmatrix}0\\1\\i\\z_2+iz_3\\0\end{pmatrix}.$$

We next determine the conditions that $F(w)=:\exp(wN)F\in D$ for $\operatorname{Im} w\gg 0$. We have

$$f_1(w)=\begin{pmatrix} z_1+w\\ z_2\\ z_3\\ wz_1+z^2/2+w^2/2\\ 1\end{pmatrix},\qquad f_2(w)=\begin{pmatrix}0\\1\\i\\z_2+iz_3\\0\end{pmatrix}.$$

Setting $|z|^2=z\cdot\bar z$ the matrix $Q(f_i(w),\overline{f_j(w)})$ is

$$\begin{pmatrix} -|w|^2-|z|^2-2\operatorname{Re} z_1\bar w+\operatorname{Re}(2wz_1+z^2+w^2) & -2i\operatorname{Im}(z_2)\\ 2i\operatorname{Im}(z_2) & -2\end{pmatrix}.$$

Conclusion. *For* $\operatorname{Im} w\geqq C_1(z)$

$$\|Q(f_i(w),\overline{f_j(w)})\|<0.$$

Remark. The conclusion becomes very transparent if we do the following. First, we may rescale to have $z_1=0$. Next, for simplicity of notation we take $z_2=u$, $z_3=v$ so that

$$f_1=\begin{pmatrix}0\\u\\v\\ \frac{u^2+v^2}{2}\\1\end{pmatrix},\qquad f_2=\begin{pmatrix}0\\1\\i\\u+iv\\0\end{pmatrix}.$$

Since for any $\zeta\in\mathbb{C}$

$$-|\zeta|^2+\operatorname{Re}\zeta^2=-2(\operatorname{Im}\zeta)^2$$

and the above matrix is

$$Q=-2\begin{pmatrix}(\operatorname{Im} s)^2+(\operatorname{Im} u)^2+(\operatorname{Im} v)^2 & 2iv\\ -2i\bar v & 1\end{pmatrix}$$

so that $Q<0$ unless $\operatorname{Im} s=\operatorname{Im} u=\operatorname{Im} v=0$.

For later reference, we observe that the basis for $F^1 = F^\perp$ may be taken to be

$$\underbrace{\underbrace{f_1 = \begin{pmatrix} 0 \\ u \\ v \\ \frac{u^2+v^2}{2} \\ 1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 \\ 1 \\ i \\ u+iv \\ 0 \end{pmatrix}}_{F=F^2}, \ f_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{F^1}$$

Then

- F^2 modulo $W_{2,\mathbb{C}}$ is spanned by e_1.
- $\begin{cases} F^2 \cap W_{2,\mathbb{C}}/W_{0,\mathbb{C}} \text{ is spanned by} \\ e_2 + ie_3 + (u+iv)e_4. \end{cases}$
- $F^1 \cap W_{2,\mathbb{C}}/W_{0,\mathbb{C}}$ spanned by $F^2 \cap W_{2,\mathbb{C}}/W_{0,\mathbb{C}}$ together with e_1.
- $N(f_1) = f_3$.
- $\mathrm{Gr}_{2,\mathrm{prim}}$ is spanned by e_2, e_3 and $F^2\mathrm{Gr}_{2,\mathrm{prim}}$ is spanned by $e_2 + ie_3$.
- The extension class in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2), H)$ is given by noting that
$$H^{2,0} = \mathbb{C}(e_2 + ie_3),$$
so that the extension class is represented by $\binom{u}{v} \in \mathbb{C}^2/\mathbb{C}\binom{1}{i}$.
- If we have a lattice, then equivalent extensions are given by
$$\begin{cases} u \to u+m \\ v \to v+n \end{cases}$$
where $m, n \in \mathbb{Z}$.

Example 2.[58] Recall that we have

$$\begin{aligned} \mathfrak{g}_{\mathbb{R}} &\cong \left\{ g \in \mathrm{End}(V_{+,C}) : {}^t[g]_\gamma[\mathbb{H}]_\gamma + [\mathbb{H}]_\gamma[g]_\gamma = 0 \right\} \\ &= \left\{ \begin{pmatrix} A & B & C \\ D & E & \overline{B} \\ G & \overline{D} & -\overline{A} \end{pmatrix} : \begin{array}{c} C, E, G \in i\mathbb{R} \\ A, B, D \in \mathbb{C} \end{array} \right\}. \end{aligned}$$

[58]This exposition is based in part on notes on [C3] written by Matt Kerr.

Here we have chosen an $\mathbb{F}$-basis for W_+ so that the matrix of $\mathbb{h}$ is $\left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$.

LEMMA: *Any nilpotent cone can be conjugated in $G_{\mathbb{R}}$ to be of the form*

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

We will see that there can be 2-dimensional spaces of commuting nilpotent matrices of this form, but due to the condition $N_\lambda(F^p) \subset F^{p-1}$ there are only 1-dimensional nilpotent cones σ that give nilpotent orbits. Let

$$\sigma \in \left\{ \begin{pmatrix} 0 & \alpha & ib \\ 0 & 0 & \overline{\alpha} \\ 0 & 0 & 0 \end{pmatrix} : \begin{array}{c} b \in \mathbb{R} \\ \alpha \in \mathbb{C} \end{array} \right\} =: \sigma_0.$$

We note that

$$\begin{aligned} \exp(\sigma_0) &= \left\{ \begin{pmatrix} 1 & \alpha & ib + \frac{|\alpha|^2}{2} \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} : \beta + \overline{\beta} = |\alpha|^2 \right\}. \end{aligned}$$

For $N_1, N_2 \in \sigma_0$

$$[N_1, N_2] = \begin{pmatrix} 0 & 0 & \alpha_1\overline{\alpha}_2 - \alpha_2\overline{\alpha}_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Recalling our notation from "9" and where we do not differentiate between non-zero points in $\mathbb{C}^3$ and their images in $\mathbb{P}^2$

$$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } l = (u, v, w)$$

and their images in $\mathbb{P}^2$ and $\mathbb{P}^{2*}$, we have for $j = 1, 2$

$$N_j p = \begin{pmatrix} \alpha_j y + ib_j z \\ \overline{\alpha}_j z \\ 0 \end{pmatrix}.$$

Since we must have

$$\exp(wN_j)p \in \mathbb{B}^c = \mathbb{P}^2 \backslash \mathbb{B}, \qquad \operatorname{Im} w \gg 0$$

because of the form of $\mathbb{H}_\gamma$ this cannot happen if $z = 0$. Thus we may assume that

$$p = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Then the condition $N_j(F^3) \subseteq F^2$, which translates into

$$\langle l, N_j p \rangle = 0,$$

forces

$$\begin{pmatrix} a_1 y + ib_1 \\ \overline{\alpha}_1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_2 y + ib_2 \\ \overline{\alpha}_2 \\ 0 \end{pmatrix}$$

to be dependent. Thus

$$\begin{aligned} 0 &= \alpha_1 \overline{\alpha}_2 y + ib_1 \overline{\alpha}_2 - \overline{\alpha}_1 \alpha_2 y - ib_2 \overline{\alpha}_1 \\ &= (\alpha_1 \overline{\alpha}_2 - \alpha_2 \overline{\alpha}_1) y + i(b_1 \alpha_2 - b_2 \overline{\alpha}_1), \end{aligned}$$

and because of $[N_1, N_2] = 0$ the first term is zero so that

$$b_1/b_2 = \overline{\alpha}_1/\overline{\alpha}_2 = \alpha_1/\alpha_2$$

which gives that N_2 is a multiple of N_1.

Thus, we may assume that $\Sigma = \{\sigma\}$ where $\sigma = \operatorname{span}_{\mathbb{R}^+}(N)$, and where there are two cases

$$\text{type (III): } N^2 \neq 0, \qquad N = \begin{pmatrix} 0 & \alpha & ib \\ 0 & 0 & \overline{\alpha} \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha \neq 0$$

$$\text{type (II): } N^2 = 0, \qquad N = \begin{pmatrix} 0 & 0 & ib \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b \neq 0.$$

We will take $\Gamma = U_{\mathbb{h}}(\mathcal{O}_{\mathbb{F}})$ and $\Gamma_\sigma = \Gamma \cap \{\text{stabilizer of } \sigma\}$. This gives

$$\text{type (III):} \qquad \Gamma_\sigma = \left\{ \begin{pmatrix} 1 & a & \beta \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} a^2/2 = \operatorname{Re}(\beta) \\ \quad \beta \in \mathcal{O}_{\mathbb{F}}, \quad a \in \mathbb{Z} \end{array} \right\}$$

$$\text{type (II):} \qquad \Gamma_\sigma = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} \beta + \overline{\beta} = |\alpha|^2 \\ \alpha, \beta \in \mathcal{O}_{\mathbb{F}} \end{array} \right\}.$$

The next step is to determine the nilpotent orbits. We take $w = is$, $s > 0$, and set

$$(p', l') = \exp(isN)(p, l).$$

The conditions that $(p', l') \in D$ are

$$(\natural) \qquad \begin{cases} |y'|^2 > 2\operatorname{Re}(x'\bar{z}) \\ |v'|^2 > 2\operatorname{Re}(u'\bar{w}'). \end{cases}$$

We shall work out what these mean for each of the two types. Type (II) will separate into the cases (B), (C) discussed above.

type (III): Then

$$p' = \begin{pmatrix} x + isy - (s^2/2) \cdot z \\ y + isz \\ z \end{pmatrix},$$

and $(\natural)$ for p' is

$$y^2 + isz\bar{y} - is\bar{z}y + s^2|z|^2 > 2\operatorname{Re}(x\bar{z}) + 2\operatorname{Re}(isy\bar{z}) - 2\operatorname{Re}\left(\frac{s^2}{2}|z|^2\right)$$

for $s \gg 0$. This gives $z \neq 0$. Next

$$l' = \left(u, -isu + v, \frac{-s^2 u}{2} - isv + w\right),$$

and $(\natural)$ for l' is

$$s^2|u|^2 - isu\bar{v} + is\bar{u}v + |v|^2 > 2\operatorname{Re}\left(-\frac{s^2}{2}|u|^2 + isu\bar{v} + u\bar{w}\right)$$

for $s \gg 0$. This gives $u \neq 0$. Projectively we may take

$$p = \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix},$$

and by moving in the nilpotent orbit we may then take

$$p = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}.$$

Next, $N(p) = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$ and the condition $\langle l, N(p)\rangle = 0$ gives $l = (1, 0, -x)$. Thus the set of nilpotent orbits, factored by rescalings, has dimension one, and in fact is just $\mathbb{C}$ with the coordinate x. To factor by Γ_σ, we take

$$g = \begin{pmatrix} 0 & a & \beta \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_\sigma.$$

Then

$$g(p) = \begin{pmatrix} x+\beta \\ a \\ 1 \end{pmatrix}$$

$$\exp(nN)g(p) = \begin{pmatrix} x+\beta+wa+w^2/2 \\ a+w \\ 1 \end{pmatrix},$$

and rescaling we take $w = -a$. Using $-a^2/2 + \beta = \beta - \mathrm{Re}(\beta)$ this last vector is

$$\begin{pmatrix} x + i\ \mathrm{Im}(\beta) \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\Gamma_\sigma \backslash \partial D_\sigma \cong \mathbb{C}/\sqrt{-d}\mathbb{Z} \cong \mathbb{C}^*$$

where $d > 0$.

type (II): We assume $b > 0$; the case $b < 0$ is similar but not the same. Then

$$p' = \begin{pmatrix} x - sbz \\ y \\ z \end{pmatrix}$$

$$l' = (u, v, w + sbu).$$

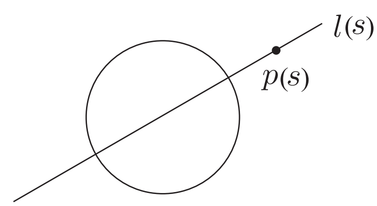
The conditions ($\natural$) are

$$\begin{cases} |y|^2 > 2\operatorname{Re}(x\bar{z}) - 2\operatorname{Re} sb|z|^2 & \iff z \neq 0 \\ |v|^2 > 2\operatorname{Re}(u\bar{w}) - 2\operatorname{Re}(sb|u|^2) & \iff u = 0, v \neq 0. \end{cases}$$

The nilpotent orbit is given by

$$p(s) = \begin{pmatrix} x - sb \\ y \\ 1 \end{pmatrix}, \quad l(s) = (0, 1, w).$$

In the usual picture we have



By rescaling we may take $x = 0$, so that

$$p = \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix}, \qquad l = (0, 1, -y)$$

and ∂D_σ has the coordinate $y \in \mathbb{C}$. For

$$g = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_\sigma$$

we have

$$g(p) = \begin{pmatrix} \beta + \alpha y \\ y + \overline{\alpha} \\ 1 \end{pmatrix},$$

and renormalizing gives $\beta + \alpha y = 0$. Then we have

$$(0, 1, -y) \begin{pmatrix} 1 & -\alpha & \overline{\beta} \\ 0 & 1 & -\overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} = (0, 1, -(y + \alpha))$$

where $\alpha \in \mathcal{O}_{\mathbb{F}}$, so that

$$\Gamma_\sigma \backslash \partial D_\sigma = \mathbb{C}/\mathcal{O}_{\mathbb{F}} = \text{CM elliptic curve } E_\sigma.$$

Remark: Earlier we listed the possible LMHS's. The connection with the description of the possible nilpotent orbits is:

$$\text{type (III)} \iff \quad \text{(A): both } p \text{ and } l \text{ move}$$

$$\text{type (II)} \iff \begin{cases} \text{(B): } b > 0, & p \text{ moves and } l \text{ remains fixed} \\ \text{(C): } b < 0, & l \text{ pivots around a fixed } p. \end{cases}$$

3. Kato-Usui extensions

When $\dim \sigma \geqq 2$, except in the classical case the glueing of boundary components into $X = \Gamma\backslash D$ is rather involved. When σ is spanned by a single N the process is relatively direct. This section has two steps.

Step one: Describe a neighborhood $\mathcal{Z}_\sigma \subset X_\sigma$ of a boundary component in general and illlustrate the construction in the case of the toy example and one classical but substantive example.

Step two: Describe the neighborhood $\mathcal{Z}_\sigma$ of E_σ in Example 1 ($D = \mathrm{SO}(4,1)/U(2,1)$) and in Example 2, type (II), and $b > 0$.[59]

Step one. We first introduce the space

$$\widetilde{\mathcal{Z}}_\sigma = \left\{ (\xi, F^\bullet) \in \mathbb{C} \times \check{D} : \begin{array}{ll} \text{if } \xi \neq 0, \text{ then } & \exp((\log \xi)/2\pi i)N)F^\bullet \in D \\ \text{if } \xi = 0, \text{ then } & \exp(\sigma_{\mathbb{C}})F^\bullet \text{ is a nilpotent orbit} \end{array} \right\}.$$

Next, we let $\mathbb{C}$ with coordinate λ act on $\widetilde{\mathcal{Z}}_\sigma$ by

$$\lambda \cdot (\xi, F^\bullet) = (\exp(2\pi i \lambda)\xi,\ \exp(-\lambda N)F^\bullet).$$

Then $\widetilde{\mathcal{Z}}_\sigma$ is acted on Γ_σ and, when factored by the action of Γ_σ the quotient is a neighborhood of the boundary component E_σ.

An intermediate step to factorizing by Γ_σ is to factor by $\Gamma(\sigma)^{gp}$ where $\Gamma(\sigma) = \Gamma \cap \exp \sigma$. Then (cf. [KU] pp. 124–5) we have that

$$\mathcal{Z}_\sigma \cong \Gamma(\sigma)^{gp} \backslash D_\sigma$$

where $D_\sigma = \partial D_\sigma \coprod D$ as above and $\Gamma(a)^{gp}$ is the group generated by $\Gamma \cap \exp \sigma$. To see this we consider the map

$$\Theta : \widetilde{\mathcal{Z}}_\sigma \twoheadrightarrow \Gamma(\sigma)^{gp} \backslash D_\sigma = \partial D_\sigma \coprod (\Gamma(\sigma)^{gp} \backslash D)$$

[59] For $\mathrm{Sp}(4)/T$ the LMHS's have been described in [GGK0] but the neighborhoods of the boundary components have yet to be worked out.

given by

$$(\xi, x) \to \begin{cases} \exp\left(\frac{\log \xi}{2\pi i}\right) N \cdot F^\bullet & \text{if } \xi \neq 0 \\ (\sigma, \exp \sigma_{\mathbb{C}} \cdot F^\bullet) & \text{if } \xi = 0\,. \end{cases}$$

The top term is in $\Gamma(\sigma)^{gp} \backslash D$, and the bottom one is in ∂D_σ. The fibres of Θ are the orbits of the action of $\mathbb{C}$ with coordinate λ given above.

Toy example (continued): Before turning to our running examples, we shall revisit the toy example to see what it gives in this framework. We first note that if $(\xi, F) \in \widetilde{\mathcal{Z}}_\sigma$, then if $F = \begin{bmatrix} x \\ y \end{bmatrix}$ in both the cases $\xi \neq 0$ and $\xi = 0$ we must have $y \neq 0$. Thus we may take $F = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$. The conditions are

$$\begin{cases} \xi \neq 0 \implies \operatorname{Im}\left(\frac{\log \xi}{2\pi i} + x\right) > 0 \\ \xi = 0 \implies F = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]. \end{cases}$$

The action of $\lambda \in \mathbb{C}$ is given by

$$\begin{cases} \quad \xi \to \exp(2\pi i \lambda)\xi \\ \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x+\lambda \\ 1 \end{smallmatrix}\right]. \end{cases}$$

We remark that taking a "slice" in the fibration

$$\widetilde{\mathcal{Z}}_\sigma \to \widetilde{\mathcal{Z}}_\sigma / \mathbb{C}$$

is related to, but not the same as, normalizing a point in a nilpotent orbit.

Toy example (continued): Here, $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Gamma(\sigma)^{gp} = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. We have described $\widetilde{\mathcal{Z}}_\sigma$ above. The map Θ is

$$\Theta : \widetilde{\mathcal{Z}}_\sigma \to \Gamma(\sigma)^{gp} \backslash D_\sigma \cong \Delta \quad (= \text{unit disc})$$

is given by

$$(\xi, x) \left\{ \begin{array}{ll} \frac{\log \xi}{2\pi i} + x & \operatorname{mod} \mathbb{Z} \\ 0 & (\text{equal to } \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1) \end{array} \right\} \to \xi e^{2\pi i x}.$$

Thus $\Gamma(\sigma)^{gp} \backslash D \cong \Delta^* = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \backslash \mathcal{H}$.

Example: Before giving the two non-classical examples, we want to give a more substantive classical example. For $V \cong \mathbb{Q}^4$ with

$$Q = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

we consider the period domain P ($\cong \mathcal{H}_2$) for weight 1 PHS's with $h^{1,0} = 2$. In the compact dual we take $F(x,y,z) \in \check{D}$ where

$$F(x,y,z) = \operatorname{span}\{f_1(x,y,z),\ f_2(x,y,z)\}$$

$$f_1(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ x \\ y \end{pmatrix},\ f_2(x,y,z) = \begin{pmatrix} 0 \\ 1 \\ y \\ z \end{pmatrix}.$$

For N we take

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\sigma = \operatorname{span}_{\mathbb{R}^+}\{N\}$. Recall that

$$\widetilde{\mathcal{Z}}_\sigma = \{(\xi; x,y,z) : \text{conditions (i) and (ii) are satisfied for } F = F(x,y,z)\}.$$

For (i), setting $\zeta = \frac{\log \xi}{2\pi i}$ we have

$$(*) \qquad \exp(\zeta N)F(x,y,z) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ x+\zeta \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ y \\ z \end{pmatrix} \right)$$

and condition (i) is

$$\left\| \begin{matrix} x+\zeta & y \\ y & z \end{matrix} \right\| > 0.$$

Writing $\xi = re^{i\theta}$ this is

$$\begin{cases} -\frac{\log r}{2\pi} + \operatorname{Im} x > 0 \\ \operatorname{Im} z > 0 \end{cases}$$

together with a condition on ζ, z, y. It is then clear that (i) is satisfied for

$$0 < r < C(x,y,z).$$

Condition (ii) is just $\operatorname{Im} z > 0$.

The action of $\lambda \in \mathbb{C}$ on $\widetilde{\mathcal{Z}}_\sigma$ is, from $(*)$,

$$\lambda(\xi; x,y,z) = (\exp 2\pi i\lambda \cdot \xi; x-\lambda, y, z).$$

From this a "natural" slice of $\widetilde{\mathcal{Z}}_\sigma \to \widetilde{\mathcal{Z}}_\sigma/\mathbb{C}$ is given by $x = \lambda$.

For the normalizer Γ_N of $\mathbb{Q}\cdot N$ in Γ,[60] computation gives that Γ_N is a semi-direct product

$$\Gamma_N = \Gamma' * \Gamma''$$

where Γ' is given in coordinates $(\xi; y, z)$ along the slice by

$$\begin{cases} \xi \to \xi \\ y \to y \\ z \to \frac{az+b}{cz+d} \end{cases}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \cap \Gamma$, and Γ'' is

$$\begin{cases} \xi \to \xi \\ y \to y + m + nz \qquad\qquad m, n \in \mathbb{Z} \\ z \to z \end{cases}$$

where this transformation is in Γ.[61]

Geometrically, we have a fibration of $\mathcal{Z}_\sigma = \Gamma\backslash\widetilde{\mathcal{Z}}_\sigma/\mathbb{C}$

$$\begin{array}{c} \mathcal{Z}_\sigma \\ \Big\downarrow \\ \Gamma'\backslash\mathcal{H} \end{array}$$

with fibres elliptic curves isogeneous to $E_z = \mathbb{C}/\mathbb{Z} + z\cdot\mathbb{Z}$.

We give this example to illustrate that in this classical case, there is a linear slice in the natural coordinates being used. This is very special, as the next example will show. However, miraculously it does occur in Carayol's $\mathrm{SU}(2,1)/T$ case.

Step two.

Example 1: To have a sense of what to expect, we recall that the LMHS's are of the form

$$(*) \qquad \begin{array}{c} \mathbb{Q}(-2) \to \mathbb{Q}(-1) \to \mathbb{Q} \\ H \end{array}$$

where H is a polarized Hodge structure of weight 2 with $H_{\mathbb{C}} = H^{2,0} \oplus H^{2,0}$. This may also be pictured as

$$\begin{array}{ccccccccc} 0 & \to & W_2/W_0 & \to & W_4/W_0 & \to & W_4/W_2 & \to & 0 \\ & & \wr\Vert & & & & \wr\Vert & & \\ & & Q(-1)\oplus H & & & & \mathbb{Q}(-2). & & \end{array}$$

---

[60] We assume there is a lattice $V_{\mathbb{Z}}$ and $\Gamma \subset \mathrm{Aut}(V_{\mathbb{Z}}, Q)$.

[61] Thus, the m, n's that appear are subgroup of finite index in $\mathbb{Z}\oplus\mathbb{Z}$.

Using the polarizing form, the two short extensions in the top row in $(*)$ are dual, and we may use the rescaling parameter to make them zero, thus normalizing the LMHS. Then the remaining extension classes are in

$$\begin{gathered}\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2),H^2)\cong H^{0,2}/H\\ \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2),\mathbb{Q}).\end{gathered}$$

If we have a lattice $V_{\mathbb{Z}}$ with $V=\mathbb{Q}\otimes_{\mathbb{Z}}V_{\mathbb{Z}}$ and we set $H_{\mathbb{Z}}=$ image of $V_{\mathbb{Z}}\cap W_2$ in H, then over $\mathbb{Z}$ the first extension class will be in $H^{0,2}/H_{\mathbb{Z}}$. Choosing an isomorphism $H_{\mathbb{Z}}\cong\mathbb{Z}^2$, then we may assume that $H^{0,2}=\mathbb{C}v_\tau$ with $v_\tau=(\tau,1)$ where $\operatorname{Im}\tau\neq 0$. If

$$Q_H: H\otimes H\to\mathbb{Q}$$

is the polarizing form, then $Q(v_\tau,v_\tau)=0$ is a quadratic equation in τ with $\mathbb{Q}$-coefficients, and this gives that

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2),H)\ \textit{is a CM elliptic curve } E.$$

The second extension class is in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2),\mathbb{Z})\cong\mathbb{C}^*$. Thus we may expect $\mathbb{C}^*\times E$ for the boundary component in this case. To carry this out, and for easy reference, we recall the following notations

- $Q=\left(\begin{array}{c|cc}-I_3&0&0\\\hline 0&0&1\\0&1&0\end{array}\right).$

- $N_0=\begin{pmatrix}0&0&0&0&a_1\\0&0&0&0&a_2\\0&0&0&0&a_3\\a_1&a_2&a_3&0&0\\0&0&0&0&0\end{pmatrix}.$

Then

- $N_a\in\mathrm{so}(4,1)$ and $N_a\in\mathfrak{g}=\mathrm{so}(4,1)$ is defined over $\mathbb{Q}$ if the $a_j\in\mathbb{Q}$.

- $[N_a,N_b]=0$.

- $N_a^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a^2/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad a^2 = a \cdot a.$

- $N_a^3 = 0.$

- $\exp N_a = \begin{pmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ a_1 & a_2 & a_3 & 1 & -a^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

- If the standard basis for $\mathbb{Q}^5$, written as column vectors, is $e_1, \dots, e_5$, then

$$\begin{cases} Q(e_i, e_j) = -\delta_{ij} & 1 \leqq i,\ j \leqq 3 \\ Q(e_4, e_5) = 1 \\ \text{all other } Q(e_\alpha, e_\beta) = 0. \end{cases}$$

For simplicity of calculation we shall take $a_1 = 1$, $a_2 = a_3 = 0$. Then

$$\begin{cases} Ne_1 = e_4 \\ Ne_5 = e_1 \Rightarrow N^2 e_1 = e_4 \\ \text{all other } Ne_\alpha = 0. \end{cases}$$

The weight filtration is then

$$\begin{aligned} &W_0 = \{e_4\} \\ &\cap \\ &W_2 = \{e_1, e_2, e_3, e_4\} \end{aligned}$$

where $\{\ \}$ denotes the span $/\mathbb{Q}$.

Glueing in the boundary component. Recall that we set

$$\widetilde{\mathcal{Z}} = \left\{ (\xi, F) \in \mathbb{C} \times \check{D} : \quad \begin{array}{ll} \text{(i)} & \xi \neq 0 \Rightarrow \left(\exp\left(\frac{\log \xi}{2\pi i}\right) N\right) F \in D \\ \text{(ii)} & \xi = 0 \Rightarrow \exp(\mathbb{C}N)F \text{ is a nilpotent orbit.}^{62} \end{array} \right.$$

We then take the quotient $\widetilde{\mathcal{Z}}/\mathbb{C}$ for the action of $\lambda \in \mathbb{C}$ given by

$$\lambda(\xi, F) = (\exp(2\pi i\lambda)\xi, \exp(-\lambda N)F)$$

[62] For simplicity of notation, in this example we drop the subscript σ on $\widetilde{\mathcal{Z}}$ and $\mathcal{Z}$.

and factor the quotient by the action of the normalizer Γ_N in Γ of $\operatorname{span}_{\mathbb{Q}}\{N\} \subset \mathfrak{g}$ to obtain the neighborhood

$$\mathcal{Z} = \Gamma_\sigma \backslash \widetilde{\mathcal{Z}} / \mathbb{C} \subset \Gamma \backslash D_\sigma$$

of the boundary component.

We have determined the F in (ii) above, and we now determine the condition on F in (i). Setting

$$f_1 = \begin{pmatrix} u \\ a \\ b \end{pmatrix}, \qquad f_2 = \begin{pmatrix} v \\ c \\ d \end{pmatrix}$$

where $u, v \in \mathbb{C}^3$, $Q(F, F) = 0$ gives

$$\begin{cases} u^2 = 2ab \\ v^2 = 2cd \\ u \cdot v = ad + bc. \end{cases}$$

Setting $w = \log \xi / 2\pi i$, computation gives

$$(*) \qquad \begin{cases} Q(f_1(w), \overline{f_1(w)}) &= -|u_1 + bw|^2 - |u_2|^2 - |u_3|^2 + 2\operatorname{Re}(a\bar{b} + |b|^2 w^2/2) \\ Q(f_2(w), \overline{f_2(w)}) &= -|v_1 + dw|^2 - |v_2|^2 - |v_3|^2 + 2\operatorname{Re}(c\bar{d} + |d|^2 w^2/2) \\ Q(f_1(w), \overline{f_2(w)}) &= -u\bar{v} + (\text{terms involving } b, d). \end{cases}$$

Case a: b or $d \neq 0$.

Then we may take $b = 1, d = 0$ and we are in case (ii) that was worked out above.

Case b: $b = d = 0$.

From $(*)$ we see that $(\exp wN)F \in D$ for any w. The conditions on u, v are

$$u^2 = v^2 = u \cdot v = 0.$$

These give a pair of orthogonal points on a conic in $\mathbb{P}^2$.

We note that in case (a), F depends on three parameters, whereas in case (b) it depends on one parameter.

We now restrict to case (a) and normalize along nilpotent orbits to take

$$f_1(x,y)=\begin{pmatrix}0\\x\\y\\\frac{x^2+y^2}{2}\\1\end{pmatrix}=xe_2+ye_3+\left(\frac{x^2+y^2}{2}\right)e_4+e_5$$

$$f_2(x,y)=\begin{pmatrix}0\\1\\i\\x+iy\\0\end{pmatrix}=e_2+ie_3+(x+iy)e_4.$$

Setting $F(x,y)=\operatorname{span}\{f_1(x,y),f_2(x,y)\}$ over $\mathbb{C}$ and $\zeta=\frac{\log\xi}{2\pi i}$, we shall determine the conditions on $(\xi;x,y)$ that $(\xi;F(x,y))\in\widetilde{\mathcal{Z}}$. Now for condition (i) we have

$$\left(\exp\left(\frac{\log\xi}{2\pi i}\right)N\right)F(x,y)=\operatorname{span}\{f_1(\zeta;x,y),f_2(\zeta;x,y)\}$$

where

$$(*)\qquad\begin{cases}f_1(\zeta;x,y)=\zeta e_1+xe_2+ye_3+\left(\frac{\zeta^2+x^2+y^2}{2}\right)e_4+e_5\\ f_2(\zeta;x,y)=f_2(x,y)=e_2+ie_3+(x+iy)e_4.\end{cases}$$

We have seen earlier that the matrix $Q(f_i(\zeta;x,y),\overline{f_j(\zeta,x,y)})$ is negative definite unless

$$\left(\operatorname{Im}\left(\frac{\log\xi}{2\pi i}\right)\right)^2+(\operatorname{Im}x)^2+(\operatorname{Im}y)^2=0.$$

If $|\xi|\neq 1$, this is satisfied for all x,y, which then gives an embedding

$$\Delta^*\times\mathbb{C}^2\hookrightarrow\widetilde{\mathcal{Z}}.$$

From $(*)$ we may think of an expansion of the periods as a quadratic polynomial in $\frac{\log\xi}{2\pi i}$ with holomorphic coefficients.[63]

When $\xi=0$ in case (ii), any $x,y\in\mathbb{C}^2$ give a nilpotent orbit. Thus the "boundary" of $\mathcal{Z}=\widetilde{\mathcal{Z}}/\mathbb{C}$ is given by

$$\xi=0,\{(x,y)\in\mathbb{C}^2\}\ .$$

It remains to factor

$$\Gamma_N\backslash\widetilde{\mathcal{Z}}/\mathbb{C}$$

[63]In general, for degenerating PHS's of weight n, if $N^{m+1}=0$ the periods are polynomials of degree m in $\frac{\log\xi}{2\pi i}$ with holomorphic coefficients.

by the normalizer in Γ of N in $\mathfrak{g}$. We have noted above that

$$\Gamma_N = \left\{ \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ a_1 & a_2 & a_3 & 1 & a^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, a_i \in \mathbb{Z} \right\}.$$

Then

$$\gamma \cdot f_1 = \begin{pmatrix} a_1 \\ x + a_2 \\ y + a_3 \\ (x+a_2)^2 + (y+a_3)^2 \\ 1 \end{pmatrix}, \ \gamma \cdot f_2 = \begin{pmatrix} 0 \\ 1 \\ i \\ x + iy + a_2 + ia_3 + a^2/2 \\ 0 \end{pmatrix}.$$

We need to now slide along the nilpotent orbit to make the first entry zero in $\gamma \cdot f_1$. The new $\gamma \cdot f_1$ is now

$$\begin{pmatrix} 0 \\ x + a_2 \\ y + a_3 \\ (x+a_2)^2 + (y+a_3)^2 + a_1^2/2 \\ 1 \end{pmatrix}.$$

We map $\mathcal{Z} \to \mathbb{C}$ by $(\xi; x, y) + x + iy$. After quotienting by Γ_N we get a copy of $\mathbb{C}/\left(\frac{1}{2}\right)\mathbb{Z} + i\mathbb{Z}$. The fibre is a copy of $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$. *Thus, the boundary component is a $\mathbb{C}^*$-bundle over a CM elliptic curve.*

Example 2. We will now use the analogous process to determine $\mathcal{Z}_\sigma$ in case (II), $b > 0$. Shifting notation to conform with [C3], we let $\beta_0 = ib$ be such that

$$\begin{pmatrix} 1 & 0 & \beta_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma, \qquad \operatorname{Im} \beta_0 > 0$$

and β_0 is chosen to have the smallest imaginary part with this property. Then

$$\exp(isN) = \begin{pmatrix} 1 & 0 & is\beta_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the above notations we have for $(p', l') = \exp(isN) \cdot (p, l)$

$$p' = \begin{pmatrix} x + is\beta_0 z \\ y \\ z \end{pmatrix}, \qquad l' = (u, v, -is\beta_0 u + w).$$

The positivity conditions that express the condition $(p', l') \in D$ are

$$\begin{cases} |y|^2 > 2\,\mathrm{Re}(x\bar{z} + is\beta_0 |z|^2) \\ |v|^2 > 2\,\mathrm{Re}(w\bar{u} - is\beta_0 |u|^2). \end{cases}$$

From the above positivity conditions we see that this is still the case in a neighborhood of E_σ. Thus we may asssume it to hold in $\widetilde{\mathcal{Z}}_\sigma$. We have seen above that when $\xi = 0$, in order to have a nilpotent orbit, using $\beta_0 = ib$ with $b \in \mathbb{R}$ we must have

$$z \neq 0, \quad v \neq 0 \quad \text{ and } \; u = 0.$$

The action of $\mathbb{C}$ is given by

$$\lambda(\xi; p, l) = (\exp(2\pi i\lambda)\xi; \tilde{p}, \tilde{l})$$

where

$$\tilde{p} = \begin{pmatrix} x - \lambda\beta_0 z \\ y \\ z \end{pmatrix}, \qquad \tilde{l} = (u, v, \lambda\beta_0 u + w).$$

Next, following Carayol we consider a 2-dimensional subvariety $\widetilde{\mathcal{U}}_\sigma \subset \widetilde{\mathcal{Z}}_\sigma$ with the following properties:

(i) $\widetilde{\mathcal{U}}_\sigma$ is invariant under the $\mathbb{C}$-action,

(ii) the quotient $\widetilde{\mathcal{U}}_\sigma/\mathbb{C}$ will contain all of the boundary component.

For this we define $\widetilde{\mathcal{U}}_\sigma$ by $\{u = 0\}$. Taking then $z = v = 1$, as in the toy example a slice of the fibration $\widetilde{\mathcal{U}}_\sigma \to \mathcal{U}_\sigma$ is obtained taking $x = 0$, which then gives from $\langle \tilde{l}, \tilde{p} \rangle = 0$ that $w = -y$.[64] Thus in $\mathcal{U}_\sigma$ we have coordinates ξ, y where for $(\xi; p, l) \in \mathcal{U}_\sigma$

$$p = \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix}, \qquad l = (0, 1, -y).$$

The boundary $\partial\mathcal{U}_\sigma$; i.e. that part of $\mathcal{U}_\sigma$ *not* coming from D, is given by

$$\partial\mathcal{U}_\sigma \cong \{y : y \in \mathbb{C}\}.$$

[64]Referring to the previous footnote, the reason that we can choose a linear slice is that $N^2 = 0$.

It remains to factor by the action of Γ_σ. If

$$\gamma = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_\sigma$$

where $\alpha, \beta \in \mathcal{O}_{\mathbb{F}}$ and $\beta + \overline{\beta} = |\alpha|^2$, then the action of γ on

$$p = \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix}, \qquad l = (u, 1, -y)$$

sending p, l to p', l' is given by

$$p' = \begin{pmatrix} \alpha y + \beta \\ y + \overline{\alpha} \\ 1 \end{pmatrix}, \qquad l' = (0, 1 - \overline{\alpha} - y).$$

This is equivalent to the action of $\lambda = \frac{1}{\beta_0}(\alpha y + \beta)$, where $\beta_0 = \operatorname{Im} \beta$, to

$$\left(\exp\left(\frac{2\pi i}{\beta_0}(\alpha y + \beta) \right) \xi; p'', l'' \right)$$

where

$$p'' = \begin{pmatrix} 0 \\ y + \overline{\alpha} \\ 1 \end{pmatrix}, \qquad \begin{aligned} l'' &= (u, 1 - \alpha u, \alpha y u + \beta u + u\overline{\beta} - \overline{\alpha} - y) \\ &= (1 - \alpha u) \left(\frac{u}{1 - \alpha u}, 1, -\overline{\alpha} - y \right). \end{aligned}$$

Thus the action of Γ_σ on $\widetilde{\mathcal{Z}}_\sigma / \mathbb{C}$ is

$$(\xi, y, u) \to \left(\exp\left(\frac{2\pi i}{\beta_0}(\alpha y + \beta) \right) \xi, y + \overline{\alpha}, \frac{u}{1 - \alpha u} \right).$$

The boundary

$$\boxed{\partial \mathcal{U}_\sigma = \Gamma_\sigma \backslash \{(0, y, 0)\} \cong \mathbb{C}/\mathcal{O}_{\mathbb{F}} = E_\sigma}\,[65]$$

is, as expected, a CM elliptic curve.

[65]The subtlety here is that $\mathcal{Z}_\sigma$ will be 3-dimensional and more complicated to describe, presumably due to its being a slit analytic variety. The 2-dimensional subspace $\mathcal{U}_\sigma \subset \mathcal{Z}_\sigma$ contains the element boundary information for the calculations below.

4. Expansion of Picard modular forms about a boundary component and relation to automorphic cohomology

We will proceed in three steps.

Step one: *Expansion of Picard modular forms.* In the case of an Hermitian symmetric domain, after suitably trivializing the canonical bundle automorphic forms are given by holomorphic functions satisfying a functional equation for each $\gamma \in \Gamma$. For the usual upper-half plane $\mathcal{H}$ and γ unipotent, this functional equation is trivial and one obtains a function $f(\tau)$ on $\mathcal{H}$ invariant under (say) $\tau \to \tau + 1$ which then leads to the Fourier expansion of $f(q)$ where $q = \exp(2\pi i \tau)$. A similar story was done by Shimura for Picard modular forms on $Y = \Gamma\backslash\Delta$. In this case, a neighborhood of a cusp in Y is given by

$$\{(x, y) \in \mathbb{C}^2 : 2\,\mathrm{Re}(x) > |y|^2\},$$

the cusp being $(0, 0)$. For

$$\gamma = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta + \overline{\beta} = |\alpha|^2$$

as above and where $\alpha, \beta \in \mathcal{O}_{\mathbb{F}}$, a Picard modular form has an expansion[66]

$$f'(x, y) = \sum_{r \in \mathbb{N}^*} g'_r(y) \exp\left(-\frac{2\pi i r}{\beta_0} x\right)$$

where β is an imaginary multiple of β_0 and $g'_r(y)$ satisfies the functional equation

$$g'_r(y + \overline{\alpha}) = g'_r(y) \exp\left(\frac{2\pi i r}{\beta_0}(\alpha y + \beta)\right).$$

To line up with the above notation we write this as

$$g'_r(y + \overline{\alpha}) = g'_r(y)\chi(\overline{\alpha}) \exp\left(\frac{2\pi i r}{\beta_0}\left(\alpha y + \frac{|\alpha|^2}{2}\right)\right)$$

where $\chi(\overline{\alpha})$ can be given explicitly. We interpret this as saying that g'_r is a *section of a line bundle* $L'_r \to E'$ (and is thus a theta-function).

For Δ^*, which we recall denotes the ball in $\mathbb{P}^{2*}$ of lines not meeting $\overline{\Delta}$, in terms of the coordinates $(1, v, w)$ in Δ^* we have

$$f''(v, w) = \sum_{r \in \mathbb{N}^*} g''_r(v) \exp\left(\frac{-2\pi i r}{\beta_0} w\right)$$

[66]Here, the primes refer to D'; below there will be given similar expressions for D''. We shall denote by E' the elliptic curve $\mathbb{C}/\mathcal{O}_{\mathbb{F}}$ where $z' \sim z' + \overline{\alpha}$, and by E'' the elliptic curve $\mathbb{C}/\mathcal{O}_{\mathbb{F}}$ where $z'' \sim z'' - \alpha$. The latter is a boundary component of X_{Σ} given by case (II), $b < 0$.

where

$$g_r''(v+\alpha) = g_r''(v)\exp\left(\frac{2\pi i r}{\beta_0}(\overline{\alpha}v+\beta)\right)$$

and $g_r'' \in H^0(E'', L_r'')$. (The reason why we have $g_r''(v+\alpha)$ and not $g_r''(v-\alpha)$ comes out of the explicit calculation in [C3]).

Remark: Geometrically, there is a *smooth* compactification

$$Y \subset \overline{Y}$$

whose boundary components include the elliptic curves E', E'' (cf. Larson, *Arithmetic compactification of some Shimura surfaces*, in Zeta functions of Picard modular surfaces, edited by Langlands and Ramakrishnan (Publications CRM, 1992)). There is also the Kato-Usui extension

$$Y \subset Y_\Sigma,$$

which in this case is a toroidal compactification. Several issues arise, which in the case at hand would provide a more conceptual framework for [C3]:

(i) Is $\overline{Y} = Y_\Sigma$ (for a suitable choice of fan Σ)?

(ii) In general, there seems to not yet be any "functoriality" theory associated to the Kato-Usui extensions. Now $\mathbb{B}$ is a Mumford-Tate domain for polarized abelian varieties A of dimension 3 with extra structure in $H^1(A)$. It is a moduli space, as is D', points of which are given by flags in $H^1(A)$'s satisfying certain Hodge-theoretic conditions.[67] Thus

$$\begin{array}{ccc} \Gamma\backslash D' & \to & \Gamma\backslash\Delta \\ \parallel & & \parallel \\ X' & \to & Y' \end{array}$$

is a map of Mumford-Tate domains, meaning that for each point of X' represented by an equivalence class of weight 3 PHS's there is canonically associated a weight 1 PHS giving a point of Y'. Fans refer to the data $(\mathfrak{g},\Gamma)$, and one may ask if there is a map

$$X'_\Sigma \to Y'_\Sigma.$$

[67]The description if very much like that given for the Mumford-Tate domain $\mathcal{U}(2,1)/\widetilde{T}$ above.

(iii) Finally, in the [EGW] framework where we have

$$
\begin{tikzcd}
 & \Gamma\backslash\mathcal{W} \arrow[dl] \arrow[dr] & \\
X & & X' \arrow[d] \\
 & & Y'
\end{tikzcd}
$$

are there Kato-Usui extensions giving a picture

$$
(*')\qquad
\begin{tikzcd}
 & (\Gamma\backslash\mathcal{W})_\Sigma \arrow[dl] \arrow[dr] & \\
X_\Sigma & & X'_\Sigma \arrow[d] \\
 & & Y'_\Sigma
\end{tikzcd}
$$

that would allow us to geometrically interpret the expansion of an automorphic cohomology class, around the boundary components E' and E'' of X_Σ in terms of the expansion of Picard modular forms around the corresponding boundary components of Y'_Σ, which as analytic varieties are the same elliptic curves E' and E''?

Step two: *The EGW transform between $H^0(E', L'_r)$ and $H^1(E'', L''_{-r})$.* We shall use the coordinate $z' \in \mathbb{C}$ to give $E' = \mathbb{C}/\mathcal{O}_\mathbb{F}$, and similarly $z'' \in \mathbb{C}$ to give E''. These notations are consistent with those from Lecture 2. We then define

$$\mathbb{W} = \mathcal{O}_\mathbb{F}\backslash\mathbb{C} \times \mathbb{C}$$

where $\mathbb{C} \times \mathbb{C}$ has coordinates (z', z'') and where $\alpha \in \mathcal{O}_\mathbb{F}$ operates by $\overline{\alpha}$ on the first factor and by $-\alpha$ on the second. There is a diagram

$$
\begin{tikzcd}
 & \mathbb{W} \arrow[dl, "\pi'"'] \arrow[dr, "\pi''"] & \\
E' & & E''.
\end{tikzcd}
$$

Carayol proves that $\mathbb{W}$ is Stein and that the fibres of the two projections are Stein and contractable. Thus the [EGW] theory applies to give isomorphisms

$$
\begin{cases}
\eta' : H^0(E', L'_r) \cong H^1(E'', L''_{-r}) \\
\eta'' : H^0(E'', L''_r) \cong H^1(E', L'_{-r}).
\end{cases}
$$

To describe η', we let $\theta' \in H^0(E', L'_r)$ be a theta-function as above and set

$$h(z', z'') = \theta'(z') \exp\left(\frac{2\pi i r}{\beta_0} z'z''\right) dz'.$$

The functional equation

$$h(z' + \overline{\alpha}, z'' - \alpha) = h(z', z'') \exp\left(\frac{2\pi i r}{\beta_0}(\alpha z'' + \beta)\right)$$

gives a section over $\mathbb{W}$ of $\pi''^{-1}(L''_{-r})$, and then

$$\eta'(\theta')(z', z'') = \theta'(z') \exp\left(\frac{2\pi i r}{\beta_0} z'z''\right) dz''$$

defines a relative differential for $\pi'' : \mathbb{W} \to E''$ with values in $\pi''^{-1}(L''_{-r})$. This gives the above map η', and η'' is defined similarly. By explicit theta function calculations, which ultimately relate back to work of Siegel in 1963, Carayol proves:

> *The line bundles $L'_r \to E'$ and $L''_{-r} \to E''$ are defined over the maximal abelian extension $\mathbb{F}^{\mathrm{ab}}$ of $\mathbb{F}$, and the isomorphisms η' and η'' are then defined over $\mathbb{F}^{\mathrm{ab}}$.*

A hint as to what is involved is the following: The line bundle $L'_r \to E'$ has an Hermitian metric relative to which the inner product of two sections θ'_1, θ'_2 is given by

$$(\theta'_1, \theta'_2) = \int_{E'} \theta'_1(z')\overline{\theta'_2(z')} \exp\left(\frac{-2\pi i r}{\beta}|z'|^2\right) dz' \wedge \overline{dz'}.$$

Then, using work of Siegel and Shimura, Carayol shows that, up to a multiplicative construct c independent of θ'_1 and θ'_2,

> *If θ'_1 and θ'_2 are defined over $\mathbb{F}^{\mathrm{ab}}$, then so is $c(\theta'_1, \theta'_2)$.*

The calculation of the integral is carried by expanding θ' and θ'' in Fourier series. Then by orthogonality-type relations, only finitely many terms appear, which are then explicitly evaluated and shown to lie in $c^{-1}\mathbb{F}^{\mathrm{ab}}$.

Step three: It remains to pull everything together. I will take poetic license and interpret the calculations in [C3] in terms of the diagram $(*')$ above, which has not (yet?) been proved to exist geometrically but does exist "in coo rdinates" in Carayol's calculations.

First, as is evident from its formulation, the [EGW] theory is functorial. Thus, although we do not know that there are actual pictures

$$(*')\qquad \begin{tikzcd} & (\Gamma\backslash\mathcal{W})_\Sigma \arrow[dl] \arrow[dr] & \\ X_\Sigma & & X'_\Sigma \arrow[d] \\ & & Y'_\Sigma \end{tikzcd}$$

and

$$(**)\qquad \begin{tikzcd} & (\Gamma\backslash\mathcal{W})_\Sigma \arrow[dl] \arrow[dr] & & & & \Gamma\backslash\mathbb{W} \arrow[dl] \arrow[dr] & \\ X'_\Sigma \arrow[d] & & X''_\Sigma \arrow[d] & \supset & E' & & E'' \\ Y'_\Sigma & & Y''^{*}_\Sigma & & & & \end{tikzcd}$$

as just mentioned in [C3] they do exist computationally. By functoriality of the [EGW] constructions, the Penrose-type transformations $\mathcal{P}'$ and $\mathcal{P}''$ from lecture I may be used to (here we drop the subscript Σ and the primes on the Y's)

(a) move automorphic cohomology in $H^1(X, L_\lambda)$ to $H^0(Y, \widetilde{L})$ and to $H^1(Y^*, \widetilde{L}^*)$;
(b) restrict everything to punctured neighborhoods of the boundary components E', E'';
(c) expand classes in $H^*(X', L'_{\lambda'})$, $H^*(X'', L''_{\lambda''})$ and $H^*(Y, \widetilde{L})$, $H^*(Y^*, \widetilde{L}^*)$ about E', E'' as above;
(d) using the isomorphism $\mathcal{P}'$, define a class $\alpha \in H^1(E, L_\lambda)$ to be *arithmetic* if the coefficients $g'_r(y)$ and $g''_r(v)$ are arithmetic;
(e) using step two show that these conditions are compatible and conclude the result stated in Section 1.

For the record, the final expression for a class in $H^1_e(X, L_\lambda)$, expressed as a holomorphic relative differential on $\Gamma\backslash\mathcal{W}$, is

$$\sum_{r\in\mathbb{N}^*} g'_r(y) \exp\left(\frac{2\pi i r}{\beta_0} xw\right) \theta^r dy$$

where

$$\theta = \exp\left(\left(\frac{2\pi i}{\beta}\right) w\right).$$

References

[AS] M. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.* **42** (1977), 1–62.

[BE] R. Baston and M. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory*, Clarendon Press, Oxford, 1989.

[Bu] N. Buchdahl, On the relative de Rham sequence, *Proc. Amer. Math. Soc.* —bf 87 (1983), 363–366.

[BBH] D. Burns, S. Halverscheid, and R. Hind, The geoemtry of Grauert tubes and complexification of symmetric spaces *Duke J. Math.* **118** (2003), 465–491.

[C1] H. Carayol, Limites dégénérées de séries discrètes, formes automorphes et variétés de Griffiths-Schmid: le case du groupe $U(2,1)$, *Compos. Math.* **111** (1998), 51–88.

[C2] ———, Quelques relations entre les cohomologies des variétés de Shimura et celles de Griffiths-Schmid (cas du group $S\mathcal{U}(2,1)$), *Compos. Math.* **121** (2000), 305–335.

[C3] ———, Cohomologie automorphe et compactifications partielles de certaines variétés de Griffiths-Schmid, *Compos. Math.* **141** (2005), 1081–1102.

[CK] H. Carayol and A. W. Knapp, Limits of discrete series with infinitesimal character zero, *Trans. Amer. Math. Soc.* **359** (2007), 5611–5651.

[CO] W. Casselman and M. S. Osborne, The $\mathfrak{n}$-cohomology of representations with an infinitesimal character, *Compos. Math.* **31** (1975), 219–227.

[CKn] E. Cattani and A. Kaplan, Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure, *Invent. Math.* **67** (1982), 101–115.

[CKS] E. Cattani, A. Kaplan, and W. Schmid, L^2 and intersection cohomologies for a polarizable variation of Hodge structure, *Invent. Math.* **87** (1987), 217–252.

[EGW] M. Eastwood, S. Gindikin, and H. Wong, Holomorphic realizations of $\bar{\partial}$-cohomology and constructions of representations, *J. Geom. Phys.* **17**(3) (1995), 231–244.

[EWZ] M. Eastwood, J. Wolf, and R. Zierau (eds.), *The Penrose Transform and Analytic Cohomology in Representation Theory*, *Contemp. Math.* **154**, Amer. Math. Soc., Providence, RI, 1993.

[FHW] G. Fels, A. Huckelberry, and J. A. Wolf, *Cycle Spaces of Flag Domains, A Complex Geometric Viewpoint*, *Progr. Math.* **245**, (H. Bass, J. Oesterlé, and A. Weinstein, eds.), Birkhäuser, Boston, 2006.

[FL] R. Friedman and R. Laza, Semi-algebraic horizontal subvarieties of Calabi-Yau type, preprint, 2011, available at http://arxiv.org/abs/1109.5632.

[Gi] S. Gindikin, Holomorphic language for $\overline{\partial}$-cohomology and respresentations of real semisimple Lie groups, in *The Penrose Transform and Analytic Cohomology in Representation Theory* (South Hadley, MA, 1992), *Comptemp. Math.* **154**, Amer. Math. Soc., Providence, RI, 1993, pp. 103–115.

[GM] S. Gindikin and T. Matsuki, Stein extensions of Riemannian symmetric spaces and dualities of orbits on flag manifolds, *Transform. Groups* **8** (2003), 333–376.

[GG] M. Green and P. Griffiths, Correspondence and Cycle Spaces: A result comparing their cohomologies, to appear in the volume in honor of Joe Harris' 60th birthday.

[GGK0] M. Green, P. Griffiths, and M. Kerr, Néron models and boundary components for degeneration of Hodge structure of mirror quintic type, in *Curves and Abelian Varieties*, *Contemp. Math.* **465**, Amer. Math. Soc., Providence, RI, 2008.

[GGK1] M. Green, P. Griffiths, and M. Kerr, Mumford-Tate Groups and Domains: Their Geometry and Arithmetic, *Annals of Math. Studies* **183**, Princeton University Press, Princeton, NJ, 2012.

[GGK2] ———, Special values of automorphic cohomology classes, available at http://www.math.wustl.edu/$\sim$matkerr/.

[GS] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta Math.* **123** (1969), 253–302.

[HMSW] H. Hecht, D. Milicĭč, W. Schmid and J. Wolf, Localization and standard modules for real semisimple Lie groups I: The duality theorem, *Invent. Math.* **90** (1987), 297–332.

[KU] K. Kato and S. Usui, *Classifying Spaces of Degenerating Polarized Hodge Structure*, *Ann. of Math. Studies* **169**, Princeton Univ. Press, Princeton, NJ, 2009.

[Ke] M. Kerr, Shimura varieties: a Hodge-theoretic perspective, preprint 2011, available at http://www.math.wustl.edu/∼matkerr/.

[KP] M. Kerr and G. Pearlstein, Boundary components of Mumford-Tate domains, to appear.

[K1] A. Knapp, *Lie Groups Beyond an Introduction*, *Progr. in Math.* **140**, Birkhäuser, Boston, 3rd printing, 2005.

[K2] ———, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton Univ. Press, Princeton, NJ, 1986.

[Ko] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.* **74** (1961), 329–387.

[M] T. Matsuki, Orbits on affine symmetric spaces under the action of parabolic subgroups, *Hiroshima Math. J.* **12** (1982), 3097–320.

[MUV] I. Mirković, T. Uzawa, and K. Vilonen, Matsuki correspondence for sheaves, *Invent. Math.* **109** (1992), 231–245.

[Sch1] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Ph.D. dissertation, Univ. California, Berkeley (1967); reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, *Math. Surveys and Monogr.* **31**, Amer. Math. Soc. (1989), 223–286.

[Sch2] ———, Discrete series, *Proc. Symp. Pure Math.* **61** (1997), 83–113.

[Sch3] ———, Geometric methods in representation theory (lecture notes taken by Matvei Libine), in *Poisson Geometry Deformation Quantisation and Group Representations*, 273–323, *London Math. Soc. Lecture Note Ser.* **323**, Cambridge Univ. Press, Cambridge, 2005.

[SW] H. Shiga and J. Wolfart, Criteria for complex multiplication and transcendence properties of autormorphic forms, *J. Reine Angew. Math.* **464** (1995), 1–25.

[Shi] G. Shimura, *Arithmeticity in the Theory of Automorphic Forms*, *Math. Surveys and Monogr.* **82**, Amer. Math. Soc, Providence, RI, 2002.

[WW1] R. Wells, Jr. and J. Wolf, Automorphic cohomology on homogeneous complex manifolds, *Rice Univ. Stud.* **6** (1979), Academic Press.

[WW2] ———, Poincaré series and automorphic cohomology of flag domains, *Ann. of Math.* **105** (1977), 397–448.

[Wi1] F. L. Williams, Discrete series multiplicities in $L^2(\Gamma\backslash G)$ (II). Proof of Langlands conjecture, *Amer. J. Math.* **107** (1985), 367–376.

[Wi2] ———, The $\mathfrak{n}$-cohomology of limits of discrete series, *J. Funct. Anal.* **80** (1988), 451–461.

[W1] J. A. Wolf, Flag Manifolds and representation theory, ESI preprint 342, 1996, available at http://www.esi.ac.at/preprints/ESI-Preprints.html .

[W2] ———, The Stein condition for cycle spaces of open orbits on complex flag manifolds, *Ann. of Math.* **136** (1992), 541–555.

[W3] ———, Exhaustion functions and cohomology vanishing theorems for open orbits on complex flag manifolds, *Math. Res. Lett.* **2** (1995), 179–191.

[WZ] J. A. Wolf and R. Zierau, Holomorphic double fibration transforms, in *The Mathematical Legacy of Harish-Chandra: A Celebration of Representation Theory and Harmonic Analysis* (R. S. Doran and V. S. Varadarajan, eds.), *Proc. Sympos. Pure Math.* **68**, 527–551, Amer. Math. Soc., Providence, RI, 2000.

Index

Notations used in the talks

$A^* =$ the dual of a vector space A
$A^{p,q}(X) = C^\infty(p,q)$ forms on a complex manifold X
$A^r = \underset{p+q=r}{\oplus} =$ polarized Hodge structure (PHS)
$(A)_\mathbb{R}$= real points in a complex vector space having a conjugation.
$\mathfrak{b} =$ Borel subalgebra
$\mathbb{B} =$ unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$
$\mathbb{B}^c = \mathbb{P}^2 \backslash$(closure of $\mathbb{B}$)
$\overline{\mathbb{B}} =$ unit ball with conjugate complex structure
$B =$ Cartan-Killing form or Borel subgroup, depending on the context
$d_\pi =$ relative differential
$D_\varphi =$ Mumford-Tate domain
$\boxtimes$ is the external tensor product
$F^p =$ Hodge filtration bundles
$G = \mathbb{Q}$-algebraic group
$G_\mathbb{R}, G_\mathbb{C} =$ corresponding real and complex Lie groups
$\mathfrak{g}^\alpha, \mathfrak{h}, X_\alpha$ etc. are standard notations from Lie theory listed in Lecture 2
$G_{\widetilde{\varphi}} =$ Mumford-Tate group of $(V, \widetilde{\varphi})$
$G_\varphi =$ Mumford-Tate group of (V, Q, φ)
$G_\mathcal{W} =$ part of $G_\mathbb{C}$ lying over $\mathcal{W}$
$\mathrm{Gr}(n, E) =$ Grassmannian of n-planes in a complex vector space E
$\mathbb{G}(n, E) =$ Grassmannian of $\mathbb{P}^{n-1}$'s in $\mathbb{P}E$
$G_L(n, E) =$ Lagrangian Grassmannian of n-planes P in a vector space E having a bilinear form Q and with $Q(P, P) = 0$
$\mathbb{G}_L(n, E) =$ Lagrangian Grassmannian of Lagrangian $\mathbb{P}^{n-1}$'s in $\mathbb{P}E$.
$h^{p,q} =$ Hodge numbers and $f^p = \sum_{p/\geqq p} h^{p',q'}$
$H^*_{\mathrm{DR}}\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big) =$ de Rham cohomology of global, relative F-valued holomorphic forms
$\mathcal{H} =$ upper half plane
$\mathcal{I} =$ incidence space
$\kappa_\mu =$ Kostant class
$\mathfrak{n} =$ direct sum of negative root spaces (except in the appendix to Lecture 6)
$\mathfrak{n}_c =$ direct sum of negative, compact root spaces
$\mathfrak{n}_{nc} =$ direct sum of negative, non-compact root spaces
$\mathcal{O}G_\mathcal{W} =$ global holomorphic functions on $G_\mathcal{W}$
$\mathcal{O}_{\mathbb{P}^n(k)} =$ standard line bundle over projective space

ω_Z = canonical line bundle for a complex manifold Z
Ω_μ = curvature form of $L_\mu \to D$
Ω_π = sheaf of relative differential forms
$\pi^*\mathcal{F}$ = pullback of a vector bundle
$\pi^{-1}\mathcal{F}$ = pullback of a coherent analytic sheaf
Φ_c^+, Φ_{nc}^+ are positive compact, respectively non-compact roots
Φ, Φ^+ = roots, respectively positive roots
$q(\mu) = \#\{\alpha \in \Phi_c^+ : (\mu, \alpha) < 0\} + \#\{\beta \in \Phi_{nc}^+ : (\mu, \beta) > 0\}$
$\rho = (1/2)$ (sum of positive roots)
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ = restriction of scalars
$s_2 \in W$ is reflection in the α root plane
σ_μ = Schmid class
$\mathbb{S}$ = $\mathbb{Q}$-algebraic group given by $\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) : a, b \in \mathbb{Q}$ and $a^2 + b^2 = 1\}$
$\mathcal{U}(\mathfrak{g}_\mathbb{C})$ = universal enveloping algebra
$\mathcal{U}$ = cycle space $\subset \check{\mathcal{U}} = G_\mathbb{C}/K_\mathbb{C}$
V = vector spaced defined over $\mathbb{Q}$
$V_\mathbb{R}, V_\mathbb{C} = V \otimes_\mathbb{Q} \mathbb{R}$, $V \otimes_\mathbb{Q} \mathbb{C}$
$V^{p,q}$ = Hodge (p,q) spaces
$(V, \widetilde{\varphi})$ = general Hodge structure
$\mathbb{V}^{p,q}$ = Hodge bundles
W = Weyl group of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h})$
W_K = Weyl group of $(\mathfrak{g}_\mathbb{R}, \mathfrak{t})$ = "compact" Weyl group
$\mathcal{W}$ = correspondence space included in its dual $\check{\mathcal{W}}$
χ_ζ = infinitesimal character
$Z_G(H)$ = centralizer in G of a subgroup $H \subset G$