Calculus II
Main Ideas

This document provides a summary of the major ideas discussed in class each day. Students are responsible for understanding and being able to apply these ideas, and there will be questions about them on quizzes and exams.

January 14, Chapter 5 review:

• If \( f(x) \) is a continuous function, the definition of \( \int_a^b f(x) \, dx \) is

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x,
\]

where \( \Delta x = \frac{b-a}{n} \) and \( c_i \) is a point in the \( i \)th subinterval of \([a, b]\). [Remark: the \( i \)th subinterval is \([a + (i - 1) \cdot \frac{b-a}{n}, a + i \cdot \frac{b-a}{n}]\). This formula can be useful when making computations from the definition].

• The definition allows for many different interpretations depending on context. For example, the most popular interpretation is area: if we just consider the graph \( y = f(x) \), then \( \Delta x \) is a width along the \( x \)-axis, \( f(c_i) \) is the height above (or below) the \( x \)-axis, and so \( f(c_i) \Delta x \) is the area of a rectangle (or negative area if \( f(c_i) < 0 \)). Then \( \sum_{i=1}^{n} f(c_i) \Delta x \) is an approximate total area, and taking the limit to get \( \int_a^b f(x) \, dx \) gives the exact area (counting area below the \( x \)-axis as negative area). Alternatively, we could interpret \( f(x) \) as giving linear density, and then \( f(c_i) \Delta x \) is mass of a small segment, \( \sum_{i=1}^{n} f(c_i) \Delta x \) is an approximate total mass, and taking the limit to get \( \int_a^b f(x) \, dx \) gives the exact mass. Or, we could have a function of time \( f(t) \) for which \( f(t) \) gives velocity at \( t \); then \( \Delta t \) is a time interval and \( f(c_i) \Delta t \) is displacement (positive or negative depending on the sign of \( f(c_i) \)), \( \sum_{i=1}^{n} f(c_i) \Delta t \) is an approximate total displacement, and taking the limit to get \( \int_a^b f(t) \, dt \) gives the exact displacement.

January 16, Chapter 5 review

• To actually compute definite integrals, the best way is the Fundamental Theorem of Calculus (FTC): If \( \frac{dF}{dx} = f \), then \( \int_a^b f(x) \, dx = F(x) \big|_a^b = F(b) - F(a) \).

• You should know the integration rules on page 366 and how to use them together with the substitution technique.

January 17, Section 7.1

• If \( f(x) \) and \( g(x) \) are functions, we can use integrals to find the area between their graphs. In particular, if \( f(x) \geq g(x) \) and we want the area between \( x = a \) and \( x = b \), then we can get the area as the Riemann sum \( \lim_{n \to \infty} \sum_{i=1}^{n} (f(c_i) - g(c_i)) \Delta x \) because now \( \Delta x \) is still the width of a subinterval between \( a \) and \( b \) and \( f(c_i) - g(c_i) \) gives us the height between the two curves at \( c_i \). In the limit, we get \( \int_a^b f(x) - g(x) \, dx \) as the area between the curves.
• If it is not always true that \( f(x) \geq g(x) \), then we have to break the integral into pieces so that we always subtract the bottom functions from the top function at each point (since we do not want to consider signed area as in Calculus I).

• Sometimes it’s useful to find areas of regions bounded by curves for which we consider \( y \) to be the independent variable and let \( x \) depend on \( y \) as \( x = f(y) \). All the ideas are exactly the same and the area between \( f(y) \) and \( g(y) \) is \( \int_a^b f(y) - g(y) \, dy \), assuming \( f(y) \) is to the right of \( g(y) \). If there are multiple regions between the graphs, be careful again to use separate integrals for each piece.

January 21, Section 7.2

• To find the volume of a solid, we can slice it up into slices perpendicular to the \( x \)-axis and approximate by \( \sum_{i=1}^n A(c_i) \Delta x \), where \( A(c_i) \) is the area of the cross section of the solid at \( x = c_i \). In the limit, we get the exact volume, which is \( \int_a^b A(x) \, dx \).

• If we rotate a curve \( y = f(x) \), \( a \leq x \leq b \) around an axis \( y = L \) and want to find the volume enclosed, the cross sections will be disks. The cross section area can then be computed using the formula for the area of a disk: \( A = \pi r^2 \).

• If we rotate the area between two curves around an axis, then the cross sections will be washers and we can compute the cross section area using that the area of a washer is \( \pi R^2 - \pi r^2 \), where \( R \) is the big radius and \( r \) is the little radius.

January 23, Section 7.3

• Instead of slicing volumes of revolution up into washers, we can instead think of the volumes as being made up of nested cylindrical shells. In this case the integrals look like \( \int_a^b A(x) \, dx \), where \( A(x) \) is the surface area of the shell the goes through \( x \). Use that the surface area of a cylinder is \( 2\pi rh \).

January 27, Section 7.4

• Suppose we want to find the length of the curve \( y = f(x) \), \( a \leq x \leq b \). We can approximate the length using straight line segments: divide \([a, b]\) into \( n \) subintervals \([x_i, x_{i+1}]\) and use the straight line segment connecting \((x_i, f(x_i))\) to \((x_{i+1}, f(x_{i+1}))\). If we let \( \Delta x = x_{i+1} - x_i = \frac{b-a}{n} \) and \( \Delta y = f(x_{i+1}) - f(x_i) \), then the length of each line segments is \( \sqrt{(\Delta x)^2 + (\Delta y)^2} \), whence we can rewrite (somewhat mysteriously) as \( \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \cdot \Delta x \). So the approximate length of the curve is \( \sum_{i=1}^n \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \cdot \Delta x \) (note that \( \Delta x \) is independent of which interval we’re looking at while \( \Delta y \) does depend on the interval, but we omit it from the notation). We get the exact length of the curve by taking

\[
\lim_{n \to \infty} \sum_{i=1}^n \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \cdot \Delta x
\]
But this looks like a definite integral and we know (from Calc I!) that \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{df}{dx} \).

So with a little work (see the book), this limit becomes the integral

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \cdot \Delta x = \int_{a}^{b} \sqrt{1 + \left( \frac{df}{dx} \right)^2} \, dx.
\]

This is our formula for arc length.

January 28-29, Section 7.5

- In physics, the work done by a constant force moving an object a given distance is defined to be Work equals Force times Distance.

- If we work with a variable force \( F \) that depends on the location \( x \), then to compute the work of something being moved from \( a \) to \( b \) through that force, we can break \([a,b]\) up into subintervals and approximate the work on the \( i \)th subinterval by \( F(c_i) \Delta x \), where \( c_i \) is in the \( i \)th subinterval. Then the total approximate work is \( \sum F(c_i) \Delta x \) and the exact amount of work is \( \lim_{n \to \infty} \sum F(c_i) \Delta x = \int_{a}^{b} F(x) \, dx \).

- Alternatively, we might need to compute the work done by moving an object where different pieces of the object move different distances. For example, if we want to pump a fluid out of a tank (assuming we always pump from the top), we can divide up the height of the tank into intervals of height \( \Delta y \). Then on the \( i \)th interval, the volume of fluid is approximately \( A(c_i) \Delta y \) for some \( c_i \) in the \( i \)th interval, where \( A(y) \) is the cross-section area, just like in volume problems. If \( d \) is the density of the fluid in weight per volume, the weight of the section of fluid in the \( i \)th interval is approximately \( dA(c_i) \Delta y \). Additionally, we need a function \( f(y) \) that tells us how far we have to move this slab of fluid (usually something like \( f(y) = h - y \) where \( y \) is the height of the tank, though this could depend on the problem). So then the amount of work done moving the slab of fluid in the \( i \)th interval is approximately \( dA(c_i) f(c_i) \Delta y \). Adding and taking the limit, we see that the work is given by \( \int_{a}^{b} d \cdot A(y) \cdot f(y) \, dy \). See the book in Section 7.5 for other examples.

January 31-February 4, Section 8.2

- Integrals are hard! Unlike derivatives, where a few formulas let us compute anything, there are some functions that have no “nice” anti-derivative formulas at all (though by the Fundamental Theorem of Calculus, if \( f(x) \) is continuous, then \( \int_{a}^{x} f(t) \, dt \) is an antiderivative of \( f(x) \)). We will try to find antiderivative formulas when we can.

- Similar looking functions might have very different processes for finding their antiderivatives, so be very careful, make sure to use correct rules, and sometimes you might need to make multiple attempts.

- Integration by parts: From the product rule \( \frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g + f \frac{dg}{dx} \). So if we take antiderivatives, we get \( fg = \int \frac{d}{dx}g \, dx + \int f \frac{dg}{dx} \, dx \), so

\[
\int f \frac{dg}{dx} \, dx = fg - \int \frac{df}{dx}g \, dx.
\]
We can use this to replace integral problems with simpler integrals.

- Example: $f(x) = x$, $g(x) = e^x$. Then $\int xe^x \, dx = xe^x - \int e^x = xe^x - e^x + C$.

- A mnemonic for integration by parts: $\int uv \, dv = uv - \int v \, du$

- Sometimes you need to use integration by parts multiple times in one problem, for example to integrate $\int x^n e^x \, dx$. You’d use the process to make $n$ smaller each time.

- It’s okay to just have $dv = dx$ if necessary!

Section 8.3

- There are various techniques for integrating products of trig functions. Be careful in using them.

- If you want to integrate $\sin^n x \cos^m x$ where $m$ is odd, save one $\cos x$, convert the rest to $\sin x$ using $\sin^2 x + \cos^2 x = 1$ and then substitute with $u = \sin x$.

- If $n$ is odd, save one $\sin x$, convert the rest to $\cos x$ using $\sin^2 x + \cos^2 x = 1$ and then substitute with $u = \cos x$. If both $m$ and $n$ are even, start with the trig identities $\cos^2 x = \frac{1+\cos(2x)}{2}$, $\sin^2 x = \frac{1-\cos(2x)}{2}$, and then work from there.

- If you want to integrate $\tan^n x \sec^m x$ where $m$ is even, save a $\sec^2 x$, convert the rest of the secants to tangents using $\tan^2 x + 1 = \sec^2 x$ and then substitute with $u = \tan x$. If $n$ is odd and $m > 0$, save a $\sec x \tan x$, convert the remaining tangents to secants using $\tan^2 x + 1 = \sec^2 x$ and then substitute with $u = \sec x$.

- For other situations, see the rules in the book.

- $\int \sec x = \ln |\sec x + \tan x| + C$. Unfortunately, the derivation requires some cleverness.

- To find $\int \sec^3 x$, do integration by parts using $u = \sec x$, $dv = \sec^2 x \, dx$. Then $du = \tan x \sec x \, dx$, $v = \tan x$ and so $\int \sec^3 x = \tan x \sec x - \int \tan^2 x \sec x = \tan x \sec x - \int (\sec^2 x - 1) \sec x = \tan x \sec x + \int \sec x - \int \sec^3 x$. Using the formula for $\int \sec x$ and solving for $\int \sec^3 x$ gives $\int \sec^3 x = \frac{1}{2} (\tan x \sec x + \ln |\sec x + \tan x|) + C$.

February 13, Section 8.4

- Trig substitution is another useful technique using the trig identities $\sin^2 \theta + \cos^2 \theta = 1$ and $\tan^2 \theta + 1 = \sec^2 \theta$. If you want to integrate something involving $1 - x^2$, use $x = \sin \theta$. If you want to integrate something involving $1 + x^2$, use $x = \tan \theta$. If you want to integrate something involving $x^2 - 1$, use $x = \sec \theta$. If there are other constants, put constant factors into your substitutions so that you can simplify to use the trig identities. Don’t forget to reverse substitute and simplify at the end.

February 17, Section 8.5

- A rational function is a quotient $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials.
To integrate rational functions, step 1: do polynomial long division to write \( \frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \), where \( q(x) \) is the quotient polynomial and \( r(x) \) is the remainder polynomial with \( \deg(r(x)) < \deg(g(x)) \). Then it’s easy to integrate \( q(x) \).

To integrate rational functions, step 2: To integrate \( \frac{f(x)}{g(x)} \) when \( \deg(f(x)) < \deg(g(x)) \), we use the method of partial fractions.

By the fundamental theorem of algebra, we can always (in principle) factor \( g(x) \) as
\[
g(x) = a(x-c_1)^{n_1}(x-c_2)^{n_2} \cdots (x-c_k)^{n_k}(x^2+d_1x+e_1)^{m_1}(x^2+d_2x+e_2)^{m_2} \cdots (x^2+d_\ell x+e_\ell)^{m_\ell},
\]
where \( a \) and each of the \( c_i, d_i, e_i \) are real numbers, the \( n_i \) are positive integer powers, the terms of the expression are all unique, and the quadratic terms cannot be factored further into linear terms.

If \( g(x) \) is factored as above, the partial fractions expansion of \( \frac{f(x)}{g(x)} \) has the form
\[
\frac{f(x)}{g(x)} = \frac{C_{1,1}}{x-c_1} + \frac{C_{1,2}}{(x-c_1)^2} + \frac{C_{1,3}}{(x-c_1)^3} + \cdots + \frac{C_{1,n_1}}{(x-c_1)^{n_1}} + \frac{C_{2,1}}{x-c_2} + \frac{C_{2,2}}{(x-c_2)^2} + \frac{C_{2,3}}{(x-c_2)^3} + \cdots + \frac{C_{2,n_2}}{(x-c_2)^{n_2}} + \cdots
\]
\[
+ \frac{D_{1,1}x+E_{1,1}}{x^2+d_1x+e_1} + \frac{D_{1,2}x+E_{1,2}}{(x^2+d_1x+e_1)^2} + \cdots + \frac{D_{1,m_1}x+E_{1,m_1}}{(x^2+d_1x+e_1)^{m_1}} + \cdots
\]
\[
+ \frac{D_{\ell,1}x+E_{\ell,1}}{x^2+d_\ell x+e_\ell} + \frac{D_{\ell,2}x+E_{\ell,2}}{(x^2+d_\ell x+e_\ell)^2} + \cdots + \frac{D_{\ell,m_\ell}x+E_{\ell,m_\ell}}{(x^2+d_\ell x+e_\ell)^{m_\ell}}
\]

Notice that for each factor of \( g(x) \) of the form \( (x-c_i)^{n_i} \), there are \( n_i \) summands of this expansion, with increasing powers in the denominators but with constant numerators. For each factor of \( g(x) \) of the form \( (x^2+d_\ell x+e_\ell)^{m_\ell} \), there are \( m_\ell \) summands of the expansion, with increasing powers in the denominators but with linear numerators.

To integrate, we must determine all these coefficients. Then we can integrate the individual terms using previous techniques.

February 18, Section 8.5

To integrate a rational function \( f(x)/g(x) \) follow these steps:

1. Do polynomial division to get \( q(x) + r(x)/g(x) \) with \( \deg(r(x)) < \deg(g(x)) \).
2. Factor \( g(x) \).
3. Use the factorization of \( g \) to write out the partial fractions form for \( r(x)/g(x) \).
4. Do algebra to find the coefficients in the partial fractions form.

5. Integrate.

February 20, Section 8.7

- Indeterminate forms: If you try to evaluate a limit in the obvious way and obtain any of these, then you can’t draw any conclusions and need to do more work: \( \frac{0}{0} \), \( \frac{\pm \infty}{\pm \infty} \), \( \infty - \infty \), \( \pm \infty \cdot 0 \), \( 1^{\pm \infty} \), \( 0^{0} \), \( \infty^{0} \).

- If \( f(c) = g(c) = 0 \) and \( f, g \) are differentiable at \( c \), then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)}
\]

\[
= \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}
\]

\[
= \lim_{x \to c} \frac{f(x)-f(c)}{g(x)-g(c)}
\]

\[
= \frac{df}{dx}(c) \frac{dg}{dx}(c)
\]

- More generally, L’Hospital’s rule states that if the obvious attempt to evaluate \( \lim_{x \to c} \frac{f(x)}{g(x)} \) yields \( \frac{0}{0} \) or \( \frac{\pm \infty}{\pm \infty} \), then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{df}{dx} \frac{dg}{dx}
\]

Here \( c \) can be a number of \( \pm \infty \).

- Important: make sure the hypothesis holds before applying the rule!

February 24, Section 8.7

- When limits yield indeterminate forms other than \( 0/0 \) or \( \infty/\infty \), it is often still possible to use L’Hospital’s rule after using some other techniques.

- If \( \lim_{x \to c} f(x)g(x) \) yields an indeterminate form of \( 0 \cdot \pm \infty \), try rewriting \( f(x)g(x) \) as \( \frac{f(x)}{g(x)} \) or \( \frac{g(x)}{f(x)} \).

- If \( \lim_{x \to c} f(x) + g(x) \) yields an indeterminate form of \( \infty - \infty \), try rewriting \( f(x) + g(x) \) as a single fraction.

- If \( \lim_{x \to c} f(x) \) yields an indeterminate form of \( 1^{\infty} \), \( 0^{0} \), or \( \infty^{0} \), try using the rule

\[
\lim_{x \to c} f(x) = \lim_{x \to c} e^{lnf(x)} = e^{\lim_{x \to c} ln f(x)}
\]
By definition, \( \int_a^\infty f(x) \, dx \) means \( \lim_{b \to \infty} \int_a^b f(x) \, dx \).

Similarly, \( \int_{-\infty}^b f(x) \, dx \) means \( \lim_{a \to -\infty} \int_a^b f(x) \, dx \).

These are called improper integrals. In either case, if the limit exists, we say the integral converges; otherwise we say it diverges.

To evaluate \( \int_{-\infty}^\infty f(x) \, dx \), break up the integral into \( \int_{a}^\infty f(x) \, dx + \int_{-\infty}^a f(x) \, dx \) and use the rules above. Do NOT try to evaluate using \( \lim_{a \to \infty} \int_a^{-\infty} f(x) \, dx \).

The second type of improper integral occurs if the function is not continuous at \( c \). In this case evaluate \( \int_c^b f(x) \, dx \) as \( \lim_{a \to c} \int_a^b f(x) \, dx \), evaluate \( \int_a^c f(x) \, dx \) as \( \lim_{b \to c} \int_a^b f(x) \, dx \), or, if \( a < c < b \), evaluate \( \int_a^c f(x) \, dx \) as \( \lim_{b \to c} \int_a^b f(x) \, dx + \lim_{a \to c} \int_a^b f(x) \, dx \).

If \( f \) is not continuous at some point in the interval you want to integrate over, the Fundamental Theorem of Calculus does not apply! Do not try to use it. You will get the wrong answer!

February 27, Section 9.1

A sequence is an ordered list of numbers. Mostly we will be interested in infinite sequences.

We often denote a sequence as \( a_1, a_2, a_3, a_4, \ldots \).

We can describe a sequence by just listing the numbers (such as \( 4, 14, 23, 34, 42, 50, 59 \)), by providing a formula (such as \( a_n = n^2 \)), or by providing a recursion (such as \( a_n = a_{n-1} + a_{n-2} \) with \( a_1 = a_2 = 1 \)).

If a sequence is infinite, we say \( \lim_{n \to \infty} a_n = L \), if \( a_n \) gets “closer and closer” to \( L \) as \( n \) gets large. The technical definition is in the book. The basic idea is the same as for functions.

Theorem: if there is a function \( f(x) \) such that \( a_n = f(n) \) and \( \lim_{x \to \infty} f(x) = L \), then also \( \lim_{n \to \infty} a_n = L \). (Be careful, the converse (look it up) does not always hold).

Limits of sequences share many properties with limits of functions, such as formulas for sums, products, and scalar products, and a squeeze theorem.

February 28, Section 9.1

\( n! \) is pronounced “\( n \) factorial” and means \( 1 \cdot 2 \cdot 3 \cdots n \). Note: \( 0! \) is defined to be 1.

Some things about sequences: A sequence is called increasing if \( a_{n+1} \geq a_n \). A sequence is called decreasing if \( a_{n+1} \leq a_n \). A sequence is called monotonic if it is increasing or decreasing. A sequence is called bounded above if there is some number \( M \) such that \( a_n \leq M \) for all \( n \). A sequence is called bounded below if there is some number \( m \) such that \( a_n \geq m \) for all \( n \). A sequence is called bounded if it is both bounded above and bounded below.
• We can talk about any of these properties happening *eventually*, meaning that they hold for $n$ large enough.

• Theorem (from real analysis): If a sequence is (eventually) increasing and bounded above, it has a limit. If a sequence is (eventually) decreasing and bounded below, it has a limit. This theorem is used in the detailed proof of the integral test (see the book or class notes for details).

March 3, 4 - Section 9.2

• A *series* is a sum of all the terms in a sequence: $a_1 + a_2 + a_3 + a_4 + \cdots$. For an infinite sequence, we can write $\sum_{i=1}^{\infty} a_i$.

• To define what this means concretely, we let $S_n$ be the partial sum $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

• If $\lim_{n \to \infty} S_n = L$, we say the series $\sum_{i=1}^{\infty} a_i$ converges and write $\sum_{i=1}^{\infty} a_i = L$. If the limit does not exist, we say that $\sum_{i=1}^{\infty} a_i$ diverges.

• A useful alternative notation is to write $S_n = \sum_{i=1}^{n} a_i$. Then, by definition, $\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$.

• It is sometimes possible to compute a series $\sum_{i=1}^{\infty} a_i$ if there is a lot of cancellation amongst terms. These are called *telescoping series*. For example, consider $\sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{i+1}$.

$$S_N = \sum_{i=1}^{N} \frac{1}{i} - \frac{1}{i+1}$$

$$= (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots + (1/N - 1/(N + 1))$$

$$= 1 - 1/(N + 1),$$

So $\sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{i+1} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - 1/(N + 1) = 1$.

• IMPORTANT OBSERVATION: Suppose $\sum_{i=1}^{\infty} a_i = L$. That means that $\lim_{N \to \infty} \sum_{i=1}^{N} a_i = L$. But notice that this is the same as $\lim_{N \to \infty} \sum_{i=1}^{N-1} a_i = L$. So then

$$\lim_{N \to \infty} \sum_{i=1}^{N} a_i - \lim_{N \to \infty} \sum_{i=1}^{N-1} a_i = L - L = 0$$

but

$$\lim_{N \to \infty} \sum_{i=1}^{N} a_i - \lim_{N \to \infty} \sum_{i=1}^{N-1} a_i = \lim_{N \to \infty} (\sum_{i=1}^{N} a_i - \sum_{i=1}^{N-1} a_i)$$

$$= \lim_{N \to \infty} a_N.$$

CONCLUSION: If $\sum_{i=1}^{\infty} a_i$ converges, then we must have $\lim_{N \to \infty} a_N = 0$. 8
• The contrapositive statement to the preceding conclusion is the important TEST FOR DIVERGENCE: If \( \lim_{N \to \infty} a_N \neq 0 \), then \( \sum_{i=1}^{\infty} a_i \) diverges.

• Note: THIS DOES NOT SAY THAT IF \( \lim_{N \to \infty} a_N = 0 \) THEN \( \sum_{i=1}^{\infty} a_i \) CONVERGES. THIS IS NOT TRUE! IF \( \lim_{N \to \infty} a_N = 0 \), THEN \( \sum_{i=1}^{\infty} a_i \) MIGHT CONVERGE OR MIGHT DIVERGE. MORE WORK IS NEEDED.

• A series of the form \( \sum_{i=0}^{\infty} ar^i \) is called a geometric series.

• Technique for find partial sums for a geometric series:
\[
\sum_{i=0}^{n} ar^i = a + ar + ar^2 + ar^3 + \cdots + ar^n
\]
\[
(1 - r) \sum_{i=0}^{n} ar^i = (1 - r)(a + ar + ar^2 + ar^3 + \cdots + ar^n)
\]
\[
= (a - ar) + (ar - ar^2) + (ar^2 - ar^3) + \cdots - ar^n + (ar^n - ar^{n+1})
\]
\[
= a - ar^{n+1}
\]

So \( \sum_{i=0}^{n} ar^i = \frac{a-ar^{n+1}}{1-r} \), and
\[
\sum_{i=0}^{\infty} ar^i = \lim_{n \to \infty} \sum_{i=0}^{n} ar^i
\]
\[
= \lim_{n \to \infty} \frac{a-ar^{n+1}}{1-r}.
\]

If \( |r| < 1 \), this converges to \( \frac{a}{1-r} \). Otherwise, it diverges.

• Note: the above formula is for precisely the case \( \sum_{i=0}^{\infty} ar^i \). If the starting index \( i \) is not 0, you might need to adjust by adding extra terms or subtracting terms that aren’t included in the series.

March 17, Section 9.3

• The convergence or divergence of a series does not depend on any finite number of terms. In other words, \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \sum_{n=M}^{\infty} a_n \) converges for every finite \( M \geq 1 \).

• Integral test: suppose \( f(x) \) is a function such that, for large enough \( x \), \( f \) is continuous, \( f(x) > 0 \), and \( f \) is a decreasing function (by saying “large enough” we mean that these properties are true on some interval \([M, \infty)\)). Suppose that \( a_n = f(n) \). Then \( \sum_{n=M}^{\infty} f(n) \) (and hence \( \sum_{n=1}^{\infty} f(n) \)) converges if and only if \( \int_{M}^{\infty} f(x) \, dx \) converges. In other words, whether the integral converges or diverges tells us whether the series converges or diverges.

• The integral test comes from comparing areas - see the book or class notes for details.
March 20, Section 9.4

- The **comparison test**: Suppose that eventually $0 \leq a_n \leq b_n$, then
  
  1. If \( \sum_{n=1}^{\infty} a_n \) diverges, so does \( \sum_{n=1}^{\infty} b_n \)
  2. If \( \sum_{n=1}^{\infty} b_n \) converges, so does \( \sum_{n=1}^{\infty} a_n \)

  See the book or class notes for the proof.

March 21

- The limit comparison test: If \( a_n, b_n \geq 0 \) (eventually) and \( \lim_{n \to \infty} \frac{a_n}{b_n} \) exists and is not 0 (or \( \infty \)), then \( \sum a_n \) and \( \sum b_n \) either both converge or both diverge.

- Reasoning for limit comparison test [optional]: Since \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0 \), for large enough \( n \), \( \frac{a_n}{b_n} \leq L + 1 \). So \( 0 \leq a_n \leq (L + 1)b_n \). Now, if \( \sum b_n \) converges, so does \( \sum (L + 1)b_n \), so by the comparison test, so does \( \sum a_n \). On the other hand, if \( \sum a_n \) diverges, so does \( \sum (L + 1)b_n = (L + 1)\sum b_n \) and so \( \sum b_n \) converges. This provides half of the necessary statements. For the others, if \( \lim_{n \to \infty} \frac{a_n}{b_n} = L = 0 \) then \( \lim_{n \to \infty} \frac{b_n}{a_n} = 1/L \neq 0 \). So eventually \( 0 \leq b_n \leq (\frac{1}{L} + 1)a_n \). So similarly, if \( \sum a_n \) converges so does \( \sum b_n \) and if \( \sum b_n \) diverges, so does \( \sum a_n \). This provides all the statements of the limit comparison test.

March 24, Section 9.5

- Suppose \( \sum a_n \) is a series and that \( \sum |a_n| \) converges. Then we say that \( \sum a_n \) is **absolutely convergent** and it follows that \( \sum a_n \) also converges. (Reason: \( 0 \leq a_n + |a_n| \leq 2|a_n| \), so if \( \sum |a_n| \) converges, so does \( \sum 2|a_n| = 2\sum |a_n| \), so by the comparison test \( \sum (a_n + |a_n|) \) converges. But then \( \sum (a_n + |a_n|) - \sum |a_n| \) converges, and it is equal to \( \sum (a_n + |a_n|) - \sum |a_n| = \sum (a_n + |a_n| - |a_n|) = \sum a_n. \) So \( \sum a_n \) converges.)

- If \( \sum a_n \) converges, but \( \sum |a_n| \) does not, we say that \( \sum a_n \) is **conditionally convergent**.

- Alternating series test: If 1) the terms \( a_n \) alternate in sign, 2) \( |a_n| \) is eventually decreasing, and 3) \( \lim_{n \to \infty} a_n = 0 \), then \( \sum a_n \) converges.

- Proof of alternating series test [optional]: assume the series is \( a_1 - a_2 + a_3 - a_4 + \cdots \), with \( a_i \geq 0 \) (otherwise, we could multiply by \(-1\) and make the same argument). Then \( S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) + \cdots - a_{2n} \). Since each \( a_i - a_{i+1} \geq 0 \) (since the terms decrease), \( S_{2n} \leq a_1 \). But also \( S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots \), which is increasing since
again each \( a_i - a_{i+1} \geq 0 \). Therefore the sequence \( S_{2n} \) is increasing and bounded, so it has a limit \( \lim_{n \to \infty} S_{2n} = L \). But now since \( S_{2n} = S_{2n-1} + a_{2n} \), we have \( S_{2n-1} = S_{2n} - a_{2n} \) and \( \lim_{n \to \infty} S_{2n-1} = \lim_{n \to \infty} S_{2n} - \lim_{n \to \infty} a_{2n} = L + 0 = L \), since \( \lim_{n \to \infty} a_n = 0 \). Therefore since \( \lim_{n \to \infty} S_{2n-1} = \lim_{n \to \infty} S_{2n} \), we must have \( \lim_{n \to \infty} S_n = L \). So the series converges.

March 25, Section 9.6

- The ratio test: If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges absolutely; if \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges; if \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 1 \), then the ratio test is inconclusive.

- The root test: If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges absolutely; if \( \lim_{n \to \infty} \sqrt[n]{|a_n|} > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges; if \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \), then the ratio test is inconclusive.

- The argument for the first part of the ratio test: If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1 \), then choose some \( R \) with \( L < R < 1 \). Then \( \frac{|a_{n+1}|}{|a_n|} < R \) once \( n > N \) for some big enough \( N \). But then \( |a_{N+1}| < R|a_N|, |a_{N+2}| < R|a_{N+1}| < R^2|a_N|, \) and so on, so that \( |a_{N+k}| < R^k|a_N| \). So then

\[
\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots
\leq |a_1| + |a_2| + \cdots + |a_N| + |a_N|R + |a_N|R^2 + |a_N|R^3 + \cdots
\]

So \( \sum_{n=1}^{\infty} |a_n| \) converges by comparison to a geometric series, and \( \sum_{n=1}^{\infty} a_n \) converges absolutely.

March 27, Sections 9.6

- Recommended order to apply convergence/divergence tests:

1. Divergence test
2. Look for special series: geometric, telescoping, \( p \)-series, alternating series
3. the ratio test and the root test; note: the ratio test is most useful when factorials are involved or \( n \) occurs in an exponent
4. comparison or limit comparison tests; usually easier to use the limit version
5. integral test

March 27, Section 9.7

- New game: choose a function \( f(x) \) and a point \( x = c \). We want to find an \( n \)th degree polynomial \( P_n(x) \) (called a Taylor polynomial) such that \( P_n(x) \) is the best possible approximation to \( f(x) \) near \( x = c \).
Generically, $P_n(x)$ will have the form

$$P_n(x) = a_0 + a_1(c-x) + a_2(x-c)^2 + \cdots + a_n(x-c)^n = \sum_{k=0}^{n} a_k(x-c)^k.$$ 

Notice that if we really wanted to we could multiply out the terms and rewrite this in the standard way to look like $P_n(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$, but it will be useful to write it as we have it. (Conversely, if we have a polynomial $b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$, we can always rewrite it as $a_0 + a_1(c-x) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ — just multiply the right side out and then solve the equations for the $a_i$ in terms of the $b_i$ — but we won’t usually need to do this by hand.)

How do we make sure $P_n(x)$ is a good approximation to $f(x)$ at $c$? We want to choose the coefficients of $P_n(x)$ in such a way that as many derivatives as possible of $f(x)$ and $P_n(x)$ agree at $c$. In other words, we want $f(c) = P_n(c)$, $\frac{df}{dx}(c) = \frac{dP_n}{dx}(c)$, and so on up to $\frac{d^nf}{dx^n}(c) = \frac{d^nP_n}{dx^n}(c)$. Notice that since $P_n(x)$ is a polynomial of degree $n$, any derivatives of higher order must be 0, so this is the best we can do. The idea is that we are trying to get the two functions to match location, velocity, acceleration, etc. at $x = c$.

So let’s see what this forces the $a_0$ to be.

1. $P_n(c) = a_0$, so if we want $f(c) = P_n(c)$, we need to have $a_0 = f(c)$.
2. $\frac{dP_n}{dx}(c) = a_1$ (check this for yourself!), so if we want $\frac{df}{dx}(c) = \frac{dP_n}{dx}(c)$, we need to have $a_1 = \frac{df}{dx}(c)$.
3. $\frac{d^2P_n}{dx^2}(c) = 2a_2$ (check this for yourself!), so if we want $\frac{d^2f}{dx^2}(c) = \frac{d^2P_n}{dx^2}(c)$, we need to have $a_2 = \frac{1}{2} \frac{d^2f}{dx^2}(c)$.
4. In general, for $k \leq n$, it turns out that $\frac{d^kP_n}{dx^k}(c) = k!a_k$ (check this for yourself!), so if we want $\frac{d^kf}{dx^k}(c) = \frac{d^kP_n}{dx^k}(c)$, we need to have $a_k = \frac{1}{k!} \frac{d^kf}{dx^k}(c)$.

So in general the coefficients $a_k$ must be $a_k = \frac{1}{k!} \frac{d^k}{dx^k}(c)$, and the Taylor polynomial is $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k}{dx^k}(c)(x-c)^k$

March 31, Section 9.7

Note: When $c = 0$, the Taylor series $P_n(c)$ is also sometimes called a Maclaurin series. Everything is the same except $c = 0$.

The remainder of a Taylor polynomial is $R_n(x) = f(x) - P_n(x)$.

We can estimate the remainder at $x$ by $|R_n(x)| \leq \frac{1}{(n+1)!}|x-c|^{n+1} \max |\frac{d^{n+1}}{dx^{n+1}}(x)|$, where the maximum is taken over the interval $[c, x]$ if $x \geq c$ or $[x, c]$ if $x \leq c$.  

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• Rarely do we need to use Calc I to actually compute the max. In many cases we can use that the derivative is increasing or decreasing on the whole interval, so the max is at an endpoint, or we can use an other estimate, such at $|\sin x| \leq 1$. (Note: since our goal in working with remainders is to find upper bounds, we need them to be true, but we don’t always need them to be as precise as possible.)

April 1, Section 9.8

• Power series are “infinite polynomials” $\sum_{n=0}^{\infty} a_n(x-c)^n$. Here $c$ is a fixed number, and we say the power series is “centered at $c$”.

• If we plug in a specific number for $x$, then $\sum_{n=0}^{\infty} a_n(x-c)^n$ might converge or diverge, and we’d like to know where it does each one.

• Theorem: there are three possibilities:
   1. $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges only when $x = c$ (we say the radius of convergence is $R = 0$), or
   2. $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges for every $x$ (we say the radius of convergence is $R = \infty$), or
   3. there is some specific positive number $R$ such that $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$. The number $R$ is called the radiance of converges. Depending on the series, there might be convergence or divergence at $x = c + R$ and $x = c - R$.

• The set of points on which a power series converges is called its “interval of convergence”. By the theorem, the interval is centered at $c$, has a radius $R$, and might or might not include each endpoint.

• The best tool for finding radius of convergence is the ratio test. Then the endpoints need to be checked “by hand”

April 3, Section 9.8

• If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series with radius of converge $R$, then $\frac{df}{dx} = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ and also has radius of convergence $R$.

• If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series with radius of converge $R$, then $\int f(x) = k + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$ and also has radius of convergence $R$.

April 5, Section 9.10

• If $a > 1$, $\lim_{n \to \infty} \frac{a^n}{n!} = 0$.

• Suppose we compute some $P_n$ but then let $n$ go to infinity. In other words, given $f(x)$ and $c$, we can form the power series $T(x) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(c)(x-c)^i$. This is called the Taylor series of $f(x)$ centered at $c$. (If $c = 0$, we can call $T(x)$ the MacLaurin series for $f(x)$.)
There are then two interesting questions: 1) What is the interval of convergence of $T(x)$? 2) If $x$ is in the interval of convergence, is $T(x) = f(x)$?

The answer to question 2) is not always yes: Notice that if two functions $f(x)$ and $g(x)$ have the exact same derivatives at $c$, their Taylor series will be the same, even if the functions are not identical elsewhere.

To determine whether or not the answer to 2) is “yes”, we can look at the Taylor polynomial remainders $R_n(x)$: Wherever $R_n(x)$ goes to 0 (as $n$ goes to infinity), then $T(x) = f(x)$ there.

April 8, Section 9.10

If you already know the Maclaurin series for $f(x)$, it’s easy to get the Taylor series for things like $f(x^2)$ (by just plugging in) or $xf(x)$ (by multiplying like polynomials).

If you already know the Taylor series for $f(x)$, it’s also easy to get the Taylor series for $\frac{df}{dx}$ and $\int f(x) \, dx$ by taking the derivative or antiderivative of the series as in section 9.8.

April 14, Section 9.9

Another easy way to get a power series: notice that if $|x| < 1$ then we already know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ from our study of geometric series.

We can use this to get a variety of power series by algebraic manipulation of this expression.

April 15, Section 10.2

A parameterized curve consists of two functions $x(t)$ and $y(t)$. We think of $t$ as representing time and then the function pair $(x(t), y(t))$ give us coordinates in the plane at each time. We can think of this as describing our position as a function of time as we walk around the plane.

The most basic way to try to understand the curve $(x(t), y(t))$ is to start plotting points. Along with connecting the dots to make a curve, we also draw in arrows to indicate the orientation, meaning which direction we are moving as time increases.

If we do some algebra to eliminate $t$, we can often get an expression just involving $x$ and $y$ (for example if $x = t^2$ and $y = t^4$, we see that $y = x^2$). All points of the the parameterized curve will lie on the curve given by the equation in $x$ and $y$; for example all points of $(x(t), y(t)) = (t^2, t^4)$ lie on the curve $y = x^2$. However, we have lost information about a) exactly which points of the curve are involved (in our example, we never travel over the points of $y = x^2$ with $x < 0$), and b) exactly when we are where on the curve. Nonetheless, this is a useful technique for helping to understand what the parameterized curve looks like.
Suppose we are given a curve and want to parameterize a walk along it. There are some easy cases where this isn’t too hard:

1. If the curve is a graph \( y = f(x), \ a \leq x \leq b \), we can simply use \( x(t) = t, \ y(t) = f(t), \ a \leq t \leq b \). This goes left to right. To go right to left, instead use \( x(t) = -t, \ y(t) = f(-t), \ -b \leq t \leq -a \). It’s also not too hard to modify these constructions to meet other requirements, like being at a certain point at a certain time.

2. To move counterclockwise along the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), use \( x(t) = a \cos t \) and \( y(t) = b \sin t \). To move clockwise, we could use \( x(t) = a \cos t \) and \( y(t) = -b \sin t \). Both these parameterizations start on the positive \( x \) axis. To start at other axis locations, try interchanging \( \pm \cos( t ) \) and \( \pm \sin( t ) \) (or shift the time variable to \( t + C \) for an appropriate \( C \)).

April 17, Section 10.3

- Suppose we have a parametrized curve \( (x(t), y(t)) \) and want to find the slope of the tangent line to the curve. We can proceed as in calculus I: the secant lines will have slopes \( \frac{y(t+\Delta t)-y(t)}{x(t+\Delta t)-x(t)} \). And we can compute

\[
\lim_{\Delta t \to 0} \frac{y(t+\Delta t)-y(t)}{x(t+\Delta t)-x(t)} = \lim_{\Delta t \to 0} \frac{\frac{y(t+\Delta t)-y(t)}{\Delta t}}{\frac{x(t+\Delta t)-x(t)}{\Delta t}} = \frac{dy/dt}{dx/dt}.
\]

- Using the same formula with \( \frac{dy}{dx} \) in place of \( y \), we get \( \frac{d^2 y}{dx^2} = \frac{d}{dx} \frac{dy}{dt} \frac{dt}{dx} \).

- We can also compute arc length of parameterized curves. If the curve is a piece of a graph, we know

\[
L = \int_{x_0}^{x_1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
= \int_{x_0}^{x_1} \sqrt{1 + \left( \frac{dy}{dt} \frac{dt}{dx} \right)^2} \, dx
= \int_{x_0}^{x_1} \frac{(dx/dt)^2 + (dy/dt)^2}{dx/dt} \, dx
= \int_{t=a}^{t=b} \frac{(dx/dt)^2 + (dy/dt)^2}{dx/dt} \, dt \quad \text{by a substitution } x = x(t)
= \int_{t=a}^{t=b} \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt
\]

This works even when the curve isn’t just a piece of a graph.
April 21-22, Section 10.4

- It is sometimes useful to work with polar coordinates. In polar coordinates, we describe a point using its distance from the origin \( r \) and its angle from the positive \( x \) axis \( \theta \). If \( r < 0 \), we interpret this as using the same angle \( \theta \) but moving backwards, so, for example \((-r, \theta) = (r, \theta + \pi)\).

- Be careful, the same point in the plane can be written many different ways using polar coordinates.

- Using some basic trigonometry, we can convert from polar coordinates to rectangular using \( x = r \cos \theta \) and \( y = r \sin \theta \).

- To convert from rectangular coordinates to polar, we use that \( r^2 = x^2 + y^2 \) and \( \tan \theta = y/x \). Warning: this does not mean that always \( \theta = \arctan(y/x) \) since \( \arctan \) only gives values between \(-\pi/2\) and \(\pi/2\). If your point is in another quadrant, make sure to adjust accordingly.

- We can graph functions that give \( r \) as functions of \( \theta \), i.e. \( r = f(\theta) \).

- To find the slopes of tangent lines to polar graphs, notice that if \( r = f(\theta) \) is the curve, then by writing \( x = r \cos \theta = f(\theta) \cos \theta \) and \( y = r \sin \theta = f(\theta) \sin \theta \), we describe the graph as a parametrized curve with \( \theta \) as the parameter. Then we can use what we know about parameterized curves to write \( \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \).

April 24, Section 10.5

- We can use Riemann sums to find area “under” a polar curve. If we break the angle interval \( a \leq \theta \leq b \) into small increments \( \Delta \theta = \frac{b-a}{n} \), then the polar version of thin rectangles is thin sectors. Since the area of a sector of radius \( r \) and angle \( \theta \) is \( \frac{r^2 \theta}{2} \), the Riemann sum becomes \( \sum_{i=1}^{n} \frac{1}{2}(f(c_i))^2 \frac{b-a}{n} \), where \( c_i \) is an angle in the \( i \)th subinterval. This is the sum of areas of sectors approximating the area under the polar curve. Taking the limit, we get \( \int_{a}^{b} \frac{1}{2}(f(\theta))^2 \, d\theta \).