Calculus I
Main Ideas

This document provides a summary of the major ideas discussed in class each day. You are responsible for understanding and being able to apply these ideas. There will be questions about them on quizzes and exams, as well as written homework assignments.

August 25:

- Suppose $s(t)$ is a function that tells us distance from a given location as a function of time; in other words, $s(t)$ is how far we've traveled as of time t. If we have two times t_0, t_1, we can find the *average velocity* between the two times using

$$\text{Average velocity} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

  If we are moving at a constant rate, this is that rate (rate $= \frac{\text{distance}}{\text{time}}$).

- We can approximate the *instantaneous velocity* at t_0 by taking $t_1 - t_0$ to be very small. As we let t_1 get closer and closer to t_0 we can hope to approach a *limiting value* that gives us the exact *instantaneous velocity* at t_0.

- Geometrically, if we graph the function $s(t)$, the average velocities $\frac{s(t_1)-s(t_0)}{t_1-t_0}$ correspond to the slopes of the *secant lines* through $(t_0, s(t_0))$ and $(t_1, s(t_1))$. As we let t_1 get closer and closer to t_0 these slopes limit to the slope of the *tangent line* to the graph through the point $(t_0, s(t_0))$.

August 25, Section 2.2

- Suppose $f(x)$ is a function. If as x gets closer and closer to a (without actually being a) there is some *number* L such that $f(x)$ gets closer and closer to L, then we say that L is the limit of $f(x)$ as x approaches a and write $\lim_{x\to a} f(x) = L$.

- Limits can tell us what the value of a function "should be" at places where the function isn't actually defined. For example, this is how we figured out instantaneous velocity yesterday.

- Limits might not exist! For example, if the function jumps or if it oscillates too much, then a limit might not exist.

- Limits depend only on what happens *near* $x = a$ and not at all on what happens *at* $x = a$.

- Limits must be numbers. Technically, ∞ and $-\infty$ are *not* limits, though later we will write $\lim_{x\to a} f(x) = \infty$ if $f(x)$ keeps getting larger and large as x gets close to a.

August 27, Section 2.3

- For suitably "nice" functions (such as continuous functions), we can compute $\lim_{x\to a} f(x)$ by just computing $f(a)$.

- If $f(x)$ is not continuous (for example, if $f(a)$ is not even defined), we need to use some technique such as algebra or trigonometry to rewrite $f(x)$ to understand it better. In particular, if $f(x) = g(x)$ for $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, so if we can find such a $g(x)$ that is "nice", we can use that to find $\lim_{x\to a} f(x)$

- For example $\frac{x^2-6x+8}{x-4}$ is not defined at $x = 4$. But by doing some algebra, we can notice that $\frac{x^2-6x+8}{x-4} = x - 2$ when $x \neq 4$. So $\lim_{x\to 4} \frac{x^2-6x+8}{x-4} = \lim_{x\to 4} x - 2$. But $x - 2$ is continuous, so $\lim_{x\to 4} x - 2 = 2$

August 28, Section 2.3

- The squeeze theorem says that if $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then also $\lim_{x\to a} g(x) = L$.

- We used the squeeze theorem and some geometry (see the book) to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

- In general, if you're not sure how to proceed on a limit, try to use some algebra or trigonometry to make it look like something you've seen before.

September 2, Section 2.4

- Definition of continuity: $f(x)$ is *continuous at* $x = a$ if $\lim_{x\to a} f(x) = f(a)$. Note that this says the limit at a must exist *and* equal the value of the function at a.

- We say a function is continuous on an interval (a, b) if $f(x)$ is continuous at all points in the interval.

- The intermediate value theorem: if $f(x)$ is continuous on an interval containing the interval $[a, b]$ then $f(x)$ must take *every* value between $f(a)$ and $f(b)$ (this is the idea - see the book or class notes for the technical statement).

- In order to have $\lim_{x\to a} f(x) = L$, we need $f(x)$ to approach L as x approaches a both for $x > a$ and for $x < a$. If we only know that $f(x)$ approaches L when x approaches a for $x > a$, then we say $\lim_{x\to a^+} f(x) = L$. If we only know that $f(x)$ approaches L when x approaches a for $x < a$, then we say $\lim_{x\to a^-} f(x) = L$. These are called one-sided limits.

- $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^+} f(x) = L$ AND $\lim_{x\to a^-} f(x) = L$.

- We can also talk about "one sided continuity." We say $f(x)$ is *continuous from the left at* $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$. We say $f(x)$ is *continuous from the right at* $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.

September 1, Section 2.5

- If $\lim_{x\to a} f(x) = L$ doesn't exist because, as x approaches a, $f(x)$ grows bigger and bigger without bound, we say $\lim_{x\to a} f(x) = \infty$ (though the limit still doesn't exist, this tells us it doesn't exist in this particular way). Similarly, we can have $\lim_{x\to a} f(x) = -\infty$, and there are one sided versions of these limits.

- To evaluate infinite limits, plug in values near $x = a$ and think about whether the function must be increasing or decreasing without bound as you move closer to a. It might also be useful to do algebraic simplifications first.

- Remember, if you plug in and get $0/0$, you can't make conclusions - you need to do more work. If you get $c/0$ for $c \neq 0$, there are infinite one-sided limits on each side. If they agree, there's an infinite limit.

September 3, Section 3.1

- If $s(t)$ represents distance traveled at time t, then we can compute the *instantaneous velocity* at time a using the formula $\lim_{\Delta t\to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$. Here $\frac{s(a+\Delta t)-s(a)}{\Delta t}$ is the average velocity between time a and time $a + \Delta t$, so we expect to get better and better approximations to the instantaneous velocity at time a by letting the time interval length Δt get smaller and smaller.

- Geometrically, $\frac{f(a+\Delta x)-f(a)}{\Delta x}$ represents the slope of the *secant line* containing $(a, f(a))$ and $(a+\Delta x, f(a+\Delta x))$. Then $\lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ represents the slope of the tangent line which goes through $(a, f(a))$ and whose slope matches the rate of increase of the graph of f at a. So this is also the a rate of change: the rate of change of the height of a graph.

- If $y = f(x)$ is any function, then $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ is a new function of x (notice that if we plug in any number for x, we get an output number that depends on x - this is the definition of a function). This new function is called the derivative of f with respect to x. There are a number of notations for this new function: $\frac{df}{dx}$, $\frac{dy}{dx}$, f', y', $\frac{d}{dx}f$, or $D_x f$.

September 4, Section 3.1-3.2

- The derivative $\frac{df}{dx}(a) = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ can be used to compute the instantaneous rate of change when $x = a$. For example, if x represents units produced by a company and $f(x)$ represents profit then $\lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ represents the rate at which profit increases per unit produced *when producing* a units; note that the rate of profit increase per unit my vary depending on how much is being produced just as your velocity (your rate of change per unit time) can vary at different times.

- Similarly, $\frac{df}{dx}(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ can be used to compute the instantaneous rate of change for all x at once. For example, suppose x is the amount of chemical in a reaction and $f(x)$ is the amount of heat given off by the reaction as a function of x. Then $\frac{df}{dx}(x)$ tells us how quickly the heat changes depending on the amount of material.

For example, if heat is in joules and amount of material is in grams, then $\frac{df}{dx}$ would tell us about the rate of change in joules per gram of the heat as we add more chemicals. So if, say, $\frac{df}{dx}(2) = 7$, then as we add chemicals, when we get to 2 grams, the heat is increasing at a rate of 7 joules per gram - meaning that the ratio of increase of heat to increase in chemicals is currently 7 to 1. But notice that $\frac{df}{dx}(x)$ is itself a function of x; we can now plug in any x we want to learn about the rate of change *at* x.

- Derivatives don't always exist because limits don't always exist! If the derivative of f exists at $x = a$, we say that f is *differentiable* at a. If f is differentiable at every point on an interval, we say f is differentiable on the interval.
- If f is not continuous at a, then f can't be differentiable at a. It follows that if f is differentiable at a then it is continuous at a (by the rule of the contrapositive).
- However, even if f is continuous at a, f still might not be differentiable at a. For example, a function with a "corner" will not be differentiable at the corner. An example is $f(x) = |x|$ at $x = 0$.
- Rather than use the definition every time, we can work out some formulas to help us compute derivatives. Basic first examples $\frac{d}{dx}c = 0$, $\frac{d}{dx}x = 1$, $\frac{d}{dx}x^n = nx^{n-1}$ for any $n \neq 0$.

September 8, Section 3.2

- Rather than use the definition every time, we can work out some formulas to help us compute derivatives. Basic first examples $\frac{d}{dx}c = 0$, $\frac{d}{dx}x = 1$, $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$, $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$,
- $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}e^x = e^x$

September 10, Section 3.3

- It is NOT true that $\frac{d}{dx}(fg) = \frac{df}{dx}\frac{dg}{dx}$. DO NOT MAKE THIS MISTAKE.
- The product rule: $\frac{d}{dx}(fg) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$
- The quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$
- Sometimes you need to use multiple rules. For example, to differentiate $\frac{x \sin x}{e^x}$ you'd probably use the quotient rule, but then you'd need to use the product rule at the point where you need to take the derivative of the numerator.
- It is sometimes useful to take more than one derivative of a function. If n is a positive integer, we use the notation $\frac{d^n f}{dx^n}$ to mean that we should take the derivative of f n times.
- If $s(t)$ is a distance function, the first derivative is *velocity* and the second derivative is the *acceleration* function.

September 11, Section 3.4

- A composite function is one that is made up of other functions with the output of one being plugged into the next. For example, if f and g are functions, $f(g(x))$ is a composite. g is the *inner function*; f is the *outer function*. Example: $\sin(x^2)$. Then $\sin x$ is the outer function and x^2 is the inner function.

- The Chain Rule: $\frac{d}{dx}f(g(x)) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$. So we take the derivative of f and plug $g(x)$ into the result. Then we multiply the whole thing by the derivative of g. Example: $\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x$

- We might need to use the chain rule in combination with other rules, such as the product and quotient rules or even the chain rule again. Be careful to use them properly.

- More formulas we get using the chain rule: $\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}a^x = a^x \ln(a)$, $\frac{d}{dx}\log_a x = \frac{1}{x\ln(a)}$

- We also have $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This is a little strange-looking, but $\ln|x|$ is a nice way to extend $\ln x$ to negative values of x while still keeping the same derivative formula (as verified using the chain rule).

September 14, Sections 3.5

- When we have a nice formula $y = f(x)$ for y (for example $y = x^2$), we say that we have an *explicit* formula for y in terms of x.

- But sometimes we have a complicated relation between x and y such as $y^3+xy+x^2 = 11$. Then we can't solve explicitly for y, but we can still say that x determines y *implicitly* near a given point on the graph.

- To find the slope of a tangent line at a point on a graph for an implicit relation, we take the x derivative of both sides of the expression, using the chain rule. Then we solve for $\frac{dy}{dx}$. For example, taking derivatives of $y^3+xy+x^2 = 11$ we get $3y^2\frac{dy}{dx}+y+x\frac{dy}{dx}+2x = 0$. Then we can solve for $\frac{dy}{dx}$ to get $\frac{dy}{dx} = \frac{-y-2x}{3y^2+x}$.

- Important point: these formulas only make sense when (x, y) is a point on the graph.

September 15, Section 1.5

- Recall that a function f can be thought of as a machine that takes inputs (x) and produces outputs (y). Question: given f and an output y, can we determine the x so that $y = f(x)$?

- Answer: not always! For example, if $f(x) = x^2$, and $y = 4$, we can't tell whether x was 2 or -2.

- A function f is called *one-to-one* (or 1-1 or injective) if for any output y there is only one input x with $f(x) = y$. A function is 1-1 if and only if its graph passes the horizontal line test: any horizontal line intersects the graph in at most one point.

- Not all functions are 1-1; example $f(x) = x^2$. But sometimes we can make them 1-1 by restricting their domains; example $f(x) = x^2$ on $x \geq 0$ or $f(x) = x^2$ on $x \leq 0$.

- If a function is 1-1 on its domain, there is an inverse function f^{-1} that, given an output of f, takes you back to the input of f. In other words $f^{-1}(f(x)) = x$. Similarly $f(f^{-1}(y)) = y$. Note: this is horrible notation but we're stuck with it.

- When we write $\sqrt{y}$, this is the inverse function to $f(x) = x^2$ with the choice of domain $x \geq 0$ for f. When we write $-\sqrt{y}$, this is the inverse function to $f(x) = x^2$ with the choice of domain $x \leq 0$ for f. (We don't always mention this choice explicitly, but we are making it every time we write $\sqrt{\ }$.

- Another function that is not 1-1 on its full domain is $f(x) = \sin x$, but $\sin x$ is 1-1 on its *principal part* $-\pi/2 \leq x \leq \pi/2$ (this is also sometimes called the *fundamental domain*). The function arcsin is the inverse function to $\sin x$ with this domain; so in particular, the output of arcsin is always in the interval $[-\pi/2, \pi/2]$.

- Similarly, the fundamental domain for cos is $0 \leq x \leq \pi$ and the fundamental domain for tan is $-\pi/2 < x < \pi/2$.

- If we limit the domain of $\sin x$ to the fundamental domain $-\pi/2 \leq x \leq \pi/2$, then $\arcsin(\sin(x)) = x$ and $\sin(\arcsin y) = y$. Similar rules hold for the other trig functions and their fundamental domains. The first formula might fail if we're not restricted to the fundamental domain.

September 21, Sections 3.6

- If we do not limit the domain of $\sin x$, then it will not always be true that $\arcsin(\sin(x)) = x$ because x might not be in the fundamental domain, but arcsin, by definition, always returns a value in the fundamental domain. For example $\arcsin(\sin(3\pi/2)) = \arcsin(-1) = -\pi/2$. This is completely analogous to $\sqrt{(-2)^2} = 2$.

- Regardless of the domain, it will always be true that $\sin(\arcsin x) = x$ since by definition $\arcsin x$ is an angle (in the fundamental domain) whose sin is x. Analogously, $(\pm\sqrt{x})^2 = x$.

- To simplify expressions like $\cos(\arctan x)$, make a right triangle and label it appropriately. (see class notes.)

- To find the derivative of f^{-1} (when it exists). First write out $f(f^{-1}(x)) = x$. Now take $\frac{d}{dx}$ of both sides using the chain rule to get $\frac{df}{dx}(f^{-1}(x)) \cdot \frac{df^{-1}}{dx} = 1$. Now solve for $\frac{df^{-1}}{dx} = \frac{1}{\frac{df}{dx}(f^{-1}(x))}$ and simplify.

- Example: $\sin(\arcsin x) = x$, so $\cos(\arcsin x)\frac{d}{dx}\arcsin x = 1$, so $\frac{d}{dx}\arcsin x = \frac{1}{\cos\arcsin x} = \frac{1}{\sqrt{1-x^2}}$

September 22 Section 3.7

- Related rates problems use derivatives to relate the rates of change of various quantities that are changing in time depending on relations among each other.
- Step to solving a related rates problem:
  1. Read the problem and understand what's going on.
  2. Draw a picture, label the relevant quantities, and identify all units.
  3. Write down a formula giving the relationship among the quantities *at an arbitrary fixed time.*
  4. Take the time derivatives of your formula using the chain rule to get a formula relating the derivatives of the various quantities in the problem.
  5. To finish solving the problem, first identify what values for quantities and their derivatives you're given in the problem. Then plug these in and solve for the quantity you want to solve for. This might involve some auxiliary computations involving the relation formula.
- Warning: never ever plug in specific numbers before you take derivatives.

September 25, Section 4.1

- If $f(x)$ is a function, it has an *absolute maximum* (or *global maximum*) at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f.
- If $f(x)$ is a function, it has an *absolute minimum* (or *global minimum*) at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f.
- A function might have multiple absolute maxima or minima, or it might have none.

September 29, Section 4.1

- If $x = c$ is an absolute maximum or minimum, then $\frac{df}{dx}(c)$ is either 0 or does not exist. Here is the argument for an absolute maximum; the argument for minimina is similar: If c is a maximum, then $f(c) \geq f(x)$ for all x. So then for $x > c$, $\frac{f(x)-f(c)}{x-c} \leq 0$ and for $x < c$, $\frac{f(x)-f(c)}{x-c} \geq 0$. Thus $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} \leq 0$ and $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \geq 0$. Since $\frac{df}{dx}(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ either doesn't exist or it equals both $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$ and $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$, if it does exist it would have to be 0. So at an absolute maximum (or minimum) either the derivative is 0 or does not exist.
- If the derivative doesn't exist or is 0 at $x = c$, then c is called a *critical point.*

- So absolute maxima and minima must occur at critical points, *but not every critical point is an absolute maximum or minimum.*
- Theorem: If $f(x)$ is continuous and its domain is a closed bounded interval $[a, b]$, then $f(x)$ definitely has at least one absolute maximum and at least one absolute minimum. (Bounded means a and b are finite numbers, not infinities.)
- How to find absolute maxima and minima on a closed bounded interval: 0. Make sure f is continuous on a closed bounded interval, 1. Find all critical points and endpoint, 2. Plug in to f (not $\frac{df}{dx}$!!) to see where the biggest and smallest values are.

October 2, Section 4.3

- $x = c$ is a *local maximum* (or relative maximum) if $f(c) \geq f(x)$ for all x in some open interval containing c. $x = c$ is a *local minimum* (or relative minimum) if $f(c) \leq f(x)$ for all x in some open interval containing c.
- It is also true that all local maxima and minima occur at critical points (by the same argument as for absolute maxima and minima).
- But a critical point might be a local maximum, a local minimum, or neither.
- If $\frac{df}{dx}$ is continuous between two critical points, then it is either positive on the whole interval or negative on the whole interval. On the intervals where $\frac{df}{dx} > 0$, the function is increasing. Where $\frac{df}{dx} < 0$, the function is decreasing.

October 5, Section 4.3

- To determine how a function behaves at critical points, we use the first derivative test:
  - If $\frac{df}{dx} > 0$ to the left of a critical points and $\frac{df}{dx} < 0$ to the right, the critical point is a local maximum.
  - If $\frac{df}{dx} < 0$ to the left of a critical points and $\frac{df}{dx} > 0$ to the right, the critical point is a local minimum.
  - If $\frac{df}{dx}$ is the same sign on the left and on the right, the critical point is neither a local maximum, nor a local minimum.

October 6, Section 4.3

- Important note: in order for $x = c$ to be a critical point, $f(c)$ must be define so that we have a point on the curve. So if both f and $\frac{df}{dx}$ are undefined at c, it still isn't a critical point.
- We can use our information about local maxima, local minima, and where the function is increasing and decreasing to sketch a graph of the function: Plot the critical points (using the original function $f(x)$ to find the y values), and then use the increasing/decreasing information to fill in the rest of the graph.

- To find absolute max/min of a differentiable function on a closed bounded interval (one of the form $a \leq x \leq b$ with a, b finite numbers), we just need to find the critical points in the interval, and plug the critical points and the endpoints into f. Where we obtain the highest number is the absolute maximum, and where we obtain the smallest number is the absolute minimum.

- To find absolute max/min of a differentiable function with a domain that is not closed and bounded, find the critical points and endpoints, then use the increasing/decreasing information along with some basic reasoning to figure out which are absolute maxima and minima (see the examples from class). Note: absolute max/min might not exist.

October 8, Section 4.4

- On intervals where $\frac{d^2f}{dx^2} > 0$, we say that $f(x)$ is *concave up*.

- On intervals where $\frac{d^2f}{dx^2} < 0$, we say that $f(x)$ is *concave down*.

- If $(c, f(c))$ is a point on the graph where $\frac{d^2f}{dx^2}(c)$ is 0 or doesn't exist *and* the graph changes from concave up to concave down (or vice versa), we say that $(c, f(c))$ is a *point of inflection*.

- See class notes for pictures.

October 9, Section 4.4

- The second derivative test: Suppose $x = c$ is a critical point for $f(x)$. If $\frac{d^2f}{dx^2}(c) > 0$, f has a local minimum at c; if $\frac{d^2f}{dx^2}(c) < 0$, f has a local maximum at c; if $\frac{d^2f}{dx^2}(c) = 0$, the test fails and we need to use another technique, such as the first derivative test, to determine what kind of critical point c is.

October 15, Section 4.5

- We say $\lim_{x\to\infty} f(x) = L$ if (roughly) as x gets bigger and bigger, $f(x)$ gets closer and closer to L. Similarly, we can define $\lim_{x\to-\infty} f(x) = L$.

- If as x gets bigger and bigger, $f(x)$ gets bigger and bigger to infinity, we can have $\lim_{x\to\infty} f(x) = \infty$, and other similar statements.

October 19, Section 4.5

- If $f(x)$ and $g(x)$ are polynomials, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{l.t(f(x))}{l.t.(g(x))}$ where $l.t.(f(x))$ is the leading term of $f(x)$, and similarly for $g(x)$ or for $x \to -\infty$. So for example, $\lim_{x\to\infty} \frac{5x^3-4x^2+2}{8x^7+4x^2} = \lim_{x\to\infty} \frac{5x^3}{8x^7}$.

- In general, for algebraic expressions, often the best technique is to focus on leading terms and ignore "lower order" terms.

- When in doubt - think logically about the properties of the function.

October 20, Section 4.6

- We can draw good graphs of functions using the techniques from this chapter and from precalculus. We should try to take as much of the following as possible into account: the domain of f, x- and y-intercepts, vertical asymptotes, continuity of f, differentiability of f, critical points, where f is increasing and decreasing, local maxima and minima, concavity, points of inflection, limits at $\pm\infty$

October 22, Section 4.7

- In this section, we'll do optimization problems. Steps: 1) Read and understand the problem, 2) draw a picture and label variables, 3) write down the *primary equation* for the quantity you want to optimize (maximize/minimize) and the *secondary equations (constraint equations)* that relate the other quantities in the problem, 4) use the constraint equations to rewrite the optimization equation in terms of a single independent variable (and figure out the domain of that variable), 5) use calculus to find the appropriate max/min (including applying the necessary reasoning to know you've found a maximum or minimum and not just a critical point), 6) make sure to provide a full answer to the original question.

- A useful technique: since squaring preserves the order of non-negative numbers (i.e. if $0 \leq a \leq b$, then also $0 \leq a^2 \leq b^2$), finding the max/min of a distance d is equivalent to finding the max/min of d^2. This can be useful for eliminating ugly square roots from some problems.

October 26, Section 5.1

- If $\frac{dF}{dx} = f(x)$, $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$. This is denoted $F(x) = \int f(x)\,dx$.

- Antiderivatives are not unique: for example the derivative of any constant is 0, so 0 has many antiderivatives.

- A useful fact: the only functions whose derivatives are 0 are constants.

- So if $\frac{dF}{dx} = \frac{dG}{dx}$, then $\frac{d}{dx}(F(x) - G(x)) = 0$, so $F(x) - G(x)$ must be a constant. In other words, $F(x) = G(x) + C$ for some constant C.

- This also makes sense looking at graphs: two functions that have the same slopes of their tangent lines at every x must differ from each other just by being shifted up or down.

- So even though antiderivatives are not unique, two antiderivatives only differ by a constant.

- There are many functions where we can find nice formulas for the antiderivatives because we know nice formulas for derivatives. See the table on page 282 of the book.

October 27, Section 5.1

- If you want to find an antiderivative of something that isn't on the table on page 282, first try using algebraic and trigonometric simplifications.

- Common errors. The following statements are not true; don't try to use them: $\int f(x)g(x)\,dx = \int f(x)\,dx \cdot \int g(x)\,dx$, $\int \frac{f(x)}{g(x)}\,dx = \frac{\int f(x)\,dx}{\int g(x)\,dx}$; for example $e^x \sin x$ is *NOT* an antiderivative of $e^x \cos x$ (take the derivative and check!)

- Recall that antiderivatives are not unique: if $F(x)$ is an antiderivative for $f(x)$, then so is any $F(x) + C$. But sometimes we can use other information about a problem to choose a specific C.

October 29, Section 5.1

- If $v(t)$ is a velocity function then $s(t) = \int v(t)\,dt$ is the position function. To find a specific position function, it suffices to know the exact position at a single time. We can then plug in that information and solve for C.

- Similarly, if we know acceleration, we can find position by taking the antiderivatives twice and using extra information to solve for the constants.

- Summation notation: $\displaystyle\sum_{i=m}^{n} f(i)$ means that we're going to add up $f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$. Here m, n must be integers with $n \geq m$. i is called the index of the summation.

- Example: $\displaystyle\sum_{i=3}^{6} \cos(i) = \cos(3) + \cos(4) + \cos(5) + \cos(6)$.

October 30, Section 5.2

- Special formulas:

  1. $\displaystyle\sum_{i=1}^{n} 1 = n$
  2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  3. $\displaystyle\sum_{i=1}^{n} cf(i) = c\sum_{i=1}^{n} f(i)$
  4. $\displaystyle\sum_{i=1}^{n} (f(i) + g(i)) = \left(\sum_{i=1}^{n} f(i)\right) + \left(\sum_{i=1}^{n} g(i)\right)$

  There are some other useful ones in the book

November 2, Section 5.2

- New topic: Suppose $f(x)$ is a function on an interval $[a, b]$ with $f(x) \geq 0$ on the interval. We want to find the area of the region R under the graph but above the x-axis.

- First approximation: Suppose M and m are the maximum and minimum values, respectively, for $f(x)$ on the interval. Then looking at the rectangles with width $b - a$ and heights M and m, we see that $m(b - a) \leq \text{area}(R) \leq M(b - a)$.

- We can do better if we increase the number of rectangles by dividing the interval $[a, b]$. Next we can do two pieces: Let R_1 be the region under the curve on $[a, a + \frac{b-a}{2}]$, and let R_2 be the region under the curve on $[a + \frac{b-a}{2}, b]$. So R is the union or R_1 and R_2, and $\text{area}(R) = \text{area}(R_1) + \text{area}(R_2)$. Now we do the same approximations with rectangles of R_1 and R_2. Let m_1, M_1 and m_2, M_2 be the minima/maxima of $f(x)$ on the first or second interval, respectively. The using rectangles (see pictures from class) we see that $m_1 \frac{b-a}{2} + m_2 \frac{b-a}{2} \leq \text{area}(R) \leq M_1 \frac{b-a}{2} + M_2 \frac{b-a}{2}$. These are better approximations.

- Next step: more rectangles.

- Suppose we have a function with $f(x) \geq 0$ on an interval $[a, b]$ and want to find the area under the curve. General concept: break $[a, b]$ up into n subintervals of width $\frac{b-a}{n}$. Let M_i be the maximum value of $f(x)$ on the ith interval and let m_i be the minimum value of $f(x)$ on the ith interval. The upper rectangle over the ith interval has area $\frac{b-a}{n} M_i$; the lower rectangle over the ith interval has area $\frac{b-a}{n} m_i$. So the total approximation of the area using upper rectangles is $U_n = \sum_{i=1}^{n} \frac{b-a}{n} M_i$ and the total approximation of the area using lower rectangles is $L_n = \sum_{i=1}^{n} \frac{b-a}{n} m_i$. We will always have
$$L_n \leq \text{area under the curve} \leq U_n.$$
If $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$, that's the area under the curve.

- For specific problems, the hard part is finding m_i and M_i, but for nice examples (such as functions that are always increasing or decreasing) these will be at the endpoints of the subintervals. So it is useful to observe that the ith interval has endpoints $[a + (i - 1) \cdot \frac{b-a}{n}, a + i \cdot \frac{b-a}{n}]$

- For example. If $f(x) = x^2$ and the interval is $[4, 8]$, and we divide into n pieces, the width of each subinterval is $\frac{8-4}{n} = \frac{4}{n}$, and the ith interval is $[4 + (i - 1)\frac{4}{n}, 4 + i\frac{4}{n}]$. Since x^2 is increasing on $[4, 8]$, the minimum on the ith interval is $m_i = (4 + (i - 1)\frac{4}{n})^2$ and the maximum is $(4 + i - \frac{4}{n})^2$. So $L_n = \sum_{i=1}^{n} \frac{4}{n}(4 + (i - 1)\frac{4}{n})^2$ and $U_n = \sum_{i=1}^{n} \frac{4}{n}(4 + i\frac{4}{n})^2$. We can compute that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \frac{448}{3}$, so this is the area under the curve.

November 4, Section 5.2

- Notation: When $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$, we denote the common limits as $\int_a^b f(x)\,dx$ and call it the *definite integral*.

- Important theorem: If $f(x)$ is continuous on $[a,b]$, then $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$ and so the area exists!

- But the area doesn't always exist. For example, if $f(x)$ is the function that takes value 0 when x is rational and 1 when x is irrational, then the upper sum on $[0,1]$ is 1 for any n and the lower sum is 0 for any n, so the limits never come together.

November 5, Section 5.3

- Notice that if we pick any point c_i in the ith subinterval, then $m_i \le f(c_i) \le M_i$ (by definition of maxima/minima). So $\frac{b-a}{n} \cdot m_i \le \frac{b-a}{n} \cdot f(c_i) \le \frac{b-a}{n} \cdot M_i$, and

$$\sum_{i=1}^{n} \frac{b-a}{n} \cdot m_i \le \sum_{i=1}^{n} \frac{b-a}{n} \cdot f(c_i) \le \sum_{i=1}^{n} \frac{b-a}{n} \cdot M_i$$

.

  So if $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$ (for example if $f(x)$ is continuous), then by the Squeeze Theorem, $\sum_{i=1}^{n} \frac{b-a}{n} \cdot f(c_i)$ has the same limit and so it also computes the area!

  This is great because it says that IF WE KNOW $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$ (for example if $f(x)$ is continuous) we can use any point in the subinterval we want for computations. A convenient choice is the right endpoint, so we can compute area as $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(a + i \cdot \frac{b-a}{n})$.

- The definition allows for many different interpretations depending on context. For example, the most popular interpretation is *area*: if we just consider the graph $y = f(x)$, then $\frac{b-a}{n}$ is a width along the x-axis, $f(c_i)$ is the height above (or below) the x-axis, and so $f(c_i) \cdot \frac{b-a}{n}$ is the area of a rectangle (or negative area if $f(c_i) < 0$). Then $\sum_{i=1}^{n} f(c_i) \cdot \frac{b-a}{n}$ is an approximate total area, and taking the limit to get $\int_a^b f(x)\,dx$ gives the exact area (counting area below the x-axis as negative area). Alternatively, we could interpret $f(x)$ as giving linear density, and then $f(c_i) \cdot \frac{b-a}{n}$ is mass of a small segment, $\sum_{i=1}^{n} f(c_i) \cdot \frac{b-a}{n}$ is an approximate total mass, and taking the limit to get $\int_a^b f(x)\,dx$ gives the exact mass. Or, we could interpret $f(x)$ as giving linear electrical charge density, and then $f(c_i) \cdot \frac{b-a}{n}$ is the total charge of a small segment, $\sum_{i=1}^{n} f(c_i) \cdot \frac{b-a}{n}$ is an approximate total charge, and taking the limit to get $\int_a^b f(x)\,dx$ gives the exact amount of charge.

November 6, Section 5.3

- We can split definite integrals into pieces: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

- From the definitions, it's easy to see $\int_a^a f(x)\,dx = 0$.

- A useful convention (also can be seen in definitions by interchanging a and b): $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

- Other easy rules: $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$, and $\int_a^b cf(x) = c\int_a^b f(x)\,dx$ for any constant c.

November 19, Section 5.4

- What is $\int_a^b \frac{df}{dt}\,dt$ when $\frac{df}{dt}$ is continuous? This turns out to be extremely important.

- By definition, $\int_a^b \frac{df}{dt}\,dt = \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n}\frac{df}{dt}(c_i)$. Now, let's suppose $f(t)$ represents position so that $\frac{df}{dt}$ represents velocity and each interval $\frac{b-a}{n}$ is a small time interval. Then since $\frac{df}{dt}$ is continuous, if n is very large, $\frac{df}{dt}$ will be approximately $\frac{df}{dt}(c_i)$ anywhere on the ith time interval. Then $\frac{b-a}{n}\frac{df}{dt}(c_i)$ (which is time times rate) is approximately the displacement of the motion on the ith time interval. So $\sum_{i=1}^n \frac{b-a}{n}\frac{df}{dt}(c_i)$ approximates the total displacement from time a to time b. It turns out that the limit then gives the exact total displacement. So $\int_a^b \frac{df}{dt}\,dt$ gives the exact total displacement. But the exact total displacement is just $f(b) - f(a)$ (ending location minus starting location). So $\int_a^b \frac{df}{dt}\,dt = f(b) - f(a)$.

- (First fundamental theorem of calculus). Using slightly different notation, suppose $f(x)$ is a function and $F(x)$ is another function with $\frac{dF}{dx} = f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

- Note: the fundamental theorem of calculus requires the hypothesis that the function $f(x)$ be continuous! Make sure to pay attention to that.

November 10, Section 5.4

- It is interesting to study the functions $A(x) = \int_0^x f(t)\,dt$. If $f(t) \geq 0$, then $A(x)$ represents the area under the graph of f from 0 to x.

- It is very interesting to compute the derivative $\frac{dA}{dx}$. We use the definition of the derivative:

$$\begin{aligned}\frac{dA}{dx} &= \lim_{\Delta x\to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0} \frac{\int_0^{x+\Delta x} f(t)\,dt - \int_0^x f(t)\,dt}{\Delta x}\\ &= \lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x}\end{aligned}$$

If Δx is sufficiently small and f is continuous on the the interval $[x, x+\Delta x]$, then $f(t) \approx f(x)$, so the area $\int_x^{x+\Delta x} f(t)\,dt$ is approximately $f(x)\Delta x$. But then

$$\begin{aligned}\lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x} &\approx \lim_{\Delta x\to 0} \frac{f(x)\Delta x}{\Delta x}\\ &= \lim_{\Delta x\to 0} f(x)\\ &= f(x).\end{aligned}$$

These arguments can be made rigorous, so if f is a continuous function, $\frac{dA}{dx} = \frac{d}{dx} \int_0^x f(t)\, dt = f(x)$.

- This result is called the second part of the Fundamental Theorem of Calculus. It tells us that continuous functions have antiderivatives (though it doesn't really help us compute them).

- Notice that the derivative of $A(x)$ at x tells us the rate of change of the area function at x but it doesn't really depend on how much area there already is to the left. So in fact, the theorem generalizes to $\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$ for any a. Changing a is the same as changing $A(x)$ by a constant.

- Using the chain rule, we can take the derivative of things like $A(x^2) = \int_a^{x^2} f(t)\, dt$

- When necessary, remember to use that $\int_x^a f(t)\, dt = -\int_a^x f(t)\, dt$

- And if there are functions in both limits of integration, such as $\int_{x^2}^{x^3} f(t)\, dt$, use that $\int_{x^2}^{x^3} f(t)\, dt = \int_{x^2}^{c} f(t)\, dt + \int_c^{x^3} f(t)\, dt$ for any constant c.

November 17, Section 5.5

- One technique for finding antiderivatives is called substitution. It's based on the chain rule.

- Recall that the chain rules says that the derivative of a composite $f(g(x))$ has the form $f'(g(x))g'(x)$. Hence $\int f'(g(x))g'(x)\, dx = f(g(x)) + C$.

- Suppose we let $u = g(x)$. Then if we write $du = g'(x)dx$, we can rewrite our integral $\int f'(g(x))g'(x)\, dx$ as $\int f'(u)du$. Now we're just finding the antiderivative of $f'(u)$, which of course is just $f(u) + C$. Now plugging back in $u = g(x)$, we get $f(g(x)) + C$ for the antiderivative, which is the right answer!

- For example if we want to find $\int \cos(x^2)2x\, dx$, we notice that x^2 looks like an inner function whose derivative is also present. So let $u = x^2$ and $du = 2x\, dx$. Then the problem becomes $\int \cos(u)\, du$, which we know is $\sin(u) + C = \sin(x^2) + C$. We can check that this is the correct answer.

- Notice that we don't need to have exactly $g'(x)$ appear in the original integral. For example, if we have $\int \cos(x^2)x\, dx$, substitution still works to help us find the antiderivative, with a little more algebra.

- Tip: in any expression you should have all xs or all us. If you have them mixed together, you're doing it wrong! Also, your final answer should always be back in terms of xs.

November 19, Section 5.5

- There are two ways to use substitution to compute definite integrals $\int_a^b f(x)\,dx$. One is to use substitution to find the antiderivative $F(x)$ of $f(x)$. Then use this in the formula $\int_a^b f(x)\,dx = F(b) - F(a)$. The other is to turn the problem completely into a u problem by changing the limits of integration. So if $u = g(x)$, the limits of integration after you do the substitution will be from $g(a)$ to $g(b)$. See the examples from class and the book.

November 20, Section 5.7

- Two nice trig integrals:
  - $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$. This can be solved by a substitution of $u = \cos x$.
  - To integrate $\int \sec x\,dx$, there's a trick: multiply by $\frac{\sec x + \tan x}{\sec x + \tan x}$ and then substitute.

November 23, Section 5.8

- From what we know about derivatives, we know that $\int \frac{1}{1+x^2}\,dx = \arctan x + C$, and $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$, and $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}|x| + C$. Using these we can compute integrals like $\int \frac{1}{a^2+x^2}\,dx$ and $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ using some algebra and substitution (see examples from class).

November 30, Section 5.8

- If we have expressions like $\int \frac{1}{ax^2+bx+c}\,dx$, we can also use substitution and inverse trig functions if we can complete the square to write $ax^2 + bx + c = a(x-h)^2 + k$.

December 1, Section 8.2

- Integration by parts: From the product rule $\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g + f\frac{dg}{dx}$. So if we take antiderivatives, we get $fg = \int \frac{df}{dx}g\,dx + \int f\frac{dg}{dx}\,dx$, so

$$\int f\frac{dg}{dx}\,dx = fg - \int \frac{df}{dx}g\,dx.$$

  We can use this to replace integral problems with simpler integrals.

- Example: $f(x) = x$, $g(x) = e^x$. Then $\int xe^x\,dx = xe^x - \int e^x = xe^x - e^x + C$.
- A mnemonic for integration by parts: $\int u\,dv = uv - \int v\,du$

December 3, Section 8.2

- It's okay to just have $dv = dx$ if necessary!
- Sometimes you need to use integration by parts multiple times in one problem, for example to integrate $\int x^n e^x\,dx$. You'd use the process to make n smaller each time.
- Sometimes when you do integration by parts a few times it looks like you've wound up back where you started. In this case, try to finish the problem with algebra!