Algebraic Topology I Homework
Spring 2014

Homework solutions will be available
`http://faculty.tcu.edu/gfriedman/algtop/algtop-hw-solns.pdf`

Due 5/1

A Do Hatcher 2.2.4

B Do Hatcher 2.2.9b (Find a cell structure)

C Do Hatcher 2.2.12 (There are a few ways to do this!)

D Do Hatcher 2.2.13a (note the "usual" cell structure on S^1 has one 0-cell and one 1-cell).

E Read about the homology of $\mathbb{R}P^n$ in Hatcher and then do Hatcher 2.2.19

Due 4/24/2014

A
1 Let M be the closed Mobius strip (i.e. including its boundary). Let ∂M be the boundary of M. Compute $H_*(M, \partial M)$ using arguments with the long exact sequence and other basic properties of homology. You can use that you already know homology computations for circles, but you shouldn't compute $H_*(M, \partial M)$ directly from the Delta complex. If necessary, you can also use that $H_*(X) = H_*^\Delta(X)$ if X can be realized as a Delta complex in order to compute $H_*(M)$ and $H_*(\partial M)$.

2 Recall that $\mathbb{R}P^2$ can be obtained from the Mobius strip by filling in the boundary circle with a disk. Using this and excision, compute $H_*(\mathbb{R}P^2)$.

B Do Hatcher 2.1.29

C Hatcher 2.2.2 (note: he means no *real* eigenvectors)

Due 4/17/2014

A Part A

1 Let A be the closed annulus, i.e. $A = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$. Let ∂A be the boundary of A. Compute $H_*(A, \partial A)$ using arguments with the long exact sequence and other basic properties of homology. You can use that you already know homology computations for circles, but you shouldn't compute $H_*(A, \partial A)$ directly from the Delta complex. If necessary, you can also use that $H_*(X) = H_*^\Delta(X)$ if X can be realized as a Delta complex in order to compute $H_*(A)$ and $H_*(\partial A)$.

2 Find generators for the homology groups $H_i(A, \partial A)$ you computed above.

Due 4/10/2014

A Part A

1 Let $A \xrightarrow{f} B \xrightarrow{h} C \xrightarrow{k} D \xrightarrow{g} E$ be an exact sequence. Suppose f is surjective and g is injective. Show that $C = 0$. Provide *all* the details.

2 Recall that if $f : G \to H$ is a map of abelian groups, then the cokernel, $\operatorname{cok}(f)$, is defined to be $H/\operatorname{im}(f)$. Suppose you have the following commutative diagram of abelian groups in which the horizontal sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & D & \xrightarrow{h} & E & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

The *serpent lemma* says that there is an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \operatorname{cok}(\alpha) \to \operatorname{cok}(\beta) \to \operatorname{cok}(\gamma) \to 0.$$

Prove the serpent lemma. Hint: write the kernels and cokernels as homology groups and use some big results you learned in class.

B Part B

1 Do Hatcher 2.1.16 and 2.1.17.a (just part with $X = S^2$)

2 Study for the exam

Due 4/3/2014

A Part A

1 Do Hatcher 2.1.11.

2 Suppose $H < G$ are abelian groups and $G/H \cong F$ is a free abelian group. Prove that $G \cong H \oplus F$. Hint: Let $q : G \to G/H$ be the natural quotient map. First construct a homomorphism $\psi : F \to G$ such that $q\psi = \mathrm{id}_F$; note: this implies that $\psi : F \to G$ is injective, so $\psi(F) \cong F$. Then show that $G \cong H \oplus \psi(F)$ by showing that $H \cap \psi(F) = \{0\}$ and that every element of G can be written as a sum $h + f$ with $h \in H$ and $f \in \psi(F)$. Double hint: it might help to notice that H is the kernel of q. Triple hint: to construct f, use the maps!

B Part B

1 Show that if $f : X \to Y$ is homotopic to a constant map, then $f_* : \tilde{H}_*(X) \to \tilde{H}_*(Y)$ is the 0 map.

2 The topologist's sine curve is the subspace of $\mathbb{R}^2$ consisting of all points $(x, \sin(1/x))$ for $0 < x \leq 1$ and all points $(0, y)$ for $-1 \leq y \leq 1$. Compute the singular homology groups of this space.

3 Read Section 2.A, then do the following problem: Suppose X is a path connected space such that $\pi_1(X)$ is a non-abelian simple group (i.e. its only normal subgroups are $\{1\}$ and the whole group). Show that $H_1(X) = 0$.

4 Do Hatcher 2.1.12

5 Do Hatcher 2.1.13

Due 3/27/2014

A Part A

1 Do Hatcher 2.1.1

2 Do Hatcher 2.10.a - note that he means that every edge is glued to precisely one other edge.

3 Find a way to realize the "two-holed torus" (what Hatcher calls M_2) as a Δ-complex.

4 Find a Δ-complex X such that $\pi_1(X) = \mathbb{Z}_3$.

B Part B

1 Do Hatcher 2.1.4, 2.1.5 ,2.1.9

Due March 20

A Part A

1 Note: the following exercise is essentially a special case of Hatcher's 1.40, which actually follows from the same sort of ideas as this by "putting more spaces in the middle". Suppose $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ path connected, locally path connected, and *simply connected*.

a Show that $\tilde{X}$ is a normal cover and that the group of deck transformations G for $\tilde{X}$ over X is isomorphic to $\pi_1(X)$.

b Let $\tilde{x}_0$ be a basepoint of $\tilde{X}$ over $x_0 \in X$. Let H be a subgroup of $G = \pi_1(X, x_0)$ and hence also a subgroup of the group of deck transformations of $\tilde{X}$ over X. Let $X_H = \tilde{X}/H$, and let $\tilde{x}_0^H$ be the image of $\tilde{x}_0$ in X_H, which we let be the basepoint of x_H. Notice that we can factor p as $\tilde{X} \xrightarrow{p_H} X_H \xrightarrow{q} X = X/G$. We know that $p_H : X \to X_H$ is a covering map by Hatcher Proposition 1.40.a. Show that $q : X_H \to X$ is also a covering space of X.

c Show that $q_*(\pi_1(X_H, \tilde{x}_0^H)) = H$ so that X_H really is the cover of X corresponding to the subgroup H.

d Suppose H is normal. Show that G/H acts freely and properly discontinuously on X_H (in particular, show that $g_1 x = g_2 x$ for $x \in X_H$ if and only if $g_1 H = g_2 H$) and then observe that $X = X/G = X_H/(G/H)$. [More precisely: recall that each point of X_H corresponds to an orbit of a point x in $\tilde{X}$ under the action of H. In particular, every point in X_H is an image $p_H(x)$ for some x in $\tilde{X}$. Define an action of G on X_H by letting $g(p_H(x)) = p_H(gx)$. Show that this is well-defined, i.e. if y is another point in $\tilde{X}$ with $p_H(y) = p_H(x)$, show that $g(p_H(x)) = g(p_H(y))$ so that this definition is consistent. Then to show that we really have an action of G/H, show that every element of the coset gH acts on $p_H(x)$ the same way. Lastly, you should argue that the action of G/H is free and properly discontinuous. It then follows that $X = X_H/(G/H)$ from the general theory.)

2 Part B

a Recall from a previous homework the spaces X_n, $n \in \{1, 2, 3, \ldots\}$ obtained from the two-dimensional disk D^2 by identifying points on the boundary that differ by an angle $2\pi/n$; let $q : D^2 \to X_n$ be the quotient map. In that problem, you should have shown $\pi_1(X_n) \cong \mathbb{Z}_n$ and determined that the universal cover $\tilde{X}_n$ is homeomorphic to n copies of D^2 with their boundaries identified to each other.

i Describe how the deck transformations act on $\tilde{X}_n$.

ii Describe what points of $\tilde{X}_4$ you would identify to get the quotient space of $\tilde{X}_4$ corresponding to the unique subgroup of index 2 of $\pi_1(X_4)$.

b Describe all connected covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^2$ (Note: some of these will be familiar, others you made need to describe as quotients of the universal cover under certain group actions).

c Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$ with basepoint the union point x_0.

i What's the universal cover $\tilde{X}$ of X?

ii Since $\tilde{X}$ is simply-connected, its group of deck transformations is $\pi_1(X, x_0) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Let a, b be the respective generators of $\pi_1(\mathbb{R}P^2 \vee x_0)$ and $\pi_1(x_0 \vee \mathbb{R}P^2)$ as subgroups of $\pi_1(X, x_0)$. Describe the covering actions of a and b. Then describe what a covering action by a general element of $\pi_1(X, x_0)$ would look like. Can you see why all such transformations are generated by those corresponding to a and b? (Note, while the group of deck transformations is isomorphic to $\pi_1(X, x_0)$, the exact isomorphism might depend upon the choice of basepoint in $\tilde{X}$.)

Due 3/6/2014 (put in my box)

1 Part A

a Describe all connected covering spaces of $\mathbb{R}P^2$, up to isomorphism.

b Describe all connected covering spaces of $\mathbb{R}P^2 \vee S^2$, up to isomorphism.

c Describe all connected covering spaces of $S^1 \vee S^2$, up to isomorphism.

d Describe all connected covering spaces of $S^1 \times S^2$, up to isomorphism.

e Let $X = S^1 \vee S^1$ with $\pi_1(X, x_0) = \langle a, b \rangle$ in the standard way. For each of the following subgroups of $\pi_1(X, x_0)$, find (draw) a covering space (with basepoint) $(\tilde{X}, \tilde{x}_0)$ of X such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is the given subgroup: [Hint: think about the pictures on page 58]

  i $\{(ab^2)a^n(ab^2)^{-1} \mid n \in \mathbb{Z}\}$

  ii $\{(ab)^n \mid n \in \mathbb{Z}\}$

  iii $\{b^3(ab)^n b^{-3} \mid n \in \mathbb{Z}\}$

  iv the subgroup freely generated by a^2 and b

  v the subgroup freely generated by a^3, b, aba^{-1} and $a^{-1}ba$.

Due 2/27/2014 (you can hand this in the week after the exam)

1 Part A

a Let $p : \tilde{X} \to S^1 \vee S^1$ be the covering space shown in box 2 of Hatcher's table of examples of coverings of $S^1 \vee S^1$. Let $f : S^1 \times S^1 \to S^1 \vee S^1$ be the map given by the composition of the projection $S^1 \times S^1 \to S^1 \times y_0$ followed by the map $S^1 \times y_0 \to S^1 \vee S^1$ described by the loop a^3b^3. Does f lift to $\tilde{X}$?

b Use covering space theory to show that every map $\phi : \mathbb{R}P^2 \to S^1$ is homotopic to a constant map.

c Suppose Y is simply connected and $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ contractible. Show that every map $f : Y \to X$ is homotopic to a constant map.

2 Part B

a Describe the universal cover of $T^2 \vee S^1$, where T^2 is the torus $S^1 \times S^1$.

b Let X_n be the space obtained from the two-dimensional disk D^2 by identifying points on the boundary that differ by an angle $2\pi/n$; let $q : D^2 \to X_n$ be the quotient map.

  i What's $\pi_1(X_n)$ (you don't need to write out the argument)?

  ii Suppose x is in the interior of D^2; describe what small neighborhoods of $q(x)$ look like. Now suppose x is in the boundary of D^2; describe what small neighborhoods of $q(x)$ look like.

  iii Find and describe the universal cover $\tilde{X}_n$ of X_n using what you figured out in the previous sections; hint: think about the familiar case $X_2 = \mathbb{R}P^2$.

Due 2/20/2014

1 Show that a covering map $p : X \to Y$ is an open map (and hence a quotient map).

2 Do Hatcher 1.3.1 and 1.3.2

3 Use a covering space of $S^1 \vee S^1$ to show that the free group on 3 generators is isomorphic to a subgroup of the free group on 2 generators. Generalize 3 to n.

4 Find a simply connected cover of the space θ (the space homeomorphic to the lowercase Greek letter theta).

Due 2/13/2014

1 Reading assignments

  a Read Section 1.2 in the book, including the proof of the van Kampen theorem and the proof of the theorem stated at the end of class on Feb. 6

  b Read this about the classification of surfaces: `http://pages.uoregon.edu/koch/math431/Surfaces.pdf`

  c Technically, the last link only talks about surfaces that can be obtained by gluing the edges of polygons and does not give a complete proof that all surfaces can be obtained that way. Another approach to the classification of surfaces that you can read (optional) is the following: `http://www.maths.ed.ac.uk/~aar/papers/francisweeks.pdf`. Even this proof assumes that surfaces can be triangulated, which is hard to prove and won't be treated in this class.

2 Part A

  a Think of the real projective plane $\mathbb{R}P^2$ as the space obtained from the unit disk by identifying opposite points on the boundary.

    i Show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

    ii Draw a picture showing a loop representing the nontrivial element $[f] \in \pi_1(\mathbb{R}P^2)$.

    iii Draw a series of pictures showing the nullhomotopy from $2[f]$ to the constant path.

  b Let X be the quotient space of the disk D^2 obtained by identifying points on the boundary that are 120 degree apart. Compute $\pi_1(X)$

  c Let X be the complement of n points in $\mathbb{R}^2$. Compute $\pi_1(X)$. Do the same for the complement of n points in $\mathbb{R}^3$. [Hint: use induction on n].

3 Part B

  a Do Hatcher 1.2.7

  b Consider the annulus. Identify antipodal (opposite) points on the outer circle with each other (as if you're forming $\mathbb{R}P^2$). Also identify antipodal points on the inner circle with each other. Call the resulting space X. Describe $\pi_1(X)$ in terms of generators and relations.

c Let X be the union of a sphere S^2 with one of its diameters. Compute $\pi_1(X)$.

Due 2/6/2014

1 Part A

a Some point set topology:

i Suppose that X is a *compact* metric space and that $\mathcal{U}$ is an open covering of X. Show that there is a number $\delta > 0$ (called a *Lebesgue number*) such that for every set $Z \subset X$ with diameter $< \delta$, there is an element $U \in \mathcal{U}$ such that $Z \subset U$.

ii Use the results of the previous exercise to show that if $f : I \to Y$ is a path in the arbitrary space Y and if $\mathcal{V}$ is an open covering of Y then there is a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ of I such that for each interval $[t_i, t_{i+1}]$, the image $f([t_i, t_{i+1}])$ is contained in some single element $V \in \mathcal{V}$.

b Do Hatcher 1.1# 13, 16b, 17

c Do Hatcher Section 0 #3a, 3c (you may use 3b), 4, 0.6a ([Hint: Here's a point set topology lemma that might be useful. Suppose $X \times Y$ is a product space, Y is compact, and N is an open set in $X \times Y$. Suppose also that $x_0 \times Y \subset N$ for some $x_0 \in X$. Then there is a neighborhood W of x_0 in X such that $W \times Y \subset N$. This is a very important lemma, sometimes called the "tube lemma".])

2 Part B

a Give a presentation in terms of generators and relations of $\mathbb{Z}_4$ involving one generator; involving two generators; involving three generators.

b Recall the fundamental theorem of finitely generated abelian groups. If G is a finitely generated abelian group, explain how you would write down a presentation for it in terms of generators and relations.

c Consider the group G with presentation $\langle x, y \mid x^4 = e, y^2 = e, (xy)^2 = e\rangle$. Show that G is a finite group and determine how many elements it has. Hint: start by rewriting the relation $(xy)^2 = e$ in a more useful form. Extra credit: can you identify G as a familiar group?

Due 1/30/2014

1 Part A

a Do Hatcher Section 1.1 #2, 3, 6

2 Part B

a Do Hatcher Section 1.1 #10

b Let G be a topological group with operation $*$ and identity e. This means that G is a group *and* a topological space and that the group operation $* : G \times G \to G$, $(g, h) \to g * h$ and the inverse map $g \to g^{-1}$ are continuous. Let $\Omega(G, e)$ denote the set of all loops in G based at e. If $f, g \in \Omega(G, e)$, define a loop $f \otimes g$ by the rule $(f \otimes g)(s) = f(s) * g(s)$.

i Show that the operation $\otimes$ makes the set $\Omega(G, e)$ into a group.

ii Show that the operation $\otimes$ induces a group operation on $\pi_1(G, e)$ (this requires showing that if $f \sim f'$ and $g \sim g'$, then $f \otimes g \sim f' \otimes g'$, so that $\otimes$ is an operation on homotopy classes).

iii Show that the usual group operation and $\otimes$ are actually the same operation on $\pi_1(G, e)$ (we write the usual operation on $f, g \in \pi_1(G, e)$ as fg, as in class). Hint: compute $(fc_e) \otimes (c_e g)$, where f, g are loops and c_e is the constant loop at e.

iv Show that $\pi_1(G, e)$ is abelian.

Due 1/23/2014

1 Start reading Hatcher Chapter 1 (since Hatcher's book has been continually evolving, different printings have different page numbers, so I can't really reference these; nonetheless, our lectures will mostly follow the book pretty closely so it should be too hard to pick out the bits that correspond to what we're doing in class)

2 Do Hatcher Exercises for Section 1.1 #1, 4

3 More general than homotopies of paths are homotopies of maps. Two maps $f, g : X \to Y$ are called *homotopic* (written $f \sim g$) if there exsits a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Show that, for any fixed X and Y, homotopy is an equivalence relation on the set of maps from X to Y.

4 Suppose $f, f' : X \to Y$ with $f \sim f'$ and $g, g' : Y \to Z$ with $g \sim g'$. Show that $g \circ f \sim g' \circ f'$, where $\circ$ denotes composition of functions.

5 Suppose $f, f' : X \to Y$ are homotopic and $g, g' : Z \to W$ are homotopic. Show that $f \times g, f' \times g' : X \times Z \to Y \times W$ are homotopic.