Sketches of solutions to selected exercises

Note: these are intended as sample solutions. There will often be alternative solutions to problems. Furthermore, solutions presented here are not intended to be 100% complete but rather to demonstrate the idea of the problem. If the solution is not clear to you, please come ask me about it!

Due April 24

Let M be the closed Mobius strip (i.e. including its boundary). Let ∂M be the boundary of M. Compute $H_*(M, \partial M)$ using arguments with the long exact sequence and other basic properties of homology. You can use that you already know homology computations for circles, but you shouldn't compute $H_*(M, \partial M)$ directly from the Delta complex. If necessary, you can also use that $H_*(X) = H_*^{\Delta}(X)$ if X can be realized as a Delta complex in order to compute $H_*(M)$ and $H_*(\partial M)$. To get at $H_*(M, \partial M)$, we first compute $H_*(M)$ and $H_*(\partial M)$. The boundary ∂A is homeomorphic to a circle, so, using what we know about circles (or spheres in general), $H_1(\partial M) = \mathbb{Z}$, and $H_0(\partial M) = \mathbb{Z}$. We'll also have $H_i(\partial M) = 0$ for $i > 1$. Similarly, M is homotopy equivalent to a circle, so we have $H_0(M) \cong H_1(M) \cong \mathbb{Z}$ and all others are 0. So far we have the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="'] & H_2(M,\partial M) \arrow[r] \arrow[d, "="'] & H_1(\partial M) \arrow[r, "g"] \arrow[d, "="'] & H_1(M) \arrow[r] \arrow[d, "="'] & H_1(M,\partial M) \arrow[r] \arrow[d, "="'] & H_0(\partial M) \arrow[r, "f"] \arrow[d, "="'] & H_0(M) \arrow[r] \arrow[d, "="'] & H_0(M,\partial M) \arrow[r] \arrow[d, "="'] & 0 \arrow[d, "="'] \\
0 \arrow[r] & H_2(M,\partial M) \arrow[r] & \mathbb{Z} \arrow[r, "g"] & \mathbb{Z} \arrow[r] & H_1(M,\partial M) \arrow[r] & \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \arrow[r] & H_0(M,\partial M) \arrow[r] & 0
\end{tikzcd}$$

Note that as all $H_i(M)$ and $H_i(\partial M)$ are 0 for $i > 1$, it is immediate that $H_i(M, \partial M) = 0$ for $i > 2$.

So what are the maps? $H_0(\partial M) = \mathbb{Z}$ is generated by any singular 0-simplex, and this maps to a generator of $H_0(M) = \mathbb{Z}$, so the map $H_0(\partial M) \to H_0(M)$ is an isomorphism and $H_0(M, \partial M)$ must be 0. We can also see this directly as any 0 simplex in M is homologous via a path to a 0-simplex in ∂M.

Similarly, $H_1(\partial M) = \mathbb{Z}$ is generated by a cycle that goes once around the circle ∂M. This can most easily be seen using simplicial homology. Furthermore, the inclusion of ∂M to M followed by the homotopy equivalence of M to the circle provides a map that wraps around the circle twice. So the map $H_1(\partial M) \cong \mathbb{Z} \to H_1(M) \cong \mathbb{Z}$ is multiplication by 2. It now follows that $H_1(M, \partial M) \cong \mathbb{Z}_2$, using also that the map $H_1(M) \to H_1(M, \partial M)$ must be surjective because the map $H_0(\partial M) \to H_0(M)$ is injective from above.

Finally, as $H_1(\partial M) \to H_1(M)$ is injective, this forces $H_2(M, \partial M)$ to be 0.

Recall that $\mathbb{R}P^2$ can be obtained from the Mobius strip by filling in the boundary circle with a disk. Using this and excision, compute $H_*(\mathbb{R}P^2)$. By excision, we have $H_*(M, \partial M) \cong H_*(\mathbb{R}P^2, \text{disk})$, which is isomorphic to $H_*(\mathbb{R}P^2, \text{pt}) \cong \tilde{H}_*(\mathbb{R}P^2)$ using homotopy equivalence and our previous result about reduced homology. Putting this all together, $\tilde{H}_*(\mathbb{R}P^2) \cong H_*(M, \partial M)$ is 0 in all dimensions except when $* = 1$, in which case it is $\mathbb{Z}_2$. So $H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, and all other homology groups are 0.

Hatcher 2.1.29 Let $T = S^1 \times S^1$. Let $X = S^1 \vee S^1 \vee S^2$ and x_0 be the union point. By Example 2.3, $H_2(T) = \mathbb{Z}$, $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, $H_0(T) = \mathbb{Z}$ and all other homology groups of T are 0. By Corollary 2.25, we have $\tilde{H}_*(X) \cong \tilde{H}_*(S^1) \oplus \tilde{H}_*(S^1) \oplus \tilde{H}_*(S^2)$. Since we know $\tilde{H}_i(S^n) = \mathbb{Z}$ if $i = n$ and 0 otherwise, we see that $H_*(X) \cong H_*(T)$ (using also basic facts about the relation between ordinary and reduced homology).

The universal cover of T is $\mathbb{R}^2$, which is contractible, so $H_2(\tilde{T}) = 0$. But the universal cover of X is like the antenna space over $S^1 \vee S^1$ but with an S^2 attached at every vertex. This is homotopy equivalent to a wedge of a (countably) infinite number of S^2s (one for each word in a, b, a^{-1}, b^{-1}). So $\tilde{H}_2(\tilde{X}) \cong \oplus_a \tilde{H}_2(S^2_a) \cong \oplus_a \mathbb{Z} \neq 0$.

Hatcher 2.2.2 Suppose for all x, $f(x) \neq x$ and $f(x) \neq -x$, then $g_t(x) = \frac{(1-t)x + tf(x)}{|(1-t)x+tf(x)|}$ is a homotopy from f to the identity and $g_t(x) = \frac{(1-t)(-x)+tf(x)}{|(1-t)(-x)+tf(x)|}$ is a homotopy from f to the antipodal map. So the identity and antipodal map are homotopic. But the identity has degree 1 and the antipodal map has here degree -1 because S^{2n} is an even sphere. So this is a contradiction, so there must be an x that gets taken to x or $-x$ by f.

Now $\mathbb{R}P^{2n}$ is covered by S^{2n}. Condsider the following diagram

$$\begin{tikzcd}
S^{2n} \arrow[r, dashed, "\tilde{f}"] \arrow[d, "p"'] & S^{2n} \arrow[d, "p"'] \\
\mathbb{R}P^{2n} \arrow[r, "f"] & \mathbb{R}P^{2n}
\end{tikzcd}$$

The map $\tilde{f}$ exists by applying the lifting theorem to fp. By the first part of the theorem, some $x \in S^{2n}$ gets taken to x or $-x$, so $fp(x) = p(x)$.

Finally consider the map $f : \mathbb{R}^{2n} \to R^{2n}$ given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in blocks along the diagonal and 0 elsewhere. This takes lines to lines, and so induces a map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$. But since there are no real eigenvectors, there are no lines fixed by this.

Due April 17

Let A be the closed annulus, i.e. $A = \{z \in \mathbb{C} \mid 1 \le |z| \le 2\}$. Let ∂A be the boundary of A. Compute $H_*(A, \partial A)$ using arguments with the long exact sequence and other basic properties of homology. You can use that you already know homology computations for circles, but you shouldn't compute $H_*(A, \partial A)$ directly from the Delta complex. If necessary, you can also use that $H_*(X) = H_*^{\Delta}(X)$ if X can be realized as a Delta complex in order to compute $H_*(A)$ and $H_*(\partial A)$.

To get at $H_*(A, \partial A)$, we first compute $H_*(A)$ and $H_*(\partial A)$. The boundary ∂A is the disjoint union of two circles, so, using what we know about circles (or spheres in general), $H_1(\partial A) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(\partial A) = \mathbb{Z} \oplus \mathbb{Z}$. We'll also have $H_i(\partial A) = 0$ for $i > 1$. Similarly, A is homotopy equivalent to a circle, so we have $H_0(A) \cong H_1(A) \cong \mathbb{Z}$ and all others are 0. So far we have the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="'] & H_2(A,\partial A) \arrow[r] \arrow[d, "="'] & H_1(\partial A) \arrow[r, "g"] \arrow[d, "="'] & H_1(A) \arrow[r] \arrow[d, "="'] & H_1(A,\partial A) \arrow[r] \arrow[d, "="'] & H_0(\partial A) \arrow[r, "f"] \arrow[d, "="'] & H_0(A) \arrow[r] \arrow[d, "="'] & H_0(A,\partial A) \arrow[r] \arrow[d, "="'] & 0 \arrow[d, "="'] \\
0 \arrow[r] & H_2(A,\partial A) \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "g"] & \mathbb{Z} \arrow[r] & H_1(A,\partial A) \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \arrow[r] & H_0(A,\partial A) \arrow[r] & 0
\end{tikzcd}$$

Note that as all $H_i(A)$ and $H_i(\partial A)$ are 0 for $i > 1$, it is immediate that $H_i(A, \partial A) = 0$ for $i > 2$.

So what are the maps? $H_0(\partial A) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by one 0-simplex in each component. These each map to a generator of $H_0(A) = \mathbb{Z}$, so the map $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is $(a, b) \to a + b$. Therefore, it is onto, so $H_0(A, \partial A) = 0$. We can also see this directly as any 0 simplex in A is homologous via a path to a 0-simplex in ∂A.

Similarly, $H_1(\partial A) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by a cycle that goes once around each circle. This can most easily be seen using simplicial homology. Furthermore, as each inclusion of a boundary circle into A is a homotopy equivalence, each of these inclusions induces an isomorphism. This implies that $g : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is also given by $(a, b) \to a + b$. So this is also surjective, and its kernel must be $H_2(A, \partial A)$. This is easily seen to be isomorphic to $\mathbb{Z}$, generated by $(1, -1)$.

That leaves $H_1(A, \partial A)$. As g is surjective, $H_1(A, \partial A)$ must inject into $H_0(A)$, and, in fact, it is the kernel of f. Again, this is $\mathbb{Z}$ generated by $(1, -1)$.

So, $H_2(A, \partial A) \cong H_1(A, \partial A) \cong \mathbb{Z}$ and $H_0(A, \partial A) = 0$.

Find generators for the homology groups $H_i(A, \partial A)$ you computed above. For each of $H_1(A, \partial A)$ and $H_2(A, \partial A)$, we need generators that map to $(1, -1)$ under ∂_*, where here the 1 and -1 are with respect to the generators of $H_0(\partial A)$ or $H_1(\partial A)$ observed above. So for $H_1(A, \partial A)$, we can use a 1-simplex that runs from one boundary component to the other. For $H_2(A, \partial A)$ we can use any 2-chain that triangulates A as a Delta-complex; in other words, form such a Δ complex and then take the sum over all the 2-simplices, oriented compatibly.

Due April 10

Let $A \xrightarrow{f} B \xrightarrow{h} C \xrightarrow{k} D \xrightarrow{g} E$ be an exact sequence. Suppose f is surjective and g is injective. Show that $C = 0$. Provide *all* the details. As g is injective, $\ker(g) = 0$, so $\mathrm{im}(k) = 0$, so $\ker(k) = C$.

On the other hand, as f is surjective, $\mathrm{im}(f) = B$, so $\ker(h) = \mathrm{im}(f) = B$. This implies that $\mathrm{im}(h) = 0$.

But $\mathrm{im}(h) = \ker(k)$, so $C = 0$.

Recall that if $f : G \to H$ is a map of abelian groups, then the cokernel, $\mathbf{cok}(f)$, is defined to be $H/\mathbf{im}(f)$. Suppose you have the following commutative diagram of abelian groups in which the horizontal sequences are exact:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

The *serpent lemma* says that there is an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \mathbf{cok}(\alpha) \to \mathbf{cok}(\beta) \to \mathbf{cok}(\gamma) \to 0.$$

Prove the serpent lemma. Hint: write the kernels and cokernels as homology groups and use some big results you learned in class. Notice that the map $A \to D$ can be included in a chain complex $\cdots \to 0 \to A \xrightarrow{\alpha} D \to 0 \to \cdots$. You can check this is indeed a chain complex. If we think of D as being in degree 0, then $H_0 = \mathrm{cok}(\alpha)$ and $H_1 = \ker(\alpha)$. Similarly for the other groups. The exact sequence of the serpent lemma is then just the long exact homology sequence from the short exact sequence of chain complexes.

Hatcher 2.1.16 a) This could be done directly but let's use the exact sequence. First, notice that if X has one component and A is not empty, then a 0-chain generating $H_0(A)$ also generates $H_0(X)$. So $H_0(A) \to H_0(X)$ is onto and $H_0(X, A)$ is 0 from the long exact sequence. More generally, suppose X has multiple connected components and that A intersects each component. If X_i is a component of X, then $H_0(A \cap X_i) \to H_0(X_i)$ is surjective by the preceding argument. But then $H_0(A) = \oplus_i H_0(A \cap X_i) \to \oplus_i H_0(X_i) = H_0(X)$ is surjective. So $H_0(X, A) = 0$.

b) If $H_1(X, A) = 0$, then $H_1(A) \to H_1(X)$ is onto immediately by the long exact sequence and $H_0(A) \to H_0(X)$ is injective. This last statement can't be true if some path component X_i of X contains multiple components of A because then $H_0(A \cap X_i) \cong \mathbb{Z}^n$ for some $n \geq 2$ while $H_0(X_i) = \mathbb{Z}$. So then $H_0(A \cap X_i) \to H_0(X_i)$ can't be 1-1, and the same follows for $H_0(A) \to H_0(X)$.

Conversely. If $H_1(A) \to H_1(X)$ is onto, then $\ker(p_1 : H_1(X) \to H_1(X,A)) = H_1(X)$, so the map $p_1 : H_1(X) \to H_1(X,A)$ is the 0 map. Similarly, if each component of X contains at most one component of A, then $H_0(A) \to H_0(X)$ is injective. So its kernel is 0, so the image of $H_1(X,A) \to H_0(A)$ is 0. But then by the first isomorphism theorem, $0 = H_0(X,A)/\ker(\partial_*) = H_0(X,A)/\mathrm{im}(p_1) = H_0(X,A)$.

Hatcher 2.1.17.a (just the sphere for now) We know from previous computations that $H_2(S^2) \cong H_0(S^2) \cong \mathbb{Z}$, and otherwise $H_i(S^2) = 0$. Also if A consists of n points, $H_0(A) \cong \mathbb{Z}^n$ and $H_i(A) = 0$ otherwise. So the long exact reduced homology sequence looks like

$$0 \longrightarrow \tilde{H}_2(S^2) \cong Z \longrightarrow H_2(S^2, A) \longrightarrow 0 \longrightarrow$$

$$0 \longrightarrow H_1(S^2, A) \longrightarrow \tilde{H}_0(A) \cong \mathbb{Z}^{n-1} \longrightarrow \tilde{H}_0(X) = 0 \longrightarrow H_0(S^2, A) \longrightarrow 0$$

So immediately, we have $H_2(S^2, A) \cong \tilde{H}_2(S^2) \cong \mathbb{Z}$, and $H_1(S^2, A)$ is isomorphic to $\tilde{H}_0(A) \cong \mathbb{Z}^{n-1}$. We also must have $H_0(S^2, A) = 0$.

Due April 3

Hatcher 2.1.11 Since A is a retract, there are maps $A \xrightarrow{i} X \xrightarrow{r} A$ such that $ri = \mathrm{id}_A$. Thus by functoriality, $r_* i_*$ is the identity, so $i_* : H_*(A) \to H_*(X)$ is injective.

Suppose $H < G$ are abelian groups and $G/H \cong F$ is a free abelian group. Prove that $G \cong H \oplus F$. Hint: Let $q : G \to G/H$ be the natural quotient map. First construct a homomorphism $\psi : F \to G$ such that $q\psi = \mathbf{id}_F$; note: this implies that $\psi : F \to G$ is injective, so $\psi(F) \cong F$. Then show that $G \cong H \oplus \psi(F)$ by showing that $H \cap \psi(F) = \{0\}$ and that every element of G can be written as a sum $h + f$ with $h \in H$ and $f \in \psi(F)$. Double hint: it might help to notice that H is the kernel of q. Triple hint: to construct f, use the maps! First we construct ψ. As F is free, it has generators, say $\{x_a\}$. As q is surjective, there is an element (not necessarily unique) $y_a \in G$ such that $q(y_a) = x_a$. Define $\psi(x_a) = y_a$. Again, ψ is not unique, but that's okay. Then clearly $q\psi(x_a) = x_a$. As F is free abelian, defining ψ on the generators determines ψ on all of F, and as q and ψ are homomorphisms, $q\psi(\sum_a c_a x_a) = \sum_a c_a q\psi(x_a) = \sum_a c_a x_a$ for any $c_a \in \mathbb{Z}$ and finite index set for the sum. So ψ is as desired.

Now, let us first show that $H \cap \psi(F) = \{0\}$. Suppose $y \in H \cap \psi(F)$. Then $q(y) = 0$ because $H = \ker q$. But also, if $y \in \psi(F)$, the $y = \psi(x)$ for some x and so $0 = q(y) = q\psi(x) = x$. So $x = 0$ and $y = \psi(x) = 0$.

Next, let's show that every element of G can be written as $h + f$ with $h \in H$ and $f \in \psi(F)$. Let $y \in G$. Notice $y = y - \psi q(y) + \psi q(y)$. Obviously $\psi q(y) \in \psi(F)$. But then $q(y - \psi q(y)) = q(y) - q\psi(q(y)) = q(y) - q(y) = 0$. so $y - \psi q(y) \in H$.

Show that if $f : X \to Y$ is homotopic to a constant map, then $f_* : \tilde{H}_*(X) \to \tilde{H}_*(Y)$ is the 0 map . Since homotopic maps induce the same map on homology (and reduce homology), we might as well assume f is the constant map. The we can write $f = gh$, where h maps X to a point and g maps a point to the image point $f(X)$. So then the image of f_* is the image of the composition $\tilde{H}_*(X) \xrightarrow{h_*} \tilde{H}_*(pt) \xrightarrow{g_*} \tilde{H}_*(Y)$. But $\tilde{H}_*(pt) = 0$, so the result follows.

The topologist's sine curve is the subspace of $\mathbb{R}^2$ consisting of all points $(x, \sin(1/x))$ for $0 < x \leq 1$ and all points $(0, y)$ for $-1 \leq y \leq 1$. Compute the singular homology groups of this space. The space X has two path components, let's say X_1 is the set of points on the $(x, \sin(1/x))$ piece and X_2 the other piece. Each of these pieces is contractible. So $H_*(X) = H_*(X_1) \oplus H_*(X_2)$ and for each i, $H_0(X_i) = \mathbb{Z}$ and $H_j(X_i) = 0$ for $j > 0$. So $H_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_i(X) =)$ otherwise.

Suppose X is a path connected space such that $\pi_1(X)$ is a non-abelian simple group. Show that $H_1(X) = 0$. Since $H_1(X)$ is the abelianization of $\pi_1(X)$, we must show that $\pi_1(X)$ abelianizes to 0. The abelianization of a group is its quotient by its commutator subgroup C. This is the normal subgroup generated by the commutators, which are elements of the form $aba^{-1}b^{-1}$. Since C is normal by definition and since $\pi_1(X)$ is simple, C is either trivial or all of $\pi_1(X)$. In the latter case, $H_1(X) \cong \pi_1(X)/\pi_1(X) = 0$. So we just have to show that C is not trivial. But if C is trivial, then every commutator $aba^{-1}b^{-1}$ is equal to the identity e. So $aba^{-1}b^{-1} = e$, whence $ab = ba$. So C will be nontrivial as long as there are two elements of $\pi_1(X)$ that don't commute.

Hatcher 2.1.12 Equivalence relation:

1. Reflexivity . Suppose $f : A_* \to B_*$ is a chain map. We need to show that f is chain equivalent to f, i.e. that there is a $P : A_* \to B_{*+1}$ such that $\partial P + P\partial = f - f = 0$. Just take $P = 0$, and that will work.

2. Suppose $f, g : A_* \to B_*$ are chain homotopic. So there is a $P : A_* \to B_{*+1}$ such that $\partial P + P\partial = f - g$. But now consider the map $-P$. Then using linearity of all the maps, $\partial(-P) + (-P)\partial = g - f$. So g is chain homotopic to f.

3. Suppose $f, g : A_* \to B_*$ are chain homotopic via $P : A_* \to B_{*+1}$ and that $g, h : A_* \to B_*$ are chain homotopic via $Q : A_* \to B_{*+1}$. Then $\partial P + P\partial = f - g$ and $\partial Q + Q\partial = g - h$. So $\partial P + P\partial + \partial Q + Q\partial = \partial(P + Q) + (P + Q)\partial = f - h$. So the relation is transitive.

Do Hatcher 2.1.13. From the proof of Theorem 2.10, there is a chain homotopy $P : C_*(X) \to C_{*+1}(Y)$ between $f_\#$ and $g_\#$, ie. $\partial P + P\partial = f_\# - g_\#$. If we want a chain homotopy $\tilde{P} : \tilde{C}_*(X) \to \tilde{C}_{*+1}(Y)$, we can try to let $\tilde{P} = P$ on $C_i(X)$ for $i > 0$. We only need to be careful about what happens at the bottom, so let's have a closer look at what $\tilde{P}_0 : \tilde{C}_0(X) \to \tilde{C}_1(Y)$ and $\tilde{P}_{-1} : \mathbb{Z} \to \tilde{C}_0(Y)$ must be.

Note: Since we need $\epsilon f_\# = f_\# \epsilon$, we must define $\tilde{f}_\# : \tilde{C}_{-1}(X) = \mathbb{Z} \to \tilde{C}_{-1}(Y) = \mathbb{Z}$ to be the identity on $\mathbb{Z}$, and similarly for any chain map.

Let's first look at $\tilde{P}_{-1}$. We need $\partial_0 \tilde{P}_{-1} + \tilde{P}_{-2}\partial_{-1} = \tilde{f}_\# - \tilde{g}_\#$. Notice that $\partial_{-1} = 0$, $\partial_0 = \epsilon$, by definition, and $\tilde{f}_\# - \tilde{g}_\# = 0$ at the degree -1. So we can just take $\tilde{P}_{-1} = 0$.

For $\tilde{P}_0$, we need $\partial_1 \tilde{P}_0 + \tilde{P}_{-1}\partial_0 = \tilde{f}_\# - \tilde{g}_\#$. We have just defined $\tilde{P}_{-1} = 0$, and from our chain homotopy P, we already have a P_0 such that $\partial_1 P_0 + P_{-1}\partial_0 = \partial_1 P_0 = f_\# - g_\#$ (because in the unreduced chain complex, $\partial_0 = 0$). So we just use $\tilde{P}_0 = P_0$, and from here we can just let $\tilde{P}_i = P_i$ for all $i \geq 0$, and everything will be consistent by the properties of P.

The homology result follows from the chain homotopy result as in the unreduced case.

Due March 27

Hatcher 2.1.1 It's a Mobius strip. (The easiest way to see this is that the instructions given an $\mathbb{R}P^2$ with a hole cut out of it. That's always a Mobius strip.)

Hatcher 2.10.a - note that he means that every edge is glued to precisely one other edge. Let X be such a space. Every point of X lies in the interior of a vertex, an edge, or a face. If it lies in the interior of a face, it clearly has a euclidean neighborhood. If it lies on the interior of an edge, then we can find a euclidean neighborhood that looks like a half disk in each of the two 2-simplices joined together along that edge. At a vertex, the space looks locally like a pie. Since there are no free edges in the quotient, it must be a "full pie" and we get a euclidean neighborhood by adjoining wedges from each of the 2-simplices

Find a way to realize the "two-holed torus" (what Hatcher calls M_2) as a Δ-complex.

Find a *Delta*-complex X such that $\pi_1(X) = \mathbb{Z}_3$. See me for a picture.

Hatcher 2.1.4 We have $\Delta_0(X) =\ll v \gg$, $\Delta_1(X) =\ll e_1, e_2, e_3 \gg$, $\Delta_2(X) =\ll f \gg$.

- $\partial_2 f = e_1 + e_2 - e_3$. So $\ker \partial_2 = 0$, so $H_2^\Delta(X) = 0$.
- We've seen $\mathrm{im}\partial_2 =\ll e_1 + e_2 - e_3 \gg$. Also, $\partial_1 e_1 = \partial_1 e_2 = \partial_1 e_3 = 0$. So $\ker \partial_1 = \Delta_1(X)$. So $H_1^\Delta(X) =\ll e_1, e_2, e_3 \mid e_1 + e_2 - e_3 \gg =\ll e_1, e_2 \gg \cong \mathbb{Z} \oplus \mathbb{Z}$.
- We know $\mathrm{im}\partial_1 = 0$ and clearly $\ker \partial_0 = \Delta_0(X) =\ll v \gg$. So $H_0^\Delta(X) \cong \mathbb{Z}$.

Hatcher 2.1.5 Using the notation from Hatcher's picture, $\Delta_0(K) =\ll v \gg$, $\Delta_1(K) =\ll a, b \gg$, and $\Delta_2(K) =\ll U, L \gg$.

- $\partial_2 U = a + b - c$ and $\partial_2 L = c + a - b$. It's easy then to check that $\ker \partial_2 = 0$, so $H_2^\Delta(K) = 0$.
- We've seen $\mathrm{im}\partial_2 =\ll a + b - c, a - b + c \gg$. Also, $\partial_1 a = \partial_1 b = \partial_1 c = 0$. So $\ker \partial_1 = \Delta_1(K)$. So $H_1^\Delta(K) =\ll a, b, c \mid a + b - c, a - b + c \gg =\ll a, b \mid a - b + (a + b) \gg =\ll a, b \mid 2a \gg \cong \mathbb{Z}_2 \oplus \mathbb{Z}$.
- We know $\mathrm{im}\partial_1 = 0$ and clearly $\ker \partial_0 = \Delta_0(X) =\ll v \gg$. So $H_0^\Delta(K) \cong \mathbb{Z}$.

Hatcher 2.1.9 Here clearly each $\Delta_j(X)$ has one generator for each $0 \leq i \leq n$. What are the maps δ_j. If d is the lone j-face (after identification), then let's represent d as $[v_0, \ldots, v_j]$ (of course all the v_i have also been identified, but we don't use that yet. Then $\partial_i[v_0, \ldots, v_j] = \sum(-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_j]$. But now each of the $[v_0, \ldots, \hat{v}_i, \ldots, v_j]$ have been identified to each other. If we call the identified face f, this is $\sum_{i=0}^{j}(-1)^i f$, which is 0 if j is odd and f if j is even. So each map ∂_{2k+1} is 0 and each map ∂_{2k} is the identity $\ll d \gg \to \ll f \gg$. We just have to be careful around $j = 0$ and $j = n$ because there are no -1 or $n+1$ simplices. Thinking about all these cases shows that $H_0^\Delta(X) \cong \mathbb{Z}$, $H_n^\Delta(X) \cong \mathbb{Z}$ if n is odd, and $H_i^\Delta(X) = 0$ otherwise.

Due March 20

Note: the following exercise is essentially a special case of Hatcher's 1.3.24, which actually follows from the same sort of ideas as this by "putting more spaces in the middle". 1. Suppose $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ path connected, locally path connected, and *simply connected*. Show that $\tilde{X}$ is a normal cover and that the group of deck transformations for $\tilde{X}$ over X is isomorphic to $\pi_1(X)$. Since $\pi_1(\tilde{X})$ is trivial, so is its image in $\pi_1(X)$. The trivial subgroup is normal, so the covering is normal, and $G(\tilde{X}) = \pi_1(X)/\{e\} = \pi_1(X)$.

2. Let $\tilde{x}_0$ be a basepoint of $\tilde{X}$ over $x_0 \in X$. Let H be a subgroup of $\pi_1(X, x_0)$ and hence also a subgroup of the group of deck transformations of $\tilde{X}$ over X. Let $X_H = \tilde{X}/H$, and let $\tilde{x}_0^H$ be the image of $\tilde{x}_0$ in X_H, which we let be the basepoint of x_H. Notice that we can factor p as $\tilde{X} \xrightarrow{p_H} X_H \xrightarrow{q} X = X/G$. Show that $q : X_H \to X$ is a covering space of X. For each $x \in X$, let U_x be a neighborhood such that $p^{-1}(U_x)$ consists of homeomorphic copies of U_x. In fact, if $\tilde{U}_x$ is one such copy, then all the other copies will have the form $g\tilde{U}_x$, where $g \in \pi_1(X, x_0)$ and $g\tilde{U}_x$ is the image of $\tilde{U}_x$ under the action of g. X_H is formed from $\tilde{X}$ by identifying equivalence classes of the form $\{hy \mid h \in H\}$ for each $y \in \tilde{X}$. So going from $\tilde{X}$ to X_H identifies all of the $h\tilde{U}_x$ for $h \in H$. Thus $q^{-1}(U_x)$ is the quotient $p^{-1}(U_x)/H$, which consists of a copy of U_x for each left coset gH.

3. Show that $q_*(\pi_1(X_H, \tilde{x}_0^H) = H$. Recall that $q_*(\pi_1(X_H, \tilde{x}_0^H)$ will be consist of those loops in X that are images of loops in X_H. Since $\tilde{X}$ is simply connected, the loops in X_H are the image from $\tilde{X}$ of the paths in $\tilde{X}$ starting at $\tilde{x}_0$ and ending at points of the form $h\tilde{x}_0$ for $h \in H$. Thus there is a correspondence between elements of H and elements of $\pi_1(X_H, \tilde{x}_0^H)$. One then checks that this is a homomorphism.

4. Suppose H is normal. Show that G/H acts properly discontinuously on X_H (in particular, show that $g_1 x = g_2 x$ if and only if $g_1 H = g_2 H$) and observe that $X = X/G = X_H/(G/H)$.

(Clarification: recall that each point of X_H corresponds to an orbit of a point x in the $\tilde{X}$ under the action of H. In particular, every point in X_H is an image $p_H(x)$ for some x in $\tilde{X}$. Define an action of G on X_H by letting $g(p_H(x)) = p_H(gx)$. Part of your job is to show that this is well-define, i.e. if y is another point in $\tilde{X}$ with $p_H(y) = p_H(x)$, show that $g(p_H(x)) = g(p_H(y))$ so that this definition is consistent. Then to show that we really have an action of G/H, show that every element of the coset gH acts on $p_H(x)$ the same way. Lastly, you should argue that the action of G/H is properly discontinuous.) Recall that $p_H : \tilde{X} \to X_H$ identifies points of $\tilde{X}$ that are in the same orbit of H. So let $x \in \tilde{X}$, and let $p_H(\bar{x})$ be its image in x_H. Define $gp_H(\bar{x})$ to be $p_H(g\bar{x})$. To see that this is well-defined, suppose $p_H(y) = p_H(x)$. Then $y = hx$ for some $h \in H$. But then $gy = ghx$, so $gp_H(y) = p_H(gy) = p_H(ghx) = gp_H(hx) = gp_H(x)$

because x and hx are in the same orbit. So the map is well-defined. Now, to show that G/H acts on X_H, observe $ghp_H(x) = p_H(ghx) = gp_H(hx) = gp_H(x)$. Therefore, each element of the coset gH acts on X_H in the same way, and we can consider this an action of G_H on X_H. This action is properly discontinuous because if U_x is as above, then g takes $p_H(\tilde{U}_x)$ to $p_H(g\tilde{U}_x)$, which is disjoint from $\tilde{U}_x$ if $g \notin H$. For freeness, if $g(p_H(x)) = p_H(x)$, that means that $p_H(gx) = p_H(x)$, which happens only if $g \in H$ for some $h \in H$. But then gH represents the identity element of G/H.

Recall from last week's homework the spaces X_n, $n \in \{1, 2, 3, \ldots\}$ obtained from the two-dimensional disk D^2 by identifying points on the boundary that differ by an angle $2\pi/n$; let $q : D^2 \to X_n$ be the quotient map. In that problem, you should have shown $\pi_1(X_n) \cong \mathbb{Z}_n$ and determined that the universal cover $\tilde{X}_n$ is homeomorphic to n copies of D^2 with their boundaries identified to each other.

1. **Describe how the deck transformations act on $\tilde{X}_n$.** The group of transformations is $\mathbb{Z}_n$. The generator 1 acts by taking the ith copy of D^2 to the $(i+1)$st copy (the nth copy goes to the first) and rotating it by an angle $2\pi/n$.

2. **Describe what points of $\tilde{X}_4$ you would identify to get the quotient space of $\tilde{X}_4$ corresponding to the unique subgroup of index 2 of $\pi_1(X_4)$.** Each point on disk i would be identified with its antipodal point on the disk $i + 2 \mod 4$

Describe all connected covering spaces of $\mathbb{R}P^2$, up to equivalence. The covering spaces are $\mathbb{R}P^2$ and S^2.

Describe all connected covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^2$ (Note: some of these will be familiar, others you made need to describe as quotients of universal cover under certain group actions). $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, so the covering spaces correspond to 0, $\mathbb{Z}_2 \times 0$, $0 \times \mathbb{Z}_2$, and $\{0, (1, 1)\}$. The covering spaces corresponding to the first three of these are $S^2 \times S^2$, $\mathbb{R}P^2 \times S^2$ and $S^2 \times \mathbb{R}P^2$. The last space is $S^2 \times S^2/(p, q) \sim (-p, -q)$.

Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$ with basepoint the union point x_0.

1. **What's the universal cover $\tilde{X}$ of X?**

   It's a string of S^2s with the north pole of one identified to the south pole of the next.

2. **Since $\tilde{X}$ is simply-connected, its group of deck transformations is $\pi_1(X, x_0) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Let a, b be the respective generators of $\pi_1(\mathbb{R}P^2 \vee x_0)$ and $\pi_1(x_0 \vee \mathbb{R}P^2)$ as subgroups of $\pi_1(X, x_0)$. Describe the covering actions of a and b. Then describe what a covering action by a general element of $\pi_1(X, x_0)$ would look like. Can you see why all such transformations are generated by those corresponding to a and b? (Note, while the group of deck transformations is isomorphic to $\pi_1(X, x_0)$, the exact isomorphism might depend upon the choice of basepoint in $\tilde{X}$.)**

Take two consecutive spheres and label them A and B. a acts as the antipodal map on sphere A and consequently flips the entire string of spheres. b acts similarly but inverts B instead. A general element would flip the string of spheres, inverting one them.

Due 3/6/2014

Describe all connected covering spaces of $\mathbb{R}P^2$, up to isomorphism. $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ which has only one subgroup - the trivial subgroup. So the only cover is the universal cover, which is S^2.

Describe all connected covering spaces of $\mathbb{R}P^2 \vee S^2$, up to isomorphism. This should look like three spheres in a row.

Describe all connected covering spaces of $S^1 \vee S^2$, up to isomorphism. This should look like $\mathbb{R}$ with a sphere attached at every integer.

Describe all connected covering spaces of $S^1 \times S^2$, up to isomorphism. Since S^2 is simply connected, we need only take products of S^2 with the covering spaces of S^1. The trivial subgroup of $\pi_1(S^1) = \mathbb{Z}$ corresponds to the covering $e^{i\theta} : \mathbb{R} \to S^1$, while the subgroup $n\mathbb{Z}$ corresponds to $e^{ni\theta} : S^1 \to S^1$. So the desired covers are $e^{i\theta} \times \mathrm{id} : \mathbb{R} \times S^2 \to S^1 \times S^2$ and $e^{ni\theta} \times \mathrm{id} : S^1 \times S^2 \to S^1 \times S^2$.

Let $X = S^1 \vee S^1$ with $\pi_1(X, x_0) = \langle a, b \rangle$ in the standard way. For each of the following subgroups of $\pi_1(X, x_0)$, find (draw) a covering space (with basepoint) $(\tilde{X}, \tilde{x}_0)$ of X such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is the given subgroup:

1. $\{(ab^2)a^n(ab^2)^{-1} \mid n \in \mathbb{Z}\}$
2. $\{(ab)^n \mid n \in \mathbb{Z}\}$
3. $\{b^3(ab)^n b^{-3} \mid n \in \mathbb{Z}\}$
4. the subgroup freely generated by a^2 and b
5. the subgroup freely generated by a^3, b, aba^{-1} and $a^{-1}ba$.

I can't draw these here. Come see me if you want to talk about them.

Due 2/28/2014

Let $p : \tilde{X} \to S^1 \vee S^1$ be the covering space shown in box 2 of Hatcher's table of examples of coverings of $S^1 \vee S^1$. Let $f : S^1 \times S^1 \to S^1 \vee S^1$ be the map given by the composition of the projection $S^1 \times S^1 \to S^1 \times y_0$ followed by the map $S^1 \times y_0 \to S^1 \vee S^1$ described by the loop a^3b^3. Does f lift to $\tilde{X}$? Yes! The image f_* in $\pi_1(S^1 \vee S^1)$ is the cyclic subgroup generated by $a^3b^3 = a^2(ab)b^2$, which is a subgroup of the image of p_* because a^2, b^2, ab are all in this image (by direct inspection or using the information given in Hatcher's table).

Use covering space theory to show that every map $\phi : \mathbb{R}P^2 \to S^1$ is homotopic to a constant map. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, and $\pi_1(S^1) = \mathbb{Z}$, the homomorphism ϕ_* must be the trivial homomorphism with image 0. So ϕ must lift to a map $\tilde{\phi} : \mathbb{R}P^2 \to \mathbb{R}$. But every map to $\mathbb{R}$ is nullhomotopic, so we have a homotopy, say $\tilde{\phi}_t$ from $\tilde{\phi}_0 = \tilde{\phi}$ to $\tilde{\phi}_1 = 0$. Then if $p : \mathbb{R} \to S^1$ is the covering map, $p \circ \tilde{\phi}_t$ is a nullhomotopy of ϕ.

Suppose Y is simply connected and $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ contractible. Show that every map $f : Y \to X$ is homotopic to a constant map. Since Y is simply connected, the criterion for the lifting theorem is automatically satisfied, so there is a lift $\tilde{f} : Y \to \tilde{X}$ of f. Since $\tilde{X}$ is contractible, say by a homotopy H_t, the composition $H_t\tilde{f}$ gives a homotopy from $\tilde{f}$ to a constant map and $pH_t\tilde{f}$ gives a homotopy from $p\tilde{f} = f$ to a constant map.

Describe the universal cover of $T^2 \vee S^1$, where T^2 is the torus $S^1 \times S^1$. This is like an antenna space, except it alternates lines and planes. Start with a line. At every integer points attach a plane. At ever $\mathbb{Z} \times \mathbb{Z}$ lattice point in each plane, attach a line (disjoint from the rest of the space otherwise). At every integer point of each new line, attach a plane. And so on.

Let X_n be the space obtained from the two-dimensional disk D^2 by identifying points on the boundary that differ by an angle $2\pi/n$; let $q : D^2 \to X_n$ be the quotient map.

What's $\pi_1(X_n)$ (you don't need to write out the argument)? $\mathbb{Z}_n$

Suppose x is in the interior of D^2; describe what small neighborhoods of $q(x)$ look like. Now suppose x is in the boundary of D^2; describe what small neighborhoods of $q(x)$ look like. For points in the interior of D^2, the corresponding points in X_n have neighborhoods homeomorphic to Euclidean 2-space $\mathbb{R}^2$. For points on ∂D^2, their neighborhoods in X_n look like n copies of the half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, all joined along the x-axis.

Find and describe the universal cover $\tilde{X}_n$ of X_n using what you figured out in the previous sections; hint: think about the familiar case $X_2 = \mathbb{R}P^2$. Recall that each point in X_n has to have n points in its preimage in $\tilde{X}_n$. Thinking about $\mathbb{R}P^2$, the space can be assembled as follows: start with n copies of D^2, then glue together their boundaries such that the point at angle θ on the ith copy ($\mod n$) gets glued to the point at angle $\theta + 2\pi/n$ on the $i+1$st copy ($\mod n$). It is not difficult to see that this space is simply connected, using the van Kampen Theorem, and then we can check that each point in $\tilde{X}_n$ has neighborhoods that get taken homeomorphically to X_n in the appropriate way: the preimage of each point in the interior of D^2 has a neighborhood that looks like n copies of D^2 (displaced in the copies of D^2 by angles of $2\pi/n$), while the preimage of each point corresponding to a boundary point of the original D^2 consists of n neighborhoods around the "edge" of $\tilde{X}_n$, each of the form described above (a union of half planes).

Due 2/20/2014

Show that a covering map $p : X \to Y$ is an open map (and hence a quotient map). Let $\{U_a\}$ be a covering of Y by open sets such that each $p^{-1}(U_a)$ is a disjoint union of open sets each homeomorphic to U_a. We can write $p^{-1}(U_a) = \cup_b \tilde{U}_a^b$, where each $\tilde{U}_a^b$ is homeomorphic to U_a. Then $\cup_{a,b}\tilde{U}_a^b$ covers X, so there is a basis $\{B_i\}$ for X for which each elements of the basis lies inside some $\tilde{U}_a^b$. But then if $B_i \subset \tilde{U}_a^b$ is an open set in the basis, then $p(B_i)$ is also open because $p|_{\tilde{U}_a^b}$ is an isomorphism. Any open set V of X is a union of some collection of the B_i, $V = \cup_{i\in J} B_i$. Then $p(V) = p(\cup_{i\in J} B_i) = \cup_{i\in J} p(B_i)$ is open. So p is an open map.

Alternative: Let U be an open set of X. Let $x \in U$. Then, as p is a covering map, $p(x) \in Y$ has a neighborhood V such that $p^{-1}(V)$ is a disjoint union of open sets homeomorphic to U. Let V_0 be the subset of $p^{-1}(V)$ that is homeomorphic to V and contains x. Then $V_0 \cap U$ is an open neighborhood of x that maps to an open neighborhood of $p(x)$ in $p(U)$. Since $x \in U$ was arbitrary and $p : U \to p(U)$ is surjective, every point of $p(U)$ has an open neighborhood in $p(U)$. So $p(U)$ is open.

Hatcher 1.3.1. If U_a is the open covering of X in the definition of what makes $p : \tilde{X} \to X$ a covering map, then $\{U_a \cap A\}$ is an open covering of A. Furthermore, it has the desired property because if $p^{-1}(U_a) = \amalg_b U_{a,b}$, then the homeomorphism $U_{a,b} \to U_a$ induced by p restricts to a homeomorphism $U_{a,b} \cap p^{-1}(A) \to U_a \cap A$.

Hatcher 1.3.1. Let $\{U_\alpha\}$ and $\{V_\beta\}$ be appropriate coverings of X_1 and X_2. Consider $(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta) = p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta)$ (this is an easy set theory exercise). Suppose $p_1^{-1}(U_\alpha) = \amalg U_{\alpha,i}$ and $p_2^{-1}(V_\beta) = \amalg V_{\beta,j}$. Then the sets $U_{\alpha,i} \times V_{\beta,j}$ are disjoint from each other (again by easy set theory). And $p_1 \times p_2$ restricts to a homeomorphism from $U_{\alpha,i} \times V_{\beta,j}$ to $U_\alpha \times V_\beta$ because the product of homeomorphisms is a homeomorphism.

Use a covering space of $S^1 \vee S^1$ to show that the free group on 3 generators is isomorphic to a subgroup of the free group on 2 generators. Generalize 3 to n.

Use a covering space that is a union of 3 (or n) circles and so has $\pi_1 = F_n$.

Find a simply connected cover of the space θ. This is the free graph of valence 3.

Due 2/13/2014

Think of the real projective plane $\mathbb{R}P^2$ as the space obtained from the unit disk by identifying opposite points on the boundary. 1. Show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. 2. Draw a picture showing a loop representing the nontrivial element $[f] \in \pi_1(\mathbb{R}P^2)$. 3. Draw a series of pictures showing the nullhomotopy from $2[f]$ to the constant path. Using the picture, we see that $\mathbb{R}P^2$ can be formed as a CW complex with one cell in each dimension $0, 1, 2$. If a represents the one-cell, then the attaching map takes the boundary of the 2-cell to a^2. So by our theorems about π_1 of cell complexes, $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/\langle a^2 \rangle \cong \mathbb{Z}_2$.

Let X be the quotient space of the disk D^2 obtained by identifying points on the boundary that are 120 degree apart. Compute $\pi_1(X)$ Just like for $\mathbb{R}P^2$, this $\pi_1 = \mathbb{Z}/3$.

Let X be the complement of n points in $\mathbb{R}^2$. Compute $\pi_1(X)$. Do the same for the complement of n points in $\mathbb{R}^3$. [Hint: use induction on n]. If $n = 1$, we can assume without loss of generality that the point is the origin. Then X deformation retracts to S^1 so $\pi_1(X) = \mathbb{Z}$. Now, suppose we've shown that for $n-1$ points $\pi_1(X) = *_{n-1}\mathbb{Z}$, the free product on $n-1$ copies of $\mathbb{Z}$. Let X be the complement of n points. It's possible to find a plane that separates the points into two groups of size m and m', each less than n. Now write $X = A \cup B$ where A and B are each halfspaces determined by the plane, but thickened up slightly so that $A \cap B \cong \mathbb{R}^{n-1} \times (-\epsilon, \epsilon)$. Then by induction $\pi_1(A) \cong *_m\mathbb{Z}$ and $\pi_1(B) \cong *_{m'}\mathbb{Z}$. $\pi_1(A \cap B) = 0$, so by the van Kampen theorem, $\pi_1(X) \cong (*_m\mathbb{Z}) * (*_{m'}\mathbb{Z}) \cong *_n\mathbb{Z}$.

For $\mathbb{R}^3$, the proof is the same except all groups are trivial.

Hatcher 1.2.7 Start with the cell structure on S^2 with two cells in each dimension. So there are two 0 cells, v_0, v_1. We can assume the two one cells e_1, e_2 are attached so that they both run from v_0 to v_1. Then to attach the two 2-cells, we can attach each one by sending its boundary to the loop $e_1e_2^{-1}$. Now form the quotient X by gluing v_0 to v_1. Then e_1 and e_2 are loops, and the 1-skeleton X^1 of X is the figure eight. So $\pi_1(X^1) = \mathbb{Z} * \mathbb{Z}$. But now using Proposition 1.26, $\pi_1(X) = \pi_1(X^1)/N$, where N is the subgroup generated by the loops along which we glue the 2-cells. But the only such loop is still $e_1e_2^{-1}$. So $\pi_1(X) = \mathbb{Z} * \mathbb{Z}/\langle e_1e_2^{-1} \rangle \cong \mathbb{Z} * \mathbb{Z}/\langle e_1 = e_2 \rangle$. So this group is generated by $\{e_1, e_2\}$ but $e_1 = e_2$. So this is just $\mathbb{Z}$.

Consider the annulus. Identify antipodal (opposite) points on the outer circle with each other (as if you're forming $\mathbb{R}P^2$). Also identify antipodal points on the inner circle with each other. Call the resulting space X. Compute $\pi_1(X)$.

Let the space be X. Suppose the annulus consists of the points in the plane with radius $1 \leq r \leq 2$. Write the annulus as $A \cup B$ where A consists of the points of radius $1 < r \leq 2$ and B consists of the points with radius $1 \leq r < 2$. Then each of A and B deformation retracts to the boundary, which is just a circle (note: if you identify the antipodal points of a circle, you get a circle). So $\pi_1(A) \cong \pi_1(B) \cong \mathbb{Z}$. So $\pi_1(X) \cong \mathbb{Z} *_{\pi_1(A \cap B)} \mathbb{Z}$. Now $A \cap B$ also retracts to a circle, so $\pi_1(A \cap B) \cong \mathbb{Z}$. The inclusions $\pi_1(A \cap B) \to \pi_1(A)$ and $\pi_1(A \cap B) \to \pi_1(B)$ can each be identified as multiplication by 2 $\mathbb{Z} \to \mathbb{Z}$. So $\pi_1(X) = \mathbb{Z} * \mathbb{Z}/\langle x^2 = y^2 \rangle \cong \mathbb{Z} * \mathbb{Z}_2$.

Let X be the union of a sphere S^2 with one of its diameters. Compute $\pi_1(X)$. $\pi_1(X) = \mathbb{Z}$ using the van Kampen theorem. Alternatively, note that this space is homotopy equivalent to the one in Hatcher 1.2.7.

Due 2/6/2014

1. Suppose that X is a *compact* metric space and that $\mathcal{U}$ is an open covering of X. Show that there is a number $\delta > 0$ (called a *Lebesgue number*) such that for every set $Z \subset X$ with diameter $< \delta$, there is an element $U \in \mathcal{U}$ such that $Z \subset U$.

   Suppose there is no such δ. Then for each $n \in \mathbb{N}$, there is a subset $Z_n \subset X$ with diameter $< 1/n$ and such that Z_n is not contained in any single element of $\mathcal{U}$. Let $z_n \in Z_n$. Then z_n has a convergent subsequence, as X is compact. Let z be the limit of the subsequence. But then there is some $U \in \mathcal{U}$ such that $x \in U$, and so there is a ball of some diameter d centered at x and contained in U. But then an easy inequality argument shows that there must be some Z_m contained in U, a contradiction.

2. Use the results of the previous exercise to show that if $f : I \to Y$ is a path in the arbitrary space Y and if $\mathcal{V}$ is an open covering of Y then there is a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ of I such that for each interval $[t_i, t_{i+1}]$, the image $f([t_i, t_{i+1}])$ is contained in some single element $V \in \mathcal{V}$.

   Let δ be the Lebesgue number for the covering $\{f^{-1}(V) : V \in \mathcal{V}\}$ of I. Then $0 \leq \delta/2 \leq\leq \delta \leq 3\delta/2 \leq \cdots \ldots N\delta/2 \leq 1$, for an appropriate N, gives a partition of the desired form.

Hatcher 1.1.13 Let $\phi : A \to X$ be the inclusion. Suppose every path in X with endpoints in A is homotopic to a path in A. Let f be a loop in X based at x_0. By the assumption, f is homotopic to a loop f' in A. But then f' represents an element $[f'] \in \pi_1(A, x_0)$ and also $\phi_*([f']) = [f] \in \pi_1(X, x_0)$ because f is homotopic to f'. So ϕ_* is surjective.

Conversely, let f be a *path* with endpoints in A. Let h_1 be a path from $f(0)$ to x_0 in A, and let h_2 be a path from x_0 to $f(1)$ in A. Since ϕ_* is onto, there is a homotopy from $h_1^{-1} f h_2^{-1}$ to a loop g contained in A, i.e. $h_1^{-1} f h_2^{-1} \sim g$. But then $f \sim h_1 g h_2$ using previous problems.

Hatcher 1.1.16b If there were a retraction $S^1 \times D^2 \to S^1 \times S^1$, then there would be homomorphisms $\pi_1(S^1 \times S^1) \to \pi_1(S^1 \times D^2) \to \pi_1(S^1 \times S^1)$ that compose to the identity. But $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}$, while $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. This leads to a contradiction as there is no surjection $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ (pick your favorite reason from abstract algebra; mine is that the image would have to consist of multiples of some $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ because the image of a cyclic group is cyclic, but then it's not hard to argue that this cannot include all possible (x, y)).

Hatcher 1.1.17 For $n \in \mathbb{Z}$ define $\phi_n : S^1 \vee S^1 \to S^1$ so that ϕ_n is the identity on the first S^1 of $S^1 \vee S^1$ and ϕ_n restricts to the angle map $\theta \to n\theta$ for $\theta \in S^1$. So ϕ_n is the identity on the first circle and wraps the second circle n times around the image circle. ϕ_n cannot be homotopic to ϕ_m for $m \neq n$ because if $[f] \in \pi_1(S^1 \vee S^1)$ is represented by the loop that generates π_1 of the second circle in $S^1 \vee S^1$, then $\phi_{n*}([f])$ and $\phi_{m*}([f])$ represent different elements of $\pi_1(S^1)$ by what we know about the fundamental group of the circle. Thus ϕ_n is not homotopic to ϕ_m using Lemma 1.19 of the book.

Hatcher 0.3a Suppose $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences. Let f' and g' be the homotopy inverses. We claim $f'g'$ is a homotopy inverse to gf. To see this, we recall that, by definition, there are homotopies, say F_t from $f'f$ to id_X, G_t from $g'g$ to id_Y, H_t from ff' to id_Y, and I_t from gg' to id_Z. Then we can obtain a homotopy from $f'g'gf$ to id_X as follows. First do the homotopy $f'G_tf$ from $f'g'gf$ to $f'\mathrm{id}_Y f = f'f$. Then do the homotopy F_t from $f'f$ to id_X. The other homotopy from $gff'g'$ to id_Z is similar.

This shows that the relation of being homotopy equivalent is transitive. But clearly for any space X, X is homotopy equivalent to X by the identity. Symmetry is evident from the definition. So homotopy equivalence is an equivalence relation.

Hatcher 0.3c (You may assume 0.3b) Suppose $f' : X \to Y$ is homotopic via F_t to the homotopy equivalence $f : X \to Y$, with homotopy inverse $g : Y \to X$. Then $f'g$ is homotopic to fg via F_tg, which is homotopic to id_Y and gf' is homotopic via gF_t to gf, which is homotopic to id_X. Thus g is also a homotopy inverse to f'.

Hatcher 0.4 Let F be the weak deformation retraction. Let $f_1(x) = F(x, 1) : X \to A$, and let $g : A \hookrightarrow X$ be the inclusion. Then for $gf_1 : X \to X$, we have $gf_1 = f_1$, which is homotopic to f_0, which is the identity. On the other hand, $f_1g : A \to A$ is the same as the restriction of f_1 to A. But then the restriction of F to $A \times I$ is a homotopy from $f_1|_A$ to $f_0|_A = \mathrm{id}|_A$.

Hatcher 0.6a [Hint to students. Here's a point set topology lemma that might be useful: Suppose $X \times Y$ is a product space, Y is compact, and N is an open set in $X \times Y$. Suppose also that $x_0 \times Y \subset N$ for some $x_0 \in X$. Then there is a neighborhood W of x_0 in X such that $W \times Y \subset N$. This is a very important lemma, sometimes called the "tube lemma".]

First we can perform a deformation retraction $f_t(x, y) = (x, ty)$. At time $t = 1$ this is the identity and at time $t = 0$ every point is on the horizontal segment. Now perform a deformation retract from the horizontal segment to your favorite point x_0. Explicitly, this can be done as $g_t(x) = tx_0 + (1 - t)x$. The desired deformation retraction comes by first doing f_t and then g_t.

It is impossible to have a deformation retract to any other point $(x_0, y_0) \in X$ for $y_0 > 0$ because this would contradict the continuity of the retraction. Suppose it were possible. Then the homotopy $F : X \times I \to X$ giving the deformation retract would have to be continuous. Pick a ball neighborhood $U = B_\epsilon((x_0, y_0))$ of (x_0, y_0) where the radius ϵ satisfies $0 < \epsilon < y_0$. Now $F^{-1}(U)$ would have to be open in $X \times I$ and contain $[X \times 1] \cup [(x_0, y_0) \times I]$. In particular, it follows from the tube lemma that $F^{-1}(U)$ would contain some neighborhood $B_\delta \times I$ around $(x_0, y_0) \times I$ in $X \times I$. But then every point in B_δ would have to stay within ϵ of (x_0, y_0) through the homotopy, which is impossible.

Give a presentation in terms of generators and relations of $\mathbb{Z}_4$ involving one generator; involving two generators; involving three generators. There are many answers to this. Here are some: $\langle x \mid x^4 = e \rangle$, $\langle x, y \mid x^4 = 1, y = x^2 \rangle$, $\langle x, y, z \mid x^4 = 1, y = x, z = x$.

Recall the fundamental theorem of finitely generated abelian groups. If G is a finitely generated abelian group, explain how you would write down a presentation for it. Every finitely generated abelian group has the form $\mathbb{Z}^r \times \mathbb{Z}_{p_1} \times \cdots \mathbb{Z}_{p_n}\}$, where $\mathbb{Z}^r$ is the product of r copies of z. So a presentation would be $\langle x_1, \ldots, x_r, y_1, \ldots, y_n \mid y_i^{p_i} = e, x_i x_j = x_j x_i, y_a y_b = y_b y_a, x_k y_\ell = y_\ell x_k \rangle$.

Consider the group G with presentation $\langle x, y \mid x^4 = e, y^2 = e, (xy)^2 = e \rangle$. Show that G is a finite group and determine how many elements it has. Hint: start by rewriting the relation $(xy)^2 = e$ in a more useful form. Extra credit: can you identify G as a familiar group? Using $(xy)^2 = xyxy = e$, we see $xy = y^{-1}x^{-1} = yx^{-1}$ (since $y^2 = e$ implies $y = y^{-1}$). So given any word, we can use this relation to pull any ys from right to left across powers of x (for negative powers, note that $x^{-1} = x^3$, so we can always write xs in terms of positive powers). Thus every word can be rewritten as $y^n x^m$, and clearly $n = 0, 1$ and $m = 0, 1, 2, 3$, so there are at most 8 elements of the group. Furthermore, this group is isomorphic to the dihedral D_4; it's not hard to find an explicit isomorphism.

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Hatcher 1.1.2. Suppose h and g are homtopic paths from x_0 to x_1 and that $[f] \in \pi_1(X, x_0)$. Then hfh^{-1} is homotopic to gfg^{-1} (see the paragraph in which Hatcher defined the product of paths), so $\beta_h([f]) = \beta_g([f])$.

Hatcher 1.1.3. First suppose β_h depends only on the endpoints of h and not the actual path. Let $x_0 \in X$ be a basepoint. Let $[h] \in \pi_1(X, x_0)$. Because h has the same endpoints as the constant path, $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the same as $\beta_c : \pi_1(X, x_0) \to \pi_1(X, x_0)$, where c is the constant path. But it is clear that $\beta_c = \mathrm{id}$. Therefore, using the definition of β_h, $[g] = \beta_c([g]) = \beta_h([g]) = [hgh^{-1}] = [h][g][h]^{-1}$. But then $[g][h] = [h][g]$. Since $[h]$ and $[g]$ were arbitrary, $\pi_1(X, x_0)$ is abelian.

Conversely, assume $\pi_1(X, x_0)$ is abelian for any x_0. Let h and g be two paths from x_0 to x_1, and let $[f] \in \pi_1(X, x_1)$. We must show that $\beta_h([f]) = \beta_g([f])$, i.e. that $[hfh^{-1}] = [gfg^{-1}]$, which is the same as showing that $hfh^{-1}gf^{-1}g^{-1}$ is homotopic to the constant map to x_0. But now notice that $h^{-1}g$ and f are both loops based at x_1, so $[h^{-1}g][f] = [f][h^{-1}g]$ as elements of $\pi_1(X, x_1)$. But now using this homotopy, we see that $hfh^{-1}gf^{-1}g^{-1}$ is homotopic to $hh^{-1}gff^{-1}g^{-1}$, which is homotopic to the constant map.

Hatcher 1.1.6. Since I can't draw pictures here, I'll just outline the ideas. Let $[g]_f$ denote the equivalence class of the loop $g : S^1 \to X$ in $[S^1, X]$ (the g stands for "free" - these are sometimes called free homotopy classes). This is an equivalence relation just like homotopy of paths is (the proof is essentially identical).

One of the keys to this problem is to notice the following: suppose $f : S^1 \to X$ can be written as the composition of paths $g_1 g_2 \dots g_k$, where the g_i are paths but not necessarily loops. Then $[g_1 g_2 \dots g_k]_f = [g_2 \dots g_k g_1]_f$; in other words, cyclically permuting the order of the paths gives the same equivalence class. This can be seen by just rotating the parametrization, i.e. if the parametrization of g_2 starts at angle ψ from s_0, then we have a homotopy $f_t(\theta) = f(\theta + \psi t)$ from $g_1 g_2 \dots g_k$ to $g_2 \dots g_k g_1$.

Now, let $f \in [S^1, X]$, and let $x_1 = f(s_0)$, where s_0 is a basepoint of S^1. Let h be any path from x_0 to x_1 (this exists by the assumption of path-connectedness). We can consider the map f as representing an element of $\pi_1(X, x_1)$ and form $\beta_h([f])$. This gives a loop, hfh^{-1} based at x_0. By our observation above, this is homotopic to $fh^{-1}h$, and this is homotopic to f by "reeling" in $h^{-1}h$ to x_1 analogously to the proof that $h^{-1}h$ is homotopic to the constant path. This shows that Φ is surjective.

Now suppose $[f], [g] \in \pi_1(X, x_0)$ are conjugate. Then by definition, $[f] = [h][g][h]^{-1}$ for some $[h] \in \pi_1(X, x_0)$. So $[f] = [hgh^{-1}]$. But then $[f]_f = \Phi([f]) = \Phi([hgh^{-1}]) = [hgh^{-1}]_f = [gh^{-1}h]_f = [g]_f$.

Finally, suppose $\Phi([f]) = \Phi([g])$, so there is a free homotopy $F : S^1 \times I \to X$ from f to g. Let h be the path followed by s_0 during this homotopy. Then f is homotopic, preserving basepoints, to hgh^{-1}. (This would be easier in pictures (come by my office if you want to see it), but the homotopy can be described by the composition $I \times I \xrightarrow[H]{S}{}^1 \times I \xrightarrow[F]{X}$, where H takes $I \times 0$ to $S^1 \times 0$ by the obvious quotient and $0 \times I$ and $1 \times I$ to $s_0 \times 0$ and $I \times 1$ up $s_0 \times I$, around $S^1 \times 1$ and then down $s_0 \times I$).

Hatcher 1.1.10 Suppose $[f] \in \pi_1(X, x_0)$ and $[g] \in \pi_1(Y, y_0)$. We will show that as a loop in $X \times Y$, fg is homotopic to the loop $s \to (f(s), g(s))$. The same argument in reverse shows that that gf is also homotopic to $s \to (f(s), g(s))$, so $fg \sim gf$. For the homotopy, we simply note that fg consists of the path $s \to (f(2s), c_{y_0}) \in X \times Y$ for $s \in [0, 1/2]$, where c_{y_0} is the constant path at $y_0 \in Y$, and similarly, $(fg)(s) = (c_{x_0}, g(2s-1))$ for $s \in [1/2, 1]$. In other words, if f' is the path that does $f(2s)$ for $s \in [0, 1/2]$ and is constant after that and g' is the path that does $g(2s-1)$ for $s \in [1/2, 1]$ and is constant before that, then $fg = (f', g')$. But clearly $f' \sim f$ and $g' \sim g$, so $(f', g') \sim (f, g)$, using the isomorphism $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Topological group problem. i. The operation is well-defined and I'll let you show it's associative (should be easy). The identity is the constant path c_e since $(f \otimes c_e)(s) = f(s) * e = f(s)$. If f is a loop, then its inverse in this group is the loop $s \to (f(s))^{-1}$.

ii. Suppose $F(s,t)$ is the homotopy from f to f' and $G(s,t)$ is the homotopy from g to g'. Let $H(s,t) = F(s,t) * G(s,t)$. Then $H(s,0) = F(s,0) * G(s,0) = f(s) * g(s) = (f \otimes g)(s)$ and $H(s,1) = F(s,1) * G(s,1) = f'(s)g'(s) = (f' \otimes g')(s)$. So H is a homotopy from $f \otimes g$ to $f' \otimes g'$.

iii. On the one hand $fc_e \sim f$ and $c_e g \sim g$. So by part ii, $(fc_e) \otimes (c_e g) \sim f \otimes g$. On the other hand, $((fc_e) \otimes (c_e g))(s) = (fc_e)(s) * (c_e g)(s)$. Notice that if $s \leq 1/2$, then $c_e g(s)$ is just e, and if $s \geq 1/2$, $fc_e(s) = e$. So $(fc_e)(s) * (c_e g)(s) = fg$. So $f \otimes g \sim fg$. So $[f \otimes g] = [fg] \in \pi_1(G, e)$.

iv. In the last problem we saw that $(fc_e) \otimes (c_e g) \sim f \otimes g$, but $fc_e \sim f \sim c_e f$ and $c_e g \sim g \sim gc_e$. So $(fc_e) \otimes (c_e g) \sim (c_e f) \otimes (gc_e) \sim g \otimes f$ (the last relation for essentially the same reason as in part iii). So $[f] \otimes [g] = [g] \otimes [f] \in \pi_1(G, e)$, and since $[f] \otimes [g] = [f][g]$ by part iii, also $[f][g] = [g][f]$.

Due 1/23/2014

Hatcher 1.1.1. Since $g_0 \sim g_1$, it follows that $\bar{g}_0 \sim \bar{g}_1$: Recall that $\bar{g}_0(s) = g_0(1-s)$ and $\bar{g}_1(s) = g_1(1-s)$. So if $G(s,t)$ is the homotopy from g_0 to g_1, then $G(1-s,t)$ is the homotopy from $\bar{g}_0$ to $\bar{g}_1$. Now, as $f_0 g_0 \sim f_1 g_1$ and $\bar{g}_0 \sim \bar{g}_1$, we have $f_0 g_0 \bar{g}_0 \sim f_1 g_1 \bar{g}_1$. But $g_0 \bar{g}_0$ is homotopic to the constant path at $f_0(1) = f_1(1)$, so $f_0 g_0 \bar{g}_0 \sim f_0$. Similarly $f_1 g_1 \bar{g}_1 \sim f_1$.

Hatcher 1.1.4. UPDATED: Lemma: if Z is a star-shaped subset of $\mathbb{R}^n$, then Z is simply connected. Proof: Let $z \in Z$ be the "center" and consider $\pi_1(Z, z)$. Let f be any loop based at z. Let $F(t,s) = tz + (1-t)f(s)$. This is a continuous homotopy from f to the constant path at z. Since every point in the homotopy lies along the straight path from $f(s) \in Z$ to z, F is contained in Z. Therefore, Z is simply connected.

Now, let $f : I \to X$ be a path. By compactness, we can divide I into subintervals $0 = s_0 \leq s_1 \cdots \leq s_n = 1$ such that each $f([s_i, s_{i+1}])$ is contained in a single star-shaped neighborhood, say S_i. Let z_i be the "center" of the star S_i. As S_i is simply connected, any two paths with the same endpoints are homotopic (we proved this in class). So $f|_{[s_i, s_{i+1}]}$ is homotopic to the path that goes straight from $f(s_i)$ to z_i and then straight from z_i to $f(s_{i+1})$.

More general than homotopies of paths are homotopies of maps. Two maps $f, g : X \to Y$ are called *homotopic* (written $f \sim g$) if there exists a map F :

$X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Show that, for any fixed X and Y, homotopy is an equivalence relation on the set of maps from X to Y.

Same proof as for path homotopies! (see class notes)

Suppose $f, f' : X \to Y$ with $f \sim f'$ and $g, g' : Y \to Z$ with $g \sim g'$. Show that $g \circ f \sim g' \circ f'$, where $\circ$ denotes composition of functions.

Let $F : X \times I \to Y$ be the homotopy from f to f' and $G : Y \times I \to Z$ be the homotopy from g to g'. Now use $G(F(x,t),t)$.

Suppose $f, f' : X \to Y$ are homotopic and $g, g' : Z \to W$ are homotopic. Show that $f \times g, f' \times g' : X \times Z \to Y \times W$ are homotopic.

Let $F : X \times I \to Y$ be the homotopy from f to f' and $G : Z \times I \to W$ be the homotopy from g to g'. Now use $(F(x,t), G(z,t))$.