Twisted Geometric Cycles

This talk is on some results of my collaborator Bai-Ling Wang in


It had its origins in:


and

Overview

There are several ingredients that need explaining from the topological and geometric side (the analytic side is essentially known).

(i) Twisted K theory.

(ii) Twisted Poincaré duality between twisted K-cohomology and twisted K-homology.

(iii) Generalising Baum-Douglas K-homology to the twisted situation.
0. Motivation

Understand Witten's ideas on D-brane charges as taking values in twisted K-theory and to get a twisted version of some of what is in his original article

hep-th/9810188 ‘D-branes and K-theory’


The upshot of Wang’s approach is that there is a way to think geometrically of D-branes, at least insofar as they relate to topological twisted K-homology, as twisted versions of the Baum-Douglas geometric cycles.

Finally I mention some additional topics to do with D-branes.
1. Twisted $K$-theory: topological and analytic definitions

Let $X$ be a paracompact Hausdorff topological space, and $\mathcal{H}$ be an infinite dimensional, complex and separable Hilbert space.

$PU(\mathcal{H})$ is the projective unitary group with norm topology. $PU(\mathcal{H})$ can be identified with an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. So the classifying space $BPU(\mathcal{H})$ is a $K(\mathbb{Z}, 3)$.

A twisting is a continuous map $\alpha : X \to K(\mathbb{Z}, 3)$. The associated $PU(\mathcal{H})$ bundle $\mathcal{P}_\alpha$ is given by pulling back the universal $PU(\mathcal{H})$-bundle over $K(\mathbb{Z}, 3)$.

The set of isomorphism classes of principal $PU(\mathcal{H})$-bundles over $X$ is the homotopy classes of maps

$$[X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).$$
Let \( \text{Fred} \) be the space of Fredholm operators with norm topology.

The ‘conjugation’ action \( PU(\mathcal{H}) \times \text{Fred} \to \text{Fred} \) defines an associated bundle with fiber the Fredholm operators

\[
\mathcal{P}_\alpha(\text{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \text{Fred}
\]

Let \( \Omega^n_X \mathcal{P}_\alpha(\text{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \Omega^n \text{Fred} \) be the fiber-wise \( n \)-iterated loop spaces.

The (topological) twisted \( K \)-groups of \( (X, \alpha) \) are defined to be

\[
K^{-n}(X, \alpha) := \pi_0 \left( C_c(X, \Omega^n_X \mathcal{P}_\alpha(\text{Fred})) \right),
\]

the set of homotopy classes of compactly supported sections. Due to Bott periodicity, we only have two different twisted \( K \)-groups, denoted by \( K^0(X, \alpha) \) and \( K^1(X, \alpha) \).

Associated with the \( PU(\mathcal{H}) \) bundle \( \mathcal{P}_\alpha \) is a continuous trace \( C^* \)-algebra and one may define the analytic twisted \( K \)-theory of \( (X, \alpha) \) as the \( K \)-theory (via Kasparov) of this algebra.
2. Twisted K-homology: Analytic and topological definitions

The analytic twisted K-homology of \((X, \alpha)\), denoted by \(K^{\text{an}}_{ev/odd}(X, \alpha)\), is defined as the K-homology (via Kasparov) of the continuous trace C*-algebra associated to \(\mathcal{P}_\alpha\).

Introduce the space \(\mathcal{P}_\alpha(\text{Fred})/X\) obtained by identifying the base points (the identity operator) in the fibers. Then the topological twisted K-homology \(K^{\text{top}}_{ev/odd}(X, \alpha)\) is defined to be

\[
K^{\text{top}}_{ev}(X, \alpha) = \lim_{k \to \infty} \pi_{2k}(\mathcal{P}_\alpha(\text{Fred})/X)
\]

and

\[
K^{\text{top}}_{odd}(X, \alpha) = \lim_{k \to \infty} \pi_{2k+1}(\mathcal{P}_\alpha(\text{Fred})/X).
\]

The proof that the topological and analytic objects are isomorphic uses twisted Poincare dualities in the topological and analytic settings and the equivalence between topological and analytic twisted K-theory.
3. The Twisted Poincaré duality

The twisted version introduces a shift in the twist
\[ \alpha \mapsto \alpha + (W_3 \circ \tau) \]
where \( \tau : X \to BSO \) is the classifying map of the stable tangent bundle and \( W_3 \) is the classifying map for the bundle \( B\text{Spin}^c \to BSO \), and \( \alpha + (W_3 \circ \tau) \) denotes the map \( X \to K(\mathbb{Z}, 3) \), representing the class \( [\alpha] + W_3(X) \) in \( H^3(X, \mathbb{Z}) \). (There is a tricky point in this definition where we proceed by fixing an isomorphism \( \mathcal{H} \otimes \mathcal{H} \cong \mathcal{H} \).)

**Theorem** Let \( X \) be a smooth manifold with a twisting \( \alpha : X \to K(\mathbb{Z}, 3) \).

(i) (Wang) There exists an isomorphism
\[ K^\text{top}_{ev/odd}(X, \alpha) \cong K^\text{ev/odd}_{top}(X, \alpha + (W_3 \circ \tau)) \]
with the degree shifted by \( \text{dim}X \mod 2 \).

(ii) (Tu, Echterhoff-Emerson-Kim) There exists an isomorphism
\[ K^\text{an}_{ev/odd}(X, \alpha) \cong K^\text{ev/odd}_{an}(X, \alpha + (W_3 \circ \tau)) \]
with the degree shifted by \( \text{dim}X \mod 2 \).
4. Twisted geometric cycles

Let \((X, \alpha)\) be a paracompact Hausdorff space with a twisting \(\alpha\).

A geometric cycle for \((X, \alpha)\) is a quintuple

\[(M, \iota, \nu, \eta, [E])\]

where \([E]\) is a \(K\)-class in \(K^0(M)\), \(M\) an oriented smooth closed manifold with a classifying map \(\nu\) of its stable normal bundle, \(\iota : M \to X\) is a continuous map such that there exists a homotopy commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BSO} \\
\iota \downarrow & & \downarrow \eta \\
X & \xleftarrow{\alpha} & K(\mathbb{Z}, 3),
\end{array}
\]

with a homotopy \(\eta\) between \(W_3 \circ \nu\) and \(\alpha \circ \iota\). We refer to this diagram of maps as an ‘\(\alpha\)-twisted Spin\(^c\) structure’.
Remarks.

1. $M$ admits an $\alpha$-twisted $Spin^c$ structure if and only if

$$\iota^*([\alpha]) + W_3(M) = 0.$$ 

If $\iota$ is an embedding, this is the anomaly cancellation condition introduced by Freed and Witten.

2. If the twists are all trivial this reduces to the Baum-Douglas definition and $\eta$ corresponds to a choice of $Spin^c$ structure.

Two geometric cycles $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$ and $(M_2, \iota_2, \nu_2, \eta_2, [E_2])$ are isomorphic if there is an isomorphism $f : (M_1, \iota_1, \nu_1, \eta_1) \to (M_2, \iota_2, \nu_2, \eta_2)$, as $\alpha$-twisted $Spin^c$ manifolds over $X$, such that $f_!([E_1]) = [E_2]$.

Let $\Gamma(X, \alpha)$ be the collection of all geometric cycles for $(X, \alpha)$. We now impose an equivalence relation $\sim$ on $\Gamma(X, \alpha)$, generated by the following three relations:
Direct sum - disjoint union

If \((M, \iota, \nu, \eta, [E_1])\) and \((M, \iota, \nu, \eta, [E_2])\) are two geometric cycles with the same \(\alpha\)-twisted \(Spin^c\) structure, then

\[
(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2]) \sim (M, \iota, \nu, \eta, [E_1] + [E_2]).
\]

Bordism

If there exists an \(\alpha\)-twisted \(Spin^c\) manifold \((W, \iota, \nu, \eta)\) and \([E] \in K^0(W)\) such that

\[
\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2)
\]

and \(\partial([E]) = [E_1] \cup [E_2]\). Here \(-(M_1, \iota_1, \nu_1, \eta_1)\) denotes the manifold \(M_1\) with the opposite \(\alpha\)-twisted \(Spin^c\) structure, then

\[
(M_1, \iota_1, \nu_1, \eta_1, [E_1]) \sim (M_2, \iota_2, \nu_2, \eta_2, [E_2]).
\]
Spin<sup>c</sup> vector bundle modification

Take a geometric cycle \((M, \iota, \nu, \eta, [E])\) and a Spin<sup>c</sup> vector bundle \(V\) over \(M\) with even dimensional fibers. Denote by \(\mathbb{R}\) the trivial rank one real vector bundle. Choose a Riemannian metric on \(V \oplus \mathbb{R}\), let \(\hat{M} = S(V \oplus \mathbb{R})\) be the sphere bundle of \(V \oplus \mathbb{R}\).

Denote by \(\rho : \hat{M} \to M\) the projection which is K-oriented. The vertical tangent bundle \(T^v(\hat{M})\) of \(\hat{M}\) admits a natural Spin<sup>c</sup> structure with an associated \(\mathbb{Z}_2\)-graded spinor bundle \(S^+_V \oplus S^-_V\). Then

\[
(M, \iota, \nu, \eta, [E]) \sim (\hat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^*E \otimes S^+_V]).
\]

**Definition.** The geometric twisted K-homology \(K_{ev/odd}^{geo}(X, \alpha)\) is defined to be \(\Gamma(X, \alpha)/\sim\) with the grading given by even or odd dimension of \(\alpha\)-twisted Spin<sup>c</sup> manifolds. Addition is given by the disjoint union - direct sum relation.
4. Twisted assembly map

There exists a natural homomorphism

$$\mu : K_{ev/odd}^{\text{geo}}(X, \alpha) \to K_{ev/odd}^{\text{an}}(X, \alpha)$$

where \(\mu(M, \iota, \nu, \eta, [E])\) is defined by composition of a sequence of maps:

$$[E] \in K^0(M) \xrightarrow{PD} K_{ev/odd}^{\text{an}}(M, W_3 \circ \tau)$$

$$\xrightarrow{I_*} K_{ev/odd}^{\text{an}}(M, \alpha \circ \iota) \xrightarrow{\eta_*} K_{ev/odd}^{\text{an}}(M, W_3 \circ \nu)$$

$$\xrightarrow{\cong} \iota_* \xrightarrow{\eta_*} K_{ev/odd}^{\text{an}}(X, \alpha).$$

Here \(PD : K^0(M) \cong K_{ev/odd}^{\text{an}}(M, W_3 \circ \tau)\) is the Kasparov’s Poincaré duality with the degree shift by \(\dim M (mod 2)\), \(\iota_*\) is the natural push-forward map in twisted K-homology, \(\eta_*\) is the isomorphism induced by the homotopy \(\eta\), and \(I_*\) is the isomorphism induced by the trivial \(Spin^c\) structure on the trivial bundle \(\tau \oplus \nu\).
**Theorem** (Wang) The twisted assembly map

\[ \mu : K_{geo}^{ev/odd}(X, \alpha) \rightarrow K_{an}^{ev/odd}(X, \alpha) \]

is an isomorphism for any **smooth** closed manifold \( X \) with a twisting \( \alpha : X \rightarrow K(\mathbb{Z}, 3) \).

The proof of this theorem is via establishing that there is a map \( \Psi : K_{ev}^{top}(X, \alpha) \rightarrow K_{0}^{geo}(X, \alpha) \) such that the following diagram

\[
\begin{array}{ccc}
K_{ev/odd}^{top}(X, \alpha) & \xrightarrow{\Psi} & K_{ev/odd}^{geo}(X, \alpha) \\
& \searrow \mu \swarrow \Phi & \\
K_{ev/odd}^{geo}(X, \alpha) & \xrightarrow{=} & K_{ev/odd}^{an}(X, \alpha)
\end{array}
\]

commutes and \( \Psi \) is surjective.
5. The twisted index theorem

One of the applications of geometric cycles is to express an index pairing between twisted K-theory and twisted K-homology in terms of an index pairing on geometric cycles.

**Theorem** Let $X$ be a smooth manifold with a twisting $\alpha: X \to K(\mathbb{Z}, 3)$. The index pairing

$$K_0(X, \alpha) \times K^0(X, \alpha) \to \mathbb{Z}$$

is given by

$$< (M, \iota, \nu, \eta, [E]), \xi > = \int_M ch_{w_2}(M) (\eta_*(\iota^* \xi \otimes E)) \hat{A}(M)$$

where $\xi \in K^0(X, \alpha)$, and the geometric cycle

$$(M, \iota, \nu, \eta, [E])$$

defines a twisted K-homology class on $(X, \alpha)$. Here

$$\eta_* : K^*(M, \iota^* \alpha) \cong K^*(M, W_3(M))$$

is an isomorphism, and $ch_{w_2}(M)$ is the Chern character on $K^0(M, W_3(M))$ which we now explain.
6. Twisted Chern character

Under the identification between $K^0(M, W_3(M))$ and the K-theory of Clifford modules over $M$,

$$ch_{w_2}(M) : K^0(M, W_3(M)) \rightarrow H^{ev}(M, \mathbb{R})$$

is given by the relative Chern character on Clifford modules as described for example in Berline-Getzler-Vergne.

The general twisted Chern character on $K^0(X, \alpha)$ requires a choice of gerbe connection and curving. A geometric definition was given in *Differential Twisted K-theory and its Applications*, C-Mickelsson-Wang. An analytical definition using the Chern-Connes character in noncommutative geometry was given Mathai-Stevenson. A topological definition was given by Atiyah-Segal.
7. Twisted Riemann-Roch

There is a Riemann-Roch theorem in C-Mickelsson-Wang *op cit*, which implies that the above index formula can be written as

\[ < (M, \iota, \nu, \eta, [E]), \xi > = \int_{M} ch_{w2}(M) (\eta^*(\iota^*\xi \otimes E)) \tilde{A}(M) \]

\[ = \int_{X} ch_{w2}(X) (\iota_!(E) \otimes \xi) \tilde{A}(X) \]

where \( \iota_! \) is the push-forward map on twisted K-theory defined by

\[
\begin{align*}
K^0(M) & \cong K_0(M, W_3(M)) \\
& \cong K_0(M, -\iota^*\alpha) \\
& \cong K_0(X, -\alpha) \\
& \cong K^0(X, -\alpha + W_3(X))
\end{align*}
\]

and \( ch_{w2}(X) \) is the canonical twisted Chern character on

\[
K^0(X, -\alpha + W_3(X)) \otimes K^0(X, \alpha) \rightarrow K^0(X, W_3(X)).
\]
8. D-branes

**Theorem.** (Wang) Given a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ on a smooth manifold $X$, every twisted $K$-class in $K^{ev/odd}(X, \alpha)$ is represented by a geometric cycle supported on an $(\alpha + (W_3 \circ \tau))$-twisted closed $Spin^c$-manifold $M$ and an ordinary $K$-class $[E] \in K^0(M)$.

Thus there are three definitions of twisted $K$-theory $K^*(X, \alpha)$ for a smooth manifold $X$:

1. A topological definition in terms of homotopy equivalence classes of sections of a bundle of $K$-theory spectra associated to $(X, \alpha)$.

2. An analytical definition in terms of the continuous trace $C^*$-algebra associated to $(X, \alpha)$.

3. A geometric definition in terms of a geometric cycle $(M, \iota, \nu, \eta, E)$ with $\nu$ the classifying map for the map $\iota : M \to X$. 
We propose that this geometric cycle is the so-called Type II D-brane for a class in $K^*(X, \alpha)$. The equivalence of these three definitions gives a candidate for the D-brane charge map on the category of D-branes:

$$\{\text{D-branes over } (X, \alpha)\} \longrightarrow K^*(X, \alpha).$$

There is a version of Type I D-branes using twisted $Spin$-manifolds over $(X, \alpha)$ with $\alpha : X \to K(\mathbb{Z}_2, 2)$.

**Remark on $T$-duality**

Given a principal $T^n$-bundle $p : Y \to X$ with a twisting $\alpha$ on $Y$ satisfying $p^! \alpha = 0 \in H^1(X, \mathbb{Z}_{2^{n(n-1)/2}})$, there is a classical $T$-dual $(Y^#, \alpha^#)$ such that

$$K^*(Y, \delta) \cong K^{*+n}(Y^#, \delta^#).$$

The dependence of twisted Chern character

$$ch_{\tilde{\alpha}} : K^*(Y, \alpha) \longrightarrow H^*(Y, \text{curv}(\tilde{\alpha}))$$

on $\tilde{\alpha}$ (a gerbe connection and curving) makes the geometric formulation of classical $T$-duality, in terms of geometric cycles with connection

$$(M, \iota, \nu, \eta, E, \nabla_E),$$

more subtle. More work is needed in this direction.
9. Remark on String structures

One may think of the obstruction to the existence of a string structure on the loop space $LM$ as an analogue of the class $W_3(M)$ except that the string class lies in $H^4(M, \mathbb{Z})$.

In Wang’s paper he draws on this analogy with the view to making a connection with elliptic cohomology. This leads to some interesting conjectures which are under investigation.