The Geometry behind Nongeometric Fluxes

Peter Bouwknegt $^{(1,2)}$

$^{(1)}$ Department of Theoretical Physics
Research School of Physics and Engineering

$^{(2)}$ Department of Mathematics
Mathematical Sciences Institute

The Australian National University
Canberra, AUSTRALIA

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By considering T-duality for strings moving in a geometric background, i.e. in the presence of curvature and H-fluxes, one arises at situations in which the string is coupled to, what is known in the literature as, non-geometric fluxes. In this talk we will consider T-duality in the context of generalized geometry and unravel the geometry behind these non-geometric fluxes.
Topological T-duality, as developed in Rosenberg’s lectures, is a mere shadow of the equivalence of certain string theories under T-duality. The full picture involves geometry.

What is the ‘geometry’ behind the ‘missing T-duals’?

As we have seen, T-duality exchanges momentum (related to $T E$), with winding (related to $T^* E$).

A natural geometric framework for T-duality is therefore a framework which treats $T E$ and $T^* E$ on equal footing.

GENERALIZED GEOMETRY
Generalized geometry

Replace structures on $TE$ by structures on $TE \oplus T^*E$

- Bilinear form on sections $(X, \Xi) \in \Gamma(TE \oplus T^*E)$

\[
\langle (X_1, \Xi_1), (X_2, \Xi_2) \rangle = \frac{1}{2} (\iota_{X_1} \Xi_2 + \iota_{X_2} \Xi_1)
\]

- (twisted) Courant bracket

\[
\begin{aligned}
\llbracket (X_1, \Xi_1), (X_2, \Xi_2) \rrbracket_H &= \\
&= ([X_1, X_2], \mathcal{L}_{X_1} \Xi_2 - \mathcal{L}_{X_2} \Xi_1 - \frac{1}{2} d (\iota_{X_1} \Xi_2 - \iota_{X_2} \Xi_1) + \iota_{X_1} \iota_{X_2} H)
\end{aligned}
\]

where $H \in \Omega^3_{\text{cl}}(E)$
Clifford algebra

\[ \{ \gamma(x_1, \Xi_1), \gamma(x_2, \Xi_2) \} = 2 \langle (X_1, \Xi_1), (X_2, \Xi_2) \rangle \]

Clifford module \( \Omega^\bullet(E) \)

\[ \gamma(x, \Xi) \cdot \Omega = \iota_x \Omega + \Xi \wedge \Omega \]

(twisted) Differential on \( \Omega^\bullet(E) \)

\[ d_H \Omega = d \Omega + H \wedge \Omega \]
Properties of the Courant bracket

For $A, B, C \in \Gamma(TE \oplus T^*E), f \in C^\infty(E)$,

(a) \[ [A, B] = -[B, A] \]

(b) \[ \text{Jac}(A, B, C) = [[[A, B], C] + \text{cycl} = dNij(A, B, C) \]

with \[ Nij(A, B, C) = \frac{1}{3} (\langle[A, B], C\rangle + \text{cycl}) \]

(c) \[ [A, fB] = f[A, B] + (\rho(A)f)B - \langle A, B \rangle df \]

where $\rho : TE \oplus T^*E \to TE$ is the projection.

[Note that isotropic, involutive subbundles $A \subset TE \oplus T^*E$ (Dirac structures) give rise to Lie algebroids.]
(d) Symmetries of $\langle \cdot , \cdot \rangle$ are given by orthogonal group $O(TM \oplus T^*M) \cong O(d, d)$.

A particular kind of orthogonal transformation is the so-called B-field transform. For $b \in \Omega^2(E)$

$$e^b(X, \Xi) = (X, \Xi + \iota_X b)$$

We have

$$e^b[A, B]_H = [e^b A, e^b B]_{H + db}$$
Courant bracket as a derived bracket

We have the following ‘Cartan formulas’

\[ \{ \gamma(x_1, \Xi_1), \gamma(x_2, \Xi_2) \} = 2 \langle (X_1, \Xi_1), (X_2, \Xi_2) \rangle \]

\[ \{ d_H, \gamma(x, \Xi) \} = \mathcal{L}(x, \Xi) \]

\[ [\mathcal{L}(x_1, \Xi_1), \gamma(x_2, \Xi_2)] = \gamma(x_1, \Xi_1) \circ (x_2, \Xi_2) \]

\[ [\mathcal{L}(x_1, \Xi_1), \mathcal{L}(x_2, \Xi_2)] = \mathcal{L}(x_1, \Xi_1) \circ (x_2, \Xi_2) = \mathcal{L}\[ (x_1, \Xi_1), (x_2, \Xi_2) \] \]

where

\[ \mathcal{L}(x, \Xi) \cdot \Omega = \mathcal{L}_x \Omega + (d\Xi + \iota_x H) \wedge \Omega \]

and the Dorfmann bracket is defined by

\[ (X_1, \Xi_1) \circ (X_2, \Xi_2) = ([X_1, X_2], \mathcal{L}_x \Xi_2 - \iota_{x_2} d\Xi_1 + \iota_{x_1} \iota_{x_2} H) \]
T-duality for principal circle bundles

Given a principal circle bundle $E$ with H-flux $H$

$$
\begin{array}{c}
S^1 \longrightarrow E \\
\pi \downarrow \\
M
\end{array}
$$

$$
H = H_3 + A \wedge H_2, \quad F = dA
$$

there exists a T-dual principal circle bundle

$$
\begin{array}{c}
S^1 \longrightarrow \hat{E} \\
\hat{\pi} \downarrow \\
M
\end{array}
$$

$$
\hat{H} = H_3 + \hat{A} \wedge \hat{F}, \quad \hat{F} = H_2 = d\hat{A}
$$
(a) We have an isomorphism of differential complexes 
\[ \tau : (\Omega^\bullet(E)_{S^1}, d_H) \rightarrow (\Omega^\bullet(\hat{E})_{S^1}, d_{\hat{H}}) \]

\[ \tau(\Omega_{(k)} + A \wedge \Omega_{(k-1)}) = -\Omega_{(k-1)} + \hat{A} \wedge \Omega_{(k)} \]

\[ \tau \circ d_H = -d_{\hat{H}} \circ \tau \]

Hence, \( \tau \) induces an isomorphism on twisted cohomology

(b) We can identify \((X, \Xi) \in \Gamma(TE \oplus T^*E)_{S^1}\) with a quadruple \((x, f; \xi, g)\)

\[ X = x + f \partial_A, \quad \Xi = \xi + gA \]

and define a map \( \phi : \Gamma(TE \oplus T^*E)_{S^1} \rightarrow \Gamma(T\hat{E} \oplus T^*\hat{E})_{S^1} \)

\[ \phi(x + f \partial_A + \xi + gA) = x + g\partial_{\hat{A}} + \xi + f\hat{A} \]

The map \( \phi \) is orthogonal wrt pairing on \( TE \oplus T^*E \), hence \( \tau \) induces an isomorphism of Clifford algebras
(c) For \((X, \Xi) \in \Gamma((TE \oplus T^*E)_{S^1})\) we have

\[
\tau(\gamma(X,\Xi) \cdot \Omega) = \gamma\phi(X,\Xi) \cdot \tau(\Omega)
\]

Hence \(\tau\) induces an isomorphism of Clifford modules

(d) For \((X_i, \Xi_i) \in \Gamma((TE \oplus T^*E)_{S^1})\) we have

\[
\phi \left( \llbracket (X_1, \Xi_1), (X_2, \Xi_2) \rrbracket_H \right) = \llbracket \phi(X_1, \Xi_1), \phi(X_2, \Xi_2) \rrbracket_{\hat{H}}
\]

Hence \(\phi\) gives a homomorphism of twisted Courant brackets
(e) Generalized metric on $TE \oplus T^*E$

$$\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

Note that $\mathcal{G}^2 = 1$. We have

$$C_+ = \text{Ker}(\mathcal{G} - 1) = \{(X, (g + b)(X)), \ X \in \Gamma(TE)\} = \text{graph}(g + b : TE \to T^*E),$$

The transformed generalized metric $\hat{\mathcal{G}}$ is given by

$$\hat{C}_+ = \text{graph}(\hat{g} + \hat{b} : T\hat{E} \to T^*\hat{E})$$

where $(\hat{g}, \hat{b})$ are given by the Buscher rules.
We have
\[ d_H = \bar{d} + H_{(3)} + F \partial_A + A \wedge H_{(2)} \]
which proves
\[ \tau \circ d_H = -d_{\hat{H}} \circ \tau \]

The isomorphism of Clifford algebra and modules follows just as easily, and the statement on the Courant bracket follows from the Cartan formulas.
Dimensionally reduced Courant bracket

\[
\begin{align*}
\llbracket (x_1, f_1; \xi_1, g_1), (x_2, f_2; \xi_2, g_2) \rrbracket_{F, H} &= \\
&= \llbracket [x_1, x_2], x_1(f_2) - x_2(f_1) + \iota_{x_1} \iota_{x_2} F; \\
&\quad (\mathcal{L}_{x_1} \xi_2 - \mathcal{L}_{x_2} \xi_1) - \frac{1}{2} d(\iota_{x_1} \xi_2 - \iota_{x_2} \xi_1) + \iota_{x_1} \iota_{x_2} H^{(3)} \\
&\quad + \frac{1}{2} (df_1 g_2 + f_2 dg_1 - f_1 dg_2 - df_2 g_1) \\
&\quad + (g_2 \iota_{x_1} F - g_1 \iota_{x_2} F) + (f_2 \iota_{x_1} H^{(2)} - f_1 \iota_{x_2} H^{(2)}), \\
&\quad x_1(g_2) - x_2(g_1) + \iota_{x_1} \iota_{x_2} H^{(2)} \rrbracket,
\end{align*}
\]
Generalization to principal torus bundles

We have

$$H = H_{(3)} + A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)}$$

such that

$$dH = \tilde{d} + H_{(3)} + F_{(2)i} \partial A_i + \frac{1}{2} F_{(1)ij} \partial A_i \wedge \partial A_j + \frac{1}{6} F_{(0)ijk} \partial A_i \wedge \partial A_j \wedge \partial A_k$$

$$+ A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)}$$

The $F_{(1)ij}$ and $F_{(0)ijk}$ are known as nongeometric fluxes.
Let \( \{e_a\} \) be a basis of \( \Gamma(TE) \), such that \([e_a, e_b] = f_{ab}^c e_c\), and \( \{e^a\} \) be a dual basis of \( \Gamma(T^*E) \), then the Courant bracket can be expressed as

\[
[e_a, e_b] = f_{ab}^c e_c + h_{abc} e^c
\]
\[
[e_a, e^b] = q^{bc} a e_c - f_{ac}^b e^c
\]
\[
[e^a, e^b] = 0 r^{abc} e_c + q^{ab} c e^c
\]

where \( H = \frac{1}{6} h_{abc} e^a \wedge e^b \wedge e^c \).

Together with certain conditions on the structure constants this defines a Courant algebroid.

Theorem [Bouwknegt-Garretson-Kao]: T-duality provides an isomorphism of (certain) Courant algebroids.
THANK YOU FOR LISTENING !!

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