1. Riemannian Manifolds

Our principal objects of study will be Riemannian manifolds, \( M^n \), without boundary. Our eventual objects of interest will be closed Riemannian manifolds (compact, without boundary). \( G \) will usually denote a compact, Lie group. Recall that, by the structure theorem for compact Lie groups, every such \( G \) admits a finite cover, say \( \tilde{G} \), which is isomorphic to a product of simple groups and possibly a torus (a product of circle groups).

1.1. Curvature.

A Riemannian manifold comes equipped with a (Levi-Civita) connection, \( \nabla \), (a way of taking derivatives) which in turn yields a curvature tensor defined on every tangent space: for every \( p \in M \), we have \( R : T_p M \otimes T_p M \otimes T_p M \to T_p M \) defined by \( R(x, y, z) = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \), where \( X, Y, Z \) are vector fields around \( p \) with \( X_p = x, Y_p = y, Z_p = z \). This allows us to define the sectional curvature of any 2-plane \( \sigma \subseteq T_p M \). If \( \sigma = \text{span}(u, v) \), then the sectional curvature of \( \sigma \) is defined as \( \text{sec}(\sigma) := \frac{R(u, v, u, v)}{\text{Area}(u, v)^2} \) (the 4-tensor \( R(u, v, u, v) \) is simply \( \langle R(u, v, u), v \rangle \)). The sectional curvature of \( \sigma \) is defined so that it is independent of choice of basis of \( \sigma \).

We say that a manifold \( M^n \) has non-negative sectional curvature if \( \text{sec}(\sigma) \geq 0 \) for every 2-plane \( \sigma \subseteq TM \). In the talk we are interested in constructing and understanding the structure of non-negatively curved manifolds. A starting point for most constructions is the following fact: if \( G \) is a simple, compact Lie group, then it naturally admits a bi-invariant metric i.e., a metric for which every left translation, \( L_g : G \to G, x \mapsto gx \), and every right right translation, \( R_g : G \to G, x \mapsto xg \) is an isometry for every \( g \in G \). This is given by, for instance, the negative of the Killing form on the Lie algebra \( g \). A useful feature of this metric is that it has non-negative sectional curvature (the zero curvatures come precisely from 2-planes tangent to a maximal torus) and a large group of isometries (\( G \times G \) almost effectively). Since non-negative curvature is preserved under products every compact Lie group admits non-negative curvature with a large group of symmetries.

1.2. Submersions.

A smooth map \( f : M \to N \) between manifolds is called a submersion if for every \( p \in M \) with \( q = f(p) \), the differential \( df_p : T_p M \to T_q N \) is surjective. A submersion between Riemannian manifolds is a Riemannian submersion if the restriction of the differential on the horizontal space is
as isometry i.e., if \( K_p = \ker(df_p) \) and if \( H_p = \ker(df_p) \), then \( f \) is a Riemannian submersion if \( df_p : H_p \to T_qN \) is an isometry for every \( p \in M \).

A useful feature of Riemannian submersions is their curvature increasing (or non-decreasing) property. Namely, the Gray–O’Neill submersion formulas for curvature show that the sectional curvature can only go up as one pushes down in a submersion. More precisely, given a Riemannian submersion \( f : M \to N \) and \( q = f(p) \), suppose \( \sigma \) is a 2-plane in \( T_qN \). Let \( \sigma \) be a horizontal 2-plane in \( H_p \subseteq T_pM \) such that \( df_p(\sigma) = \tilde{\sigma} \). Then \( \sec(\tilde{\sigma}) \geq \sec(\sigma) \).

2. GROUP ACTIONS

Let \( G \) be a group acting on a space \( X \) (where we keep in mind that in our setting \( G \) will be a Lie group and \( X \) a smooth manifold). This means that there is a homomorphism \( \Phi : G \to \text{Aut}(X) \), where \( \text{Aut}(X) \) could be a group of homeomorphisms, diffeomorphisms or some other group of invertible self maps. (For example, when \( X \) is a vector space and \( \text{Aut}(X) \) is a subgroup of invertible linear self maps, then this is a representation of \( G \)). In particular, the identity \( e \in G \) acts as the identity map and multiplication of group elements corresponds to composition of maps.

For \( x \in X \), the set \( G(x) := \{ y = g \cdot x : g \in G \} \) is called the orbit of \( x \in X \). Note that \( G(x) \subseteq X \). The subgroup \( G_x := \{ g \in G : g(x) = x \} \) is called the isotropy group or the stabilizer of \( x \in X \). Note that \( G_x \subseteq G \). The orbit space of an action is the quotient space \( X/G := X/\sim \), where \( x_1 \sim x_2 \) if and only if \( x_2 = g(x_1) \) for some \( g \in G \).

If \( G_x = \{ e \} \) for every \( x \in X \), then the action if said to be free. If \( G_x \subseteq G \) for every \( x \in X \), then the action is said to fixed point free. If \( G \) acts freely on a manifold \( M \), then the orbit space \( M/G \) admits a canonical smooth structure as a manifold.

2.1. Transitive actions and homogeneous space.

We look at an important special case where we consider the action of a subgroup \( H \subseteq G \) on \( G \) given by \( H \times G \to G, h(g) = gh^{-1} \). This is a free action and every orbit looks like a copy of \( H \) inside \( G \) (these orbits are also called cosets). A simple example of this is \( H = S^1 \times \{ e \} \subseteq S^1 \times S^1 \), the 2-torus. The action rotates the circle in the first coordinate and provides a foliation of the 2-torus into a bunch of circles (orbits). The orbit space in this case is a circle (imagine an orthogonal circle meeting every orbit exactly once).

In the setting of \( H \) acting on \( G \), the orbit space or the space of (left) cosets is denoted as \( G/H := \{ gH : g \in G \} \). Note that this orbit space admits an action of \( G \) via \( G \times G/H \to G/H, g(g'H) = gg'H \). This is a transitive action i.e., any two points in \( G/H \) lie in the same orbit (equivalently there is only one orbit): if \( g_1H, g_2H \) are any two points in \( G/H \), then \( g_2g_1^{-1}(g_1H) = g_2H \). Thus, the orbit space is a single point \( \{ * \} \).
When a group $G$ acts transitively on a space $X$, then $X$ is called a **homogeneous** space and is homeomorphic to $G/H$ for some subgroup $H \subseteq G$. For the $G$ action on $G/H$ the isotropy group at any point $gH$ is $gHg^{-1}$ i.e., a subgroup conjugate to $H$.

### 2.2. Some facts about group actions.

(i) If $G$ acts on $X$, then we can stratify $X$ into disjoint $G$-orbits.

(ii) Any orbit $G(x)$ is naturally homeomorphic to the homogeneous space $G/G_x$.

(iii) Two orbits $G(x_1), G(x_2)$ (disjoint or coincident) are said to be of the same type if all isotropy groups of points in either orbit are conjugate in $G$ to a fixed subgroup $H$. An important fact that we will need is that for a compact group action there are only finitely many orbit types.

(iv) For a compact group actions there exists a principal orbit type i.e., a unique orbit type such that the set of points in this orbit type form an open, dense subset of the space $X$ (and project to an open dense subset of the orbit space). The other orbits are usually called **singular orbits**.

### 2.3. An Example.

Let $G = \text{SO}(3)$ act via the canonical orthogonal linear transformations on $X = \mathbb{R}^3$. Since the action preserves the lengths of vectors, one can see that the orbit of any point $(x, y, z)$ except the origin is the 2-sphere of radius $\sqrt{x^2 + y^2 + z^2}$ with isotropy group a circle. For instance, the orbit of the point $(0, 0, 1)$ is the unit sphere with isotropy group the (circle) group of rotations of the $XY$-plane. The origin is a fixed point i.e., the orbit is a single point with isotropy group $G$. In this case we see that there are two orbit types (the origin and everything else).

### 2.4. Cohomogeneity.

Now we restrict to compact Lie groups acting on smooth manifolds. An action of $G$ on $M^n$ is said to be of **cohomogeneity** $k$ if a principal orbit has codimension $k$. With this terminology, one can see that the transitive $G$ action on the coset space $G/H$ has cohomogeneity zero. The $S^1$ action on the torus $T^2 = S^1 \times S^1$ that we saw earlier is cohomogeneity one (every orbit is a circle which has codimension 1). The $\text{SO}(3)$ action on $\mathbb{R}^3$ above is also cohomogeneity 1. It is not hard to construct actions of higher cohomogeneity using these examples.

We already looked at the cohomogeneity 0 setting (homogeneous space) and saw that the orbit space is a single point. In the cohomogeneity 1 setting there are several possibilities for the orbit space: an open interval $(-1, 1)$, a half open interval $[-1, 1)$, a circle $S^1$ or a closed interval $[-1, 1]$. If $M$ is compact, then we are limited to $S^1$ or $[-1, 1]$. In the case when $M/G \approx S^1$, it follows that there is exactly one orbit type, say with isotropy group type $H$ and there is a fibration $G/H \to M \to S^1$. This in turn implies that $\pi_1(M)$ is infinite. Since we will focus on simply connected manifolds, we consider the orbit space $[-1, 1]$. 
2.5. **Cohomogeneity One manifolds.**

Now we look at the setting of a closed, connected Riemannian manifold $M$ admitting the isometric action of a compact Lie group $G$ such that the orbit space is $[−1,1]$ i.e., a cohomogeneity one action. Let $π : M → M/G$ be the quotient map. Fix a point $x₀ ∈ π⁻¹(0)$ and let $c : [−1,1] → M$ be the unique minimal geodesic orthogonal to each orbit such that $c(0) = x₀$ and $π ∘ c = Id_{[−1,1]}$. Then $c : R → M$ intersects all orbits orthogonally. Let $B_− = π⁻¹(−1) = G(x_−)$ and $B_+ = π⁻¹(1) = G(x_+)$ be the two non-principal orbits with isotropy groups $K_−$ and $K_+$ respectively, where $x_− = c(−1)$ and $x_+ = c(1)$. Let $H$ be the principal isotropy group. If we write the tubular neighborhoods of $B_−$ and $B_+$ as $D(B_−) = π⁻¹([−1,0])$ and $D(B_+) = π⁻¹([0,1])$ respectively, then we have the following description by the slice theorem,

$$D(B_±) = G × K_± D^{l±+1},$$

where $D^{l±+1}$ is the unit normal disk to $B_±$ at $x±$. As a consequence, we may write $M$ as,

$$M = D(B_−) ∪ E D(B_+),$$

where $E = π⁻¹(0) = G(x₀) = G/H$ is canonically identified with the boundaries, $∂D^{l±+1} = S^{l±} = K_±/H$. Thus we see that $M$ can be completely described in terms of $G$ and the inclusions of the subgroups, $K_±$ and $H$.

On the other hand, suppose we are given a compact Lie group $G$ with closed subgroups $K_−$, $K_+$ and $H$ such that $K_±/H = S^{l±}$ are spheres. Then the diagram of inclusions shown below determines a cohomogeneity one manifold.

$$\begin{array}{c}
  & G \downarrow & \\
  & \downarrow j_− & \downarrow j_+ \\
 K_− \quad & \quad & \quad \quad & K_+ \downarrow & \downarrow h_− & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \ quasi
isometries, the most general subgroup of the isometry group looks like $U \subseteq G \times G$ and acts on $G$ as follows:

$$U \times G \to G$$

$$(u_1, u_2) \cdot g \mapsto u_1 gu_2^{-1}$$

Unlike the case of a subgroup $H$ acting on $G$, there is no guarantee that the action is free (most of the time it is not). However, when it is free, then we potentially have a larger class of manifolds arising as quotients of Lie groups. The orbit spaces in this case are called biquotients or double coset manifolds. Here is an instructive example.

Let $G = \text{Sp}(2)$, the simple compact group of $2 \times 2$ matrices $A$ with quaternionic entries such that $AA^* = \text{Id}$. Here $A^*$ is the conjugate transpose (the quaternionic analog of orthogonal matrices). Let $U \subseteq G \times G$ be the subgroup

$$U = \left\{ \left( \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right), \text{ where } q \text{ is a unit quaternion} \right\}$$

Then Gromoll and Meyer showed in '72 that the double coset space $G/U$ (the double slash indicates that this is not just a coset space) is a 7 dimensional exotic or homotopy sphere. Since Borel showed that a homogeneous manifold that is homeomorphic to a standard sphere must, in fact, be diffeomorphic to it, this shows that the class of biquotients is strictly larger than homogeneous spaces and contains interesting examples.

---

*E-mail address: Krishnan.Shankar-1@ou.edu*