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BACKGROUND MATERIAL

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1. Riemannian Manifolds

Our principal objects of study will be Riemannian manifolds, M^n, without boundary. Our eventual objects of interest will be closed Riemannian manifolds (compact, without boundary). G will usually denote a compact, Lie group. Recall that, by the structure theorem for compact Lie groups, every such G admits a finite cover, say $\widetilde{G}$, which is isomorphic to a product of simple groups and possibly a torus (a product of circle groups).

1.1. **Curvature.**

A Riemannian manifold comes equipped with a (Levi-Civita) connection, ∇, (a way of taking derivatives) which in turn yields a *curvature tensor* defined on every tangent space: for every $p \in M$, we have $R : T_pM \otimes T_pM \otimes T_pM \to T_pM$ defined by $R(x, y, z) := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, where X, Y, Z are vector fields around p with $X_p = x, Y_p = y, Z_p = z$. This allows us to define the *sectional curvature* of any 2-plane $\sigma \subseteq T_pM$. If $\sigma = \text{span}(u, v)$, then the sectional curvature of σ is defined as $\mathbf{sec}(\sigma) := \dfrac{R(u, v, u, v)}{\text{Area}(u, v)^2}$ (the 4-tensor $R(u, v, u, v)$ is simply $\langle R(u, v, u), v\rangle$). The sectional curvature of σ is defined so that it is independent of choice of basis of σ.

We say that a manifold M^n has non-negative sectional curvature if $\mathbf{sec}(\sigma) \geqslant 0$ for every 2-plane $\sigma \subseteq TM$. In the talk we are interested in constructing and understanding the structure of non-negatively curved manifolds. A starting point for most constructions is the following fact: if G is a simple, compact Lie group, then it naturally admits a bi-invariant metric i.e., a metric for which every left translation, $L_g : G \to G, x \mapsto gx$, and every right right translation, $R_g : G \to G, x \mapsto xg$ is an isometry for every $g \in G$. This is given by, for instance, the negative of the Killing form on the Lie algebra $\mathfrak{g}$. A useful feature of this metric is that it has non-negative sectional curvature (the zero curvatures come precisely from 2-planes tangent to a maximal torus) and a large group of isometries ($G \times G$ almost effectively). Since non-negative curvature is preserved under products every compact Lie group admits non-negative curvature with a large group of symmetries.

1.2. **Submersions.**

A smooth map $f : M \to N$ between manifolds is called a *submersion* if for every $p \in M$ with $q = f(p)$, the differential $df_p : T_pM \to T_qN$ is surjective. A submersion between Riemannian manifolds is a *Riemannian submersion* if the restriction of the differential on the horizontal space is as isometry i.e., if $K_p = \ker(df_p)$ and if $H_p = K_p^{\perp}$, then f is a Riemannian submersion if $df_p : H_p \to T_qN$ is an isometry for every $p \in M$.

A useful feature of Riemannian submersions is their curvature increasing (or non-decreasing) property. Namely, the Gray–O'Neill submersion formulas for curvature show that the sectional curvature can only go up as one pushes down in a submersion. More precisely, given a Riemannian submersion $f : M \to N$ and $q = f(p)$, suppose $\widetilde{\sigma}$ is a 2-plane in T_qN. Let σ be a horizontal 2-plane in $H_p \subseteq T_pM$ such that $df_p(\sigma) = \widetilde{\sigma}$. Then $\mathbf{sec}(\widetilde{\sigma}) \geqslant \mathbf{sec}(\sigma)$.

2. Group Actions

Let G be a group acting on a space X (where we keep in mind that in our setting G will be a Lie group and X a smooth manifold). This means that there is a homomorphism $\Phi : G \to \text{Aut}(X)$, where $\text{Aut}(X)$ could be a group of homeomorphisms, diffeomorphisms or some other group of invertible self maps. (For example, when X is a vector space and $\text{Aut}(X)$ is a subgroup of invertible linear self maps, then this is a representation of G). In particular, the identity $e \in G$ acts as the identity map and multiplication of group elements corresponds to composition of maps.

For $x \in X$, the set $G(x) := \{y = g \cdot x : g \in G\}$ is called the *orbit* of $x \in X$. Note that $G(x) \subseteq X$. The subgroup $G_x := \{g \in G : g(x) = x\}$ is called the *isotropy group* or the *stabilizer* of $x \in X$. Note that $G_x \subseteq G$. The *orbit space* of an action is the quotient space $X/G := X/\sim$, where $x_1 \sim x_2$ if and only if $x_2 = g(x_1)$ for some $g \in G$.

If $G_x = \{e\}$ for every $x \in X$, then the action if said to be *free*. If $G_x \subsetneq G$ for every $x \in X$, then the action is said to *fixed point free*. If G acts freely on a manifold M, then the orbit space M/G admits a canonical smooth structure as a manifold.

2.1. Transitive actions and homogeneous space.

We look at an important special case where we consider the action of a subgroup $H \subseteq G$ on G given by $H \times G \to G, h(g) = gh^{-1}$. This is a free action and every orbit looks like a copy of H inside G (these orbits are also called cosets). A simple example of this is $H = S^1 \times \{e\} \subseteq S^1 \times S^1$, the 2-torus. The action rotates the circle in the first coordinate and provides a foliation of the 2-torus into a bunch of circles (orbits). The orbit space in this case is a circle (imagine an orthogonal circle meeting every orbit exactly once).

In the setting of H acting on G, the orbit space or the space of (left) cosets is denoted as $G/H := \{gH : g \in G\}$. Note that this orbit space admits an action of G via $G \times G/H \to G/H$, $g(g'H) = gg'H$. This is a *transitive* action i.e., any two points in G/H lie in the same orbit (equivalently there is only one orbit): if g_1H, g_2H are any two points in G/H, then $g_2g_1^{-1}(g_1H) = g_2H$. Thus, the orbit space is a single point $\{*\}$.

When a group G acts transitively on a space X, then X is called a *homogeneous* space and is homeomorphic to G/H for some subgroup $H \subseteq G$. For the G action on G/H the isotropy group at any point gH is gHg^{-1} i.e., a subgroup conjugate to H.

2.2. **Some facts about group actions.**

(*i*) If G acts on X, then we can stratify X into disjoint G-orbits.

(*ii*) Any orbit $G(x)$ is naturally homeomorphic to the homogeneous space G/G_x.

(*iii*) Two orbits $G(x_1), G(x_2)$ (disjoint or coincident) are said to be of the same *type* if all isotropy groups of points in either orbit are conjugate in G to a fixed subgroup H. An important fact that we will need is that for a compact group action there are only finitely many orbit types.

(*iv*) For a compact group actions there exists a principal orbit type i.e., a unique orbit type such that the set of points in this orbit type form an open, dense subset of the space X (and project to an open dense subset of the orbit space). The other orbits are usually called *singular orbits*.

2.3. **An Example.**

Let $G = \mathrm{SO}(3)$ act via the canonical orthogonal linear transformations on $X = \mathbf{R}^3$. Since the action preserves the lengths of vectors, one can see that the orbit of any point (x, y, z) except the origin is the 2-sphere of radius $\sqrt{x^2 + y^2 + z^2}$ with isotropy group a circle. For instance, the orbit of the point $(0, 0, 1)$ is the unit sphere with isotropy group the (circle) group of rotations of the XY-plane. The origin is a fixed point i.e., the orbit is a single point with isotropy group G. In this case we see that there are two orbit types (the origin and everything else).

2.4. **Cohomogeneity.**

Now we restrict to compact Lie groups acting on smooth manifolds. An action of G on M^n is said to be of *cohomogeneity* k if a principal orbit has codimension k. With this terminology, one can see that the transitive G action on the coset space G/H has cohomogeneity zero. The S^1 action on the torus $\mathrm{T}^2 = S^1 \times S^1$ that we saw earlier is cohomogeneity one (every orbit is a circle which has codimension 1). The $\mathrm{SO}(3)$ action on $\mathbf{R}^3$ above is also cohomogeneity 1. It is not hard to construct actions of higher cohomogeneity using these examples.

We already looked at the cohomogeneity 0 setting (homogeneous space) and saw that the orbit space is a single point. In the cohomogeneity 1 setting there are several possibilities for the orbit space: an open interval $(-1, 1)$, a half open interval $[-1, 1)$, a circle S^1 or a closed interval $[-1, 1]$. If M is compact, then we are limited to S^1 or $[-1, 1]$. In the case when $M/G \approx S^1$, it follows that there is exactly one orbit type, say with isotropy group type H and there is a fibration $G/H \to M \to S^1$. This in turn implies that $\pi_1(M)$ is infinite. Since we will focus on simply connected manifolds, we consider the orbit space $[-1, 1]$.

2.5. **Cohomogeneity One manifolds.**

Now we look at the setting of a closed, connected Riemannian manifold M admitting the isometric action of a compact Lie group G such that the orbit space is $[-1,1]$ i.e., a cohomogeneity one action. Let $\pi : M \to M/G$ be the quotient map. Fix a point $x_0 \in \pi^{-1}(0)$ and let $c : [-1,1] \to M$ be the unique minimal geodesic orthogonal to each orbit such that $c(0) = x_0$ and $\pi \circ c = \mathrm{Id}_{[-1,1]}$. Then $c : \mathbf{R} \to M$ intersects all orbits orthogonally. Let $B_- = \pi^{-1}(-1) = G(x_-)$ and $B_+ = \pi^{-1}(1) = G(x_+)$ be the two non-principal orbits with isotropy groups K_- and K_+ respectively, where $x_- = c(-1)$ and $x_+ = c(1)$. Let H be the principal isotropy group. If we write the tubular neighborhoods of B_- and B_+ as $D(B_-) = \pi^{-1}([-1,0])$ and $D(B_+) = \pi^{-1}([0,1])$ respectively, then we have the following description by the slice theorem,

$$D(B_\pm) = G \times_{K_\pm} D^{l_\pm+1},$$

where $D^{l_\pm+1}$ is the unit normal disk to $B_\pm$ at $x_\pm$. As a consequence, we may write M as,

$$M = D(B_-) \cup_E D(B_+),$$

where $E = \pi^{-1}(0) = G(x_0) = G/H$ is canonically identified with the boundaries, $\partial D^{l_\pm+1} = \mathbf{S}^{l_\pm} = K_\pm/H$. Thus we see that M can be completely described in terms of G and the inclusions of the subgroups, $K_\pm$ and H.

On the other hand, suppose we are given a compact Lie group G with closed subgroups K_-, K_+ and H such that $K_\pm/H = \mathbf{S}^{l_\pm}$ are spheres. Then the diagram of inclusions shown below determines a cohomogeneity one manifold.

(2.1)
$$\begin{tikzcd}
 & G & \\
K_- \arrow[ur, "j_-"] & & K_+ \arrow[ul, "j_+"'] \\
 & H \arrow[ul, "h_-"] \arrow[ur, "h_+"'] &
\end{tikzcd}$$

The manifold may be written as $M = G \times_{K_-} D^{l_-+1} \cup_{G/H} G \times_{K_+} D^{l_++1}$ on which the G action is of cohomogeneity one. The above was first described by **[?]**.

In this setting Grove and Ziller showed that when $K_\pm/H$ are both $\mathbf{S}^1$, then (M, G) admits an invariant metric of non-negative sectional curvature.

3. Biquotients

We saw already that there is a large class of spaces one can construct as quotient of Lie groups by subgroups namely coset spaces. In particular, they yield Riemannian submersions $G \to B$, where $B = G/H$ for some (closed) subgroup $H \subseteq G$. However, this is not the most general Riemannian submersion from a Lie group with a bi-invariant metric. Since both left and right translations are isometries, the most general subgroup of the isometry group looks like $U \subseteq G \times G$ and acts on G as follows:

$$U \times G \to G$$

$$(u_1, u_2) \cdot g \mapsto u_1 g u_2^{-1}$$

Unlike the case of a subgroup H acting on G, there is no guarantee that the action is free (most of the time it is not). However, when it is free, then we potentially have a larger class of manifolds arising as quotients of Lie groups. The orbit spaces in this case are called *biquotients* or *double coset manifolds*. Here is an instructive example.

Let $G = \mathrm{Sp}(2)$, the simple compact group of 2×2 matrices A with quaternionic entries such that $AA^* = \mathrm{Id}$. Here A^* is the conjugate transpose (the quaternionic analog of orthogonal matrices). Let $U \subseteq G \times G$ be the subgroup

$$U = \left\{ \left(\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right), \text{ where } q \text{ is a unit quaternion} \right\}$$

Then Gromoll and Meyer showed in '72 that the double coset space $G /\!\!/ U$ (the double slash indicates that this is not just a coset space) is a 7 dimensional exotic or homotopy sphere. Since Borel showed that a homogeneous manifold that is homeomorphic to a standard sphere must, in fact, be diffeomorphic to it, this shows that the class of biquotients is strictly larger than homogeneous spaces and contains interesting examples.

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