An introduction to intersection homology
(without sheaves)

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February 26, 2015

typeset=February 26, 2015

Abstract

This is a preliminary (incomplete) manuscript of an introductory book on intersection homology from the simplicial/PL/singular chain point of view, inspired by a series of talks given in Lille in May 2013. The existing chapters are mostly complete, except where noted (though this isn’t to say that there won’t be future additions, changes, or corrections or that they wouldn’t still benefit from further proofreading and editing). The largest current omissions (which will be added in the future) are the sections on cup and cap products and Poincaré duality. For now, the reader should “insert there” the original intersection homology paper of Goresky and MacPherson [42] (for the PL intersection pairing and PL duality) and the paper [38] by myself and Jim McClure (for cup products, cap products, and topological singular chain duality). It is planned for the finished version of the book to reconcile the PL and cup product pairings without the use of sheaves (a reconciliation using sheaves appears in work in preparation [37]).

As one can see comparing the table of contents to the actual contents, there are some other sections and topics planned, but at the moment these plans should be considered tenuous. Figures will definitely be added at some point, as well as an appropriate (and much needed) introduction. I apologize for the lack of proper introduction to anyone who gets their hands on this, though presumably this will mostly be those who don’t need much introduction or motivation to dive in.

Any comments, corrections, or suggestions are very welcome. And I would be particularly grateful to anyone who can reconcile the sign discrepancy in Section 10.4.

IMPORTANT NOTE: ANY SECTION LABELED AS “NEW!” MAY NOT YET HAVE RECEIVED ANY PROOFREADING OR EDITING WHATSOEVER. ALSO,

*During the writing of this book, in addition to primary support my home institution, Texas Christian University, I received support from a grant from the Simons Foundation (#209127 to Greg Friedman), a grant from the National Science Foundation (DMS-1308306), and funding from Universit Lille 1 and Universit Artois for my visits.
SECTIONS WITH NOTES TO MYSELF ALONG THE LINES OF “DOUBLE CHECK THIS” SHOULD BE REGARDED WITH SOME SUSPICION.

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0.1 Notations and conventions

This section describes some conventions, notational and otherwise, we attempt to use throughout the book, though we make no claim to complete consistency of notation.

1. Spaces

   (a) Manifolds are usually denoted $M$ or $N$. 

Remark 0.1. A note on prerequisites: Most of the material in the book should be accessible following a first course in algebraic topology that treats homology and cohomology theory. In fact, many of the elementary proofs of algebraic topology are, by necessity, repeated here in generalized form adapted to intersection homology and cohomology. As we progress into later sections of the book, our homological algebra becomes a bit more sophisticated, due in some part to our struggle to use Dedekind domains (as opposed to the slightly less general principal ideal domains (PIDs)) where possible. There is also some occasional need for facts from elementary sheaf theory or more advanced algebraic or geometric topology. At these later points, rather than attempt to remain completely self contained, we instead provide copious references to other sources, with a preference for textbooks when at all possible. Our favored sources include topology text books by Hatcher [53], Dold [23], Munkres [77] and [78], and Spanier [97]; algebra books by Lang [64] and Bourbaki [10]; homological algebra books by Hilton and Stammbach [55] and Weibel [105]; and introductions to sheaf theory by Swan [100] and Bredon [13].
(b) Arbitrary spaces have letters from the end of the alphabet such as $Z$, though sometimes also other letters.

(c) Open subsets get letters such as $U, V, W$.

(d) Subsets will be denoted $A \subset X$, rather than $A \subseteq X$; in other words $A \subset X$ includes the possibility that $A = X$.

(e) Generic maps between spaces will be denoted with letters such as $f$ or $g$. The letter $i$, or variants such as $i$, generally denotes an inclusion. The map $d$ is the diagonal map $d : Z \to Z \times Z$, $d(z) = (z, z)$.

(f) When working with product spaces, we may write elements of $X \times Y$ as either $(x, y)$ or $x \times y$. Products maps are usually written $f \times g$.

(g) While we will attempt to parenthesize fairly thoroughly, we will occasionally rely on a few simplifying conventions. In particular, expressions of the form $A - B$ should be understood as $(A) - (B)$. So, for example, $X \times Y - A \times B$ means $(X \times Y) - (A \times B)$ and not $X \times (Y - A) \times B$.

(h) For a compact space $Z$, $cZ$ denotes the open cone $cZ = [0, 1) \times Z/\sim$, where $\sim$ is the relation $(w, 0) \sim (z, 0)$ for all $w, z \in Z$. We typically denote the vertex of a cone by $v$. Similarly, the closed cone is $\bar{c}Z = [0, 1] \times Z/\sim$. More generally, for $r > 0$, we let $c_r Z = [0, r) \times Z/\sim$ and $\bar{c}_r Z = [0, r] \times Z/\sim$; in particular, $cZ = c_1 Z$. Then $c_r Z \subset \bar{c}_r Z \subset c_s Z \subset \bar{c}_s Z$ whenever $r < s$.

(i) For a compact space $Z$, the (unreduced) suspension is $SZ = [-1, 1] \times X/\sim$, where the relation $\sim$ is such that $(-1, w) \sim (-1, z)$ and $(1, w) \sim (1, z)$ for any $w, z \in Z$.

(j) When taking the product of a space with a Euclidean space, interval, or sphere, we usually put the Euclidean space, interval, or sphere on the left, e.g. $\mathbb{R} \times Z$ instead of $Z \times \mathbb{R}$. This has some ramifications for signs. For example, if $x$ is a singular cycle in $Z$ and $\bar{c}x$ denotes the singular cone on $x$ in $\bar{c}Z$ (see Example 3.38), this is the convention that is consistent with adding the cone vertex as the first vertex and so gives us $\partial(\bar{c}x) = x$.

(k) Filtered spaces (our main object of study) are generally denoted by capital letters near the end of the alphabet, in particular $X$ (or $Y$ when we talk about multiple filtered spaces at the same time); the filtrations are usually left implicit in the sense that we say “the filtered space $X$”. When we need to refer to the filtration explicitly, we let $X^i$ denote the $i$th skeleton of the filtration, and we let $X_i = X^i - X^{i-1}$; see Section 2.2. The formal dimension of a filtered space is generically denoted $n$ (or $m$ for a second filtered space). When we wish to emphasize the dimension of $X$, we write $X = X^n$. Subspaces have letters like $A$ or $B$, so we tend to have filtered pairs $(X, A)$ or $(Y, B)$.

(l) If we wish to consider the underlying topological space of a filtered space $X$, i.e. we wish to explicitly disregard the filtration, we write $|X|$.
(m) The singular locus of a filtered space $X = X^n$ is defined to be $X^{n-1}$ and can also be written $\Sigma_X$, or simply $\Sigma$ if the space is clear.

(n) Generic strata (see Section 2.2) of a filtered space have letters such as $S$ and $T$. Regular strata are sometimes denoted $R$.

(o) The links occurring in locally-conelike spaces (see Section 2.3), in particular CS sets or stratified pseudomanifolds, are denoted $L$ or, occasionally, $\ell$. We let $\text{Lk}(x)$ denote the polyhedral link of a point in a piecewise linear space, i.e. if $x$ is contained in the piecewise linear space $X$, then $\text{Lk}(x)$ is the unique PL space such that $x$ has a neighborhood piecewise linearly homeomorphic to $c\text{Lk}(x)$; see [86, Section 1.1].

(p) If $X$ is a piecewise linear space, we let $X^*$ denote the filtered space with the underlying space of $X$ but with its intrinsic stratification; see Section 2.8. Similarly, if $X$ is a CS set, $X^*$ will denote the underlying space of $X$ with its intrinsic filtration as a CS set.

2. Algebra

(a) $G$ will always be an abelian group, $R$ a commutative ring with unity. In some contexts, $R$ will be assumed to be a Dedekind domain, though this will be established at the relevant time.

(b) Subgroups (or submodules) will be denoted $H \subset G$, rather than $H \subseteq G$; in other words $H \subset G$ includes the possibility that $H = G$.

(c) We use the standard notations for standard algebraic objects: $\mathbb{Z}$ for integers, $\mathbb{Q}$ for rational numbers, $\mathbb{R}$ for real numbers (which also notates the space of real numbers, i.e. 1-dimensional Euclidean space).

(d) When working with $R$-modules in the context of a fixed ring $R$, we write $\text{Hom}(A, B)$ and $A \otimes B$ rather than $\text{Hom}_R(A, B)$ and $A \otimes_R B$.

(e) Dedekind domains have cohomological dimension $\leq 1$ (see [85, Proposition 8.1] and use that Dedekind domains are hereditary by definition [85, page 161]). Therefore, if $R$ is a Dedekind domain, $\text{Ext}^n_R(A, B) = 0$ for $n > 1$ and for any $A, B$. Therefore, we write simply $\text{Ext}(A, B)$ instead of $\text{Ext}_R^1(A, B)$.

(f) Generic purely algebraic chain complexes are denoted $C_*$, $D_*$, etc. Cohomologically graded complexes can be denote $C^*$, $D^*$, etc.

(g) For almost all chain complexes, the boundary maps are all denoted $\partial$. For cohomologically graded complexes, we use $d$ for the coboundary maps. If we wish to emphasize that $\partial$ is the boundary map of a chain complex $C_*$, we can write $\partial_{C_*}$, and analogously for coboundary maps of cochain complexes.

(h) Elements of chain complexes are denoted with lowercase Greek letters such as $\xi, \zeta, \eta$, though we sometimes also use $x, y, z$. N.B. we generally abuse notation by

\footnote{We will see an exception in Section 6.2 for $\mathbb{P}S_*(X)$.}
using the same symbol to refer to both a homology class and a chain representing it. For example, \( \xi \in H_i(C_\ast) \) means that \( \xi \) is a homology class that we also think of as being represented by a cycle in \( C_i \) that we also denote \( \xi \). In most contexts, this should not cause much confusion, though in those instances where confusion might reasonably occur, we use \( \xi \) just to denote the chain and \( [\xi] \) to specify the homology class. We will indicate this notation specifically when it occurs. More generally, \([\cdot]\) indicates some sort of equivalence class, so, depending on context, \([\xi]\) might reference an element \( \xi \in S_\ast(X) \) representing an element \([\xi] \in S_\ast(X,A)\) or an element \([\xi] \in H_\ast(X)\) or \([\xi] \in H_\ast(X,A)\). Similarly, if \( \xi \) is a simplicial chain, \([\xi]\) might denote the class in \( C_\ast(X) \) represented by \( \xi \).

(i) Elements of cochain complexes are denoted with lowercase Greek letters such as \( \alpha, \beta, \gamma \). Again, we generally abuse notation by using the same symbol to refer to both a cohomology class and a cochain representing it. For example, \( \alpha \in H^i(C^\ast) \) means that \( \alpha \) is a cohomology class that we also think of as being represented by a cocycle in \( C^i \) that we also denote \( \alpha \). In most contexts, this should not cause much confusion, though in those instances where confusion might reasonably occur, we use \( \alpha \) just to denote the chain and \( [\alpha] \) to specify the cohomology class. We will indicate this notation specifically when it occurs. More generally, \([\cdot]\) indicates an equivalence class.

(j) The connecting morphisms in long exact homology sequences are denoted \( \partial \). The connecting morphisms in long exact cohomology sequences are denoted \( d^\ast \).

(k) Augmentation maps of chain complexes are denoted \( a \), e.g. we might have \( a : S_\ast(X) \to \mathbb{Z} \).

(l) If \( x \) is an element of a chain or cochain complex, then \(|x|\) will always denote the degree of \( x \). For example, if \( x \in C_i \) or \( X \in C^n \), then \(|x| = i\).

3. Algebraic topology

(a) \( \Delta^i \) denotes the standard \( i \)-dimensional simplex

(b) Lowercase Greek letters such as \( \sigma, \tau \), and often others can denote either simplices in a simplicial complex or singular simplices, depending on context.

(c) Simplicial chain complexes are denoted \( C_\ast(X) \), singular chain complexes are denoted \( S_\ast(X) \), PL chain complexes are denoted \( \mathfrak{C}_\ast(X) \). When there are subspaces or coefficients involved, the notations look like \( C_\ast(X,A;G) \) for a subspace \( A \) and a coefficient group \( G \). We use the same notation \( H_\ast(X) \) for both homology groups \( H_\ast(C_\ast(X)) \) or \( H_\ast(S_\ast(X)) \), letting context determine which is meant. Since simplicial and PL chains often occur in the same context, we use \( \mathfrak{S}_\ast(X) \) for \( H_\ast(\mathfrak{C}_\ast(X)) \).

\(^2\)Technically, this is not quite the right thing to do as the standard equivalence between homological and cohomological gradings tells us that the notation \( C^i \) should be equivalent to the notation \( C_{-i} \). However, matters of degree will arise only when working with signs, and so \(|x|\) will really only have significance mod \( 2 \). Therefore, we will live with this inconsistency. This observation should also alleviate concern that our degree notation might conflict with standard absolute value notation.
(d) If $f : X \to Y$ is a map of spaces, we abuse notation by letting $f$ also denote both
the induced chain maps of chain complexes defined on the spaces and the induced
maps on homology, e.g. we write $f : S_*(X) \to S_*(Y)$ and $f : H_*(X) \to H_*(Y)$.
The dualized maps of cochain complexes and cohomology groups are denoted $f^*$, e.g.
$f^* : S^*(Y) \to S^*(X)$ and $f : H^*(Y) \to H^*(X)$. Similarly, if $f : C_* \to D_*$
is a purely algebraic map of chain complexes of $R$-modules, we also write $f : H_*(C_*) \to H_*(D_*)$ for the induced homology map and $f^* : H^*(\text{Hom}(D_*, R)) \to H^*(\text{Hom}(C_*, R))$ for the induced cohomology map.

(e) For Mayer-Vietoris sequences, the map $H_*(U) \oplus H_*(V) \to H_*(U \cup V)$ will take
$(\xi, \eta)$ to $\xi + \eta$. Therefore, the map $H_*(U \cap V) \to H_*(U) \oplus H_*(V)$ will take $\xi$ to
$(\xi, -\xi)$.

(f) The cross product chain map $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ (and its variants) can
be written either as $\epsilon$ or $\times$. For example, we tend to write $\epsilon : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$, but given two specific chains $x, y$, we may write $x \times y$. Unfortunately, it
is common in algebraic topology to use the symbol $\times$ for both chain cross products
and cochain cross products. We perpetuate this ambiguity, though context should
make clear which is meant.

4. Intersection homology and cohomology

(a) Perversities (see Section 3.1) are denoted $\bar{p}$, $\bar{q}$, $\bar{r}$, etc. In general, perversities will
always have bars, with the exception$^3$ of the special perversities $\bar{Q}$ that occur in
the discussion of the Künneth theorem; see Section 6.4.

(b) $\bar{0}$ denotes the perversity that always evaluates to 0. $\bar{t}$ is the top perversity $\bar{t}(S) =
codim(S) - 2$. $\bar{m}$ and $\bar{n}$ are respectively the lower middle perversity and upper
middle perversity, i.e.

$$
\bar{m}(S) = \left\lfloor \frac{\text{codim}(S) - 2}{2} \right\rfloor,
$$

$$
\bar{n}(S) = \left\lceil \frac{\text{codim}(S) - 2}{2} \right\rceil.
$$

(c) For a perversity $\bar{p}$, $D\bar{p}$ is the dual or complementary perversity with $D\bar{p}(S) =
\bar{t}(S) - \bar{p}(S)$ for all singular strata $S$; see Definition 3.5.

(d) Throughout the first part of the book, simplicial, PL, and singular perversity
$\bar{p}$ intersection chain complexes are written $I^{\bar{p}}C_*^{\text{GM}}(X)$, $I^{\bar{p}}C_*^{\text{GM}}(X)$, $I^{\bar{p}}S_*^{\text{GM}}(X)$,
with corresponding homology groups $I^{\bar{p}}H_*^{\text{GM}}(X)$, $I^{\bar{p}}S_*^{\text{GM}}(X)$, $I^{\bar{p}}H_*^{\text{GM}}(X)$. The
GM here stands for “Goresky-MacPherson”. In Section 6, we introduce the variant
“non-GM” intersection homology and the notation becomes simply $I^{\bar{p}}C_*^{\text{GM}}(X)$,
$I^{\bar{p}}C_*^{\text{GM}}(X)$, and $I^{\bar{p}}S_*^{\text{GM}}(X)$ with corresponding homology groups $I^{\bar{p}}H_*^{\text{GM}}(X)$, $I^{\bar{p}}S_*^{\text{GM}}(X)$,
and $I^{\bar{p}}H_*^{\text{GM}}(X)$.

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$^3$This special case is partly historical, partly because there is little risk of confusion since $\bar{Q}$ is not used
for anything else, and party idiosyncratic. Probably we should use $\bar{Q}$.
For intersection cohomology, we raise the index and lower the perversity marking, e.g. $I_p S^*(X)$ and $I_p H^*(X)$. Lowering the perversity symbol has no intrinsic meaning; it is meant as a further distinguishing aid between homology and cohomology.

5. Miscellaneous conventions

**Signs:**
- We utilize throughout the Koszul sign conventions, so that interchange of elements of degrees $i$ and $j$ usually results in a sign $(-1)^{ij}$.
- The standard exception to the Koszul rule, necessary for evaluation to be a chain map, is that the sign occurring in the coboundary map of the chain complex $E^* = \text{Hom}^*(C_*, D_*)$ has the form
  \[
  (d^*_E f)(c) = \partial_{D_*}(f(c)) - (-1)^{|f|}f(\partial_{C_*}(c))
  \]
  for $c \in C_*$ and $f \in \text{Hom}(C_*, D_*)$. In particular, if $\alpha$ is a cochain in $\text{Hom}^i(C_*, R) = \text{Hom}(C_i, R)$, then $d\alpha = (-1)^{i+1}f\partial$.
- The connecting morphisms of long exact homology sequences has degree $-1$ and so can generate signs upon interchanges.

**id:** The expression $id$ is used for the identity function. It can be either a topological or algebraic identity. Context will usually make clear which identity function is meant, though we can make it precise with subscripts such as $id_X : X \to X$ or $id_{C_*} : C_* \to C_*$.

**Parentheses:**
- When a function $f$ acts on an element $x$ of a set, group, etc., which generally write $f(x)$. The general exception will be boundary maps $\partial$ acting on chains $\xi$ for which we will generally write $\partial \xi$.
- To avoid the ambiguity inherent in writing expressions such as $\partial (\xi \otimes \eta)$, we will write either $\partial (\xi \otimes \eta)$ or $(\partial \xi) \otimes \eta$, as appropriate. We also use $\xi \otimes \partial \eta$, as there is no ambiguity here.
- When parentheses are omitted, expressions compile from the right. For example, if $f : X \to Y$ and $g : Y \to Z$, then, as usual, $gf(x)$ means $g(f(x))$. As a more complex example, $\Phi(id \otimes \beta)\partial(\xi \otimes \eta)$ means $\Phi((id \otimes \beta)(\partial(\xi \otimes \eta)))$.
- We will use an obnoxious number of parentheses to describe spaces as clearly as possible. One place where we will sometimes avoid this is when considering complements. So, for example, if $K, L \subset X$, then $X - K \cup L$ means $X - (K \cup L)$. 

1 Background/motivation

1.1 Poincaré duality on manifolds

1.2 Intersection pairings on PL manifolds

2 Stratified Spaces

2.1 First examples of stratified spaces

In this section, we attempt to provide the reader with some beginning intuition concerning the spaces that shall be our main object of study. The idea is to consider spaces that are not manifolds but that in some sense are not too far from being manifolds. We shall begin with technical definitions in Section 2.2; for now we simply mention some first examples in attempt to describe the flavor of the types of non-manifold spaces that we will be thinking about. We shall see in our examples that a natural notion of “layering” occurs, and this leads to the notion of a stratified space.

As a first example, let $M^n$ be a closed manifold of dimension $n \geq 2$ and consider the (unreduced) suspension $SM$ of $M$. Recall that the suspension $SY$ of the space $Y$ is obtained by taking the product of $Y$ with an interval and “collapsing the ends”. More formally $SY$ is the quotient space $[-1,1] \times Y / \sim$, where the relation $\sim$ is such that $(-1,x) \sim (-1,y)$ and $(1,x) \sim (1,y)$ for any $x,y \in Y$. Alternatively, if we define the closed cone $\overline{c}Y$ on the space $Y$ to be the quotient space $[0,1] \times Y / \sim$, where $\sim$ is the relation $(1,x) \sim (1,y)$ for all $x,y \in Y$ then the suspension is just the union of two cones along $Y$: $SY = \overline{c}Y \cup_{Y} \overline{c}Y$. Now, assuming that our manifold $M$ is not a homology sphere $\Sigma^n$, $X$ will not be a manifold.

Surprisingly, it is perhaps easiest to see this using algebraic topology: Let $\{N,S\}$ be the two vertices of the cones in our suspension and note that $X - \{N,S\} \cong (-1,1) \times M$ is a manifold of dimension $n+1$, so if $X$ were to be a manifold, it would have to be an $n+1$ manifold. However, neither $N$ nor $S$ can have a neighborhood homeomorphic to euclidean space: if $N$ had a neighborhood $U \cong \mathbb{R}^{n+1}$ then by excision, homotopy equivalence, and the exact sequence of the pair, $H_* (X, X - N) \cong H_* (\overline{U}, \overline{U} - N) \cong H_* (\mathbb{R}^n, \mathbb{R}^n - 0) \cong \tilde{H}_{* - 1} (S^{n-1})$. But $N$ does have a neighborhood homeomorphic to the open cone $cM = \overline{c}M - 1 \times Y$, and so $H_* (X, X - N) \cong H_* (cM, cM - N) \cong \tilde{H}_{* - 1} (cM - N) \cong \tilde{H}_{* - 1} (M)$, where we have used that the reduced homology of a cone is trivial and that $cM - N$ is homotopy equivalent to $M$. We have assumed that $M$ does not have the homology of a sphere, so $X$ cannot be a manifold. We refer to points that fail to have euclidean neighborhoods as singular points or singularities. When this set of points is discrete, we say that the space has isolated singularities; more generally, we refer to the union of such points as the singular set or singular locus.

Notice that the space $X$ is a manifold except at the two singular points $N$ and $S$. Yet this is enough to ruin one of the most prized properties of manifolds, Poincaré duality. To illustrate this point, let’s suppose $M = T^2$, the 2-dimensional torus. Standard algebraic topology computations show that
\[ H_2(T^2) \cong \mathbb{Z}, \]
\[ H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}, \]
\[ H_0(T^2) \cong \mathbb{Z}, \]

and so, recalling that suspending a space shifts its reduced homology up one degree (by an easy Mayer-Vietoris argument), we have that

\[ H_3(ST^2) \cong \mathbb{Z}, \]
\[ H_2(ST^2) \cong \mathbb{Z} \oplus \mathbb{Z}, \]
\[ H_1(ST^2) = 0, \]
\[ H_0(ST^2) \cong \mathbb{Z}. \]

But if \( ST^2 \), which is three-dimensional, possessed Poincaré duality, \( H_1(ST^2) \) and \( H_2(ST^2) \) would have to have the same rank.

Already in this example, there are several interesting features:

1. Even though \( X = SM \) is not a manifold, it is “mostly” a manifold, meaning that it is a manifold on the dense set \( X - \{N, S\} \cong (-1,1) \times M \).

2. The two “bad points” \( \{N, S\} \) are already enough to ruin Poincaré duality.

3. But \( \{N, S\} \) is itself a fairly “nice” set: it is a manifold, though of a lower dimension than \( X \).

These features will be typical of the spaces we intend to study: they are assembled from manifold pieces of various dimensions, including a dense top dimensional piece, but this is not enough for the space to possess Poincaré duality in the usual sense. Instead we will need intersection homology to obtain duality results.

It is not difficult to construct other, more elaborate, non-manifold spaces with similar features. For example, we could construct a space with a single non-manifold (or singular) point by taking a compact manifold with boundary and attaching a cone on the boundary, or we could obtain multiple isolated singular points by starting with a manifold with multiple boundary components and coning each off separately.

It is also not difficult to form spaces with singular sets of higher-dimension. For example, take our suspension \( SM \) and another manifold \( M' \) and form the product space \( SM \times M' \). Then \((-1,1) \times M \times M'\) is our dense manifold, while the singular set (of non-manifold points) will be homeomorphic to two copies of \( M' \).

How about a space in which the singular set is not itself a manifold but can be similarly disassembled into manifold pieces? We could start with the torus \( T^2 \), suspend it to form \( ST^2 \), cross it with a circle to get \( Y = ST^2 \times S^1 \), and then suspend this whole space to arrive at \( X = S(ST^2 \times S^1) \). Keeping track of the singular points, \( Y \) has a singular set consisting of two copies of the circle, and \( X \) has a singular set consisting of the suspension of the two
circles, which is a set $\Sigma$ homeomorphic to two copies of the sphere $S^2$ with their north poles attached to each other and their south poles attached to each other. Notice that $X - \Sigma$ is a manifold homeomorphic to $(-1, 1) \times (-1, 1) \times T^2 \times S^1$. Meanwhile $\Sigma$ is not a manifold, but it is a suspension of a manifold, so if we let $\{N, S\}$ be the north and south poles of the last suspension, $\Sigma - \{N, S\} \cong (-1, 1) \times (S^1 \amalg S^1)$, which is a two-dimensional manifold. So we begin to see spaces whose singular sets are also "nearly" manifolds, except for their own singular subsets!

The reader is invited to think through more examples obtained by repeated applications of suspension or crossing with a manifold (or even crossing with a singular space such as $SM$!). What is the dense manifold? What is the singular set? What is the dense manifold of the singular set, and what are its singularities? And so on.

Some less artificial examples come from algebraic geometry. For example, if $X$ is an irreducible complex algebraic variety, i.e. a subset of $\mathbb{C}^n$ determined as the zero set of a collection of polynomials, then the singular set $\Sigma$ of $X$ is itself an algebraic variety described by a larger set of polynomial equations. $\Sigma$ itself will be a union of irreducible complex varieties (possibly of different dimensions), each of which will itself consist of a dense manifold set and lower-dimensional singular sets, which will be a union of irreducible varieties, and so on. Eventually this decomposition process bottoms out and we see that the singular set of $X$ is naturally layered, or stratified, in a way similar to our artificial constructions involving suspensions and products.

**Example 2.1.** Consider the subspace of $\mathbb{R}^3$ determined by the polynomial equation $xyz = 0$. This is the union of the three coordinate planes. It is a manifold except along the three axes, whose union is described by the system of equations $\{xyz = 0, xy + yz + xz = 0\}$. This is in turn a manifold except at the origin, which we can describe by the system $\{xyz = 0, xy + yz + xz = 0, x + y + z = 0\}$. (These systems of equations are not unique!)

Although our work will be completely topological, we will briefly discuss stratifications of algebraic spaces further in Section 2.6. The study of algebraic varieties through stratified space methods has consistently been a leading motivation for development in the field of study and remains an active area of research.

### 2.2 Filtered and stratified spaces

So far, we have seen spaces with various layers of singularities such that each layer is the closure of a manifold. We wish to make this concept more precise through a series of definitions. We will begin with a much more general notion and then define more and more specific types of spaces until we arrive at the definition we want. The reader should be aware that the definitions in the literature are inconsistent and that it is not always clear what the canonical versions should be (it probably depends on what theorems one wants to prove); thus the reader should exercise care in reading other sources. Ultimately, however, the various general notions all seem to converge together and mostly agree for the definition of topological pseudomanifolds, which will eventually be our primary objects of study.

The most general level of definition is that of a filtered space.

We will assume all spaces are Hausdorff, usually without further mention.
Remark 2.2. It is common to find additional point-set topological assumptions, such as paracompactness or second countability, in the definitions of the various types of spaces we will be considering. One of the benefits of our approach is that these additional assumptions do not seem to be necessary for any of the results we will encounter. In particular, we do not need to assume that manifolds are paracompact or second countable.

Filtered spaces.

Definition 2.3. A \textit{filtered space} is a Hausdorff topological space \( X \) together with a sequence of closed subspaces

\[
\{ \emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^{n-1} \subseteq X^n = X \}.
\]

We will generally refer to “the filtered space \( X \)”, leaving the filtration tacit. We will also cease to mention explicitly \( X^{-1} \) since it is always empty (though it is notationally useful to have it available).

The space \( X^i \) is called the \( i \)-skeleton, and the index \( i \) is called the \textit{formal dimension of the skeleton}.\footnote{N.B. We use the phrase “formal dimension” differently from, e.g., Siebenmann \cite{94}, page 127, who defines the formal dimension to be \( \max\{i \mid X^i - X^{i-1} \neq \emptyset \} \). This will be necessary when dealing with subsets; see Remark 2.15 and Section 4.3 for more details. We will often omit the word “formal” when no confusion can arise, especially when formal dimension agrees with topological dimension.} This notion of formal dimension does not necessarily have anything to do with other concepts of dimension, though for most of the spaces we consider below, the skeleton dimension will be the same as the topological dimension. Notice that it is possible to have \( X^i = X^{i-1} \); if \( X = X^n \neq X^{n-1} \), we say that \( X \) has formal dimension \( n \).

We will use throughout the notation \( X_i = X^i - X^{i-1} \). The connected components\footnote{The intuition in this case is similar to that for CW complexes, where we would say that the \( k \)-skeleton of \( X \) contains all cells of dimension \( \leq k \), but the \( k \) and \( k+1 \) skeletons are equal if there are no \( k+1 \) cells.} of \( X_i \) are called the \textit{strata} of \( X \) of formal dimension \( i \). Note that, together, the strata of all codimensions partition \( X \); in fact recall that two points \( x, y \in X \) are defined to be in the same connected component if and only if there is a connected subspace of \( X \) containing both \( x \) and \( y \) \cite[Section 25]{75}. If \( X \) has formal dimension \( n \), and \( S \subset X_{n-k} = X^{n-k} - X^{n-k-1} \) is a stratum of \( X \), we say that \( S \) is a stratum of formal \textit{codimension} \( k \).

Example 2.4. Any finite-dimensional simplicial or CW complex is a filtered space, filtered by its simplicial or cellular skeleta. The strata are, respectively, the open simplices (i.e. the interiors of the simplices) of \( X \) or the interiors of cells of \( X \). CW complexes may satisfy the condition that \( X^i = X^{i-1} \). For example, if we think of the \( n \)-sphere \( X = S^n \) as being composed of one 0-cell and one \( n \)-cell, attached in the unique way to the 0-cell, then \( X^0 = X^1 = \cdots = X^{n-1} \).

Example 2.5. Let \( X \) be a finite simplicial complex filtered by its simplicial skeleta, end let \( p : E \to X \) be a fibration. Then we can filter \( E \) with the filtration \( E^i = p^{-1}(X^i) \).
When we use filtrations for which \( X^i = X^{i-1} \) for some \( i \), it is inconvenient to have to list all the skeleta, so we often employ an abbreviated notation. For example, suppose \( X \) is a filtered space for which \( X^i = \emptyset \) for \( i < 3 \), \( X^3 = X^4 \), and \( X = X^5 \). Then we will write the filtration of the space simply as \( X^3 \subset X^5 \). Note also that the statement \( X = X^5 \) is meant to imply that \( X \) has formal dimension 5; we will continue to use this convention below.

**Example 2.6.** Let \( X = X^5 \) be a 5-dimensional simplicial complex, and let \( X^2 \) be its simplicial 2-skeleton. Then \( X^2 \subset X^5 \) is a filtration of \( X \).

One particular point of care (and possibly of confusion) with the shortened notation is that we could just as well have defined \( X^2 \) to be the simplicial 3-skeleton of \( X \), since the definition of a filtered space does not require that the formal dimension of a skeleton necessarily have any connection with the topological dimension of the space. Then the notation \( X^2 \subset X^5 \) has some ambiguity: does \( X^2 \) refer to the simplicial 2-skeleton or to the formal 2-skeleton of the filtration. In nearly all situations below in which there is a topological notion of dimension available, the topological and formal definitions of a skeleton will coincide, and so the reader is free to trust his or her instincts. When the formal and topological dimensions do not coincide, we will be explicit.

**Example 2.7.** If \( M^m \) is a smooth manifold and \( N^n \) is a closed smooth submanifold of \( M \), we have the filtered space \( N^n \subset M^m \). As in the previous example, and in lieu of statements to the contrary, this notation is taken to mean that \( M \) has formal dimension \( m \) (corresponding to its topological dimension), \( N \) is taken to be the \( n \)-skeleton (as well as the \( k \)-skeleton for all \( n \leq k < m \)), and the skeleta of formal dimension less than \( n \) are empty.

The \( n \)-dimensional strata in this example are the connected components of \( N \), and the \( m \)-dimensional strata are the connected components of \( M - N \). All other strata are empty.

**Example 2.8.** Let \( X = X^n \) be a finite dimensional simplicial complex filtered by its simplicial skeleta \( X^i \). We can form a filtration on the path space \( PX \), which is the space of maps \( \gamma : [0,1] \to X \) with the compact-open topology. We can define a filtration by letting \( (PX)^i = \{ \gamma \in PX \mid \gamma(1) \in X^i \} \). Notice that in this case the skeleton dimension \( i \) of \( (PX)^i \) will usually not correspond to the dimension in any geometric sense since path spaces are usually infinite dimensional. A stratum \( E \) of \( PX \) is a set of paths such that for each \( \gamma_1, \gamma_2 \in E \), \( \gamma_1(1) \) and \( \gamma_2(1) \) lie in the same open simplex of \( X \). In fact, given two such paths, it is not difficult to verify that they are homotopic through paths all of which lie in the same stratum, and so with these filtrations, there is a bijection between strata of \( X \) and strata of \( PX \).

**Example 2.9.** Let \( X = X^2 \) be the union of the open upper half-plane \( \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \) with the \( y \)-axis \( \{(x, y) \in \mathbb{R}^2 \mid x = 0\} \). If we let \( X^1 \) be the \( y \)-axis \( Y \), we can filter \( X \) as \( Y \subset X \). Here the strata are \( Y \) and the two components of \( X - Y \).

**Example 2.10 (Product filtrations).** If \( X, Y \) are filtered spaces of respective formal dimensions \( n, m \), then \( X \times Y \) has a natural filtration of formal dimension \( m + n \) such that \( (X \times Y)^i = \bigcup_{j+k=i} X^j \times Y^k \). In this case, the strata have the form \( S \times T \) where \( S \subset X \) and \( T \subset Y \) are strata of \( X \) and \( Y \), respectively. Naturally, the formal dimension of \( X \times Y \) is the sum of the formal dimensions of \( X \) and \( Y \).
Example 2.11 (Cones). An extremely important way to create new filtered spaces from old ones is by taking cones. If $X$ is a (non-empty) compact filtered space of formal dimension $n - 1$, there is a natural filtration of formal dimension $n$ on the open cone $cX = [0, 1) \times X / \sim$, where $\sim$ is the relation $(w, 0) \sim (z, 0)$ for all $w, z \in Z$. We typically denote the vertex of a cone by $v$. Similarly, the closed cone is $cZ = [0, 1] \times Z / \sim$. More generally, for $r > 0$, we let $c_rZ = [0, r) \times Z / \sim$ and $\bar{c}_rZ = [0, r] \times Z / \sim$; in particular, $cZ = c_1Z$. Then $c_rZ \subset \bar{c}_rZ \subset c_sZ \subset \bar{c}_sZ$ whenever $r < s$.

In the study of filtered spaces, the strata of the highest possible dimension often play an important role; for example, if $X$ is an $n$-dimensional stratified pseudomanifold (see Definition 2.13 below), then $X$ is the closure of the union of its $n$-dimensional strata. Hence the following definition is useful:

Definition 2.12. For a compact space $Z$, $cZ$ denotes the open cone $cZ = [0, 1) \times Z / \sim$, where $\sim$ is the relation $(w, 0) \sim (z, 0)$ for all $w, z \in Z$. We typically denote the vertex of a cone by $v$. Similarly, the closed cone is $cZ = [0, 1] \times Z / \sim$. More generally, for $r > 0$, we let $c_rZ = [0, r) \times Z / \sim$ and $\bar{c}_rZ = [0, r] \times Z / \sim$; in particular, $cZ = c_1Z$. Then $c_rZ \subset \bar{c}_rZ \subset c_sZ \subset \bar{c}_sZ$ whenever $r < s$.

In the study of filtered spaces, the strata of the highest possible dimension often play an important role; for example, if $X$ is an $n$-dimensional stratified pseudomanifold (see Definition 2.13 below), then $X$ is the closure of the union of its $n$-dimensional strata. Hence the following definition is useful:

Definition 2.13. If $X$ is a filtered space of (formal) dimension $n$, the components of $X_n = X^n - X^{n-1}$ are called the regular strata of $X$ and all other strata are called singular strata. We sometimes let $\Sigma_X$ denote the union of the singular strata; $\Sigma_X$ is called the singular set or the singular locus of $X$. Of course $\Sigma_X = X^{n-1}$, but the notation $\Sigma_X$ is a convenient way to refer to the singular locus without explicitly referencing the formal dimension of $X$.

Example 2.14. Let $X$ be an $n$-dimensional simplicial complex, filtered by its simplicial skeleta as in Example 2.4. Then the interiors of the $n$-simplices are the regular strata, and the interiors of all other faces are the singular strata.

Remark 2.15. As we have defined filtered spaces, it is possible for there to be no regular strata. For example, it is allowable within the definitions to have $X$ be homeomorphic to the circle $S^1$ and to have $X^1 = S^1$ but to consider $X$ as having formal dimension 2. In this case $X^2 - X^1 = \emptyset$, and $S^1$ is a stratum of dimension 1. While we do not generally intend to study spaces with no regular strata, unfortunately they become unavoidable (and something of a nuisance) in arguments that require us to consider subsets of more well-behaved spaces.

For example, let $X$ be the disjoint union $X = S^1 \amalg S^2$. A fairly reasonable filtration here would be to let $X^1 = S^1$, $X^2 = S^2$, and to consider $X$ has having formal dimension 2. The problem is that when we want to think of $X^1 = S^1$ as a subspace of $X$, for reasons that will
become clear below, we will want to continue to think of $X^1$ as having codimension one, and
so our subspace will really be a subspace of formal dimension 2 corresponding precisely to
the example of the previous paragraph.

Since there’s a lot of room for confusion here, we encourage the reader for now to treat
all spaces as though they have regular strata (in fact it would not be too problematic on a
first pass through the book to imagine that the regular strata are always dense in $X$). We
will assume that all spaces have this property unless stated explicitly otherwise or required
when working with subspaces.

See Section 4.3 below for more details concerning these issues, especially Remark 4.17.

Working with spaces without regular strata sometimes adds technicalities to results (see,
for example, Theorem 4.12 below), so it is useful to have some language available to recognize
these situations. This is provided by the following definition.

2.2.1 Stratified spaces

From a certain point of view, the space of Example 2.9 is a bit pathological compared to
some of the other examples in that it doesn’t satisfy what is called the Frontier Condition.

Definition 2.16. We say that the filtered set $X$ satisfies the Frontier Condition if for any
two strata $S, S'$ of $X$, $S \cap \overline{S'} \neq \emptyset$ implies that $S \subset \overline{S'}$, where $\overline{S'}$ is the closure of $S'$.

This condition does not hold for Example 2.9 as $Y$ here certainly intersects the closure
of $X - Y$ in $X$ (which is the union of the upper half space and the origin), but it is not
contained in that closure. The frontier condition does not always hold for CW complexes
with their natural filtrations either: Let $X$ be the CW complex obtained by starting with
an interval (with its standard CW structure with two 0-cells and one 1-cell) and attaching
a 2-cell by gluing its boundary to the midpoint of the interval. Then the interiors of the 1-
and 2-cells are strata and the 1-cell intersects the closure of the 2-cell but is not contained
in that closure.

By contrast, the Frontier Condition does hold for our other examples:

1. An embedded $k$-dimensional smooth submanifold of a smooth $n$-manifold is certainly
   contained in the closure of its complement if $k < n$.

2. Each open simplex of a simplicial complex intersects only the closures of the interiors
   of the simplices of which it is a face, and it is then contained in those simplices of
   which it is a face

3. For our path space example, Example 2.8 suppose $\gamma$ is in a stratum $S$ of $PX$ consisting
   of paths with endpoint in the interior of the $i$-simplex $\sigma$ of $X$ and that $\gamma$ is also in
   the closure of another stratum $S'$ of $PX$. Let $\sigma'$ be the $j$-simplex such that $\gamma'(1)$ is
   in the interior of $\sigma'$ if $\gamma' \in S'$. Then there are paths in $S'$ arbitrarily close to $\gamma$
   in the compact-open topology of $PX$. But then we must have that $\sigma$ is a face of $\sigma'$, since
   $\gamma(1)$ must be a limit point of the $\gamma'(1)$. So now to show that $S \subset \overline{S'}$, we need only
   observe that if now $\eta$ is any path in $X$ with $\eta(1)$ in the interior of $\sigma$, then there are
arbitrarily nearby paths \( \eta' \) with \( \eta'(1) \) in the interior of \( \sigma' \) (just extend \( \eta \) a bit into the interior of \( \sigma' \)). Thus \( S \subset \bar{S}' \).

**Definition 2.17.** We say that a filtered space \( X \) is a **stratified space** if it satisfies the Frontier Condition.

**Remark 2.18.** This is not a standard definition in all sources. Other conditions are sometimes required, for example that the collection of strata be locally finite (i.e. that every point of the space has a neighborhood that intersects only finitely many strata). Alternatively, stratified spaces are sometimes defined without reference to a filtration by simply declaring a space to be a disjoint union of a locally finite collection of subsets called strata such that the frontier condition holds and the strata are locally closed\(^7\) see, for example, [60]. As we’ve defined them here, strata are automatically locally closed: the space \( X_i = X^i - X^{i-1} \) is the intersection of the closed subspace \( X^i \) and the open subspace \( X - X^{i-1} \). If \( S \) is a stratum of \( X \) in \( X_i \), then \( S \) is a connected component of \( X_i \) and \( X_i \) can be decomposed as the union of two open subsets of \( X_i \), \( X_i = S \cup (X_i - S) \). But this means that there are two open sets \( U, V \subset X \) (not necessarily disjoint) so that \( S = X_i \cap U \) and \( X_i - S = X_i \cap V \). But then \( S = X^i \cap ((X - X^{i-1}) \cap U) \), so our strata are locally closed.

All of the spaces we will work with below will naturally be filtered, so we will not need this more general definition, and assuming the existence of a filtration makes some statements simpler.

The benefit of working with stratified spaces, rather than general filtered spaces, is that the set of strata possesses some nice structure. The following proposition and its proof are adapted from [18] (though these facts are certainly much older).

**Proposition 2.19.** If \( X \) is a stratified space, then the set of strata of \( X \) is partially ordered by the relation \( S < S' \) if \( S \subset \bar{S}' \). Furthermore, the closure of any stratum is a union of strata of lower dimension, in fact \( \bar{S} = \cup_{S' \prec S} S' \).

**Proof.** Reflexivity of the relation is evident, and transitivity follows from basic topological properties of closure. To demonstrate anti-symmetry, we need to see that if \( S \subset \bar{S}' \) and \( S' \subset \bar{S} \), then \( S = S' \). For this, suppose \( S \subset X_i = X^i - X^{i-1} \) and that \( S' \subset X_j \). Without loss of generality, assume \( i \leq j \). Since \( S \subset X^i \) and \( X^i \) is closed in \( X \), \( \bar{S} \subset X^i \). Furthermore, since \( S \) is a connected component of \( X^i - X^{i-1} \), \( S \) is closed in \( X^i - X^{i-1} \); so if \( x \in X^i - X^{i-1} \) but \( x \notin S \), there is an open set \( U \) of \( X^i - X^{i-1} \) containing \( x \) but not intersecting \( S \). Let \( U' \) be an open set of \( X \) such that \( U' \cap (X - X^{i-1}) = U \), let \( W \) be the open set \( U' \cap (X - X^{i-1}) \). Note that \( x \in W \). Then \( S \) is in the complement of the open set \( W \), and so the closure of \( S \) is also disjoint from \( W \), implying that \( x \notin \bar{S} \). It follows that \( \bar{S} - S \subset X^{i-1} \), and also that \( S = \bar{S} \cap (X - X^{i-1}) \). So if \( S' \subset \bar{S} \), then \( S' - (S' \cap X^{i-1}) \subset S \). But since \( j \geq i \), \( S' \cap X^{i-1} = \emptyset \), so \( S' \subset \bar{S} \). But then \( S' \subset X^i \), so \( i = j \). The same argument then shows that also \( \bar{S} \subset S' \), so \( S = S' \).

\(^7\)Recall that a subset \( Z \subset X \) is **locally closed** if it is the intersection of an open set in \( X \) and a closed set in \( X \).
Finally, it is clear from the definitions that $\cup_{S \prec S'} S' \subset \bar{S}$. Now suppose $x \in \bar{S}$. Then since the strata partition $X$, $x$ is in some stratum $S'$ and by the Frontier Condition, $S' \subset \bar{S}$ and so $S' \prec S$. So $x \in \cup_{S \prec S'} S'$, and therefore $\bar{S} \subset \cup_{S \prec S'} S'$.

Example 2.20. Let $X$ be a simplicial complex filtered by its simplicial skeleta. Then the strata are open simplices, and $s \prec s'$ if and only if $s$ is an open face of the closed simplex $\bar{s}'$, so $X$ is a stratified space. In fact, the closure of an open simplex is the disjoint union of all its open faces.

**Manifold stratified spaces.** Our preferred stratified spaces will be those all of whose strata are manifolds.

**Definition 2.21.** A manifold stratified space is a stratified space all of whose $i$-dimensional strata are $i$-dimensional manifolds.

**Example 2.22.** A finite dimensional simplicial complex filtered as in Example 2.11 is a manifold stratified space. Its strata are the open faces of its simplices, each of which is homeomorphic to some Euclidean space.

**Example 2.23.** An (topological) $m$-dimensional manifold $M$ with the trivial filtration (i.e. the filtration whose only strata are the components of $M$ in dimension $m$) is naturally an $m$-dimensional manifold stratified space. If $M^n$ is a smooth manifold with smooth closed submanifold $N^n \subset M^m$, $n < m$, then $M$ with the filtration $N \subset M$ is naturally an $m$-dimensional manifold stratified space whose $n$-dimensional strata are the components of $N$ and whose $m$ dimensional strata are the components of $M - N$.

**Example 2.24 (Product filtrations).** If $X$ is a manifold stratified space and $M$ is an $m$-dimensional manifold, then $M \times X$ has a natural structure as a manifold stratified space with skeleta $(M \times X)^i = M \times X^{i-m}$. Clearly each stratum of $M \times X$ has the form $M \times S$ for some stratum $S \subset X$ and so is also a manifold, and it is easy to verify the Frontier Condition for $M \times X$ using that the Frontier condition holds on $X$.

**Example 2.25 (Cones).** If $X$ is a manifold stratified space, there is a natural manifold stratified space structure on the open cone $cX = [0,1) \times Y$ where $\sim$ is the relation $(x,0) \sim (y,0)$ for all $x,y \in X$. We define the filtration on $cX$ as in Example 2.11 so that $(cX)^0$ is the cone point (represented in the quotient by any point $(x,0)$), and so that for $i > 0$, $(cX)^i = (0,1) \times X^{i-1}$. The strata are then the cone point and the products of the strata of $X$ with the interval, each of which is a manifold if $X$ is manifold stratified. Note that the Frontier Condition holds on $cX$ as a consequence of it holding for $X$ and that the cone point is in the closure of every non-empty stratum.

Similarly, the suspension of a manifold stratified space is a stratified space.

**Remark 2.26.** Unless stated otherwise or when working with subspaces, we will assume that if $X$ is an $n$-dimensional manifold stratified space then $X^n - X^{n-1}$ is non-empty so that $X$ possesses regular strata. Unfortunately, as mentioned in Remark 2.15, it will be necessary within some arguments to utilize manifold stratified spaces without regular strata.

See Section 4.3, especially Remark 4.17, for more details.
Depth. For running induction arguments, it is helpful to have the notion of depth, which is a measure of how many layers of strata there are in a stratified space. Before giving the definition, let us give some motivating examples.

Example 2.27. Suppose $X$ is a stratified space of formal dimension $n$ such that $X^i = \emptyset$ for all $i < n$. In this case, we often say that $X$ is unfiltered or has the trivial filtration, and we define its depth to be 0. Similarly, if we have a space as in Remark 2.15 such $X$ has formal dimension $n$ and is such that $X^i = \emptyset$ for all $i < m$ and $X^m = X^{m+1} = \cdots = X^n$, then again there is only one dimension possessing non-empty strata (namely dimension $m$), and so we say that $X$ has depth 0.

If we have a stratified space $X^k \subset X^n$, $k \neq n$, meaning that $X^i = \emptyset$ for $i < k$, $X^i = X^k$ for $k \leq i < n$, and $X^{n-1} \neq X^n$, then we say that $X$ has depth 1. The stratified space in Examples 2.7 has depth 1.

Building on these examples, we present the formal definition:

Definition 2.28. Let $S$ be a stratum of the stratified space $X$. Suppose that $S = S_d < S_{d-1} < S_{d-2} < \cdots < S_0$, where $<$ is the partial ordering of the strata in $X$, $S_i \neq S_j$ for $i \neq j$, and this is the (not necessarily unique) longest such chain of strata containing $S$ as its minimal element. Then we call $d$ the depth of $S$.

The depth of a stratified space $X$ is defined to be the maximum of the depth of its strata. This is well-defined as all filtered spaces are assumed to have finite formal dimension.

Example 2.29. If $X$ is an $n$-dimensional simplex filtered by its simplicial skeleta, then each open $i$-face has depth $n - i$, and $X$ has depth $n$.

If $X$ is filtered as $N \subset M = X$, where $N$ is a smooth nonempty $n$-dimensional submanifold of a smooth $m$-manifold $M$, $n < m$, then the stratum $N$ has depth 1, the stratum $M - N$ has depth 0, and $X$ has depth 1.

If $X$ is the disjoint union of spheres $S^2$ and $S^3$, filtered by $S^2 \subset S^3$, each each stratum has dimension 0 and $X$ has dimension 0.

Suppose $X$ is the simplicial complex $\Delta^2 \vee \Delta^1$, with the two simplices attached at a common vertex, filtered by its simplicial skeleta. Each vertex of $\Delta^2$ has depth 2, while the vertex of $\Delta^1$ that is not shared with $\Delta^2$ has depth 1. The stratified space $X$ has depth 2.

Notice that the formal dimension of $X$ is irrelevant for considerations of depth.

Remark 2.30. There are other reasonable definitions of depth. This one will work well for us. In [94], Siebenmann defines depth of a stratified space $X$ to be

$$\sup\{m - n \mid X^m - X^{m-1} \neq \emptyset \neq X^n - X^{n-1}\}.$$

It is also sometimes useful to define depth to be one less than the number of distinct non-negative formal dimensions such that $X$ has a nonempty stratum of that dimension.

Both of these alternative definitions have the benefit that they can be applied to arbitrary filtered spaces without requiring a stratification.
2.3 Locally conelike spaces and CS sets

So at this point we have manifold stratified spaces: spaces that decompose into partially ordered sets of strata such that each stratum is a manifold. It turns out that even this is too general a setting to expect nice results. For example, consider a wild arc embedded in \( \mathbb{R}^3 \). PICTURE. The arc itself is a 1-manifold, and its complement is a 3-manifold, but this example is nonetheless somewhat pathological; for example, the arc cannot have anything like a tubular neighborhood in \( \mathbb{R}^3 \). To avoid these sorts of issues, we refine our spaces of interest yet further. To avoid wild arcs in topological knot theory, one usually imposes a condition of \textit{local flatness} by requiring that each point in the embedded arc \( K \) have a neighborhood \( (U, U \cap K) \) that is homeomorphic to the standard pair \( (\mathbb{R}^3, \mathbb{R}) \). The following definition does something analogous for stratified spaces by imposing a local topological structure at each point.

\textbf{Definition 2.31.} A filtered space \( X^n \) is \textit{locally cone-like} if for each \( x \in X_i \) there is an open neighborhood \( U \) of \( x \) in \( X_i \), a neighborhood \( N \) of \( x \) in \( X \), a compact filtered space \( L \) (which may be empty), and a homeomorphism \( h : U \times cL \to N \) such that \( h(U \times c(L^k)) = X^{i+k+1} \cap N \). In this case \( L \) is called a \textit{link} of \( x \) and \( N \) is called a \textit{distinguished neighborhood} of \( x \). For a given \( x \), the spaces \( L \) and \( N \) are not necessarily unique (see Example 2.36, below).

Locally cone-like manifold stratified spaces are called \textit{CS sets}; see \cite{94, 61}. In this case every distinguished neighborhood \( N \cong U \times c(L^k) \) of \( x \) contains a possibly smaller distinguished neighborhood homeomorphic to \( \mathbb{R}^i \times c(L^k) \), so we will generally restrict attention to distinguished neighborhoods of this form.

Notice that the condition \( h(U \times c(L^k)) = X^{i+k+1} \cap N \) is consistent with the product and cone filtrations introduced in Examples 2.10 and 2.11. Since \( U \) is an \( i \)-dimensional stratum, we expect \( U \times c(L^k) \) to have dimension \( i + k + 1 \). When we have such a homeomorphism of spaces that takes the \( i \)-skeleton of the domain onto the \( i \)-skeleton of the target for all \( i \), we say that the homeomorphism is \textit{filtration preserving}. These definitions remain consistent when \( L \) is the empty set: by definition filtration dimensions always range from \(-1 \) (which can apply only to the empty set) to the formal dimension of the space; hence always \( i + k + 1 \geq i \), and when \( L \) is empty, \( U \times c(L^k) = U \) is a subset of the \( i + k + 1 \) skeleton as \( i + k + 1 \) ranges from \( i \) to the dimension of \( X \).

The locally cone-like condition functions as a sort of homogeneity condition for filtered spaces. Unlike manifolds, which are completely homogeneous in the sense that any two points have homeomorphic neighborhoods, two points of a locally cone-like filtered space do not have to have homeomorphic neighborhoods, especially if they are contained in different strata. However, two points in the same stratum and sufficiently close together to be contained in the same distinguished neighborhood will have homeomorphic neighborhoods.

\textbf{Example 2.32.} Let \( M \) be a compact connected manifold of dimension \( n - 1 \). Suppose we filter the suspension \( X = SM \) so that \( X^0 = \{ N, S \} \) consists of the cone points of the suspension, and the full filtration is \( \{ N, S \} \subset SM \). This is a CS set: \( X - \{ N, S \} \) is the manifold \( \mathbb{R} \times M \),

\footnotesize{\textsuperscript{8}In these sources CS sets are assumed to be metrizable but we will not need this here.}
and so each point in this stratum has a distinguished neighborhood of the form $\mathbb{R}^n \times c\emptyset = \mathbb{R}^n$. The strata $\{N\}$ and $\{S\}$ are each 0-manifolds and each has a distinguished neighborhood homeomorphic to $\mathbb{R}^0 \times cM = cM$. So the points $N$ and $S$ each have links homeomorphic to $M$.

If we consider the space $Y = S^1 \times SM$, filtered by the product filtration as in Example 2.10, then again we obtain a CS set with manifold strata $Y - S^1 \times \{N, S\} \cong S^1 \times \mathbb{R} \times M$, $S^1 \times \{N\}$ and $S^1 \times \{S\}$. Points in $Y - S^1 \times \{N, S\}$ have distinguished neighborhoods filtered homeomorphic to $\mathbb{R}^{n+1} \times c\emptyset$, while points in the other strata have distinguished neighborhoods filtered homeomorphic to $\mathbb{R}^1 \times cM$.

**Example 2.33.** Here is an example to demonstrate why it is not possible in a CS set to refer to “the” link of a point: it is possible for two non-homeomorphic spaces to have homeomorphic cones. This example comes from the famous double suspension theorem [14] which states that a double suspension of a homology sphere is homeomorphic to a sphere. Let $\Sigma$ be a homology sphere; then $\Sigma$ and $\Sigma^\prime$ must be a closed set in $X$. Then there is a neighborhood $U$ of $x$ in $X$ and a neighborhood $N$ of $x$ in $X$ such that $N \cong U \times cL$ for some compact filtered link $L$. The link $L$ cannot be empty or $x$ could not be in the closure of $\bar{S}^i$. Note that the strata of $N$ that do not intersect $S$ all have the form $(U \times cs) - U$ for some stratum $s$ of $L$, though some of these strata may be parts...
of the same stratum of $X$ that “join up” outside of $N$. In particular $\tilde{S}' \cap N$ is the union of some strata $(U \times cs) - U$ of $N$, and the closure of these strata of $N$ contains all of $U$. Thus $U \subset S \cap \tilde{S}'$, and $U$ is open in $S$. Since each point of $S \cap \tilde{S}'$ has such an open neighborhood in $S$, this shows that $S \cap \tilde{S}'$ is open in $S$.

A common strengthening of the definition of a CS set requires that each link also be a CS set (and hence that the links of the links be CS sets, and so on). We will refer to spaces with this stronger condition as recursive CS sets.

Siebenmann intentionally did not assume that the links of a CS set be themselves CS sets; see the second remark on page 128 of [94]. However, it is common in many sources (e.g. [60, 18]) to include such a condition, making the definition inductive over dimension: 0-dimensional CS sets are discrete sets of points, and then the definition of $n$-dimensional CS sets for $n > 0$ involves CS sets of dimension $0 \leq k < n$, which play the role of the links. This setting can be useful for making inductive arguments. However, unless stated explicitly, we assume CS sets are not necessarily recursive.

Example 2.35. The CS sets of Examples 2.32 and 2.33 were both recursive CS sets, since all links were manifolds or suspensions of manifolds.

Example 2.36. In the introduction to [94], Siebenmann provides an example of a CS set that is not evidently a recursive CS set. This is a compact non-manifold $X$ such that $X \times \mathbb{R} \cong S^3 \times \mathbb{R}$. It follows that the cone on $X$ is a CS set with two non-trivial skeleta — the cone point and $X \times \mathbb{R}$. But the obvious link of the cone vertex is $X$, which we know is not a manifold. Siebenmann notes that conjecturally it might be possible to find a manifold link that provides $cX$ the structure of a recursive CS set, but the question is not settled there.

Lemma 2.37. Let $X$ be a CS set.

1. The stratification of $X$ is locally finite, i.e. every point has a neighborhood intersecting only finitely many strata of $X$.

2. If $X$ is a compact CS set then it possesses a finite number of strata.

Proof. The second statement follows easily from the first: as every point of $X$ has a neighborhood that intersects only finitely many strata and as $X$ is covered by a finite number of such neighborhoods, $X$ has only finitely many strata.

To prove the second statement, we will argue by contradiction. Suppose $x \in X$ is a point such that every neighborhood of $x$ intersects infinitely many strata. Let $U$ be a neighborhood of $x$ that is filtration-preserving homeomorphic to $\mathbb{R}^i \times cL$ for some compact filtered space $L$, and let $h : U \to \mathbb{R}^i \times cL$ be the homeomorphism. Then $U$ intersects infinitely many strata of $X$, say the indexed collection $\{S_\alpha\}$. We note that, for a given $\alpha$, $S_\alpha \cap U$ might no longer be connected and so is not necessarily a stratum of $U$ (by analogy, think of the stratification of $S^1$ as $\{x_0\} \subset S^1$ for any $x_0 \in S^1$; then $S^1 - \{x_0\}$ is a single stratum that intersects any distinguished neighborhood of $x_0$ in two components). However, each $S_\alpha \cap U$ will be a union

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9While this notion is standard in the literature, this is not standard nomenclature. Authors who desire this condition usually simply define “CS sets” this way.
of strata of $U$, and as the collection $\{S_\alpha\}$ is infinite, so will be the disjoint collection of sets $\{S_\alpha \cap U\}$. Let $T_\alpha = S_\alpha \cap U$.

Next, consider $\mathbb{R}^i \times cL$. Due to the conical structure and the requirement that the homeomorphism $h$ preserve the filtration, we see that, aside from the stratum $\mathbb{R}^1 \times \{v\}$, where $v$ is the cone vertex, all the strata of $\mathbb{R}^i \times cL$ have the form $\mathbb{R}^i \times (cS - \{v\})$, where $S$ is a stratum of $L$. In particular, each set $h(T_\alpha)$ must be a union of sets of this form. Since $h(T_\alpha)$ is a union of strata of $\mathbb{R}^i \times cL$ and there are an infinite number of $T_\alpha$, it follows that $L$ must itself possess infinitely many strata. Let us identify $L$ with the subspace $\{0\} \times \{1/2\} \times L \subset \mathbb{R}^2 \times cL$, where (abusing notation) $\{1/2\} \times L$ denotes the image of $\{1/2\} \times L \subset [0,1) \times L$ under the quotient map to $cL$. Let $\{y_\alpha\}$ be an infinite set of points such that $y_\alpha \in L \cap h(T_\alpha)$. Then the compactness of $L$ implies that $\{y_\alpha\}$ must have a limit point, say $y$. But now back in $X$, $h^{-1}(y)$ must be a limit point of the $h^{-1}(y_\alpha)$, and, by construction, $h^{-1}(y_\alpha)$ is contained in the stratum $S_\alpha$. So every neighborhood of $h^{-1}(y)$ also intersects infinitely many strata. But again from the construction, we see that $h^{-1}(y)$ must be contained in a stratum of $X$ of higher dimension than the stratum containing $x$.

We have shown that if $x \in X$ is a point such that every neighborhood of $x$ intersects infinitely many strata and if $x$ is contained in the stratum $S$, then there is a point $x' \in X$ contained in a higher-dimensional stratum $T$ such that $S \leq T$ and such that $x'$ also has this property. But since we assume all of our spaces are finite dimensional, this eventually leads to a contradiction. If we use the procedure of the preceding paragraph to construct a sequence of points with this property, each contained in a stratum of higher dimension, eventually we wind up with a point $z$ in a stratum $M$ of maximal dimension, meaning that there is no stratum $S$ of $X$ such that $M \leq X$. So sufficiently small neighborhoods of $z$ intersect only the stratum $M$, a contradiction.

**Example 2.38.** Here is an example of a compact manifold stratified set that is not locally cone-like and possesses an infinite number of strata. For example, Let $X$ be $\{x \in \mathbb{R} \mid x = \frac{1}{n}, n \in \mathbb{Z}\} \cup \{0\}$ endowed with the trivial filtration with a single skeleton $X^0$. The connected components of $X$ are the points of $X$, which are each embedded 0-manifolds, and the Frontier Condition holds, but no neighborhood of the point $\{0\}$ is homeomorphic to a cone.

The following lemma contains the useful fact that an open subset of a CS set is a CS set, and similarly for recursive CS sets.

**Lemma 2.39.** If $X$ is a (recursive) CS set and $V \subset X$ is an open subspace filtered by $V^i = V \cap X^i$, then $V$ is a (recursive) CS set.

**Proof.** It is only necessary to show the $V$ is locally cone-like. It will then follow from Lemma 2.34 that $V$ is a stratified space, and since any open subset of a manifold is a manifold, it will be manifold stratified.

Suppose $x \in V_i$. By assumption, $x$ has a neighborhood $N$ filtered homeomorphic to $\mathbb{R}^i \times cL$ in $X$. Let $U = N \cap V \cap X_i$. Then $U$ is a neighborhood of $x$ in $X_i$, and since $U$ can be identified with an open subset of $\mathbb{R}^i$, we can choose a set $D^i$ such that $x \in D^i \subset U$, $D^i \cong \mathbb{R}^i$, and $D^i$ is compact. Now consider $D^i \times cL \subset \mathbb{R}^i \times cL$. The idea now is that if $cL = [0,1) \times L/\sim$ then there is some $t \in (0,1)$ such that the subcone $c_tL = [0,t) \times L/\sim \subset cL$.
satisfies $D^i \times c_L \subset V \cap (\mathbb{R}^i \times cL)$. Since $V \cap N$ is an open neighborhood of $\bar{D}^i$, by the Tube Lemma [78, Lemma 26.8], $\bar{D}^i$ has a neighborhood in $V \cap N$ of the form $\bar{D}^i \times W$, where $W$ is an open neighborhood of the vertex of $cL$. But now again applying the Tube Lemma and the definition of the quotient topology, we see that we can find a $c_L$ such that $c_L \subset W$. Then $D^i \times c_L$ is a neighborhood of $x$ in $N \cap V$.

Since $X$ and $V$ use the same links, if $X$ is recursive, so is $V$. 

Remark 2.40. Notice that in our applications of the Tube Lemma in the proof of the preceding lemma we have made critical use of the assumption that links are compact.

We close this section with some observations about the point-set topology of CS sets that will be needed below.

**Lemma 2.41.** CS sets are locally compact.

**Proof.** If $X$ is a CS set, then, by definition, every point $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^k \times cL$, where $L$ is a compact space and the image of $x$ under the homeomorphism has the form $(z,v)$ with $z \in \mathbb{R}^k$ and $v$ the cone point of $cL$. Let $D$ be the closed disk of radius 1 about $z$ in $\mathbb{R}^k$, and let $\bar{c}_r Z$ be as in Definition 2.12 for $0 < r < 1$. Then $D \times \bar{c}_r Z$ is a compact neighborhood of $(z,v)$ in $\mathbb{R}^k \times cL$, and it follows that $x$ has a compact neighborhood in $X$.

**Corollary 2.42.** CS sets are completely regular. In particular, they are regular.

**Proof.** It is a general fact of point-set topology that locally compact Hausdorff spaces are completely regular, and so regular. See [110, Theorem 19.3 and Definition 14.8].

**Corollary 2.43.** If $X$ is a CS set, $Z_1 \subset X$ is compact, $Z_2 \subset X$ is closed, and $Z_1 \cap Z_2 = \emptyset$, then there are disjoint open subspaces $U_1, U_2 \subset X$ such that $Z_1 \subset U_1$ and $Z_2 \subset U_2$.

**Proof.** Suppose $x \in Z_1$, and notice that $Z_1$ is contained in the open subset $X - Z_2$. As $X$ is locally compact Hausdorff, there is a neighborhood $V_x$ of $x$ in $X - Z_2$ such that $\bar{V}_x \subset X - Z_2$ [78, Theorem 29.2]. As $x$ ranges over the elements of $Z_1$, the $V_x$ provide an open cover of $Z_1$; as $Z_1$ is compact, there is a finite subcover $\{V_{x_i}\}_{i=1}^m$. Then $U_1 = \bigcup_i V_{x_i}$ is an open subset of $X$ containing $Z_1$, while $U_2 = \bigcap_i (X - \bar{V}_{x_i})$ is an open subset of $X$ containing $Z_2$ and $U_1 \cap U_2 = \emptyset$.

**Corollary 2.44.** If $X$ is a CS set and $K \subset W \subset X$ with $K$ compact and $W$ open, then there is a neighborhood $V$ of $K$ in $X$ with $\bar{V} \subset W$.

**Proof.** This follows from the preceding corollary by letting $Z_1 = K$, $Z_2 = X - W$, and $V = \bar{U}_1$.

\[\text{[10] We abuse notation by identifying } N \text{ identically with } \mathbb{R}^i \times cL \text{ so that this expression makes sense.}\]
2.4 Pseudomanifolds

We now arrive at the definition of a stratified pseudomanifold; these are the spaces that we shall eventually show possess an intersection homology version of Poincaré duality. Stratified pseudomanifolds are a special kind of recursive CS set. The additional idea is that a pseudomanifold should have a sort of dimensional homogeneity. To illustrate the idea, suppose $M^m$ and $N^n$ are compact manifolds of different dimensions, and consider the cone on the disjoint union $X = c(M \amalg N)$. It is easy to verify that this is a manifold stratified space, and in fact a recursive CS set. However, $X$ is essentially made up of two pieces of different dimensions; there are points whose neighborhoods are homeomorphic to $\mathbb{R}^{m+1}$ and others whose neighborhoods are homeomorphic to $\mathbb{R}^{n+1}$. What dimension could a fundamental class be? The definition of a stratified pseudomanifold is designed to avoid this sort of problem.

**Definition 2.45.** An $n$-dimensional recursive CS space $X^n$ is a (topological) stratified pseudomanifold if $X_n = X^n - X^{n-1}$ is dense in $X$.

A space is called simply a pseudomanifold if it possesses a filtration with respect to which it is a stratified pseudomanifold.

**Remark 2.46.** It is much more common throughout the literature to also assume that a pseudomanifold must satisfy $X^{n-1} = X^{n-2}$, i.e. that $X$ not have any codimension one strata. It will be useful for us not to assume this. We will refer to a stratified pseudomanifold such that $X^{n-1} = X^{n-2}$ as a classical stratified pseudomanifold. A space is called a classical pseudomanifold if it possesses a filtration with respect to which it is a classical stratified pseudomanifold.

It is also often part of the definition to assume that the links of a stratified pseudomanifolds must be stratified pseudomanifolds, however we will show in Lemma 2.54 that this follows automatically from our definition.

**Remark 2.47.** Notice that, by definition, a stratified pseudomanifold always has regular strata, cf. Remarks 2.15 and 2.26.

The following lemma seems somewhat obvious, although Example 2.38 shows that some care is necessary.

**Lemma 2.48.** Suppose $X$ is an $n$-dimensional stratified pseudomanifold. Then the top stratum $X_n = X^n - X^{n-1}$ is homeomorphic to a disjoint union of connected $n$-manifolds.

**Proof.** By definition, each connected component of the open set $X_n$ is a manifold. Let $X_n = \amalg S_i$ be the set-wise decomposition of $X_n$ into strata. We must show that a set $W$ is open in $X_n$ if and only if its restriction to each $S_i$ is open $S_i$. One direction is trivial: if $W$ is open in $X_n$, then $W \cap S_i$ is open in $S_i$ by definition of the subspace topology on $S_i$. Conversely, suppose $W \cap S_i$ is an open set in $S_i$ and suppose $x \in W \cap S_i$. From the definitions, $x$ must have a distinguished neighborhood $N_x$ in $X$ filtered homeomorphic to $U \times cL$, where $U$ is an open neighborhood of $x$ in $S_i$, and since $S_i$ is already a top dimensional stratum, it follows that $L = \emptyset$. Using the Euclidean topology of the distinguished neighborhood, any smaller Euclidean neighborhood of $x$ in $N$ is also a neighborhood of $x$ in $X$ and in $S_i$. Since
$W \cap S_i$ is an open set in $S_i$ in its subspace topology as a manifold, we can then choose a Euclidean neighborhood $N'_x$ of $x$ such that $N'_x$ is a neighborhood of $x$ in $X$ and $N'_x \subset W \cap S_i$. Taking the union of these neighborhoods over all points in $W$ provides an open set in $X$.

Since the disjoint union of manifolds is also a manifold, we thus see that an $n$-dimensional stratified pseudomanifold $X$ is the closure of an $n$-dimensional manifold contained in $X$.

**Definition 2.49.** If $X$ is an $n$-dimensional stratified pseudomanifold, the components of $X_n$ are called the regular strata of $X$ and all other strata are called singular strata. Once again, we sometimes refer to $X^{n-1}$ as the singular locus and refer to it as $\Sigma_X$, since this notation has the benefit of not explicitly requiring the dimension to be stated. This notation reinforces the idea that a stratified pseudomanifold is often thought of as a manifold with some “bad” points. However, this notation can also be somewhat misleading in that the “singular strata” do not necessarily consist of topological singularities. For example, a smooth submanifold $N$ of codimension $> 0$ in a smooth manifold $M$ will be designated a singular stratum if we filter $M$ as $N \subset M$.

**Example 2.50.** The suspension of a compact manifold is a stratified pseudomanifold, as are the other stratified spaces in Example 2.32.

In fact, if we begin with a compact manifold and then engage in any finite iterated process of suspensions and taking products with other compact manifolds, the resulting space will be a stratified pseudomanifold.

**Example 2.51.** Our next example is somewhat controversial as it does not yield a classical pseudomanifold: Let $M^n$ be a compact $n$-manifold with boundary $\partial M$, and consider $M$ filtered by $\partial M \subset M$. Then $M$ is a stratified pseudomanifold. The interior points of $M$ have distinguished neighborhoods of the form $\mathbb{R}^n \times c\emptyset$ and the boundary points have distinguished neighborhoods of the form $\mathbb{R}^{n-1} \times cL$, where $L$ is a single point. Note that this is not a classical stratified pseudomanifold because the components of $\partial M$ are strata of codimension one.

**Remark 2.52.** The last example also illustrates the important point that the choice of filtration is critical. If $M$ is considered as a space with the trivial filtration $\emptyset \subset M$, then $M$ is not a stratified pseudomanifold; in fact it is not even a manifold stratified space as $M$ is not a manifold (manifolds with boundary are technically not manifolds!).

**Lemma 2.53.** An open subset $U$ of an $n$-dimensional stratified pseudomanifold $X$ filtered by $U^i = U \cap X^i$ is an $n$-dimensional stratified pseudomanifold.

**Proof.** By Lemma 2.39 it is only necessary to verify that $U_n$ is dense in $U$. But if $x \in U_i$, then every sufficiently small neighborhood of $x$ in $X$ is contained in $U$ since $U$ is open, but also since $X_n$ is dense in $X$, each such neighborhood intersects $X_n$. So every neighborhood of $x$ intersects $U_n$.

**Lemma 2.54.** If $X^n$ is a stratified pseudomanifold, its links are also stratified pseudomanifolds. Furthermore, if $X^n$ is a classical stratified pseudomanifold, so are its links.
Proof. Suppose $L$ is a link of a stratified pseudomanifold $X$; in other words there is a point $x \in X_i$ such that $x$ has a distinguished neighborhood $N$ filtered homeomorphic to $\mathbb{R}^i \times cL$ with $L = L^{n-i-1}$ a compact CS set (since $X$ is recursive by definition). Therefore we need only show that the regular stratum of $L$ is dense in $L$. Suppose $y \in L$ is a point such that $y$ has a neighborhood $U$ that does not intersect $L_{n-i-1}$. If we identify $N \subset X$ with $\mathbb{R}^i \times cL$ and think of $L$ as embedded as, say, $\{x\} \times \{1/2\} \times L \subset \mathbb{R}^i \times cL$, then $\mathbb{R}^i \times (0,1) \times U$ is a neighborhood of the image of $y$ in $N$ that does not intersect $\mathbb{R}^i \times cL_{n-i-1}$. But $\mathbb{R}^i \times cL_{n-i-1} \cong N \cap X_n$ under the filtered homeomorphism. So the image of $y$ in $X$ has a neighborhood that does not intersect $X_n$, which contradicts $X$ being a stratified pseudomanifold. Hence $L_{n-i-1}$ is dense in $L$.

For the claim about classical stratified pseudomanifolds, we need only notice that if $L = L^{n-i-1}$ has a codimension one stratum $L^{n-i-2} - L^{n-i-3}$, then $\mathbb{R}^i \times (cL - \{v\}) \cong \mathbb{R}^{i+1} \times L$ has a stratum $\mathbb{R}^{i+1} \times (L^{n-i-2} - L^{n-i-3})$ of dimension $n-1$. If so, then $X$ cannot be a classical pseudomanifold.

\begin{remark}
Following on from the proof of the preceding lemma, let $L$ be a link of a point $x \in X_i$ for the stratified pseudomanifold $X$, and let $\ell$ be a link of $L$. So $L$ has a point $y$ in some $L_k$ such that $y$ has a distinguished neighborhood $\mathbb{R}^k \times c\ell$. Then as observed in the proof of the lemma, the image of $y$ under an embedding of $L$ within a distinguished neighborhood in $X$ has a neighborhood of the form $\mathbb{R}^i \times (0,1) \times \mathbb{R}^k \times c\ell \cong \mathbb{R}^{i+k+1} \times c\ell$. These homeomorphisms preserve the filtrations, and this demonstrates that $\ell$ is a link of $y$ in $X$. In other words, a link in a link of a stratified pseudomanifold is a link in the stratified pseudomanifold.
\end{remark}

Given Lemma \ref{lem:filtered_homeomorphism}, it is natural to formulate an alternative definition of topological stratified pseudomanifolds that does not directly refer to CS sets; this version of the definition is common in the literature (e.g., c.f. \cite{43}). The definition is recursive on dimension.

\begin{definition}[Alternative definition of stratified pseudomanifold] A 0-dimensional (topological) stratified pseudomanifold is a discrete set of points.

For $n > 0$, an $n$-dimensional (topological) stratified pseudomanifold $X^n$ is an $n$-dimensional filtered space such that

1. Each connected component of $X^i - X^{i-1}$ is an $i$-dimensional manifold,

2. $X_n = X^n - X^{n-1}$ is dense in $X$,

3. for all $i$ and for each $x \in X_i$, there is an open neighborhood $U$ of $x$ in $X_i$, a neighborhood $N$ of $x$ in $X$, a compact stratified pseudomanifold $L$ (which may be empty), and a homeomorphism $h : U \times cL \to N$ such that $h(U \times c(L^k)) = X^{i+k+1} \cap N$.

We call an $n$-dimensional (topological) stratified pseudomanifold classical if $X^{n-1} = X^{n-2}$.

\end{definition}
2.4.1 Piecewise linear and simplicial pseudomanifolds.

Recall that a space $X$ is called piecewise linear or PL if it can be triangulated, that is it is homeomorphic to a locally finite simplicial complex NEED REF. We think of such an $X$ as endowed with a class of admissible triangulations that are compatible in the sense that any pair of admissible triangulations has a common admissible subdivision$^{11}$ and any subdivision of an admissible triangulation is admissible.

A PL space is a filtered PL space if $X$ has an admissible triangulation such that each skeleton of $X$ is a simplicial subcomplex. Similarly, we can define PL stratified spaces, PL manifold stratified spaces, PL CS spaces, etc. All the definitions remain the same with the additional requirements that the filtration should be compatible with the PL structure, that all manifolds should be PL manifolds, and that all structural homeomorphisms should be PL homeomorphisms.

For example, here is the full definition of a PL stratified pseudomanifold.

**Definition 2.57.** A PL stratified pseudomanifold $X$ is a (topological) stratified pseudomanifold such that

1. $X$ is a filtered PL space,
2. the strata of $X$ are PL manifolds$^{12}$
3. each point $x$ has a distinguished neighborhood $N \cong \mathbb{R}^i \times cL$ such that the link $L$ is a recursive PL CS set and the filtered homeomorphism $N \rightarrow \mathbb{R}^i \times cL$ is piecewise linear (i.e. it is simplicial with respect to some admissible triangulations of the spaces).

If $X$ is a PL stratified pseudomanifold that has been given a fixed triangulation (compatible with the PL structure) such that each $X^i$ is triangulated as a subcomplex, we will call $X$ a simplicial stratified pseudomanifold.

We will call $X$ a PL pseudomanifold if it possesses some filtration with respect to which it is a PL stratified pseudomanifold.

**Remark 2.58.** Again, our definition is uncommon in that it is usually assumed for a PL stratified pseudomanifold of dimension $n$ that $X^{n-1} = X^{n-2}$. Once again we will refer to PL stratified pseudomanifolds satisfying this condition as classical.

**Remark 2.59.** By the same argument as in Lemma 2.54, the links of a PL stratified pseudomanifold will themselves be PL stratified pseudomanifolds, and the links of a classical PL stratified pseudomanifold will be classical PL stratified pseudomanifolds.

Once again, it is nice to have a direct definition that does not refer directly to CS sets, at the expense of the definition becoming recursive:

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$^{11}$We assume the reader is familiar with the basic notions of simplicial complexes and subdivision as can be found, for example, in [77, 97].

$^{12}$Recall that this means that each point has a neighborhood PL homeomorphic to Euclidean space - see [86] for more details.
Definition 2.60 (Alternative definition of PL stratified pseudomanifold). A 0-dimensional PL stratified pseudomanifold is a discrete set of points.

For \( n > 0 \), an \( n \)-dimensional PL stratified pseudomanifold \( X^n \) is an \( n \)-dimensional filtered PL space such that

1. Each connected component of \( X^i - X^{i-1} \) is an \( i \)-dimensional PL manifold,
2. \( X_n = X^n - X^{n-1} \) is dense in \( X \),
3. for all \( i \) and for each \( x \in X_i \), there is an open neighborhood \( U \) of \( x \) in \( X_i \), a neighborhood \( N \) of \( x \) in \( X \), a compact PL stratified pseudomanifold \( L \) (which may be empty), and a PL homeomorphism \( h : U \times cL \to N \) such that \( h(U \times c(L^k)) = X^{i+k+1} \cap N \).

We call an \( n \)-dimensional PL stratified pseudomanifold classical if \( X^{n-1} = X^{n-2} \).

In a number of ways, PL spaces are more nicely behaved than their topological counterparts, which is not surprising given their additional structure. The next few results illustrate some of these niceties.

Lemma 2.61. Every finite dimensional PL space has a filtration with respect to which it is a PL CS space.

Proof. Let \( X \) be a PL space, and fix an admissible triangulation of \( X \) by a locally-finite simplicial complex \( K \); in particular, we have a homeomorphism \( h : K \to X \). Let \( K^i \) be the simplicial skeleta of \( K \). We claim the images \( X^i = h(K^i) \) provide a CS space filtration of \( X \). Indeed, the strata of \( X^i \) are then the images of the interiors of simplices of \( K \), which are manifolds. Furthermore, by basic simplicial topology (see e.g. [77, 86]), if \( \sigma \) is a simplex of \( K \), then \( \sigma \) has a (relative) star neighborhood \( St(\sigma) \) obtained by taking the union of all simplices in the barycentric subdivision of the triangulation that include the barycenter of \( \sigma \) as a vertex. The simplicial link \( L \) of \( \sigma \) is then the union of the simplices in the star neighborhood that do not intersect \( \sigma \). In this case, \( St(\sigma) \) is PL homeomorphic to the join \( \sigma * L \). The interior of \( St(\sigma) \) is then PL homeomorphic to \( \sigma \times cL \). Filtering \( L \) by its intersection with the skeleta of \( K \) provides the necessary PL filtration of \( L \).

The lemma is certainly not true in the topological category; see Example 2.38.

As we saw in Example 2.33, the links in CS sets are not necessarily unique, and that example also applies to topological stratified pseudomanifolds. However, the extra rigidity in the PL setting does yield uniqueness results for links.

Lemma 2.62. Let \( X \) be a PL CS set, and let \( S \) be a stratum of \( X \). Then the links of any two points in \( S \) are PL homeomorphic.

Proof. Suppose that \( X \) is a PL CS set and that \( x \in X \) is contained in an \( i \)-dimensional stratum \( S \) so that \( x \) has a neighborhood PL homeomorphic to \( \mathbb{R}^i \times cL \). By basic PL topology (see [86, Exercise 2.24(3)] or the argument on [3, page 419]), \( \mathbb{R}^i \times cL \cong c(S^{i-1} * L) \), where \( S^{i-1} * L \) is the join of \( L \) with \( S^{i-1} \) or, equivalently, this is the \( i \)th suspension of \( L \). In other
words, \( S^{i-1} \ast L \) is the polyhedral link\(^{13}\) \([80]\). Section 1.1] of \( x \) in \( X \). If \( L' \) were another possible link for \( x \), then we would similarly have a neighborhood of \( x \) that is PL homeomorphic to \( c(S^{i-1} \ast L') \). But since polyhedral links are unique up to PL homeomorphism \( S^{i-1} \ast L \cong S^{i-1} \ast L' \), we must have \( S^{i-1} \ast L \cong S^{i-1} \ast L' \), and in the PL category, this implies\(^{14}\) that \( L \cong L' \) \([75\), Theorem 1].

It is also true that the links of any two points in the same stratum are PL homeomorphic. Since strata are connected, it suffices to show that the set of points in a stratum \( S \) with links homeomorphic to the link at a given point \( z \in S \) is both open and closed in \( S \). So let \( L \) be the link of \( z \), let \( A \) be the set of points in \( S \) with link PL homeomorphic to \( L \), and suppose \( x \in A \). Then \( x \) has a neighborhood \( N \) in \( X \) that is PL homeomorphic to \( \mathbb{R}^i \times L \) and where \( \mathbb{R}^i \times \{v\} \) is taken by the homeomorphism to a neighborhood \( U \) of \( x \) in \( S \). Any point in \( U \) also has \( N \) as a neighborhood in \( X \), and so also has \( L \) as link. Therefore, \( A \) is open in \( S \). Now, suppose \( x \) is in the closure of \( A \) in \( S \). Then \( x \) has a neighborhood PL homeomorphic to \( \mathbb{R}^i \times L' \) for some \( L' \). But since \( x \) is in the closure of \( A \), there is a point \( y \in A \) that is in the image of \( \mathbb{R}^i \times \{v\} \) under the homeomorphism. Hence the link of \( y \) is both \( L' \) and PL homeomorphic to \( L \), we must have \( L' \cong L \) by the arguments of the preceding paragraph. Therefore, \( x \in A \). So \( A \) is closed and open in \( S \) and so must be all of \( S \).

In the PL category, it is also much simpler to recognize which spaces are pseudomanifolds. The following proposition shows that any PL space which is dimensionally homogeneous can be stratified as a PL pseudomanifold.

**Proposition 2.63.** Suppose \( X \) is any PL space of dimension \( n \) containing a closed PL subspace \( \Sigma \) of dimension \( < n \) such that \( X - \Sigma \) is an \( n \)-dimensional PL manifold that is dense in \( X \). Then \( X \) is a PL pseudomanifold. If, additionally, \( \dim(\Sigma) < n - 1 \), then \( X \) can be stratified as a classical PL stratified pseudomanifold.


**Remark 2.64.** In the setting of the proposition, it is tempting to attempt to define a stratification of \( X \) by fixing a triangulation of \( X \) with respect to which \( \Sigma \) is a subcomplex and \( X - \Sigma \) is a PL manifold and then letting \( X^n = X \) and, for \( i < n \), letting \( X^i \) be the union of the \( i \)-simplices of \( \Sigma \) in the triangulation. This is the approach suggested, for example, in [50, Proposition 1.4]. However, it is not proven there, and it is not obvious to the author, that the resulting links satisfy the required condition of being themselves PL stratified pseudomanifolds. Regardless, the approach of Proposition [2.111] via intrinsic stratifications is in many ways more natural as it does not depend on a choice of triangulation.

**Example 2.65.** If \( M \) is a compact PL manifold, then the suspension \( SM \) is a compact PL stratified pseudomanifold if we filter by \( \{N,S\} \subset SM \). If \( \dim(M) > 0 \), \( SM \) is a classical PL stratified pseudomanifold. Of course we could also determine that these examples are PL stratified pseudomanifolds directly from the definitions.

\(^{13}\)In PL topology, the space \( \text{Lk}(x) \) such that \( x \) has a neighborhood of the form \( c\text{Lk}(x) \) is often called simply the “link”, but since we have another meaning for “link”, we use the expression “polyhedral link”.

\(^{14}\)Warning: this fact is not true in the topological category.
Remark 2.66. The condition of the proposition is equivalent to the assumption that \( X \) can be triangulated as a union of \( n \)-simplices. Indeed, if \( X \) is the union of \( n \)-simplices then the union of the interiors of those simplices constitutes a dense PL manifold, and so the proposition shows that \( X \) is a PL pseudomanifold.

Conversely, and more generally, if \( X \) is a stratified PL pseudomanifold, then any simplex in any triangulation of \( X \) must be a face of an \( n \)-simplex in order for the condition to be fulfilled that \( X \) possesses a dense \( n \)-dimensional subspace. Hence every simplex is a face of an \( n \)-simplex, and since \( X \) is \( n \)-dimensional, \( X \) is a union of \( n \)-simplices as a simplicial complex.

Now, suppose that \( X \) is a union of \( n \)-simplices such that every \( n - 1 \) dimensional face is glued to exactly one other \( n - 1 \) dimensional face. Then if we let \( \Sigma \) be the simplicial \( n - 2 \) skeleton of \( X \), Proposition 2.11 provides a filtration of \( X \) as a classical PL stratified pseudomanifold.

Again conversely, and more generally, if \( X \) is a classical stratified PL pseudomanifold, then not only must \( X \) be a union of \( n \)-simplices, but every \( n - 1 \) dimensional face must be glued to exactly one other \( n - 1 \) dimensional face. For if not, then any point of any \( n - 1 \) face \( \tau \) that is glued either to no other \( n - 1 \) face or to more than one other \( n - 1 \) face cannot have a Euclidean neighborhood. Thus the entire \( n - 1 \) dimensional interior of \( \tau \) must be part of a stratum of dimension at least \( n - 1 \) that is not contained in the \( n \)-manifold \( X - \Sigma \). Therefore \( \tau \) must be contained in some codimension one stratum of \( X \).

Classical simplicial pseudomanifolds. Classically, simplicial pseudomanifolds were defined precisely as simplicial complexes such that 1) every simplex is the face of an \( n \)-simplex for some fixed \( n \) and 2) every \( n - 1 \) simplex is the face of exactly two \( n \)-simplices. It is worth pausing to consider this original definition and to see how it motivates the more modern definitions as well as work to come. We will also see why classical stratified pseudomanifolds cannot have codimension one strata.

Consider the most primitive possible approach to constructing an \( n \)-dimensional manifold. Certainly one must begin with all simplices of the same dimension \( n \) since a manifold is dimensionally homogeneous — all points have neighborhoods of the same dimension. So there can be no simplices of dimension \( > n \), and any simplices of dimension \( < n \) would eventually have to be attached as faces of \( n \)-simplices. Next, there can be no free \( n - 1 \) dimensional faces (since we are constructing manifolds, not manifolds with boundary), so each \( n - 1 \) dimensional face must be attached to another. Furthermore, each such face must be attached to exactly one other, since we would not have a manifold if we attached more than two \( n - 1 \) faces. So, now suppose we have taken all our \( n \)-simplices and attached each \( n - 1 \) face to exactly one other \( n - 1 \) face. Do we have a manifold? We might or we might not. Any point in the interior of an \( n \)-simplex or the interior of an \( n - 1 \) face now has a Euclidean neighborhood, but things can go wrong around lower dimensional faces. The reader can verify that a suspended torus can also be constructed this way by gluing together 2-dimensional faces of 3-simplices. We do see, however, that any non-manifold points must

\[^{15}\text{There is also typically a third condition, which we will discuss momentarily.}\]
occur in the $n−2$ skeleton of the triangulation, and so in some sense things are not quite too bad — we have constructed a classical PL pseudomanifold, which we know we can give the structure of a classical PL stratified pseudomanifold.

In fact, classical PL pseudomanifolds are yet a bit more general than this discussion so far indicates because we may also make other gluings of our $n$-simplices along lower dimensional faces. For example, if we have a PL $n$-sphere, $n ≥ 2$, that we have constructed by gluing $n$-simplices along $n−1$ faces, we might yet glue two distinct vertices together and still have a classical pseudomanifold.

Nonetheless, despite these extra gluings, our classical PL pseudomanifold is not too far from being a manifold in the following sense. Suppose $X$ is a compact $n$-dimensional classical PL stratified pseudomanifold, and that the manifold $X − Σ$ is oriented. By Remark 2.66, $X$ can be triangulated as a union of a finite collection of $n$-simplices $\{σ_i\}$ so that each $n−1$ face attached to exactly one other $n−1$ face. The global orientation condition then implies that it is possible to orient each of these $n$-simplices so that the chain $Γ = \sum_i σ_i$ is a cycle, i.e. its boundary as an element of the simplicial chain complex associated to the triangulation is 0. So $Γ$ is a fundamental class for $X$ in the same sense as for the homology of closed manifolds.

Just as for manifolds, we can eliminate the orientation assumption if we are willing to work with $\mathbb{Z}_2$ coefficients. Precisely these fundamental classes will arise later when we consider intersection homology Poincaré duality for PL pseudomanifolds.

Another condition one typically sees in the definition of a classical $n$-dimensional PL pseudomanifold (e.g. [90, Section 24]) is that it should be possible to embed a path from any point in the interior of an $n$-simplex to any point in the interior of any other $n$-simplex such that the path only intersects interiors of $n$-simplices and $n−1$ simplices. This condition ensures that the PL manifold $X − Σ$ is connected and thus that $H_n(X) ≅ \mathbb{Z}$, generated by $Γ$ (more generally, $H_n(X)$ will be a direct sum of $\mathbb{Z}$ summands corresponding to connected components of $X − Σ$). This condition is sometimes called a “strong connectedness” condition. An example of a PL pseudomanifold that is connected but not strongly connected would be two $n$-spheres that are attached at a vertex.

Now, what about our PL stratified pseudomanifolds that do have $\dim(Σ) = \dim(X)−1$? In this case, we are not requiring that each $n−1$ face attaches to exactly one other $n−1$ face. It might attach to none or to more than one. We can still form the chain $Γ = \sum_i σ_i$, but it is clear that it will no longer be a cycle; it will only be a cycle in the relative chain group $C_*(X, Σ)$. In this sense, one might then expect that stratified pseudomanifolds, as we have defined them here, are more analogous to manifolds with boundary. As we will see below in discussing Poincaré duality results for intersection homology of pseudomanifolds, this is not necessarily the case either, and in fact, with the proper definitions of the intersection homology groups, they really do behave more like manifolds than like manifolds with boundary. However, they also do need to be handled in somewhat different way, which is why it is not unusual for some sources to restrict attention only to classical pseudomanifolds.

As we have observed above in Remark 2.66, the classical pseudomanifolds that can be assembled from $n$-simplices can all be given filtrations making them PL stratified pseudomanifolds (though the filtration is not necessarily unique). Given the properties of such spaces, as they appear, for example, in Definition 2.57, we can then see how the definitions
of topological stratified pseudomanifolds and CS spaces constitute natural generalizations. In fact, we will even see below in Section that topological stratified pseudomanifolds possess fundamental classes, though we must use singular homology rather than simplicial.

### 2.4.2 Normal pseudomanifolds.

**NOTE:** THIS SECTIONS IS MUCH SHAKIER THAN OTHERS AND NEEDS TO BE REWRITTEN. PROCEED AT RISK.

Difficulties can sometimes arise in working with stratified pseudomanifolds that are connected but that have multiple regular strata. In these situations it is technically simpler to work with *normal* stratified pseudomanifolds.

**Definition 2.67.** A (topological or PL) stratified pseudomanifold is called **normal** if the link of any point is connected.

**Remark 2.68.** Of course in the topological case a point does not have a unique link, but if one link is connected, they all must be. This is not completely obvious geometrically, but it follows, for example, from Corollary [5.41] which says that all links of a point have the same intersection homology groups and from the fact that intersection homology can be used to detect connectivity, just as ordinary homology groups can be. In fact, \( \overline{pH^i_0} \left( L \right) = \overline{H^i_0} \left( L \right) \) for a large enough perversity \( \overline{p} \) (all of this will be explained once we get into intersection homology below).

The following lemma makes the case that, in some sense, normal stratified pseudomanifolds should play a role analogous to connected closed manifolds:

**Lemma 2.69.** If \( X \) is a normal stratified pseudomanifold, then

1. every link of \( X \) is a normal stratified pseudomanifold, and
2. if \( X \) is connected, then \( X \) has only one regular stratum.

**Proof.** The first statement follows from our earlier observation in Remark 2.55 that a link \( \ell \) in a link \( L \) of a of a stratified pseudomanifold \( X \) is also a link in \( X \).

We prove the second statement by induction. It is true for 0-dimensional pseudomanifolds. Now suppose that \( X = X^n \) is connected for \( n > 0 \) but that \( X \) has more than one regular stratum. Let \( S \) be one of the regular strata, and let \( T = X^n - (X^{n-1} \cup S) \), the union of the other regular strata. Since any locally cone-like space is clearly locally path connected, \( X \) is path connected [78, Theorem 25.5]. Let \( x \in S \), \( y \in T \), and let \( \gamma \) be a path from \( x \) to \( y \). Let \( t_0 \in [0,1] \) be the maximum of \( \{ t \in [0,1] \mid \gamma(t) \in \overline{S} \} \), and let \( z = \gamma(t_0) \). Then \( z \in \overline{S} \cap \overline{T} \), so \( z \) is contained in some singular stratum \( s \) such that \( s \subset \overline{S} \cap \overline{T} \). Now consider a distinguished neighborhood \( N \cong \mathbb{R}^i \times cL \) of \( z \). The intersections of \( S \) and \( T \) with \( N \) have the respective forms \( \mathbb{R}^i \times cU - \mathbb{R}^i \times \{ 0 \} \) and \( \mathbb{R}^i \times cV - \mathbb{R}^i \times \{ 0 \} \), where \( \{ 0 \} \) is the cone point and \( U \) and \( V \) are distinct regular strata of the link \( L \). But now by the first statement of the lemma, \( L \) is normal, so by induction this is impossible. Hence \( T \) must be empty. \( \square \)
Though it is often technically simpler to work with normal stratified pseudomanifolds, results can often be transferred to stratified pseudomanifolds that are not normal from normal stratified pseudomanifolds using the fact that every stratified pseudomanifold has a normalization.

**Definition 2.70.** Let $X$ be an $n$-dimensional stratified pseudomanifold. A *normalization* of $X$ is an $n$-dimensional normal stratified pseudomanifold $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that

1. $p$ is a proper surjection,
2. $p$ is stratification preserving in the sense that $i$-dimensional strata of $\tilde{X}$ map to $i$-dimensional strata of $X$,
3. the restriction of $p$ to $\tilde{X} - \Sigma_{\tilde{X}}$ is a homeomorphism onto $X - \Sigma_X$,
4. for any point $x \in \Sigma_X$, $p^{-1}(x)$ is a disjoint union of points and the number of such points is equal to the number of regular components of any link of $x$.

It is a standard abuse of language to refer to $\tilde{X}$ alone as the normalization, leaving the map $p$ tacit.

The existence and uniqueness of normalizations is proven in [80]. We will provide the proof for PL stratified pseudomanifolds below. It is useful first, however, to obtain some intuition by considering examples.

**Example 2.71.** Suppose $\coprod_{i=1}^k M_i$ is a finite disjoint union of closed connected $n-1$ manifolds. Then $c(\coprod_{i=1}^k M_i)$ with the natural stratification is an $n$-dimensional stratified pseudomanifold, but it is not normal unless $k = 1$. The normalization is $\coprod_{i=1}^k (cM_i)$, with normalization map given by the quotient map that identifies the cone points together.

Similarly, something like $N \times c(\coprod_{i=1}^k M_i)$ normalizes to $\coprod_{i=1}^k (N \times cM_i)$.

**Example 2.72.** If $M$ is an $n$-manifold, $n > 0$, then the quotient map $M \to X$, where $X$ is $M$ with a finite number of points identified together, is a normalization map. More generally, if $M$ is a smooth manifold containing disjoint embeddings $N_i$ of the same manifold $N$, then we can obtain a stratified pseudomanifold by gluing the $N_i$ together. The quotient map is a normalization.

Even more generally, suppose $X, Y$ are stratified pseudomanifolds with $\Sigma_X = \Sigma_Y$. Then the union of $X$ and $Y$ along the common $\Sigma$ will be a non-normal pseudomanifold, and the quotient will be a normalization if $X$ and $Y$ are normal.

**Proposition 2.73.** An $n$-dimensional stratified PL pseudomanifold $X$ possesses a unique normalization.

**Proof.** The following construction is from [42, p. 151].

Choose a triangulation of $X$, and let $Y$ be the disjoint union of the $n$-simplices of $X$. Glue together the $n-1$ faces of the simplices of $Y$ according to how they are glued together in $X$ and call the resulting space $\tilde{X}$. We filter $\tilde{X}$ so that $\tilde{X}^i = p^{-1}(X^i)$. 
We need to check the conditions of Definition 2.70 and verify that \( \tilde{X} \) is a stratified pseudomanifold. The map \( p \) is clearly a surjection. That it is proper will follow from the fact that the inverse image of every simplex will be a finite union of simplices. Also, \( p \) is stratification preserving from the definition of the filtration on \( \tilde{X} \).

\[ \square \]

Remark 2.74. Notice that a PL stratified pseudomanifold with codimension one strata can be normal only if each \( n-1 \) face in each \( n-1 \) dimensional stratum is not glued to any other \( n-1 \) faces of any \( n \)-simplices of \( X \).

### 2.5 Pseudomanifolds with boundaries

Stratified pseudomanifolds constitute a generalization of manifolds. Since one also wants to consider the important class of “manifolds with boundary”, it is reasonable to ask for “pseudomanifolds with boundary”. Before getting into the details, we first digress regarding terminology.

The English phrase “manifold with boundary” sounds as though it should be picking out a specific kind of manifold, but really the concept “manifold with boundary” generalizes the concept “manifold”. In particular, if a “manifold with boundary” has a non-empty boundary, then it is not a manifold! This is because points on the boundary fail to satisfy the property that they should have Euclidean neighborhoods, which is part of the definition of being a manifold. The other problem is that “manifold with boundary” implies that there is a boundary and it is tempting to think then that the boundary cannot be empty. As an alternative, some authors have taken to using the notation “\( \partial \)-manifold” as a replacement for “manifold with boundary”. This seems to avoid both issues as well as eliminate some clunky phrasing. Following \([38]\), we will take the same approach to what might otherwise be called “stratified pseudomanifolds with boundary”.

**Definition 2.75.** An \( n \)-dimensional \( \partial \)-stratified pseudomanifold is a pair \((X, B)\) together with a filtration on \( X \) such that

1. \( X - B \), with the induced filtration \( (X - B)^i = (X - B) \cap X^i \), is an \( n \)-dimensional stratified pseudomanifold,

2. \( B \), with the induced filtration \( B^{i-1} = B \cap X^i \), is an \( n-1 \) dimensional stratified pseudomanifold,

3. \( B \) has an open collar neighborhood in \( X \), that is, a neighborhood \( N \) with a filtered homeomorphism \( N \to [0, 1) \times B \) (where \([0, 1)\) is given the trivial filtration) that takes \( B \) to \( \{0\} \times B \).

\( B \) is called the boundary of \( X \) and is also denoted \( \partial X \).

We will often abuse notation by referring to the “\( \partial \)-stratified pseudomanifold \( X \),” leaving \( B \) tacit.
Definition 2.76. A space is called simply a \( \partial \)-pseudomanifold if it possesses a filtration with respect to which it is a \( \partial \)-stratified pseudomanifold. We will refer to a stratified \( \partial \)-pseudomanifold such that \( X^{n-1} = X^{n-2} \) as a classical \( \partial \)-stratified pseudomanifold; a space is called a classical \( \partial \)-pseudomanifold if it possesses a filtration with respect to which it is a classical \( \partial \)-stratified pseudomanifold.

Example 2.77. A stratified pseudomanifold \( X \) is a \( \partial \)-stratified pseudomanifold with \( \partial X = \emptyset \).

If \( X \) is a stratified pseudomanifold, then \( X \times [0, 1] \) is a \( \partial \)-stratified pseudomanifold with \( \partial X = (X \times \{0\}) \cup (X \times \{1\}) \).

If \( X \) is a compact stratified pseudomanifold, then the closed cone \( \bar{c}X \) is a \( \partial \)-stratified pseudomanifold with \( \partial X \cong X \).

If \( M \) is a compact \( \partial \)-manifold with the trivial stratification \( \emptyset \subset M = M^n \), then \( M \) is a \( \partial \)-stratified pseudomanifold with \( \partial M \) being the usual boundary of \( M \) in the manifold sense. Note that the choice of filtration is crucial here. See Example 2.80 below for more details.

Remark 2.78. When working with \( \partial \)-manifolds, the existence of collared boundaries is not part of the definition but is rather a theorem (at least for paracompact manifolds); see e.g. [53, Proposition 3.42 and remarks following]. For pseudomanifolds, however, it is necessary to make this part of the definition.

For example, let \( M \) be an \( n - 1 \) dimensional \( \partial \)-manifold with \( \partial M \neq \emptyset \). Consider the closed cone \( X = cM \). If we filter \( X \) by \( \{v\} \subset X \), where \( v \) is the cone vertex, then \( X \) is a stratified space (though it is not manifold stratified as \( X - \{v\} \) is not a manifold). If we let \( B = M \cup_{\partial M} c(\partial M) \), then \( X - B \) is an \( n \)-manifold homeomorphic to \( M \times (0, 1) \), and \( B \) is a stratified pseudomanifold. However, \( B \) will not necessarily have a collar neighborhood \( N \). If so, under the filtered homeomorphism \( B \times [0, 1] \cong N \), the image \( z \) of \( \{v\} \times \frac{1}{2} \) in \( N \) must have a neighborhood \( W \) homeomorphic to \( c(\partial M) \times (0, 1) \). But then \( H_*(X, X - z) \cong H_*(W, W - z) \cong H_{*-1}(W - z) \), since \( W \) is contractible, and \( W - z \) is homotopy equivalent to the suspension of \( \partial M \). So if \( S\partial M \) does not have the homology of \( S^{n-1} \), \( z \) cannot have a Euclidean neighborhood in \( X - B \), a contradiction.

Remark 2.79. The strata of a \( \partial \)-stratified pseudomanifold \( X \) will not necessarily be manifolds, thus \( \partial \)-stratified pseudomanifolds are not necessarily manifold stratified spaces. For example, a trivially filtered \( \partial \)-manifold with non-empty boundary is not a manifold, though it is a \( \partial \)-stratified pseudomanifold. However, the strata will be \( \partial \)-manifolds with the boundary of the stratum \( S \) consisting of \( S \cap \partial X \).

A critical point to observe is that the boundary of a \( \partial \)-stratified pseudomanifold depends strongly on the stratification. This is demonstrated by the following example.

Example 2.80. Let \( M \) be a paracompact \( n \)-dimensional \( \partial \)-manifold, and let \( P \) be its boundary (in the usual manifold-with-boundary sense). Suppose \( P \neq \emptyset \).

1. Suppose we filter \( M \) trivially so that \( M \) itself is the only non-empty stratum. Then \( (M, P) \) is a \( \partial \)-stratified pseudomanifold. Note that all the conditions of Definition 2.75 are fulfilled: \( M - P \) is an \( n \)-manifold, \( P \) is an \( n - 1 \) manifold, and \( P \) is collared in \( M \) by classical manifold theory (see [53, Proposition 3.42]).
2. On the other hand, suppose $X$ is the filtered space $P \subset M$. Then it is easy to check that $(X, \emptyset)$ is a $\partial$-stratified pseudomanifold; that is, $X$ is a stratified pseudomanifold. But with this filtration, we cannot have $\partial X = P$ because condition (3) of Definition 2.75 would not be satisfied: $P$ has a collared neighborhood in $X$ but the collar homeomorphism does not preserve the filtration.

Despite this dependence of the boundary upon the choice of stratification in general, our next result (from [38]) shows that when there are no codimension one strata $\partial X$ depends only on the underlying space $X$ and not on the choice of stratification (without codimension one strata). Unfortunately, the proof is fairly technical and relies on several outside references concerning dimension theory.

**Proposition 2.81.** Let $(X, B)$ and $(X', B')$ be $\partial$-stratified pseudomanifolds of dimension $n$ with no codimension one strata, and let $h : X \to X'$ be a homeomorphism (which is not required to be filtration preserving). Then $h$ takes $B$ onto $B'$.

**Proof.** It suffices to show that $h$ takes $B$ to $B'$, as the equivalent result for $h^{-1}$ shows then that $h$ takes $B$ onto $B'$.

It thus suffices to show that $h$ takes the union of the regular strata of $B$ to $B'$, since the regular strata are dense in $B$ and $B'$ is closed. So let $x$ be in a regular stratum of $B$ and suppose that $h(x)$ is not in $B'$. Then there is a Euclidean neighborhood $E$ of $x$ in $B$ such that $h(E) \subset X' - B'$. The existence of an open collar neighborhood of $B$ shows that the local homology group $H_n(X, X - \{y\}) = 0$ for each $y \in E$, so by topological invariance of homology $h(E)$ must be contained in the singular set $S$ of $X' - B'$, for otherwise each point $h(y)$ would have a Euclidean neighborhood and $H_n(X, X - h(y)) \cong \mathbb{Z}$.

Next we use the dimension theory of [13, Section II.16]. We will use the fact that each skeleton of a pseudomanifold (and in particular the singular set) is locally compact, as follows from the definition of a CS set.

As defined in [13, Definition II.16.6], $\dim_Z E$ is $n - 1$ by [13, Corollary II.16.28], so $\dim_Z h(E)$ is also $n - 1$, and by [13, Theorem II.16.8] (using the fact that $S$ is locally compact) this implies that $\dim_Z S \geq n - 1$. To obtain a contradiction it suffices to show that $\dim_Z$ of the $i$-skeleton of a pseudomanifold is $\leq i$.

So let $Y$ be a pseudomanifold and assume by induction that $\dim_Z Y^i \leq i$ for some $i$. Let $c$ denote the family of compact supports and let $\dim_{c,Z}$ be as in [13, Definition 16.3]. Then $\dim_Z$ is equal to $\dim_{c,Z}$ for any locally compact space by [13, Definition II.16.6]. Since $Y^i$ is a closed subset of $Y^{i+1}$ and $Y^{i+1} - Y^i$ is a (possible empty) $(i + 1)$-manifold, [13, Exercise II.11 and Corollary II.16.28] imply that $\dim_{c,Z} Y^{i+1} \leq i + 1$ as required.

The two cases of Example 2.80 together with the identity map of the underlying spaces, shows that Proposition 2.81 is not true if codimension one strata are allowed.

### 2.6 Other species of stratified spaces

In this section we mention some other types of manifold stratified spaces. The Whitney stratified and Thom-Mather stratified spaces possess extra conditions beyond those required...
for a space to be a stratified pseudomanifold. On the other hand, manifold homotopically stratified spaces are not necessarily CS sets — rather than satisfying a local cone-like condition, there are homotopy theoretic conditions imposed on the interaction of strata. We will not utilize any of these spaces in what follows, except to mention when certain results extend to manifold homotopically stratified space. We include mention of them simply for the interested reader; for a more detailed survey, see [60].

**Whitney stratified spaces.** The following geometric conditions on a manifold stratified space are due to Whitney and assume that our stratified space $X$ is a closed subspace of a smooth manifold $M$. This will always be the case, for example, if one studies algebraic or analytic varieties.

**Definition 2.82.** The stratified space $X \subset M$ is **Whitney stratified** if

1. each stratum of $X$ is a locally closed smooth submanifold of $M$,

2. (Whitney’s condition A) if $\{x_i\} \subset S'$ is a sequence of points in the $k$-dimensional stratum $S'$ converging to a point $x$ in a stratum $S \subset \bar{S}'$ and if the $k$-dimensional tangent spaces $T_{x_i}S'$ to $S'$ at $x_i$ converge to a $k$-dimensional subspace $V$ of $T_x M$, the tangent plane to $M$ at $x$, then $V$ contains the tangent space to $S$ at $x$, i.e. $T_x S \subset V$.

3. (Whitney’s condition B) if the hypotheses of condition A hold and $\{y_i\} \subset S$ is a sequence of points also converging to $x$ such that the sequence of secant lines between $x_i$ and $y_i$ converges to a line $\ell$, then $\ell \subset V$.

Note: to understand the condition on secant lines, one should choose a local coordinate chart for $M$ around $x$. It can be shown that the condition is independent of the choice.

In fact, it was shown by Mather [68] that Condition B implies Condition A.

Whitney’s conditions were formulated with algebraic varieties in mind, however not every algebraic variety satisfies Whitney’s conditions with its natural filtration, in which $X^{i-1}$ is the set of singular (non-smooth) points of the subvariety $X^i$ .

The standard example is the **Whitney umbrella** $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = z y^2\}$, which is an irreducible real algebraic variety. The space $W$ is the union of the $z$-axis, $Z$, with the 2-dimensional manifold that is the graph of the surface $z = \frac{1}{y^2}x^2$ for $y \neq 0$ (when $y = 0$, also $x = 0$ and we are on the $z$-axis). Notice that each point along the negative $z$-axis has a neighborhood whose intersection with $W$ is equal to its intersection with the $z$-axis. By contrast, any slice of $W$ determined by $z = c$, for $c$ a positive constant, is the union of two lines. When $z = 0$, then also $x = 0$ and $y$ can be arbitrary, so the intersection of $W$ with the $x$-$y$ plane is the $y$-axis. As $W - Z$ is a smooth 2-dimensional manifold, the standard filtration of $W$ is $Z \subset W$, since $Z$ is the set of points at which $W$ is not a smooth 2-dimensional manifolds, and $Z$ itself is the smooth 1-dimensional manifold corresponding to the subvariety $x = y = 0$. However, notice that this is not even a stratification as $Z \cap \overline{W - Z} \neq \emptyset$, but $Z \not\subset \overline{W - Z}$. Whitney’s conditions are also violated as we see by letting $\{x_i\}$ be a sequence of points along the positive $y$-axis converging to the origin $\vec{0}$. These are
points of the stratum \( W - Z \), and their tangent spaces can all be identified with the \( x\)-\( y \) plane (notice that for \( y \neq 0 \), \( W \) is the graph of the surface \( z = \frac{1}{y^2}x^2 \) and both partial derivatives vanish at points on the \( y \)-axis). Hence the limit of \( T_x(W_Z) \) at the origin is also the \( x\)-\( y \) plane. But clearly this plane does not contain \( T_0Z \).

Nonetheless, it is possible to choose a different filtration of \( W \) with respect to which it is a stratified space satisfying the Whitney conditions. In fact, algebraic sets, semi-algebraic sets (finite unions of sets determined by finitely many polynomial equations or inequalities), analytic sets and semi-analytic sets (these are defined as for algebraic and semi-algebraic sets but using analytic functions rather than just polynomials), and sub-analytic sets (which we will not define here) all can be filtered so as to possess Whitney stratifications. An expository reference is [92].

Thom-Mather spaces. These spaces often arise in settings where one wants to be able to make analytic arguments concerning stratified spaces. The idea is that each stratum should have an analogue of a tubular neighborhood but that the different tubes around the different strata should interact compatibly. We adapt our version of the definition from [60].

**Definition 2.83.** For \( 0 \leq k \leq \infty \), the manifold stratified space \( X \) is a Thom-Mather \( C^k \) stratified space if

1. each stratum of \( X \) is a \( C^k \) manifold,
2. there is a tube system \( \{T_i, \pi_i, \rho_i\} \) such that \( T_i \) is an open neighborhood of \( X_i \) in \( X \) (called a tubular neighborhood), \( \pi_i : T_i \to X_i \) is a retraction (called the local retraction), and \( \rho_i : T_i \to [0, \infty) \) is a map such that \( \rho_i^{-1}(0) = X_i \),
3. for each pair \( X_i, X_j \), if \( T_{ij} = T_i \cap X_j \) and the restriction of \( \pi_i, \rho_i \) to \( T_{ij} \) are denoted \( \pi_{ij}, \rho_{ij} \), then the map \( (\pi_{ij}, \rho_{ij}) : T_{ij} \to X_i \times (0, \infty) \) is a \( C^k \) submersion,
4. If \( S_i, S_j, S_k \) are strata and \( x \in T_{jk} \cap T_{ik} \cap \pi_{jk}^{-1}(T_{ij}) \), then \( \pi_{ij}\pi_{jk}(x) = \pi_{ik}(x) \) and \( \rho_{ij}\pi_{jk}(x) = \rho_{ik}(x) \).

The idea here is that each \( \pi_i \) plays a role analogous the projection of a tubular neighborhood to a submanifold in manifold theory, while each \( \rho_i \) is a measure of radial distance from a stratum. Condition (3) says that the \( \pi_i \) and \( \rho_i \) are not too wild as functions. The first equation of Condition (4) says that the image of a point under two successive local retractions, from the stratum \( S_k \) to the stratum \( S_j \) and then from \( S_j \) to \( S_i \) is the same as its image under the local retraction directly from \( S_k \) to \( S_i \) when the point is close enough to \( S_i \) and \( S_j \) to be contained in all the relevant tubes. The second equation of Condition (4) says roughly that local retraction from \( S_k \) to \( S_j \) should not change the radial distance of a point from the stratum \( X_i \), again when points are close enough to all relevant strata.

These sorts of conditions are relevant when one wants to study stratified spaces using techniques of global analysis; see for example [5].

Whitney stratified spaces always possess \( C^\infty \) Thom-Mather stratifications [65]. Furthermore, by a theorem of Goresky [17], \( C^\infty \) Thom-Mather stratified spaces can be (smoothly) triangulated by triangulations compatible with their filtrations. Hence if \( X \) is an \( n \)-dimensional
Thom-Mather space such that $X - \Sigma X$ is dense, then $X$ is a PL pseudomanifold by Propositions 2.63 and 2.111. So, for example, irreducible complex algebraic varieties can be stratified as classical PL stratified pseudomanifolds.

**Homotopically stratified spaces.** The Whitney and Thom-Mather conditions in some sense prescribe fairly rigid local conditions on stratified spaces. In fact any point of a locally cone-like space must possess a neighborhood of a given form. Hence one might wonder whether it is possible to work effectively with manifold stratified spaces that do not possess such conditions. This is indeed the case for a class of spaces we refer to as manifold homotopically stratified spaces.

The spaces were introduced by Quinn in [83] to provide “a setting for the study of purely topological stratified phenomena, particularly group actions on manifolds.” Quinn’s spaces are manifold stratified spaces that are not necessarily locally cone-like. Instead, they must satisfy certain homotopy theoretic conditions concerning how the strata fit together. To explain these conditions, we will need to introduce some definitions.

If $X$ is a filtered space, a map $f : Z \times A \rightarrow X$ is stratum-preserving along $A$ if, for each $z \in Z$, $f(z \times A)$ lies in a single stratum of $X$. If $A = I = [0, 1]$, we call $f$ a stratum-preserving homotopy. If $f : Z \times I \rightarrow X$ is only stratum-preserving when restricted to $Z \times [0, 1)$, we say $f$ is nearly stratum-preserving.

If $X$ is a filtered space, then $Y \subset X$ is forward tame in $X$ if there is a neighborhood $U$ of $Y$ in $X$ and a nearly-stratum preserving deformation retraction $R : U \times I \rightarrow X$ retracting $U$ to $Y$ rel $Y$. So $R$ is a deformation retraction that keeps each point in its original until time 1 when everything collapses into $Y$.

The stratified homotopy link of $Y$ in $X$ is the space (with compact-open topology) of nearly stratum-preserving paths with their tails in $Y$ and their heads in $X - Y$:

$$\text{holink}_s(X, Y) = \{\omega \in X^I \mid \omega(0) \in Y, \omega((0, 1]) \subset \text{a single stratum of } X - Y\}.$$

The holink evaluation map takes a path $\omega \in \text{holink}_s(X, Y)$ to $\omega(0)$. For $x \in X_i$, the local holink, denoted $\text{holink}_s(X, x)$, is simply the subset of paths $\omega \in \text{holink}_s(X, X_i)$ such that $\omega(0) = x$. Holinks inherit natural filtrations from their defining spaces, as in Example 2.8:

$$(\text{holink}_s(X, Y))^j = \{\omega \in \text{holink}(X, Y) \mid \omega(1) \in X^j\}.$$

Using these notions, we can now provide the definition of manifold homotopically stratified spaces:

**Definition 2.84.** A filtered space $X$ is a manifold homotopically stratified space (MHSS) if the following conditions hold:

- $X$ is locally-compact, separable, and metric,
- each $X_i$ is an $i$-manifold and is locally-closed in $X$,
- for each $k > i$, $X_i$ is forward tame in $X_i \cup X_k$, 

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• for each $k > i$, the holink evaluation $\text{holink}_k(X_i \cup X_k, X_i) \to X_i$ is a fibration.

• for each $x$, there is a stratum-preserving homotopy

$$\text{holink}(X, x) \times I \to \text{holink}(X, x)$$

from the identity into a compact subset of $\text{holink}(X, x)$.

While these spaces may seem complex, they have important applications. For example, manifold homotopically stratified spaces can arise as quotient spaces of manifolds under topological group actions and they have been utilized in this context by Yan [112], Beshears [7], and Weinberger and Yan [107, 108] to study topological group actions on manifolds. They also arise naturally in categories with more structure — for example, Cappell and Shaneson showed that they occur as mapping cylinders of maps between smoothly stratified spaces [15]. MHSSs even show up when simply studying manifolds and their submanifolds; for example a locally-flat topological submanifold of a higher-dimensional manifold may not possess a mapping cylinder neighborhood, but such a pair does satisfy the homotopy conditions required to constitute a manifold homotopically stratified space [59] (it is also a stratified pseudomanifold). Generalizations of this neighborhood property to MHSSs with more strata have been developed by Hughes [58], and there is even a surgery theory for MHSSs that has been developed by Weinberger [106]. A further survey of MHSSs in such geometric settings can be found in Hughes and Weinberger [60]. More recently, the homotopy properties of manifold homotopically stratified spaces have been studied in the work of Miller [70, 71] and Woolf [111].

Intersection homology of manifold homotopically stratified spaces has been studied by Quinn [82] and the author [27, 28, 30]. We will have a bit more to say about this below.

2.7 Maps of stratified spaces

Of course a central tenet of topology is that even if one is interested only in studying a specific space, it is important to be able to consider maps into and out of that space. When working with stratified spaces, it is natural to work with maps that are in some sense compatible with the stratification. For example, the definition of a distinguished neighborhood uses only homeomorphisms that preserve the filtration between the distinguished neighborhood $N$ and its “model” $\mathbb{R}^i \times cL$. But of course it is too limiting to work only with homeomorphisms, and so it is necessary to define more general “stratified maps”. There are various definitions in the literature. We will use the following ones.

The key property of a map $f : X \to Y$ between stratified spaces is that the image of a stratum of $X$ should not cross between multiple strata of $Y$.

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This condition, requiring compactly dominated local holinks, was not part of the original definition of Quinn [83]. It first appears in the work of Hughes leading towards his Approximate Tubular Neighborhood Theorem in [58].

Here “map” always means “continuous function”.

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Definition 2.85. If $X, Y$ are stratified spaces and $f : X \to Y$ is a continuous function (map), we say that $f$ is a stratified map if for each stratum $S \subset X$ there is a unique stratum $S' \subset Y$ such that $f(S) \subset S'$.

Example 2.86. If $X$ is a stratified space and $Y$ is filtered trivially as $\emptyset \subset Y$, then any map $f : X \to Y$ is a stratified map. In particular, any map between trivially filtered spaces is stratified (trivially).

If $U$ is an open subset of a CS set of a stratified pseudomanifold endowed with the filtration $U^i = U \cap X^i$, then by Lemmas 2.39 and 2.53 the inclusion $U \hookrightarrow X$ is a stratified map.

Suppose now that $f : X \to Y$ is a stratified map that is also a homeomorphism. If $f(S) \subset S'$ for a stratum $S \subset X$, then in fact the restriction of $f$ to $S$ must take $S$ homeomorphically onto $S'$ (since the restriction of $f$ to $S$ is certainly injective, and if it were not surjective, then the inverse $f^{-1}$ could not be a stratified map as $f^{-1}(f(S)) = S$ but there would be other points of $S'$ that could not map to $S$). It follows that $f$ sets up a bijection between strata of $S$ and strata of $S'$. Note, however, that we have not yet made any assumptions about the dimension of $S$ in $X$ or the dimension of $S'$ in $Y$ (which might be different, as we recall that dimensions in filtered spaces may be purely formal). For the purposes of intersection homology, it will be useful to impose upon what we will call stratified homeomorphisms one further condition: that $f$ preserves codimensions of strata.

Definition 2.87. We call a stratified map $f : X \to Y$ a stratified homeomorphism if it is a topological homeomorphism and, for each stratum $S \subset X$, the codimension of the stratum $f(S) \subset Y$ is equal to the codimension of $S \subset X$.

Example 2.88. Let $X$ be a filtered space, and let $Y$ be the filtered space with the same underlying space as $X$ but such that $Y^i = X^{i-k}$ for some $k \geq 0$ (this includes the assumption that if $X$ has formal dimension $n$, then $Y$ has formal dimension $n+k$). Then the identity map on the underlying space provides a filtered homeomorphism between $X$ and $Y$.

Example 2.89. Suppose $X = X^n = \mathbb{R}^n$ filtered by $\{0\} \subset \mathbb{R}^n$ as a manifold stratified space (so $\{0\} = X^0$). Then $X$ is not stratified homeomorphic to the unfiltered $\mathbb{R}^n$.

Remark 2.90. The reader might find it somewhat surprising that we require stratified homeomorphisms to preserve codimension and not dimension. Ultimately, preserving codimension is a weaker requirement, as we saw in Example 2.88, and one that will prove useful, as the definition of intersection homology depends not on dimension but on codimensions.

However, some care must be taken due to some of the oddities associated with this greater flexibility. For example, a space stratified homeomorphic to a manifold stratified space may not be manifold stratified, as the definition of a manifold stratified space depends on the strata having the proper dimensions. That said, if $f$ is a stratified homeomorphism between manifold stratified spaces, then $f$ must preserve dimensions of strata as well as codimensions, so in particular the formal dimensions of $X$ and $Y$ must agree.

We also have a notion of stratified homotopy. For stratified homotopy equivalences, we will again want codimension to be preserved appropriately.
Definition 2.91. Let $X, Y$ be stratified spaces, and let $I$ be the unit interval with the trivial stratification. Endow $X \times I$ with its product stratification. Then a stratified map $H : X \times I \to Y$ is called a stratified homotopy; in particular for each stratum $S \subset X$, $H|_{S \times I}$ is contained in a single stratum of $Y$. If $f = H|_{X \times 0}$ and $g = H|_{X \times 1}$, we say that $f$ and $g$ are stratified homotopic stratified maps.

If $f : X \to Y$ and $g : Y \to X$ are stratified maps such that

1. there exist stratum stratified homotopies between $fg$ and $\text{id}_Y$ and between $gf$ and $\text{id}_X$,
2. for each stratum $S \subset X$, the codimension of the stratum $f(S)$ in $Y$ is equal to the codimension of $S$ in $X$,
3. for each stratum $S' \subset Y$, the codimension of the stratum $g(S')$ in $X$ is equal to the codimension of $S'$ in $SY$,

then we say that $f$ and $g$ are stratified homotopy equivalences, that $f$ and $g$ are stratified homotopy inverses to each other, and that $X$ and $Y$ are stratified homotopy equivalent.

Remark 2.92. Note that a stratified homotopy equivalence is possible if and only if there is a bijection between the set of strata of $X$ and the set of strata of $Y$ such that if $S \subset X$ and $T \subset Y$ correspond under the bijection then $f(S) \subset T$ and $g(T) \subset S$. Of course this remark also applies to the special case of a stratified homeomorphism.

Remark 2.93. Some care must be taken not to confuse stratified homotopies, as just defined, with stratum-preserving homotopies, as in defined in Section 2.6. Our stratum-preserving homotopies did not require the domains to be filtered spaces and only required that $H(\{x\} \times I)$ is contained in a single stratum of the codomain for each $x \in X$. The reader should note, however, that the two terms are generally interchangeable in the existing literature, with the context usually making it clear which is meant. We will attempt, however, to stay consistent with the definitions given here.

While we will not use it here, the reader should also note that if $f : X \to Y$ and $g : Y \to X$ are stratified maps whose compositions are stratum-preserving homotopy equivalent to the respective identity maps, we obtain a notion of stratum-preserving homotopy equivalence that does not impose the requirement on codimensions of strata.

Example 2.94. Suppose $X$ is a filtered space, that $\mathbb{R}^n$ is given the trivial filtration and that $\mathbb{R}^n \times X$ is given the product filtration as in Example 2.10. Then the inclusion $X \to \mathbb{R}^n \times X$ given by $x \mapsto (0, x)$ is a stratified homotopy equivalence.

If $cX$ is the cone on $X$ with vertex $v$, filtered as in Example 2.11, then $cX - \{v\}$ is stratified homeomorphic to $X \times \mathbb{R}$ and so stratified homotopy equivalent to $X$.

Definition 2.95. Suppose $X$ is a stratified space and that the inclusion $i : Z \hookrightarrow X$ is a stratified map such that, for some $m$, $i$ extends to a stratified homeomorphism from $\mathbb{R}^m \times Z$ onto some neighborhood of $i(Z)$. Then $i$ is called a normally nonsingular inclusion, and we will call the image $i(Z)$ a normally nonsingular subspace of $X$.

Example 2.96. Any of the inclusions $X \hookrightarrow cX$ determined by $x \mapsto (t, x)$ for some fixed $t > 0$ is a normally nonsingular inclusion.
2.8 Advanced topic: the topology of CS sets and intrinsic stratifications

In order to prove some of our more advanced results later, we will need a deeper understanding of CS sets, including results about intrinsic stratifications of CS sets and of PL pseudomanifolds. For reference purposes, this chapter is the most natural place to include such results, but we strongly urge the first-time reader to proceed on to our discussion of intersection homology in Section 3.6 and return here for results as they are needed. The first such necessity will be in our discussion of the invariance theorem, Theorem 5.52 in Section 3.6.

Our first lemma will be a general theorem of point-set topology concerning conical neighborhoods. In [61], the theorem is attributed to Stallings with references to [98] and [93]. The precise statement of the lemma does not seem to be contained in those references, though the proof we give is certainly a direct application of their techniques; in particular it is a slick combination of Stallings’s “invertible cobordisms” with an infinite process trick.

**Lemma 2.97.** If $X$ and $Y$ are compact topological spaces and there is a neighborhood $U$ of the vertex $v$ of $cX$ such that $(U, v) \cong (cY, v)$, then $(cX, v) \cong (cY, v)$.

**Proof.** For $0 < t \leq 1$, let $c_t X = [0, t) \times X / \sim$, which we identify as a subset of $cX = c_1 X = [0, 1) \times X / \sim$. Each $c_t X$ is a retraction of $cX$ along its cone lines. Similarly, for $0 < t < 1$, let $\tilde{c}_t X = [0, t) \times X / \sim \subset cX$. Note that $c_1 X$ is the interior of $\tilde{c}_1 X$.

We need to build a nested collection of neighborhoods of $v$. Since $U$ is a neighborhood of $v$ and $X$ is compact, there is a $\delta \in (0, 1)$ such that $\tilde{c}_\delta X \subset U$. This follows from the Tube Lemma [78, Lemma 26.8]: Let $\pi : [0, 1) \times X \to cL$ be the quotient map. By definition of the quotient topology, $U \subset cX$ is open if and only if $\pi^{-1}(U) \subset [0, 1) \times X$ is open. In order for $U$ to contain the cone point, $\pi^{-1}(U)$ must contain $\{0\} \times X$, and since $X$ is compact, the Tube Lemma tells us that there is an open subset of $[0, 1) \times X$ of the form $[0, s) \times X$ contained in $\pi^{-1}(U)$. But then if $\delta = s/2$, $\tilde{c}_\delta X \subset U$. Similarly, using the homeomorphism $h : (cY, v) \to (U, v)$, there is a $\mu$ such that $h(\tilde{c}_\mu Y)$ is contained in $c_\delta X$. Further applications of the argument then provide $\gamma, \mu$ such that $\tilde{c}_\gamma X \subset h(c_\mu Y)$ and $h(\tilde{c}_\mu Y) \subset c_\gamma X$.

Let $P = \tilde{c}_\gamma X - h(c_\mu Y)$, $Q = h(c_\mu Y) - c_\gamma X$ and $R = \tilde{c}_\delta X - h(c_\mu Y)$. Observe that $P$ has disjoint boundary components homeomorphic to $X$ and $Y$ and that the boundary components can be taken to have disjoint collars; similar statements hold for $Q$ and $R$.

We will write $PQ$ to stand for the union of $P$ and $Q$ along their common boundary that is homeomorphic to $X$, and $QR$ for the union of $Q$ and $R$ along the common boundary that is homeomorphic to $Y$. Notice that

\[ PQ \cong h(\tilde{c}_\mu Y) - h(c_\mu Y) \cong h(\tilde{c}_\mu Y - c_\mu Y) \cong h([0, 1) \times Y) \cong [0, 1] \times Y \]

and

\[ QR \cong \tilde{c}_\delta X - c_\gamma X \cong [0, 1] \times X. \]

\[ \text{Note: there is a typo in the conclusion of this theorem in its statement as Proposition 1 of [61]: in King’s notation, the last symbol in the statement should be } (\tilde{C}Y, *). \]

\[ \text{The assumption of compactness does not occur in [61]. I am not sure whether or not the lemma still holds without this assumption, but the compact case will be sufficient for our purposes.} \]
We next claim that also $RQ \cong Y \times [0, 1]$, where $RQ$ is the union of $Q$ and $P$, now along their common copy of $X$. To see this, we observe (using the collars of the boundaries and continuing to use concatenation to represent union along common boundaries) that

$$RQ \cong ([0, 1] \times Y)RQ \cong (PQ)RQ \cong P(QR)Q \cong P([0, 1] \times X)Q \cong PQ \cong [0, 1] \times Y.$$

Now consider the infinite union

$$(\bar{c}_\gamma X)(QR)(QR)(QR) \cdots \cong (\bar{c}_\gamma X)([0, 1] \times X)([0, 1] \times X) \cdots \cong cX.$$  

But if we regroup, this is the same as

$$(\bar{c}_\gamma X)(RQ)(RQ) \cdots \cong h(\bar{c}_\mu Y)([0, 1] \times Y)([0, 1] \times Y) \cdots \cong \bar{c}_\mu Y([0, 1] \times Y)([0, 1] \times Y) \cdots \cong cY,$$

This completes the proof.

**Corollary 2.98.** Let $X$ and $X'$ be two CS set stratifications of the same underlying topological space, say $|X|$. Let $x \in |X|$, and let $N, N'$ be distinguished neighborhoods of $x$ in $X$ and $X'$, respectively. Then $N$ and $N'$ are homeomorphic as topological spaces.

**Proof.** Any distinguished neighborhood of $x$ in some CS set stratification has the form $\mathbb{R}^k \times cL$ by definition. Suppose $N \cong \mathbb{R}^i \times cL$ and $N' \cong \mathbb{R}^j \times cL'$ are two such distinguished neighborhoods. Notice that $(N, x) \cong (\mathbb{R}^i \times cL, x) \cong (c(S^{i-1} \ast L), x)$ and $(N', x) \cong (\mathbb{R}^j \times cL', x) \cong (c(S^{j-1} \ast L'), x)$, where $\ast$ indicates the join of two spaces. By contracting along cone lines, we may assume, up to homeomorphism, that $N' \subset N$. But now, by the lemma, $(N, x) \cong (N', x)$.

Next we discuss the intrinsic topology of CS sets. In particular, we will need to utilizes the fact that all stratifications of a CS set have a common coarsening, which can then be used for comparison amongst stratifications. More precisely, if $X$ is a CS set, then a coarsening of a filtration $\{X^i\}$ of $X$ is a second filtration $\{Y^j\}$ (of the same underlying space and of the same formal dimension) such that each stratum of the $Y$ filtration is a union of strata of the $X$ filtration. We will see that each CS set possesses an *intrinsic coarsest stratification*, which is denoted in [61] by $X^*$. 

So we first must construct $X^*$. Again we follow King [61], who credits Dennis Sullivan with the construction, though see also [51].

**Definition 2.99.** Let $X$ be a CS set of formal dimension $n$, and define an equivalence relation $\sim$ on $X$ such that two points $x_0, x_1 \in X$ are equivalent if they possess neighborhoods $U_0, U_1$ such that $(U_0, x_0) \cong (U_1, x_1)$.

It is clear that this is an equivalence relation; we next develop a needed property.

**Lemma 2.100.** If $x_0, x_1$ are both in the same stratum of $X$, then $x_0 \sim x_1$. 

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Proof. Let \( S \) be a stratum of \( X \) of dimension \( i \), and let \( x_0 \in S \). We will show that the set \( W \) of points of \( S \) that are equivalent to \( x_0 \) is both open and closed in \( S \). Since \( S \) is connected, it will follow that \( W = S \), which will prove the lemma.

First, suppose \( x \in W \) and notice that, by definition of CS sets, \( x \) has a neighborhood \( U \) homeomorphic to \( \mathbb{R}^i \times cL \) with \( x \) corresponding to the point \( 0 \times v \) and with \( S \cap U \cong \mathbb{R}^i \times \{v\} \). But since \( \langle \mathbb{R}^i \times cL, 0 \times v \rangle \cong \langle \mathbb{R}^i \times 0L, z \times v \rangle \) for any \( z \in \mathbb{R}^i \), we see that we must have \( x \sim y \) for any \( y \in S \cap U \). Hence \( W \) is open in \( S \).

Next, suppose \( y \in \hat{W} \), where \( \hat{W} \) is the closure of \( W \) in \( S \). Then again \( y \) has a distinguished neighborhood \( V \) homeomorphic to \( \mathbb{R}^i \times cL \) with \( y \) corresponding to the point \( 0 \times v \) and with \( S \cap V \cong \mathbb{R}^i \times \{v\} \). Since \( y \in \hat{W} \), there must be some \( x \in S \) corresponding to \( z \times v \in \mathbb{R}^i \times \{v\} \) and contained in \( W \). But now again since \( \langle \mathbb{R}^i \times cL, 0 \times v \rangle \cong \langle \mathbb{R}^i \times cL, z \times v \rangle \), we have \( y \sim x \sim x_0 \). So \( W \) is also closed, and it follows that \( W = S \). \( \Box \)

Now, since any two points in a stratum of \( X \) are equivalent, it follows that for any CS set filtration of \( X \), the equivalence classes under \( \sim \) must be unions of strata of the filtration. We let \( Y^i \) be the union of the equivalence classes that only contain strata of dimension \( \leq i \).

**Lemma 2.101.** Given a CS set \( X \), let \( Y^i \) be the union of the equivalence classes that only contain strata of \( X \) of dimension \( \leq i \). The spaces \( Y^i \) filter \( X \) as a CS set. The resulting CS set stratification\(^{20}\}{Y^i} \) does not depend on the initial stratification of \( X \) as a CS set.

**Definition 2.102.** Given a space \(|X|\) that can be stratified as a CS set, let \( X^* \) denote the stratification constructed in Lemma 2.101. This is called the *intrinsic stratification* of \( X \).

**Proof of Lemma 2.101.** Let us first demonstrate that the \( Y^i \) defined in the lemma do not depend on the initial filtration of \( X \) as a CS set. Let \( \hat{X} \) denote an alternative CS set stratification of \( X \), and let \( \hat{Y}^i \) denote the corresponding stratification using equivalence classes, i.e. each \( Y^i \) is defined to be the union of the equivalence classes that only contain strata of \( \hat{X} \) of dimension \( \leq i \). The key to the demonstration is the following observation: Suppose \( x \in X \) has a neighborhood \( N \) that is stratified homeomorphic to \( \mathbb{R}^j \times cL \) for some compact filtered \( L \). Then \( x \) is equivalent to a point in some stratum of \( X \) of dimension at least \( j \). Indeed, consider the points of \( N \) contained in the homeomorphic image in \( X \) of \( \mathbb{R}^j \times \{v\} \). All the points in this \( j \)-dimensional set are clearly equivalent. We know that each equivalence class is a union of strata; so, if the equivalence class of \( x \) contained only strata of dimension \( < j \), this would create a contradiction, as the union of strata of dimension \( < j \) cannot cover a \( j \)-dimensional set due to the niceness of the local conical structures. Therefore, \( x \) must be equivalent to a point in a \( j \)-dimensional stratum. Now, suppose \( x \in Y^i \). By definition, \( x \) is not equivalent to any point in a stratum of \( X \) of dimension \( > i \). Therefore, by our immediately preceding argument, \( x \) cannot have a neighborhood homeomorphic to \( \mathbb{R}^j \times cL \) for any \( j > i \). In particular, \( x \) must be contained in a skeleton of \( \hat{X} \) of dimension \( \leq i \), and the same must be true of all points equivalent to \( x \), i.e. all points equivalent to \( x \) are contained in \( \hat{X}^i \). But this implies that \( x \in \hat{Y}^i \). So \( Y^i \subseteq \hat{Y}^i \). The equivalent argument then shows that \( \hat{Y}^i \subseteq Y^i \), so the filtrations are the same, justifying the claim that \( X^* \) is intrinsic to \( X \).

\(^{20}\)Recall that a filtration yielding a CS set is automatically a stratification by Lemma 2.34.
Next we show that the $Y^i$ are closed sets of $X$. Suppose $x \in X - Y^i$. It is not necessarily true that $x \in X - X^i$, but $x$ must be equivalent to a point in a stratum of dimension greater than $i$ (or else $x \in Y^i$), and so $x$ has a neighborhood $U$ homeomorphic to a neighborhood of some point in $X - X^i$. But then every point of $U$ must be equivalent to a point in $X - X^i$ by the definition of the equivalence relation. So no point of $U$ is contained in $Y^i$, so $Y^i$ is closed. Thus $Y^0 \subset \ldots \subset Y^n = X$ is a filtration of $X$.

Now, suppose $x \in Y^i - Y^{i-1}$. Then $x$ is equivalent to a point in an $i$-dimensional stratum of $X$ since, by definition of $Y^i$, $x$ is in an equivalence class containing strata of $X$ of dimension at most $i$, but if $x$ is in an equivalence class containing only strata of dimension at most $i - 1$, then $x$ would be in $Y^{i-1}$. So $x$ has a neighborhood $U$ that is homeomorphic to a neighborhood $V$ of a point $z$ of an $i$-dimensional stratum of $X$. But by assumption $z$ has a neighborhood homeomorphic to $\mathbb{R}^i \times cL$ for some compact filtered $L$. By contracting $\mathbb{R}^i$ and $cL$ by homeomorphisms if necessary, we may assume that $z$ has a neighborhood $W \cong \mathbb{R}^i \times cL \subset V$. But then by inverting the homeomorphism $U \to V$, we see that $x$ must also have a neighborhood homeomorphic to $\mathbb{R}^i \times cL$. Furthermore, the induced homeomorphism $h$ from $\mathbb{R}^i \times cL$ to a neighborhood $N$ of $x$ must have $h(\mathbb{R}^i \times \{v\}) \subset Y^i$ or else we could extend the equivalence relation from $x$ to other points not in $Y^i$. Now, let us identify $L$ with, say, $\{0\} \times \{1/2\} \times L \subset \mathbb{R}^i \times cL$; we refilter $L$ according to its intersections with $h^{-1}(Y^i)$ (which must be closed subsets of $L$ as $Y^i$ is closed in $X$ and $L$ is closed in $\mathbb{R}^i \times cL$). Let $L^*$ denote this filtration on $L$. Then $N$ is stratified homeomorphic to $\mathbb{R}^i \times cL^*$, since if $s$ is a point of $L$, then every point of $h(\mathbb{R}^i \times (0,1) \times s)$ must lie within a single equivalence class in $X$ (since they all have homeomorphic neighborhoods). This shows that $X$, filtered by the $Y^i$ is a CS set.

**Remark 2.103.** The proof of the lemma demonstrates that, no matter what stratification of $X$ as a CS set we begin with, $X^*$ is a coarsening of $X$. Therefore, $X^*$ is the coarsest possible CS set stratification of $X$.

**Example 2.104.** Suppose $X = X^n = M$ is a smooth $n$-dimensional manifold and that $V \subset M$ is a smooth submanifold so that $X$ is filtered as $V \subset M$. Then every point $x \in M$ has a neighborhood homeomorphic to $(\mathbb{R}^n, x)$, so every point of $M$ is equivalent and the intrinsic stratification is the trivial stratification on $M$.

**Lemma 2.105.** Let $U$ be an open subset of the CS set $X$. Then $U$ is a CS set and $U^*$ agrees with the restriction of the stratification of $X^*$ to $U$. In other words, $(U^*)^i = (X^*)^i \cap U$.

**Proof.** If we filter $U$ by $U^i = (X^i \cap U)$, then the $i$-dimensional strata of $U$ are the connected components of the intersection of $U$ with the strata of $X$. These will be open subsets of $i$-manifolds and so $i$-manifolds themselves. The locally cone-like condition is a local condition, and so it is satisfied in $U$ at each point since it is satisfied in $X$ at each point. By Lemma 2.34 it follows that $U$ is a CS set, and therefore $U$ possesses an intrinsic stratification $U^*$ by Lemma 2.101. As the equivalence relation of Definition 2.99 is also determined entirely by local conditions, we see that two points in $U$ are equivalent if and only if they are equivalent in $X$. The last claim of the lemma therefore follows from the constructions of $U^*$ and $X^*$. 

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We will need one more lemma, which is a slight (obvious) generalization of 
Lemma 2.

**Lemma 2.106.** Let \( M \) be a manifold and \( Y \) a filtered space such that \( M \times Y \) is a CS set. Then \((M \times Y)^*\) is stratified homeomorphic to \( M \times Z \), where \( Z \) is some coarsening of \( Y \). Furthermore, if \( M_1, M_2 \) are two \( n \)-manifolds and \( Z_1, Z_2 \) are coarsenings of \( Y \) such that \((M_i \times Y)^*\) is stratified homeomorphic to \( M \times Z_i \), for \( i = 1, 2 \), then \( Z_1 = Z_2 \). In other words, the coarsening of \( Y \) does not depend on the specific choice of \( n \)-manifold.

**Proof.** Let \( y \in Y \), and notice that all points of \( M \times \{y\} \) are equivalent under \( \sim \). So \((M \times Y)^*\) must have the form \( M \times Z \) for some filtration \( Z \) of \( Y \). Furthermore, since \((M \times Y)^*\) is coarser than \( M \times Y \), \( Z \) must be coarser than \( Y \). The last statement of the first paragraph follows because the definition of the intrinsic coarsest stratification is local, and so the filtration \( Z \) depends only on the subspace \( \mathbb{R}^n \times Y \).

Notice that the filtered space \( Z \) of the lemma is not necessarily a cone, and so if \( c \) is the cone point of \( cW \), \( M \times \{v\} \) might not be a stratum; it may be a subset of a larger stratum. For example, if \( W = S^{n-1} \) and \( M = \mathbb{R}^k \), then \( M \times cW \) is homeomorphic to \( \mathbb{R}^{k+n} \), whose intrinsic stratification is trivial and so stratified homeomorphic to \( \mathbb{R}^{n+k} \cong \mathbb{R}^k \times \mathbb{R}^n \). Here the trivial filtration of \( \mathbb{R}^n \) provides the intrinsic stratification of \( cW \).

**Remark 2.107.** We will see below in Lemma 2.115 that, in the piecewise-linear world, Lemma 2.106 can be strengthened to the statement that if \( M \) is a PL \( n \)-manifold and \( Y \) a PL filtered space. Then \((M \times Y)^*\) is PL homeomorphic to \( M \times Y^* \). The proof relies strongly on facts of PL topology, so it is not clear that there are versions of this available in the topological world, even with additional restrictions.

**2.8.1 Intrinsic PL stratifications**

One can also consider intrinsic PL stratifications of PL spaces. These have been a historically important tool in the study of PL spaces (e.g. [3]), and they will be particularly important for us in our construction of L-classes in Section 10.4. They are also useful in studying bordism groups of PL pseudomanifolds and the resulting bordism homology theories, see [4, 26].

Recall that every PL space is a CS set with respect to some filtration; see Lemma 2.61. We can then set up an equivalence relation \( \sim_{PL} \) analogous to that of Definition 2.99 but requiring PL homeomorphisms of neighborhoods; we will say that points satisfying the PL version of Definition 2.99 are PL equivalent. Then PL analogues of Lemmas 2.100 and 2.101 hold with the identical proofs, assuming each homeomorphism is a PL homeomorphism. For reference purposes, we restate them in this context:

**Lemma 2.108.** If \( X \) is a PL CS set and \( x_0, x_1 \) are both in the same stratum of \( X \), then \( x_0 \sim_{PL} x_1 \).

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\(^{21}\) One could potentially also study bordism groups of topological pseudomanifolds or CS spaces, but the surgery techniques that are involved in the PL setting (see, e.g., [95, 81, 45]) would be very difficult to emulate topologically.
Lemma 2.109. Given a PL CS set $X$, let $Y^i$ be the union of the PL equivalence classes that only contain strata of $X$ of dimension $\leq i$. The spaces $Y^i$ filter $X$ as a PL CS set. The resulting CS set stratification $\{Y^i\}$ does not depend on the initial stratification of $X$ as a CS set.

One additional observation is needed for the second lemma: the intrinsic PL skeleta, say $Y^i_{PL}$, are PL subsets, i.e. they will be subcomplexes in an admissible triangulation of the PL CS set $X$. To see this, let $K$ be a triangulation of $X$, and let $\sigma$ be a simplex of $K$. Then all points in $\tilde{\sigma}$ have the same PL neighborhood, the open star simplicial star of $\sigma$ in $K$. So if $Y^i_{PL}$ intersects $\tilde{\sigma}$, it contains $\tilde{\sigma}$, and hence all of $\sigma$, since each $Y^i_{PL}$ is closed by the PL analogue of the arguments of Lemma 2.101. Therefore, every $Y^i_{PL}$ is a union of simplices, and so it is a subcomplex of the chosen triangulation.

Definition 2.110. If $X$ is a PL set, let $X^*$ denote the underlying space $X$ filtered with skeleta $\{Y^i_{PL}\}$. We call $X^*$ the intrinsic PL stratification of $X$.

Intrinsic PL stratifications are better behaved than purely topological intrinsic stratifications, as the following lemma demonstrates. This lemma will be useful in our development of L classes in Section 10.4.

Proposition 2.111. Let $X$ be an $n$-dimensional PL space that contains a dense $n$-dimensional PL manifold $M$. Then $X^*$ is a PL stratified pseudomanifold. If $X - M$ has dimension $\leq n-2$, then $X^*$ is a classical PL stratified pseudomanifold.

Proof. We will prove the proposition by induction on the dimension of $X$. If $X$ has dimension 0, then we must have that $X = X^*$ is a discrete set of points, and the proposition is immediate. Now, we assume we have proven the lemma for dimensions $< n$ and suppose that $X$ is $n$-dimensional.

By Lemma 2.61, $X$ can be filtered as a PL CS set, so, using that filtration to get started, we can apply the PL analogue of Lemma 2.101 to see that $X^*$ is a PL CS set. To show that $X^*$ is a stratified pseudomanifold as defined in Definition 2.57, we need to verify that the union of the regular strata of $X^*$ is dense in $X$ and that the links in $X^*$ are themselves recursive PL CS sets.

For the density requirement, we have assumed that $X$ possesses a dense manifold $M$. We claim that $M \subset (X^*)^n - (X^*)^{n-1}$ so that $(X^*)^n - (X^*)^{n-1}$ must also be dense in $X$. To verify the claim, let’s utilize the filtration, say $X'$, of $X$ guaranteed by Lemma 2.61; this was simply the simplicial filtration with respect to some admissible triangulation $K$ of $X$. Since $X^*$ coarsens all other filtrations by the PL analogue of Remark 2.103 every skeleton of $X^*$ is a union of skeleta of $X'$. By definition, we take as the $n-1$ skeleton of $X^*$ the union of the equivalence classes of points on $X$ that contain only strata of $X'$ of dimension $\leq n-1$. So if $x \in (X^*)^{n-1}$, $x$ is contained in an $n-1$ simplex of $K$ and $x$ is not equivalent to any point in $(X')^n - (X')^{n-1}$. But $(X')^n - (X')^{n-1}$ is a union of open $n$-simplices, so this means that $x$ cannot have an $n$-dimensional Euclidean neighborhood, so $x \notin M$. Therefore, if $x \notin (X')^n - (X')^{n-1}$ then $x \notin M$, and so $x \in M$ implies $x \in (X^*)^n - (X^*)^{n-1}$. This is the desired result. We also observe here that if $\text{dim}(X-M) \leq n-2$, then $\text{dim}((X^*)^{n-1}) \leq n-2$, etc.
and so $X^*$ can then have no codimension one strata, making it a classical PL stratified pseudomanifold once we have finished showing it is a PL stratified pseudomanifold.

Now we must consider the links of $X^*$ and show that, with the filtrations that are compatible with $X^*$, they are recursive $CS$ spaces.

Suppose $x \in X^*$ has a distinguished neighborhood $U$ filtered PL homeomorphic to $\mathbb{R}^i \times cL$ for some PL filtered space $L$. Since $X^*$ is a CS set, every point has some such neighborhood, and this assumption implies that $x$ is in an $i$-dimensional stratum of $X^*$. Let us identify $L$ as the subset $\{0\} \times \{1/2\} \times L \subset \mathbb{R}^i \times cL$, and let us identify $\mathbb{R}^i \times cL$ with $U$ so that we can think of $L$ as embedded in $X^*$. We claim that the given filtration on $L$ (the one compatible with it being a subset of $X$ in this way) is the intrinsic filtration on $L$. Notice that $\mathbb{R}^i \times (cL - \{v\}) \cong \mathbb{R}^{i+1} \times L$ is an open set of $X$, and since the intrinsic filtration is determined by local properties, its intrinsic filtration is the restriction of the intrinsic filtration on $X$. Suppose by way of contradiction that there is a coarser filtration $L^*$ of $L$. Since each stratum of $L^*$ is a union of strata of $L$, if $L^* \neq L$, there are points $y,z \in L$ that are in the same stratum of $L^*$ but not in the same stratum of $L$. Since $y,z$ are in the same stratum of $L^*$, they are equivalent in $L$ in the sense of the PL analogue of Definition 2.99 by Lemma 2.100. But then it is clear that $0 \times 1/2 \times y$ and $0 \times 1/2 \times z$ must be equivalent in $X$, which contradicts that $y$ and $z$ are in different strata of $L$. It follows that we must in fact have $L = L^*$, so the links of $X^*$ automatically inherit from $X^*$ their intrinsic stratifications. Therefore, it now suffices to show that $L^*$ contains a dense PL $n - i - 1$-manifold, and the result will follow by the induction hypothesis, which will tell us that $L^*$ is in fact a stratified pseudomanifold.

To see that $L$ possesses a dense PL $n - i - 1$ manifold, fix an admissible triangulation of $L$, and let $M_L$ be the union of the interiors of the $n - i - 1$ simplices of the triangulation. We claim that $M_L$ is dense in $L$. By way of contradiction, assume that $x \in L$ is a point that has no neighborhood that intersects $M_L$; this implies that $x$ is not a face of any $n - i - 1$ simplex of $L$, so $x$ has a neighborhood $V$ in $L$ that has dimension $< n - i - 1$. But continuing to think of $L$ as a subspace of $X$, if $x$ has a neighborhood $V$ of dimension $< n - i - 1$ in $L$, then $0 \times 1/2 \times x$ has a neighborhood homeomorphic to $\mathbb{R}^{i+1} \times V$ in $X$ that must have dimension $< n$. But this is a contradiction with $X$ containing a dense $n$-manifold. Therefore, every point of $L$ must be in a face of an $n - i - 1$ simplex and $M_L$ is dense in $L$. Thus, we can apply the induction argument and see that $L = L^*$ is a PL stratified pseudomanifold, and, in particular, a PL recursive $CS$ set.

The following corollaries are immediate:

**Corollary 2.112.** If $X$ is an $n$-dimensional PL pseudomanifold, then $X^*$ is a PL stratified pseudomanifold. If $X$ is a classical PL pseudomanifold, then $X^*$ is a classical PL stratified pseudomanifold.

**Corollary 2.113.** If $X$ is a PL space with a triangulation in which every simplex is a face of an $n$-simplex, then $X$ is an $n$-dimensional PL pseudomanifold. If $X$ is a PL space with a triangulation in which every simplex is a face of an $n$-simplex and such that every $n - 1$ simplex is the face of exactly two $n$-simplices, then $X$ is a classical $n$-dimensional PL pseudomanifold.
In the last case, the manifold for Proposition \([2.111]\) is the union of the interiors of the \(n\) and \(n-1\) simplices of the triangulation.

The PL version of Lemma \([2.105]\) holds by the same arguments used to prove that lemma:

**Lemma 2.114.** Let \(U\) be an open subset of the PL pseudomanifold \(X\). Then \(U\) is a PL pseudomanifold and \(U^*\) agrees with the restriction of the stratification of \(X^*\) to \(U\). In other words, \((U^*)^i = (X^*)^i \cap U\).

In the PL setting, we also have a stronger version of Lemma \([2.106]\).

**Lemma 2.115.** Let \(M\) be a PL \(n\)-manifold and \(Y\) a PL filtered space. Then \((M \times Y)^*\) is PL homeomorphic to \(M \times Y^*\).

**Proof.** Since the intrinsic stratification of a PL CS set is the coarsest stratification, and since \((M \times Y)^*\) and \(M \times Y^*\) are both PL stratified pseudomanifolds, \((M \times Y)^*\) must be a coarsening of \(M \times Y^*\). Suppose these are not PL CS sets. Then there must be two points, say \((t, x)\) and \((s, y)\), that are in the same stratum of \((M \times Y)^*\) but different strata of \(M \times Y^*\). This implies that \(x\) and \(y\) are in different strata of \(Y^*\). Since \((t, x)\) and \((s, y)\) are in the same stratum of \((M \times Y)^*\), they have PL homeomorphic star neighborhoods. Let \(\ell\) be the polyhedral link of \(x\) in \(Y\), i.e. \(x\) has a neighborhood \(\ell\) in \(Y\). The space \(\ell\) is unique up to PL homeomorphism by basic PL topology [86, Lemma 2.19]. Owing to the product structure on \(M \times Y^*\) and basic PL topology, the point \((t, x)\) then has a neighborhood of the form \(c(S^{n-1} \ast \ell)\), where \(S^{n-1} \ast \ell\) is the PL join of \(S^{n-1}\) with \(\ell\). Notice that \(S^{n-1} \ast \ell\) is also the nth suspension of \(\ell\). Similarly, if \(y\) has polyhedral link \(\ell'\) in \(Y\), then \((s, y)\) has a neighborhood in \(M \times Y\) of the form \(c(S^{n-1} \ast \ell')\). But since \((t, x)\) and \((s, y)\) are in the same stratum of \((M \times Y)^*\), by the uniqueness of polyhedral links, we must have \(S^{n-1} \ast \ell \cong S^{n-1} \ast \ell'\), where \(\cong\) denotes PL homeomorphism. But now we can invoke another basic result of PL topology to conclude that \(\ell \cong \ell'\) [75, Theorem 1]. This implies that \(c\ell \cong c\ell'\), so that \(x\) and \(y\) have PL homeomorphic neighborhoods in \(Y\). This is not quite enough yet to conclude that \(x\) and \(y\) are in the same stratum of \(Y^*\) as \(x\) and \(y\) could have homeomorphic neighborhoods but lie in different strata. However, now let us consider a path from \((t, x)\) to \((s, y)\) in \((M \times Y)^*\) in the stratum containing the two points. This is possible because the stratum is a connected PL set. The same arguments we employed above apply to any two points along the path, so if \((u, z)\) is such a point, then \(z\) is PL equivalent to \(x\) and \(y\) in \(Y\). Projecting the path to \(Y\) provides a path between \(x\) and \(y\) consisting entirely of points that are PL equivalent to both \(x\) and \(y\). Therefore, \(x\) and \(y\) must be in the same stratum of \(Y^*\). We have reached a contradiction, and so \((M \times Y)^* \cong M \times Y^*\).  

\(\boxdot\)

Notice that the proof of Lemma \([2.115]\) leans heavily upon PL topology.

**Intrinsic stratifications of PL pseudomanifolds with boundary.** It is also useful to have a notion of an intrinsic stratification for a PL pseudomanifold with boundary. This is a bit more delicate, as we know from Example \([2.80]\) that the notion of “boundary” itself can depend upon the stratification. Let us reconsider Example \([2.80]\) which applies just as
well if we assume everything PL. There, we considered a (now PL) \( \partial \)-manifold \( M \) with non-empty boundary (in the manifold sense) \( P \). If we let \( X \) be the PL filtered space \( P \subset M \), then we have a PL stratified pseudomanifold (without boundary!). In fact, we can easily verify that this is the intrinsic stratification of \( X \). But if we instead think of \( M \) as a PL \( \partial \)-stratified pseudomanifold with boundary \( P \), then it is reasonable to ask for a version of intrinsic stratification that continues to “know” that there is a boundary present.

In fact, no intrinsic PL stratification, following our previous definitions, could have a boundary. The reason for this is that we have defined PL \( \partial \)-stratified pseudomanifolds so that the boundary has a stratified collar. In other words, if \( X \) is a PL \( \partial \)-stratified pseudomanifold, \( \partial X \) must have a neighborhood in \( X \) PL stratified homeomorphic to the product \([0, 1) \times \partial X \), with \( \{0\} \times \partial X \) being taken to \( \partial X \subset X \) by the homeomorphism. In particular, then, if \( x \in \partial X \), then all the points in \([0, 1) \times \{x\}\) live in a single stratum of \( X \). However, the points \( \{t\} \times \{x\}, \) for \( 0 < t < 1 \), will have identical neighborhoods in \( X \), while \( \{0\} \times \{x\}\) will have a different neighborhood. Thus \( \{0\} \times \{x\}\) and the \( \{t\} \times \{x\}, \) for \( t > 0 \) cannot live in the same stratum of any intrinsic stratification. Thus no intrinsic stratification on \( X \) could have a boundary.

Nonetheless, there is still a way that we can usefully introduce intrinsic stratifications into the context of PL \( \partial \)-stratified pseudomanifolds.

**Definition 2.116.** Let \( X \) be a PL \( \partial \)-stratified pseudomanifold. We will say that \( X \) is naturally stratified if \( X - \partial X \) and \( \partial X \) are intrinsically stratified PL stratified pseudomanifolds.

**Example 2.117.** Consider again a trivially stratified PL \( \partial \)-manifold \( M \) with \( \partial M \neq \emptyset \). Then \( M \) is naturally stratified, as \( M \) and \( \partial M \) are both intrinsically stratified.

**Proposition 2.118.** Let \( X \) be a PL \( \partial \)-pseudomanifold. Then \( X \) can be naturally stratified. In other words, there exists a stratification \( \tilde{X} \) of \( X \) such that

1. \( \tilde{X} \) is a PL \( \partial \)-stratified pseudomanifold,
2. \( (\tilde{X}, \partial \tilde{X}) \) and \( (X, \partial X) \) have the same underlying PL space pairs,
3. \( \tilde{X} - \partial \tilde{X} \) and \( \partial \tilde{X} \) are intrinsically stratified PL pseudomanifolds.

**Proof.** By the definition of \( \partial \)-stratified pseudomanifolds, \( X - \partial X \) and \( \partial X \) are each stratified pseudomanifolds, and so by Corollary 2.112 each has an intrinsic stratification. Also by the definition of \( \partial \)-stratified pseudomanifolds, \( \partial X \) has a collar neighborhood in \( X \) that is PL homeomorphic to \([0, 1) \times X \). Consider the subspace of \( X \) PL homeomorphic to \([0, 1) \times \partial X \). If we let \( (X - \partial X)^* \) denote the intrinsic stratification, then by Lemma 2.114, the restriction of this stratification to \([0, 1) \times \partial X \) is \( (\partial X)^* \), which by Lemma 2.115 is \( (0, 1) \times (\partial X)^* \). Therefore, if we take \( (X - \partial X)^* \) and \([0, 1) \times (\partial X)^* \), we can glue these spaces along their common PL stratified subset \((0, 1) \times (\partial X)^* \). The resulting space is the desired \( \tilde{X} \). Notice that \( \tilde{X} \) does indeed meet the requirements to be a PL \( \partial \)-stratified pseudomanifold with \( \partial \tilde{X} = (\partial X)^* \).

\[22\text{Here we use the collar homeomorphism to provide coordinates for points in the collar.}\]
Remark 2.119. Notice that the proof of Proposition 2.118 depends on Lemma 2.115, and, as noted in Remark 2.107, we do not necessarily have this available in the topological world. This thwarts our attempts to define appropriate analogous natural stratifications of $\partial$-stratified pseudomanifolds in terms of intrinsically stratified CS sets.

**Corollary 2.120.** Suppose $X$ is a PL space possessing a triangulation such that

1. every simplex is a face of an $n$-simplex,
2. every $(n - 1)$-simplex is a face of either one or two $n$-simplices
3. if $B$ is the union of all $(n - 1)$-simplices of $X$ that are a face of only one $n$-simplex, then $B$ has a collar, meaning that there is a PL embedding of $[0, 1) \times B$ into $X$ taking $\{0\} \times B$ to $B \subset X$.

Then $X$ is an $n$-dimensional PL $\partial$-pseudomanifold. If each $n - 2$ simplex of $B$ is a face of exactly two $n - 1$ simplices of $B$, then $X$ is an $n$-dimensional classical $\partial$-stratified pseudomanifold.

**Proof.** If we let $M$ denote the union of the interiors of the $n$-simplices of the triangulation and the interiors of the $n - 1$ simplices of the triangulation that are not in $B$, then $M$ is a manifold that is dense in the PL set $X - B$ and $(X - B) - M$ has dimension $\leq n - 2$. So by Proposition 2.111, $X - B$ is a classical PL pseudomanifold. Similarly, the interiors of the $n - 1$ simplices of $B$ are dense in $B$, so $B$ is a PL pseudomanifold, also by Proposition 2.111. By assumption, $B$ is collared in $X$, so by the same arguments as used in Proposition 2.118, we can glue together the intrinsic stratifications of $X - B$ and $[0, 1) \times B$ to obtain a PL $\partial$-stratified pseudomanifold with $X$ as its underlying space. If $B$ satisfies the extra condition, then both $X - B$ and $B$ will be classical PL pseudomanifolds by Proposition 2.111, so $X$ will be a classical PL $\partial$-stratified pseudomanifold. \[\square\]

### 2.9 Advanced topic: products and joins of filtered spaces, CS sets, and pseudomanifolds

This section contains proofs that products and joins of CS sets and stratified pseudomanifolds are again CS sets and stratified pseudomanifolds. We also show that the product of $\partial$-stratified pseudomanifolds are $\partial$-stratified pseudomanifolds and that the product of intrinsically stratified PL pseudomanifolds is intrinsically stratified. These are obviously desirable results, and, as for the previous section, this material is included here because it fits naturally with our chapter on stratified spaces. However, once again, the first-time reader is encouraged to skip this material for now in order to “get on with it” and to come back to this section as needed later on. In fact, most of the results of this section are the expected ones, so this section should serve more as a reference for the purposes of completeness.

As we observed in Example 2.10, if $X$ and $Y$ are filtered spaces, then $X \times Y$ has a natural product filtration such that $(X \times Y)^j = \bigcup_{j+k=i} X^j \times Y^k$. If $X$ and $Y$ have respective formal dimensions $n$ and $m$, then this product has formal dimension $m + n$. 54
Lemma 2.121. The strata of $X \times Y$ all have the form $S \times T$, where $S$ is a stratum of $X$ and $T$ is a stratum of $Y$.

Proof. By a basic set-theoretic argument, which we leave to the reader,

$$(X \times Y)^i - (X \times Y)^{i-1} = (\bigcup_{j+k=i} X_j \times Y^k) - (\bigcup_{j+k=i-1} X^j \times Y^k) = \bigcup_{j+k=i} ((X^j \times X^{j-1}) \times (Y^k \times Y^{k-1}))).$$

Now consider $S \times T$ for $S$ a stratum of $X$ and $T$ a stratum of $Y$. To be specific, suppose $S$ has formal dimension $j$ and $T$ has formal dimension $k$. Since $S$ and $T$ are connected, the set $S \times T$ is connected. We must show that each such $S \times T$ is in fact a connected component of $(X \times Y)^i - (X \times Y)^{i-1}$. As $S \times T$ is connected, it suffices to show that $S \times T$ is separated from each other $S' \times T'$ with $S'$ a stratum of $X$, $T'$ a stratum of $y$, and $\dim(S') + \dim(T') = j + k = i$, i.e. that $(S \times T) \cup (S' \times T')$ is not connected. By [73, Lemma 23.1], we need to show that neither of $S \times T$ or $S' \times T'$ contains a limit point of the other. The arguments are symmetric, so we will show that $S' \times T'$ cannot contain a limits point of $S \times T$.

Suppose $(x, y)$ is a limit point of $S \times T$. If $(x, y) \in S \times T$, then clearly $(x, y) \notin (S' \times T')$ as $S \cap S' = T \cap T' = \emptyset$, so $(S \times T) \cap (S' \times T')$ is non-empty only if $S = S'$ and $T = T'$. So, suppose $(x, y) \notin S \times T$. Then either $x$ is a limit point of $S$ not contained in $S$ or $y$ is a limit point of $T$ not contained in $T$. Suppose $x \notin S$. As $X^j$ is closed, $x \in X^j$, but $x$ cannot be contained in a $j$-dimensional stratum because $S$ is a connected component of $X^j \times X^{j-1}$, and so $S$ is separated from any other $j$-dimensional stratum of $X$. Therefore, $x \in X^{j-1}$. Applying the same argument to $y$, we have that either $x \in X^{j-1}$ or $y \in Y^{k-1}$, so $(x, y) \in (X \times Y)^{i-1}$. Thus $(x, y)$ is not contained in any of the other strata of dimension $i$, in particular $S' \times T'$.

We will see that taking products preserves other nice structure. For example the products of CS sets are CS sets and the products of pseudomanifolds are pseudomanifolds. In order to verify these claims, it is necessary to study not just products of filtered spaces, but also their joins, as the joins arise as links in product spaces.

We recall the construction of the join of two spaces $X$ and $Y$; see, e.g., [53, Sections 0 and 4.G]. In all of our applications, $X$ and $Y$ will be compact. Conceptually, the join $X \ast Y$ of two spaces is the union of all line segments connecting a point of $X$ to a point of $Y$. A more constructive definition is that $X \ast Y$ is the quotient space of $X \times [0, 1] \times Y$ under the relations $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$, where $x, x' \in X$ and $y, y' \in Y$. As for cones, it is convenient to parameterize points of the join with coordinates $(x, t, y)$, noting that the coordinate system is degenerate when $t = 0$ or $t = 1$. We can observe that $X \ast Y$ contains canonical copies of $X$ and $Y$ as the respective images of $X \times \{0\} \times Y$ and $X \times \{1\} \times Y$ of the quotient map. We have $X \ast Y - X \cong cX \times Y$, while $X \ast Y - Y \cong X \times cY$. Of course, $X \ast Y - (X \times Y) \cong X \times (0, 1) \times Y$. If we identify $X \times Y$ with the subset $X \times \{1/2\} \times Y \subset X \ast Y$, then we can also identify $X \ast Y$ as the union of closed subsets by

$$X \ast Y \cong (X \times cY) \cup_{X \times Y} (cX \times Y).$$
These descriptions allow us to introduce a filtration for $X \ast Y$ if $X$ and $Y$ are filtered. If the respective formal dimensions of $X$ and $Y$ are $n$ and $m$, then $X \ast Y$ will have formal dimension $m + n + 1$. Looking at $X \times (0, 1) \times Y$, we can use the product filtration, letting $(0, 1)$ be filtered trivially, i.e.

$$(X \times (0, 1) \times Y)^i = \cup_{j + k = i - 1} X^j \cup (0, 1) \cup Y^k.$$

On $X \ast Y - Y$, the product filtration on $X \times cY$ is $(X \times cY)^i = \cup_{j + k = i} X^j \times (cY)^k$, but if $k > 0$, each $X^j \times (cY)^k$ has the form $X^j \times c(Y^{k-1})$. Therefore, if we consider $X \ast Y - Y$ with this product filtration, the induced filtration on the subset $((X \ast Y) - Y) - X$ is consistent with the product filtration on $X \times (0, 1) \times Y$. Similarly, looking at the product filtration on $X \ast Y - X \cong cX \times Y$ with its product filtration, the induced filtration on the subset $((X \ast Y) - X) - Y$ is consistent with the product filtration on $X \times (0, 1) \times Y$. Therefore, assembling $X \ast Y$ as the union of $X \times cY$ and $cX \times Y$ with their product filtrations provides a natural join filtration on $X \ast Y$. Even better, we observe that the union in $X \ast Y$ of the set $X^j \times (cY^{k-1}) \subset X \times cY$ with the set $c(X^j) \times Y^{k-1} \subset cX \times Y$ is itself the joint $X^j \ast Y^{k-1}$, so we can write the $i$-skeleton of $X \ast Y$ as $\cup_{a+b=i} X^a \ast Y^b$. Notice that the reason we have $a + b = i - 1$ instead of $a + b = i$ is that the $I$ factor in the join adds a dimension that is only implicit in the notation. Also observe that we allow in the formula the possibility that $a$ or $b$ is equal to $-1$, letting $X^{-1} = Y^{-1} = \emptyset$; by convention $Z \ast \emptyset = \emptyset \ast Z = Z$. So the $i$-skeleton of $X \ast Y$ includes the $i$-skeleta in $X$ and $Y$. One can also compute that the set of $i$-dimensional strata of $X \ast Y$ comprises the $i$-strata of $X$, the $i$-strata of $Y$, and the $i$-strata of $X \times (0, 1) \times Y$.

**Lemma 2.122.** If $X$ and $Y$ are (recursive) CS sets, then so is $X \times Y$ with the product filtration. If $X$ and $Y$ are compact (recursive) CS sets, then so is $X \ast Y$ with the join filtration.

**Proof.** Since the strata of $X \times Y$ have the form $S \times T$, where $S$ is a stratum of $X$ and $T$ is a stratum of $Y$, the strata of $X \times Y$ are manifolds. Similarly, as the strata of $X \ast Y$ are strata of $X \times Y$, or have the form $S \times (0, 1) \times T$ where $S$ and $T$ are respective strata of $X$ and $Y$, the strata of $X \ast Y$ are manifolds.

To verify the locally-cone-like property, we will proceed by a simultaneous induction on $\dim(X) + \dim(Y)$ for $X \times Y$ and $X \ast Y$. We first dispense with some trivial cases by noting that if either $X$ or $Y$ is empty, then so is $X \times Y$, and if $X$ is empty, then $X \ast Y = Y$, while if $Y$ is empty, $X \ast Y = X$. So the result is established whenever $\dim(X)$ or $\dim(Y)$ is $< 0$. If $\dim(X) = \dim(Y) = 0$, then both $X$ and $Y$ are discrete unions of points, and hence so is $X \times Y$, which is a recursive CS set. If $\dim(X) = \dim(Y) = 0$ and $X$ and $Y$ are compact (and so finite), then $X \ast Y$ is the union of all intervals between $X$ and $Y$, with $(X \ast Y)^0$ consisting of the $\#X + \#Y$ points in $X$ and $Y$ and $(X \ast Y)^1$ consisting of $(\#X)(\#Y)$ open intervals. The link of each point of $X$ is homeomorphic to $Y$, and the link of each point of $Y$ is homeomorphic to $X$. So $X \ast Y$ is a recursive CS set.

We will also need to consider separately the case of $X \times Y$ with $\dim(X) + \dim(Y) = 1$. If either $\dim(X)$ or $\dim(Y)$ is $< 0$, the product is empty and there is nothing to prove.
Otherwise, one of $X$ or $Y$ must be 0-dimensional. Choosing $\dim(X) = 0$ without loss of generality, we then have $X \times Y \cong \Pi_{\neq} X Y$. In other words, $X \times Y$ is a disjoint collection of copies of $Y$, one for each point of $X$, and again the conclusion is trivial.

Now, let $A_n$ be the statement that if $\dim(X), \dim(Y) \geq 0$, and $\dim(X) + \dim(Y) \leq n$ then $X \times Y$ is a (recursive) CS set if $X$ and $Y$ are, and let $B_n$ be the statement that if $X$ and $Y$ are compact, $\dim(X), \dim(Y) \geq 0$, and $\dim(X) + \dim(Y) \leq n$ then $X \ast Y$ is a (recursive) CS set if $X$ and $Y$ are. We will show that $A_{n+1} \Rightarrow B_n$ if $n \geq 0$ and that $B_{n-2} \Rightarrow A_n$ if $n - 2 \geq 0$. Thus we have the chain of implications

$$B_0 \Rightarrow A_2 \Rightarrow B_1 \Rightarrow A_3 \Rightarrow B_2 \Rightarrow A_4 \Rightarrow B_3 \Rightarrow \ldots .$$

Together with our low dimensional cases, this will demonstrate the lemma.

First, we assume $A_{n+1}$, with $n \geq 0$, and consider $X \ast Y$ with $X$ and $Y$ compact and $\dim(X) + \dim(Y) = n \geq 0$. In our initial discussion, we observed that $X \ast Y$ is the union of the open subsets $X \times cY$ and $cX \times Y$. If $X$ is a (recursive) CS set, then so is $cX$, and similarly for $Y$. Thus $X \times cY$ and $cX \times Y$ are (recursive) CS sets by our assumption that $A_{n+1}$ holds, as $\dim(X) + \dim(cY) = \dim(cX) + \dim(Y) = n + 1$. Since the (recursive) locally cone-like condition is a local condition, it follows then for all points in $X \ast Y$.

Next, let us assume $B_{n-2}$, with $n - 2 \geq 0$, and let $\dim(X) + \dim(Y) = n$. We will demonstrate $A_n$. Let $(x, y) \in X \times Y$ with $x$ in a stratum $S \subset X$ and $y$ in a stratum $T \subset Y$ with $\dim(S) = j, \dim(T) = k$. Then $x$ has a distinguished neighborhood $N$ in $X$ with a homeomorphism $h_N : U \times cL \rightarrow N$ such that $h_N(U \times cL^a) = X \ast X^{a+1} \cap N = N^j \ast N^j$ for all $a \geq -1$, and $y$ has a distinguished neighborhood $M$ in $Y$ with a homeomorphism $h_M : V \times cK \rightarrow M$ such that $h_M(V \times cK^b) = Y \ast Y^{b+1} \cap M = M^k \ast M^k$ for all $b \geq -1$. By the definition of distinguished neighborhoods, we assume $L$ and $K$ are compact and that they are recursive if $X$ and $Y$ are. Then $N \times M$ is a neighborhood of $(x, y)$ in $X \times Y$, and $(h_N \times h_M)^{-1}$ provides an isomorphism between $N \times M$ and

$$U \times cL \times V \times cK \cong U \times V \times cL \times cK \cong (U \times V) \times (L \ast K),$$

where we have used the basic topological fact that $cL \ast cK \cong c(L \ast K)$, where $L \ast K$ is the join of $L$ and $K$. Since $\dim(L) \leq \dim(X) - 1$ and $\dim(K) \leq \dim(Y) - 1$, we have $\dim(L) + \dim(K) \leq \dim(X) + \dim(Y) - 2 = n - 2$. So $L \ast K$ is a (recursive) CS set by the hypothesis that $B_{n-2}$ holds. This provides a distinguished neighborhood of $(x, y)$ provided we verify the compatibility of the filtrations.

From the definitions, the product of the skeleta $N^{j+a+1}$ and $M^{k+b+1}$ are contained in the $j + k + a + b + 2$ skeleton of $N \times M$. Via $(h_N \times h_M)^{-1}$, this product corresponds to

$$U \times cL^a \times V \times cK^b \cong U \times V \times cL^a \times cK^b \cong U \times V \times c(L^a \ast K^b).$$

This formula holds even if one or both of $a$ and $b$ are $-1$. So we have

$$N^{j+a+1} \ast M^{k+b+1} \cong U \times V \times c(L^a \ast K^b).$$

Taking the unions over all $a, b \geq -1$ such that $a + b = i$, we see that $(h_N \times h_M)^{-1}$ takes the $j + k + i + 2$-skeleton of $N \times M$ homeomorphically onto

$$U \times V \times c(\{i + 1 \text{ skeleton of } K \ast L\}),$$

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as desired.

**Corollary 2.123.** If \( X \) and \( Y \) are topological stratified pseudomanifolds then so are \( X \times Y \), and, if \( X \) and \( Y \) are compact, so is \( X \star Y \). Similarly, if \( X \) and \( Y \) are PL stratified pseudomanifolds then so are \( X \times Y \) and, if \( X \) and \( Y \) are compact, so is \( X \star Y \).

**Proof.** If \( U \) and \( V \) are respectively the unions of the regular strata of \( X \) and \( Y \), then \( U \) is dense in \( X \) and \( V \) is dense in \( Y \). The union of the regular strata of \( X \times Y \) is \( U \times V \), and this is dense in \( X \times Y \) by basic point-set topology. Similarly, unless one of \( X \) or \( Y \) is empty, in which case the result is trivial, the union of the regular strata of \( X \times Y \) is \( U \times (0, 1) \times V \), which is again easily seen to be dense. So, by Lemma 2.122 and Definition 2.45, \( X \times Y \) is a topological stratified pseudomanifold.

For the PL case, we only need to additionally note that, using the definition of PL stratified pseudomanifolds, the homeomorphisms of the proof of Lemma 2.122 can all be taken to be PL, and, for PL spaces, \( c(L) \times c(K) \cong c(L \ast K) \) piecewise linearly by Exercise 2.24(3) or the argument on page 419.

It is also true that the product of \( \partial \)-stratified pseudomanifolds is a \( \partial \)-stratified pseudomanifold. Given Corollary 2.123, the main additional technicality is the need to demonstrate the collaring of the boundary.

**Lemma 2.124.** If \( X \) and \( Y \) are \( \partial \)-stratified pseudomanifolds, then so is \( X \times Y \). If \( X \) and \( Y \) are PL \( \partial \)-stratified pseudomanifolds, then so is \( X \times Y \).

**Proof.** The interior of \( X \times Y \) is \((X - \partial X) \times (Y - \partial Y)\), which is a stratified pseudomanifold (or PL stratified pseudomanifold) by Corollary 2.123. The boundary of \( X \times Y \) will be \((\partial X \times Y) \cup (X \times \partial Y)\) once we show that it is a (PL) pseudomanifold and that it is collared in \( X \times Y \). So we will provisionally label this set \( \partial(X \times Y) \). Again by Corollary 2.123, \( \partial(X \times Y) - \partial X \times \partial Y = (\partial X \times (Y - \partial Y)) \cup ((X - \partial X) \times \partial Y) \) is a stratified pseudomanifold, so to see that \( \partial(X \times Y) \) is a (PL) stratified pseudomanifold, we only have to be careful near the “corner” \( \partial X \times \partial Y \). Now, we have stratified collars that we can identify as \([0, 1) \times \partial X \times Y \) in \( X \) and \((-1, 0] \times Y \) in \( Y \); in the latter cases, the only difference from our standard conventions is a choice of a different parameterization on the collar, which will be useful below. So in \( X \times \partial Y \), \( \partial X \times \partial Y \) has a neighborhood \([0, 1) \times \partial X \times \partial Y \) with the product filtration, treating \([0, 1) \) as trivially filtered. Similarly, \( \partial X \times \partial Y \) has a neighborhood \( \partial X \times (-1, 0] \times \partial Y \cong (-1, 0] \times \partial X \times \partial Y \) with the product filtration, treating \((-1, 0] \) as trivially filtered. So we then see that \( \partial X \times \partial Y \) has a neighborhood in \( \partial(X \times Y) \) that is stratified homeomorphic to \((-1, 1) \times \partial X \times \partial Y \), which is a (PL) stratified pseudomanifold by Corollary 2.123. We have shown that \( \partial(X \times Y) \) is covered by open sets that are stratified pseudomanifolds, and so \( \partial(X \times Y) \) must be a stratified pseudomanifold.

Slightly more complex is the issue of demonstrating that \( \partial(X \times Y) \) has a stratified collar. For this, let \( C \cong [0, 1) \times \partial X \) and \( D \cong (-1, 0] \times \partial Y \) be the respective collars of \( \partial X \) and \( \partial Y \) in \( X \) and \( Y \). It will be useful to assume that the closures have the form \( \bar{C} \cong [0, 1) \times \partial X \) and \( \bar{D} \cong [-1, 0] \times \partial Y \); this entails no loss of generality as we could, for example, form a new \( C \) from the subset \([0, 1/2) \times \partial X \) of the original \( C \). Then

\[
N = (C \times Y) \cup (X \times D) \cong ([0, 1) \times \partial X \times Y) \cup (X \times (-1, 0] \times \partial Y)
\]
is a neighborhood of $\partial(X \times Y)$ in $X \times Y$. Here, the product stratifications of the pieces of this neighborhood are consistent with those inherited from $X \times Y$. We need to show that $N \cong [0, 1) \times \partial(X \times Y)$. For this, we notice that

$$V = (C \times Y) \cap (X \times D) \cong [0, 1) \times \partial X \times (-1, 0] \times \partial Y \cong [0, 1) \times (-1, 0] \times \partial X \times \partial Y,$$

and the closure of this intersection in $N$ has the form

$$(([0, 1) \times [-1, 0]) - ([1) \times \{-1\})) \times \partial X \times \partial Y.$$

In particular, we can think of $N$ as consisting of three closed pieces: $[0, 1) \times \partial X \times (Y - D)$, $(X - C) \times (-1, 0] \times \partial Y$, and $V$, glued along

$$[0, 1) \times \partial X \times \partial(Y - D) \cong [0, 1) \times \partial X \times \{-1\} \times \partial Y \cong [0, 1) \times \{-1\} \times \partial X \times \partial Y$$

and

$$\partial(X - C) \times (-1, 0] \times \partial Y \cong \{1\} \times \partial X \times (-1, 0] \times \partial Y \cong \{1\} \times (-1, 0] \times \partial X \times \partial Y.$$

But $(([0, 1) \times [-1, 0]) - ([1) \times \{-1\})$ is simply a solid square with one corner point removed, and this space is piecewise-linearly homeomorphic to $[-1, 1) \times [0, 1)$. We can see this using the accompanying figure FIGURE!! Thus

$$V \cong [-1, 1) \times [0, 1) \times \partial X \times \partial Y,$$

and the two boundary pieces glued to the rest of $N$ get taken under this homeomorphism to $\{-1\} \times [0, 1) \times \partial X \times \partial Y$ and $\{1\} \times [0, 1) \times \partial X \times \partial Y$.

Now, we re-glue this image of $V$ back to our two other pieces of $N - \hat{V}$, identifying $\{-1\} \times [0, 1) \times \partial X \times \partial Y$ in our homeomorphed $V$ with $[0, 1) \times \{-1\} \times \partial X \times \partial Y$ in $[0, 1) \times \partial X \times (Y - D)$ and identifying $\{-1\} \times [0, 1) \times \partial X \times \partial Y$ in our homeomorphed $V$ with $\{1\} \times (-1, 0] \times \partial X \times \partial Y$ in $(X - C) \times (-1, 0] \times \partial Y$. Together, we obtain a space that has the form $[0, 1) \times \partial(X \times Y)$, using our earlier observation that $\partial(X \times Y) \cong (X - C) \cup (Y - D) \cup ([-1, 1) \times \partial X \times \partial Y).$

This is the desired collar.

**Products of intrinsic stratification.** A reasonable question to ask is whether the products of intrinsically stratified spaces are intrinsically stratified in their product stratifications. In the topological category, results like the Double Suspension Theorem demonstrate that this is too much to ask. However, in the PL category, and even here with an extra assumption, we can have such results.

**Proposition 2.125.** Let $X$ and $Y$ be two PL stratified pseudomanifolds with intrinsic stratifications $X^*$ and $Y^*$. If at least one of $X, Y$ is a classical pseudomanifold, then $(X \times Y)^* \cong X^* \times Y^*$.  

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Remark 2.126. The assumption that at least one of \( X \) or \( Y \) be classical is necessary, as demonstrated by the following example: Let \( I = [0, 1] \) be filtered as \( \{0, 1\} \subset I \). This is a PL stratified pseudomanifold filtration, and it is intrinsic. Then \( I \times I \) is filtered as

\[
\{(0,0), (0,1), (1,0), (1,1)\} \subset \{(0,1) \times I \} \cup (I \times \{0,1\}) \subset I \times I.
\]

This is not an intrinsic stratification as all of the boundary points of the square \( I \times I \) have PL homeomorphic relative neighborhoods. This follows from, among other arguments, the PL unbending procedures that we utilized in the proof of Lemma 2.124.

The proof of the proposition relies upon the following lemma:

Lemma 2.127. A stratification of a PL pseudomanifold \( X \) is the intrinsic stratification if and only if for every link \( L \), \( L \) is not PL homeomorphic to a suspension; i.e. there is no compact PL space \( Z \) such that \( L \) is PL homeomorphic to \( SZ \).

Furthermore, if \( X \) is a classical PL pseudomanifold, then no link of the intrinsic stratification is PL homeomorphic to a closed cone.

Proof. Throughout this proof, \( \cong \) will denote PL homeomorphism without regard for stratifications. First, suppose \( L \) is a link of a point in an \( i \)-dimensional stratum of \( X^* \), i.e. there is some \( x \in X^* \) with a distinguished neighborhood stratified PL homeomorphic to \( \mathbb{R}^i \times L \). Suppose \( L \) is PL homeomorphic to a suspension, so that \( L \cong SZ \) for some compact PL space \( Z \). Then we have

\[
\mathbb{R}^i \times L \cong \mathbb{R}^i \times c(SZ) \cong \mathbb{R}^i \times \mathbb{R}^1 \times cZ \cong \mathbb{R}^{i+1} \times cZ,
\]

using again that \( c(A) \times c(B) \cong c(A \ast B) \) with \( A = S^0 \) and \( B = Z \). Then if \( w \) is the cone vertex of \( cZ \), all the points in \( \mathbb{R}^{i+1} \times \{w\} \), including \( x \), have PL homeomorphic neighborhoods, contradicting that \( x \) is contained in an \( i \)-dimensional stratum of \( X^* \). Thus \( L \) is not a suspension. So no link in \( X^* \) can be a suspension.

Conversely, suppose \( X \) is a PL stratified pseudomanifold such that no link is a suspension. We claim that \( X \) is stratified by the intrinsic stratification. Again, suppose not. Since \( X^* \) is the coarsest stratification, there must be a stratum \( T \) of \( X \) that is contained in a higher-dimensional stratum \( S \) of \( X^* \) with \( \dim(S) > \dim(T) \). Let \( \dim(T) = i \) and \( \dim(S) = j \), and suppose \( x \in T \). In \( X \), \( x \) has a distinguished neighborhood of the form \( \mathbb{R}^i \times cL \cong c(S^{i-1} \ast L) \), while in \( X^* \), \( x \) has a distinguished neighborhood of the form \( \mathbb{R}^j \times cL' \cong c(S^{j-1} \ast L') \). Note: if \( i = 0 \), we let \( S^{-1} = \emptyset \) and \( \emptyset \ast L = L \); similarly, the formulas apply if \( L = \emptyset \). By the uniqueness of polyhedral links\(^{23}\) \([57\text{ Corollary 1.15}]\), this implies that \( S^{i-1} \ast L \cong S^{j-1} \ast L' \), or, written in terms of iterated suspensions, \( S^i L \cong S^j L' \). Since \( i < j \), \( L \cong S^{j-i} L' \) by \([75\text{ Theorem 1}]\), so \( L \) is a suspension, contradicting the assumption. Therefore, \( X \) must actually be \( X^* \).

Lastly, suppose \( X \) is a classical PL pseudomanifold and \( X^* \) its intrinsic stratification. By Remark 2.59, each link \( L \) is also a classical PL pseudomanifold. By Remark 2.66, this implies that if \( L \) has dimension \( k \), then no triangulation of \( L \) can have a dimension \( k - 1 \) face.

\(^{23}\)See Footnote 13 on page 31.
that does not bound exactly two \( k \)-simplices. This implies that \( L \) cannot have the structure of a closed cone \( \partial Z \) unless \( Z \) is empty. But if \( Z \) is empty, \( \partial Z \) is a point. However, since \( X \) is classical, it has a PL pseudomanifold stratification \( X' \) with no codimension one strata, and since \( X^* \) is coarser than \( X' \), \( X^* \) also has no codimension one strata. Therefore, \( L \) also cannot be a point. Therefore, \( L \) is not a cone.

Proof of Proposition \( 2.125 \). By Lemma \( 2.127 \) it suffices to show that the links of \( X^* \times Y^* \) are not suspensions. But every link of \( X^* \times Y^* \) has the form \( L \times K \), where \( L \) and \( K \) are respective links of \( X \) and \( Y \), by our computations in the proof of Lemma \( 2.125 \). In particular, \( L \) and \( K \) cannot be suspensions, again by Lemma \( 2.127 \). We must show that \( L \times K \) is not a suspension.

In [75], Morton defines a compact polyhedron (PL space) to be reduced if it is not a (closed) cone or a suspension. By [75] Corollary to Theorems 1 and 2], every compact polyhedron (PL space) factors uniquely as the join of a ball or sphere to a reduced polyhedron. Since \( L \) and \( K \) are not suspensions, their unique factorizations must have the form \( L = B^a \times D \) and \( K = B^b \times E \), where \( D \) and \( E \) are reduced and \( B^a \) and \( B^b \) are balls of respective dimensions \( a \) and \( b \) such that \( a, b \leq 0 \). The last condition is due to the fact that any ball of dimension \( > 0 \) is a suspension, so if, for example, \( L \cong B^a \times D \) with \( a > 0 \), then \( L \cong (S^0 \times B^{a-1}) \times D \cong S^0 \times (B^{a-1} \times D) \), presenting a contradiction. Here we let \( B^a = \emptyset \) if \( a < 0 \). It now follows that \( L \times K \cong B^a \times D \times B^b \times E \cong B^{a+b+1} \times D \times E \), and again by [75] Corollary to Theorems 1 and 2], this decomposition is unique up to possibly rewriting \( B^{a+b+1} \) as a suspension. But if \( a + b + 1 \leq 0 \), then \( B^{a+b+1} \) is also uniquely written as \( \emptyset \) or \( B^0 \), and so \( L \times K \) is not a suspension, and the proposition follows. Since we know that \( a, b \leq 0 \), to have \( a + b + 1 \leq 0 \), it is only necessary for either \( a \) or \( b \) to be \( < 0 \), or equivalently, to have \( B^a \) or \( B^b \) empty. This in turn is equivalent to having \( K \) or \( L \) not be PL homeomorphic to a cone. But we have assumed that one of \( X \) or \( Y \) is classical, and so by Lemma \( 2.127 \) one of \( L \) or \( K \) is not a cone. 

We also have a product result for naturally stratified PL \( \partial \)-stratified pseudomanifolds; see Definition \( 2.116 \).

Proposition 2.128. Let \( X, Y \) be naturally stratified PL \( \partial \)-stratified pseudomanifolds such that at least one of \( X \) or \( Y \) is a classical PL \( \partial \)-stratified pseudomanifold. Then \( X \times Y \) is naturally stratified.

Proof. By Lemma \( 2.124 \) \( X \times Y \) is a \( \partial \)-stratified pseudomanifold. According to Definition \( 2.116 \) we must show that \( X \times Y - \partial(X \times Y) \) and \( \partial(X \times Y) \) are intrinsically stratified.

First, we see that \( X \times Y - \partial(X \times Y) = (X - \partial X) \times (Y - \partial Y) \) is intrinsically stratified by Proposition \( 2.125 \). By the same proposition, \( (X - \partial X) \times \partial Y, \partial X \times (Y - \partial Y), \) and \( \partial X \times \partial Y \) are intrinsically stratified. For this we note that if \( X \) is a classical PL stratified \( \partial \)-pseudomanifold, then \( \partial X \) is also classical, as if \( \partial X \) has codimension one strata, then so does the collar neighborhood of \( \partial X \) in \( X \); the same is, of course, true of \( Y \). From the proof of Lemma \( 2.124 \) we observe that \( \partial(X \times Y) \) is the union of the open subsets \((X - \partial X) \times \partial Y, \partial X \times (Y - \partial Y), \) and \((-1, 1) \times \partial X \times \partial Y \), where the latter space is also intrinsically stratified.
by Proposition 2.125. Piecing these spaces together to form $\partial(X \times Y)$ and using that intrinsic stratification is a local property, we see that $\partial(X \times Y)$ must be intrinsically stratified.

3 Intersection homology

In this chapter, we will define the intersection homology groups and compute some examples. ADD MORE LATER.

3.1 Perversities

The idea for defining intersection homology groups is that we should consider the simplices and chains that are ordinarily used to define homology groups but that we should place some limitations on how such chains are allowed to interact with different strata. In practice limitations are placed on the dimension of intersection with the strata. Of course there can be many different ways to do this: we could forbid a chain from intersecting a stratum altogether, we could pose no limitation with a given stratum, or we could make various choices in between. These choices are encoded in a perversity parameter (most often referred to simply as a perversity). As there are many choices for these parameters, there are many different intersection homology complexes. In this section, we provide the precise definition of perversity.

In fact, the definition of perversity has evolved. We begin with the most general definition and then discuss some of the other limitations that were originally imposed on the definition.

**Definition 3.1.** Let $X$ be a filtered space of formal dimension $n$, and let $\mathcal{S}$ be the set of strata of $X$. A perversity on $X$ is a function

$$\bar{p} : \mathcal{S} \to \mathbb{Z}$$

such that $\bar{p}(S) = 0$ if $S \subset X - \Sigma_X$.

**Remark 3.2.** Given the generality of the definition, one might wonder whether we could do without the requirement that $\bar{p}(S) = 0$ if $S \subset X - \Sigma_X$, i.e. if $S$ is a regular stratum. It will turn out that we could just as well require that $\bar{p}(S) \geq 0$ if $S \subset X - \Sigma_X$ without changing the definition of the intersection homology groups (or even of the intersection chain complexes), but it is occasionally simpler in technical statements to have that $\bar{p}(S) = 0$ for such strata. On the other hand, if $\bar{p}(S) < 0$, the definition of intersection homology either becomes trivial or simply doesn’t see the regular strata at all, depending on the definition of intersection homology being used. See Remarks 3.16 and 3.36 for more details.

The original definition of perversity was more complex owing to the properties Goresky and MacPherson wanted for their intersection homology groups. In particular, they wanted intersection homology of stratified pseudomanifolds to possesses a Poincaré duality theorem and to be invariant of the stratification of the space. We will see below in detail how these requirements force additional conditions on the perversity parameters. In fact, they will also place certain requirements on the space in that pseudomanifolds for which these
properties hold simultaneously cannot have codimension one strata, so they must be classical pseudomanifolds.

For now, we will simply provide the definition for what we will call *Goresky-MacPherson perversities* or *GM perversities*. We will come to understand the additional requirements later on. We will sometimes refer to perversities as defined in Definition 3.1 as *general perversities* when we wish to distinguish them from GM perversities.

One further point worth mentioning before providing the definition is that GM perversities assign the same value to all strata of the same codimension. Hence given that perversities always evaluate to 0 on codimension 0 strata and that we will assume when using GM perversities that there are no codimension one strata, it is standard to write GM perversities as functions of codimension with domain \( \{2, 3, 4, \ldots \} \).

**Definition 3.3.** A *Goresky-MacPherson perversity* (or *GM perversity*) is a function

\[
\bar{p} : \{2, 3, 4, \ldots \} \to \mathbb{Z}
\]

such that

1. \( \bar{p}(2) = 0 \),

2. \( \bar{p}(k) \leq \bar{p}(k + 1) \leq \bar{p}(k) + 1 \).

The conditions of the definition say that a perversity is a function defined on the integers \( \geq 2 \) that “starts” at 0 and then for each transition in the domain from \( k \) to \( k + 1 \) the perversity value either stays the same or increases by 1. So a GM perversity is sort of a “sub-step” function. One convenient way to describe GM perversities is to think of them as sequences \( [\bar{p}(2), \bar{p}(3), \bar{p}(4), \ldots] \). So, for example, a GM perversity might look like

\[
\bar{p} = [0, 1, 1, 2, 3, 3, 4, 5, 5, \ldots].
\]

There are a few particular perversities that have special importance:

**Example 3.4.** The minimal GM perversity is the *zero perversity* \( \bar{0} \), which takes the smallest possible values at each step by starting at 0 and then never increasing:

\[
\bar{0} = [0, 0, 0, 0, \ldots].
\]

On the other hand, the maximal GM perversity is the *top perversity* \( \bar{t} \), which always takes the step up

\[
\bar{t} = [0, 1, 2, 3, \ldots].
\]

Both of these GM perversities can be extended to general perversities in somewhat obvious ways. On strata, the zero perversity continues always to take the values 0, ie. \( \bar{0}(S) = 0 \) for all strata \( S \) of a filtered space \( X \). Similarly, the general top perversity can be defined by the function \( \bar{t}(S) = \text{codim}(S) - 2 \), where \( \text{codim}(S) \) is the codimension of the singular stratum \( S \). Note that we always have \( \bar{t}(S) = 0 \) if \( S \) is a regular stratum; from a purely
philosophical viewpoint, this makes it somewhat unclear whether $\bar{t}(S)$ should equal 0 or $-1$ when codim($S$) = 1, but we shall see by our duality results that the definition we have given here is indeed the most reasonable one.

The top perversity plays an especially important role in intersection homology versions of Poincaré duality, as intersection homology groups dualize with respect not only to dimension index but with respect to dualization of perversities. The top perversity plays the key role in dualizing perversities.

**Dual perversities.**

**Definition 3.5.** Given a perversity $\bar{p}$, its dual perversity (or complementary perversity) is the perversity $D\bar{p}$ defined so that

$$D\bar{p}(S) = \bar{t}(S) - \bar{p}(S) = \text{codim}(S) - 2 - \bar{p}(S)$$

for all singular strata $S$, and $D\bar{p}(S) = 0$ if $S$ is a regular stratum. We will often abbreviate this property by saying $D\bar{p} = \bar{t} - \bar{p}$ or $\bar{p} + D\bar{p} = \bar{t}$.

**Example 3.6.** It is immediate to verify that $D\bar{0} = \bar{t}$ and $D\bar{t} = \bar{0}$.

The following lemma, whose proof is immediate, shows that dualization acts as an involution on the set of perversities.

**Lemma 3.7.** For any perversity, $D(D\bar{p}) = \bar{p}$.

Manifold theory, and in particular the study of the cup product pairing or the intersection pairing, might lead us to expect that if duality between two objects is important, then objects that are dual to themselves are even more important. For example, the symmetric self-dual cup product pairing on $H^{2k}(M; \mathbb{Q})$ of a closed oriented $4k$-manifold yields signature invariants. This theme will be developed in detail as we progress. To see now what sorts of perversities might have this property, we observe that since

$$\bar{p}(S) + D\bar{p}(S) = \text{codim}(S) - 2$$

for all singular strata $S$, it will only be possible ever to have $\bar{p}(S) = D\bar{p}(S)$ on strata that are of even codimension, in which case we would want $\bar{p}(S) = D\bar{p}(S) = \frac{\text{codim}(S) - 2}{2}$. We would be justified in calling this a middle perversity since it is halfway between the perversities $\bar{0}$ and $\bar{t}$ (which are the extreme possibilities of GM perversities).

What about spaces that do possess odd codimension strata. If our goal is to continue to have $\bar{p}(S)$ and $D\bar{p}(S)$ remain as close in value as possible, we would want $|\bar{p}(S) - D\bar{p}(S)| = 1$. In fact, one of the perversities must have value $\left\lfloor \frac{\text{codim}(S) - 2}{2} \right\rfloor$ and the other one have value $\left\lceil \frac{\text{codim}(S) - 2}{2} \right\rceil$. Such perversities were defined by Goresky and MacPherson so that one would always take the higher value and the other would always take the lower value:

$^{24}$Here $[x]$ denotes the greatest integer less than or equal to the real number $x$ and that $\lfloor x \rfloor$ denotes the least integer greater than or equal to $x$.  

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**Definition 3.8.** The lower middle GM perversity $\bar{m}$ and the upper middle GM perversity $\bar{n}$ are defined by

$$\bar{m} = [0, 0, 1, 1, 2, 2, 3, \ldots]$$
$$\bar{n} = [0, 1, 1, 2, 2, 3, 3, \ldots].$$

These extend to general perversities with the definitions

$$\bar{m}(S) = \left\lfloor \frac{\text{codim}(S) - 2}{2} \right\rfloor$$
$$\bar{n}(S) = \left\lceil \frac{\text{codim}(S) - 2}{2} \right\rceil.$$

Notice that if $X$ has no strata of odd codimension then $\bar{m}$ and $\bar{n}$ take the same value on all singular strata. In this case it is customary to use either symbol $\bar{m}$ or $\bar{n}$ to stand for a single “middle perversity”. The author tends to prefer $\bar{m}$ for this notation.

**Remark 3.9.** The reader might well ask why we want to make a choice such that always $\bar{n}(S) \geq \bar{m}(S)$. For example, why is this better than two dual GM perversities such as $[0, 0, 1, 2, 2, \ldots]$ and $[0, 1, 1, 2, 2, \ldots]$ in which sometimes one and sometimes the other perversity is allowed to be the greater one. Beyond it being pleasing to have made a definite choice, we will see that it is possible in general to have maps between intersection homology groups of a given space of the form $I^pH_* (X) \to I^qH_* (X)$ only when $\bar{p}(S) \leq \bar{q}(S)$ for all singular strata $S$ (we will abbreviate this requirement by $\bar{p} \leq \bar{q}$). Thus it is possible to have such a comparison map $I^mH_* (X) \to I^nH_* (X)$, but we would not generally be able to do this for other pairs of dual perversity “near” the middle. This comparison map will be critical for finding intersection homology groups that are self-dual under the intersection pairing.

### 3.2 Intersection homology

Now that we have defined perversities on stratified spaces, we are prepared to provide a first definition of intersection chains $I^pC_*(X)$ and intersection homology $I^pH_* (X)$. As for classical homology, there are in fact several different types of chain complexes — simplicial, singular, piecewise linear, etc.\footnote{This might be a good place to note that a good CW theory of intersection homology does not seem to have been worked out, perhaps for reasons that will become evident as we proceed.} — that lead to the same homology groups, at least with the proper assumptions. We will need to consider each of these in turn. However, the reader should also be aware that there are at least two competing definitions in another sense, reflecting certain challenges that arise when perversities take values that are “too high”, meaning that they take values on strata that exceed the value of the top perversity $\bar{t}$, or when working with spaces with strata of codimension one. When either of these conditions arise, the definitions of intersection homology become incompatible. However, they do all agree when considering, e.g. Goresky-MacPherson perversities and classical pseudomanifolds.

In order to ease the exposition, we will begin with the definitions of intersection homology closest to the original definition of Goresky and MacPherson\cite{GoreskyMacPherson1988}, though the reader should...
be aware that this is not the definition that we will ultimately want when $\bar{p}(S) \geq \bar{t}(S)$ for some stratum or when discussing stratified pseudomanifolds with codimension one strata (though it is still defined in those cases). Once we have gotten used to the basic ideas, we will proceed to discuss how the definition should be modified to obtain the best results in the general settings beginning in Chapter 8. Since we will eventually want to use the notations $I^pC_s(X)$ and $I^pH^*_s(X)$ for the modified definition we will present later, for now we use the notations $I^pC^{GM}_s(X)$ and $I^pH^{GM}_s(X)$.

3.2.1 Simplicial intersection homology

We will begin with a simplicial version. For this we let $X$ be a simplicial filtered space, meaning that $X$ is a filtered space with a fixed triangulation such that each skeleton of the filtration is a simplicial subcomplex (not necessarily a simplicial skeleton). We could perhaps begin with simplicial stratified spaces of a more general type (i.e. simplicial filtered spaces), but for a first introduction it will useful for the formal dimensions and codimensions arising from the stratification to correspond to the familiar geometric/simplicial dimensions. We will not need the greater generality later.

**Definition 3.10.** Let $X$ be a simplicial filtered space endowed with a perversity $\bar{p}$, and let $C_*(X)$ be the chain complex of oriented simplices of $X$ (as in [77, Section 5]). We deem an $i$-simplex $\sigma$ of $X$ to be $\bar{p}$-allowable if, for each stratum $S \subset X$,

$$\dim(\sigma \cap S) \leq i - \text{codim}(S) + \bar{p}(S).$$

(1)

For this purpose, we define $\dim(\emptyset) = -\infty$. If inequality (1) is satisfied for some $\sigma$ and some $S$, we say that $\sigma$ is $\bar{p}$-allowable with respect to the stratum $S$. If the perversity $\bar{p}$ has been fixed in advance, we will sometimes simply say that $\sigma$ is allowable.

A chain $\xi \in C_*(X)$ is $\bar{p}$-allowable if all of the simplices of $\xi$ and all of the simplices of $\partial \xi$ (with non-zero coefficient) are $\bar{p}$-allowable.

Let $I^pC^{GM}_s(X) \subset C_*(X)$ be the chain complex of $\bar{p}$-allowable chains, which we call the (perversity $\bar{p}$) intersection chain complex. Let the (perversity $\bar{p}$) intersection homology groups be the homology groups $H_*\left(I^pC^{GM}_s(X)\right)$.

A number of observations are in order. First of all, we should note that the $I^pC^{GM}_s(X)$ is well-defined as a chain complex. In particular, if $\xi$ and $\eta$ are $\bar{p}$-allowable chains, then every simplex in $\xi + \eta$, $\partial(\xi + \eta) = \partial \xi + \partial \eta$, or $\partial(-\xi) = -\partial \xi$ must be a simplex already.

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26 The initial Goresky-MacPherson intersection homology of [42] is defined with respect to PL chains, meaning the direct limit complex of simplicial chains over all triangulations compatible with the stratification. We will address this version shortly.

27 For a more general simplicial space, this might not be the case. Consider, for example, the simplicial space $X$ constructed from a triangulated 2-sphere and an interval by gluing them along a vertex, and stratify this space by $X^0 \subset X = X^2$, where $X^0$ consists of the gluing vertex. This is a stratified space, and in fact it is locally conelike. However, the set of regular strata of $X - X^0$ comprises $S^2 - \text{pt}$ and a half open interval. These each have formal dimension 2, which disagrees with the geometric dimension for the interval.

28 We do not assume that $\bar{p}$ is a GM perversity.
contained in $\xi$, $\eta$, $\partial\xi$, or $\partial\eta$ and so must be $\bar{p}$-allowable. Thus each $I^\bar{p}C^GM_s(X)$ is a well-defined subgroup of $C_s(X)$. It is also a well-defined chain complex by fiat, owing to the declaration that in order for $\xi$ to be $\bar{p}$-allowable so must be all of the simplices of $\partial\xi$ and that $\partial^2 = 0$ as a consequence of working within the chain group $C_s(X)$.

We will discuss motivation for this definition in Section 3.2.3 after computing a few examples. The interested reader may feel free to skip ahead to that section now and come back to the examples here.

### 3.2.2 First examples

In order to get a feel for working with intersection homology, let us compute some elementary examples.

**Example 3.11.** Let $X = X^0$ be a point. In this case there is only one stratum, $X$ itself, and it is a regular stratum so $\bar{p}(X) = 0$ for any perversity $\bar{p}$. There is also only one simplex to work with, a 0-simplex we shall denote $v$. The allowability condition then becomes that

$$\dim(v) = \dim(v \cap X) \leq \dim(v) - \text{codim}(S) + \bar{p}(S) = 0 - 0 + 0 = 0.$$  

This is evidently true, so $v$ is allowable, $\partial v = 0$ is allowable, and $I^\bar{p}C^GM_s(X) = C_s(X)$, the ordinary chain complex. So in this case intersection homology yields nothing new.

**Example 3.12.** Put a figure here

For a more interesting example, let $X = X^1 = S^1$, the circle, triangulated as the boundary of a 2-simplex $\Delta^2 = [0, l, 2]$. Suppose $X$ is stratified as $X^0 = [0] \subset X$, where $[0]$ is the vertex of $\Delta^2$ labelled by 0. Then $X$ has two strata: the regular stratum $X - [0]$ and the singular stratum $[0]$. For any perversity, we must have $\bar{p}(X - [0]) = 0$, but $\bar{p}([0])$ could be any integer. Let us generically use the notation $v$ for a vertex and $e$ for an edge. Then a vertex $v$ is allowable if it satisfies the conditions

$$\dim(v \cap (X - [0])) \leq \dim(v) - \text{codim}(X - [0]) + \bar{p}(X - [0]) = 0 - 0 + 0 = 0$$

$$\dim(v \cap [0]) \leq \dim(v) - \text{codim}([0]) + \bar{p}([0]) = 0 - 1 + \bar{p}([0]) = \bar{p}([0]) - 1.$$  

Since $\dim(v \cap (X - [0]))$ must always be $\leq 0$ (as $v$ is a vertex), we see that any vertex in $X - [0]$ is allowable. By contrast, the vertex $[0]$ itself is allowable only if $\bar{p}([0]) \geq 1$.

Similarly, for an edge $e$, the allowability conditions are

$$\dim(e \cap (X - [0])) \leq \dim(e) - \text{codim}(X - [0]) + \bar{p}(X - [0]) = 1 - 0 + 0 = 1$$

$$\dim(e \cap [0]) \leq \dim(e) - \text{codim}([0]) + \bar{p}([0]) = 1 - 1 + \bar{p}([0]) = \bar{p}([0]).$$  

Again $\dim(e \cap (X - [0]))$ must always be $\leq 1$ since $e$ is an edge, and so the first condition always holds. Additionally, the edge $[1, 2]$ does not intersect the vertex $[0]$, so it is allowable for any $\bar{p}$. The edges $[0, 1]$ and $[0, 2]$ both intersect $[0]$ with $\dim(e \cap [0]) = 0$, so they will be allowable only if $\bar{p}([0]) \geq 0$.

So already we see that there are three distinct cases according to whether $\bar{p}([0])$ is $\geq 1$, $= 0$, or $< 0$:  

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If \( \bar{p}([0]) \geq 1 \), then we have seen that all simplices will be allowable, and hence all chains will be allowable. In this case \( \mathcal{P}C_{\ast}^GM(X) = C_{\ast}(X) \), and we recover the standard simplicial chain complex, so \( \mathcal{P}H_{\ast}^GM(X) = H_{\ast}(X) \).

If \( \bar{p}([0]) < 0 \) then neither the vertex \([0]\) nor either of the edges of \(X\) intersecting \([0]\) will be allowable, but the edge \([1,2]\) and the vertices \([1],[2]\) are allowable. So \( \mathcal{P}C_{\ast}^GM(X) = C_{\ast}([1,2]) \), the simplicial chain complex of 1-simplex \([1,2]\). Correspondingly \( \mathcal{P}H_{\ast}^GM(X) \) is the homology of the interval.

If \( \bar{p}([0]) = 0 \), then our analysis concerning vertices is the same as in the previous case: \([0]\) is not allowable, but \([1]\) and \([2]\) are. Now, however, all the edges are allowable. So what is the intersection chain complex \( \mathcal{P}C_{\ast}^GM(X) \)? Here for the first time we must pay attention to boundaries. The edge \([0,1]\) is allowable as a simplex, but it is not allowable as a chain because its boundary \([1] - [0]\) contains a vertex that is not allowable. However, the chain \([0,1] - [0,2]\) is allowable because its boundary is \([1] - [0] - ([2] - [0]) = [1] - [2]\), which is allowable. In fact, we can see that \( \mathcal{P}C_{1}^GM(X) \cong \mathbb{Z} \oplus \mathbb{Z} \) generated by \([1,2]\) and \([0,1] - [0,2]\). We have also seen that \( \mathcal{P}C_{0}^GM(X) \cong \mathbb{Z} \oplus \mathbb{Z} \), generated by \([1]\) and \([2]\). One can then compute by hand, using the boundary map, that \( \mathcal{P}H_{1}^GM(X) \cong \mathbb{Z} \), generated by \([0,1] - [0,2] + [1,2]\), just as for the standard homology, while \( \mathcal{P}H_{0}^GM(X) \cong \mathbb{Z} \) generated by either \([1]\) or \([2]\), which are homologous via the edge \([1,2]\). So ultimately \( \mathcal{P}H_{\ast}^GM(X) \cong H_{\ast}(X^{1}) \) again here.

In the last case, \( \bar{p}([0]) = 0 \), there are some shortcuts we could have used to compute the groups \( \mathcal{P}H_{\ast}^GM(X) \) without having to compute the complexes \( \mathcal{P}C_{\ast}^GM(X) \). For example, since \( \mathcal{P}C_{\ast}^GM(X) \subset C_{\ast}(X) \) and since we know from familiar homology computations that the only cycles in \( C_{1}(X) \) are the multiples of \([0,1] - [0,2] + [1,2] \), these are also the only possible cycles in \( \mathcal{P}C_{1}^GM(X) \). Therefore, since \( \mathcal{P}C_{2}^GM(X) = 0 \) trivially, to compute \( \mathcal{P}H_{1}^GM(X) \) we need only determine whether \([0,1] - [0,2] + [1,2]\) is allowable. Once we have determined that it is, then we must have \( \mathcal{P}H_{1}^GM(X) \cong \mathbb{Z} \). Similarly, since all vertices are cycles, once we have noticed that \([0]\) is not allowable but that \([1]\), \([2]\), and \([1,2]\) are allowable, we can quickly conclude that \( \mathcal{P}H_{0}^GM(X) \cong \mathbb{Z} \). Such computational techniques will prove very useful.

**Example 3.13.** Let \(X\) again be the circle \(S^{1}\) triangulated as the boundary of the 2-simplex \([0,1,2]\), but this time suppose the stratification is the simplicial stratification, i.e. \(X^{0} = \{[0],[1],[2]\}\) and \(X^{1} = X\). Suppose \( \bar{p}([0]) = \bar{p}([1]) = \bar{p}([2]) = 0 \). Now, by the same analysis as in Example 3.12, none of the vertices are allowable but all of the edges are. Hence the cycle \([0,1] - [0,2] + [1,2]\) is allowable and \( \mathcal{P}H_{1}^GM(X) \cong \mathbb{Z} \), but since no vertices are allowable, \( \mathcal{P}H_{0}^GM(X) = 0 \).

**Allowability with respect to regular strata.** One observation we might conjecture from these first examples is that the allowability condition is vacuous when it comes to regular strata. This is indeed the case as we formalize in the following lemma, which will help shorten the computations in our further examples.

**Lemma 3.14.** Let \(\sigma\) be an \(i\)-simplex of a simplicial filtered space \(X\) and let \(S\) be a regular stratum of \(X\). Then the allowability condition \(\text{[I]}\) is always satisfied.
Proof. Since \( \sigma \) is an \( i \)-simplex, for any subspace \( Z \subset X \) it must be true that \( \dim(\sigma \cap Z) \leq i \), and since \( \operatorname{codim}(S) = \bar{p}(S) = 0 \), the righthand side of the inequality (1) reduces to \( i \).

Example 3.15. Suppose \( X \) is an \( n \)-dimensional simplicial filtered space that is filtered trivially so that there is only one stratum of dimension \( n \) (which will therefore be a triangulated \( n \)-manifold) and that \( \bar{p}(X) = 0 \). Then it follows from the preceding lemma that \( I^pC^\bullet_{GM}(X) = C^\bullet(X) \).

Remark 3.16. Lemma 3.14 allows us to provide some justification for setting \( \bar{p}(S) = 0 \) for all regular strata. We see from the lemma that with \( \bar{p}(S) = 0 \) all simplices are allowable with respect to all regular strata. Furthermore, if \( \bar{p}(S) = m \) for any \( m \geq 0 \), then it is easy to see that the same conclusion will hold, so as mentioned in Remark 3.2 any choice of \( \bar{p}(S) \geq 0 \) for regular strata would provide the same intersection homology, but we choose \( \bar{p}(S) = 0 \) for definiteness and convenience.

By contrast if \( S \) is regular and \( \bar{p}(S) \leq -1 \), then for an \( i \)-simplex to be allowable with respect to \( S \), we would need

\[
\dim(\sigma \cap S) \leq i - \operatorname{codim}(S) + \bar{p}(S) = i - 0 + \bar{p}(S) \leq i - 1.
\]

But in a simplicial complex, this implies that the interior of \( \sigma \) cannot be completely contained in any regular stratum. But since the triangulation is compatible with the stratification and since the skeleta (with respect to the stratification) of \( X = X^n \) are closed, this implies that \( \sigma \subset \Sigma_X \). So if \( \bar{p}(S) \leq -1 \) for any regular stratum, no simplex can intersect that stratum at all. In other words, \( I^pH^\bullet_{GM}(X) \) does not see that stratum, so it is equal to \( I^pH^\bullet_{GM}(X - S) \), noting that \( X - S \) is a subcomplex of \( X \) so that this group is defined. Therefore, having regular strata with negative perversities is the same as working on spaces without those strata\(^{29}\), and we could just as well have taken that view from the beginning and worked on a different space.

Altogether, this makes it reasonable to always have \( \bar{p}(S) = 0 \) for regular strata.

See Remark 3.36 below for the analogous considerations for singular intersection chains.

**Effects of subdivision.** Next we explore the effects of subdivision on the computation of intersection homology groups.

Example 3.17. Let \( X = X^1 \) again be the boundary of the 2-simplex \([0, 1, 2]\) as in Example 3.13. Now consider \( X' \), the first barycentric subdivision of \( X \) but with the same stratification as in Example 3.13 so that the 0-skeleton of the stratification consists only three of the six vertices. The barycenters of the edges will not be allowable vertices, and these vertices will be homologous via simplicial paths which cross through \([0]\), \([1]\), or \([2]\). Thus \( I^pH^0_{GM}(X') \cong \mathbb{Z} \).

Example 3.18. Let \( Y = Y^2 \) be the boundary of an octahedron with its standard triangulation as the union of two square pyramids (equivalently this is the suspension of the boundary of a square), and let \( X^0 \) be the union of the vertices of the square. Let \( \bar{p} \) be a perversity that

\(^{29}\) This is slightly untrue as \( \Sigma_X = X^{n-1} \) has different total dimension than \( X = X^n \), and so if we remove all regular strata the codimensions of strata will also be different; however, we could adjust for this by modifying the perversities by the same shift.
takes the same value \( \leq 1 \) on all four vertices of \( X^0 \), which we label \( v_i, 1 \leq i \leq 4 \). Then we can easily check that the two cone vertices are the only allowable 0-simplices. Suppose \( 1 - 2 + \overline{p}(\{v_i\}) < 0 \), i.e. if \( \overline{p}(\{v_i\}) < 1 \), then no 1-simplex of \( Y \) is allowable, and we must have \( I^pH^0_{GM}(Y) = \mathbb{Z} \oplus \mathbb{Z} \). Yet if \( Y' \) is the first barycentric subdivision of \( Y \), then there are allowable paths connecting any vertices and that don’t contain any of the \( v_i \), so in this case \( I^pH^0_{GM}(Y) = \mathbb{Z} \).

These two examples show that the intersection homology groups are not independent of the triangulation, and in fact maps on intersection homology induced by subdivision maps will in general be neither injective nor surjective. This might raise some reasonable concerns; however, we will show below in Theorem 3.26 that there is independence of the triangulation assuming some minor conditions. In particular, the groups will stabilize with respect to subdivision.

**Some more advanced examples.** The next examples involves computation of the intersection homology of stratified spaces build by coning off the boundary of a manifold. This example provides an intriguing first glimpses of the duality results we shall study later.

**Example 3.19.** Let \( M \) be an \( n \)-dimensional triangulated manifold with boundary \( \partial M \). Let \( c\partial M \) be the simplicial cone on the boundary of \( M \) (see [77, Section 8]), and let \( X \) be the space obtained by coning off the boundary of \( M \):

\[
X = X^n = M \cup_{\partial M} c\partial M.
\]

If \( v \) is the cone vertex, let \( X \) be stratified as \( \{v\} \subset X \). We compute \( I^pH^*_{GM}(X) \).

We have already seen that every simplex is allowable with respect to the regular stratum \( X - \{v\} \), so we need only check which simplices are allowable with respect to \( \{v\} \). This is only an issue for those simplices containing \( v \), for which \( \dim(\sigma \cap \{v\}) = 0 \). So if \( \sigma \) is an \( i \)-simplex, we need

\[
0 \leq i - \text{codim}(\{v\}) + \overline{p}(\{v\}) = i - n + \overline{p}(\{v\}).
\]

In other words, an \( i \) simplex is allowed to contain \( v \) if only if \( i \geq n - \overline{p}(\{v\}) \).

From this computation, we deduce that for \( i < n - \overline{p}(\{v\}) \), no allowable simplex, and hence no allowable chain, may contain \( v \). Hence in this range every simplex must be a simplex in \( M \) itself and \( I^pC^i_{CM}(X) = C^i(M) \). On the other hand, since every simplex is allowable for \( i \geq n - \overline{p}(\{v\}) \), every chain is allowable for \( i > n - \overline{p}(\{v\}) \), and so in this range \( I^pC^i_{CM}(X) = C^i(M) \). The complicated case arises for \( i = n - \overline{p}(\{v\}) \). Now each \( i \)-simplex is allowable, but no \( i - 1 \) simplex may contain \( v \). So \( I^pC^i_{CM}(X) \) consists of all the \((n - \overline{p}(\{v\}))-\)chains of \( X \) whose boundaries are in \( M \).

Now, let us use these computations to find the intersection homology groups. Since \( I^pC^i_{CM}(X) \) in particular includes all cycles of \( C^i_{CM}(X) \), we have \( I^pH^i_{CM}(X) = H^i(X) \) for \( i \geq (n - \overline{p}(\{v\})) \). It also follows readily from the above discussion that \( I^pH^i_{CM}(X) = H^i(M) \) for \( i < n - \overline{p}(\{v\}) - 1 \). To compute \( I^pH^i_{CM}(X) \), we observe that the cycles in \( I^pC^i_{CM}(X) \) are precisely the cycles in \( M \), but they may bounded any chain in \( X \). This
is a description of the image group of the homomorphism \(H_{n-\bar{p}(\{v\})-1}(M) \to H_{n-\bar{p}(\{v\})-1}(X)\) induced by the inclusion \(M \hookrightarrow X\).

Summarizing, we have computed

\[
I^\bar{p}H_i^{GM}(X) \cong \begin{cases} H_i(X), & i \geq n - \bar{p}(\{v\}), \\ \text{im}(H_i(M) \to H_i(X)), & i = n - \bar{p}(\{v\}) - 1, \\ H_i(M), & i < n - \bar{p}(\{v\}) - 1. \end{cases}
\]

In particular, if \(\bar{p}(\{v\}) \geq n\), then \(I^\bar{p}H_\ast^{GM}(X) \cong H_\ast(X)\), and if \(\bar{p}(\{v\}) \leq -2\) then \(I^\bar{p}H_\ast^{GM}(X) \cong H_\ast(M)\). In fact, the latter isomorphism is also true when \(\bar{p}(\{v\}) = -1\), since then \(H_n(M) = \text{im}(H_n(M) \to H_n(X)) = 0\).

It is interesting to observe that if \(i > 0\) then \(H_i(X) \cong H_i(M, \partial M)\) by an easy homological argument (employ the exact sequence of the pair, the contractibility of cones, and excision), so that in this case we can identify \(I^\bar{p}H_i^{GM}(X)\) with \(H_i(M, \partial M)\) if \(i \geq n - \bar{p}(\{v\}) > 0\) and \(I^\bar{p}H_i^{GM}(X)\) with \(\text{im}(H_{n-\bar{p}(\{v\})-1}(M) \to H_{n-\bar{p}(\{v\})-1}(M, \partial M))\) if \(n - \bar{p}(\{v\}) - 1 > 0\). In particular, if \(n > \bar{p}(\{v\}) + 1\), we can reformulate our computation as

\[
I^\bar{p}H_i^{GM}(X) \cong \begin{cases} H_i(M, \partial M), & i \geq n - \bar{p}(\{v\}), \\ \text{im}(H_i(M) \to H_i(M, \partial M)), & i = n - \bar{p}(\{v\}) - 1, \\ H_i(M), & i < n - \bar{p}(\{v\}) - 1. \end{cases}
\]

This innocent-seeming computation is actually fairly remarkable. The intersection homology groups incorporate both the groups \(H_\ast(M)\) and the groups \(H_\ast(M, \partial M)\) with a transition point depending on the perversity. At the transition point, we have \(\text{im}(H_i(M) \to H_i(M, \partial M))\). This is already reminiscent of duality results, as Lefschetz duality provides pairings (in appropriate dimensions) between \(H_\ast(M)\) and \(H_\ast(M, \partial M)\) and self pairings on \(\text{im}(H_\ast(M) \to H_\ast(M, \partial M))\).

**Example 3.20.** The last example involved considering what happens at a cone vertex of a cone on a manifold. This next example, in which we compute the intersection homology of a suspension of a simplicial filtered space, is somewhat similar in terms of the involvement of cones, though the computations become much more involved. We present this computation as an extended example of the kinds of computations that might arise when working with simplicial intersection homology. We note that such computations become much simpler once further tools are developed, such as singular intersection homology and the Mayer-Vietoris sequence. Singular intersection homology might give a different computation depending on the properties of the triangulation; however, singular and simplicial intersection homology will agree for suitably fine triangulations by Corollary 3.28 and Theorem 5.47 below.

Let \(X\) be a compact \((n-1)\)-dimensional simplicial filtered space endowed with a stratification, and let \(SX\) be the simplicial suspension of \(X\) obtained by adjoining two closed cones on \(X\) along \(X\). Then \(SX\) has a natural filtration given by \((SX)^{i+1} = S(X^i)\) for \(i \geq 0\) and by letting \((SX)^0 = \{N, S\}\), the “north and south poles” given by the two cone vertices. This filtration determines a stratification.
Let $\bar{p}$ be a perversity on $SX$. For simplicity, we will compute the example for which $\bar{p}(N) = \bar{p}(S)$, but we encourage the reader consider the more general case an instructive exercise. Notice that this perversity induces a perversity $\bar{p}_X$ on $X$ itself whose value on the stratum $T \subset X$ is $\bar{p}(ST - \{N,S\})$.

To begin to compute $IPC^G_M(X)$, we first observe that since the codimension of a stratum $T$ in $X$ is the same as the codimension of $ST - \{N, S\}$ in $SX$, the condition for a simplex contained in $X$ to be $\bar{p}_X$-allowable with respect to a stratum $T \subset X$ is exactly the same as the condition for the simplex to be $\bar{p}$-allowable with respect to $ST - \{N, S\}$ in $SX$.

Next we consider simplices containing $N$ or $S$. Notice that any simplex of dimension $> 0$ containing one of these vertices can be written as a cone on a simplex contained in $X$. To explain, recall that we obtain the simplicial cone of a simplex $\sigma = [v_0, \ldots, v_j]$ by appending an initial vertex, $w$, to get $[w, v_0, \ldots, v_j]$ (see [77], Section 8). If $\sigma$ is a simplex of $X$, then we can form the cones $\bar{c}_N\sigma = [N, v_0, \ldots, v_j]$ or $\bar{c}_S\sigma = [S, v_0, \ldots, v_j]$. It is easy to check that these generate homomorphisms $C_*(X) \to C_{*+1}(SX)$ and so maps on chains, though these are not chain maps. In fact $\partial(\bar{c}_N\xi) = \xi - \bar{c}_N(\partial\xi)$ if $\xi$ is a chain of dimension greater than 0; if $v$ is a vertex, then $\partial(\bar{c}_Nv) = v - N$. Similarly formulas hold for $\bar{c}_S$. Conversely, if $\tau$ is a simplex containing $N$, then $\tau$ can be written (up to sign) as $[N, v_0, \ldots, v_j]$, and we recognize $\tau$ as a cone on $\pm\sigma = \pm[v_0, \ldots, v_j] \in X$. Again the analogous statement holds for simplices containing $S$. No simplex contains both $N$ and $S$.

Now, let $\sigma = [v_0, \ldots, v_j] \in X$. Then $\sigma$ is $\bar{p}_X$-allowable in $X$ with respect to a stratum $T \subset X$ if and only if $\bar{c}_N\sigma$ and $\bar{c}_S\sigma$ are $\bar{p}$-allowable in $SX$ with respect to $ST - \{N, S\}$. The argument is the same for each of $\bar{c}_N\sigma$ and $\bar{c}_S\sigma$ so we provide only the former. Clearly,
\[
\dim(\bar{c}_N\sigma \cap (ST - \{N, S\})) = \dim(\sigma \cap T) + 1,
\]
which we can rewrite at
\[
\dim(\bar{c}_N\sigma \cap (ST - \{N, S\})) \leq j + 1 - \text{codim}_{SX}(ST - \{N, S\}) + \bar{p}(ST - \{N, S\}).
\]
Hence we see that the conditions for the allowability of $\sigma$ and $\bar{c}_N\sigma$ with respect to $T$ and $ST - \{N, S\}$ are equivalent.

Next we look at allowability with respect to the strata $\{N\}$ and $\{S\}$. As in our previous computation, if a simplex contains the vertex $N$ then $\dim(\sigma \cap \{N\}) = 0$. So for an $i$-simplex $\sigma$ containing $N$ to be allowable with respect to $\{N\}$, we need
\[
0 \leq i - \text{codim}(\{N\}) + \bar{p}(\{N\}) = i - n + \bar{p}(\{N\}).
\]
In other words, an $i$ simplex is allowed to contain $N$ only if $i \geq n - \bar{p}(\{N\})$, and similarly for the vertex $S$.

\[30\]We assume here that $\sigma$ is not the “empty simplex”. 

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So if \( i < n - \bar{p}(\{N\}) \), no allowable simplex, and hence no allowable chain, may contain \( N \) (or \( S \)), and so all such chains must be contained in \( X \). But we have already noted that a simplex in \( X \) is \( \bar{p} \)-allowable in \( SX \) if and only if it is \( \bar{p}_X \)-allowable in \( X \). So for \( i < n - \bar{p}(\{N\}) \), \( I^pC^G_i(X) = I^pC^G_i(SX) \), and it follows that \( I^pH^G_i(SX) = I^pH^G_i(X) \) for \( i < n - \bar{p}(\{N\}) - 1 \).

We also have that any cycle in \( I^pC^G_{n-\bar{p}(\{N\})-1}(SX) \) is contained in \( X \), but chains in \( I^pC^G_{n-\bar{p}(\{N\})}(SX) \) may contain the suspension vertices. So suppose \( \xi \) is a cycle in \( I^pC^G_{n-\bar{p}(\{N\})-1}(SX) \) and that \( n - \bar{p}(\{N\}) - 1 > 0 \). Then \( \xi \in I^pC^G_{n-\bar{p}(\{N\})-1}(X) \) and \( \bar{e}_N\xi \in I^pC^G_{n-\bar{p}(\{N\})}(SX) \) will be allowable — we have seen above that the cone on the simplices of \( \xi \), which are each \( \bar{p}_X \)-allowable, will be \( \bar{p} \)-allowable with respect to all strata \( ST - \{N, S\} \) but we have also just seen allowability of simplices of this dimension with respect to \( \{N\} \). Since \( \partial\bar{e}_N\xi = \xi \), the boundary of \( \bar{e}_N\xi \) also consists of allowable simplices. So \( \bar{e}_N\xi \) is allowable, and \( \xi \) represents 0 in intersection homology. Since this argument applies to any cycle, we have \( \bar{e}_N\xi \in \bar{p}C_{n-\bar{p}(\{N\})-1}(X) = 0 \).

If \( n - \bar{p}(\{N\}) - 1 = 0 \), we have a slightly more delicate situation as if \( v \) is a single vertex in \( X \), then \( \partial\bar{e}_N v = v - N \), and the point \( N \) is not allowable. However, for any two allowable vertices \( v, w \) in \( X \), \( \bar{e}_N(v - w) \) is an allowable 1-chain and has boundary \( v - w \). Hence any two allowable 0-simplices in \( SX \) are homologous. So in this case \( I^pH^G_0(SX) \) is either \( \mathbb{Z} \) or 0 according as there are or are not any allowable vertices in \( X \).

Finally, we must compute the intersection homology for degrees \( i \geq n - \bar{p}(\{N\}) \). In this case all cycles are allowable with respect to \( \{N, S\} \), and the argument becomes more delicate. We will see that, except in low-dimensional cases, these intersection homology groups are equal to the intersection homology groups of \( X \) in dimension \( i - 1 \). This agrees with what one might expect from the suspension formula for ordinary homology \( \tilde{H}_i(SX) = \tilde{H}_{i-1}(X) \).

Also as for ordinary homology, we will see that the isomorphism can be given by suspending chains in \( X \). We will consider the possibilities \( i = 0, 1 \), which must be handled separately, below. For now assume that \( i \geq n - \bar{p}(\{N\}) \geq 2 \).

First suppose \( \xi \in I^pC^G_i(SX), i \geq n - \bar{p}(\{N\}) \geq 2 \), is a cycle that does not intersect \( \{N, S\} \). Then the same coning argument used above shows that \( \xi \) is the boundary of the allowable chain \( \bar{e}_N\xi \). Similarly, suppose \( \xi \) is a cycle containing a simplex that includes, say \( N \), but not \( S \). Then we can write \( \xi \) uniquely as \( \xi = x + \bar{e}_N y \), where \( x \) and \( y \) are each contained in \( X \); \( \partial\bar{e}_N y \) contains exactly those simplices of \( \xi \) containing \( N \) and then \( x = \xi - \bar{e}_N y \). We do not assume that \( x \) or \( \bar{e}_N y \) are allowable chains, as their boundaries might not be allowable, though of course all simplices of \( x \) and \( \bar{e}_N y \) are allowable, as \( \xi \) is. This implies, as above, that all simplices of \( \bar{e}_N x \) are allowable, and we next show that \( \partial\bar{e}_N x = \xi \). Since \( \xi \) is a cycle, we have \( \partial x = -\partial\bar{e}_N y = -y + \bar{e}_N(\partial y) \). As \( x \) and \( y \) are contained in \( X \), so must be \( \partial x + y = \bar{e}_N(\partial y) \). But if \( \partial y \neq 0 \), then \( \bar{e}_N(\partial y) \) must contain \( N \), which would be a contradiction. So we must
have \( \partial y = 0 \). Now we compute that

\[
\begin{align*}
\partial \bar{c}_N x &= x - \bar{c}_N \partial x \\
&= x + \bar{c}_N \partial \bar{c}_N y \\
&= x + \bar{c}_N (y - \bar{c}_N \partial y) \\
&= x + \bar{c}_N y \\
&= \xi.
\end{align*}
\]

So we see that the only cycles that might not be trivial in intersection homology in this dimension range are those containing both \( N \) and \( S \). We can write any such intersection cycle as \( \xi = x + \bar{c}_N y - \bar{c}_S z \), where \( x, y, z \) are contained in \( X \) and composed of \( \bar{p} \)-allowable simplices, though their individual boundaries might not be. By arguments similar to those above, \( \bar{c}_N x \) is composed of allowable simplices but now with boundary

\[
\begin{align*}
\partial \bar{c}_N x &= x - \bar{c}_N \partial x \\
&= x + \bar{c}_N \partial (\bar{c}_N y - \bar{c}_S z) \\
&= x + \bar{c}_N (y - \bar{c}_N (\partial y) - z + \bar{c}_S (\partial z)) \\
&= x + \bar{c}_N (y - z).
\end{align*}
\]

For the last equation, we have used that \( \partial x = y - z - \bar{c}_N (\partial y) + \bar{c}_S (\partial z) \) and so, since \( x, y, z \in X \), we must have \( \partial y = \partial z = 0 \), as in the argument above. So now

\[
\begin{align*}
\xi - \bar{c}_N z + \bar{c}_S z &= x + \bar{c}_N y - \bar{c}_S z - \bar{c}_N z + \bar{c}_S z \\
&= x + \bar{c}_N (y - z) \\
&= \partial \bar{c}_N x.
\end{align*}
\]

\( \xi \) and \( z \) are allowable and hence so is the cycle \( \bar{c}_N z - \bar{c}_S z \), which we denote \( Sz \), the suspension of \( z \). This also shows that \( \partial \bar{c}_N x \) is allowable, so \( \bar{c}_N x \) is an allowable chain. We have therefore shown that every cycle of \( \bar{p} C_i^{GM} (SX) \) for \( i \geq n - \bar{p} (\{N\}) \geq 2 \) is a suspension of an allowable cycle of \( \bar{p} C_{i-1}^{GM} (X) \), i.e. suspension induces a surjective homomorphism \( S : \bar{p} H_{i-1}^{GM} (X) \to \bar{p} H_i^{GM} (SX) \).

We next show that the intersection homology homomorphism \( S \) is also injective in this dimension range. Suppose \( z \) is an allowable cycle in \( \bar{p} C_i^{GM} (X) \) so that \( Sz \) is a cycle in \( \bar{p} C_{i-1}^{GM} (SX) \) for \( i \geq n - \bar{p} (\{N\}) \), and suppose \( Sz \) bounds an allowable \( i+1 \) chain \( Z \). Again we can write \( Z = \bar{c}_N A - \bar{c}_S B + D \) for chains \( A, B, D \) contained in \( X \) and composed of allowable simplices; and again \( A, B, D \) need not be allowable as chains, a priori. However, we have

\[
Sz = \partial Z \\
= \partial (\bar{c}_N A - \bar{c}_S B + D) \\
= A - \bar{c}_N \partial A - B + \bar{c}_S \partial B + \partial D.
\]

But since \( Sz = \bar{c}_N z - \bar{c}_S z \), we can identify the subchains of \( Sz \) in each equation that contain \( N \) to obtain \( \bar{c}_N z = -\bar{c}_N \partial A \), and it follows that \( z = -\partial A \). As \( A \) was assumed to consist of
allowable cycles, this shows that \( z \) bounds an intersection chain in \( X \). So the suspension homomorphism is injective for \( i \geq n - \bar{p}(\{N\}) \geq 2 \).

To conclude we must consider the cases \( i \geq n - \bar{p}(\{N\}) \) and \( i = 0, 1 \). For \( i = 0 \), we note that \( N \) and \( S \) are now both allowable 0-cycles. If there is any allowable vertex \( v \) in \( X \), then \( \partial c_{N} \) and \( \partial c_{S} \) are allowable and show that \( v, N, \) and \( S \) are all intersection homologous so \( I^{p}H_{0}^{GM}(SX) = \mathbb{Z} \). However, if there is no allowable vertex \( v \) in \( X \), then \( N \) and \( S \) are not intersection homologous as any 1-chain with boundary \( N - S \) would have to include an edge \([N, v]\) for some vertex \( v \) in \( X \), and we know this will be allowable only if \( v \) is allowable by our work way back at the beginning of the example. So in this case \( I^{p}H_{0}^{GM}(SX) = \mathbb{Z} \oplus \mathbb{Z} \).

Finally, we consider \( i = 1 \geq n - \bar{p}(\{N\}) \). The main concern here is that 0-chains will be involved in the argument, but cones on 0-chains must be treated carefully since, for example, if \( v \) is a vertex in \( X \), \( \partial \bar{c}_{N}v = v - N \), and, similarly, \( \partial Sv = S - N \). So allowability considerations at the strata \( \{N\} \) and \( \{S\} \) come into play in ways that they haven’t for higher dimensional cycles; in particular \( c(\bar{c}_{N}v) \) might not be an allowable chain as \( N \) might not be an allowable vertex. Additionally, we see that the suspension of a 0-cycle will only be a cycle if the augmentation of the 0-cycle (the sum of the coefficients of all vertices of the 0 cycle) is 0. Nevertheless, the argument above continues to hold to show that an element of \( I^{p}H_{1}(SX) \) is intersection homologous to a suspension of a dimension 0-intersection cycle; notice that the argument only requires taking cones on 0-chains that are boundaries (and hence have trivial augmentation) and in this case there are no allowability problems at \( N \) or \( S \). But since we have seen that the suspension of a 0-cycle \( z \) will be a 1-cycle only if \( z \) has trivial augmentation, we obtain only a surjection \( \tilde{S} : I^{p_{x}}\tilde{H}_{0}^{GM}(X) \to I^{p}H_{1}^{GM}(SX) \), where \( I^{p}H_{0}^{GM}(X) \) is the reduced intersection homology group, which is defined analogously to reduced ordinary homology as the kernel of the augmentation \( I^{p_{x}}\tilde{C}_{0}^{GM}(X) \to \mathbb{Z} \) modulo the image of the boundary \( I^{p_{x}}\tilde{C}_{1}^{GM}(X) \to I^{p_{x}}\tilde{C}_{0}^{GM}(X) \). The injectivity argument already provided for higher dimensions also continues to hold, and so \( \tilde{S} \) is an isomorphism.

At last we can provide now the full formula\(^{31}\):

\[
I^{p}H_{i}^{GM}(SX) = \begin{cases} 
I^{p_{x}}\tilde{H}_{i-1}^{GM}(X), & i \geq n - \bar{p}(\{N\}), i \neq 0, \\
0, & i = n - \bar{p}(\{N\}) - 1, i \neq 0, \\
I^{p_{x}}H_{i}^{GM}(X), & i < n - \bar{p}(\{N\}) - 1, \\
\mathbb{Z} \oplus \mathbb{Z}, & i = 0 \geq n - \bar{p}(\{N\}), \text{X does not have an allowable vertex}, \\
\mathbb{Z}, & i = 0 \geq n - \bar{p}(\{N\}) - 1, \text{X has an allowable vertex}, \\
0, & i = 0 = n - \bar{p}(\{N\}) - 1, \text{X does not have an allowable vertex}.
\end{cases}
\]

Notice that, except for some quirks in dimension 0, \( I^{p}H_{i}^{GM}(SX) \) agrees with \( I^{p_{x}}H_{i}^{GM}(X) \) in lower dimensions and with \( I^{p_{x}}H_{i-1}^{GM}(X) \) in higher dimensions; it is 0 at the transition dimension, which depends on the perversity. For \( \bar{p}(\{N\}) = \bar{p}(\{S\}) \geq n \), \( I^{p}H_{i}^{GM}(SX) \) behaves like a suspension in ordinary homology in all dimensions, except possibly dimension 0, by always being isomorphic to \( I^{p_{x}}H_{i-1}^{GM}(X) \). For \( \bar{p}(\{N\}) = \bar{p}(\{S\}) \leq -2 \), \( I^{p}H_{i}^{GM}(SX) = I^{p_{x}}H_{i}^{GM}(X) \), as if no suspension took place.

\(^{31}\)Note, as for ordinary homology, \( I^{p_{x}}H_{i}^{GM}(X) = I^{p_{x}}H_{i}^{GM}(X) \) if \( i \neq 0 \).
3.2.3 The motivation for the definition of intersection homology

We should next briefly discuss the curious condition that $\dim(\sigma \cap S) \leq i - \operatorname{codim}(S) + \bar{p}(S)$ for an $i$-simplex $\sigma$. For definiteness, let us assume that $\dim(X) = n$ and that $\dim S = k$. Then the condition $\dim(\sigma \cap S) \leq i - \operatorname{codim}(S) + \bar{p}(S)$ becomes precisely

$$\dim(\sigma \cap S) \leq i - (n - k) + \bar{p}(S) = i + k - n + \bar{p}(S).$$

If we were to ignore the perversity and replace this with the condition

$$\dim(\sigma \cap S) \leq i - (n - k) + \bar{p}(S) = i + k - n,$$

and assume that $X$ is a manifold and $S$ a submanifold then this would precisely be the requirement for $\sigma$ to be in general position with respect to $S$. Actually, this is not quite correct since a lone simplex will rarely be in general position with respect to a submanifold of a manifold (for example, no 1-simplex in a triangulation of the plane compatible with having the $x$-axis as a stratum can be in general position with respect to the $x$-axis), but the statement does begin to make sense once we begin instead to talk about chains. In our plane example, a 1-chain corresponding to an embedded arc will be in general position with respect to the $x$-axis precisely when its intersection with the $x$-axis has dimension $\leq 0$ and the intersection of its boundary with the $x$-axis has dimension $\leq -1$ (i.e. the intersection is empty). These are precisely the conditions of (2).

Within a manifold $M$ it is always possible to push a simplicial chain into general position with respect to a simplicial submanifold $N$ by an ambient isotopy, so placing requirements that chains be in general position with respect to $N$ presents no undue burden and in fact does not alter the homology: by such isotopies, ever cycle is homologous to a cycle in general position with respect to $N$, and similarly if two cycles in general position with respect to $N$ are homologous then a relative isotopy can be used to push the homology into general position with respect to $N$ without altering the two cycles (see [86, Chapter 5]). Therefore, if it suited our purposes, we might define the ordinary homology of a manifold using only chains in general position with respect to certain submanifolds or even more general subcomplexes.

Now, if $S$ is an $n - k$ stratum of a filtered $n$-manifold, and $\xi$ is an $i$ chain, the general position requirement would be that

$$\dim(\xi \cap S) = i + (n - k) - n = i - k.$$ 

So we see that the condition

$$\dim(\xi \cap S) \leq i - k + \bar{p}(S)$$

is just a generalization. If $\bar{p}(S) = 0$, we recover the general position condition, but having $\bar{p}(S) > 0$ lets us relax the general position requirements by a degree controlled by the perversity parameter. By contrast, taking $\bar{p}(S) < 0$, which is less common, strengthens the general position requirement!

Now, suppose $X$ is a PL manifold stratified space. If $X$ is not a manifold, we no longer expect it to be possible to achieve general position with respect to subspaces. For example,
let ξ be a 1-cycle that runs through the pinch point v of a pinched 2-dimensional torus. General position on a 2-manifold would require a 1-cycle and a 0-manifold to be disjoint, but there is no ambient isotopy of the pinched torus that can achieve that. So should we allow ξ in the chain group for our pinched torus? The perversity decides! If \( \bar{(\{v\})} \leq 0 \), then no, that cycle is not allowed. But if \( \bar{(\{v\})} > 0 \), then it is. So the perversity provides stratum-by-stratum control over how much deviation from general position we are willing to allow in defining the intersection chains. This is the origin of the term “perversity” - in some sense it is perverse that we are not requiring general position.

Later, we will see how this loosening of general position requirements comes into play concerning intersection pairings. REF???

### 3.3 PL intersection homology

As one learns in an introductory text on algebraic topology, there are many benefits to working with simplicial homology. For example, unlike singular homology, if the space is compact then all of the chain groups are finitely generated. Additionally, there are no simplices of dimension higher than that of the space. Simplicial homology is also completely combinatorial, and so, at least in theory, computations can be done by a computer. Nonetheless, there are drawbacks to working with a fixed triangulation. Even given a triangulable manifold there is not necessarily any canonical choice of triangulation, and, as we have seen, intersection homology might depend on the choice of triangulation. There are also theorems that become difficult to prove when locked into a specific triangulation.

In this section we turn to a homology theory that takes away the choice of triangulation but that still takes advantage of piecewise linear structure when it exists. This is piecewise linear (PL) homology. PL homology could be presented in any introductory algebraic topology text that treats simplicial homology, though usually it is not considered as most texts have little reason not to jump straight to singular homology. Nonetheless, for the purposes of treating transversality and intersections in manifolds (or pseudomanifolds), which is a topic that will concern us later, PL chains come in handy. Additionally, PL intersection homology will help us prove theorems about simplicial homology. This is not as big of an issue for ordinary homology where less care needs to be taken to make sure that simplicial maps don’t inadvertently change the allowability of simplices with respect to strata. The language of PL chains was also the original language in which intersection homology was formulated by Goresky and MacPherson in [42].

So let X be a PL space. Recall again that this means that X is endowed with a family of admissible locally-finite triangulations (meaning that any two admissible triangulations have a common subdivision and any subdivision of an admissible triangulation is admissible). The collection of triangulations of X form a partially ordered set \( \mathcal{T} \). If T is a triangulation of X, let \( C^*_T(X) \) denote the simplicial chain complex of X with respect to the triangulation T. If \( T, T' \in \mathcal{T} \) and \( T' \) is a subdivision of T, then there is a subdivision chain map

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32 The historical survey [62] contains a bit more regarding the origins of the terminology.

33 Note that subdivisions are not necessarily barycentric subdivisions.
$C^T_*(X) \to C^T_*(X)$ that takes each $i$-simplex of $T$ to the sum of $i$-simplices of $T'$ contained in $T$ (with the compatible orientations).

**Definition 3.21.** If $X$ is a PL space, define the PL chain complex $\mathcal{C}_*(X)$ as

$$\mathcal{C}_*(X) = \lim_{T \rightarrow \hat{T}} C^T_*(X).$$

So each chain in $\mathcal{C}_i(X)$ can be described as a simplicial chain with respect to some admissible triangulation $K$ of $X$, and two such simplicial chains defined with respect to possibly different triangulations $T_1, T_2$ represent the same PL chain if their images agree on some common subdivision $T$ of $T_1$ and $T_2$.

Since PL chain complexes are much less familiar than simplicial chain complexes, we pause to record an important property that will be needed below.

**Lemma 3.22.** Let $X$ and $Y$ be PL spaces, and let $f : X \to Y$ be a proper PL map. Then $f$ induces a chain map $f : \mathcal{C}_*(X) \to \mathcal{C}_*(Y)$.

**Proof.** Recall that a map $f : X \to Y$ is proper if the inverse image of every compact subspace of $Y$ is compact. In particular, $f$ will automatically be proper if $X$ is compact.

Now, let $[\xi] \in \mathcal{C}_i(X)$. By the definition, this means that $[\xi]$ can be represented as a simplicial chain with respect to some admissible triangulation $K$ of $X$. Let $\xi$ be such a simplicial chain. Let $L$ be any admissible triangulation of $Y$. By [57, Theorem 3.6.C], there are subdivisions $K'$ and $L'$ of $K$ and $L$ with respect to which $f : K' \to L'$ is simplicial. So then $f$ induces a map of simplicial chain complexes $f' : C^K_*(X) \to C^L_*(Y)$. Let $\xi'$ be the image of $\xi$ in the subdivision $K'$, and note that $\xi$ and $\xi'$ represent the same chain in $\mathcal{C}_i(X)$. We can now set $f([\xi]) = [f'(\xi')]$. To see that this is well-defined, we need to verify that the construction does not depend on the choices of $K, L, K'$, and $L'$.

We first observe that we obtain the same element $f([\xi])$ if we further subdivide $K'$ to $K''$ and $L'$ to $L''$ in such a way that $f : K'' \to L''$ remains a simplicial map. In fact, let $\sigma$ be a (geometric) $i$-simplex of $K'$. As a simplicial map $K' \to L'$, $f$ either takes $\sigma$ to a lower dimensional simplex of $L'$, or it takes it linearly homeomorphically onto some $i$-simplex $\tau$ of $L'$. As a map of chain complexes $f' : C^K_*(X) \to C^L_*(Y)$, in the first case $f'(\sigma) = 0 \in C^L_i(Y)$, and in the second case, $f'(\sigma)$ is $\mp \tau \in C^L_i(Y)$. Now, suppose we further subdivide $K'$ to $K''$ and $L'$ to $L''$ in such a way that $f : K'' \to L''$ remains a simplicial map, inducing $f'' : C^{K''}_*(X) \to C^{L''}_*(Y)$. Under the subdivision, $\sigma$ is subdivided into a chain $\sigma' \in C^L_i(X)$. In the case where $f$ collapses $\sigma$ to a lower-dimensional face, this must be true of every $i$-simplex in $\sigma'$, and so $f''(\sigma) = 0$. And if $f$ takes $\sigma$ homeomorphically to an $i$-simplex $\tau$ of $L'$, then our assumptions imply that $f$ must also take $\sigma'$ homeomorphically onto the subdivision $\tau'$ of $\tau$ in $L''$. If $f''$ sends all the simplices of $\sigma'$ coherently with the orientation of $\sigma$ and similarly orient all the simplices of $\tau'$ coherently with the orientation of $\tau$, then we must have $f''(\sigma') = \mp \tau'$, where the sign agrees with the sign in $f'(\sigma) = \pm \tau$. In other words, $f''s_K = s_Lf'$, where $s_K$ and $s_L$ are respectively the subdivision maps taking $K'$ to $K''$ and $L'$ to $L''$.

Now, let us return to our initial assumptions that we have chosen to represent $[\xi]$ in a triangulation $K$ of $X$, we have chosen an arbitrary triangulation $L$ of $Y$, and that we have
found $K'$ and $K'$ with respect to which $f$ is simplicial. Now suppose we instead choose a representative $\tilde{\xi}$ for $[\xi]$ in an admissible triangulation $\tilde{K}$ of $X$, then there is a common subdivision $\tilde{K}'$ of $K'$ and $\tilde{K}$, and we can now instead represent $[\xi]$ by $\tilde{\xi}'$ in this subdivision. Also, if we choose a subdivision $\tilde{L}$ of $Y$ instead of $L$, then there is a common subdivision $\tilde{L}'$ of $\tilde{L}$ and $L'$. Again by [57, Theorem 3.6.C], there are subdivisions $K''$ of $K'$ and $L''$ of $\tilde{L}'$ with respect to which $f$ is simplicial. Note that $K''$ is a subdivision of $K'$ and $L''$ is a subdivision of $L'$. We let $f'': C_*^{K''}(X) \to C_*^{L''}(Y)$ denote the induced map of simplicial chain complexes. If we let $\xi''$ denote the image of $\xi$ in $K''$, then $\xi$ and $\xi''$ both represent $[\xi] \in \mathcal{C}_i(X)$, and the arguments of the last paragraph show that $f''(\xi'')$ is a subdivision of $f'(\xi')$ and so represents the same element as $f'(\xi')$ in $\mathcal{C}_i(U)$.

We have now shown that our choices of $K$ and $L$ do not effect the class $[f'(\xi')]$. Similarly, suppose we proceed as in the second paragraph of the proof but instead of choosing $f : K' \to L'$ as our simplicial map we choose some $f : \tilde{K}' \to \tilde{L}'$; really we are not choosing the map to be different but just different triangulations. Let $\tilde{f}' : C_*^{\tilde{K}'}(X) \to C_*^{\tilde{L}'}(Y)$ be the induced map on the simplicial chain complex, and let $\tilde{\xi}'$ denote the image of $\xi$ in $C_*^{\tilde{K}'}(X)$. Again, there are common subdivisions $K''$ of $K'$ and $\tilde{K}'$ and $L''$ of $L'$ and $\tilde{L}'$, a simplicial complex map $f'' : C_*^{K''}(X) \to C_*^{L''}(Y)$, and an image $\xi''$ of $\xi$ in $C_*^{K''}(X)$. The above arguments now show that $[\tilde{f}'(\xi')] = [f''(\xi'')] = [\tilde{f}''(\tilde{\xi}')$, which shows that $[f'(\xi')]$ is also independent of the choice of subdivisions in the application of [57] Theorem 3.6.C.

To see that the map $f : \mathcal{C}_i(X) \to \mathcal{C}_i(Y)$ we have just defined is a chain map, we need only observe that each of the maps of simplicial complexes used in the above discussion are chain maps, as are the subdivision operators. So, continuing with the above notation,

$$\partial(f[\xi]) = \partial[f'(\xi')]$$

$$= [\partial f'(\xi')]$$

$$= [f'(\partial \xi')]$$

$$= [f'(\partial \xi')]$$

$$= [f'(\partial \xi')]$$

$$= f([\partial \xi])$$

$$= f(\partial \xi).$$

\[\square\]

### 3.3.1 PL intersection homology

Now, suppose that $X$ is a PL filtered space so that each skeleton of the filtration is a subcomplex of any admissible triangulation. We would like to define the PL intersection chain complex as

$$I^p \mathcal{C}_*^{GM}(X) = \lim_{T \in \mathcal{T}} I^p C_*^{GM,T}(X),$$

where $I^p C_*^{GM,T}(X)$ is the simplicial intersection chain complex with respect to the triangulation $T$. However, in order to ensure that this definition makes sense, we need to show that subdivision provides a well-defined chain map $I^p C_*^{GM,T}(X) \to I^p C_*^{GM,T'}(X)$ when $T'$ is a subdivision of $T$. This is the content of the next lemma.
Lemma 3.23. For any perversity \( \bar{p} \) and for any admissible triangulations \( T, T' \) of the PL manifold stratified space \( X \) such that \( T' \) is a subdivision of \( T \), the subdivision chain map \( \nu : C^T_*(X) \to C_{* \nu}(X) \) restricts to a chain map \( \nu : I^\bar{p}C^\nu_* GM,T(X) \to I^\bar{p}C^\nu_* GM,T'(X) \).

Proof. As the subdivision map is already a chain map \( C^T_*(X) \to C^T_*(X) \), it is only necessary to show that if the \( \sigma \)-simplex \( \sigma \in T \) is allowable then so is each \( \sigma \)-simplex \( \sigma' \in T' \) such that \( \sigma' \) is contained in \( \sigma \). But if \( \sigma \) is allowable, then for each singular stratum \( S \),

\[
\dim(\sigma \cap S) \leq i - \text{codim}(S) + \bar{p}(S).
\]

If \( \sigma' \) is contained in \( \sigma \), we must have \( \dim(\sigma' \cap S) \leq \dim(\sigma \cap S) \), so \( \sigma' \) is also allowable. \( \Box \)

Definition 3.24. Given the preceding lemma, we can now define

\[
I^\bar{p}\mathcal{C}^{GM}_i(X) = \lim_{T \in \mathcal{T}} I^\bar{p}\mathcal{C}^{GM,T}_i(X)
\]

and

\[
I^\bar{p}\mathcal{H}^{GM}_i(X) = H_*(\lim_{T \in \mathcal{T}} I^\bar{p}\mathcal{C}^{GM,T}_i(X)) = \lim_{T \in \mathcal{T}} I^\bar{p}H^{GM,T}_i(X).
\]

Lemma 3.25. Let \( \xi \in \mathcal{C}_i(X) \), and let \( |\xi| \subset X \) be the support of \( \xi \), i.e. the union of the simplices with non-zero coefficient in some representation of \( \xi \) with respect to some triangulation. Then \( \xi \in I^\bar{p}\mathcal{C}^{GM}_i(X) \) if and only if, for each stratum \( S \) of \( X \),

\[
\dim(|\xi| \cap S) \leq i - \text{codim}(S) + \bar{p}(S)
\]

and

\[
\dim(|\partial \xi| \cap S) \leq i - 1 - \text{codim}(S) + \bar{p}(S).
\]

Proof. Suppose \( \xi \in \mathcal{C}_i(X) = \lim_{T \in \mathcal{T}} C^T_i(X) \). Then \( \xi \) is represented by a chain \( \xi^T \) in \( C^T_i(X) \) for some \( T \), and if the given conditions on \( |\xi| \) and \( |\partial \xi| \) hold, then they clearly also hold on each simplex of \( \xi^T \) and \( \partial \xi^T \). Thus \( \xi^T \in I^\bar{p}C^{GM,T}_i(X) \), and it follows that \( \xi \in I^\bar{p}\mathcal{C}^{GM}_i(X) \).

Conversely, suppose \( \xi \in I^\bar{p}\mathcal{C}^{GM}_i(X) = \lim_{T \in \mathcal{T}} I^\bar{p}C^{GM,T}_i(X) \). Then again \( \xi \) is represented by a chain \( \xi^T \in I^\bar{p}C^{GM,T}_*(X) \) for some \( T \), and each simplex of \( \xi^T \) and \( \partial \xi^T \) must be allowable. But if \( \dim(\sigma \cap S) \leq i - \text{codim}(S) + \bar{p}(S) \) for each simplex \( \sigma \) of \( \xi^T \), then \( \sigma \cap S \) is contained in the \( i \)-skeleton of \( T \), and therefore so will be \( |\xi| \cap S = \cup_{\sigma \text{ in } \xi} (\sigma \cap S) \). Therefore \( \dim(|\xi| \cap S) \leq i - \text{codim}(S) + \bar{p}(S) \). The same argument holds for \( \partial \xi \) and shows that \( |\xi| \) and \( |\partial \xi| \) satisfies the required conditions. \( \Box \)

3.3.2 Relation between simplicial and PL intersection homology

Notice that, for the ordinary PL chains, the homology of \( \mathcal{C}_*(X) \) agrees with the simplicial homology of \( X \) with respect to any triangulation because

\[
H_i(\mathcal{C}_*(X)) = H_* \left( \lim_{\mathcal{T}} C^T_*(X) \right) \cong \lim_{\mathcal{T}} H_*(C^T_*(X)),
\]

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as homology computes with direct limits, and every map in the direct system \( \lim_{T \in T} H_*(C^T_*(X)) \) is an isomorphism, as simplicial homology is preserved by subdivision (see Theorem 17.2). However, in Example 3.17 we saw that intersection homology is not preserved by simplicial subdivision. In this section we will show that preservation of intersection homology under subdivision does hold provided we impose some reasonable conditions on the triangulation. It will follow that \( I^p\mathcal{S}_*^{GM}(X) \) does in fact agree with \( H_*(I^pC_*^{GM,T}(X)) \) for “most” triangulations. T.

Recall that a subcomplex \( L \subset K \) of a simplicial complex is called a full subcomplex if for any simplex \( \sigma \in K \) it is true that if the vertices of \( K \) are all in \( L \) then \( \sigma \) itself is contained in \( L \). If a subcomplex \( L \subset K \) is full, it remains full under any further subdivisions \( L' \subset K' \) such that \( |L| = |K'|, |L' = |L| \) (see [66, Lemma 3.3.b]). If \( T \) is an admissible triangulation of a PL filtered space \( X \), we will say that \( T \) is full if each skeleton of the filtration of \( X \) is triangulated as a full subcomplex of \( T \).

Our goal now will be to prove the following theorem.

**Theorem 3.26.** Suppose \( T \) is a full triangulation of a PL filtered space and that \( T' \) is any subdivision of \( T \). Then the canonical maps induced by subdivision \( I^pH_*^{GM,T}(X) \to I^pH_+^{GM,T'}(X) \) and \( I^pH_*^{GM,T}(X) \to I^p\mathcal{S}_*^{GM}(X) \) are isomorphisms.

The second assertion was first proven by Goresky and MacPherson in an appendix to [67]. Our proof of Theorem 3.26 is based upon an elaboration of their argument.

The theorem will depend on two key lemmas. We will first state the lemmas and prove immediate corollaries, which will provide the proof of the theorem. Then we will prove the lemmas, which will involve some fairly technical work.

The first lemma provides a slightly stronger statement of part of Theorem 3.26 and implies part of the theorem.

**Lemma 3.27.** Suppose \( T \) is a full triangulation of a PL filtered space and that \( T' \) is any subdivision of \( T \). Then the subdivision chain map \( \nu : I^pC_*^{GM,T}(X) \to I^pC_*^{GM,T'}(X) \) has a left inverse chain map \( \mu : I^pC_*^{GM,T'}(X) \to I^pC_*^{GM,T}(X) \) so that \( \mu \nu = \text{id} \). In particular, \( \nu \) induces injection on intersection homology.

**Corollary 3.28.** If \( T \) is a full triangulation of \( X \), then the canonical map \( \beta : I^pH_*^{GM,T}(X) \to I^p\mathcal{S}_*^{GM}(X) \) is injective.

**Proof.** Since the subdivisions of \( T \) form a cofinal system \( T' \subset T \), we can compute \( I^p\mathcal{S}_*^{GM}(X) \) as \( \lim_{T' \in T} I^pH_+^{GM,T'}(X) \). Since \( T \) is a full triangulation, so will be any subdivision of \( T \). So by Lemma 3.27 each map of this direct set of groups will be injective, and it follows that each map from any \( I^pH_*^{GM,T'}(X) \), including \( T' = T \), to \( \lim_{T' \in T} I^pH_+^{GM,T'}(X) \) will be injective.

Unfortunately, the proof that each \( \nu : I^pH_*^{GM,T}(X) \to I^pH_*^{GM,T'}(X) \) is surjective for full \( T \) will need to be a bit more roundabout and utilize the PL chain complexes as an intermediary.

---

34When we think of a simplicial complex \( K \) as a combinatorial object, we use \( |K| \) to denote the geometric realization.
Lemma 3.29. If $T$ is a full triangulation of $X$, then the canonical map $\beta : I^pH_*^{GM,T}(X) \to I^p\bar{\mathcal{F}}_*^{GM}(X)$ is surjective.

Corollary 3.30. Suppose $T$ is a full triangulation of a PL filtered space and that $T'$ is any subdvision of $T$. Then the subdivision maps $\nu : I^pH_*^{GM,T}(X) \to I^pH_*^{GM,T'}(X)$ and $\beta : I^pH_*^{GM,T}(X) \to I^p\bar{\mathcal{F}}_*^{GM}(X)$ are isomorphisms.

Proof. $\nu$ is injective by Lemma 3.27, and $\beta$ is injective by Corollary 3.28. By Lemma 3.29, $\beta$ is onto. To obtain surjectivity of $\nu$, notice that $\beta$ factors through $\nu$, i.e. $\beta$ is the composite $I^pH_*^{GM,T}(X) \xrightarrow{\nu} I^pH_*^{GM,T'}(X) \xrightarrow{\cong} I^p\bar{\mathcal{F}}_*^{GM}(X)$, where the second map is also the canonical map to the direct limit and hence an isomorphism by Lemma 3.29 (replacing $T$ with $T'$).

Since $\beta$ is surjective, so must be $\nu$. This corollary includes the statement of Theorem 3.26 and so proves the theorem.

Now we turn to proving Lemmas 3.27 and 3.29. We will need one further small lemma before we begin.

Lemma 3.31. Suppose $T$ is a full triangulation of a PL filtered space. The interior of a simplex $\sigma$ is contained in the stratum $S$ if and only if

1. all vertices of $\sigma$ are contained in $\bar{S}$ (in particular every vertex is in a stratum $R$ such that $R \leq S$), and

2. at least one vertex of $\sigma$ is contained in $S$.

Proof. Suppose the interior of $\sigma$ is contained in $S$. Then the closure of $\sigma$ (which includes all the vertices of $\sigma$) is contained in the closure of $S$. Furthermore, if no vertex of $\sigma$ is contained in $S$, then every vertex is contained in some stratum $R$ with $R < S$, $R \neq S$. So every vertex of $\sigma$ is contained in some skeleton that does not contain $S$. But this is a contradiction as $T$ is a full triangulation.

Conversely, suppose the two conditions are met. Let $X^i$ be the skeleton of $X$ that contains $S$. Since every vertex of $\sigma$ is contained in $X^i$, the fullness of $T$ implies that $\sigma$ is contained in $X^i$. Since $\sigma$ has at least one vertex in $S$, this implies that $\sigma$ is not contained in $X^{i-1}$. It follows that the interior must actually be in $S$ as a simplex cannot intersect multiple strata of the same dimension without violating connectedness of the simplex.

Proof of Lemma 3.27. We have already seen in Lemma 3.23 that $\nu$ takes allowable chains to allowable chains. We must construct $\mu$. We will first define $\mu$ as a chain map, and then we will check that $\mu$ preserves allowability. Finally, we will see that it provides a left inverse to $\nu$.

We can assume that the vertices of $T$ have been given a total order by the Well-Ordering Principle.

Let $\sigma$ be a simplex of $T$, and let $S_\sigma$ be the stratum of $X$ containing the interior of $\sigma$. By Lemma 3.31, there must be some vertex $v$ of $\sigma$ such that $v \in S_\sigma$. For each $\sigma$, let the vertex of $\sigma$ in $S_\sigma$ that is greatest in the order be called $v_\sigma$. 

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Now, each vertex $w$ of $T'$ is contained within the interior of some simplex $\sigma_w$ of $T$. We define $\bar{\mu}$ on vertices by $\bar{\mu}(w) = v_{\sigma_w}$. Since, by the definition of subdivision, each simplex of $T'$ is contained within some simplex of $T$, this description of $\bar{\mu}$ on vertices is enough to extend $\bar{\mu}$ to a simplicial map $T' \to T$, and hence $\bar{\mu}$ induces a chain map $\bar{\mu}_*: C^{T'}(X) \to C_T^I(X)$.

Let us verify that $\bar{\mu}$ restricts to a well-defined chain map $\mu : I^\#C^{GM,T'}(X) \to I^\#C^{GM,T}(X)$. For this it suffices to show that the image of an allowable simplex in $T'$ is an allowable simplex of $T$.

Suppose $\sigma' \in T'$, and assume $\bar{\mu}(\sigma')$ is nondegenerate (if $\bar{\mu}(\sigma')$ is degenerate, it represents 0 in the chain group and is automatically allowable). As $T$ is already full, so is $T'$, and we have shown in Lemma 3.31 that, for any simplex, the stratum containing the interior of that simplex is determined entirely by the vertices of the simplex. But by construction, if $v$ is a vertex of $\sigma'$, then $v$ and $\bar{\mu}(v)$ are contained in the same stratum of $X$. Therefore the data assigning to each vertex the stratum containing it is the same for $\sigma'$ and $\bar{\mu}(\sigma')$. Thus, the interiors of the faces (of all dimensions) of $\sigma'$ and $\bar{\mu}(\sigma')$ are contained in corresponding strata, and so the dimension of intersection of $\bar{\mu}(\sigma')$ with each stratum of $X$ must be the same as the dimension of intersection of $\sigma$ with that stratum. So $\bar{\mu}(\sigma')$ must be allowable if $\sigma'$ is, and the restriction of $\bar{\mu}$ to intersection chains provides the intersection chain map $\mu$.

Lastly, we will verify that $\bar{\mu} \nu = \text{id}$. This will imply that $\mu \nu = \text{id}$ upon restricting to the intersection chains. The argument will be inductive over dimension of simplices. In fact, we claim that if an $i$-simplex $\sigma \in T$ is triangulated as a complex $K_\sigma$ in $T'$, then exactly one $i$-simplex of $K_\sigma$, call it $\eta_\sigma$, maps onto $\sigma$ under $\mu$ (compatibly with orientation), while all other $i$-simplices of $K_\sigma$ map to degenerate simplices and hence have trivial image under $\mu$.

To start the induction, it is evident by construction that $\bar{\mu} \nu(v) = v$ for each vertex of $T$, and this is consistent with the claimed properties. Now, suppose we have verified the claim for all simplices of dimension up through $i-1$, and let $\sigma$ be an $i$-simplex of $T$. Recall that $v_\sigma$ is the vertex of $\sigma$ that all vertices of $T'$ in the interior of $\sigma$ will map to under $\bar{\mu}$. Let $\tau$ be the $i-1$ face of $\sigma$ that does not contain $v_\sigma$, and let $\eta_\tau$ be the $i-1$ simplex of $T'$ contained in $\tau$ that maps onto $\tau$ under $\bar{\mu}$ (recall that such an $\eta_\tau$ exists by the induction hypothesis). $\eta_\tau$ must be an $i-1$ face of a unique $i'$-simplex $s$ of $T'$, which we claim must be $\eta_\sigma$. Let $w$ be the vertex $s$ that is not contained in $\tau$. We claim that $w$ must map to $v_\sigma$, which would verify that $s$ maps onto $\sigma$; it would also provide the necessary compatibility with orientation as, by assumption, $\eta_\tau$ and $s$ are oriented compatibly with $\sigma$ and $\tau$ and $\bar{\mu}$ is orientation preserving from $\eta_\tau$ to $\tau$. Now, $w$ certainly maps to $v_\sigma$ if $w$ is in the interior of $\sigma$ or if $w = v_\sigma$. Otherwise, $w$ is contained in another $i-1$ face of $\sigma$ that is not $\tau$. But since $v_\sigma$ is contained in the stratum $S_\sigma$, in fact so must be at least all of $\sigma - \tau$ (using that $S_\sigma$ is an open set). But since $v_\sigma$ is the highest in the ordering of vertices among all vertices of $\sigma$ contained in $S_\sigma$, then $v_\sigma$ must also be the image of all vertices of $T'$ contained in $\sigma - \tau$ (as all faces of $\sigma$ containing such vertices also contain $v_\sigma$). Thus $\bar{\mu}(w) = v_\sigma$.

It remains to show that no other $i$-simplex of $T'$ contained in $\sigma$ maps onto $\sigma$ by $\bar{\mu}$. By the preceding argument, any simplex, say $t$ with more than one vertex in $\sigma - \tau$ must map

\footnote{Notice that any such $s$ must contain a neighborhood in $\sigma$ of the barycenter of $\eta_\tau$, and it is impossible for two disjoint simplices to contain such a neighborhood.}
multiple vertices to \( v_\alpha \), and hence must be degenerate. But the only other possibility is to have all but at most one vertex in \( \tau \). Clearly no \( i \)-simplex \( t \) can have all of its vertices in \( \tau \), thus the only possibility is to have an \( i-1 \) face of \( t \) in \( \tau \). But now except for the \( \eta_\sigma \) we have constructed, no such simplex can have its \( i-1 \) face in \( \tau \) be \( \eta_\tau \), and so for any other \( i \)-simplex \( t \), the \( i-1 \) face in \( \tau \) must degenerate under \( \bar{\mu} \) by the induction hypothesis, and so \( \bar{\mu}(\tau) = 0 \).

This completes the proof. \( \square \)

**Proof of 3.29.** Let \([\xi]\) be an element of \( I\bar{\mathfrak{CS}}_i^{GM}(X) \). We can assume that \([\xi]\) is represented by a cycle \( \xi \in I\bar{\mathfrak{CS}}_i^{GM,T}(X) \) for some subdivision \( T' \) of \( T \). Let \( \nu \) be the subdivision map \( I\bar{\mathfrak{CS}}_i^{GM,T}(X) \to I\bar{\mathfrak{CS}}_i^{GM,T'}(X) \), and let \( \mu \) be the left inverse constructed in the proof of Lemma 3.27. By the proof of 3.27, \( \mu(\xi) \) is an allowable cycle in \( I\bar{\mathfrak{CS}}_i^{GM,T}(X) \). We claim that the images of \( \xi \) and \( \mu(\xi) \) are homologous in \( I\mathfrak{CS}^{GM}_i(X) \), which then demonstrates that \([\xi]\) is in the image of \( \beta : I\mathfrak{H}_i^{GM,T}(X) \to I\bar{\mathfrak{CS}}_i^{GM}(X) \). The reason we need to go into \( I\mathfrak{CS}^{GM}_i(X) \) to find the homology, rather than finding it directly in \( I\bar{\mathfrak{CS}}_i^{GM,T'}(X) \), will become clear from the construction.

Let \([\xi]\) be the support of \( \xi \) in \( T' \). In other words, let \([\xi]\) be the union of the \( i \)-simplices in \( T' \) for which \( \xi \) has a non-zero coefficient, though we continue to think of \([\xi]\) as triangulated as a subcomplex of \( T' \). Consider the space \( I \times [\xi] \), and provide it with the standard prism triangulation (see, e.g., [24] Section 2.1). In particular, suppose \( \{0\} \times [\xi] \) and \( \{1\} \times [\xi] \) are triangulated just as \([\xi]\) is triangulated in \( T' \), and for each simplex \( \sigma = [v_0, \ldots, v_j] \) of \([\xi]\), let \( I \times \tau \) comprise simplices of the form \([u_0, \ldots, u_\ell, w_\ell, \ldots, w_i] \), where \( u_j, w_j \) are respectively the copies of \( v_j \) in \( \{0\} \times \tau \) and \( \{1\} \times \tau \). If the \( i \)-simplex \( \sigma = [v_0, \ldots, v_i] \) has coefficient \( m \) in \( \xi \), then let the \( i+1 \) simplex \([u_0, \ldots, u_\ell, w_\ell, \ldots, w_i] \) have coefficient \((-)^{\ell}m \) in order to obtain a simplicial chain \( \Xi \) on \( I \times [\xi] \). Then \( \partial \Xi = \{1\} \times [\xi] - \{0\} \times [\xi] \) (recalling that \( [\xi] \) is a cycle).

Now we construct a piecewise linear map \( \gamma \) from \( I \times [\xi] \) to \( X \). For each vertex \( v \) of \( [\xi] \), set \( \gamma(0 \times v) = v \in X \) and let \( \gamma(1 \times v) = \mu(v) \in X \). Since every \( I \times \tau \) gets mapped into a single simplex of \( T \), this determines \( \gamma \) as a linear map on each simplex of \( I \times [\xi] \), and so overall we obtain a piecewise linear map. The image of \( \Xi \) as a PL chain represents a chain \([\gamma(\Xi)]\) in \( \mathfrak{C}_{i+1}(X) \) such that

\[
\partial [\gamma(\Xi)] = [\mu([\xi])] - [\xi] \in \mathfrak{C}_i(X),
\]

where \([\cdot]\) denotes the class of a chain in the direct limit \( \mathfrak{C}_*X \). We notice, by the way, that the reason we need to descend all the way to \( \mathfrak{C}_*X \) is that \( \gamma \) will not necessarily be simplicial with respect to \( T \) or \( T' \). However, it is possible to make \( \gamma \) simplicial with respect to further subdivisions of \( I \times [\xi] \) and \( T \); see [24] Lemma 2.13 and Theorem 2.14, and note that, since \([\xi]\) must be compact, it is sufficient to restrict attention to a finite subcomplex of \( X \), which we may assume embedded in some \( \mathbb{R}^M \) (we also note that a subdivision of a subcomplex of \( T \) can always be extended to the rest of \( T \), for example by replacing simplices not contained in the finite subcomplex by cones on their boundaries, inductively with respect to dimension).

It only remains to show that \( \gamma(\Xi) \) is allowable. In order to do this, it suffices by Lemma 3.25 to show that \([\gamma(\Xi)]\) is allowable. As we already know that \( \partial [\gamma(\Xi)] = [\mu([\xi])] - [\xi] \) is allowable, we need only check the allowability condition with respect to \([\gamma(\Xi)]\), itself. For
this, it will suffice to show that if $\sigma$ is an $i+1$ simplex of $I \times |\xi|$ then $|\gamma(\sigma)|$ satisfies the allowability conditions in $X$, as then the required dimension conditions will also be true over a finite union of such $\sigma$. To do so, we claim that $\gamma$ is stratum-preserving in the sense that if $x$ is a point in $|\xi|$, then $\gamma$ maps all of $I \times \{x\}$ to the same stratum of $X$. This will suffice to finish the proof as follows: Assume that $\tau$ is an $i$-simplex of $\xi$. We know that $\dim(\tau \cap S) \leq i - \text{codim}(S) + \bar{p}(S)$ for any stratum $S$ because $\tau$ is allowable. But assuming the claim, we will have $\dim(\gamma(I \times |\tau|) \cap S) \leq \dim(\tau \cap S) + 1$, as only points $(t, y) \in I \times |\tau|$ such that $y \subset S$ can map to $S$ under $\gamma$. Thus, in particular, if $\sigma$ is an $i+1$ simplex of $I \times |\tau|$, we have

$$\dim(\gamma(\sigma) \cap S) \leq \dim(\gamma(I \times |\tau|) \cap S) \leq \dim(\tau \cap S) + 1 \leq i + 1 - \text{codim}(S) + \bar{p}(S).$$

So $\sigma$ is an allowable $i+1$ simplex!

Now we must prove the claim. So, for any dimension $k$, let $\tau$ be a $k$-dimensional face of a simplex of $|\xi|$ in the triangulation $T'$, and let $\tau : I \times |\xi| \rightarrow |\xi|$ be the projection. By Lemma 3.31, the interior of $\tau$ is contained in whatever stratum $S_\tau$ of $X$ has the property that all vertices of $\tau$ are contained in $S_\tau$, and at least one vertex of $\tau$ is contained in $S_\tau$. Consider now the simplices of $I \times \tau \subset I \times |\xi|$ that intersect $\pi^{-1}(\tilde{\tau})$, where $\tilde{\tau}$ is the interior of $\tau$. These are the $k+1$ simplices of the form $[u_0, \ldots, u_\ell, w_\ell, \ldots, w_k]$ and the $k$-simplices of the form $[u_0, \ldots, u_\ell, w_{\ell+1}, \ldots, w_k]$; all other simplices in the triangulation of $I \times |\tau|$ are contained in $\pi^{-1}(I \times |\partial\tau|)$ (the vertices of these other simplices all project to vertices of a proper face of $\tau$). Now recall $\gamma(u_j)$ is simply the corresponding vertex $v_j$ of $\tau$ in $X$ and $\gamma(w_j) = \mu(v_j)$, and by construction $v_j$ and $\mu(v_j)$ always lie in the same stratum of $X$. Therefore all of the $\gamma(u_j)$ and $\gamma(w_j)$ lie in $\tilde{S}_\tau$ and, for at least one index $m$, $\gamma(w_m)$ and $\gamma(u_m)$ are contained in $S_\tau$. Therefore, again by Lemma $\text{L vertrec}$, the interiors of the $k+1$ and $k$ simplices that intersect $\pi^{-1}(I \times \tilde{\tau})$ are all contained in $S_\tau$. Furthermore, the interiors of $\tau$ and $\mu(\tau)$ are contained in $S_\tau$ and thus all of $\gamma(I \times \hat{\tau})$ is contained in $S_\tau$. As the interiors of the faces (of all dimensions) of the the simplices of $|\xi|$ partition $|\xi|$, the claim follows.

### 3.4 Singular intersection homology

For ordinary homology, singular homology presents many advantages over simplicial homology, at the cost of trading a manageable number of simplices (finite on a compact simplicial space) for an uncountable number of simplices (on a space that is not a point) and thus of not being computable combinatorially. That said, many other properties, particularly homotopy properties, become much more transparent for singular homology, and of course singular homology applies to more general classes of spaces that might not even be triangulable. Singular intersection homology faces many of the same trade-offs. Singular intersection homology applies to more general spaces, and it will become easier to prove some theorems, at the expense of computation at first becoming more complicated. Ultimately, singular intersection homology will provide a setting for our most general duality results.

In this section, $X$ will be any filtered space, not necessarily simplicial or PL. Let $S_*(X)$ be the complex of singular chains of $X$. Recall that a singular simplex $\sigma \in S_*(X)$ is a continuous function $\sigma : \Delta^i \rightarrow X$, where $\Delta^i$ is the standard $i$-dimensional simplex.
Just as in the simplicial case, we wish to define a subcomplex $I_{\bar{p}}S^*_{GM}(X) \subset S_*(X)$ for each perversity $\bar{p}$. A little thought will make the reader leery of trying to use dimension of intersection to measure allowability since the images of singular simplices might now be quite complex (think of pathological things like space-filling curves). Instead, we have the following pleasant adaptation of the simplicial notion of allowability introduced by Henry King\(^{36}\)\(^{61}\).

**Definition 3.32.** Let $X$ be a filtered space endowed with a perversity $\bar{p}$, and let $S_*(X)$ be the simplicial chain complex of $X$.

We deem a singular $i$-simplex $\sigma: \Delta^i \to X$ to be $\bar{p}$-allowable if, for all strata $S$ of $X$,

$$
\sigma^{-1}(S) \subset \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i\}.
$$

(3)

Here $\text{codim}(S)$ is the formal codimension of $S$ determined by the formal dimensions of the filtration, and we refer to the simplicial skeleta of $\Delta^i$. If inequality (3) is satisfied for some $\sigma$ and some $S$, we say that $\sigma$ is $\bar{p}$-allowable with respect to the stratum $S$. If the perversity $\bar{p}$ has been fixed in advance, we will sometimes simply say that $\sigma$ is allowale.

A chain $\xi \in S_*(X)$ is $\bar{p}$-allowable if all of the simplices in $\xi$ and all of the simplices of $\partial \xi$ are $\bar{p}$-allowable.

Let $I_{\bar{p}}S^*_{GM}(X) \subset S_*(X)$ be the chain complex of $\bar{p}$-allowable chains, which we call the (perversity $\bar{p}$ singular) intersection chain complex. Let the (perversity $\bar{p}$ singular) intersection homology groups be the homology groups $H_*(I_{\bar{p}}S^*_{GM}(X))$. At the risk of some possible confusion with the simplicial homology groups, we will generally denote these by $I_{\bar{p}}H^*_{GM}(X)$. The analogous notation is always justified when working with ordinary homology because the simplicial and singular homology groups always agree on simplicial spaces. While we have already seen by Example 3.17 that this cannot always be the case here, we will have agreement with the simplicial intersection homology of “most” triangulations via Theorem 5.47. Between this fact and contextual clues, we hope the reader will not be too misled by the notation.

Notice that if $X$ is a simplicial stratified space and the singular simplex $\sigma \to X$ is simply the inclusion of one of the $i$ simplices in the triangulation of $X$, then this definition of allowability corresponds exactly to our simplicial allowability conditions.

Let us compute some examples:

**Example 3.33.** Let $X = X^0$ be a point. In this case, there is only one stratum, $X$ itself, and it is a regular stratum so $\bar{p}(X) = 0$ for any perversity $\bar{p}$. There is exactly one simplex in each

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\(^{36}\)As in the original work of Goresky-MacPherson in \([42, 43]\), King assumes that all strata of the same (co)dimension take the same perversity value. However, he dispenses with the other requirements of a Goresky-MacPherson perversity (except that it should be 0 on regular strata) and calls these loose perversities.

\(^{37}\)We do not assume that $\bar{p}$ is a GM perversity.

\(^{38}\)By saying that “$\sigma$ is a simplex in $\xi$” or that “$\sigma$ belonging to $\xi$”, we mean that $\sigma$ is a simplex in $\xi$ with non-zero coefficient. In other words, if we write $\xi = \sum_j n_j \sigma_j$ for $n_j \in \mathbb{Z}$ and the $\sigma_j$ singular simplices such that $\sigma_j \neq \sigma_\ell$ if $j \neq \ell$, then we mean that $\sigma = \sigma_k$ for some $k$ such that $n_k \neq 0$. 

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dimension, the unique map \( \sigma : \Delta^i \to X \). In this case, \( \sigma^{-1}(X) = \Delta^i \), and the allowability condition then becomes that

\[
\Delta^i \subset \{ i - \text{codim}(X) + \bar{p}(X) \text{ skeleton of } \Delta^i \} = \{ i \text{ skeleton of } \Delta^i \}
\]

So every singular simplex is allowable, and \( \bar{I}^\bar{p} \mathcal{S}_{GM}^*(X) = S_*(X) \), the ordinary chain complex.

Once again, we can observe that the allowability condition is vacuous when it comes to regular strata:

**Lemma 3.34.** Let \( \sigma \) be a singular \( i \)-simplex of a filtered space \( X \) and let \( S \) be a regular stratum of \( X \). Then the allowability condition is always satisfied.

**Proof.** In this case the requirement becomes that \( \sigma^{-1}(S) \subset \{ i \text{ skeleton of } \Delta^i \} \), which is satisfied trivially. \( \square \)

**Example 3.35.** Suppose \( X = X^n \) is a filtered space that is filtered trivially so that there is only one stratum of dimension \( n \). Then it follows from the preceding lemma that \( \bar{I}^\bar{p} \mathcal{S}_{GM}^*(X) = S_*(X) \).

**Remark 3.36.** Lemma 3.34 allows us to provide further justification for setting \( \bar{p}(S) = 0 \) for all regular strata even in the singular case. We see from the lemma that with \( \bar{p}(S) = 0 \) all simplices are allowable with respect to all regular strata. Furthermore, if \( \bar{p}(S) = m \) for any \( m \geq 0 \), then it is easy to see that the same conclusion will hold, so as mentioned in Remark 3.2, any choice of \( \bar{p}(S) \geq 0 \) for regular strata would provide the same intersection homology, but we choose \( \bar{p}(S) = 0 \) for definiteness and convenience.

By contrast if \( S \) is regular and \( \bar{p}(S) \leq -1 \), then for an \( i \)-simplex to be allowable with respect to \( S \), we would need

\[
\dim(\sigma \cap S) \subset \{ i + \bar{p}(S) \text{ skeleton of } \Delta^i \},
\]

where \( i + \bar{p}(S) \leq i - 1 \). In other words, at most the \( i - 1 \) skeleton of \( \Delta^i \) could map to \( S \) (and less of it if \( \bar{p}(S) < -1 \)). But since \( X - \Sigma_X \) is an open subset of \( X \), \( \sigma^{-1}(X - \Sigma_X) \) must be an open subset of \( \Delta^i \), and so the allowability condition can only be satisfied if \( \sigma^{-1}(X - \Sigma) = \emptyset \), i.e. if the image of \( \sigma \) is in \( \Sigma_X \). In other words, \( I^\bar{p} \mathcal{H}_{GM}^*(X) \) would not see that regular stratum, so it is equal to \( I^\bar{p} \mathcal{H}_{GM}^*(X - S) \). Therefore having regular strata with negative perversities is the same as working on spaces without those strata, and we could just as well have taken that view from the beginning and worked on a different space.

Altogether, this makes it reasonable to always have \( \bar{p}(S) = 0 \) for regular strata.

**Example 3.37.** Suppose \( \bar{p}(S) \leq \text{codim}(S) - 2 \) for all singular strata \( S \) (this is a common condition to require for a perversity). Then \( i - \text{codim}(S) + \bar{p}(S) \leq i - 2 \) and so the allowability condition requires

\[
\sigma^{-1}(S) \subset \{ i - 2 \text{ skeleton of } \Delta^i \}
\]

\[39\]As in the simplicial case (see footnote 29), this is slightly untrue as \( \Sigma_X \) has different total dimension than \( X = X^n \), and so if we remove all regular strata the codimensions of strata will also be different; however, again we could adjust for this by modifying the perversities by the same shift.

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for all singular strata. So if \( i = 0 \) or \( 1 \), no \( i \)-simplex may intersect any singular stratum. Consequently, we must have that \( \bar{I}^pH_0^{\text{GM}}(X) \cong \mathbb{Z}^m \), where \( m \) is the number of path components of \( X - \Sigma X \).

**Example 3.38.** Let \( M \) be a compact \( n-1 \) dimensional manifold, and let \( X = X^n = cM \) stratified by \( \{v\} \subset X \), where \( v \) is the vertex of the cone. Since all simplices are allowable with respect to the regular stratum, the allowability condition for an \( i \)-simplex becomes

\[
\bar{I}^{-1}(\{v\}) \subset \{i - n + \bar{p}(\{v\}) \text{ skeleton of } \Delta^i\}.
\]

If \( i < n - \bar{p}(\{v\}) \) then the image of \( \sigma \) cannot contain \( v \) at all, and so for\(^{40}\) we have \( \bar{I}^pS^i_{\text{GM}}(X) = S_i(X - \{v\}) \), and so \( \bar{I}^pH^i_{\text{GM}}(X) = H_i(X - \{v\}) \cong H_i(M) \).

For \( i \geq n - \bar{p}(\{v\}) \), each \( i \)-simplex is allowed to map at least a vertex to \( v \), and possibly more depending on dimension. To compute \( \bar{I}^pH^i_{\text{GM}}(X) \), suppose that \( \sigma \) is such an allowable \( i \)-simplex. Let \( \bar{c}\sigma \) be the (singular) cone on \( \sigma \). This is defined as follows: Since \( X \) is the cone \( cX = [0, 1] \times M/\sim \), every point in \( X \) can be described as a pair \((t, z)\), where \( t \in [0, 1] \), \( z \in M \) (with \( z \) non-unique if \( t = 0 \)). In particular, if \( x \in \Delta^i \), then \( \sigma(x) = (\sigma_I(x), \sigma_M(x)) \), where \( \sigma_I \) is the composition of \( \sigma \) with the projection to \( I \) and similarly for \( \sigma_M \). Now, think of \( \Delta^{i+1} \) as the closed cone on \( \Delta^i \), i.e. \( \Delta^{i+1} = \bar{c}\Delta^i = [0, 1] \times \Delta^i/\sim \), and each point of \( \Delta^{i+1} \) can be written \((s, x)\) with \( s \in [0, 1] \) and \( x \in \Delta^{i+1} \) (again non-uniquely if \( s = 0 \)). Define the cone \( \bar{c}\sigma \) so that \( \bar{c}\sigma(s, x) = (s\sigma_I(x), \sigma_M(x)) \). Then \( \bar{c}\sigma(1, x) = (\sigma_I(x), \sigma_M(x)) = \sigma(x) \), and \( \bar{c}\sigma(0, x) = (0, \sigma_M(x)) = v \). This map is well-defined despite the ambiguity in coordinates because if \( \sigma(x) = (0, z) = v \), then \( z = \sigma_M(x) \) is not well-defined but regardless we have \( \bar{c}\sigma(s, x) = (0, z) = v \), which is well-defined. This map is also readily seen to be continuous, and so \( \bar{c}\sigma \) is an \( i + 1 \) simplex.

We next claim that if \( \sigma \) is a \( \bar{p} \)-allowable \( i \)-simplex then so is \( \bar{c}\sigma \), provided \( i - n + \bar{p}(\{v\}) \geq -1 \). The key issue, of course, is to compute \((\bar{c}\sigma)^{-1}(\{v\})\). This set certainly includes the cone vertex \((0, x) \in \Delta^{i+1} \). Otherwise, it consists of the points \((s, x)\) such that \( \sigma(x) = v \). Suppose \( x \) is contained in the \( j \)-skeleton of \( \Delta^i \). Then each point \((s, x)\) is contained in at most the \( j + 1 \) skeleton of \( \Delta^{i+1} \). So if \( \sigma^{-1}(\{v\}) \) is contained in the \( j \)-skeleton of \( \Delta^i \) for \( j \geq -1 \), then \((\bar{c}\sigma)^{-1}(\{v\}) \) is contained in the \( j + 1 \)-skeleton of \( \Delta^{i+1} \) for \( j \geq -1 \). Now if \( \sigma \) is allowable, then

\[
\sigma^{-1}(\{v\}) \subset \{i - n + \bar{p}(\{v\}) \text{ skeleton of } \Delta^i\},
\]

and we have just see that if \( i - n + \bar{p}(\{v\}) \geq -1 \) then

\[
(\bar{c}\sigma)^{-1}(\{v\}) \subset \{i + 1 - n + \bar{p}(\{v\}) \text{ skeleton of } \Delta^i\}.
\]

But this is okay because \( \bar{c}\sigma \) is an \( i + 1 \) simplex! So the increase in the dimension of intersection with the skeleton of the model simplex is offset by the increase in the dimension of the simplex itself!

Notice that we do need to be careful to have \( i - n + \bar{p}(\{v\}) \geq -1 \) since there is no difference between \( j \)-skeletons of \( \Delta^i \) for \( j < 0 \); in fact in all these cases saying that \( \sigma^{-1}(\{v\}) \subset \{i - n + \bar{p}(\{v\}) \text{ skeleton of } \Delta^i\} \)

\(^{40}\)The extra \(-1\) is because homology in dimension \( i \) depends on chains in dimension \( i \) and \( i + 1 \).
$\{j\ \text{skeleton of } \Delta^i\}$ simply means that $v$ isn’t in the image of $\sigma$. However in all these cases $(\bar{c}\sigma)^{-1}(\{v\})$ is contained in the 0 skeleton of $\Delta^{i+1}$, regardless.

Now, let us see what this tells us about intersection homology. Let $\bar{c}$ act on chains as the linear extension of its action on simplices. Suppose $\xi \in I^p S^s_{GM}(X)$ is an allowable $i$ cycle for $i \geq n - \bar{p}(\{v\}) - 1$. Notice that this also puts us in the situation $i - n + \bar{p}(\{v\}) \geq -1$. Thus $c\xi$ is allowable. Furthermore, if $i > 0$ then since $\xi$ is a cycle we will have $\partial(c\xi) = \xi$. In fact, notice that on each $i$-simplex, $i > 0$, $\partial(c\sigma) = \sigma - \bar{c}(\partial\sigma)$, just as for simplicial simplices, and if $\xi$ is an $i$-cycle, $i > 0$, all of the $\bar{c}(\partial\sigma)$ terms must cancel in $\partial(c\xi)$. So $\xi$ bounds, and $I^p H^s_{GM}(X) = 0$, as we’d expect for a cone!

There is one last case to be careful about: when $i = 0$. This case is fundamentally different even for ordinary homology because while the cone on an $i$-cycle $\xi$, $i > 0$, always has $\partial \xi = \xi$, and so $c\xi$ provides a null-homology of $\xi$, this is not always true in the 0-cycle case, because if $\sigma$ is a singular 0-simplex, then $\partial(c\sigma) = \sigma - \sigma_v$, where $\sigma_v$ is the singular 0-simplex with image $v$. So for ordinary homology we just get a homology from any singular simplex to $\sigma_v$. But we can’t even do this in intersection homology because even if $\sigma$ and $c\sigma$ are allowable as simplices, the chain $c\sigma$ might not be allowable, as $\sigma_v$ might not be allowable. That said, if we continue to assume that $0 \geq n - \bar{p}(\{v\}) - 1$ so that $c\sigma$ is allowable as a simplex, then if $\sigma_1$, $\sigma_2$ are any two allowable 0 simplices, then the cone $c(\sigma_2 - \sigma_1)$ will have allowable boundary $\sigma_2 - \sigma_1$, and so any two allowable 0 simplices are allowable homologous. Since there are allowable simplices in the regular stratum $X - \{v\}$, we have $I^p H^s_{GM}(X) \cong \mathbb{Z}$.

Altogether, we have computed the following:

$$I^p H^s_{GM}(X) \cong \begin{cases} 
0, & i \geq n - \bar{p}(\{v\}) - 1, i \neq 0, \\
\mathbb{Z}, & i \geq n - \bar{p}(\{v\}) - 1, i = 0, \\
H_i(M), & i < n - \bar{p}(\{v\}) - 1.
\end{cases}$$

**Example 3.39. PUT A FIGURE HERE**

Now let $X = X^1 = S^1$, the circle and let $x_0 \in S^1$ be any point. Suppose $X$ is filtered as $\{x_0\} \subset X$. Then $X$ has two strata: the regular stratum $X - \{x_0\}$ and the singular stratum $\{x_0\}$. We wish to compute $I^p H^s_{GM}(X)$. As this computation will become much simpler once we have established some general properties of singular intersection homology in the next section, we limit ourselves here to the calculations that can be carried out in fairly short order.

We have already seen, in Lemma 3.34, that all simplices are allowable with respect to regular strata, so we have to check allowability at $\{x_0\}$, where $\bar{p}(\{x_0\})$ could be any integer. An $i$-simplex $\sigma: \Delta^i \to X$ is allowable if it satisfies the conditions

$$\sigma^{-1}(\{x_0\}) \subset \{i - \text{codim}(\{x_0\}) + \bar{p}(\{x_0\})\} \text{ skeleton of } \Delta^i$$

$$= \{i - 1 + \bar{p}(\{x_0\})\} \text{ skeleton of } \Delta^i$$

Already things have become much more complicated for singular intersection homology!

If $i - 1 + \bar{p}(\{x_0\}) \geq i$, i.e. if $\bar{p}(\{x_0\}) \geq 1$, then any simplex is allowable and so $I^p S^s_{GM}(X) = S_s(X)$ and $I^p H^s_{GM}(X) = H_s(X)$. 

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If $i - 1 + \bar{p}(\{x_0\}) < 0$, i.e. if $i < 1 - \bar{p}(\{x_0\})$, then no simplex whose image contains $x_0$ is allowable. In this case $I^pS^GM_i(X) = S_i(X - \{x_0\})$, which implies that $I^pH^GM_i(X) \cong H_i(X - \{x_0\})$ for $i$ in the range $i < -\bar{p}(\{x_0\})$ (we cannot draw this conclusion regarding $i = -\bar{p}(\{x_0\})$ because we do not know yet about $I^pS^GM_{-1-\bar{p}(\{x_0\})}(X)$).

What about the more general situation when $\bar{p}(\{x_0\}) \leq 0$? This is more complex, and we will defer the full computation for now. Note, however, that in the case $\bar{p}(\{x_0\}) \leq -1$ the allowability condition becomes $\sigma^{-1}(\{x_0\}) \subset \{i - 2 \text{ skeleton of } \Delta^i\}$. This is sufficient to be able to lift simplices and their boundaries to the covering interval $[0, 2\pi]$, and from there one could kill cycles of degree $> 0$ by “coning off” to the center point $\pi \in [0, 2\pi]$. Thus one would expect $I^pH^GM_i(X) = 0$ for $i > 0$ and $I^pH^GM_0(X) = \mathbb{Z}$. We will see that this is indeed the case below in Example 4.45.

As for ordinary singular homology, we have begun to see that singular intersection homology can be difficult to compute “by hand”. Therefore, we would like to have some of the standard tools of homology available to us — long exact sequences, homotopy invariance, excision, etc. We will begin to explore these properties, and whether or not they carry over, in the next section.

Remark 3.40. Before moving on to investigate properties of intersection homology groups, it is useful to make an important observation about the chain complexes $I^pS^GM_\ast(X)$ that we will need to keep in mind. The ordinary singular chain groups $S_i(X)$ are free groups generated by the $i$-dimensional singular simplices. Since $I^pS^GM_i(X) \subset S_i(X)$ by definition, each $I^pS^GM_\ast(X)$ is also a free group; see [64. Theorem III.7.1]. However, $I^pS^GM_\ast(X)$ does not necessarily have a basis of singular simplices since we know that an allowable simplex of $S_i(X)$ is not necessarily allowable as a chain. Hence $I^pS^GM_\ast(X)$ has some basis of allowable chains, but in general we will not know what it is. This necessitates some care.

Similar remarks apply for simplicial intersection chain complexes. On the other hand, PL intersection chains are already more complex because the groups $\mathcal{C}_\ast(X)$ are themselves only direct limits of free groups and so do not have evident bases even in the non-intersection case.

4 Basic properties of singular and PL intersection homology

In this chapter we establish the basic properties of intersection homology. Ultimately, we wish to acquire enough tools to be able to perform computations. We will see that many of the axioms of ordinary homology persist, though often in modified forms that are suitable to the study of stratified spaces.

In treatments of homology in most textbooks, it is sufficient to develop properties for singular homology and then call upon the equivalence of singular and simplicial homology on simplicial spaces to transfer the properties to simplicial homology, sometimes with some minor modification to account for the difficulty of dealing with arbitrary open subsets of simplicial spaces. Unfortunately, even establishing the equivalence of singular and PL inter-
section homology will require knowing that certain properties hold for both theories. Thus we will have to develop these properties independently. Fortunately, however, the proof techniques in the two settings often complement each other, so we will be able to proceed in close parallel.

In the piecewise linear world, we will restrict our attention to PL intersection homology rather than any simplicial intersection homology. This is justified by the isomorphism we have already established in Theorem 3.26 between PL and simplicial intersection homology for most triangulations, but it is also necessary due to the limitations of the simplicial theory. For example, if $X$ is a simplicial stratified pseudomanifold, and $X$ is the union of two subcomplexes, the subcomplexes will almost certainly not be pseudomanifolds themselves, except under limited conditions. This makes it difficult, for example, to construct simplicial versions of the Mayer-Vietoris sequence for simplicial intersection homology. By contrast, PL intersection homology can be treated more analogous to the singular theory, and we can get Mayer-Vietoris sequences in that vein by covering a space with open subsets (which are automatically PL subsets). See Section 4.4 for more details about Mayer-Vietoris sequences.

4.1 Stratified maps, homotopies, and homotopy equivalences

A key point about ordinary singular homology is that it is a functor from the category of spaces to the category of sequences of abelian groups (and of course it can be generalized to homology of pairs or in other ways). So far we have set up intersection homology of filtered spaces, but we have not yet considered maps of spaces.

If try to define a homomorphism $f_* : I^p H^{GM}_*(X) \to I^p H^{GM}_*(Y)$ for an arbitrary continuous map $f : X \to Y$ of filtered spaces, we immediately run into trouble. For one thing, as we have defined them, perversities are dependent upon the stratification of the space, so without further conditions it does not necessarily make sense to have the same perversity $\bar{p}$ defined on both $X$ and $Y$. Even if we take $\bar{p}$ to be a GM-perversity, so that it depends only on the codimensions of the strata and so can be applied to multiple spaces, there are still difficulties. As the simplest example, suppose $X$ is a point and $Y$ is any space with a nonempty singular stratum $S \subset Y = Y^n$. Let $f : X \to Y$ be any map that takes the point $X$ into $S$. We have previously computed in Example 3.33 that any singular $i$-simplex $\sigma$ is allowable in $X$ with respect to any perversity. But there is no reason to expect that $f(\sigma) \in S_i(Y)$ is allowable with respect to $\bar{p}$ and $S$. In fact, this will only be possible if $i - \text{codim}(S) + \bar{p}(S) \geq i$, i.e. if $\bar{p}(S) \geq \text{codim}(S)$. Thus in general it is not possible to set up a fully general functoriality with respect to any single perversity and map between filtered spaces.

Similarly, intersection homology will not be a homotopy invariant of spaces. For example, let $X = cM$ be the open cone on the $n-1$ manifold $M$. Then $X$ is contractible to a point, for which we have seen in Example 3.33 that every intersection homology group is the same as the ordinary homology group of the point. However, as seen in Example 3.38 the intersection homology of a cone is not always the homology of a point.

41Really, the issue is one of excisive couples, which are tricky but possible to obtain in the PL or singular setting but which are more problematic in the simplicial setting.
That said, certainly one should be able to set up some reasonable situations in which one obtains homomorphisms of intersection homology groups, and that is what we turn to now.

Of course in the most general situation, one could simply consider all maps \( f : X \to Y \) between filtered spaces with respective perversities \( \bar{\rho} \) and \( \bar{q} \) such that if \( \sigma \in I^\bar{\rho} S^\text{GM}_i(X) \) then \( f(\sigma) \in S_i(Y) \) is allowable so that \( f(\sigma) \in I^\bar{q} S^\text{GM}_i(Y) \). This would certainly yield maps of intersection homology groups. In practice, however, there are more specific classes of maps that seem sufficiently useful for the required purposes.

Recall from Definition \[2.85\] that \( f : X \to Y \) is a stratified map of filtered spaces if for each stratum \( S \subset X \) there is a unique stratum \( S' \subset Y \) such that \( f(S) \subset S' \). We impose further limitations as follows:

**Definition 4.1.** A map \( f : X \to Y \) is stratified with respect to \( \bar{\rho}, \bar{q} \) (or \((\bar{\rho}, \bar{q})\)-stratified) if

1. the image of each stratum of \( X \) is contained in a single stratum of \( Y \) of the same codimension, i.e. if \( Z' \subset Y \) is a stratum of codimension \( k \), then \( f^{-1}(Z') \) is a union of strata of \( X \) of codimension \( k \), and

2. if the stratum \( Z \subset X \) maps to the stratum \( Z' \subset Y \), then \( \bar{\rho}(Z) \leq \bar{q}(Z') \).

While this seems somewhat restrictive, here are two important examples:

**Example 4.2.** Let \( X \) be an open subset of the filtered space \( Y \), and let \( \bar{p} \) be the perversity on \( X \) inherited by the perversity \( \bar{q} \) on \( Y \). In other words, if \( S \) is a stratum of \( X \) and \( S \subset S' \) for a stratum \( S' \) of \( Y \), then \( \bar{p}(S) = \bar{q}(S') \). Then the inclusion \( X \hookrightarrow Y \) is stratified with respect to \( \bar{p}, \bar{q} \).

**Example 4.3.** Let \( Y = X \times Z \), where \( Z \) is any trivially filtered space and \( Y \) has the product filtration. Let \( f : X \to X \times Z \) be the inclusion \( f(x) = (x, z_0) \) for some fixed point \( z_0 \in Z \), and suppose \( Y \) has the perversity \( \bar{q} \) induced by the perversity \( \bar{p} \) on \( X \), i.e. \( \bar{q}(S \times Z) = \bar{p}(S) \) for any stratum \( S \) in \( X \). Then \( f \) is \((\bar{p}, \bar{q})\)-stratified. More generally, if \( f \) is a normally nonsingular inclusion (recall Definition \[2.95\]), then the same considerations apply if the perversities are compatible in this way in a neighborhood of the image of \( X \).

**Proposition 4.4.** If \( X,Y \) are filtered spaces and \( f : X \to Y \) is \((\bar{p}, \bar{q})\)-stratified, then \( f \) induces a chain map of singular intersection chain complexes \[^{42}\] \( f : I^\bar{p} S^\text{GM}_i(X) \to I^\bar{q} S^\text{GM}_i(Y) \). If, furthermore, \( X,Y \) are PL filtered spaces and \( f \) is a PL map that is \((\bar{p}, \bar{q})\)-stratified, then \( f \) induces a chain map \( f : I^\bar{p} \mathcal{C}^\text{GM}_i(X) \to I^\bar{q} \mathcal{C}^\text{GM}_i(Y) \) of PL intersection chain complexes. In either case, we obtain corresponding maps of intersection homology groups.

**Proof.** In both cases, there are the usual maps of chain complexes induced on the ordinary singular and PL chains by maps of spaces. Since the intersection chain complexes are subcomplexes, we need only check that allowability is preserved.

\[^{42}\] We abuse notation by letting the same symbol \( f \) stand for maps of spaces and for the algebraic homomorphisms they induce. We hope context will reduce the confusion while this practice will slightly reduce notational clutter.
First consider the singular intersection chains. If $\sigma : \Delta^i \to X$ is an allowable simplex in $I^pS_i(X)$, then $f(\sigma) \in I^pS_i(Y)$ is literally the composition $f\sigma$. So we must consider $(f\sigma)^{-1}(S') = \sigma^{-1}f^{-1}(S')$ for singular strata $S'$ of $Y$. Now

$$\sigma^{-1}f^{-1}(S') \subset \cup_{\{S | f(S) \subset S'\}} \sigma^{-1}(S),$$

but by assumption, $f^{-1}(S')$ is the union of strata of $X$ of the same codimension in $X$ as the codimension of $S'$ in $Y$. So, for each such $S$,

$$\sigma^{-1}(S) \subset \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i\}$$

$$= \{i - \text{codim}(S') + \bar{p}(S) \text{ skeleton of } \Delta^i\}$$

$$\subset \{i - \text{codim}(S') + \bar{q}(S') \text{ skeleton of } \Delta^i\},$$

where we have used the definition of being $(\bar{p}, \bar{q})$-stratified. But then the union of such spaces is also in the $i - \text{codim}(S') + \bar{q}(S')$ skeleton of $\Delta^i$, and so $\sigma$ is allowable.

The PL version is perhaps even simpler. If $\sigma \in I^p\mathcal{C}_i^{GM}(X)$ (meaning $\sigma$ is a simplex in some triangulation), then the image of $\sigma$ under $f$ is a PL subset of $Y$ (not necessarily a simplex because $f$ might be simplicial only with respect to some subdivision of the triangulation with respect to which $\sigma$ is given). But whatever the dimension of $\sigma \cap S$ is for a stratum $S \subset X$, $\dim(f(\sigma \cap S)) \leq \dim(\sigma \cap S)$ simply by the properties of PL maps. So

$$\dim(f(\sigma) \cap S') \leq \dim(\cup_{\{S | f(S) \subset S'\}} f(\sigma \cap S))$$

$$\leq \dim(\cup\{S \mid f(S) \subset S'\} \sigma \cap S)$$

$$\leq i - \text{codim}(S) + \max_{\cup\{S | f(S) \subset S'\}} \bar{p}(S)$$

$$\leq i - \text{codim}(S') + \bar{q}(S'),$$

again utilizing the definition. Thus each $i$-simplex of $f(\sigma)$ must be allowable, which suffices for the proof.

**Remark 4.5.** Although we have not strictly set up a categorical structure, we remark that basic functorial properties do apply. In particular, if $f : X \to Y$ is $(\bar{p}, \bar{q})$-stratified and $g : Y \to Z$ is $(\bar{q}, \bar{r})$-stratified, then we easily verify that $gf : X \to Z$ is $(\bar{p}, \bar{r})$-stratified and that we can therefore also compose the resulting chain maps and maps of intersection homology groups to obtain composition maps that agree that those induced by $gf$. Similarly, the identity map $X \to X$ is $(\bar{p}, \bar{p})$-stratified for any perversity on $X$ and induces the identity map on intersection chains and intersection homology.

**Remark 4.6.** In later chapters, we will occasionally have to deal with maps between stratified spaces $f : X \to Y$ that satisfy the property of taking each stratum of $X$ into a stratum of $Y$ but such that codimension is not necessarily preserved. In this case, there can still be induced maps of the form $f : I^pH^*_i^{GM}(X) \to I^qH^*_i^{GM}(X)$, and the arguments of Proposition 4.4 still show us how to determine whether such an induced map exists. In particular, if $\sigma$ is a $\bar{p}$-allowable $i$-simplex, we still need to check that if $S'$ is a singular stratum of $Y$ then $\sigma^{-1}f^{-1}(S')$ is contained in the $i - \text{codim}_Y(S') + \bar{q}(S')$ skeleton of $\Delta^i$, which requires showing...
that if \( S \subset f^{-1}(S') \), then \( \sigma^{-1}(S) \) is contained in the \( i - \text{codim}_Y(S) + \bar{q}(S') \) skeleton of \( \Delta^i \). The \( \bar{p} \)-allowability condition on \( \sigma \) tells us that \( \sigma^{-1}(S) \) is contained in the \( i - \text{codim}_X(S) + \bar{p}(S) \) skeleton of \( \Delta^i \), so if we know that \( i - \text{codim}_X(S) + \bar{p}(S) \leq i - \text{codim}_Y(S') + \bar{q}(S') \), i.e. that \( \bar{p}(S) - \bar{q}(S') \leq \text{codim}_X(S) - \text{codim}_Y(S') \) for every pair of strata \( S, S' \) with \( f(S) \subset S' \), this will suffice to provide a map on intersection homology. Of course a similar argument applies for PL chains.

Given the proposition, there are evident corollaries, such as the following:

**Corollary 4.7.** If \( f : X \to Y \) is a stratified homeomorphism\(^{43}\) and the perversities \( \bar{p} \) on \( X \) and \( \bar{q} \) on \( Y \) correspond (i.e. \( \bar{p}(S) = \bar{q}(S') \) if \( f(S) = S' \) — see Remark \( 2.92 \)), then \( \bar{P}^PH_{\ast}^{\text{GM}}(X) \cong \bar{I}^qH_{\ast}^{\text{GM}}(Y) \). The corresponding fact holds for PL spaces, PL stratified homeomorphisms, and PL intersection homology.

**Proof.** In this case, the maps \( f \) and \( f^{-1} \) are respectively stratified with respect to the appropriate perversities, and since \( ff^{-1} \) and \( f^{-1}f \) are identity maps, functoriality implies that \( f \) and \( f^{-1} \) induce isomorphisms of the intersection chain complexes. \( \square \)

Adapting definition \( 2.91 \) if \( X, Y \) be are filtered spaces and \( X \times I \) is given the product filtration with the trivially filtered unit interval \( I \), then a stratified map (with respect to \( \bar{p}, \bar{q} \) \( H : I \times X \to Y \) is called a stratified homotopy (with respect to \( \bar{p}, \bar{q} \)) and if \( f : H|_{X \times 0} \) and \( g : H|_{X \times 1} \) then \( f \) and \( g \) are stratified homotopic (with respect to \( \bar{p}, \bar{q} \)) stratified maps.

The proof that \( (\bar{p}, \bar{q}) \)-stratified homotopies induce the same maps of intersection homology is precisely the same as the proof for ordinary homology provided we can demonstrate allowability.

**Proposition 4.8.** Suppose \( f, g : X \to Y \) are \( (\bar{p}, \bar{q}) \)-stratified homotopic \( (\bar{p}, \bar{q}) \)-stratified maps. Then \( f \) and \( g \) induce chain homotopic maps \( \bar{P}^PH_{\ast}^{\text{GM}}(X) \to \bar{I}^qH_{\ast}^{\text{GM}}(Y) \) and so \( f = g : \bar{P}^PH_{\ast}^{\text{GM}}(X) \to \bar{I}^qH_{\ast}^{\text{GM}}(Y) \). If \( X, Y, f, g \) are PL and \( f, g \) are PL \( (\bar{p}, \bar{q}) \)-stratified homotopic, then they induce chain homotopic maps \( \bar{P}^PH_{\ast}^{\text{GM}}(X) \to \bar{I}^qH_{\ast}^{\text{GM}}(Y) \) and so \( f = g : \bar{P}^PH_{\ast}^{\text{GM}}(X) \to \bar{I}^qH_{\ast}^{\text{GM}}(Y) \).

**Proof.** Again we start with the singular case. The standard proof for ordinary homology (e.g. \( 53 \)) creates a chain homotopy from \( f \) to \( g \) using a prism construction \( P : C_i(X) \to C_{i+1}(Y) \) such that, for a singular \( i \)-simplex \( \sigma \), \( P(\sigma) = \sum \tau_j H(\text{id} \times \sigma)\tau_j \), where the \( \tau_j \) are the (appropriately oriented) \( i + 1 \) simplices in a prism triangulation of \( I \times \Delta^i \); see \( 53 \) Proof of Theorem \( 2.10 \) or the proof of Proposition \( 3.29 \) above. Then \( \partial P = g - f - P\partial \), so \( P \) is a chain homotopy operator between the chain maps induced by \( f \) and \( g \). The same proof goes through for intersection chains once we have shown that if \( \sigma \) is an allowable simplex, then so are all the simplices \( H(\sigma \times \text{id})\tau_j \) of \( P(\sigma) \). This is sufficient, because if \( \xi \) is an allowable chain, then we will have \( P(\xi) \) and \( P(\partial \xi) \) consisting of allowable simplices, but also \( \partial P(\xi) = g(\xi) - f(\xi) - P(\partial \xi) \) will consist of allowable simplices using the hypotheses that \( f \) and \( g \) these be \( (\bar{p}, \bar{q}) \)-stratified maps and Proposition \( 4.4 \). So \( P \) takes intersection chains to intersection chains.

\(^{43}\)See Definition \( 2.87 \)
We will show that $(\text{id} \times \sigma)|_{\tau_j} : \tau_j \to I \times X$ is an allowable singular simplex of $I \times X$ with its product filtration and the corresponding perversity, and then it follows that the images under $H$ are also allowable as $H$ is a $(\bar{\eta}, \bar{q})$-stratified map; notice that the codimension of $I \times S$ in $I \times X$ is the same as the codimension of $S$ in $X$.

So we need to consider $(\text{id} \times \sigma)|^{-1}(I \times S)$. We first see that

$$(\text{id} \times \sigma)^{-1}(I \times S) \subset I \times \sigma^{-1}(S)$$

due to basic point-set topology (in fact, just set theory). So if $\sigma$ is allowable, then $\sigma^{-1}(S) \subset \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i\}$, so

$$(\text{id} \times \sigma)^{-1}(I \times S) \subset I \times \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i\}.$$  

As $\tau_j$ is an $i+1$ simplex, it suffices to show that the intersection of $\tau_j$ with this latter set is contained in the $i+1 - \text{codim}(S) + \bar{p}(S)$ skeleton of $\tau_j$. In general, we show that the intersection of $\tau_j$ with $I \times \{k \text{ skeleton of } \Delta^i\}$ must be contained in the $k+1$ skeleton of $\tau_j$.

This will complete the proof for singular chains.

Now recall as in the proof of Proposition 3.29 that if $\Delta^i = [v_0, \ldots, v_k]$, then the $\tau_j$ have the form $[u_0, \ldots, u_j, w_j, \ldots, w_k]$, where $u_k, w_k$ are respectively the copies of $v_k$ in $\{0\} \times \Delta^i$ and $\{1\} \times \Delta^i$. In particular, if $\eta = [v_{\ell_0}, \ldots, v_{\ell_k}]$ is some $k$-simplex of $\Delta^i$, then every simplex of $I \times \eta$ is a face of one of the $k+1$ simplices $\eta = [u_{\ell_0}, \ldots, u_{\ell_n}, w_{\ell_m}, \ldots, v_{\ell_k}]$. So every simplex in the prism triangulation of the product of $I$ with the $k$ skeleton of $\Delta^i$ can have at most $k+1$ vertices. This must also be true then of the intersection of $\tau_j$ with such a product, and so this intersection must lie in the $k+1$ skeleton of $\tau_j$, as desired.

The proof in the PL case follows essentially the same idea. In this case we may assume that we begin with a full triangulation of $X$, which we know by Theorem 3.26 is sufficient to compute the PL intersection homology of $X$. We may also assume $I \times X$ itself is triangulated as a prism based upon the initial triangulation of $X$. The map $H : I \times X \to Y$ may not be simplicial with respect to the this triangulation, but nonetheless it suffices as in the argument above to show that each $i+1$ simplex of $I \times \sigma$ satisfies the allowable condition on $I \times X$ for each allowable $i$-simplex $\sigma \subset X$. Then allowability of the image of $H$ as a PL chain in $X$ follows as in the argument for the proof of Proposition 4.4. Now analogously to the singular case, $I \times \sigma$ is triangulated as a sum of $\tau_j$ of the form $[u_0, \ldots, u_j, w_j, \ldots, w_k]$ (this time right within $I \times X$ and not just in some model space!), and trivially $\tau_j \cap (I \times S) \subset I \times (\sigma \cap S)$. If $\sigma$ is allowable, $\dim(\sigma \cap S) \leq i - \text{codim}(S) + \bar{p}(S)$, and $\dim(I \times (\sigma \cap S)) \leq i + 1 - \text{codim}(S) + \bar{p}(S)$. But then certainly

$$\dim(\tau_j \cap (I \times S)) \leq \dim(I \times (\sigma \cap S)) \leq i + 1 - \text{codim}(S) + \bar{p}(S).$$

So each $\tau_j$ is allowable. Now, as in the singular case, if $\xi$ is an allowable chain in $X$, then applying $H$ to the prism PL chain $I \times \xi$ in $I \times X$ provides via Proposition 4.4 an allowable chain in $Y$ that is a homology between $f(\xi)$ and $g(\xi)$. \qed

The next corollary follows immediately just as it does for ordinary homology.
Corollary 4.9. Suppose \( f : X \rightarrow Y \) is a stratified map with a stratified homotopy inverse (see Definition 2.91). In this case, there is a bijection between strata of \( X \) and strata of \( Y \) by Remark 2.92. Suppose that the values of \( \bar{p} \) on \( X \) and \( \bar{q} \) on \( Y \) agree on corresponding strata. Then \( f \) induces an isomorphism \( I^\bar{p}H^\GM_*(X) \cong I^\bar{q}H^\GM_*(Y) \). The analogous result holds in the PL category.

Remark 4.10. In such situations, especially when \( X \) is a subset of \( Y \), we will tend to abuse notation and use the same perversity symbol \( \bar{p} \) for the perversities on both spaces. Then the result of the previous corollary would be written \( I^\bar{p}H^\GM_*(X) \cong I^\bar{p}H^\GM_*(Y) \).

Example 4.11. As \( X \) is stratified homotopy equivalent to \( \mathbb{R}^n \times X \) when \( \mathbb{R}^n \) is given the trivial filtration and \( \mathbb{R}^n \times X \) is given the product filtration, we have \( I^\bar{p}H^\GM_*(\mathbb{R}^n \times X) \cong I^\bar{p}H^\GM_*(X) \), with the isomorphism induced either by inclusion \( X \hookrightarrow \mathbb{R}^n \times X \), \( x \rightarrow (z, x) \) for some fixed \( z \in \mathbb{R}^n \), or by collapse \( \mathbb{R}^n \times X \rightarrow X \), \( (z, x) \rightarrow x \).

4.2 Cone formula

In Example 3.38, we computed the intersection homology of the open cone on a manifold. In this section, we will extend this example to the cone on a filtered space. This example turns out to be phenomenally important: we know that every point of a CS space has a neighborhood of the form \( \mathbb{R}^k \times cL \), so once we know how to compute the intersection homology of a cone, the stratified homotopy invariance of Corollary 4.9 tells us how to compute the intersection homology of all these distinguished neighborhoods. A general principle of topology is that to understand a space it is often only necessary to understand the pieces it is made of and how these pieces fit together; for example one sees this principle at work in the Mayer-Vietoris sequence. Another, more powerful, example of this principle is at work in sheaf theory, which is precisely a machine for piecing local information together into global information. The intersection homology in neighborhoods of points constitutes the local information, and so this computation provides the foundation for the sheaf theoretic approach to intersection homology. While we will not travel down the sheaf-theoretic road here, we will nonetheless see that the computation of intersection homology of cones provides a critical stepping stone for almost all of our major theorems.

So, let \( X = X^{n-1} \) be a compact \( n - 1 \) dimensional filtered space, and consider the open cone \( cX \) filtered such that \( (cX)^0 = \{v\} \), where \( v \) is the vertex of the cone, and \( (cX)^i = X^{i-1} \times [0,1)/ \sim \) for \( i > 0 \). If \( \bar{p} \) is a perversity on \( cX \), define a perversity \( \bar{p}_X \) on \( X \) such that if \( S \) is a stratum of \( X \) then \( \bar{p}_X(S) = \bar{p}((0,1) \times S) \). Then we have a stratified inclusion map \( X \hookrightarrow cX \) that takes \( x \in X \) to \( (x,t) \) for some fixed \( t \), \( 0 < t < 1 \). This map respects the codimensions and perversities, so it induces \( I^\bar{p}_X S^\GM_* (X) \rightarrow I^\bar{p}S^\GM_*(cX) \).

We will demonstrate the following theorem, which turns out to be completely analogous to the computation when \( X \) is a manifold:
Theorem 4.12. If \( X = X^{n-1} \) is a compact filtered space of formal dimension \( n - 1 \), then

\[
I^pH^GM_i(cX) \cong \begin{cases} 
0, & i \geq n - \bar{p}(\{v\}) - 1, i \neq 0, \\
\mathbb{Z}, & i \geq n - \bar{p}(\{v\}), i = 0, \\
\mathbb{Z}, & i = n - \bar{p}(\{v\}) - 1, i = 0, I^pH^GM_0(X) \neq 0, \\
0, & i = n - \bar{p}(\{v\}) - 1, i = 0, I^pH^GM_0(X) = 0, \\
I^pH^GM_i(X), & i < n - \bar{p}(\{v\}) - 1.
\end{cases}
\]

Furthermore, the isomorphisms of the last case are induced by inclusion. An equivalent conclusion holds for PL intersection homology when \( X \) is a compact PL filtered space.

Remark 4.13. The special case where \( i = n - \bar{p}(\{v\}) - 1, i = 0, \) and \( I^pH^GM_0(X) = 0 \) is not usually noted in the literature. Presumably this is because one is usually most interested in spaces that possess regular strata, and so this case does not arise, as regular strata must contain allowable 0-simplices, implying that \( I^pH^GM_0(X) \neq 0 \). So, for example, this case is unnecessary when working only with stratified pseudomanifolds, as observed in Remark 2.47. However, as noted in Remarks 2.15 and 2.26, spaces with no regular strata seem unavoidable in general; see Section 4.3, below, for more details.

It is easy to overlook this special case, and, indeed, the author is not aware of any prior reference to it, including in his own work or in [61], where singular intersection homology was first introduced.

Remark 4.14. Notice that in the special case where \( 0 \geq n - \bar{p}(\{v\}) \) (or where \( 0 \geq n - \bar{p}(\{v\}) - 1 \) and the other special condition of the second case in the formula holds), then \( I^pH^GM_i(cX) \cong \mathbb{Z} \) is the only non-trivial intersection homology group. Here the allowability at \( \{v\} \) is sufficiently high that the intersection homology behaves completely analogously to ordinary homology of the cone.

Proof of Theorem 4.12. The proof mirrors the argument of Example 3.38 nearly completely.

We begin by checking the allowability condition at \( \{v\} \), for which the allowability condition for an \( i \)-simplex becomes

\[
\sigma^{-1}(\{v\}) \subset \{i - n + \bar{p}(\{v\}) \text{ skeleton of } \Delta^i\}.
\]

If \( i < n - \bar{p}(\{v\}) \) then the image of \( \sigma \) cannot contain \( v \) at all, and so in this range we have \( I^pS^GM_i(cX) = I^pS^GM_i(cX - \{v\}) \). Therefore, for \( i < n - \bar{p}(\{v\}) - 1 \), we obtain \( I^pH^GM_i(cX) = I^pH^GM_i(cX - \{v\}) \cong I^pH^GM((0, 1) \times X) \). Note: the extra \(-1\) is because homology in dimension \( i \) depends on chains in dimension \( i \) and \( i + 1 \). But now by Corollary 4.9 and Example 4.11 the inclusion \( X \to I \times X \) induces an isomorphism \( I^pH^GM_i(X) \to I^pH^GM_i((0, 1) \times X) \).

\[\text{See Remark 4.13 immediately below for a discussion of the second case of the formula.}\]

\[\text{Here we slightly abuse notation and use } \bar{p} \text{ also to stand for the perversity on } cX - \{v\} \text{ that evaluates on strata exactly as it would thinking of them as strata of } cX; \text{ see Section 4.3, below, for a more general discussion of subsets of filtered spaces.}\]

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For \( i \geq n - \bar{p}(\{v\}) - 1 \), we again consider \( \bar{c}\sigma \), the (singular) cone on \( \sigma \) (see Example 3.38). We claim that if \( \sigma \) is a \( \bar{p} \)-allowable \( i \)-simplex then so is \( \bar{c}\sigma \), provided \( i - n + \bar{p}(\{v\}) \geq -1 \). The allowability at \( \{v\} \) is verified exactly as it is in Example 3.38 \((\bar{c}\sigma)^{-1}(\{v\})\) includes the cone vertex \((0, x) \in \Delta^{i+1}\) and also the points \((s, x)\) such that \( \sigma(x) = v \). If \( x \) is contained in the \( j \)-skeleton of \( \Delta^{i} \). Then each point \((s, x)\) is contained in at most the \( j + 1 \) skeleton of \( \Delta^{i+1} \). So if \( \sigma^{-1}(\{v\}) \) is contained in the \( j \)-skeleton of \( \Delta^{i} \) for \( j \geq -1 \), then \((\bar{c}\sigma)^{-1}(\{v\})\) is contained in the \( j + 1 \)-skeleton of \( \Delta^{i+1} \) for \( j \geq -1 \). If \( \sigma \) is allowable, then

\[
\sigma^{-1}(\{v\}) \subset \{i - n + \bar{p}(\{v\})\} \text{ skeleton of } \Delta^{i},
\]

and we have just see that if \( i - n + \bar{p}(\{v\}) \geq -1 \) then

\[
\sigma^{-1}(\{v\}) \subset \{i + 1 - n + \bar{p}(\{v\})\} \text{ skeleton of } \Delta^{i}.
\]

As \( \bar{c}\sigma \) is an \( i + 1 \) simplex, \( \bar{c}\sigma \) is allowable at \( \{v\} \).

Next let \( S \) be a stratum of \( X \), and let us consider \((\bar{c}\sigma)^{-1}((0, 1) \times S)\). Owing to the cone construction, a point \((s, x), s \neq 0\) of \( \Delta^{i+1} \) maps to \((0, 1) \times S\) if and only if \( \sigma(x) \in S \). As we have already noted that \((\bar{c}\sigma)(0, x) = v\), it follows that

\[
(\bar{c}\sigma)^{-1}((0, 1) \times S) = (0, 1) \times \sigma^{-1}(S) \subset \Delta^{i+1}.
\]

But if \( \sigma \) is allowable, \( \sigma^{-1}(S) \) lies in the \( i - \text{codim}(S) + \bar{p}(S) \) skeleton of \( \Delta^{i} \), so \((0, 1) \times \sigma^{-1}(S) \) is contained in the \( 1 + i - \text{codim}(S) + \bar{p}(S) \) skeleton of \( \Delta^{i+1} \). But this shows that \( \bar{c}\sigma \) is allowable. It now follows just as in Example 3.38 that if \( i \geq n - \bar{p}(\{v\}) - 1 \) and \( i > 0 \), then \( I^pH^{GM}_i(cX) = 0 \).

Finally, when \( i = 0 \geq n - \bar{p}(\{v\}) - 1 \), we again have to be careful, as in Example 3.38 because the cone on a singular vertex generally has two boundary vertices, even though a vertex is a cycle. But again if \( \sigma_1, \sigma_2 \) are any two allowable 0-simplices, then the cone \( \bar{c}(\sigma_2 - \sigma_1) \) will have allowable boundary \( \sigma_2 - \sigma_1 \), and so any two allowable 0-simplices are allowably homologous. So if there exists an allowable 0-simplex in \( cX \), then we have \( I^pH^{GM}_0(cX) \cong \mathbb{Z} \). This will occur if either

1. \( \bar{p}(T) \geq \text{codim}(T) \) for any stratum \( T \subset cX \) (in particular if \( X \), and hence \( cX \), has a regular stratum), in which case there is an allowable 0-simplex in \( T \) and so \( I^pH^{GM}_0(X) \neq 0 \),

2. if \( \bar{p}(\{v\}) \geq n \) (i.e. if \( 0 \geq n - \bar{p}(\{v\}) \)), in which case the unique 0-simplex with image \( \{v\} \) is itself allowable.

Otherwise, if \( 0 = n - \bar{p}(\{v\}) - 1 \) and \( \bar{p}(T) < \text{codim}(T) \) for all other strata of \( cX \), then using that these other strata of \( cX \) have the form \((0, 1) \times S\) for some stratum \( S \subset X \) and that \( \bar{p}_X(S) = \bar{p}((0, 1) \times S) \) by definition, we see that neither \( cX \) nor \( X \) can have an allowable 0-simplex. Thus we must have \( I^pH^{GM}_0(cX) = I^pH^{GM}_0(X) = 0 \). This finishes the proof for singular intersection homology.

Now suppose \( X \) is a PL filtered space. The argument here is almost completely the same! If \( i < n - \bar{p}(\{v\}) \), then no allowable PL simplex can intersect the cone vertex \( v \),
and we have \( I^p\mathfrak{C}^\text{GM}(X) = I^p\mathfrak{C}_i(cX - \{v\}) \), so that \( I^p\mathfrak{H}_i^\text{GM}(cX) = I^p\mathfrak{H}_i^\text{GM}(cX - \{v\}) \cong I^p\mathfrak{H}_i^\text{GM}((0, 1) \times X) \cong I^p\mathfrak{H}_i^\text{GM}(X) \) with the isomorphism induced by the inclusion map by Corollary 4.3 and Example 4.11. For \( i \geq n - \bar{p}(\{v\}) - 1 \), \( i \neq 0 \), we can again take closed cones on allowable simplices, except now we utilize PL cones. To make sense of this, suppose \( \sigma \) is an \( i \)-simplex in some admissible triangulation of \( cX \). Recall that every PL chain is a linear combination of such simplices. We can associate \( \sigma \) with some embedding \( j : \Delta^i = [0, \ldots, i] \hookrightarrow cX \). Now extend \( j \) linearly to \( [z, 0, \ldots, i] = \Delta^{i+1} = \bar{c}\Delta^i \) by taking the new vertex \( z \) of \( \bar{c}\Delta^i \) to the vertex \( v \) of \( cX \). This yields a map \( \bar{c}j : \Delta^{i+1} \to cX \). Using Lemma 3.22 if \( [\Delta] \) is the class of \( \Delta^{i+1} \) as an element of \( \mathfrak{C}_{i+1}(\Delta^{i+1}) \), then \( (\bar{c}j)[\Delta] \) is a chain we denote \([\bar{c}\sigma]\) in \( \mathfrak{C}_{i+1}(cX) \) with boundary \( \partial(\bar{c}\sigma) = [\sigma] - [\bar{c}\partial\sigma] \). Again using Lemma 3.22 this formula holds even if \( \bar{c}\sigma \) is degenerate\footnote{This can happen! For instance suppose the 1-simplex \( \sigma \) in some triangulation of \( cX \) is such that \( \sigma \) lies along a ray from the cone vertex. Then \( \bar{c}\sigma \) must be a 2-chain contained in a 1-dimensional space, and so is degenerate, i.e. it is 0 in \( \mathfrak{C}_2(cX) \). However, it is still true that \( \partial(\bar{c}\sigma) = \sigma - \bar{c}(\partial\sigma) \). For example, let \( \partial\sigma = u - w \), with \( u \) closer to \( v \) in \( cX \) so that \( \sigma \) is oriented toward the cone vertex. Then \( \bar{c}\sigma = 0 \) is degenerate and so \( \partial(\bar{c}\sigma) = 0 \). But now, letting \([v, u]\) and \([v, w]\) denote the 1-simplices from the vertex \( v \) to the vertices \( u \) and \( w \) and letting \( \sigma = [w, u] \) denote, we have
\[
\sigma - \bar{c}(\partial\sigma) = [w, u] - \bar{c}(u - w) \\
= [w, u] - [v, u - w] \\
= [w, u] - [v, u] + [v, w] \\
= [v, w] + [w, u] - [v, u] \\
= [v, u] - [v, u] \\
= 0,
\]
using that chains in the PL complex are equal to their subdivisions, so \([v, w] + [w, u] = [v, u] \).
}
and so the \( i \)-chain \( \bar{c}\sigma \) is allowable as a chain, only that each \( \partial\sigma \) is allowable at \( \{v\} \). It is not necessarily true that \( \bar{c}\sigma \) is allowable as a chain, only that each \( i + 1 \) simplex is allowable, however if \( \xi \) is an allowable PL cycle, then \( \bar{c}\xi \) will be an allowable PL chain with boundary \( \xi \). Hence in this range \( I^p\mathfrak{H}_i^\text{GM}(cX) = 0 \). Finally, the \( i = 0 \geq n - \bar{p}(\{v\}) - 1 \) case is also analogous to the singular situation and shows that \( I^p\mathfrak{H}_0^\text{GM}(cX) \cong \mathbb{Z} \) if \( 0 \geq n - \bar{p}(\{v\}) - 1 \), except in the special case when \( 0 = n - \bar{p}(\{v\}) - 1 \) and \( \bar{p}_X(S) < \text{codim}(S) \) for all strata \( S \subset X \). Since the other strata of \( cX \) have the form \((0, 1) \times S\) for some stratum \( S \subset X \), in which case \( I^p\mathfrak{H}_0^\text{GM}(cX) = 0 \). 

The reader might be interested to work out by hand in a similar fashion exactly why \( \partial(\bar{c}\xi) = \xi \) in the case where \( \xi \) is a 1-cycle represented by a chain that contains a 1-simplex lying along a ray from the cone vertex.
4.3 Relative intersection homology

Just as for ordinary homology, we can form relative intersection homology groups. However, there is some need to be careful. We will first explain the difficulty and then discuss how we will deal with it.

Recall that the allowability condition on intersection chains depends on both the codimension of strata and the perversities of strata. Suppose \( X \) is a filtered space and that \( x \in X \) is a point, which we can view as a subspace \( \{x\} \subset X \). How do we treat \( \{x\} \) as a filtered space? What formal dimension should \( \{x\} \) have, and in what dimension should the stratum \( \{x\} \) live (recall that \( \{x\} \) as a stratum does not have to have the same dimension as \( \{x\} \) as a space; see Remark 2.15)? What value should a perversities take on \( \{x\} \) as a stratum of \( \{x\} \)? If \( X \) is a manifold stratified space and we want to treat \( \{x\} \) in the most natural way as a manifold stratified space with dimension 0, then \( \{x\} \) will also have codimension 0 as a stratum and any perversity will take value 0 on \( \{x\} \). Then \( I^pS^*_{\text{GM}}(\{x\}) = S_*(\{x\}) \) for any perversity \( \bar{q} \). But what if \( x \) is contained in a singular stratum of \( X \)? Then these singular chains might not be allowable in \( I^pS^*_{\text{GM}}(X) \). Thus we will not have \( S_*(\{x\}) \subset I^pS^*_{\text{GM}}(X) \), and so we cannot define relative intersection chains by a quotient. This is just a single example of a difficulty with defining intersection homology of subspaces. It is not hard to think of other problems.

The best way to handle these issues will be to begin with a naive definition of a subcomplex \( I^pS^*_{\text{GM}}(Y \subset X) \subset I^pS^*_{\text{GM}}(X) \). We will first provide these definitions, and then we will provide an equivalent alternative based on the idea of inherited filtrations and perversities. In Section 4.3.1 we will see that, for particularly nice examples of \( Y \), these constructions agree with what we might expect to get more naturally based on intrinsic properties of \( Y \).

**Definition 4.15.** Let \( X \) be a filtered space, and let \( Y \) be any subspace . If \( X \) is endowed with a perversity \( \bar{p} \), define \( I^pS^*_{\text{GM}}(Y \subset X) \) to be the subcomplex of \( I^pS^*_{\text{GM}}(X) \) consisting of chains in \( I^pS^*_{\text{GM}}(X) \) all of whose singular simplices have their images in \( Y \). Similarly, if \( X \) is a PL manifold stratified space and \( Y \) is any PL subspace, we can define \( I^p\mathcal{C}^*_{\text{GM}}(Y \subset X) \) as the subcomplex of \( I^p\mathcal{C}^*_{\text{GM}}(X) \) consisting of chains in \( I^p\mathcal{C}^*_{\text{GM}}(X) \) all of whose singular simplices are contained in \( Y \).

At first glance, this definition avoids having to reconsider perversities and codimension on \( Y \) by simply using the corresponding properties of \( X \). We obtain \( I^pS^*_{\text{GM}}(Y \subset X) \subset I^pS^*_{\text{GM}}(X) \) by definition, and we thus obtain a quotient group

\[
I^pS^*_{\text{GM}}(X,Y) = I^pS^*_{\text{GM}}(X)/I^pS^*_{\text{GM}}(Y \subset X),
\]

a short exact inclusion/quotient sequence

\[
0 \rightarrow I^pS^*_{\text{GM}}(Y \subset X) \rightarrow I^pS^*_{\text{GM}}(X) \rightarrow I^pS^*_{\text{GM}}(X,Y) \rightarrow 0,
\]

and, hence, a long exact intersection homology sequence

\[
\cdots \rightarrow I^pH^*_{i\text{GM}}(Y \subset X) \rightarrow I^pH^*_{i\text{GM}}(X) \rightarrow I^pH^*_{i\text{GM}}(X,Y) \rightarrow \cdots .
\]
Analogous statements of course hold for PL intersection chains.

The complex $I^\bar{p}S^*_X(Y \subset X)$ does have an alternative description as the intersection chain complex on $Y$, provided we choose the proper filtration and perversity.

**Lemma 4.16.** Endow $Y$ with the filtration $Y^i = Y \cap X^i$; in particular, suppose $Y$ has the same formal dimension as $X$. Define a perversity $\bar{p}_Y$ so that if $S$ is a stratum of $Y$ contained in the stratum $T \subset X$, then $\bar{p}_Y(S) = \bar{p}(T)$. Then

$$I^\bar{p}S^*_X(Y \subset X) = I^\bar{p}_Y S^*_X(Y).$$

**Proof.** First, notice that if $S$ is a stratum of $Y$ contained in $Y_i$, then, as $S$ is connected, $S$ must me contained in a connected component $T$ of $X_i$, and so $\bar{p}_Y$ is well-defined. Also, since $X$ and $Y$ have the same formal dimension, the formal codimension of $S$ in $Y$ is the same as the formal codimension of $T$ in $X$. Therefore, $\bar{p}_{Y}(S)$ + $\bar{p}_Y(S) = \bar{p}(T)$.

Now suppose $\sigma$ is a $\bar{p}$-allowable singular $i$-simplex of $X$ whose image is contained in $Y$. By assumption then

$$\sigma^{-1}(T) \subset \{i - \text{codim}_X(T) + \bar{p}(T)\} \text{ skeleton of } \Delta^i,$$

but since $S \subset T$, we also have $\sigma^{-1}(S) \subset \sigma^{-1}(T)$, and so

$$\sigma^{-1}(S) \subset \{i - \text{codim}_Y(S) + \bar{p}_Y(S)\} \text{ skeleton of } \Delta^i.$$

Therefore $\sigma$ is allowable in $Y$. Making the same argument with the simplices in boundaries of chains, we see then that

$$I^\bar{p}S^*_X(Y \subset X) \subset I^\bar{p}_Y S^*_X(Y).$$

Conversely, if $\sigma$ is a $\bar{p}_Y$-allowable singular simple of $Y$ with the filtration we have described, then $\sigma$ is certainly contained in $Y^i$ and we need only show that it is $\bar{p}$-allowable as a simplex in $X$. Let $T$ be a stratum of $X$. By assumption, if $S$ is any stratum of $Y$ contained in $T$, then

$$\sigma^{-1}(S) \subset \{i - \text{codim}_Y(S) + \bar{p}_Y(S)\} \text{ skeleton of } \Delta^i.$$

But $\sigma^{-1}(T) = \bigcup_{S \subset T} \sigma^{-1}(S)$, where the union is over all strata of $Y$ contained in $T$. Since each $\sigma^{-1}(S)$ is contained in the $\{i - \text{codim}_Y(S) + \bar{p}_Y(S)\}$ skeleton of $\Delta^i$, the same is true of the union. As we have already noted that $i - \text{codim}_Y(S) + \bar{p}_Y(S) = i - \text{codim}_X(T) + \bar{p}(T)$, we conclude that $\sigma$ is $\bar{p}$-allowable in $X$, and it follows that

$$I^\bar{p}S^*_X(Y \subset X) \supset I^\bar{p}_Y S^*_X(Y).$$

$\square$

**Remark 4.17.** This lemma justifies the claims in Remarks 2.15, 2.26, and 4.13 that filtered spaces with no regular strata are unavoidable when treating subsets of more reasonable spaces. Consider, for example, the space $X = X^2 \sqcup S^1$ of Remark 2.15. This is a 2-dimensional manifold stratified space filtered by $X^1 = S^1 \subset S^2 \sqcup S_1 = X$. If we wish to
consider for some perversity \( \bar{p} \) the subcomplex of \( I^\bar{p}S^GM(X) \) consisting of chains contained in \( Y \), the lemma shows that this is the same as computing \( I^\bar{p}s^1S^GM(S^1) \), where \( S^1 \) is treated as a 1-dimensional stratum of a formally 2-dimensional manifold stratified space. With this filtration, \( S^1 \) has no regular strata.

It is similarly not hard to come up with connected examples. For example, we could connect \( S^2 \) to \( S^1 \) by an interval with a vertex on each of \( S^1 \) and \( S^2 \). Suppose we filter such a space so that the 0-stratum consists of the two attaching vertices, the 1-stratum consists of union of \( S^1 \) and the interval, and the 2-stratum is the full space \( X = X^2 \). This is a manifold stratified space, and, in fact, a CS set. But again the subspace consisting of \( S^1 \) has no regular strata if we filter it as in Lemma 4.16.

**Definition 4.18.** Suppose \( X \) is a filtered space endowed with a perversity \( \bar{p} \), and suppose \( Y \subset X \) is an arbitrary subspace. The filtration \( Y^i = Y \cap X^i \) giving \( Y \) the same formal dimension as \( X \) and the perversity \( \bar{p}_Y \), defined so that \( \bar{p}_Y(S) = \bar{p}(T) \) if \( S \) is a stratum of \( Y \) contained in the stratum \( T \subset X \), will be referred to as the filtration and perversity on \( Y \) inherited from \( X \). Another common phrasing is that \( \{Y^i\} \) and \( \bar{p}_Y \) are the restriction of the filtration and perversity from \( X \) to \( Y \). We will use the two terminologies interchangeably.

Whenever we discuss a subspace \( Y \subset X \), we will automatically assume it is endowed with the inherited filtration and perversity. We will also generally denote \( \bar{p}_Y \) simply as \( \bar{p} \) unless there is some danger of confusion\(^{47}\); generally there is little risk, as there are no other contenders for the notation \( I^\bar{p}S^GM(Y) \) once we have established that \( Y \) is a subspace of a space \( X \) with perversity \( \bar{p} \).

With these conventions, our long exact sequence takes on the more comfortable form

\[
\cdots \rightarrow I^\bar{p}H_i^GM(Y) \rightarrow I^\bar{p}H_i^GM(X) \rightarrow I^\bar{p}H_i^GM(X,Y) \rightarrow \cdots \tag{6}
\]

**Remark 4.19.** The assumption that subsets inherit their dimensions does have the possibility to cause some semantic confusion. For example, one of our standard constructions is the cone construction. The cone formula of Theorem 4.12 assumes we have taken the cone on an \( n-1 \) dimensional space \( X \) in order to compute the intersection homology of the \( n \)-dimensional space \( cX \). These dimension conventions are first presented in the definition of the filtered cone in Example 2.11. Now we are saying that if we identify \( X \) with, say the image image of \( \{1/2\} \times X \) in \( cX \) then we should treat \( X \) as having dimension \( n \), since that is the dimension that the subspace \( X \) inherits from \( cX \). Well which is it? Does \( X \) have dimension \( n-1 \) or dimension \( n \)?

Unfortunately, we will have to leave this up to a matter of context. If we construct the cone on an \( n-1 \) dimensional space \( X \), we will have a cone of dimension \( n \). If we want to consider \( X \) as a subspace of \( cX \) to compute, say, \( I^\bar{p}H_i^GM(cX,X) \), then we will have to consider \( X \) with dimension \( n \), as discussed above. While this looks like it will create problems, fortunately it will rarely be an issue, primarily because we have seen that when it comes to computing intersection homology, it’s really codimensions that matter, and whether we consider \( X \) as having formal dimension \( n \) or \( n-1 \), there is no confusion\(^{47}\).

\(^{47}\) Though we will continue to be rather pedantic in this section while first working out the details.
about the codimensions of strata: if we start with $X$ as an $n-1$ dimensional space with a stratum $S$ of dimension $k$, then $S$ has codimension $n-k-1$ in $X$. If we form the cone, $cX$, this space has dimension $n$ and the stratum $(0, 1) \times S$ (or $cS - \{v\}$) has dimension $k+1$, so the codimension of $(0, 1) \times S$ in $cX$ is still $n-k-1$. Now if we identify $X$ as the subspace $\{1/2\} \times X \subset cX$ with the inherited filtration, then $X$ inherits the formal dimension $n$ and $S$, identified with $\{1/2\} \times S$, inherits the dimension $k+1$ by Definition 4.18 again the codimension of the stratum $S$ in $X$ is $n-k-1$. So at least our intersection homology computations will be consistent, even if our labeling of dimensions is not!

Therefore, let us not get too caught up in this point, though we will be careful when necessary.

**Stratified maps revisited.** Suppose $(X,A)$ and $(Y,B)$ are space pairs each consisting of a filtered space and a subspace and that $f: X \rightarrow Y$ is a $(\bar{p}, \bar{q})$-stratified map that takes $A$ into $B$. Using Proposition 4.4, $f$ takes $\bar{p}$ intersection chains on $X$ supported in $A$ to $\bar{q}$ intersection chains on $Y$ supported in $B$. Thus $f$ induces maps $I^pH^G_*(X,A) \rightarrow I^qH^G_*(Y,B)$. In other words, we have the following relative versions of Proposition 4.4.

**Proposition 4.20.** If $X,Y$ are filtered spaces, $f: X \rightarrow Y$ is $(\bar{p}, \bar{q})$-stratified, and $A \subset X$ and $B \subset Y$ with $f(A) \subset B$, then $f$ induces a chain map $f : I^pS^G_*(X,A) \rightarrow I^qS^G_*(Y,B)$. If, furthermore, $X,Y$ are PL filtered spaces, $A,B$ are PL subspaces, and $f$ is a PL map that is $(\bar{p}, \bar{q})$-stratified, then $f$ induces a chain map $f : I^p\mathcal{C}^G_*(X,A) \rightarrow I^q\mathcal{C}^G_*(Y,B)$ of PL intersection chain complexes. In either case, we obtain corresponding maps of intersection homology groups.

From here, it is not difficult to modify the arguments of the various results of Section 4.1 so that they hold for such maps of relative intersection homology groups. Thus we have the following:

**Corollary 4.21.** If $f: X \rightarrow Y$ is a stratified homeomorphism that is also a homeomorphism of pairs $f : (X,A) \rightarrow (Y,B)$ and the perversities $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ correspond, then $I^pH^G_*(X,A) \cong I^qH^G_*(Y,B)$. The corresponding fact holds for PL spaces, PL stratified homeomorphisms, and PL intersection homology.

This follows from the naturality of long exact sequences, Corollary 4.7, and the five lemma.

**Proposition 4.22.** Suppose $f, g : X \rightarrow Y$ are $(\bar{p}, \bar{q})$-stratified maps that are $(\bar{p}, \bar{q})$-stratified homotopic via a $(\bar{p}, \bar{q})$-stratified homotopy taking the pair $(I \times X, I \times A)$ to $(Y,B)$. Then $f$ and $g$ induce chain homotopic maps $I^pS^G_*(X,A) \rightarrow I^qS^G_*(Y,B)$ and so $f = g : I^pH^G_*(X,A) \rightarrow I^qH^G_*(Y,B)$. The analogous result holds in the PL category.

The proof here is the same as that of Proposition 4.22 by using prism operators as in the classical case and noting that the prism operator takes chains contained in $A$ to chains contained in $B$.  

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Corollary 4.23. Suppose $f : X \to Y$ is a stratified map with a stratified homotopy inverse $g$ such that $f, g$ are also maps of pairs $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ and the homotopies used to demonstrate that $f$ and $g$ are stratified homotopy inverses are also homotopies of pairs $I \times (X, A) \to (X, A)$ and $I \times (Y, B) \to (Y, B)$. Suppose that the values of $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ agree on corresponding strata. Then $f$ induces an isomorphism $I^p H_*^{GM}(X, A) \cong I^q H_*^{GM}(Y, B)$. The analogous result holds in the PL category.

4.3.1 Nice subsets

In this subsection, we will see that there are a number of common situations for which the intersection chain complex $I^p S_*^{GM}(Y \subset X)$ can be defined not only directly on $Y$, as is always the case by Lemma 4.16, but directly on $Y$ using a reasonable filtration on $Y$, not just the one inherited from $X$.

Example 4.24. We have already seen in Lemma 2.53 that an open subset $U$ of an $n$-dimensional stratified pseudomanifold $X$ is an $n$-dimensional stratified pseudomanifold with its inherited filtration.

More generally, suppose $X$ is any $n$-dimensional manifold stratified space such that the union of its $n$-dimensional strata is dense in $X$, and let $U \subset X$ be an open set. In this case, the intersection of $U$ with the $i$-dimensional strata of $X$ will be $i$-dimensional manifolds, so $U$ is also a manifold stratified space. The union of the $n$-dimensional strata of $U$ will be dense in $U$, and so the filtration of $U$ inherited from $X$ is $n$-dimensional, topologically and not just formally.

Such examples conform best to our intuition regarding dimension and inherited filtration, and indeed they will be our most important examples.

Example 4.25. Normally non-singular subspaces provide another useful class of subspaces for which intersection homology can often be computed using the intuitive dimensions of the subspace.

First, suppose $Z$ is a $k$-dimensional filtered space, and consider the product $\mathbb{R}^m \times Z$ with the product filtration, where $\mathbb{R}^m$ is trivially filtered with one regular stratum of dimension $m$ (comprising all of $\mathbb{R}^m$). The formal dimension of $\mathbb{R}^m \times Z$ is $m + k$. Recall that, as in Example 2.10, there is a bijection between strata $S \subset Z$ and the strata $\mathbb{R}^m \times S \subset \mathbb{R}^m \times Z$; if $S$ is a $j$-dimensional stratum of $Z$, then $\mathbb{R}^m \times Z$ is a $j + k$-dimensional stratum of $\mathbb{R}^m \times Z$. It follows that the codimension of $S$ in $Z$ is equal to the codimension of $\mathbb{R}^m \times S$ in $\mathbb{R}^m \times Z$.

Now, suppose instead that we began with the product filtration on $\mathbb{R}^m \times Z$ and wanted to consider $\hat{Z} = \{0\} \times Z$ as a subset. The prescription above states that we should consider $\hat{Z}$ to have formal dimension $m + k$, and similarly the the $i$-dimensional strata of $\hat{Z}$ will be the intersection of $\hat{Z}$ with the $i$-dimensional strata of $\mathbb{R}^m \times Z$. This is disconcerting: for example, if $Z$ were a manifold stratified space, then an $i$-dimensional manifold stratum in $Z$ thought of as a stratum, say $\hat{S}$, in $\hat{Z}$ would have to have dimension $m + i$, and so $\hat{Z}$ could not be a manifold stratified space!

However, we notice that the codimension of a stratum $S$ in $Z$ is nonetheless the same as the codimension of the corresponding stratum in $\hat{Z}$ $(m + k - (m + i) = k - i)$. Furthermore, suppose that $\bar{p}$ is a perversity on $Z$ that we extend to a perversity $\bar{p}^\times$ on $\mathbb{R}^m \times Z$ so that
\[\bar{p}^\times (\mathbb{R}^n \times S) = \bar{p}(S).\] Then, as defined above, the restriction of \(\bar{p}^\times\) to \(\bar{Z}\), which we will denote \(\bar{p}_\bar{Z}\), must have \(\bar{p}_\bar{Z}(\bar{S}) = \bar{p}^\times (\mathbb{R}^n \times S) = \bar{p}(S)\). Given this correspondence of perversity values and codimensions, we see that \(Z\) is stratified homeomorphic to \(\bar{Z}\) and

\[I^\bar{p} S_*^{GM}(Z) = I^{\bar{p}_\bar{Z}} S_*^{GM}(\bar{Z}).\]

In fact, the inclusion map \(Z \to \mathbb{R}^n \times Z\) a stratified homotopy equivalence, so altogether we have

\[I^\bar{p} H_*^{GM}(\mathbb{R}^n \times Z) \cong I^\bar{p} H_*^{GM}(Z) \cong I^{\bar{p}_\bar{Z}} H_*^{GM}(\bar{Z}).\]

In other words, if we start with a filtered space \(Z\) and then want to treat \(Z\) as a subspace of \(\mathbb{R}^n \times Z\), this is equivalent, for intersection homology purposes, to working with \(Z\) itself.

At first this example might seem somewhat artificial, but recall from Definition 2.95 that a normally nonsingular inclusion \(i : Z \hookrightarrow X\) is a stratified map such that, for some \(m\), \(i\) extends to a stratified homeomorphism \(\bar{i}\), from \(\mathbb{R}^m \times Z\) onto some neighborhood of \(i(Z)\). Let \(Y = i(Z)\), the normally nonsingular subspace. If the stratified homeomorphism \(\bar{i}\) also preserves formal dimension and \(\bar{p}\) is a perversity on \(X\), then our work just above shows that the intersection chain complex of the subspace \(I^\bar{p} S_*^{GM}(Y \subset X) = I^\bar{p}_Y S_*^{GM}(Y)\) is equal to the intersection chain complex \(I^{\bar{p}_\bar{Z}} S_*^{GM}(Z)\), where \(\bar{p}_\bar{Z}\) in the latter express is suitably interpreted to be compatible with \(\bar{p}\) on \(X\). In fact, as \(Y\) and \(Z\) are stratified homeomorphic, the only real differences between these two expressions are the formal dimensions of the spaces and the strata (though the codimensions of corresponding strata are the same).

We have already seen this example in play when computing the the intersection homology of a cone in Theorem 4.12. There, we saw that \(I^\bar{p} H_*^{GM}(cX) \cong I^\bar{p}_X H_*^{GM}(X)\) in the appropriate dimension range, thinking of \(X\) as the initially-given space with the given filtration. Now, however, we see that we can also consider \(X\) as a subspace of \(cX\) via a normally nonsingular inclusion

\[X \to \{t_0\} \times X \subset cX = [0, 1) \times X/\sim,\]

for some choice of \(t_0 \subset (0, 1)\). In this context, letting \(\bar{X} = \{t_0\} \times X\), we have

\[I^\bar{p}_X S_*^{GM}(X) = I^\bar{p}_X S_*^{GM}(\bar{X}) \subset I^\bar{p} S_*^{GM}(cX),\]

and the isomorphism \(I^\bar{p} H_*^{GM}(cX) \cong I^\bar{p}_X H_*^{GM}(X)\) (in the appropriate dimensions) is induced by this subspace/subcomplex inclusion.

Example 4.26. Perhaps the nicest (and most natural) examples of all occur when \(X\) is an \(n\)-dimension stratified pseudomanifold (endowed with perversity \(p\)) and \(U\) is a distinguished neighborhood of a point \(x \in X^{n-k}\). Then, as already observed, \(U\) is itself an \(n\)-dimensional stratified pseudomanifold by Lemma 2.53. Furthermore, \(U\) is stratified homeomorphic to the \(n\)-dimensional stratified pseudomanifold \(\mathbb{R}^{n-k} \times cL^{k-1}\), where \(L^{k-1}\) is a \(k - 1\) dimensional stratified pseudomanifold, which we can identify with a normally nonsingular subspace of \(\mathbb{R}^{n-k} \times cL^{k-1}\). By the preceding examples, the perversity \(\bar{p}\) intersection chain complex of the link \(L^{k-1}\), thought of as a subspace of \(X\) via the normally nonsingular inclusion and the stratified homeomorphism \(U \cong \mathbb{R}^{n-k} \times cL^{k-1}\), is isomorphic to the intersection chain complex \(I^\bar{p}_X S_*^{GM}(L^{k-1})\), thinking of \(L\) as a \(k - 1\) dimensional filtered space in its own right.

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and letting $\bar{p}_L$ be the perversity that takes a stratum of $L^{k-1}$ to the value of $\bar{p}$ evaluated on the stratum of $X$ into which $S$ embeds. In later sections, we will simply label this latter complex $I^\bar{p}S^GM(L^{k-1})$.

Notice that, in this example, all notions of dimension agree completely with what we would expect topologically; there is no need for formal dimensions, expect perhaps in the intermediate steps that we can now bury.

**Example 4.27.** Finally, another useful example of a “reasonably behaved” subset occurs when $Y = \partial X$ is the boundary of an $n$-dimensional $\partial$-stratified pseudomanifold. This isn’t quite a normally nonsingular subspace, as $Y$ only has a collar neighborhood stratified homeomorphic to $[0,1) \times Y$ in $X$. However, exactly the same sorts of arguments apply as in the previous example and demonstrate that the intersection chain complex $I^\bar{p}\times S^GM(\partial \subset X)$ obtained by thinking of $\partial X$ as a subspace is equal to the intersection chain complex $I^\bar{p}v S^GM(Y)$ obtained by thinking of $\partial X = Y$ as an $n-1$ dimensional stratified pseudomanifold in its own right. This example is particular pleasing as the density of the union of the regular strata is one of the key defining properties of a stratified (partial-)pseudomanifold, and so we would certainly rather think of $\partial X$ as being $n-1$ dimensional than as inheriting the formal dimension $n$ from $X$.

### 4.3.2 The relative cone formula

Next, let us provide a computation of an important relative intersection homology group.

Let $X$ be a compact $n-1$ dimensional filtered space, and let $cX$ be the open cone on $X$. In Theorem 4.12, we computed the intersection homology of $cX$. In this example, we will compute the intersection homology of $I^\bar{p}H^GM(cX, cX - \{v\})$, where $v$ is the cone point. As $cX - \{v\} \cong (0,1) \times X$, we have

$$I^\bar{p}H^GM(cX - \{v\}) \cong I^\bar{p}H^GM((0,1) \times X) \cong I^\bar{p}H^GM(X),$$

by the preservation of intersection homology under stratified homotopy equivalence$^{48}$ Thus the long exact sequence of the pair is isomorphic to

$$\cdots \to I^\bar{p}H_i^GM(X) \xrightarrow{\text{inc}} I^\bar{p}H_i^GM(cX) \to I^\bar{p}H_i^GM(cX, cX - \{v\}) \to I^\bar{p}H_{i-1}^GM(X) \to \cdots,$$

where inc stands for the inclusion map into a level set $x \to (t_0, x)$ for fixed $t_0 \in (0,1)$. By Theorem 4.12 and Example 4.25, this inclusion is an isomorphism for $i < n - \bar{p}(\{v\}) - 1$, and so $I^\bar{p}H_i^GM(cX, cX - \{v\}) = 0$ for $i < n - \bar{p}(\{v\}) - 1$.

For $i \geq n - \bar{p}(\{v\}) - 1$, $i \neq 0$, Theorem 4.12 tells us that $I^\bar{p}H_i^GM(cX) = 0$. And so for $i \geq n - \bar{p}(\{v\}) - 1$, $i > 0$, $I^\bar{p}H_i^GM(X) \cong I^\bar{p}H_{i+1}^GM(cX, cX - \{v\})$. Alternatively stated, for $i > n - \bar{p}(\{v\}) - 1$, $i > 1$, we have $I^\bar{p}H_i^GM(cX, cX - \{v\}) \cong I^\bar{p}H_{i+1}^GM(X)$.

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48As observed in Example 4.25, if we think of $X$ as a normally nonsingular subspace of $cX$ or of $cX - \{v\}$, we are free to think of $I^\bar{p}S^GM(X)$ either as the subspace intersection chain complex $I^\bar{p}S^GM(X \subset cX)$ or as the intersection chain complex of the $n-1$ dimensional space $X$ with the corresponding perversity. Hence there is no ambiguity in writing $I^\bar{p}H^GM(X)$. Furthermore, we have observed that in the relevant dimension ranges, the maps $I^\bar{p}H^GM(X) \to I^\bar{p}H^GM((0,1) \times X) \to I^\bar{p}H^GM(cX)$ induced by inclusion will all be homeomorphisms, in the first case by stratum-preserving homotopy equivalence, in the second case by the computations in the proof of Theorem 4.12.
Next we consider $I^p H^\text{GM}_{n-\bar{p}(\{v\})-1}(cX, cX - \{v\})$, $n - \bar{p}(\{v\}) - 1 > 0$. In this case, $I^p H^\text{GM}_{n-\bar{p}(\{v\})-1}(cX) = 0$ and $I^p H^\text{GM}_{n-\bar{p}(\{v\})-2}(X) \to I^p H^\text{GM}_{n-\bar{p}(\{v\})-2}(cX)$ is an isomorphism. Thus $I^p H^\text{GM}_{n-\bar{p}(\{v\})-1}(cX, cX - \{v\}) = 0$.

This leaves the following low-dimensional cases to check:

1. $I^p H^\text{GM}_1(cX, cX - \{v\})$, when $1 > n - \bar{p}(\{v\}) - 1$,
2. $I^p H^\text{GM}_0(cX, cX - \{v\})$, when $0 \geq n - \bar{p}(\{v\}) - 1$.

(Notice that, in both cases, $0 \geq n - \bar{p}(\{v\}) - 1$.)

In all of these cases, $I^p H^\text{GM}_1(cX) = 0$, and so the tail of the exact sequence is

$$0 \to I^p H^\text{GM}_1(cX, cX - \{v\}) \to I^p H^\text{GM}_0(cX) \to I^p H^\text{GM}_0(cX, cX - \{v\}) \to 0.$$ 

In the special case when $I^p H^\text{GM}_0(cX) = 0$, which, given the current assumption of the special cases, can only happen if $0 = n - \bar{p}(\{v\}) - 1$ and $I^p H^\text{GM}_0(X) = 0$, we must have $I^p H^\text{GM}_1(cX, cX - \{v\}) = I^p H^\text{GM}_0(cX, cX - \{v\}) = 0$.

Otherwise $I^p H^\text{GM}_0(cX) \cong \mathbb{Z}$. If the only allowable 0-simplex of $cX$ is contained in $\{v\}$, which will happen if $0 \geq n - \bar{p}(\{v\})$ and $I^p H^\text{GM}_0(X) = 0$, then we must have $I^p H^\text{GM}_1(cX, cX - \{v\}) = 0$ and $I^p H^\text{GM}_0(cX, cX - \{v\}) \cong \mathbb{Z}$.

Finally, if $I^p H^\text{GM}_0(cX) \cong \mathbb{Z}$ but there are allowable 0-simplices in $X$, then the map $I^p H^\text{GM}_0(X) \to I^p H^\text{GM}_0(cX)$ is a surjection, which must split since $I^p H^\text{GM}_0(cX) \cong \mathbb{Z}$ is free. This splitting is induced by choosing a particular generating 0-simplex for $I^p H^\text{GM}_0(cX)$ in $X$. So in this case, we must have $I^p H^\text{GM}_0(cX, cX - \{v\}) = 0$ and, if $I^p H^\text{GM}_0(X) \cong \mathbb{Z}^r$, then $I^p H^\text{GM}_1(cX, cX - \{v\}) \cong \mathbb{Z}^{r-1}$. We can label this as $I^p H^\text{GM}_1(cX, cX - \{v\}) \cong I^p H^\text{GM}_0(X)$ in analogy with the computations of reduced ordinary homology. In fact, we will also adopt the conventions that $I^p \tilde{H}^\text{GM}_1(X) = 0$, that $I^p \tilde{H}^\text{GM}_i(X) = I^p H^\text{GM}_i(X)$ for $i > 0$, and that $I^p \tilde{H}^\text{GM}_i(X) = 0$ if $I^p H^\text{GM}_i(X) = 0$ for any $i$.

We have now computed the following results for the special low-dimensional cases:

1. If $1 > n - \bar{p}(\{v\}) - 1$, then $I^p H^\text{GM}_1(cX, cX - \{v\}) \cong I^p \tilde{H}^\text{GM}_0(X)$; notice that this also includes the cases computed above where $I^p H^\text{GM}_1(cX, cX - \{v\}) = 0$, since in these case $I^p H^\text{GM}_0(X)$ (and hence $I^p \tilde{H}^\text{GM}_0(X)$) must also be 0.
2. If $0 \geq n - \bar{p}(\{v\}) - 1$, then $I^p H^\text{GM}_0(cX, cX - \{v\}) = 0 = I^p \tilde{H}^\text{GM}_i(X)$ unless $0 \geq n - \bar{p}(\{v\})$ and $I^p H^\text{GM}_0(X) = 0$, in which case $I^p H^\text{GM}_0(cX, cX - \{v\}) \cong \mathbb{Z}$.

So altogether we have shown the following.

**Theorem 4.28.** If $X$ is a compact $n-1$ dimensional filtered space then

$$I^p H^\text{GM}_i(cX, cX - \{v\}) \cong \begin{cases} I^p \tilde{H}^\text{GM}_{i-1}(X), & i \geq n - \bar{p}(\{v\}), i \neq 0 \\ I^p \tilde{H}^\text{GM}_i(X) = 0, & i \geq n - \bar{p}(\{v\}), i = 0, I^p H^\text{GM}_0(X) \neq 0, \\ \mathbb{Z}, & i \geq n - \bar{p}(\{v\}), i = 0, I^p H^\text{GM}_0(X) = 0, \\ 0, & i < n - \bar{p}(\{v\}). \end{cases}$$

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An equivalent conclusion holds for PL intersection homology when $X$ is a compact PL manifold stratified space.

Remark 4.29. As for Theorem 4.12 (see Remark 4.13), the oddities in the low-dimensional cases $P^x H^G_0(X) = 0$ do not seem to have been previously noticed in the literature. However, such cases do not arise if all spaces possess regular strata, for example when working only with stratified pseudomanifolds.

4.4 Mayer-Vietoris sequences and excision

As we have already seen many times, properties of intersection homology can often be developed quite analogously to the corresponding properties for ordinary homology with just some extra care to ensure that allowability of chains is not compromised. In some sense this is also true when treating Mayer-Vietoris sequences and excision, however in this case the extra care needed is a bit more subtle and complex. Unfortunately, this is a point that is often glossed over, or overlooked completely, in much of the existing literature, leading quite often to what should probably be called the standard error of intersection homology.

The issue is the following: Suppose $\xi \in S_i(X)$ is an ordinary singular $i$-chain. We may write $\xi = \sum_{j=1}^m c_j \sigma_j$ for some collection of singular simplices $\{\sigma_j\}$ and some coefficients $c_j \in \mathbb{Z}$. It is quite usual in chain arguments to break $\xi$ into pieces, for example $\xi = (\sum_{j=1}^k c_j \sigma_j) + (\sum_{j=k+1}^m c_j \sigma_j)$. Suppose now that $\xi$ is a chain that is allowable with respect to some perversity $\bar{p}$. Certainly, by definition, each $\sigma_j$ is an allowable simplex and, furthermore, each $i - 1$ simplex of $\partial \xi$ is allowable. However, there is no reason to suppose that all the $i - 1$ simplices of either $\partial(\sum_{j=1}^k c_j \sigma_j)$ or $\partial(\sum_{j=k+1}^m c_j \sigma_j)$ are allowable. There might be $i - 1$ simplices of each of these that are not allowable but that cancel each other out in $\partial \xi$.

Here is a simple example: Consider the real line stratified as the 1-dimensional manifold $\Delta^1 \to [-1,0] \subset \mathbb{R}$, and let $\sigma_1$ be the orientation-preserving linear homeomorphism $\Delta^1 \to [0,1] \subset \mathbb{R}$. Then for each singular simplex, $\sigma_j^{-1}$ lies in the 0-skeleton of $\Delta^1$, and so is allowable, as $i - \text{codim}(\{0\}) + \bar{p}(\{0\}) = 1 - 1 + 0 = 0$. Furthermore, the chain $\sigma_1 + \sigma_2$ is allowable since each simplex is allowable and $\partial(\sigma_1 + \sigma_2) = \tau_1 - \tau_{-1}$, where $\tau_0$ is the singular 0-simplex mapping $\Delta^0$ to $0 \in \mathbb{R}$. The 0-simplices $\tau_1$ and $\tau_{-1}$ have image in the regular stratum and so are allowable. However, neither $\sigma_1$ nor $\sigma_2$ are allowable as chains because each of their boundaries contains $\tau_0$, which is not allowable as in this case $i - 1 - \text{codim}(\{0\}) + \bar{p}(\{0\}) = 0 - 1 + 0 = -1$.

This is clearly a difficulty when discussing excision. Suppose $K \subset U \subset X$ and the closure of $K$ is contained in the interior of $U$. The idea behind the excision isomorphism $H_*(X,U) \cong H_*(X-K, U-K)$ in ordinary homology is that one can first perform subdivisions to make simplices of a chain as small as necessary and then “throw away” the simplices of the chain that intersect $K$; proofs of excision (e.g. in [53]) make this intuition precise. However, we must be careful when throwing away simplices not to leave exposed boundaries that are not allowable.

Similarly, in proving the existence of the Mayer-Vietoris exact sequence for a pair $U,V$ with $U \cup V = X$, it is necessary to demonstrate that the inclusion $S_*(U) + S_*(V) \to$
$S_*(X)$ induces an isomorphism on homology. Again the basic idea of the proof first involves subdividing chains of $S_*(X)$ to make them small enough so that every simplex fits inside one of $U$ or $V$ (which does not affect homology, which is preserved under subdivisions) and then showing that in fact $S_*(U) + S_*(V)$ is isomorphic to the complex of such chains of small simplices. But this requires showing that every chain made of small simplices can be written as the sum of a chain in $U$ and a chain in $V$. For ordinary chains, there is no problem — just split the chain up into two chains, say one containing all the simplices that are contained completely in $U$ and one containing all the rest (so all the simplices contained in both $U$ and $V$ get grouped into the chain in $S_*(U)$). But again intersection chains require much more care to make sure we are not creating unallowable boundary faces.

In this section we work through the intersection homology details. Ultimately, analogues of the ordinary homology arguments can be made to work out, but only with a good deal of care. We will first work through the PL intersection homology to get a feel for the arguments. Then we will turn to singular intersection homology, which will require a deeper investigation of singular subdivision.

### 4.4.1 PL excision and Mayer-Vietoris

We begin with the PL theory. Throughout this section, $X$ is a PL filtered space with perversity $\bar{p}$. By an allowable simplex of $C_*(X)$ we mean an allowable simplex with respect to some triangulation of $X$ and the perversity $\bar{p}$. Of course as an element of $C_*(X)$, $\sigma$ is identified with any chain obtained from $\sigma$ via subdivision, but we will not mean by this language that $\sigma$ is allowable as a chain, and so we cannot write $\sigma \in I_{\bar{p}} C_{GM}^*(X)$. If we want $\sigma$ to be allowable as a chain, we will say so explicitly or we will write $\sigma \in I_{\bar{p}} C_{GM}^*(X)$.

The key to avoiding the aforementioned perils of breaking up chains is provided by the following lemma.

**Lemma 4.30.** Let $\sigma$ be an allowable simplex of $X$. Suppose $\tau$ is an $i-1$ simplex of some subdivision of $\sigma$ such that, for each face $\eta$ of $\sigma$, $\dim(\tau \cap \eta) < \dim(\eta)$. Then $\tau$ is allowable.

**Proof.** Let $S$ be a stratum of $X$, and for a simplex $\gamma$, let $\check{\gamma}$ denote the interior of $\gamma$.

We want to show that for each stratum $S$ of $X$, $\dim(\tau \cap S) \leq i - 1 + \text{codim}(S) + \bar{p}(S)$. Suppose $\gamma$ is the face of $\tau$ of highest dimension such that $\check{\gamma} \cap S \neq \emptyset$. Then $\dim(\tau \cap S) = \dim(\gamma)$. Let $\eta$ be the face of $\sigma$ such that $\check{\gamma} \subset \check{\eta}$, so $\dim(\eta \cap S) = \dim(\eta)$. Then we must also have $\check{\eta} \cap S \neq \emptyset$. But since $\sigma$ is allowable, $\dim(\eta) \leq i + \text{codim}(S) + \bar{p}(S)$. By assumption, $\dim(\tau \cap \eta) < \dim(\eta)$, so $\dim(\gamma) < \dim(\eta)$, and this means that we must have $\dim(\gamma) \leq i - 1 + \text{codim}(S) + \bar{p}(S)$. This is what we needed to show.

So now the idea is to always break up chains in such a way that any new boundary face so created has the form described in the lemma. Luckily, there are plenty such boundary faces due to our next lemma.

**Lemma 4.31.** Let $\sigma$ be a simplicial $i$ simplex of $X$, and let $\sigma'$ be its barycentric subdivision. Let $\tau$ be an $i-1$ simplex of $\sigma'$ that does not contain a vertex of $\sigma$. Then $\tau$ satisfies the condition of the previous lemma, i.e. for each face $\eta$ of $\sigma$, $\dim(\tau \cap \eta) < \dim(\eta)$. 

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Proof. Each such simplex must have the form $\tau = [\hat{\sigma_1}, \ldots, \hat{\sigma_i}]$, where $\sigma_j$ is a $j$-dimensional face of $\sigma$ and $\hat{\sigma}_j$ is its barycenter (see, e.g. [77, Section 15]). Thus $\tau \cap \sigma_k = [\hat{\sigma}_1, \ldots, \hat{\sigma}_k]$, which has dimension $k - 1$, and if $\eta$ is a face of $\sigma$ such that $\eta \neq \sigma_i$ for any $i$, then $\tau \cap \eta = \emptyset$. 

We can now demonstrate PL excision. As for singular homology, the excision isomorphism will have the form $I^p\mathcal{S}^{GM}_i(X - K, U - K) \xrightarrow{\sim} I^p\mathcal{S}^{GM}_i(X, U)$ for subsets $K \subset U \subset X$. Since we do not make any special assumptions concerning the compatibility of $K$ and $U$ with the PL structure of $X$, in particular we don’t assume that $K$ is a closed PL subspace or that $U$ is open, $X - K$, $U - K$, and $U$ do not necessarily inherit PL structures from $X$. So we interpret these groups in the same spirit in which we interpret the restriction of intersection homology to subspaces, e.g. we define $I^p\mathcal{C}^{GM}_* (X - K, U - K) = I^p\mathcal{C}^{GM}_* (X - K)/I^p\mathcal{C}^{GM}_* (U - K)$, where $I^p\mathcal{C}^{GM}_* (X - K)$ consists of the elements of $I^p\mathcal{C}^{GM}_* (X)$ that are supported in $X - K$ and $I^p\mathcal{C}^{GM}_* (U - K)$ consists of the elements of $I^p\mathcal{C}^{GM}_* (X)$ that are supported in $U - K$. The complex $I^p\mathcal{C}^{GM}_* (X, U)$ is understood similarly.

Theorem 4.32. Let $X$ be a PL filtered space, and suppose $K \subset U \subset X$ are subsets such that $\bar{K} \subset \bar{U}$. Then inclusion induces an isomorphism $I^p\mathcal{S}^{GM}_i(X - K, U - K) \xrightarrow{\sim} I^p\mathcal{S}^{GM}_i(X, U)$, where all relevant chain complexes are treated as subcomplexes of $I^p\mathcal{C}^{GM}_* (X)$.

Proof. The inclusion $(X - K, U - K) \hookrightarrow (X, U)$ is a pair of inclusions of open subsets, so it is stratified for our given perversity $\bar{p}$.

Let us first show that the induced map on intersection homology is surjective. Let $[\xi] \in I^p\mathcal{S}^{GM}_i(X, U)$ be an allowable relative cycle. It will suffice to show that if $z$ is a chain representing $[\xi]$ in some triangulation of $X$ then we can write $z = x + y$, where $x, y$ are allowable chains, $x$ is supported in $X - K$, and $y$ is supported in $U$. In this case, $z$ and $x$ represent the same element of $I^p\mathcal{C}^{GM}_* (X, U)$ and $x$ is in the image of $I^p\mathcal{S}^{GM}_i (X - K, U - K)$.

Let us choose a full triangulation $T$ of $X$ so that $[\xi]$ can be represented by a chain $\xi$ in $T$; such a triangulation exists by Theorem 3.26. Suppose $\sigma$ is an $i$-simplex with non-zero coefficient in $\xi$. The simplex $\sigma$ is covered by the two open sets $A = \sigma \cap \bar{U}$ and $B = \sigma \cap (X - \bar{K})$. By a Lebesgue number argument (cf. the proof of [77, Theorem 16.1]), there exists a number $d(\sigma)$, such that if $\sigma'$ is the $d(\sigma)$th barycentric subdivision of $\sigma$, then every $i$-simplex of $\sigma'$ is contained in either $A$ or $B$. Let $D = \max_{\sigma \in \xi} \{d(\sigma)\}$, where the maximum is taken over all $i$-simplices with non-zero coefficient in $\xi$. Since $\xi$ has only a finite number of such simplices, $D$ is well defined.

Now, let us consider the $D$th barycentric subdivision $T'$ of $T$. Let $\xi'$ denote the image of $\xi$ under this subdivision; then $[\xi] = [\xi'] \in I^p\mathcal{S}^{GM}_i (X, U)$. Let $z$ consist of the simplices of $\xi'$ (with their coefficients) in $T'$ that intersect $\bar{K}$. By construction, $z$ must be contained in $\bar{U}$ and $\xi' - z$ is contained in $X - K$. So if we were looking to prove excision in ordinary homology, we’d be done. However, we have no reason to expect that $z$ and $\xi' - z$ will be allowable. This requires another level of work.

Let $|z|$ be the support of $z$, i.e. the union of the simplices of $z$ with non-zero coefficient. Since $|z| \subset \bar{U}$, we can emulate our previous argument to obtain a further barycentric subdivision $T''$ such that every simplex of $\xi''$ is contained in $\bar{U}$ or $X - |z|$. Potentially, we could
have \(T = T' = T''\), but we make sure by fiat that the passage from \(T'\) to \(T''\) involves at least one barycentric subdivision.

Let \(\xi''\) and \(z''\) denote the images of \(\xi''\) and \(z''\) under the subdivision operator. Let \(y\) consist of the simplices (with coefficients) of \(\xi''\) that are either contained in \(z''\) or share a vertex with a simplex in \(z''\). By the construction, every simplex of \(y\) intersects \(|z|\) and so is contained in \(\hat{U}\). Let \(x = \xi'' - y\). Then each simplex of \(x\) is subdivided from \(\xi' - z\) and so is contained in \(X - K\). It therefore only remains to show that \(y\) is an allowable chain as \(\xi''\) is an allowable chain by Lemma 3.23 and so this will also imply that \(x = \xi'' - y\) is allowable.

Now, by the proof of Lemma 3.23, since each simplex of \(\xi''\) is allowable, so is each simplex of \(y\). So we need only check the simplices of \(\partial y\). Some of the simplices of \(\partial y\) are simplices of \(\partial \xi''\), and so these are allowable. The only simplices of concern, then, are the simplices that are in \(\partial y\) but not in \(\partial \xi''\). In other words, these are simplices that must occur in canceling pairs in \(\partial y\) and \(\partial x\). What simplices might these be? Let \(\tau\) be such an \(i-1\) simplex occurring in \(\partial y\) and \(\partial x\). The simplex \(\tau\) cannot intersect \(|z|\), as no \(i\)-simplex of \(x\) can intersect \(|z|\), or it would share a vertex with \(z''\) and so be in \(y\) and not in \(x\). Additionally, if \(\tau\) is contained in a simplex \(\sigma'\) of \(T'\), then \(\tau\) cannot contain a vertex \(v\) of \(\sigma'\) by the following argument: We have already seen that \(\tau\) cannot contain such a vertex if the vertex is in \(|z|\). If \(v\) is not in \(|z|\), any \(i\)-simplex \(\sigma''\) of \(T''\) that contains \(\tau\) as a face would have to also contain \(v\) and so be contained in a barycentric star neighborhood of \(v\). But since \(v\) is not in \(|z|\), which underlies a subcomplex of \(T'\), no such \(\sigma''\) can intersect \(|z|\). But then \(\tau\) cannot be a face of any \(i\)-simplex of \(T''\) that intersects \(|z|\), which is a contradiction. Thus the only possibility is that \(\tau\) does not contain any vertices of \(T'\). So by Lemmas 4.31 and 4.30 \(\tau\) is allowable.

This completes the proof that the inclusion map induces a surjective map on intersection homology.

But the proof of injectivity is completely analogous! Suppose \(\xi\) represents an element of \(I^pS^G_X(X - K, U - K)\) and that \(\xi\) is a relative boundary in \(X\), i.e. there is an allowable chain \(\zeta\) such that \(\partial \xi = \xi + \rho\), with \(\rho\) an allowable chain supported in \(U\). Suppose all these chains, are subcomplexes of a triangulation \(T\). By an argument analogous to that above, we can find a subdivision \(\xi''\) of \(\zeta\) such that \(\xi'' = \mu + \nu\), \(|\nu| \subset \hat{U}\), \(\mu \subset X - K\), and \(\mu\) and \(\nu\) are allowable. Then \(\partial \mu = \partial \xi'' - \partial \nu = \xi'' + \rho'' - \partial \nu\). Both \(\rho''\) and \(\partial \nu\) are contained in \(U\), and, in fact, since \(\mu\) and \(\xi''\) are contained in \(X - K\), then so is \(\rho - \partial \nu\). So \(\xi''\) must be a relative boundary in \((X - K, U - K)\), and so represent 0 in \(I^pS^G_X(X - K, U - K)\).

Similar arguments allow us to formulate a Mayer-Vietoris sequence. Once again, for an arbitrary subset \(U \subset X\), we interpret \(I^pS^G_X(U)\) to be the subcomplex of \(I^pS^G_X(X)\) consisting of chains supported in \(X\).

**Theorem 4.33.** Suppose \(U, V \subset X\) such that \(\hat{U} \cup \hat{V} = X\). Then there is an exact Mayer-Vietoris sequence

\[
\rightarrow I^pS^G_X(U \cap V) \rightarrow I^pS^G_X(U) \oplus I^pS^G_X(V) \rightarrow I^pS^G_X(U \cup V) \rightarrow I^pS^G_X(U \cap V) \rightarrow .
\]

Here \(U, V\), and \(U \cap V\) inherit their filtrations (including formal dimension) and perversities from \(X\).
Proof. The standard arguments demonstrate that there is a short exact sequence

$$0 \to I_i^p \mathcal{C}^{GM}(U \cap V) \to I_i^p \mathcal{C}^{GM}(U) \oplus I_i^p \mathcal{C}^{GM}(V) \to I_i^p \mathcal{C}^{GM}(U) + I_i^p \mathcal{C}^{GM}(V) \to 0,$$

and this yields a long exact sequence. What needs to be shown is that the inclusion map $\psi : I_i^p \mathcal{C}^{GM}(U) + I_i^p \mathcal{C}^{GM}(V) \to I_i^p \mathcal{C}^{GM}(X)$ yields an isomorphism on homology.

The proof is basically the same as the argument we used to prove excision. Notice that if $x \in X$ is contained in the closure of $X - V$, it cannot be contained in the interior of $V$, so it must be contained in the interior of $U$. Therefore $\overline{X - V} \subset \check{U}$. Thus, the argument of Theorem 4.32 shows how we can take an allowable cycle $\xi$ in $X$, subdivide it to an appropriate $\xi''$, and then break it into two allowable pieces $\xi'' = x + y$, where $y$ is contained in $\check{U}$ and $x$ is contained in $X - \overline{X - V} \subset V$. So $y \in I_i^p \mathcal{C}^{GM}(U)$ and $x \in I_i^p \mathcal{C}^{GM}(V)$, and this shows that $\psi$ is surjective on homology.

Similarly, if $x + y$ is a cycle in $I_i^p \mathcal{C}^{GM}(U) + I_i^p \mathcal{C}^{GM}(V)$, that bounds a chain $\zeta$ in $X$, then we can similarly split up an appropriate $\zeta''$ as $\zeta'' = \mu + \nu$ and still have $\partial \zeta'' = \partial (\mu + \nu) = x'' + y''$, so $x + y$ is homologically trivial in $I_i^p \mathcal{C}^{GM}(U) + I_i^p \mathcal{C}^{GM}(V)$ and $\psi$ is injective. 

4.4.2 Singular excision and Mayer-Vietoris

We now turn to the singular versions of excision and the Mayer-Vietoris sequence. The basic ideas are similar to those we have already explored in the PL case, but there are additional technicalities. For one thing, we have not yet explored subdivision of singular simplices, which will be a necessary component of the proof. Fortunately, while singular subdivision includes some necessary difficulties by virtue of working with singular simplices rather than geometric simplices, it may actually have fewer technicalities arising from the allowability conditions. For example, when working simplicially to prove Theorem 3.26, we needed to construct maps from chains built on the simplicial subdivision triangulation $T'$ back to chains in the original triangulation $T$. We then had to be careful to make sure these maps preserved allowability of chains. By contrast, the subdivision of a singular chain is itself a singular chain, and so the homological inverse to a subdivision map is simply the identity homomorphism.

The results in this section is based on the author’s work in [29].

Singular subdivision. We begin by considering what it should mean for a singular chain to have a subdivision. The basic idea is that we understand subdivisions of simplicial complexes, including the geometric simplices used as the domains for singular simplices. We use these geometric subdivisions to define subdivisions of singular simplices. The technical challenge in doing this for a chain is in making sure that all the simplices of the chain are subdivided in some compatible manner to ensure that the “boundary of the subdivision is the subdivision of the boundary.”

Let’s start with the first step, which is to define singular subdivisions of geometric simplices.

49By “geometric simplex”, we mean a simplex $\Delta^i$ thought of as a simplicial complex.
Next, let $\Delta^i = [0, \ldots, i]$, $i \geq 0$, be the standard ordered $i$-simplex, thought of as a simplicial complex. Let $\hat{\Delta}^i$ be a simplicial subdivision of $\Delta^i$ with a partial ordering on its vertices such that the vertices of each simplex $\hat{\Delta}^i$ are totally ordered; we may assume that the ordering is chosen such that

1. the ordering on the vertices of $\Delta^i$ is preserved,

2. if $w_1$, $w_2$ are vertices of $\hat{\Delta}^i$ and $w_2$ is contained in the interior of a face of $\Delta^i$ of higher dimension than that of the open face in which $w_1$ is contained, then $w_1 < w_2$.

The simplicial subdivision $\hat{\Delta}^i$ is a triangulation of $\Delta^i$, and we can let $\Omega_* (\hat{\Delta}^i)$ be its ordered chain complex, which we take to be generated by the nondegenerate ordered simplices $[v_0, \ldots, v_k]$. In other words, we assume that an ordered simplex corresponds to a $[v_0, \ldots, v_k]$ for which the $\{v_j\}$ are all vertices of a simplex of $\hat{\Delta}^i$, $v_j \neq v_\ell$ if $j \neq \ell$, and the order of the simplices matters so that, for example, $[v_0, v_1, \ldots, v_k] \neq [v_1, v_0, \ldots, v_k]$. We remark that it is common to allow also as generators of an ordered chain complex the degenerate simplices in which the vertices of a generator $[v_0, \ldots, v_k]$ are not necessarily unique. We will not need this level of generality (though it would not hurt to allow it either). See [77, Section 13] for comparison. If $\tau$ is an ordered $i$-simplex of $\Omega_* (\hat{\Delta}^i)$, let $\text{sgn}(\tau)$ be 1 if the orientation of $\tau$ induced by the ordering of its vertices (given by the ordering of the vertices in $\tau$, not the partial order on all vertices of $\hat{\Delta}^i$) agrees with the orientation of $\Delta^i$ and $-1$ if they disagree.

Now, let $\{\delta_j^i\}$ be the collection of non-degenerate ordered $i$-simplices $[v_0, \ldots, v_k]$ of $\hat{\Delta}^i$ such that $v_j < v_\ell$ if $j < \ell$, using the partial ordering on the vertices of $\hat{\Delta}^i$. Note that there is a bijection between the set of $i$-simplices in the subdivision $\hat{\Delta}^i$ and $\{\delta_j^i\}$. Define $\sigma \in \Omega_i (\hat{\Delta}^i)$ by $\sigma = \sum_j \text{sgn}(\delta_j^i)\delta_j^i$. We claim that $\sigma$ is a fundamental chain for $\Delta^i$ in the sense that $\sigma$ represents a generator of $H_i (\partial \Delta^i) \cong \mathbb{Z}$. In fact, more precisely, if $D_k$, $0 \leq k \leq i$, is the $k$th $i-1$ face of $\Delta^i$ (obtained by deleting the $(k + 1)$st vertex of $\Delta^i$ in the ordering of the vertices of $\Delta^i$), then the subdivision $\hat{\Delta}^i$ induces a subdivision $\hat{D}_k$ of $D_k$ and hence a chain $\sigma_k$ defined analogous to $\sigma$. We claim that $\partial \sigma = \sum_{k=0}^i (-1)^k \sigma_k$. Then $\sigma$ must represent an element of $H_i (\partial \Delta^i)$ that must certainly be a generator.

**Lemma 4.34.** $\partial \sigma = \sum_{k=0}^i (-1)^k \sigma_k$.

**Proof.** Clearly the boundaries of the ordered $i$-simplices of $\sigma$ are ordered $i-1$ simplices of the triangulation $\hat{\Delta}^i$. We must show that the proper cancellations and orientations occur.

Suppose $\tau_v, \tau_w$ are two unique nondegenerate oriented $i$-simplices $\hat{\Delta}^i$ that share a common $i-1$ dimensional face. We can identify these faces as $\tau_v = [v_0, \ldots, v_i]$ and $\tau_w = [w_0, \ldots, w_i]$, where the notation does not imply that the $\{v_j\}$ are unique from the $\{w_j\}$. In fact, the common face of $\tau_v$ and $\tau_w$ must have the form $[v_0, \ldots, \hat{v}_k, \ldots, v_i] = [w_0, \ldots, \hat{w}_\ell, \ldots, w_i]$. Then in the boundary formula for $\tau_v$, $[v_0, \ldots, \hat{v}_k, \ldots, v_i]$ has a sign $(-1)^k$, and in the boundary formula for $\tau_w$, $[w_0, \ldots, \hat{w}_\ell, \ldots, w_i]$ has a sign $(-1)^\ell$. As in the definition of subdivision, let $\text{sgn}(\tau)$ be 1 if the orientation of the ordered $i$-simplex $\tau$ agrees with the orientation of $\Delta^i$ and $-1$ otherwise. Then we claim

$$\text{sgn}(\tau_v) = (-1)^k \text{sgn}([v_k, v_0, \ldots, \hat{v}_k, \ldots, v_i]) = (-1)^{k+\ell+1} \text{sgn}([w_\ell, w_0, \ldots, \hat{w}_\ell, \ldots, w_i]) = (-1)^{k+\ell+1} \text{sgn}(\tau_w).$$
The outer equalities are a consequence of the permutations involved, while the middle equality is easiest to see by thinking of $\Delta^i$ embedded in $i$-dimensional Euclidean space; since both simplices $\tau_v$ and $\tau_w$ must be embedded and disjoint apart from their common face, vectors from the common face to $v_k$ or to $w_\ell$ must point to opposite sides of the plane containing the common face. Thus $\partial(-1)^{\text{sgn}(\tau_v)}\tau_w$ has a term

\[
(-1)^{\text{sgn}(\tau_v)+k}[v_0, \ldots, \hat{v}_k, \ldots, v_i] = (-1)^{\ell+1+\text{sgn}(\tau_w)}[w_0, \ldots, \hat{w}_\ell, \ldots, w_i] \\
\equiv (-1)^{\ell+\text{sgn}(\tau_w)}[w_0, \ldots, \hat{w}_\ell, \ldots, w_i],
\]

which is the negative of the corresponding term in $\partial(-1)^{\text{sgn}(\tau_w)}\tau_v$. Thus internal $i-1$ faces of $\Delta^i$ cancel in $\partial \sigma$ (noting that an internal $i-1$ face of $\Delta^i$ is the face of exactly two $i$-simplices).

Next, we observe that every $i-1$ simplex of $\hat{\Delta}^i$ contained in $\partial \Delta^i$ is a face of exactly one $i$-simplex $\hat{\Delta}^i$, and, $\delta_j^i$ provides the ordering on the vertices of this simplex that are compatible with the partial order on the vertices of $\hat{\Delta}^i$. It is clear that any face of $\delta_j^i$ also has its vertices in the proper order. So it only remains to check that each such $i-1$ simplex appears with the correct sign in $\partial \sigma$.

Suppose $\delta_j^i = [v_0, \ldots, v_i]$, and suppose that some face $\tau$ of $\delta_j^i$ is contained in the $k$th face $D_k$ of $\Delta^i$, $0 \leq k \leq i$. Suppose that $v_\ell$ is the vertex of $\delta_j^i$ that is not contained in $\tau$, so that $\tau = [v_0, \ldots, \hat{v}_\ell, \ldots, v_i]$. Let us first suppose that the ordering of the vertices on $\tau$ gives an orientation that up to the factor $(-1)^r$ with the orientation on the face $D_k$ of $\Delta^i$ given by the ordering of the vertices of $\Delta^i$, in other words that $\text{sgn}(\tau) = (-1)^r$ (we can assume $r = 0$ or $r = 1$). To compare then the orientation on $\delta_j^i$ with the orientation of $\tau$, it is useful to look at the ordered simplex\(^50\) $\alpha = [v_0, \ldots, v_{k-1}, v_\ell, v_k, \ldots, \hat{v}_\ell, \ldots, v_i]$. Since, up to an even permutation, the ordered vertices of $\tau$ give an orientation that is compatible with $(-1)^r$ times that of $D_k$, the ordered simplex $\alpha$ simply adds the vertex $v_\ell$ into the $k$th slot to stand in for the vertex that has been removed from $\delta_j^i$ to provide an orientation for $\alpha$ that agrees with $(-1)^r$ times that of $\Delta^i$. (Suppose, for example, that $\tau = D_k$, in which case clearly putting the extra vertex of $\delta_j^i$ in the $k$th slot gives an ordering compatible with the orientation of $\Delta^i$. This is basically the same idea, except that we only ask for the orientation of $\tau$ to $(-1)^r$-agree with that of $D_k$, not come from the exact same ordering of vertices.) Then we must have $\text{sgn}(\alpha) = (-1)^r$. Furthermore, to obtain $\alpha$ from $\delta_j^i$ requires moving $v_\ell$ to the $v_k$ slot, which requires $|k-\ell|$ transpositions of neighboring vertices. So $\text{sgn}(\delta_j^i) = (-1)^r(-1)^{k-\ell}$.

Now we compute: $\tau$ appears in the boundary of $\sigma$ as

\[
\text{sgn}(\delta_j^i)(-1)^{\ell}[v_0, \ldots, \hat{v}_\ell, \ldots, v_i] = (-1)^{r+k-\ell}(-1)^{\ell}[v_0, \ldots, \hat{v}_\ell, \ldots, v_i] \\
= (-1)^{r+k}[v_0, \ldots, \hat{v}_\ell, \ldots, v_i] \\
= (-1)^{r+k}\tau.
\]

But since we have assumed the orientation of $\tau$ agrees up to $(-1)^r$ with that of $D_k$, this is exactly how the term corresponding to $\tau$ appears in the expression for $(-1)^k\sigma_k$. \(\square\)

\(^50\)Note: we mean for the notation to indicate the generic situation for which $\ell$ might have any integer value in the interval $[0, i]$. The point is that we move $v_\ell$ to the $k$th slot.
Let φ : O∗(ˆΔ) → S∗(Δ) be the chain map that takes an ordered simplex [v₀, ..., vₖ] to the singular simplex σ that maps Δ = [0, ..., k] to [v₀, ..., vₖ] linearly with σ(ℓ) = vₖ; see [77] Section 34. We define the singular subdivision of Δ corresponding to Δ to be s = φ(σ). So if we let φ(δj) = iₖ, which is the unique order-preserving linear homeomorphism iₖ: Δ → δj, and if we let sgn(iₖ) = sgn(δj), then s = ∑ sgn(iₖ)iₖ.

If σ : Δ → X is a singular i-simplex, then define the singular subdivision η of σ with respect to the subdivision ˆΔ of Δ to be the singular chain σs, i.e. if s = ∑ sgn(iₖ)iₖ, then σs = ∑ sgn(iₖ)σ o iₖ.

Suppose now that ξ is a singular i-chain of X, ξ = ∑ nₖσₖ, nₖ ∈ Z, each σₖ a singular i-simplex of X. For each σₖ, let ˆΔₖ represent a copy of the standard model i-simplex so that σₖ : ˆΔₖ → X. We say that subdivisions { ˆΔₖ} are compatible with respect to ξ if the following condition holds: suppose that σₖ and σ₇ are singular simplices in ξ and that they have faces τₖ and τ₇ such that τₖ = τ₇ as singular simplices (i.e. τₖ : Δ⁻¹ → Δ⁻¹ σₖ X equals τ₇ : Δ⁻¹ → Δ⁻¹ σ₇ X, where the first map in each composition is the order-preserving embedding of the appropriate face). Then the induced singular subdivision τₖ and τ₇ should agree as chains. Note that k may equal l so this condition may impose non-trivial relations among faces of the same i-simplex. Given such compatible subdivisions, we can form the chain ˆξ = ∑ nₖηₖ and have d ˆξ = (dξ), where the latter term indicates the induced subdivision of i - 1 chains in the boundary of ξ. We call ˆξ a singular subdivision of ξ.

The standard example of a subdivision ξ′ is given by the barycentric subdivision of singular chains (see [77]). In this case, there is a natural partial ordering on the vertices of the subdivided model simplices obtained by letting the barycenter of each face have a place in the order greater than the original vertices of each face and greater than each of the barycenters of the lower-dimensional faces. The uniformity of the construction ensures compatibility among simplices in any chain. Similarly, we can find such natural orderings for generalized barycentric subdivisions, in which not every face is subdivided at each step, though it still takes some effort to ensure that compatibility among simplices is maintained. In general, if T : S(X) → S(X) is any chain map that restricts to a singular subdivision on each simplex, then the compatibility condition is automatically satisfied (DOUBLE CHECK).

We note for future use that the idea of using a singular subdivision of a simplex Δ modeled upon some subdivision ˆΔ in order to obtain a subdivided singular i-chain in X can be extended to define singular chains starting with any map f : Z → X such that Z can be triangulated with an i-dimensional triangulation. In particular, we will utilize below singular chains based on triangulations of Δ⁻¹ × [0, 1] in order to create homologies.

Of course it will be important to know that a subdivision of a p-allowable chain remains allowable:

Lemma 4.35. Let ˆξ be a singular subdivision of the i-chain ξ ∈ IGS(X). Then ˆξ ∈ IGS(X).

Proof. By assumption, for each σ in ξ and each stratum S of X, σ⁻¹(S) is contained in the i - codim(S) + p(S) skeleton of Δ, and similarly for each i - 1 simplex τ in ∂ξ, τ⁻¹(S) is contained in the i - 1 - codim(S) + p(S) skeleton of Δ. Now ˆξ is composed of the singular i simplices of the form σiₖ where iₖ : Δ → Δ is linear and injective.
So let $\sigma : \Delta^i \to X$ be an allowable singular simplex of $\xi$, let $\hat{\Delta}^i$ be its subdivision, and let $i_j$ be a simplex in the singular subdivision of $\hat{\Delta}^i$. We first notice that $i^{-1}_j$ of the $r$ skeleton of $\Delta^i$ must lie in the $r$ skeleton of $\Delta^i$. So now using the allowability of $\hat{\xi}$,

$$(\sigma i_j)^{-1}(S) = i^{-1}_j \sigma^{-1}(S)$$

$$\subset i^{-1}_j(\{i - \text{codim}(S) + \tilde{p}(S) \text{ skeleton of } \Delta^i\})$$

$$\subset \{i - \text{codim}(S) + \tilde{p}(S) \text{ skeleton of } \Delta^i\}.$$ 

Thus each $\sigma i_j$ is allowable, and $\hat{\xi}$ is composed of allowable $i$-simplices. Similarly, the simplices in $\partial \hat{\xi}$ are allowable since $\partial \hat{\xi}$ is a singular subdivision of $\partial \xi$, so the above arguments hold in the same manner.

Using this lemma, we show that an intersection cycle and its subdivisions define the same intersection homology class.

**Proposition 4.36.** Let $\xi$ be a $\tilde{p}$-allowable chain representing an element of $\bar{I}^pH^*_GM(X,U)$, where $U$ is a possibly empty subset of $X$. Then $\xi$ is intersection homologous to any singular subdivision $\hat{\xi}$, so $\xi$ and $\hat{\xi}$ represent the same element of $\bar{I}^pH^*_GM(X,U)$.

**Proof.** We can construct the homology explicitly by constructing an allowable $i + 1$ chain $D$ such that $\partial D = \hat{\xi} - \xi + E$, where $E$ is an allowable chain in $X$ with support in $U$. We use a prism construction, though one slightly different from the prisms we have used previously in the proofs of Lemma 3.29 and Proposition 4.8.

Suppose that $\xi = \sum n_k \sigma_k$, and let $\Delta^i_k$ be the domain simplex for $\sigma_k$. By definition of $\hat{\xi}$, $\hat{\xi} = \sum k \sum j \sgn(i_{k,j}) \sigma_k \circ i_{k,j}$, where each $i_{k,j}$ is a linear injection $\Delta^i \to \Delta^i_k$ determined by a simplicial subdivision $\hat{\Delta}^i_k$ of $\Delta^i_k$.

We begin by triangulating the set $B = [0,1] \times \Pi_k \Delta^i_k$, where the index $k$ ranges over the simplices of $\xi$. Suppose that each $0 \times \Delta^i_k$ is subdivided trivially (i.e. it is triangulated via the identity map to itself) and that each $1 \times \Delta^i_k$ is triangulated using $\Delta^i_k$. We want to extend this triangulation to the whole space $B$. We do so inductively on the dimensions of faces of $\Delta^i_k$:

1. For each 0-simplex $v$ in $0 \times \Pi \Delta^i_j$, triangulate $v \times [0,1]$ as $\bar{c}((0 \times v) \cup (1 \times v))$, where $\bar{c}$ represents the closed cone. In this triangulation, we obtain a partial order by letting the cone vertex be greater than $0 \times v$ and $1 \times v$.

2. Now for each $m$ simplex $\gamma$ in $\Pi \Delta^i_k$, assume that we have inductively constructed a partially ordered triangulation of $[0,1] \times \partial w$. Together with the trivial triangulation of $0 \times w$ as a face of $\Delta^i_k$ and the triangulation of $1 \times w$ as a subcomplex of $\Delta^i_k$, we then have a triangulation of $C = (0 \times w) \cup (1 \times w) \cup ([0,1] \times \partial w)$. Now triangulate $[0,1] \times w$ by taking the closed cone on $C$ with the new cone vertex contained in the interior of $[0,1] \times w$. Again the new cone vertex is placed higher in the partial ordering than the pre-existing vertices.

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51Note that the number of $j$ indices is dependent on the subdivision of the corresponding $\Delta^i_k$, but it would be awkward to include this in the notation.
3. Since the dimension of the chain $\xi$ is finite, this process terminates with a triangulation of $B$. Notationally, we denote $B$ with this triangulation as $\hat{B}$.

Now for each $\Delta^i_k$, we have constructed a triangulation $\Gamma_k$ of $[0, 1] \times \Delta^i_k$ as a subcomplex of $B$ (with an appropriate partial order on its vertices), and so we can construct the ordered chain $\mathfrak{D}_k = \sum_{a} sgn(\gamma_{k,a}) \gamma_{k,a}$, where the sum is over the nondegenerate ordered $i + 1$ simplices of $\Gamma_k$, with the order on each simplex determined by that of $\Gamma_k$, and the sign corresponds to agreement or disagreement between the orientation of $\gamma_{k,a}$ and that of $[0, 1] \times \Delta^i_k$. Letting $\psi : \mathfrak{D}_s([0, 1] \times \Delta^i_k) \to S_*([0, 1] \times \Delta^i_k)$ be the homomorphism that takes an ordered simplex of the triangulation to the corresponding singular simplex, thought of as an embedding, we then have a singular chain $S_k = \psi(\mathfrak{D}_k) \in S_{i+1}([0, 1] \times \Delta^i_k)$ with $\partial S_k = 1 \times s_k - 0 \times \Delta^i_k + F_k$, where $s_k$ is the singular subdivision of $\Delta^i_k$ corresponding to $\hat{\Delta}^i_k$ and $F_k$ is a fundamental class of the singular subdivision of $[0, 1] \times (\partial\Delta^i_k)$ (the argument for this is analogous to the proof of Lemma 4.34).

Finally, let $p : [0, 1] \times B \to B$ be projection and define

$$D = \sum_k n_k(\sigma_k p S_k),$$

where $p$ and $\sigma_k$ act as chain maps

$$S_*( [0, 1] \times \Delta^i_k) \xrightarrow{p} S_*(\Delta^i_k) \xrightarrow{\sigma_k} S_*(X).$$

Now, by the assumption that $\hat{\xi}$ is a singular subdivision of $\xi$, the subdivision of the $i$-simplices of $\xi$ are compatible along their boundaries. Furthermore, our construction of the triangulation of $B$ was also consistent in that if corresponding $i - 1$ faces of $\Delta^i_k$ and $\Delta^i_l$ are subdivided in the same way in $\hat{\Delta}^i_k$ and $\hat{\Delta}^i_l$, then $[0, 1] \times \Delta^i_k$ and $[0, 1] \times \Delta^i_l$ will be triangulated equivalently in $B$. Thus faces that cancel each other in $\partial \xi$ will yield cancellations in $D$. It follows that we must have

$$\partial D = \hat{\xi} - \xi + E,$$

where $E$ is supported in the support of $\partial \xi$. In particular, $E$ is supported in $U$.

It remains to check that $D$ is allowable. Since $\xi$ and $\hat{\xi}$ are allowable and since $\partial E = \partial \xi - \partial \hat{\xi}$, it remains only to check that the $i + 1$ simplices of $D$ are allowable and that the $i$ simplices of $E$ are allowable. So let $j : \Delta^{i+1} \to B$ be a singular $i + 1$ simplex that maps linearly onto a nondegenerate $i + 1$ simplex of $B$, and let $\eta = \sigma p j$ be the corresponding singular simplex of $\Delta^{i+1}$. Then for any singular simplex $\sigma : \Delta^i \to X$ of $\xi$, we have $\eta^{-1}(S) = j^{-1} p^{-1} \sigma^{-1}(S)$. Since $\sigma$ is allowable, $\sigma^{-1}(S)$ is in the $i - \text{codim}(S) + \bar{p}(S)$ skeleton of $\Delta^i$, and so

$$p^{-1} \sigma^{-1}(S) \subset [0, 1] \times \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta_j\},$$

which must lie in the $i - \text{codim}(S) + \bar{p}(S) + 1$ skeleton of our triangulation of $[0, 1] \times \Delta^i \subset B$. But then any $i + 1$ simplex of our triangulation of $B$ can intersect this skeleton in at most its $i + 1 - \text{codim}(S) + \bar{p}(S)$ skeleton. And since $j$ is a linear homeomorphism from $\Delta^{i+1}$ onto
such a simplex, $j^{-1}p^{-1}\sigma^{-1}(S)$ is contained in the $i + 1 - \operatorname{codim}(S) + \bar{p}(S)$ skeleton of $\Delta^{i+1}$. Thus $\eta$ is allowable.

The argument that $E$ is allowable follows by exactly the same arguments one dimension lower by using the allowability of the simplices of $\partial\xi$. $\square$

The following generalization will be useful below in Section 7.1.

**Corollary 4.37.** Suppose that $T : S_*(X) \to S_*(X)$ is a chain map that restricts to a singular subdivision on each singular simplex. Then the induced map $T : I^pS_*^{GM}(X,U) \to I^pS_*^{GM}(X,U)$ is chain homotopic to the identity for any subset $U \subset X$.

**Proof.** First, assume $U = \emptyset$. By Lemma 4.35, the image under $T$ of each allowable simplex is allowable, and since $T$ is a chain map, if $\xi \in I^pS_*^{GM}(X)$, then $T(\xi) \in I^pS_*^{GM}(X)$. The argument that $T$ is chain homotopic to the identity follows from the construction of the proof of Proposition 4.36. For a simplex $\sigma$, let $D(\sigma)$ denote the corresponding chain constructed from $\sigma$ in the proof of Proposition 4.36. Then, by that construction, for each singular simplex $\sigma, \partial D(\sigma) = T(\sigma) - \sigma - D(\partial\sigma)$. Here the identification of the $E$ of the proof of the proposition with $D(\partial\sigma)$ here is a direct result of the iterative construction of $D$ over skeleta of simplices; to check the signs, we observe the consistency:

$$0 = \partial\partial D(\sigma) = \partial T(\sigma) - \sigma - \partial D(\partial\sigma).$$

We also showed in the proof of the proposition that if $\sigma$ is allowable, then so is $D(\sigma)$. Therefore, if $\xi \in I^pS_*^{GM}(X)$, all of the simplices of $D(\xi)$ are allowable. Furthermore, extending $D$ linearly, we obtain that $\partial D(\xi) = T(\xi) - \xi - D(\partial\xi)$, and we know that all the simplices of $\xi, T(\xi)$, and $D(\partial\xi)$ are allowable, using again Lemma 4.35 together with the allowability by assumption of the simplices of $\xi$ and $\partial\xi$. This completes the argument if $U$ is empty.

If $U \neq \emptyset$, then since subdivision takes simplices supported in $U$ to chains supported in $U$, $T$ therefore induces a map from $I^pS_*^{GM}(X,U)$ to itself. Furthermore, since $D$ also preserves support, $D$ is well-defined as a map $S_*(X,U) \to S_{*+1}(X,U)$. But if $\xi \in S_*(X)$ represents an element of $I^pS_*^{GM}(X,U)$, then $\xi$ is composed of simplices that are either allowable or contained in $U$, so the same is true of $D(\xi)$. Similarly, $\partial D(\xi) = T(\xi) - \xi - D(\partial\xi)$ also consists entirely of simplices that are allowable or contained in $U$, and so $D$ induces a well-defined chain homotopy $I^pS_*^{GM}(X,U) \to I^pS_*^{GM}(X,U)$. $\square$

**Excision.** Now that we have established that subdivision of singular chains preserves intersection homology classes, we can establish excision by using barycentric subdivision to ensure that chains are composed of small simplices and then breaking the chains into pieces, being careful to ensure that the pieces are each allowable chains.

**Definition 4.38.** Suppose $\sigma : \Delta^i \to X$ is a singular simplex. Let $\hat{\Delta}^i$ be the barycentric subdivision of $\Delta^i$. Let $\gamma$ be an $i - 1$ simplex of $\hat{\Delta}^i$ that does not contain any of the vertices of
Proof. We first show that inclusion induces a surjection on intersection homology. Let \( X \) be a completely interior simplex of \( \sigma \).

More generally, we call a singular simplex arising from a singular subdivision of a singular simplex interior if the corresponding \( \gamma \) is not contained in \( \partial \Delta^i \).

**Lemma 4.39.** If \( \sigma \) is an allowable singular \( i \)-simplex, and \( \tau \) is a completely interior \( i - 1 \) simplex of \( \sigma \), then \( \tau \) is a allowable.

Proof. The proof is completely analogous to that of Lemma 4.30. By Lemma 4.31 the intersection of \( \gamma \) with every face \( \eta \) of \( \Delta^i \) has dimension less than \( \dim(\eta) \). So the intersection of \( \gamma \) with the \( k \)-skeleton of \( \Delta^i \) must be contained in the \( k - 1 \) skeleton of \( \Delta^i \), or in other words,

\[
i^{-1}(k \text{ skeleton of } \Delta^i) \subset \{k - 1 \text{ skeleton of } \Delta^{i-1}\}.
\]

By assumption, if \( S \) is a stratum of \( X \), \( \sigma^{-1}(S) \) is contained in the \( i - \text{codim}(S) + \bar{p}(S) \) skeleton of \( \Delta^i \), so it follows that \( \tau^{-1}(S) = i^{-1}\sigma^{-1}(S) \) is contained in the \( i - 1 - \text{codim}(S) + \bar{p}(S) \) skeleton of \( \Delta^i \). So \( \tau \) is allowable.

**Theorem 4.40.** Let \( X \) be a filtered stratified space, and suppose \( K \subset U \subset X \) such that \( \bar{K} \subset \bar{U} \). Then inclusion induces an isomorphism \( I^\beta H_i^{\text{GM}}(X - K, U - K) \xrightarrow{\sim} I^\beta H_i^{\text{GM}}(X, U) \), where all subspaces of \( X \) inherit their filtrations (including formal dimension) and perversities from \( X \).

Proof. We first show that inclusion induces a surjection on intersection homology. Let \( \xi \) be a relative cycle representing an element of \( I^\beta H_i^{\text{GM}}(X, U) \). The strategy is essentially the same as for PL subdivision. We first replace \( \xi \) by a subdivision \( \hat{\xi} = x + y \) where \( x \) and \( y \) are allowable singular chains, \( x \) is supported in \( X - \bar{K} \) and \( y \) is supported in \( U \). Then, applying Proposition 4.36, \( x \) and \( \xi \) represent the same element of \( I^\beta H_i^{\text{GM}}(X, U) \), but \( x \) is in the image of \( I^\beta H_i^{\text{GM}}(X - K, U - K) \).

We will perform an iterated barycentric subdivision of \( \xi \) via the singular subdivision technique discussed above. Let \( \beta \) be the operator that replaces a chain with its singular barycentric subdivision. As \( \xi \) has only a finite number of simplices and as simplices are compact, an easy Lebesgue number argument as in [77, Theorem 31.3] suffices to show that there is an \( m \) such that \( \beta^m \xi \) consists entirely of singular simplices with image in \( X - \bar{K} \) or \( \bar{U} \), which together constitute an open cover of \( X \). If we were working with ordinary homology, this would be sufficient. However, we will need to employ techniques analogous to those we used to demonstrate PL excision, and so we need a slight buffer between simplices that intersect \( K \) and simplices that intersect \( X - U \). Let \( Z \) be the union of the images of all the simplices of \( \beta^m \xi \) whose images intersect \( \bar{K} \); note that \( Z \subset \bar{U} \). We can now further subdivide \( \beta^m \xi \) to obtain \( \beta^M \xi \) such that every simplex of \( \beta^M \xi \) lies in \( X - (\bar{K} \cup Z) \) or \( U \). We let \( \xi = \beta^{M+1} \xi \).

To explain our motivation so far, if we were working with ordinary homology, we’d be content to let \( y \) consist of the sum of all the simplices of \( \beta^m \xi \) (with their coefficients) whose images intersect \( \bar{K} \). The problem here is that this might create unallowable boundaries. So
the purpose of the extra subdivisions is to make sure that we have enough extra singular simplices forming a “halo” around those touching \( \bar{K} \), but still inside \( \hat{U} \), that we can cut the halo simplices along completely interior faces (see Definition 4.38) of one further subdivision, ensuring allowability of the new boundaries by Lemma 4.39. This is the program we now undertake in detail.

Let \( \mathcal{A} \) be the set of singular simplices \( \sigma_j \) in \( \beta^M \xi \) such that the image of \( \sigma_j \) intersects \( \bar{K} \), and let \( \mathcal{B} \supset \mathcal{A} \) be the set of singular simplices of \( \beta^M \xi \) that share a vertex with a simplex in \( \mathcal{A} \). By sharing a singular vertex, we mean that there is a point of \( X \) that is the common image of some vertex of each of the domain simplices. Since every simplex of \( \mathcal{A} \) must be contained in a singular subdivision of a simplex with support in \( Z \), the support of every simplex of \( \mathcal{B} \) must intersect \( Z \), and it follows from the construction that every simplex of \( \mathcal{B} \) has image in \( U \). Furthermore, every simplex of \( \beta^M \xi \) not in \( \mathcal{B} \) is contained in \( \hat{X} - \hat{K} \). Conceptually, \( \mathcal{A} \) is the core of simplices that intersect \( \bar{K} \), while \( \mathcal{B} - \mathcal{A} \) is the “halo”.

We now let \( y \) consist of the following simplices of \( \hat{\xi} = \beta^{M+1} \xi \) (along with the coefficients they have in \( \hat{\xi} \):

1. If \( \sigma \in \mathcal{A} \), then all \( i \)-simplices of the singular barycentric subdivision of \( \sigma \) are in \( y \).

2. Suppose \( \sigma \in \mathcal{B} - \mathcal{A} \) is an \( i \)-simplex with a vertex \( v \) that is shared with a simplex in \( \mathcal{A} \) in the sense described above. Then every \( i \)-simplex of the singular barycentric subdivision of \( \sigma \) that contains the vertex \( v \) is in \( y \). (Since there is some room for confusion, let us describe this in more detail. We are assuming that that \( \sigma : \Delta^i \to X \) is a simplex of \( \beta^M \xi \) and \( v \) is a vertex of \( \Delta^i \) such that there is some other simplex \( \sigma_2 : \Delta^i_2 \to X \) of \( \beta^M \xi \) with \( \sigma_2(\Delta^i_2) \cap \bar{K} \neq \emptyset \) and a vertex \( v_2 \in \Delta^i_2 \) such that \( \sigma(v) = \sigma_2(v_2) \). Then we consider all the \( i + 1 \) simplices in the barycentric subdivision of \( \Delta^i_2 \) that contain \( v_2 \), and we let the corresponding singular simplices in the singular barycentric subdivision of \( \sigma \) be in \( y \) (with the coefficients and signs they inherit from \( \sigma \)). This does not mean that an \( i \)-simplex of the subdivision gets to be in \( y \) if one of its new vertices (one that arises from the subdivision of \( \Delta^i_2 \) and isn’t an original vertex of \( \Delta^i_2 \)) gets mapped to the same image as some vertex of some simplex of \( \beta^M \xi \).

In either case, each simplex \( \sigma' \) of \( \hat{\xi} \) that qualifies for \( y \) is given the same coefficient it would have in \( \hat{\xi} \), up to the sign determined by the relation between \( \sigma' \) and the \( \sigma \) it is subdivided from. We claim that \( y \) is allowable. If so, it follows from the construction that \( y \) contains all the simplices of \( \hat{\xi} \) that intersect \( \bar{K} \), and these must all be contained in \( \hat{U} \). Hence \( x = \hat{\xi} - y \) is also allowable and is contained in \( \hat{X} - \hat{K} \), so we will be finished with the proof of surjectivity.

Since \( \xi \) is allowable, it follows from Lemma 4.35 that \( \hat{\xi} = \beta^{M+1} \xi \) is allowable, and hence each \( i \)-simplex of \( y \) is allowable. We need to check \( \partial_y \). Let \( \tau \) be an \( i - 1 \) chain in \( \partial_y \). Then \( \tau \) occurs in the boundary of a singular simplex of \( \hat{\xi} \) that might arise in either of the two ways described above, and, in fact, it might occur as the boundary of multiple such simplices. We must show that either \( \tau \) is allowable or that, in fact, the coefficient of \( \tau \) in \( \partial \hat{\xi} \) is 0 so that \( \tau \) is not in \( \partial \hat{\xi} \) after all.

First, let \( \sigma' \) be an \( i \)-simplex of \( y \) of which \( \tau \) is a boundary face, and suppose that \( \sigma' \) is an \( i \)-simplex of \( \beta^M \xi \) of which \( \sigma'_\tau \) is a simplex of the singular subdivision \( \beta \sigma^\tau \). If \( \tau \) is a
completely interior simplex to $\sigma_\tau$, then by Lemma 4.39, $\tau$ is allowable. Therefore, we may assume that $\tau$ shares a vertex $v$ with $\sigma_\tau$. By construction, either $\sigma_\tau \in \mathcal{A}$ or $\sigma_\tau \in \mathcal{B}$ and $v$ is one of the vertices shared by $\sigma_\tau$ and some simplex of $\mathcal{A}$.

Next, suppose that $\tau$ is interior to $\sigma_\tau$ but not completely interior. As noted, then $\tau$ is a face of a $\sigma'_\tau$ that contains one of the vertices of $\sigma_\tau$ that is shared with a simplex in $\mathcal{A}$ (that simplex possibly being $\sigma_\tau$ itself). But all singular $i$-simplices in the subdivision of $\sigma_\tau$ containing that vertex are in $y$, and so, $\tau$ being internal, there are actually two such $i$-simplices in the singular subdivision of $\sigma_\tau$ that possess $\tau$ as a common face, and these copies of $\tau$ cancel out in computing $\partial y$.

Finally, we must consider the case in which $\tau$ appears non-externally in the singular subdivision of some simplex $\sigma_\tau$. In this case, $\tau$ is contained in some face $F_\tau$ of $\sigma_\tau$. If $F_\tau$ is a simplex (with non-zero coefficient) in $\partial \beta^M \xi$, then $F_\tau$, and hence $\tau$ are allowable. More generally, $\tau$ is allowable if $F_\tau$ is allowable for any reason, by Lemma 4.35. If $F_\tau$ is not allowable, then all the copies of $F_\tau$ must cancel in $\partial \beta^M \xi$. But since $F_\tau$ contains $\tau$, $F_\tau$ must contain the vertex $v$ that $\sigma_\tau$ shares with some simplex in $\mathcal{A}$. But then this implies that every simplex of $\beta^M \xi$ that has $F_\tau$ as a face is in $\mathcal{B}$. Since the coefficients of $F_\tau$ cancel out to 0 in $\partial \beta^M \xi$, it follows also that all coefficients of $\tau$ arising from its appearance in subdivisions of $F_\tau$ cancel out (since all $i$-simplices containing $v$ of the subdivisions of the $i$-simplices that have $F_\tau$ as a face appear in $y$). Considering then all possible faces $F_\tau$ in which $\tau$ appears, the same arguments show overall that, if $\tau$ is not allowable, its coefficient in $\partial y$ must be 0.

This completes the proof of surjectivity. The proof of injectivity now follows from the proof of surjectivity, just as in the PL case in Theorem 4.32.

Suppose $\xi$ represents an element of $I^pH_{i}^{GM}(X-K, U-K)$ and that $\xi$ is a relative boundary in $X$, i.e. there is an allowable chain $\zeta$ such that $\partial \zeta = \xi + \rho$, with $\rho$ an allowable chain supported in $\hat{U}$. We can now subdivide $\zeta$ as in the proof of surjectivity: construct analogous $\mathcal{A}$ and $\mathcal{B}$, and let $\nu$ be the part of $\hat{\zeta} = \beta^{M+1} \zeta$ consisting simplices that share a vertex with a simplex of $\beta^M \zeta$ in $\mathcal{A}$. Let $\mu = \hat{\zeta} - \nu$. Then by exactly the same arguments as above, $\mu$ and $\nu$ are allowable and $\nu$ is supported in $\hat{U}$. Then $\partial \mu = \partial \zeta - \partial \nu = \beta^{M+1} \xi + \beta^{M+1} \rho - \partial \nu$. Both $\beta^{M+1} \rho$ and $\partial \nu$ are contained in $\hat{U}$, and, in fact, since $\mu$ and $\beta^{M+1} \xi$ are contained in $X - \hat{K}$, then so is $\beta^{M+1} \rho - \partial \nu$. So $\beta^{M+1} \xi$ must be a relative boundary in $(X-K, U-K)$, and so represents 0 in $I^pH_i^{GM}(X-K, U-K)$. Therefore, $\xi = 0$ in $I^pH_{i}^{GM}(X-K, U-K)$ by Proposition 4.36.

Once again, similar arguments allow us to formulate a Mayer-Vietoris sequence:

**Theorem 4.41.** Suppose $X = U \cup V$, where $U, V$ are subspaces such that $X = \hat{U} \cup \hat{V}$. Then there is an exact Mayer-Vietoris sequence

$$
\rightarrow I^pH_{i}^{GM}(U \cap V) \rightarrow I^pH_{i}^{GM}(U) \oplus I^pH_{i}^{GM}(V) \rightarrow I^pH_{i}^{GM}(U \cup V) \rightarrow I^pH_{i-1}^{GM}(U \cap V) \rightarrow .
$$

**Proof.** The standard arguments (see, e.g. [53]) demonstrate that there is a short exact sequence

$$
0 \rightarrow I^pS_{i}^{GM}(U \cap V) \rightarrow I^pS_{i}^{GM}(U) \oplus I^pS_{i}^{GM}(V) \rightarrow I^pS_{i}^{GM}(U) + I^pS_{i}^{GM}(V) \rightarrow 0,
$$

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and this yields a long exact sequence. What needs to be shown is that the inclusion map
\[ \psi : I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V) \to I^p S^i_{\text{GM}}(X) \]
yields an isomorphism on homology.

The proof is basically the same as the argument we used to prove excision. The argument of Theorem [4.40] shows how we can take an allowable cycle \( \xi \) in \( X \), subdivide it into a chain \( \hat{\xi} \) representing the same intersection homology class, and then divide \( \hat{\xi} \) into two allowable pieces \( x + y \), where \( y \) is contained in \( \hat{U} \) and \( x \) is contained in \( X \setminus (X \setminus \hat{V}) = \hat{V} \). Then \( y \in I^p S^i_{\text{GM}}(U) \) and \( x \in I^p S^i_{\text{GM}}(V) \). This shows that \( \psi \) is surjective on homology.

Similarly, if \( y + x \) is a cycle in \( I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V) \) that bounds a chain \( \zeta \) in \( X \), then we can similarly split up a subdivision \( \hat{\zeta} \) as \( \zeta = \nu + \mu \) with \( \nu \in I^p S^i_{\text{GM}}(\hat{U}) \) and \( \mu \in I^p S^i_{\text{GM}}(\hat{V}) \). Then \( \partial \hat{\zeta} = \partial(\mu + \nu) = \hat{x} + \hat{y} \), where \( \hat{x}, \hat{y} \) are the induced subdivisions of \( x \) and \( y \). So \( \hat{x} + \hat{y} = 0 \in H_*(I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V)) \). But we can show that \( \hat{x} + \hat{y} \) represents the same homology class as \( x + y \) in \( H_*(I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V)) \) using the same argument from the proof of Lemma [4.35] because that argument provides a homology from \( x + y \) to \( \hat{x} + \hat{y} \) via a chain \( D \) such that the part of the homology from \( x \) to \( \hat{x} \), say \( D_x \), is contained in the support of \( x \) (hence in \( V \)) and the part of the homology from \( y \) to \( \hat{y} \), say \( D_y \), is contained in the support of \( y \) (hence in \( \hat{U} \)). In particular, \( \partial D_x = \hat{x} - x + E \) for some chain \( E \) contained in \( V \) and \( \partial D_y = \hat{y} - y + F \) for some chain \( F \) contained in \( U \). But if \( D = D_x + D_y \) is the homology from \( \hat{x} + \hat{y} = x + y \), we in fact have \( E = -F \) Furthermore, \( E = -F \) provides a homology from \( \partial x = -\partial y \) to \( \partial \hat{x} = -\partial \hat{y} \) that again agrees with the construction of Lemma [4.35] and so is allowable because \( \partial x = -\partial y \) is allowable. Thus by the arguments in Lemma [4.35], the chain \( D = D_x + D_y \in I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V) \). Since we have shown that \( \hat{x} + \hat{y} \) is trivial in \( H_*(I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V)) \), so is \( x + y \). Thus \( \psi \) is injective.

Within the proof of the theorem, we demonstrated that the inclusion map \( \psi : I^p S^i_{\text{GM}}(U) + I^p S^i_{\text{GM}}(V) \to I^p S^i_{\text{GM}}(X) \) induces an isomorphism on homology. Below, in Proposition [7.5] of Section [7.1] we will prove the stronger statement that if \( \mathcal{V} \) is a covering of \( X \) such that the interiors of the elements of \( \mathcal{V} \) constitute an open covering of \( X \) and \( I^p S^i_{\text{GM}}(X) = \bigcup_{V \in \mathcal{V}} I^p S^i_{\text{GM}}(V) \subseteq I^p S^i_{\text{GM}}(X) \), then the inclusion \( I^p S^i_{\text{GM}}(X) \hookrightarrow I^p S^i_{\text{GM}}(X) \) is a chain homotopy equivalences. For PL chains, the corresponding inclusion \( I^p \mathcal{C}_{\text{GM}}^i(X) \hookrightarrow I^p \mathcal{C}_{\text{GM}}^i(X) \) will be shown to be an isomorphism.

**Examples.**

*Example 4.42.* Let us use the Mayer-Vietoris sequence to compute the intersection homology of the suspension of a compact filtered space. Let \( X \) be an \( n - 1 \) dimensional compact filtered space, and let \( SX = [-1, 1] \times X/\sim \) be the suspension of \( X \). We filter \( SX \) so that \( (SX)^i = S(X^{i-1}) \) and \( (SX)^0 = \{N, S\} \), the north and south suspension vertices; in particular, \( SX \) has dimension \( n \). Let \( \bar{p} \) be a perversity on \( SX \); for simplicity, let us also assume that \( \bar{p}(\{N\}) = \bar{p}(\{S\}) = p \) and that \( I^p H_{0 \text{GM}}(X) \neq 0 \) (for example if \( X \) and hence also \( SX \), possesses a regular stratum). We leave it as a fun exercise for the reader to consider the cases \( \bar{p}(\{N\}) \neq \bar{p}(\{S\}) \) or \( I^p H_{0 \text{GM}}(X) = 0 \). We will also use \( \bar{p} \) to denote the perversity restricted to \( X \).
We will use the Mayer-Vietoris sequence for the two pieces $U = [-1, 1] \times X/ \sim$ and $V = (-1, 1] \times X/ \sim$ of the suspension. Then the intersection of these two pieces is $U \cap V = (-1, 1) \times X$, and we know from stratified homotopy invariance, Corollary 4.9, that $I^\delta H^G_*((-1, 1) \times X) \cong I^\delta H^G_*(X)$, induced by inclusion. We also know from Theorem 4.12 that, since we’ve assumed $X$ has regular strata and hence $I^\delta H^G_0(X) \neq 0$,

$$I^\delta H^G_i(cX) \cong \begin{cases} 0, & i \geq n - p - 1, i \neq 0, \\ \mathbb{Z}, & i = 0 \geq n - p - 1, \\ I^\delta H_i(X), & i < n - p - 1, \end{cases}$$

where the isomorphisms in dimensions $i < n - p - 1$ are induced by inclusions, and this gives us the computations of $I^\delta H^G_*(U)$ and $I^\delta H^G_*(V)$. Thus in this range the map $I^\delta H^G_*(U \cap V) \to I^\delta H^G_*(U) \oplus I^\delta H^G_*(V)$ is the diagonal inclusion (or the inclusion $x \to (x, -x)$ depending on one’s conventions for Mayer-Vietoris sequences), and we can conclude that $I^\delta H^G_i(SX) \cong I^\delta H_i(X)$ for $i < n - p - 1$, induced by inclusion.

For $i \geq n - p - 1$, $i \neq 0$, we see that $I^\delta H^G_*(U) = I^\delta H^G_*(V) = 0$, and so in this range we must have $I^\delta H^G_{i+1}(SX) \cong I^\delta H^G_i(X)$, via the Mayer-Vietoris boundary map and the stratified homotopy equivalence $(-1, 1) \times X \sim X$. For $i = 0 \geq n - p - 1$, $I^\delta H^G_0(U) \cong I^\delta H^G_0(V) \cong \mathbb{Z}$, and the map $I^\delta H^G_0(U \cap V) \to I^\delta H^G_0(U) \oplus I^\delta H^G_0(V)$ is again a diagonal map, so we must have that $I^\delta H^G_0(SX) \cong \mathbb{Z}$, and $I^\delta H^G_1(SX) \cong \mathbb{Z}^{r-1}$, where $I^\delta H^G_0(X) \cong \mathbb{Z}^r$.

Finally, we must compute $I^\delta H^G_{n-p-1}(SX)$. We have seen that $I^\delta H^G_{n-p-2}(U \cap V) \to I^\delta H^G_{n-p-2}(U) \oplus I^\delta H^G_{n-p-2}(V)$ is an injection, and for $n-p-1 \geq 1$, $IH^G_{n-p-1}(U) = IH^G_{n-p-1}(V) = 0$, so $IH^G_{n-p-1}(SX) = 0$ for $n - p - 1 \geq 1$. On the other hand, if $n - p - 1 = 0$, we have already computed $IH^G_0(SX) \cong \mathbb{Z}$.

Altogether, we have shown the following.

$$I^\delta H^G_i(SX) = \begin{cases} I^\delta H^G_{i-1}(X), & i > n - p - 1, i \neq 0, 1, \\ \mathbb{Z}^{r-1}, & i > n - p - 1, i = 1, \\ 0, & i = n - p - 1, i \neq 0, \\ I^\delta H^G_i(X), & i < n - p - 1, \\ \mathbb{Z}, & i = 0 \geq n - p - 1. \end{cases}$$

If we again let $I^\delta \hat{H}^G_*(X)$ denote reduced intersection homology, we can rewrite a bit to obtain the following:

**Theorem 4.43.** If $X$ is an $n - 1$ dimensional compact filtered space with $I^\delta H^G_0(X) \neq 0$ and $\hat{p}$ is a perversity on $SX$ that takes the same value $p$ at the two suspensions points, then

$$I^\delta H^G_i(SX) \cong \begin{cases} I^\delta \hat{H}^G_{i-1}(X), & i > n - p - 1, i \neq 0, \\ 0, & i = n - p - 1, i \neq 0, \\ I^\delta H^G_i(X), & i < n - p - 1, \\ \mathbb{Z}, & i = 0 \geq n - p - 1. \end{cases}$$
Let there are always allowable singular 0-simplices on the boundary of $U$. We can employ the Mayer-Vietoris sequence by letting $\delta$, such that the dual groups line up properly. This intuition will be validated below in Section I.

$I$-sequences of Lefschetz duality. Thus it would appear that, at least with field coefficients, $\text{im}(\partial M)$ is trivial, as we have already seen that the inclusion map is injective in this range. Hence $\text{im}(\partial M)$ vanishes, and in this range the Mayer-Vietoris sequence is isomorphic to the exact sequence $\text{im}(\partial M)$, $i > 0$. For $i = n - \bar{p}(\{v\}) - 1$, the sequence is still isomorphic to the exact sequence of the pair $(M, \partial M)$ except that the Mayer-Vietoris boundary map to $I^pH_{i-1}^G(M, \partial M)$ vanishes and the group $\text{im}(\partial M)$ is trivial.

$I^pH_i^G(M, \partial M)$ is isomorphic to $\text{Hom}(\partial M)$ if $n - \bar{p}(\{v\}) - 1 > 0$. From the ordinary long exact sequence of the pair and excision, this last group is isomorphic to $\text{im}(\partial M)$, $i > 0$, then singular 1-simplices can run through $v$ by the definition of allowability, and so all allowable 0-simplices are homologous, and $I^pH_0^G(M) \cong \text{Z}$. But this is also isomorphic to $H_0(X)$.

So, summarizing, we have computed

$$I^pH_i^G(M) \cong \begin{cases} H_i(X) \cong H_i(M, \partial M), & i \geq n - \bar{p}(\{v\}), \\ \text{im}(H_i(M) \to H_i(X)) \cong \text{im}(H_i(M) \to H_i(M, \partial M)), & i = n - \bar{p}(\{v\}) - 1, \\ H_i(M), & i < n - \bar{p}(\{v\}) - 1. \end{cases}$$

This example provides a tantalizing glimpse of Poincaré duality results to come as, for a compact oriented $n$-manifold with boundary, $H_i(M; \text{Q}) \cong \text{Hom}(H_{n-i}(M, \partial M; \text{Q}), \text{Q})$ and $\text{im}(H_i(M; \text{Q}) \to H_i(M, \partial M; \text{Q})) \cong \text{Hom}(\text{im}(H_{n-i}(M; \text{Q}) \to H_{n-i}(M, \partial M; \text{Q})), \text{Q})$ as consequences of Lefschetz duality. Thus it would appear that, at least with field coefficients, finding a duality between $I^pH_i^G(M)$ and $I^pH_{n-i}^G(M)$ is a matter of choosing perversities such that the dual groups line up properly. This intuition will be validated below in Section I. Example 3.20. Another important example is the singular intersection homology version of Example 3.19. Let $M$ be an $n$-dimensional manifold with boundary $\partial M$, let $\bar{c}(\partial M)$ be the $0$-dimensional manifold with boundary $\partial M$, and let

$$X = M \cup_{\partial M} \bar{c}(\partial M).$$

We can employ the Mayer-Vietoris sequence by letting $U = c\partial M$ and $V = X - \{v\}$, where $v$ is the cone point. Then $U \cap V \cong (0, 1) \times \partial M$, so that for any perversity $\bar{p}$,

$$I^pH_i^G(U \cap V) \cong I^pH_i^G((0, 1) \times \partial M) \cong H_i((0, 1) \times \partial M) \cong H_i(\partial M),$$

since $(0, 1) \times \partial M$ is an unfiltered manifold. Similarly, $I^pH_i^G(V) \cong H_i(M)$. By the cone formula, since $\partial M$ has regular strata as an $n - 1$ dimensional filtered space,

$$I^pH_i^G(c(\partial M)) \cong \begin{cases} 0, & i \geq n - \bar{p}(\{v\}) - 1, i \neq 0, \\ \text{Z}, & i = 0 \geq n - \bar{p}(\{v\}) - 1, \\ I^pH_i^G(\partial M), & i < n - \bar{p}(\{v\}) - 1. \end{cases}$$

Notice that this agrees with the computation of Example 3.20, as we have assumed that there are always allowable singular 0-simplices.
Example 4.45. With the machinery now developed, we can at last complete the computation of Example 3.39 as a special case of Example 4.44. Recall that Example 3.39 dealt with $X = X^1 = S^1$, the circle, and a point $x_0 \in S^1$. The space $X$ is filtered as $\{x_0\} \subset X$. Such a space arises as in Example 4.44 letting $M = I$, the standard interval. Plugging the relevant details into our computation above, we obtain:

$$I^pH^i_{S^1}(X) \cong \begin{cases} H_i(S^1), & i \geq 1 - \bar{p}(\{x_0\}), \\
\text{im}(H_i(I) \to H_i(S^1)), & i = -\bar{p}(\{x_0\}), \\
H_i(I), & i < -\bar{p}(\{x_0\}). 
\end{cases}$$

So, in particular, it is not difficult to check that if $\bar{p}(\{x_0\}) \geq 0$, $I^pH^i_{S^1}(X) \cong H_i(S^1)$ for all $i$, while if $\bar{p}(\{x_0\}) < 0$, $I^pH^i_{S^1}(X) \cong H_i(I)$ for all $i$. The determining factor in the two situations is whether or not a singular 1-simplex can contain $x_0$ in its image; it can for $\bar{p}(\{x_0\}) \geq 0$, though if $\bar{p}(\{x_0\}) = 0$, only the endpoints of the 1-simplex can map to $x_0$.

**Relative Mayer-Vietoris sequences.** Once one has a Mayer-Vietoris sequence, it is not difficult to formulate a relative version.

**Theorem 4.46.** Suppose $X = U \cup V$, where $U,V$ are subspaces such that $X = \hat{U} \cup \hat{V}$. Let $A \subset X$, let $C = A \cap U$, and let $D = A \cap V$. Then there is an exact Mayer-Vietoris sequence

$$\to I^pH^i_{S^1}(U \cap V, C \cap D) \to I^pH^i_{S^1}(U, C) \oplus I^pH^i_{S^1}(V, D) \to I^pH^i_{S^1}(X, A) \to .$$

**Proof.** Consider the diagram

$$\begin{array}{c}
0 \longrightarrow I^pS^i_{S^1}(C \cap D) \longrightarrow I^pS^i_{S^1}(C) \oplus I^pS^i_{S^2}(D) \longrightarrow I^pS^i_{S^1}(C) + I^pS^i_{S^2}(D) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow I^pS^i_{S^1}(U \cap V) \longrightarrow I^pS^i_{S^1}(U) \oplus I^pS^i_{S^2}(V) \longrightarrow I^pS^i_{S^1}(U) + I^pS^i_{S^2}(V) \longrightarrow 0.
\end{array}$$

The top and bottom rows are Mayer-Vietoris short exact sequences of chain complexes, and each vertical map is an inclusion of complexes. Therefore, the snake lemma yields a short exact sequence

$$0 \longrightarrow I^pS^i_{S^1}(U \cap V, C \cap D) \longrightarrow I^pS^i_{S^1}(U, C) \oplus I^pS^i_{S^2}(V, D) \longrightarrow I^pS^i_{S^1}(U) + I^pS^i_{S^2}(V) \longrightarrow 0$$

and a corresponding long exact sequence. It only remains to show that

$$H_* \left( \frac{I^pS^i_{S^1}(U) + I^pS^i_{S^2}(V)}{I^pS^i_{S^1}(C) + I^pS^i_{S^2}(D)} \right) \cong I^pH^i_{S^1}(U \cap V, C \cap D).$$

But now we have yet another diagram of short exact sequences

$$\begin{array}{c}
0 \longrightarrow I^pS^i_{S^1}(C) + I^pS^i_{S^2}(D) \longrightarrow I^pS^i_{S^1}(U) + I^pS^i_{S^2}(V) \longrightarrow I^pS^i_{S^1}(U) + I^pS^i_{S^2}(V) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow I^pS^i_{S^1}(A) \longrightarrow I^pS^i_{S^1}(X) \longrightarrow I^pS^i_{S^1}(X, A) \longrightarrow 0.
\end{array}$$
We have already established that the middle vertical map induces an isomorphism on homology in the proof of Theorem 4.41, and similarly the lefthand vertical map induces an isomorphism on homology using that the union of the interiors of \( C \) and \( D \) in \( A \) cover \( A \) (this is an easy exercise in point set topology). Therefore, by the Five Lemma, the righthand vertical map also induces homology isomorphisms.

\[\square\]

## 5 Mayer-Vietoris arguments and further properties of intersection homology

### 5.1 Mayer-Vietoris arguments

A basic question in topology is how to compute invariants of a space from invariants of subspaces. A basic tool for this purpose is the Mayer-Vietoris sequence. In this section, we examine more general techniques that will be useful for proving many of the theorems that occur later in the book, including Künneth theorems and Poincaré duality. Following Bott and Tu [9], we refer to these techniques as “Mayer-Vietoris arguments”. The basic idea is that we want to know that if two homology theories agree on small pieces of a space, then they agree on the space as a whole. Roughly speaking, this is a generalization of the principle that if we know that \( X = U \cup V \) and that two homology theories agree on \( U \), \( V \), and \( U \cap V \), then they agree on \( X \) by applying the Five Lemma to a diagram of Mayer-Vietoris sequences (assuming sufficiently natural compatibility amongst the theories). This concept underpins the proofs in this section, but we will need to augment the setting to handle more general collections of subspaces. Ultimately, these arguments will provide a means to prove theorems without the need to rely on acyclic models arguments, which are often used for ordinary homology but that have no appropriate analogue for intersection homology, or on sheaf theory, which we would prefer to avoid for pedagogical reasons.

More formally, suppose \( X \) is a space and that \( \mathcal{T} \) is the category whose objects are open subsets of \( X \) (so \( \text{Ob}(\mathcal{T}) \) is the topology of \( X \)) and whose morphisms are inclusion maps of these subsets. Suppose that one has two functors \( F, G \) from \( \mathcal{T} \) to some other category, such as the category of abelian groups, and that one wants to know whether these two functors are equivalent in the sense that \( F(U) \cong G(U) \) for all \( U \in \mathcal{T} \). In intersection homology settings, one most often sees this question at work within sheaf theory (see [43, 8, 6], among others). Without getting into the details of sheaf theory, one often starts with data showing (roughly stated) that two types of sheaf cohomology agree at each point of \( X \) and then uses this to conclude that more global sheaf cohomology groups must be isomorphic. Since we do not want to introduce sheaves, we will need Mayer-Vietoris arguments. For ordinary homology and cohomology, such techniques are put to much use for “good covers” of manifolds by Bott and Tu in [9]; see in particular [9 Section I.5] and also [12, Lemma V.9.5]. The first use of similar arguments for intersection homology seem to be in King [61], where arguments of this type were used to compute Künneth theorems (for which one factor is a manifold), to prove topological invariance of intersection homology (given certain conditions on perversities), and to provide a general comparison principle for intersection homology theories [61]. Theorem
10]. Saralegi [89] later used these techniques to prove a de Rham theorem for intersection homology, and Friedman and McClure made use of similar ideas in providing a non-sheaf theoretic proof of Poincaré duality for singular intersection homology on pseudomanifolds [88]. The following theorems are modifications of these prior arguments that we hope will prove to be more general; in particular, they incorporate modifications that will be necessary for our applications.

We begin with a Mayer-Vietoris argument for manifolds in Theorems 5.1 and 5.2. We then turn to a version for CS sets in Theorem 5.3.

**Theorem 5.1.** Let $\mathcal{M}$ be the category whose objects are manifolds and whose morphisms are open inclusions, and let $\mathcal{Ab}_*$ be the category of graded abelian groups. Let $F_*, G_* : \mathcal{M} \to \mathcal{Ab}_*$ be functors and let $\Phi : F_* \to G_*$ be a natural transformation. Suppose that $\Phi$ has the following three properties:

1. $\Phi : F_*(U) \to G_*(U)$ is an isomorphism for $U$ homeomorphic to $\mathbb{R}^n$ or $\emptyset$,

2. $F_*$ and $G_*$ admit exact Mayer-Vietoris sequences, i.e. if $U, V$ are open submanifolds of a manifold then there is an exact Mayer-Vietoris sequence

   $$\rightarrow F_i(U \cap V) \rightarrow F_i(U) \oplus F_i(V) \rightarrow F_i(U \cup V) \rightarrow F_{i-1}(U \cap V) \rightarrow,$$

   and similarly for $G_*$, such that $\Phi$ induces a commutative diagram of such sequences,

3. if $\{U_\alpha\}$ is an increasing collection of open submanifolds of a manifold $M$ (meaning that the indices $\alpha$ are taken from a totally ordered set and $\alpha < \beta$ implies $U_\alpha \subset U_\beta$) and $\Phi : F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each $\alpha$, then $\Phi : F_*(\cup_\alpha U_\alpha) \to G_*(\cup_\alpha U_\alpha)$ is an isomorphism.

Then $\Phi : F_*(M) \to G_*(M)$ is an isomorphism for every manifold $M$.

The theorem remains true using instead the category $\mathcal{M}_{PL}$ of PL manifolds and inclusion of open subsets, using in condition (1) the requirement that $U$ be PL homeomorphic to $\mathbb{R}^n$.

**Proof.** For a manifold $M$, let $P(M)$ be the statement that $\Phi : F_*(M) \to G_*(M)$ is an isomorphism.

We will first demonstrate the conclusion of the theorem for manifolds that are open subsets of $\mathbb{R}^n$. Note that since $\mathbb{R}^n$ is a PL manifold, so are all such open submanifolds [86 Example 1.9].

So let $M$ be an open subset of $\mathbb{R}^n$. $M$ must possess a countable dense set, and taking open convex PL balls about the points of that dense set provides a countable covering $\mathcal{V}$ by open convex sets, each PL homeomorphic to $\mathbb{R}^n$. Furthermore, as the intersection of open

\footnote{For our “Mayer-Vietoris sequence”, it would be sufficient to use here any long exact sequence of the given form as $\Phi$ induces an isomorphism of such sequences for the functors $F_*$ and $G_*$. In practice, however, the maps $F_i(U \cap V) \to F_i(U)$, $F_i(U \cap V) \to F_i(V)$, $F_i(U) \to F_i(U \cup V)$, and $F_i(V) \to F_i(U \cup V)$, will always be induced (up to sign) functorially by the inclusion maps of subsets.}
PL convex sets is open PL convex, each non-empty finite intersection \( V_{\beta_1} \cap \cdots \cap V_{\beta_m} \) of elements of \( \mathcal{V} \) is PL homeomorphic\(^{53}\) to \( \mathbb{R}^n \).

By assumption \( \text{(1)} \), \( P(V_{\beta_1} \cap \cdots \cap V_{\beta_m}) \) is true for each such intersection of elements of \( \mathcal{V} \), and in particular \( P(V_{\beta}) \) is true for each \( V_{\beta} \in \mathcal{V} \).

Next we will show that \( P(U) \) is true for any \( U \) that is the finite union of finite intersections of elements of \( \mathcal{V} \), i.e. for \( U = \bigcup_{i=1}^k U_i \), where each \( U_i \) has the form \( U_i = \bigcap_{j=1}^{\ell_i} V_{i,j} \), \( V_{i,j} \in \mathcal{V} \). As the base case, we have already seen that \( P(U) \) is true when \( k = 1 \). Now assume that \( P(U) \) is true for the union of fewer than \( k \) finite intersections of elements of \( \mathcal{V} \). We notice that

\[
U_k \cap (\bigcup_{i=1}^{k-1} U_i) = \bigcup_{i=1}^{k-1} (U_k \cap U_i),
\]

and since each \( U_i \) is a finite intersection of elements of \( \mathcal{V} \), the same is true of each \( U_k \cap U_i \), \( i < k \). So \( P(U_k \cap (\bigcup_{i=1}^{k-1} U_i)) \) holds by induction, as does \( P(\bigcup_{i=1}^{k-1} U_i) \). Since \( P(U_k) \) holds by assumption \( \text{(1)} \), it follows now from assumption \( \text{(2)} \) and the five lemma that \( P(U) \) is true.

Now, let \( W_k = \bigcup_{i \leq k} V_i \), where the indices now reflect that \( \mathcal{V} \) is countable, so we can choose a bijection of \( \mathcal{V} \) with the natural numbers to obtain an order. It follows from the last paragraph that each \( P(W_k) \) is true, and hence by assumption \( \text{(3)} \), \( P(\bigcup W_k) = P(M) \) must be true. This completes the proof for the case where \( M \) is an open subset of \( \mathbb{R}^n \).

Now, let \( M \) be an arbitrary (non-empty) \( n \)-dimensional manifold (which we assume to be Hausdorff, but not necessarily second countable). Let \( \mathcal{U} \) be the collection of open sets of \( M \) for which \( P(U) \) holds. Since every point of \( M \) has a neighborhood homeomorphic to \( \mathbb{R}^n \), \( \mathcal{U} \) is non-empty by condition \( \text{(1)} \). The set \( \mathcal{U} \) is partially ordered by inclusion, and assumption \( \text{(3)} \) implies that every totally ordered set \( \{U_\alpha\} \) has an upper bound in \( \mathcal{U} \), namely \( \bigcup_\alpha U_\alpha \). By Zorn’s lemma, it follows that \( \mathcal{U} \) has a maximal element. If this maximal element is \( M \), then we are finished, so suppose that there is a maximal element \( W \) of \( \mathcal{U} \) such that \( W \neq M \). Suppose \( x \in M - W \), and let \( V \) be a neighborhood of \( x \) homeomorphic to \( \mathbb{R}^n \). Then \( V \cap W \) is homeomorphic to an open subset of \( \mathbb{R}^n \) (possibly empty) and so \( P(V \cap W) \) holds.

\(^{53}\)It turns out, as of the time of writing, that even a proof of homeomorphism is hard to find in the literature, so much so that math reference web sites seem to comment on the obscurity \( \text{[79]} \). A proof of topological homeomorphism written in 2012 appears as \( \text{[11]} \). We require the stronger PL statement, which follows\(^{54}\) from the yet stronger statement that every open star-shaped region of \( \mathbb{R}^n \) is \( C^\infty \)-diffeomorphic (and hence PL homeomorphic) to \( \mathbb{R}^n \). This is also apparently a well-known folk theorem, though, as observed by Bruce Evans in a Mathematics Stack Exchange post\( \text{[24]} \), “Most books don’t prove it. Some say that it is hard and others give it as an exercise.” Evans outlines an argument in his post, while an explicit proof due to Stefan Born appears as \( \text{[25]} \) Theorem 237. Since, for our purposes, it is sufficient to have only a \( C^1 \)-diffeomorphism from a bounded convex open subset of \( \mathbb{R}^n \) to \( \mathbb{R}^n \), we can also cite Gromov \( \text{[48]} \) I.4.C1.

\(^{54}\)This is yet another seemingly well-known fact that is difficult to pin down authoritatively. It seems to follow from J.H.C. Whitehead’s proof of the existence and uniqueness of smooth triangulations of \( C^1 \) manifolds \( \text{[102]} \), an expository treatment of which can be found in Munkres’s \( \text{[76]} \). Here is an argument: let \( f : M \to N \) be a diffeomorphism of \( C^1 \) manifolds. By \( \text{[76]} \) Theorem 10.6, \( M \) and \( N \) each possess \( C^1 \) triangulations, say via maps \( k : K \to M \) and \( \ell : L \to N \), where \( K \) and \( L \) are simplicial complexes. Furthermore, the composition \( fk \) provides another \( C^1 \) triangulation of \( N \); see \( \text{[76]} \) Theorem 8.4]. Now, by \( \text{[76]} \) Theorem 10.5, since \( fk : K \to N \) and \( \ell : L \to N \) are two \( C^1 \) triangulation of \( N \), there exist subdivisions of \( K \) and \( L \) that are “linearly isomorphic”. Via the definitions given on page 70 of \( \text{[76]} \), this means precisely that \( K \) and \( L \) are PL homeomorphic, and such a PL homeomorphism between triangulations is exactly what we mean by a PL homeomorphism of \( M \) and \( N \).
Furthermore, $P(W)$ is true by assumption and $P(V)$ holds because $V \cong \mathbb{R}^n$. Therefore $P(V \cup W)$ holds by assumption (2), contradicting the maximality of $W$. It follows that in fact $W = M$, and $P(M)$ is true.

The argument given in the last paragraph continues to hold in $\mathcal{M}_{PL}$ if we assume $M$ to be a (non-empty) PL manifold, using that every point has a neighborhood PL homeomorphic to $\mathbb{R}^n$.

The technique of the proof yields the following variant of the theorem:

**Theorem 5.2.** Let $\mathcal{M}_M$ be the category whose objects are (homeomorphic to) open subsets of a given $n$-dimensional manifold $M$ and whose morphisms are inclusions, and let $\text{Ab}_*$ be the category of graded abelian groups. Let $F_*, G_* : \mathcal{M} \to \text{Ab}_*$ be functors and let $\Phi : F_* \to G_*$ be a natural transformation such that $F_*, G_*, \Phi$ satisfy the conditions of Theorem 5.1 with respect to $\mathcal{M}_M$. Then $\Phi : F_*(M) \to G_*(M)$ is an isomorphism.

The theorem remains true using the category $\mathcal{M}_{M, PL}$ of open subsets of a given PL n-manifold $M$ using in condition (1) the requirement that $U$ be PL homeomorphic to $\mathbb{R}^n$.

**Proof.** Since $M$ is a manifold, it has an open subset homeomorphic to $\mathbb{R}^n$ and so all open subsets of $\mathbb{R}^n$ are homeomorphic to open subsets of $M$, so the first part of the proof of Theorem 5.2 goes through unchanged. Then we observe that the conclusion of Theorem 5.2 for the space $M$ only utilizes open subsets of $M$. □

We will use Theorem 5.1 below to prove Theorem 5.28, which is a Künneth theorem for intersection homology of spaces of the form $X \times M$, where $X$ is a filtered space and $M$ is an unfiltered manifold. The reader who is interested in seeing an immediate application of Theorem 5.1 could safely peek ahead to that theorem at this point.

We next provide a Mayer-Vietoris argument for CS sets. This theorem is a variation of a theorem of King [61, Theorem 10] that we have adapted a bit to suit the purposes for which we will need it.

**Theorem 5.3.** Let $\mathcal{F}_X$ be the category whose objects are (homeomorphic to) open subsets of a given CS set $X$ and whose morphisms are stratified homeomorphisms and inclusions. Let $\text{Ab}_*$ be the category of graded abelian groups. Let $F_*, G_* : \mathcal{F}_X \to \text{Ab}_*$ be functors, and let $\Phi : F_* \to G_*$ be a natural transformation such that $F_*, G_*, \Phi$ satisfy the conditions listed below.

1. $F_*$ and $G_*$ admit exact Mayer-Vietoris sequences (with respect to open subsets of $X$) and $\Phi$ induces a commutative diagram of these sequences,

2. if $\{U_\alpha\}$ is an increasing collection of open subspaces of $X$ and $\Phi : F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each $\alpha$, then $\Phi : F_*(\bigcup_\alpha U_\alpha) \to G_*(\bigcup_\alpha U_\alpha)$ is an isomorphism,

3. if $L$ is a compact filtered space such that $X$ has an open subset stratified homeomorphic to $\mathbb{R}^i \times cL$ and $\Phi : F_*(\mathbb{R}^i \times (cL - \{v\})) \to G_*(\mathbb{R}^i \times (cL - \{v\}))$ is an isomorphism (where $v$ is the cone vertex), then so is $\Phi : F_*(\mathbb{R}^i \times cL) \to G_*(\mathbb{R}^i \times cL), \quad 129$
4. if $U$ is empty or an open subset of $X$ contained within a single stratum and homeomorphic to Euclidean space\footnote{Note that it is possible for a CS set to have open Euclidean subsets of various dimensions; for example, let $X = S^2 \vee S^1$, filtered by $\{x_0\} \subset S^1 \subset X$, where $x_0$ is the basepoint of the wedge.}, then $\Phi : F_* (U) \to G_* (U)$ is an isomorphism.

Then $\Phi : F_* (X) \to G_* (X)$ is an isomorphism.

If $X$ is a PL CS set and $\mathcal{F}_X^{PL}$ is the category whose objects are (stratified PL homeomorphic to) open subsets of $X$ and whose morphisms are stratified PL homeomorphisms and inclusions, the theorem remains true if we replace the homeomorphisms in the conditions with PL homeomorphisms.

**Proof.** Suppose $X$ is a CS set. Let $\mathcal{M}$ be the union of the strata of $X$ with depth 0. Then we can think of $\mathcal{M}$ as a disjoint union of (not necessarily connected) manifolds $M^i$, one for each dimension $i$ such that $X$ has non-empty strata of dimension $i$ of depth 0. Each $M^i$ itself is a disjoint union of strata of dimension $i$. Then every point in $\mathcal{M}$ has a neighborhood homeomorphic to a Euclidean space. By assumption (4), for any such set neighborhood $U$, $\Phi : F_* (U) \to G_* (U)$ is an isomorphism. It follows now from Theorem 5.2 and assumptions (2) and (1) that $\Phi$ is an isomorphism on each $M^i$. But since $X$ must have finite formal dimension, property (1) and the $\emptyset$ hypothesis of property (4) imply that $\Phi$ is an isomorphism on all of $\mathcal{M}$, or in fact any open subspace of $\mathcal{M}$.

The proof for arbitrary open $Y \subset X$ will now proceed by induction on the depth of $Y$; recall Definition 2.28. We have just established the theorem for all $Y$ of depth 0, so we assume that we have verified the theorem for open subsets of $X$ of depth $< K$ for some $K > 0$. We must show that this implies the theorem for $Y$ of depth $K$. The proof will then be completed by induction up through the depth of $X$. For the remainder of the proof, let $Y$ be an open subspace of $X$ of depth $K$.

As in the proof of Theorem 5.1, the condition on unions of chains of subspaces allows us to conclude by Zorn’s Lemma that there is a largest open subset $W$ of $Y$ on which $\Phi$ is an isomorphism. Using the induction assumption, if $Y_{\min}$ is the union of minimal strata in the partial ordering on strata (i.e. the strata $S$ for which there does not exist a stratum $T$ with $T < S$) that have depth $> 0$, then $Y - Y_{\min}$ has depth less than $K$, so $\Phi$ is an isomorphism on $Y - Y_{\min}$. This implies $Y - Y_{\min} \subset W$, since if not there would be a point $y \in Y - Y_{\min}$, $y \notin W$. But then $y$ has an open neighborhood $U$ in $Y$ of depth $< K$ and $W \cap U$ then also has depth $< K$, so by the Mayer-Vietoris sequences and the Five Lemma, $\Phi$ would be an isomorphism on $W \cup U$, a contradiction.

Now we want to show that $W = Y$, again by a contradiction argument, assuming there is some $y \in Y_{\min}$, $y \notin W$. By the definition of a CS set, $y$ has an open neighborhood $N$ that is stratified homeomorphic to $\mathbb{R}^m \times cL$ if $y$ is contained in a stratum of dimension $m$; note that $L \neq \emptyset$, as we have assumed $y$ is in a stratum of depth $> 0$. Since $\mathbb{R}^m \times (cL - \{v\})$ has depth $< K$, $\Phi$ is an isomorphism on it by the assumption on depth. But then $\Phi$ is an isomorphism on $N$ by assumption (3). So if we can show that $\Phi$ is an isomorphism on $W \cap N$, then by the Mayer-Vietoris sequences and the Five Lemma, it will follow that $\Phi$ is an isomorphism on $W \cup N$, contradicting the maximality of $W$. 

}\footnote{Note that it is possible for a CS set to have open Euclidean subsets of various dimensions; for example, let $X = S^2 \vee S^1$, filtered by $\{x_0\} \subset S^1 \subset X$, where $x_0$ is the basepoint of the wedge.}
So we consider $W \cap N$. Let $V = Y_{\min} \cap W \cap N$. Since $W$ includes all of $Y - Y_{\min}$, $W \cap N$ is homeomorphic to the disjoint union of $\mathbb{R}^m \times (cL - \{v\})$ and $V$. We can also describe $W \cap N$ as the (not disjoint) union of $\mathbb{R}^m \times (cL - \{v\})$ with $V \times cL$. Both $\mathbb{R}^m \times (cL - \{v\})$ and the intersection

$$(\mathbb{R}^m \times (cL - \{v\})) \cap (V \times cL) \cong V \times (cL - \{v\})$$

are open subsets of $Y - Y_{\min}$ and so have depth $< K$; thus $\Phi$ is an isomorphism on these sets by the induction hypothesis. So to use the Mayer-Vietoris sequences and the Five Lemma to show that $\Phi$ is an isomorphism on $W \cap N$, we only need to show that $\Phi$ is an isomorphism on $V \times cL$. For this we will use Theorem 5.2 with the manifold in the statement of the theorem being $V$ and, for open $U \subset V$, the functors will be $F_*(U) = F_*(U \times cL)$ and $G_*(U) = G_*(U \times cL)$ and $\hat{\Phi}$ will be $\Phi$ restricted to sets of this form after identifying them with their homeomorphic images in $X$. The second and third hypotheses of Theorem 5.2 follow immediately from the corresponding statements for $\Phi$, and the first hypothesis is satisfied since we have already seen in the preceding paragraph that $\Phi$ must be an isomorphism on any $U \times cL$ with $U \subset Y_{\min}$ homeomorphic to $\mathbb{R}^m$. Theorem 5.2 then provides that $\hat{\Phi}$ is an isomorphism on all of $V$, which is equivalent to the statement that $\Phi : F_*(V \times cL) \to G_*(V \times cL)$ is an isomorphism.

The proof in the PL case is identical.

Remark 5.4. Theorem 5.3 implies, and in some ways generalizes, [61, Theorem 10]. Let us explain the relation.

The first main difference is that King assumes his functors are defined on a category of all filtered spaces whose maps are open inclusions and inclusions $0 \times X \to \mathbb{R}^k \times X$. His conclusion then hold for all CS sets. By contrast, our theorem essentially proves the theorem one CS set at a time. An immediate benefit is that we do not need to make sure our functors are defined for all filtered sets but only on subsets of the particular CS set $X$. This does not necessarily weaken the conclusion since the arbitrariness of the $X$ in the statement of Theorem 5.3 allows for the possibility that we might draw conclusions for all CS sets, provided $F_*$, $G_*$, and $\Phi$ possess the hypothesized properties in this generality. However, this version of the theorem is in some sense more flexible, since if one can produce $F_*$, $G_*$, and $\Phi$ that satisfy the hypothesized conditions on some particular class of CS set possessing a property that is preserved by taking open subsets (e.g. PL CS sets, pseudomanifolds, oriented CS sets, or locally torsion free CS sets (see Definition 5.37)), then one can use Theorem 5.3 to draw conclusions just for the spaces in this class.

The second main difference is that in place of condition (3), King has the conditions

- the inclusion $0 \times X \hookrightarrow \mathbb{R}^k \times X$ induces isomorphisms on $F_*$ and $G_*$,
- if $L$ is a compact filtered space and $\Phi$ is an isomorphism on $L$, then $\Phi$ is an isomorphism on $cL$,
- if $\Phi$ is an isomorphism on the filtered space $L$, then $\Phi$ is an isomorphism on $M \times L$ for a manifold $M$.

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The first two of these properties can be used to imply our hypothesis (3), though not necessarily vice versa. The last property here would be used in King’s version of the theorem to provide the last isomorphism of the proof of the Theorem 5.3, but here we instead use (3), induction, and an appeal to Theorem 5.2.

The following lemma will be a useful way, in practice, to conclude that condition (2) of Theorem 5.3 holds.

Lemma 5.5. Let $F_X$ be the category whose objects are (homeomorphic to) open subsets of a given CS set $X$ and whose morphisms are stratified homeomorphisms and inclusions. Let $\text{Ab}_s$ be the category of graded abelian groups. Let $F_s, G_s : F_X \to \text{Ab}_s$ be functors, and let $\Phi : F_s \to G_s$ be a natural transformation. Suppose that if $\{U_\alpha\}$ is an increasing collection of open subspaces of $X$ then the natural maps $\lim_{\to \alpha} F_s(U_\alpha) \to F_s(\bigcup_\alpha U_\alpha)$ and $\lim_{\to \alpha} G_s(U_\alpha) \to G_s(\bigcup_\alpha U_\alpha)$ are isomorphisms. Then if $\Phi : F_s(U_\alpha) \to G_s(U_\alpha)$ is an isomorphism for each $\alpha$, the map $\Phi : F_s(\bigcup_\alpha U_\alpha) \to G_s(\bigcup_\alpha U_\alpha)$ is an isomorphism.

Proof. The naturality of $\Phi$ implies that we have a commutative diagram

$$
\begin{array}{cccc}
\lim_{\to \alpha} F_s(U_\alpha) & \longrightarrow & F_s(\bigcup_\alpha U_\alpha) \\
\downarrow \Phi & & \downarrow \Phi \\
\lim_{\to \alpha} G_s(U_\alpha) & \longrightarrow & G_s(\bigcup_\alpha U_\alpha).
\end{array}
$$

By assumption, the horizontal maps are isomorphism, and the left vertical map is an isomorphism since we have assumed each $\Phi : F_s(U_\alpha) \to G_s(U_\alpha)$ is. Hence the righthand vertical arrow is also an isomorphism.

To accompany this lemma, it is worth adding another lemma:

Lemma 5.6. If $X$ is a filtered space with perversity $\bar{p}$ and $\{U_\alpha\}$ is an increasing collection of open subspaces of $X$ then the natural map $f : \lim_{\to \alpha} I^\bar{p}H^*_{GM}(U_\alpha) \to I^\bar{p}H^*_{GM}(\bigcup_\alpha U_\alpha)$ is an isomorphism.

Proof. This lemma is well-known for ordinary homology and the proof for intersection homology is identical: if $[\xi] \in I^\bar{p}H^*_{GM}(\bigcup_\alpha U_\alpha)$, then $[\xi]$ is represent by some specific cycle $\xi$, which has compact support. Hence $\xi$ is contained in $U_k$ for some $k$. It follows that the image of the element of $I^\bar{p}H^*_{GM}(U_k)$ represented by $\xi$ under the natural maps $I^\bar{p}H^*_{GM}(U_k) \to \lim_{\to \alpha} I^\bar{p}H^*_{GM}(U_\alpha) \to I^\bar{p}H^*_{GM}(\bigcup_\alpha U_\alpha)$ represents $[\xi]$, so $f$ is surjective.

Similarly, if $[\xi] \in \lim_{\to \alpha} I^\bar{p}H^*_{GM}(U_\alpha)$ and $f([\xi]) = 0$, then $\xi$ is represented by a cycle $\xi$ contained in some $U_\lambda$, and $\xi$ bounds some $\Xi$ in $I^\bar{p}S^*_{GM}(\bigcup_\alpha U_\alpha)$. But $\Xi$ must also have compact support and so is contained in $U_\ell$ for some $\ell \geq k$. But it then follows that $[\xi] = 0 \in \lim_{\to \alpha} I^\bar{p}H^*_{GM}(U_\alpha)$. 

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A sample application. As a first application of Mayer-Vietoris arguments on CS sets, we prove the following proposition:

**Proposition 5.7.** Let $X$ be a CS set and $\bar{p}$ a perversity such that

1. every point has a neighborhood stratified homeomorphic to $\mathbb{R}^k \times cL$ such that $I^{\bar{p}}H^0_{GM}(cL) \cong \mathbb{Z}$ and $I^{\bar{p}}H_i^{GM}(cL) = 0$ for $i > 0$, and

2. the only strata of depth 0 are regular strata.

Then $I^{\bar{p}}H^*_s^{GM}(X) \cong H_*(X)$, and similarly for PL intersection homology.

Before proving the proposition, we observe that the conditions required are not as extraordinary as they might at first seem. In fact, we have the following corollary, the first part of which was first demonstrated for PL stratified pseudomanifolds in [42].

**Corollary 5.8.** The conditions of Lemma 5.7 hold if $X$ is a normal stratified pseudomanifold and $\bar{p}$ is the top perversity $\bar{t}$ such that $\bar{t}(S) = \text{codim}(S) - 2$ for each singular stratum.

Furthermore, the conditions continue to hold for non-normal stratified pseudomanifolds if $\bar{p}(S) > \bar{t}(S)$ for any stratum containing a point that has a link that is not connected.

**Proof.** The condition that the only depth 0 strata are regular strata holds for all stratified pseudomanifolds by the definition, Definition 2.45, which requires that the union of the regular strata of $X$ be dense in $X$.

For the first condition, assuming $X$ is normal, we utilize the cone formula, Theorem 4.12. If $X$ has dimension $n$ and $x$ is in a stratum of dimension $i$ (hence codimension $n - i$), then $L$ has dimension $n - i - 1$, so $I^\bar{t}H^*_s^{GM}(\mathbb{R}^i \times cL) \cong I^\bar{t}H^*_{GM}(cL)$ is trivial except when

$$* < n - i - \bar{t}(S) - 1 = n - i - (\text{codim}(S) - 2) - 1 = n - i - (n - i) + 1 = 1.$$ 

Therefore, the only non-trivial group can be $I^\bar{t}H^0_{GM}(cL) \cong I^\bar{t}H^0_{GM}(L)$. As follows from Example 3.37, $I^\bar{t}H^0_{GM}(L) \cong \mathbb{Z}^m$, where $m$ is the number of regular strata of $L$. But by Lemma 2.69, $L$ is itself a normal stratified pseudomanifolds, and since it is connected, it has only one regular stratum, again by Lemma 2.69. Therefore, $m = 1$.

Finally, if there are strata for which the links are not connected, we need only observe that

- if $\bar{p}(S) > \bar{t}(S)$, then $0 \geq n - i - \bar{p}(S) - 1$, and
- $I^{\bar{p}}H^0_{GM}(L) \neq 0$, as $L$ is a stratified pseudomanifold by Lemma 2.54 and so possesses regular strata (and therefore allowable 0-simplices).

Thus again $I^\bar{t}H^0_{GM}(L) \cong \mathbb{Z}$ by Theorem 4.12. 

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56 Recall from Definition 2.67 that this means that the link of any point is connected.
Proof of Proposition 5.7] We apply Theorem 5.3 with $F_*(U) = I^pH_*^{GM}(X)$, $G_*(U) = H_*(U)$, and $\Phi$ induced by the inclusion $I^pS_*^{GM}(U) \to S_*^{GM}(U)$; the proof in the PL case is equivalent. The fulfillment of the first conditions of the theorem is evident. Also, since the only strata of depth 0 are regular strata, the only Euclidean open sets of $X$ must all be subsets of regular strata, and so also the last condition is fulfilled tautologically, as all simplices are allowable by Lemma 5.34. For the second condition, we will invoke Lemmas 5.6 and 5.5 noting that the version of Lemma 5.6 for ordinary homology is standard (the proof is identical).

For the third condition, consider for a distinguished neighborhood the map $\Phi : I^pH_*^{GM}(\mathbb{R}^i \times cL) \to H_*(\mathbb{R}^i \times cL)$ induced by inclusion, under the assumption that $I^pH_*^{GM}(\mathbb{R}^i \times (cL - \{v\})) \to H_*(\mathbb{R}^i \times (cL - \{v\}))$ is an isomorphism. We wish to show that $I^pH_*^{GM}(\mathbb{R}^i \times cL) \to H_*(\mathbb{R}^i \times cL)$ is an isomorphism; in fact, we will not even need the assumption. By homotopy invariance of homology, $H_*(\mathbb{R}^i \times cL)$ is trivial except for $H_0(\mathbb{R}^i \times cL) \cong \mathbb{Z}$. By stratified homology invariance $I^pH_*^{GM}(\mathbb{R}^i \times cL) \cong I^pH_*^{GM}(cL)$, and by assumption, $I^pH_0^{GM}(cL) \cong \mathbb{Z}$ and $I^pH_i^{GM}(cL) = 0$ for $i > 0$. So abstractly $I^pH_0^{GM}(\mathbb{R}^i \times cL) \cong \mathbb{Z} \cong H_0(\mathbb{R}^i \times cL)$, but it is now not difficult to observe that a generator for the former group, given by some 0-simplex, maps to a generator of the latter group. Technically, this is not quite enough yet to finish the proof as Theorem 5.3 requires the third condition to hold for all neighborhoods of the form $\mathbb{R}^k \times cL$, while we have only assumed here that each point of $X$ has one distinguished neighborhood satisfying the hypothesis. However, there are two solutions to this problem: One is to invoke Lemma 5.40 to be proven below, which states that every distinguished neighborhood of a point in a CS set has the same perversity $\bar{p}$ intersection homology. The other is to observe that, in fact, the proof of Theorem 5.3 only requires its condition (3) to hold for some distinguished neighborhood of each point, so in fact this hypothesis of the theorem could be weakened.

Remark 5.9. Corollary 5.8 is not true for a general CS set with perversity $\bar{t}$, even for ones for which every points has a connected link. For example, let $X = S^2 \amalg S^1$, the disjoint union filtered by $S^1 \subset X$. Then by the computations of Example 3.37 no allowable 0 or 1 simplices may intersect $S^1$. It follows that we must have $I^tH_*^{GM}(X) = 0$, while of course $H_1(X) \cong \mathbb{Z}$.

The normality condition is also critical, assuming perversity $\bar{t}$. For example the cone $X$ on the disjoint union $S^2 \vee S^2$ is a stratified pseudomanifold (without codimension one strata, even). But $H_0(X) \cong \mathbb{Z}$, while $I^tH_0^{GM}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

### 5.2 Cross products

In this section, we develop cross products for singular intersection chains. This will lead in the next section, Section 5.3, to a Künneth theorem that holds for products $M \times X$, where $X$ is a filtered space, $M$ is an $n$-manifold with the trivial filtration, and $M \times X$ is given the product filtration, i.e. $(M \times X)^i = M \times X^{i-n}$. We will show that the perversity $\bar{p}$ intersection homology of $M \times X$ is related by a Künneth formula to the intersection homology of $X$ with the corresponding perversity and the ordinary homology of $M$. The precise statement is in Theorem 5.28. Later, in Section 6.56, we will consider another, more general, Künneth
For ordinary homology theory, the key morphism for topological Künneth theorems is the Eilenberg-Zilber cross product map

$$\varepsilon : S_\ast (X) \otimes S_\ast (Y) \to S_\ast (X \times Y)$$

(or its homotopy inverse). In modern algebraic topology texts, the preference seems to be to construct this map abstractly using the method of acyclic models (see [77]). Unfortunately, this is insufficient for our purposes, as we have seen that even contractible spaces might have non-trivial intersection homology, depending on how they are filtered. Luckily, there do exist concrete versions of the cross product, sometimes called Eilenberg-Zilber “shuffle products”. Most sources (e.g. [65]) seem to prefer to describe this map from the point of view of simplicial sets. However, a statement purely from the point of view of singular homology can be found as an exercise in Dold [23, Exercise VI.12.26.2].

The basic idea is analogous to that of the prism construction used in homotopy arguments: if \( \sigma \in S_i(X) \) and \( \tau \in S_j(Y) \) are singular simplices, then together they provide a product map \( \sigma \times \tau : \Delta^i \times \Delta^j \to X \times Y \). If we have a suitable triangulation \( T \) of \( \Delta^i \times \Delta^j \) with an ordering of the vertices, if \( \{ \Delta_k^{i+j} \} \) is the collection of \( i+j \) simplices of \( T \), and if \( \delta_k : \Delta_k^{i+j} \to \Delta^i \times \Delta^j \) is the order-preserving inclusion map, then this yields an element of \( S_{i+j}(X \times Y) \) by

$$\varepsilon(\sigma \otimes \tau) = \sum \pm (\sigma \times \tau) \circ \delta_k.$$

Here the signs must be chosen appropriately, and of course the triangulations for various \( p \) and \( q \) must be chosen in such a way that there is enough compatibility across dimensions so that the construction extends to a chain map. The standard such set of triangulations is described in terms of shuffles.

Let \( p, q \) be non-negative integers. Then a \((p, q)\)-shuffle is a partition of the set \( \{1, 2, \ldots, p+q\} \) into two disjoint ordered sets \( \mu = \{ \mu_i \}_{i=1}^p \) and \( \nu = \{ \nu_j \}_{j=1}^q \) with \( \mu_i < \mu_{i+1} \) for each \( i \) and similarly for the \( \nu_j \). The idea is that this partition \((\mu, \nu)\) tells us how to shuffle together two ordered sets, of respective cardinalities \( p \) and \( q \), to form a new ordered set of cardinality \( p+q \): the elements of the first set occupy the spots corresponding to the slots labeled by the \( \mu \)s and the elements of the second set are placed in the spots corresponding to the \( \nu \)s. So, for example, if we have ordered sets \((A, B, C)\) and \((\alpha, \beta)\), and a \((3,2)\)-shuffle \(\{(2,3,5,\{1,4\}\right)\), then we can shuffle our sets by this prescription to get the ordered set \(\{\alpha, A, B, \beta, C\}\).

If \((\mu, \nu)\) is a \((p, q)\)-shuffle, we let \(\text{sgn}(\mu, \nu)\) denote the sign of the permutation from \((1, 2, \ldots, p+q)\) to \((\mu_1, \mu_2, \ldots, \mu_p, \nu_1, \nu_2, \ldots, \nu_q)\), i.e. \(1\) if the permutation is even and \(-1\) if the permutation is odd. Let \(\eta^\mu : \Delta^{p+q} \to \Delta^p\) take the vertex \(w_i \in \Delta^{p+q}\) to the vertex \(u_j \in \Delta^p\) if \(\mu_j \leq i < \mu_{j+1}\) (letting \(\mu_0 = 0\) and \(\mu_{p+1} = p+q+1\)). Similarly, there is an \(\eta^\nu : \Delta^{p+q} \to \Delta^q\) (we will label vertices of \(\Delta^q\) as \(v_k\)). The product map \(\eta_{\mu\nu} = (\eta^\mu, \eta^\nu) : \Delta^{p+q} \to \Delta^p \times \Delta^q\) is a linear embedding, and the sum over all \((p, q)\) shuffles \(\sum \text{sgn}(\mu, \nu)\eta_{\mu\nu}\) is a sum of singular simplices whose image we claim provide a singular triangulation \(T\), with the necessary orientations.

\[^{57}\text{We will not develop a cross product for PL intersection homology. However, we will see in Section 5.5 that singular and PL intersection homology are isomorphic on PL spaces, so a singular chain cross product will be sufficient for all our later purposes.}\]
To try to understand this triangulation better, let us see explicitly where the vertices \(\{w_i\}\) of a \(p+q\) simplex get mapped by \(\eta_{\mu\nu}\). Since \(v_0 = \mu_0 = 0\) by definition, \(w_0\) gets mapped \((u_0, v_0)\). Now, if \(1 \in \mu\), \(w_1\) gets mapped \((u_1, v_0)\) and if \(1 \in \nu\), \(w_1\) gets mapped \((u_0, v_1)\). In general, if \(w_i = w_{j+k}\) goes to \((u_j, v_k)\), then \(w_{i+1}\) will go to either \((u_{j+1}, v_k)\) or \((u_j, v_{k+1})\) depending respectively on whether \(i+1\) is in \(\mu\) or \(\nu\). This is the same principle we saw at work in constructing prisms for homotopy arguments, except those always have the form \(\Delta^1 \times \Delta^q\), so there are only two choices for vertices of \(\Delta^1\).

Another way to think of a \((p, q)\)-shuffle is to imagine a walk on a \(p \times q\) grid, where columns are labeled left to right by \(\{0, \ldots, p\}\) and the rows are labeled bottom to top by \(\{0, \ldots, q\}\). Then there is a bijection between \((p, q)\)-shuffles and walks along the grid from \((0, 0)\) to \((p, q)\) in which each step must move one unit either up or to the right: on the \(i\)th step, if \(i \in \mu\) we move to the right and if \(i \in \nu\) we move up (or, conversely, given such a path, if we move right on \(i\)th step then put \(i \in \mu\) and if we move up on the \(i\)th step, put \(i \in \nu\). Then the sequence of labels of the the points of the grid along our path is the sequence of vertices that the vertices of \(\Delta^{p+q}\) map to by \(\eta_{\mu\nu}\).

Proving that this construction does indeed yield a triangulation and that it leads to a chain map is not completely trivial, but it is also a bit of a diversion from our main development, so we include the details in an appendix to this section, below. For now we turn back to intersection homology considerations after recording the necessary conclusion that will be proven in the appendix.

**Proposition 5.10.** Suppose \(\sigma_1 \in S_p(X)\) and \(\sigma_2 \in S_q(Y)\). Then the sum over \((p, q)\) shuffles

\[
\epsilon(\sigma_1 \otimes \sigma_2) = \sum \text{sgn}(\mu, \nu)(\sigma_1 \times \sigma_2) \circ \eta_{\mu\nu}
\]

extends linearly to a chain map

\[
\varepsilon : S_*(X) \otimes S_*(Y) \to S_*(X \times Y).
\]

**Definition 5.11.** We refer to the map \(\varepsilon\) as the **cross product**. It will be useful to also employ the (somewhat abusive) notation \(\epsilon(x \otimes y) = x \times y\).)

We now discuss the cross product in intersection homology.

**Lemma 5.12.** If \(X\) and \(Y\) are filtered spaces, the cross product restricts to a map \(I^p S^G_*(X) \otimes I^q S^G_*(Y) \to I^{p+q} S^G_*(X \times Y)\) if \(Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')\) for all strata \(S \subset X\) and \(S' \subset Y\).

**Proof.** First recall that we know that \(I^p S^G_*(X)\) and \(I^q S^G_*(X)\) are free complexes; see Remark 3.40. Therefore \(I^p S^G_*(X) \otimes I^q S^G_*(Y) \subset S_*(X) \otimes S_*(Y)\), so it makes sense to restrict the cross product to a map \(I^p S^G_*(X) \otimes I^q S^G_*(Y) \to S_*(X \times Y)\). Our claim is that if \(Q\) satisfies the given hypotheses then the image lies in \(I^{p+q} S^G_*(X \times Y)\).

Suppose \(\sigma_1 \in S_i(X)\) and \(\sigma_2 \in S_j(Y)\) are respectively \(\bar{p}\) and \(\bar{q}\) allowable simplices, and consider \(\sigma_1 \times \sigma_2\). We want to show that each \(i + j\) simplex of the chain \(\sigma_1 \times \sigma_2\) is allowable (and then we will consider boundaries). Such a simplex corresponds to the composition \(\Delta^i_k \overset{\eta_{\mu\nu}}{\longrightarrow} \Delta^i \times \Delta^j \overset{\sigma_1 \times \sigma_2}{\longrightarrow} X \times Y\). Now, if \(S \subset X\) and \(S' \subset Y\) are strata, by the allowability
assumptions, $\sigma_1^{-1}(S) \subset \{i - \text{codim}(S) + \bar{p}(S)\}$ skeleton of $\Delta^i$ and $\sigma_2^{-1}(S') \subset \{j - \text{codim}(S') + \bar{q}(S')\}$ skeleton of $\Delta^j$. If we let $a = i - \text{codim}(S) + \bar{p}(S)$ and $b = j - \text{codim}(S') + \bar{q}(S')$, then $\sigma_1^{-1} \times \sigma_2^{-1}(S \times S')$ lies in $(\Delta^i)^a \times (\Delta^j)^b$, where $(\Delta^i)^a$ is the $a$-skeleton of $\Delta^i$ and similarly for $(\Delta^j)^b$. But the triangulation of $\Delta^i \times \Delta^j$ coming from the cross product construction triangulates $(\Delta^i)^a \times (\Delta^j)^b$ as a subcomplex which must have dimension $a + b$. Thus any $i + j$ simplex in our triangulation of $\Delta^i \times \Delta^j$ can intersect $(\Delta^i)^a \times (\Delta^j)^b$ in at most its $a + b$ skeleton. But this implies that $(\sigma_1 \times \sigma_2)^{-1}(S \times S')$ must lie in the $a + b$ skeleton of $\Delta^{i+j}$. But now
\[
a + b = i - \text{codim}(S) + \bar{p}(S) + j - \text{codim}(S') + \bar{q}(S') = i + j - \text{codim}(S \times S') + \bar{p}(S) + \bar{q}(S').\]

So $\sigma_1 \times \sigma_2$ is allowable with respect to any perversity $Q$ such that $Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')$.

This shows that the cross product of two allowable simplices is allowable with the given assumptions on perversities. Now, suppose $\xi_1, \xi_2$ are allowable chains in the respective spaces. Since $\xi_1 \otimes \xi_2$ can be written as a sum (with coefficients) of terms of the form $\sigma_1 \otimes \sigma_2$ in $S_\ast(X) \otimes S_\ast(Y)$ and since $\epsilon$ in general is defined as the linear extension of how it acts on tensor products of simplices, the above argument therefore implies that each singular simplex of $\xi_1 \times \xi_2$ is $Q$-allowable.

Next, consider that $\partial(\xi_1 \times \xi_2) = (\partial \xi_1) \otimes \xi_2 + (-1)^{\ell_1 \ell_2} \xi_1 \otimes (\partial \xi_2)$. Since $\partial \xi_1$ and $\partial \xi_2$ are allowable, we see that $(\partial \xi_1) \otimes \xi_2$ and $\xi_1 \otimes (\partial \xi_2)$ are each contained in $I^pS_\ast^{GM}(X) \otimes I^qS_\ast^{GM}(Y)$. Therefore, by the preceding argument, each singular simplex of $(\partial \xi_1) \times \xi_2$ is $Q$-allowable, and similarly for $\xi_1 \times (\partial \xi_2)$. But since $\epsilon$ is a chain map, $\partial(\xi_1 \times \xi_2) = (\partial \xi_1) \times \xi_2 + (-1)^{\ell_1 \ell_2} \xi_1 \times (\partial \xi_2)$, so $\partial(\xi_1 \times \xi_2)$ also consists of $Q$-allowable simplices. Therefore, we conclude that $\xi_1 \times \xi_2$ is a $Q$-allowable chain.

**Corollary 5.13.** Under the assumptions of Lemma 5.12, if also $A \subset X$ and $B \subset Y$, the cross product induces maps

\[
I^pS_\ast^{GM}(X, A) \otimes I^qS_\ast^{GM}(Y, B) \rightarrow \frac{I^qS_\ast^{GM}(X \times Y)}{I^qS_\ast^{GM}(X \times Y) + I^qS_\ast^{GM}(X \times B)} \rightarrow I^qS_\ast^{GM}(X \times Y, (A \times Y) \cup (X \times B)).
\]

**Proof.** By the lemma, the cross product takes $I^pS_\ast^{GM}(A) \otimes I^qS_\ast^{GM}(Y)$ to $I^qS_\ast^{GM}(A \times Y)$ and $I^pS_\ast^{GM}(X) \otimes I^qS_\ast^{GM}(B)$ to $I^qS_\ast^{GM}(X \times B)$. So by basic algebra and the multilinearity of the cross product, the image in $\frac{I^qS_\ast^{GM}(X \times Y)}{I^qS_\ast^{GM}(X \times Y) + I^qS_\ast^{GM}(X \times B)}$ of the cross product of generators of $I^pS_\ast^{GM}(X, A)$ and $I^qS_\ast^{GM}(Y, B)$ is independent of the choice of coset representative. The second map is just a quotient map that is well defined because clearly $I^qS_\ast^{GM}(A \times Y) + I^qS_\ast^{GM}(X \times B) \subset I^qS_\ast^{GM}((A \times Y) \cup (X \times B))$. Abusing notation, we will also use the symbol $\epsilon$ to refer to the map of the corollary.

---

[58] We will see this in the detailed construction of the cross product in the appendix to this section. See, in particular, Corollary 5.26.
Remark 5.14. If a space $Y$ is filtered trivially and $Y \subset B$, then there is only one perversity $\bar{q}$ on $Y$ and $I^qS_*^{GM}(Y,B) = S_*(Y,B)$. Also in this case, the requirements of the form $Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ become $Q(S \times Y) \geq \bar{p}(S)$. In particular, if we define $Q$ on $X \times Y$ as the product perversity with $Q(S \times Y) = \bar{p}(S)$ (and then abusively relabel $Q$ to $\bar{p}$), we obtain from Corollary 5.13 cross products of the form

$$I^pS_*^{GM}(X,A) \otimes S_*(Y,B) \xrightarrow{\epsilon} I^pS_*^{GM}(X \times Y, (A \times Y) \cup (X \times B))$$

$$S_*(Y,B) \otimes I^pS_*^{GM}(X,A) \xrightarrow{\epsilon} I^pS_*^{GM}(Y \times X, (B \times X) \cup (Y \times A))$$

Remark 5.15. As for ordinary homology, the chain cross product induces a product on homology

$$I^pH_*^{GM}(X,A) \otimes I^qH_*^{GM}(Y,B) \rightarrow I^QH_*^{GM}(X \times Y, (A \times Y) \cup (X \times B)).$$

This comes from basic algebra by noticing that if $x \in C_*$ and $y \in D_*$ are cycles in chain complexes, then $x \otimes y$ is a cycle in $C_* \otimes D_*$, while altering $x$ and $y$ in their homology classes does not alter the homology class of $x \otimes y$. In particular, $\partial z \otimes y = \partial (z \otimes y)$ if $y$ is a cycles, and $x \otimes \partial z = (-1)^{|z|}\partial (x \otimes z)$ if $x$ is a cycle. Therefore an element of $H_*(C_*) \otimes H_*(D_*)$ yields a well-defined element of $H_*(C_* \otimes D_*)$.

### 5.2.1 Properties of the cross product

In this section, we will develop some of the basic properties of the cross product $I^pS_*^{GM}(X,A) \otimes I^qS_*^{GM}(Y,B) \rightarrow I^QS_*^{GM}(X \times Y, (A \times Y) \cup (X \times B))$. As

**Lemma 5.16** (Naturality). Let $(X,A), (Y,B), (X',A')$ and $(Y',B')$ be pairs of filtered spaces and subsets. Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be maps with $f(A) \subset C$ and $f(B) \subset (D)$. Suppose $\bar{p}, \bar{q}, \bar{p}', \bar{q}'$ are respective perversities on $X,Y,X',Y'$ and that $P$ and $Q$ are respective perversities on $X \times Y$ and $X' \times Y'$ such that $P(S \times T) \geq \bar{p}(S) + \bar{q}(T)$ for all strata $S \subset X$ and $T \subset Y$ and $Q(S' \times T') \geq \bar{p}'(S') + \bar{q}'(T')$ for all strata $S' \subset X'$ and $T' \subset Y'$. Finally, suppose that $f$ is $(\bar{p}, \bar{p}')$-stratified, $g$ is $(\bar{q}, \bar{q}')$-stratified, and $f \times g$ is $(P,Q)$-stratified. Then the following diagram commutes:

$$\begin{array}{ccc}
I^pS_*^{GM}(X \times Y, (A \times Y) \cup (X \times B)) & \xrightarrow{\epsilon} & I^pS_*^{GM}(X,A) \otimes I^qS_*^{GM}(Y,B) \\
\downarrow f \times g & & \downarrow f \otimes g \\
I^QS_*^{GM}(X' \times Y', (A' \times Y') \cup (X' \times B')) & \xleftarrow{\epsilon} & I^qS_*^{GM}(X',A') \otimes I^qS_*^{GM}(Y',B').
\end{array}$$

In other words, if $x \in I^pS_*^{GM}(X,A)$ and $y \in I^qS_*^{GM}(Y,B)$, then $f(x) \times g(y) = (f \times g)(x \times y)$.

**Remark 5.17.** Notice that we are abusing notation in the above diagram by allowing the one symbol $\epsilon$ to stand for all relevant cross product maps. Unfortunately, this will become somewhat of a habit (one that we are not alone in pursuing). Using additional notation to distinguish the various $\epsilon$ maps would become cumbersome and confusing, while the possibility for confusion without extra decorations should be alleviated by context clues.
Remark 5.18. Notice that it is possible Lemma 5.16 to have $X = X'$ and $Y = Y'$ with the maps being identity maps. In this case, the lemma becomes a statement about naturality with respect to change of perversity.

Proof. The hypotheses of the lemma guarantee that all the maps in the diagram are well-defined. For commutativity, as all the maps involved are restrictions of the corresponding chain maps on complexes involving ordinary singular chains, it suffices to verify commutativity of the diagram for ordinary singular chains. But $S_*(X, A) \otimes S_*(Y, B)$ is generated by elements represented by tensor products of singular simplices of the form $\sigma \otimes \tau$ with $\sigma : \Delta^i \to X$ and $\tau : \Delta^j \to Y$ for some $i, j$. But acting on $\sigma \otimes \tau$, the image of the maps left then down is represented by applying $(f \times g)(\sigma \times \tau) = (f\sigma) \times (g\tau)$ to the singular triangulation of $\Delta^i \times \Delta^j$ coming from the Eilenberg-Zilber shuffle product, while the map down then left similarly applies $(f\sigma) \times (f\tau)$ to the same singular triangulation.

Lemma 5.19 (Associativity). Suppose $X$, $Y$, and $Z$ are filtered spaces with respective perversities $\bar{p}$, $\bar{q}$, and $\bar{r}$ and subspaces $A \subset X$, $B \subset Y$, $Z \subset C$. Suppose, furthermore, that

- $P$ is a perversity on $X \times Y$ such that $P(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ for all strata $S \subset X$ and $S' \subset Y$,
- $Q$ is a perversity on $Y \times Z$ such that $Q(S' \times S'') \geq \bar{q}(S') + \bar{r}(S'')$ for all strata $S' \subset Y$ and $S'' \subset Z$,
- $R$ is a perversity on $X \times Y \times Z$ such that $R(S \times S' \times S'') \geq \bar{p}(S) + Q(S' \times S'')$ for all strata $S \subset X$, $S' \subset Y$, and $S'' \subset Z$.

Then the following diagram commutes

$$
\begin{align*}
I^P S_*^{GM}(X, A) \otimes I^P S_*^{GM}(Y, B) \otimes I^P S_*^{GM}(Z, C) & \xrightarrow{\epsilon \otimes \text{id}} I^P S_*^{GM}(X, (A \times Y) \cup (X \times B)) \otimes I^P S_*^{GM}(Z, C) \\
\text{id} \otimes \epsilon & \xrightarrow{\epsilon} I^P S_*^{GM}(X \times Y \times Z, (A \times Y \times Z) \cup (X \times B \times Z) \cup (X \times Y \times C))
\end{align*}
$$

In other words, if $x \in I^P S_*^{GM}(X, A)$, $y \in I^P S_*^{GM}(Y, B)$, $z \in I^P S_*^{GM}(Z, C)$, then $(x \times y) \times z = x \times (y \times z)$.

Proof. We first observe that all of the maps of the diagram are well-defined by Lemma 5.12 and Corollary 5.13, noting that

$$[A \times (Y \times Z)] \cup [X \times ((B \times Z) \cup (Y \times C))] = (A \times Y \times Z) \cup (X \times B \times Z) \cup (X \times Y \times C)$$

and analogously for the terms on the right.

To verify the commutativity, we recall that each group appearing in the diagram is a subgroup of the corresponding ordinary singular chain group, so it suffices to verify commutativity in that setting. There, $S_*(X, A) \otimes S_*(Y, B) \otimes S_*(Z, C)$ is generated by elements of
the form \( \sigma \otimes \tau \otimes \eta \), where \( \sigma : \Delta^p \to X, \tau : \Delta^q \to Y \), and \( \eta : \Delta^r \to Z \) are singular simplices. So the lemma reduces to showing that \((\sigma \times \tau) \times \eta = \sigma \times (\tau \times \eta)\).

Tracing through the definitions, \((\sigma \times \tau) \times \eta : \Delta^p \times \Delta^q \times \Delta^r \to X \times Y \to Z\) is the singular chain that is obtained by applying the chain map corresponding to the spatial map \(\sigma \times \tau \times \eta\) to a singular triangulation of \(\Delta^p \times \Delta^q \times \Delta^r\) corresponding to first applying the Eilenberg-Zilber shuffle triangulation to \(\Delta^p \times \Delta^q\) and then applying the Eilenberg-Zilber shuffle triangulation to each \(\delta^{p+q} \times \Delta^r\), where \(\delta^{p+q}\) is one of the resulting \(p+q\) simplices of the triangulation of \(\Delta^p \times \Delta^q\). Similarly, \(\sigma \times (\tau \times \eta)\) applies the chain map corresponding to the spatial map \(\sigma \times \tau \times \eta\) to the singular triangulation of \(\Delta^p \times \Delta^q \times \Delta^r\) that comes by applying the Eilenberg-Zilber process first to \(\Delta^q \times \Delta^r\). Therefore, it suffices to verify that it does not matter what in what order we perform the iterated Eilenberg-Zilber shuffle processes.

From the work below in Section 5.2.2, we know that the Eilenberg-Zilber shuffle process does indeed yield singular triangulations of chains and that the orientations work out properly, so we come down to verifying only that both iterative procedures for triangulating \(\Delta^p \times \Delta^q \times \Delta^r\) result in the same triangulation, which we can check by looking at the \(p+q+r\) simplices of this triangulation. But consider our earlier descriptions of this triangulation process, in which we provide the vertices of the simplices of the triangulation. Looking at this prior description, we know that if we first triangulate \(\Delta^p \times \Delta^q\), each \(p+q\) simplex will have its vertices determined by a \((p,q)\)-shuffle. We have noted that there is a bijection between shuffles and walks in a \(p \times q\) grid consisting of steps up and steps right and with the labels of the grid points determining the vertices of a \(p+q\) simplex in \(\Delta^p \times \Delta^q\). Now, the triangulation of \(\Delta^p \times \Delta^q \times \Delta^r\) comes by taking each of these \(p+q\) simplices \(\delta\) of \(\Delta^p \times \Delta^q\) and triangulating \(\delta \times \Delta^r\) using \((p+q,r)\)-shuffles. If we have fixed a \((p,q)\)-shuffle and think of it as a path in a \(p \times q\) grid, then the resulting \((p+q,r)\)-shuffles correspond to walks in a \(p \times q \times r\) grid where each step is either a horizontal step dictated by the fixed \((p,q)\)-shuffle or a vertical step. The corresponding triangulation of \(\delta \times \Delta^r\) comes by considering the \(p+q+r\) simplices with vertices labeled by the grid coordinates. As we work through all \((p,q)\)-shuffles and all corresponding \((p+q,r)\)-shuffles, we see that the collection of all \(p+q+r\) simplices of the triangulation of \(\Delta^p \times \Delta^q \times \Delta^r\) correspond to the collection of all paths on a \(p \times q \times r\) grid in which each step increases a single coordinate by 1. An analogous argument beginning with \((q,r)\)-shuffles and then, for each such shuffle, considering \((p,q+r)\)-shuffles results in the same symmetrical description of what we should call \((p,q,r)\)-shuffles. So we see that both procedures yield the same triangulation of \(\Delta^p \times \Delta^q \times \Delta^r\).

\(\square\)

**Lemma 5.20 (Commutativity).** Suppose \(X\) and \(Y\) are filtered spaces with respective perversities \(\bar{p}\) and \(\bar{q}\) and subspaces \(A \subset X\) and \(B \subset Y\). Suppose, furthermore, that \(P\) is a perversity on \(X \times Y\) such that \(P(S \times S') \geq \bar{p}(S) + \bar{q}(S')\) for all strata \(S \subset X\) and \(S' \subset Y\) and that \(Q\) is
a perversity on \( Y \times X \) with \( Q(S' \times S) = P(S \times S') \). Then the following diagram commutes

\[
\begin{array}{ccc}
I^pS^G_{s}(X,A) \otimes I^qS^G_{s}(Y,B) & \xrightarrow{\epsilon} & I^pS^G_{s}(X \times Y, (A \times Y) \cup (X \times B)) \\
\tau & & t \\
I^qS^G_{s}(Y,B) \otimes I^pS^G_{s}(X,A) & \xrightarrow{\epsilon} & I^QS^G_{s}(Y \times X, (X \times B) \cup (A \times Y)),
\end{array}
\]

where \( \tau \) is the standard (signed!) interchange map of tensor product factors and \( t \) is induced by the topological map \( t : X \times Y \to Y \times X \) given by \( t(x,y) = (y,x) \). In other words, \( t(x \times y) = (-1)^{|x||y|} y \times x \).

**Proof.** Notice that \( \tau \) is a chain map: if \( x \otimes y \) is a generator of the tensor product, then we have

\[
\tau \partial(x \otimes y) = \tau((\partial x) \otimes y + (-1)^{|x|}x \otimes \partial y)
\]

\[
= (-1)^{|(|x| - 1)||y|| \partial x + (-1)^{|x| + |y| - 1}(|y| - 1)}(\partial y) \otimes x
\]

\[
= (-1)^{|(|x| - 1)||y|| \partial x + (-1)^{|x| + |y| - 1}}(\partial y) \otimes x
\]

\[
= (-1)^{|x| |y|}(\partial y) \otimes x + (-1)^{|x| + |y|}y \otimes \partial x
\]

\[
= (-1)^{|x| |y|}(\partial y) \otimes x + (-1)^{|y|}y \otimes \partial x
\]

\[
= (-1)^{|x| |y|}\partial(y \otimes x)
\]

\[
= \partial(\tau(x \otimes y)).
\]

It is also clear that \( t \) takes \( P \)-allowable chains to \( Q \) allowable chains and so induces a well-defined chain map.

Once again, all of the maps of the diagram are well-defined by Lemma 5.12 and Corollary 5.13 and, as in the proof of Lemma 5.19, it suffices to verify commutativity for a tensor product of ordinary singular simplices \( \sigma \otimes \eta \) with \( \sigma : \Delta^p \to X \) and \( \eta : \Delta^q \to Y \).

We know that \( \epsilon \) acts on such a representative by applying \( \sigma \times \eta \) to the Eilenberg-Zilber shuffle triangulation of \( \Delta^p \times \Delta^q \) and that the map on intersection chains is a linear extension of this. So, mapping right then down from the top left corresponds to applying \( t \circ (\sigma \times \eta) \) to a triangulation of \( \Delta^p \times \Delta^q \). In particular, if \((x,y) \in \Delta^p \times \Delta^q \), its image under \( t \circ (\sigma \times \eta) \) is \((\eta(y), \sigma(x))\). On the other hand, the map \( \tau \) takes \( \sigma \otimes \eta \) to \((-1)^{|\eta|} \sigma \otimes \eta \), and \( \epsilon(\eta \otimes \sigma) \) takes points \((y, x) \in \Delta^q \times \Delta^p \) to \((\eta(y), \sigma(x))\). So, for the moment ignoring triangulations, we see that the two maps from \( \Delta^p \times \Delta^q \) to \( Y \times X \) obtained by going either way around the square agree. Now, consider the triangulations of \( \Delta^p \times \Delta^q \) and \( \Delta^q \times \Delta^p \) arising from the Eilenberg-Zilber shuffle construction. Notice that corresponding to every \((p,q)\)-shuffle, each of which determines a simplex of the subdivision of \( \Delta^p \times \Delta^q \), there is a corresponding \((q,p)\)-shuffle corresponding to a simplex in the triangulation of \( \Delta^q \times \Delta^p \). In fact, if we identify \((p,q)\)-shuffles as walks with only steps up or right on a \( p \times q \) grid, then the corresponding \((q,p)\)-shuffle is the walk on the the \( q \times p \) grid obtained by flipping the \( p \times q \) grip along its southwest to northeast axis. The map \( t \) then takes collections of vertices of \( \Delta^p \times \Delta^q \) given
by a \((p, q)\)-shuffle to the corresponding vertices of \(\Delta^q \times \Delta^p\) arising from the corresponding \((q, p)\)-shuffle. As \(t\) must be a linear map, we see that \(t\) takes our \((p, q)\)-shuffle triangulation of \(\Delta^p \times \Delta^q\) to our \((q, p)\)-shuffle triangulation of \(\Delta^q \times \Delta^p\).

We now see that the two routes around the diagram yield corresponding collections of singular simplices, and it only remains to consider orientations. The singular triangulation of \(\Delta^p \times \Delta^q\) obtained by the Eilenberg-Zilber shuffle process is constructed to conform to the natural orientation of \(\Delta^p \times \Delta^q\). Similarly, the triangulation of \(\Delta^q \times \Delta^p\) conforms to the natural orientation of \(\Delta^q \times \Delta^p\). Identifying the two spaces via \(t\), their orientations differ by a sign of \((-1)^{pq}\), owing to the interchange of factors, but this extra sign is precisely canceled by the corresponding sign implicit in \(\tau\). Therefore, the two singular chains obtained by chasing around the diagram in the two different ways agree.

Lemma 5.21 (Unitality). Suppose \(X\) is a filtered space with perversity \(\bar{p}\) and subspace \(A \subset X\). Let \(\sigma_0 : \Delta^0 \to \text{pt}\) be the unique singular 0 simplex in \(S_0(\text{pt})\). Then if \(\xi \in I^{\bar{p}} S^{GM}_i(X, A)\), we have

\[\sigma_0 \times \xi = \xi \times \sigma_0 = \xi \in I^{\bar{p}} S^{GM}_i(\text{pt} \times X, \text{pt} \times A) = I^{\bar{p}} S^{GM}_i(X \times \text{pt}, A \times \text{pt}) = I^{\bar{p}} S^{GM}_i(X, A)\]

Proof. This follows immediately from the definitions, noting that the Eilenberg-Zilber triangulation of \(\Delta^p \times \Delta^0 = \Delta^0 \times \Delta^p = \Delta^p\) is the obvious one.

Remark 5.22. The next property of cross products involves the boundary map \(\partial_*\) of long exact homology sequences. As this map lowers degree by one, we will treat it as a degree \(-1\) map for the purposes of sign conventions. So, for example, if \(\xi \in H_i(X)\) and \(\eta \in H_j(Y, B)\), for appropriate spaces, then \((\text{id} \otimes \partial_*)(\xi \otimes \eta) = (-1)^i \xi \otimes \partial_*(\eta) \in H_i(X) \otimes H_{j-1}(B)\). We note that this convention is not always followed in the literature; compare, for example, the following results with Statements VI.2.11, VI.2.12, and VI.2.13 in [23].

Lemma 5.23 (Stability). Suppose \(X\) and \(Y\) are filtered spaces with respective perversities \(\bar{p}\) and \(\bar{q}\) and subspaces \(A \subset X\) and \(B \subset Y\). Suppose that \(Q\) is a perversity on \(X \times Y\) such that \(Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')\) for all strata \(S \subset X\) and \(S' \subset Y\). Then the following diagram commutes:
Here, the unlabeled maps of the diagram are induced by the inclusions \(((A \times Y) \cup (X \times B), \emptyset) \rightarrow ((A \times Y) \cup (X \times B), X \times B)\) and \((A \times Y, A \times B) \rightarrow ((A \times Y) \cup (X \times B), X \times B)\).

In other words, if \(\xi \in I^\partial H^G_i(X, A)\) and \(\eta \in I^\partial H^G_j(Y, B)\), then
\[
(\partial_* \xi) \times \eta = \partial_* (\xi \times \eta) \in I^Q H^G_{i+j-1}((A \times Y) \cup (X \times B), X \times B).
\]

Analogously, and via a similar diagram,
\[
\xi \times \partial_* \eta = (-1)^j \partial_* (\xi \times \eta) \in I^Q H^G_{i+j-1}((A \times Y) \cup (X \times B), A \times Y).
\]

**Proof.** Recall that, via the standard “zig-zag” construction of the boundary map of the long exact sequence of a pair, if \(\xi\) is a chain representing an element of \(I^\partial H^G_i(X, A)\), then the image of this element in \(I^\partial H^G_{i-1}(A)\) under the map \(\partial_*\) is represented by \(\partial \xi\), and similarly for any other \(\partial_*\) map. Now, \(I^\partial H^G_i(X, A) \otimes I^\partial H^G_j(Y, B)\) is generated by elements of the form \(\xi \otimes \eta\) with \(\xi\) a chain representing an element of \(I^\partial H^G_i(X, A)\) and \(\eta\) a chain representing an element of \(I^\partial H^G_j(Y, B)\). The image of \(\xi \otimes \eta\) in \(I^Q H^G_{i+j-1}((A \times Y) \cup (X \times B), X \times B)\) working counterclockwise around the diagram, is represented by \((\partial \xi) \times \eta\). The image working around the diagram clockwise is \(\partial(\xi \times \eta)\), which, as the cross product is a chain map, is equal to \((\partial \xi) \times \eta + (-1)^i \xi \times \partial \eta\). But now \(\xi \times \partial \eta\) is contained in \(X \times B\). Therefore, \((\partial \xi) \times \eta\) and \((\partial \xi) \times \eta + (-1)^i \xi \times \partial \eta\) represent the same element in \(I^Q H^G_{i+j-1}((A \times Y) \cup (X \times B), X \times B)\).

Alternatively, \((\partial \xi) \times \eta\) is contained in \(A \times Y\), and so \(\partial(\xi \times \eta)\) and \((-1)^i \xi \times \partial \eta\) represent the same element in \(I^Q H^G_{i+j-1}((A \times Y) \cup (X \times B), A \times Y)\). But the latter also represents \((-1)^i\) times the cross product of \(\xi\) with \(\partial_* \eta\).

Similarly, we have the following lemma:
Lemma 5.24 (Stability). Suppose \( X \) and \( Y \) are filtered spaces with respective perversities \( \bar{p} \) and \( \bar{q} \) and subspaces \( A \subset X \) and \( B \subset Y \). Suppose that \( Q \) is a perversity on \( X \times Y \) such that \( Q(S \times S') \geq \bar{p}(S) + \bar{q}(S') \) for all strata \( S \subset X \) and \( S' \subset Y \). Then the following diagram commutes:

\[
\begin{array}{ccc}
I^pH_i^{GM}(X,A) \otimes I^qH_j^{GM}(Y,B) & \overset{\epsilon}{\longrightarrow} & I^qH_{i+j}^{GM}(X \times Y,(A \times Y) \cup (X \times B)) \\
\downarrow \quad \quad \quad \quad \downarrow \partial_i & & \downarrow \partial_{i+j-1} \\
(I^pH_{i-1}^{GM}(A) \otimes I^qH_j^{GM}(Y,B)) \oplus (I^pH_i^{GM}(X,A) \otimes I^qH_{j-1}^{GM}(B)) & \overset{\epsilon \otimes \epsilon}{\longrightarrow} & I^qH_{i+j}^{GM}(A \times Y,A \times B) \oplus I^qH_{i+j-1}^{GM}(X \times B,A \times B).
\end{array}
\]

Here, the unlabeled map of the diagram is induced by the inclusions \( ((A \times Y) \cup (X \times B),\emptyset) \to ((A \times Y) \cup (X \times B),A \times B) \) and \( i_1 + i_2 \) denotes the sum of the two inclusion maps \( (A \times Y,A \times B) \to ((A \times Y) \cup (X \times B),A \times B) \) and \( (X \times B,A \times B) \to ((A \times Y) \cup (X \times B),A \times B) \).

In other words, if \( \xi,\eta \in I^qH_j^{GM}(Y,B) \), then

\[
\partial_\ast(\xi) \times \eta + (-1)^i\xi \times \partial_\ast(\eta) = \partial_\ast(\xi \times \eta) \in I^qH_{i+j-1}^{GM}((A \times Y) \cup (X \times B),A \times B).
\]

Proof. As in the proof of Lemma 5.23, we consider a tensor product \( \xi \otimes \eta \) of chains representing a generator of \( I^pH_i^{GM}(X,A) \otimes I^qH_j^{GM}(Y,B) \). Chasing the diagram counterclockwise and using that id \( \otimes \partial_\ast(\xi \otimes \eta) = (-1)^i\xi \otimes \partial_\ast(\eta) \), the chain \( \xi \otimes \eta \) gets taken first to \((\partial\xi) \otimes \eta) + (-1)^i(\xi \otimes \partial\eta)\) then to \((\partial\xi) \times \eta) + (-1)^i(\xi \times \partial\eta)\), and finally to \((\partial\xi) \times \eta) + (-1)^i(\xi \times \partial\eta) \in I^qH_{i+j-1}^{GM}((A \times Y) \cup (X \times B),A \times B)\). Chasing the other way, and using that \( \epsilon \) is a chain map, we get \( \partial(\xi \times \eta) = (\partial\xi) \times \eta + (-1)^i\xi \times \partial\eta \). So the chain representatives agree. \( \square \)

5.2.2 Appendix: The cross product triangulation

In this appendix we provide the technical details concerning the Eilenberg-Zilber shuffle product.

Recall that we have defined in Definition 5.11

\[
\varepsilon(\sigma_1 \otimes \sigma_2) = \sum \text{sgn}(\mu,\nu)(\sigma_1 \times \sigma_2) \circ \eta_{\mu\nu},
\]

where the sum is over all \((p,q)\) shuffles \((\mu,\nu)\), and, in particular, \(\sum \text{sgn}(\mu,\nu)\eta_{\mu\nu}\) is a singular triangulation of \(\Delta^p \times \Delta^q\). We need to verify Proposition 5.10 which states that \(\varepsilon\) is a chain map.
We begin by showing that the shuffle product construction really does provide a triangulation of $\Delta^p \times \Delta^q$, i.e. that $\Delta^p \times \Delta^q$ is homeomorphic to the simplicial complex whose nondegenerate $p + q$ simplices correspond bijectively to the images of the $\eta_{\mu\nu}$, ranging over over the index set of shuffles. For this, we will utilize the nice development by Ramras in [84].

We must first briefly discuss a specific realization construction for abstract simplicial complexes. It will suffice for the present discussion for us to restrict to (abstract) simplicial complexes with finite vertex sets. We assume that the reader already has a solid background concerning simplicial complexes, both abstract and “geometric” (by which we really mean “topological”), though we will briefly (and somewhat informally) provide reminders of relevant concepts. References include [36, 77, 97].

Recall that a finite abstract simplicial complex $\Lambda$ consists of a vertex set $V(\Lambda)$ together with a collection (which we also denote $\Lambda$) of subsets of $V(\Lambda)$ such that 1) for each $v \in V(\Lambda)$, $\{v\} \in \Lambda$, and 2) if $B \subset A$ and $A \in \Lambda$, then $B \in \Lambda$. The elements of $\Lambda$ are called the simplices of $\Lambda$. An abstract simplicial complex contains the same combinatorial information as a geometric simplicial complex. In fact, given a geometric simplicial complex $\Delta$ it determines a geometric simplicial complex whose vertex set is the set of vertices of $K$, $V(K)$, and whose simplices are the subsets of $V(K)$ spanned by simplices of $K$. Conversely, every abstract simplicial complex $\Lambda$ determines a geometric simplicial complex $|\Lambda|$ obtained by taking a copy of $\Delta^i$ for each simplex of $\Lambda$ containing $i + 1$ vertices and gluing these together along appropriate faces via the combinatorial data (i.e. if $B \subset A$, $|B| = i$, $|A| = j$, then we glue the copy of $\Delta^i$ corresponding to $B$ to the appropriate face of the $\Delta^j$ corresponding to $A$). Here we will describe a specific concrete realization of $\Lambda$ as a subset of Euclidean space. For this section only, we will use the notation $|\Delta^i|$ to refer to the standard $i$-simplex as a topological space to differentiate it from the abstract simplicial complex notation.

Let $V$ be a set. Then we can let the $v_i \in V$ be generators of a real vector space isomorphic to $\mathbb{R}^{|V|}$, and we can identify $|\Delta^{|V|-1}|$ with the subset of $\mathbb{R}^{|V|}$ described as

$$\left\{ t \in \mathbb{R}^{|V|} \left| \sum_{i=1}^{|V|} t_i v_i \right. \text{ with } \sum_{i=1}^{|V|} t_i = 1, t_i \geq 0 \right\}.$$  

This is an alternative to the other standard embedding of $\Delta^i$ in $\mathbb{R}^i$ utilizing also the origin as a vertex; instead we use the unit basis vectors as vertices. Now, suppose $\Delta$ is any finite abstract simplicial complex with vertex set $V$. Then its realization $|\Delta|$ can be realized as a subset of $|\Delta^{|V|-1}|$, as $|\Delta^{|V|-1}|$ contains faces corresponding to all possible subsets of $V$. In particular, if we define the support of $t = \sum_{i=1}^{|V|} t_i v_i$ to be $\text{supp}(t) = \{v_i \mid t_i \neq 0\}$, then

$$|\Delta| \cong \left\{ t \in \mathbb{R}^{|V|} \left| \sum_{i=1}^{|V|} t_i v_i \right. \text{ with } \sum_{i=1}^{|V|} t_i = 1, t_i \geq 0, \text{supp}(t) \in \Delta \right\}.$$  

For example, the vectors with support $\{v_i, v_j\}$, $i \neq j$, are precisely those that lie on the open interval between the basis vectors corresponding to $v_i$ and $v_j$.  

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We will be particular interested in abstract simplicial complexes coming from partially ordered sets. If $P$ is a partially ordered set, we let $\Delta(P)$ be the abstract simplicial complex whose vertex set is $P$ and whose simplices are the subsets corresponding to the chains $x_0 < x_1 < \cdots < x_k$. For example, if our partially ordered set is the set $P = \{0, 1, \cdots, n\}$ with its standard order coming from the integers, then every subset corresponds to a chain and $\Delta(P) = \Delta^n$, the abstract $n$-simplex. To shorten the notation, we let $|P| = |\Delta(P)|$.

It will be useful to introduce another geometric construction associated to a partially ordered set $P$: let

$$[P] = \left\{ z \in \mathbb{R}^{[P]} \mid z = \sum_{i=1}^{n} \lambda_i x_{j_i} \text{ for some } n \text{ and some } x_{j_1} < \cdots < x_{j_n} \text{ such that } \lambda_i > 0, \sum_{i=1}^{n} \lambda_i \leq 1 \right\}.$$

Here the $x_i \in P$ are identified with standard basis vectors in $\mathbb{R}^{[P]}$, but we note that it is not necessary for all elements of $P$ to occur in each sum. To understand this set, let $[P]_n$ denote the set of elements of $[P]$ for which $n = m$. Then $[P]_1$ consists of the $|P|$ unit segments from the origin to $x_i$, omitting the origin; $[P]_2$ consists of the 2-simplices spanned by the origin, $x_i$, and $x_j$ when $x_i < x_j$, minus their intersections with the rays from the origin along the standard basis vectors (if $x_i, x_j$ are not comparable in the partial order, they don’t contribute a simplex); and so on. Continuing in this way, we see that the $[P]_m$ are disjoint, so if $z \in |P|$, then $z \in [P]_m$ for some unique $m$ that we denote $\nu(z)$. Additionally, each $[P]_m$ is further partitioned by the choice $\{x_{j_i}\}_{i=1}^{m}$, and since these vertices represent linearly independent vectors in $\mathbb{R}^{[P]}$, each $z$ determines uniquely an ordered set $\{\lambda_1(z), \ldots, \lambda_m(z)\}$ so that $z = \sum_{i=1}^{m} \lambda_i(z)x_{j_i}$. Let $\lambda(z) = \sum_{i=1}^{\nu(z)} \lambda_i(z)$; then we observe that

$$|P| = \{ z \in [P] \mid \lambda(z) = 1 \}.$$

Now, suppose $P, Q$ are two finite partially ordered sets. Let the product $P \times Q$ have the partial ordering defined by $(x, y) \leq (u, w)$ if $x \leq u$ in $P$ and $y \leq w$ in $Q$. We will show that $|P \times Q| \cong |P| \times |Q|$. In the case where $\Delta(P) = \Delta^p = \{x_0, \ldots, x_p\}$ (with each $x_i < x_j$ if $i < j$) and $\Delta(Q) = \Delta^q = \{y_0, \ldots, y_q\}$, the isomorphism therefore provides a triangulation of $|\Delta^p| \times |\Delta^q|$ by the simplicial complex realization $|P \times Q|$. Let us observe how this triangulation corresponds to the shuffle product: by definition, the $p + q$ simplices of $P \times Q$ will correspond to chains of elements $\{v_0, v_1, \ldots, v_{p+q}\} \in P \times Q$ with $v_i < v_{i+1}$. Each $v_i \in P \times Q$ has the form $(x_{j_i}, y_{j_i})$ for $x_{j_i} \in P$ and $y_{j_i} \in Q$, and to have $v_i < v_{i+1}$, we must have $x_{j_i} \leq x_{j_{i+1}}$ and $y_{j_i} \leq y_{j_{i+1}}$, not both equalities. But there are only $p + 1$ elements in $P$ and $q + 1$ elements in $Q$, so the only way to obtain a chain of length $p + q + 1$ with these properties is to have $v_0 = (x_0, y_0)$, $v_{p+q} = (x_p, y_q)$, and, for each $i$, $(x_{j_{i+1}}, y_{j_{i+1}})$ is either $(x_{j_{i+1}}, y_{j_i})$ or $(x_{j_i}, y_{j_{i+1}})$. So these simplices correspond precisely to shuffles!

Now we prove the claimed homeomorphism, following [84, Lemma 2.2.9]:

**Lemma 5.25.** For finite partially ordered sets $P, Q$, $|P \times Q| \cong |P| \times |Q|$.

**Proof.** The projections $P \times Q \to P$ and $P \times Q \to Q$ induce corresponding maps of the abstract simplicial complexes, and hence piecewise linear maps of the geometric realizations $|P \times Q| \to |P|$ and $|P \times Q| \to |Q|$. Together this gives a piecewise linear map $f : |P \times Q| \to$
$|P| \times |Q|$. Since the spaces are all compact Hausdorff spaces and piecewise linear maps are continuous, to show that $f$ is a homeomorphism, we need only show that $f$ is a bijection.

We can describe $f$ quite explicitly. Suppose $\{v_i = (p_i, q_i)\}_{i=1}^n$ is any collection of vertices of $P \times Q$ (possibly with redundancies), and $x = \sum_{i=1}^n \lambda_i v_i$ is an elements of $|P \times Q| \subset \mathbb{R}^N$, where $N$ is the number of vertices of $P \times Q$ and we identify the vertices with basis vectors. Note that not every such sum is in $|P \times Q|$, but every element of $|P \times Q|$ has this form. Then since $f$ is induced by the simplicial projections, we have

$$f(x) = \left( \sum_{i=1}^n \lambda_i p_i, \sum_{i=1}^n \lambda_i q_i \right).$$

**Proof that $f$ is surjective.** Suppose it is true that every $(x, y) \in |P| \times |Q|$ can be written in the form

$$(x, y) = \left( \sum_{i=1}^n \lambda_i p_i, \sum_{i=1}^n \lambda_i q_i \right),$$

for some $\lambda_i \in (0, 1]$, $p_1 \leq \cdots \leq p_n \in P$, and $q_1 \leq \cdots \leq q_n \in Q$ (again note that the $p_i$ and $q_i$ are not necessarily unique, and neither necessarily are the pairs $(p_i, q_i)$). Then we’ll have $(p_i, q_i) \leq (p_{i+1}, q_{i+1})$ for all $1 \leq i < n$, so that the collection $\{(p_i, q_i)\}$ spans a simplex of $P \times Q$, and hence $\sum_{i=1}^n \lambda_i (p_i, q_i) \in |P \times Q|$. And then $f(\sum_{i=1}^n \lambda_i (p_i, q_i)) = (\sum_{i=1}^n \lambda_i p_i, \sum_{i=1}^n \lambda_i q_i) = (x, y)$. So this would imply that $f$ is surjective. We will show that any $(x, y) \in |P| \times |Q|$ can indeed be written in this form.

In fact, we will show that if $(x, y) \in |P| \times |Q|$ and $\lambda(x) = \lambda(y)$ (in the notation of page 146), then there exist $\lambda_1, \ldots, \lambda_n \in (0, 1]$, $p_1 \leq \cdots \leq p_n \in P$, and $q_1 \leq \cdots \leq q_n \in Q$ such that

$$\left( x, y \right) = \left( \sum_{i=1}^n \lambda_i p_i, \sum_{i=1}^n \lambda_i q_i \right).$$

Now since $|P| = \{ x \in |P| \mid \lambda(x) = 1 \}$ and $|Q| = \{ y \in |Q| \mid \lambda(y) = 1 \}$, this will imply the needed fact.

Let $\nu(x, y) = \nu(x) + \nu(y)$ (using again the notation of page 146), and let us induct on this number. The minimum value for $\nu(x, y)$ is 2, when $\nu(x) = \nu(y) = 1$. In this case we simply have $x = \lambda(x)p \in P$ and $y = \lambda(y)q \in Q$ for some $(p, q) \in P \times Q$. But then we just let $p_1 = p$, $q_1 = q$, $\lambda_1 = \lambda(x) = \lambda(y)$ (the last equality by assumption), and we are done. Now assume we have proven the claim for $\nu(x, y) \leq r$, and choose $(x, y)$ with $\nu(x, y) = r + 1$ and $\lambda(x) = \lambda(y)$. Since $x \in |P|$ and $y \in |Q|$, we may write $x = \sum_{i=1}^{\nu(x)} t_i p_i$ for some $p_i \in P$ with $p_1 < \cdots < p_{\nu(x)}$ and all $t_i > 0$, and similarly $y = \sum_{i=1}^{\nu(y)} s_i q_i$ with $q_1 < \cdots < q_{\nu(y)}$ and all $s_i > 0$. By assumption, $\nu(x) + \nu(y) = r + 1$. Furthermore, we note that this implies that $x$ is contained in the span of $p_1, \ldots, p_{\nu(x)}$, and that no other element of $P$ can appear non-trivially in a linear combination representing $x$; similarly for $y$.

Now assume without loss of generality that $t_1 \leq s_1$, and write

$$(x', y') = (x, y) - (t_1 p_1, t_1 q_1) = \left( \sum_{i=2}^{\nu(x)} t_i p_i, (s_1 - t_1) q_1 + \sum_{i=2}^{\nu(y)} s_i q_i \right).$$
The right hand side now has \( \nu(x', y') \leq r \) (it could be \( r - 1 \) if \( s_1 = t_1 \)). Furthermore, 
\[
\lambda(x') = \sum_{i=2}^{\nu(x)} t_i - \lambda(x) - t_1, \quad \text{and} \quad \lambda(y') = (s_1 - \lambda(x) - t_1) + \sum_{i=2}^{\nu(y)} s_i - (\sum_{i=1}^{\nu(y)} s_i) - t_1 = \lambda(y) - t_1 = \lambda(x) - t_1. 
\]
So by induction hypothesis, there are \( \lambda_1, \ldots, \lambda_n \in (0, 1) \), \( p'_1 \leq \cdots \leq p'_n \), and \( q'_1 \leq q'_n \) such that
\[
(x', y') = \left( \sum_{i=1}^{n} \lambda_i p'_i, \sum_{i=1}^{n} \lambda_i q'_i \right).
\]
So now,
\[
(x, y) = (t_1 p_1 + \sum_{i=1}^{n} \lambda_i p'_i, t_1 q_1 + \sum_{i=1}^{n} \lambda_i q'_i)
\]
has the desired form, provided \( p_1 \leq p'_1 \) and \( q_1 \leq q'_1 \). But since the \( \lambda_i \) are all non-zero, each \( p'_i \) must be one of the \( \{p_i\} \) (by the earlier observation that no element not in \( \{p_i\} \) can appear non-trivially in a linear combination representing \( x \)), so as \( p_1 \) is the least of the \( \{p_i\} \) in the partial order, we also have \( p_1 \leq p'_1 \), and similarly \( q_1 \leq q'_1 \).

**Proof that \( f \) is injective.** Let \( x, y \in [P \times Q] \subset [P \times Q] \). Then \( x = \sum_{i=1}^{\nu(x)} \lambda_i (p_i, q_i) \) and \( y = \sum_{i=1}^{\nu(y)} \lambda'_i (p'_i, q'_i) \) for some unique choices \( \lambda_i, \lambda'_i > 0, \sum \lambda_i = \sum \lambda'_i = 1 \), \( (p_i, q_i) < (p_{i+1}, q_{i+1}) \), and \( (p'_i, q'_i) < (p'_{i+1}, q'_{i+1}) \) for all relevant \( i \); notice that we assume that vertices are not repeated within each of these representations of \( x \) and \( y \). We will show that if \( f(x) = f(y) \), then \( x = y \).

If \( f(x) = f(y) \), then \( \sum_{i=1}^{\nu(x)} \lambda_i p_i = \sum_{i=1}^{\nu(y)} \lambda'_i p'_i \) and \( \sum_{i=1}^{\nu(x)} \lambda_i q_i = \sum_{i=1}^{\nu(y)} \lambda'_i q'_i \). Since all the \( \lambda_i, \lambda'_i \) are \( > 0 \) and the elements of \( P \) and \( Q \) form respective bases of the Euclidean spaces containing \( |P| \) and \( |Q| \), it follows that we must have \( \{p_1, \ldots, p_{\nu(x)}\} = \{p'_1, \ldots, p'_{\nu(y)}\} \) and \( \{q_1, \ldots, q_{\nu(x)}\} = \{q'_1, \ldots, q'_{\nu(y)}\} \). These lists might each contain repeated elements, but we must have \( p_1 = p'_1 \) and \( q_1 = q'_1 \) since these are each the smallest elements in their respective sets under the order.

Our next goal is to begin to show that \( \lambda_i = \lambda'_i \) for each \( i \). Since \( (p_1, q_1) < (p_2, q_2) \), assume without loss of generality that \( p_1 < p_2 \) (otherwise \( q_1 < q_2 \) and we reverse the roles of \( P \) and \( Q \) in the following argument). Since \( \sum_{i=1}^{\nu(x)} \lambda_i p_i = \sum_{i=1}^{\nu(y)} \lambda'_i p'_i \), we must have that \( \lambda_1 = \sum_{\{i|p_i = p_1\}} \lambda'_i \geq \lambda'_1 \). If \( p'_1 \neq p'_2 \), then since \( p_1 < p'_1 \leq \cdots \leq p'_i \), we would have only \( p'_1 = p_1 \), and so \( \lambda_1 = \lambda'_1 \). If \( p'_1 = p'_2 \), then \( q'_1 < q'_2 \), and by a symmetric argument to the above, \( \lambda'_1 = \sum_{\{i|q_i = q'_1\}} \lambda_i \geq \lambda_1 \). But we already know \( \lambda_1 \geq \lambda'_1 \), so \( \lambda_1 = \lambda'_1 \). This argument also implies that while \( p_1 = p'_1 \), no \( p_i \) or \( p'_i \) with \( i > 1 \) is equal to this element of \( P \), and, symmetrically, while \( q_1 = q'_1 \), no \( q_i \) or \( q'_i \) with \( i > 1 \) is equal to this element of \( Q \). It now follows that \( p_2 = p'_2 \) are the smallest terms in \( \{p_2, \ldots, p_{\nu(x)}\} \) and \( \{p'_2, \ldots, p'_{\nu(y)}\} \) and similarly for \( Q \), and we can run the same argument inductively to eventually show that \( \nu(x) = \nu(y) \), that \( \lambda_i = \lambda'_i \) for all \( i \), and that \( p_i = p'_i \) and \( q_i = q'_i \) for all \( i \).

The lemma, together with the discussion preceding it, verifies that we have a triangulation of \( |\Delta^p| \times |\Delta^q| \) whose \( p \times q \) simplices correspond to the collection \( \{(\mu, \nu)\} \) of \( (p, q) \) shuffles. As above, if we let \( \eta_{\mu \nu} : |\Delta^{p+q}| \rightarrow |\Delta^p| \times |\Delta^q| \) be the embedding of the simplex corresponding to the shuffle \( (\mu, \nu) \) and determined using the ordering on the vertices in \( \Delta^p \times \Delta^q \), we obtain a singular triangulation of \( |\Delta^p| \times |\Delta^q| \). It remains to show that
\[
\varepsilon(\sigma \otimes \tau) = \sum \text{sgn}(\mu, \nu)(\sigma \times \tau) \circ \eta_{\mu \nu},
\]
is a chain map.

First, we notice the following easy corollary of Lemma 5.25 and its proof.

**Corollary 5.26.** If \( A \subset P \) and \( B \subset Q \) are subsets inheriting the partial ordering, then \(|A \times B| \subset |P \times Q|\) is a subcomplex, and the triangulating homeomorphism \(|P \times Q| \to |P| \times |Q|\) restricts to a triangulation of \(|A| \times |B|\).

**Proof.** By Lemma 5.25, the subspace \(|A| \times |B|\) is homeomorphic to the simplicial complex \(|A \times B|\). But, more than this, the construction of the proof demonstrates that this homeomorphism is compatible with the larger one \(|P| \times |Q| \cong |P \times Q|\): clearly points in \(|P \times Q|\) that involve only the vertices in \(A \times B\) have their image in \(|A| \times |B|\) under the constructed homeomorphism \(|P \times Q| \to |P| \times |Q|\), and the restriction of the proof of Lemma 5.25 shows that the induced map \(|A \times B| \to |A| \times |B|\) is surjective. \(\square\)

Now suppose, in particular, that \( F \) is a face of \( \Delta^p \), then the abstract simplicial complex \( F \times \Delta^q \) is a subcomplex of \( \Delta^p \times \Delta^q \) and, by the corollary, the homeomorphism \(|\Delta^p \times \Delta^q| \cong |\Delta^p| \times |\Delta^q|\) restricts to provide a triangulation of \(|F \times \Delta^q| \cong |F| \times |\Delta^q|\). Extrapolating, we see that if \( F_0, \ldots, F_p \) are the \( p \) 1-dimensional faces of \( \Delta^p \) and \( G_0, \ldots, G_q \) are the \( q \) 1-dimensional faces of \( \Delta^q \), then the restriction of \( f : |\Delta^p \times \Delta^q| \cong |\Delta^p| \times |\Delta^q|\) to each \( F_i \times \Delta^q \) and \( \Delta^p \times G_j \) provides the desired triangulations of \(|F_i| \times |\Delta^q|\) or \(|\Delta^p| \times |G_j|\), and collectively these give a triangulation of \( (|\partial \Delta^p|) \times |\Delta^q|) \cup (|\Delta^p| \times (|\partial \Delta^q|)) \cong |\partial (|\Delta^p| \times |\Delta^q|)|\). It follows from this observation that the summands of \( \varepsilon(\partial(\sigma \otimes \tau)) \) correspond (up to signs) to the restrictions of \( \sigma \times \tau \) to the \( p + q - 1 \) simplices in the complexes \(|F_k \times \Delta^q|\) and \(|\Delta^p \times G_l|\). In fact, this completely determines \( \varepsilon(\partial(\sigma \otimes \tau)) \) up to the signs of the terms on each \( p + q - 1 \) simplex.

We need to compare with \( \partial \varepsilon(\sigma \otimes \tau) \), which is easily seen to equal \( (\sigma \otimes \tau)_# \partial(\sum \text{sgn}(\mu, \nu) \eta_{\mu \nu}) \), where \((\sigma \otimes \tau)_# \) is the induced chain map \( S_\ast(|\Delta^p| \times |\Delta^q|) \to S_\ast(\Delta \times Y) \). So it will suffice to show that \( \partial(\sum \text{sgn}(\mu, \nu) \eta_{\mu \nu}) \) consists entirely of \( p + q - 1 \) simplices on \( \partial(|\Delta^p| \times |\Delta^q|) \), and that the corresponding coefficients agree with what we get from \( \varepsilon(\partial(\sigma \otimes \tau)) \).

We first tackle showing that \( \partial(\sum \text{sgn}(\mu, \nu) \eta_{\mu \nu}) \) consists entirely of \( p + q - 1 \) simplices on \( \partial(|\Delta^p| \times |\Delta^q|) \).

**Lemma 5.27.** \( \partial(\sum \text{sgn}(\mu, \nu) \eta_{\mu \nu}) \) is supported in \( \partial(|\Delta^p| \times |\Delta^q|) \). In fact \( \sum \text{sgn}(\mu, \nu) \eta_{\mu \nu} \) is a fundamental orientation class for \(|\Delta^p \times \Delta^q|\).

**Proof.** Each \( \eta_{\mu \nu} \) corresponds to an embedding of a \( p + q \) simplex in \(|\Delta^p| \times |\Delta^q|\) with vertices of the form \((x_i, y_j), \ x_i \in \Delta^p, y_j \in \Delta^q\). When we look at a face \( \mathcal{F} \) of \( \eta_{\mu \nu} \) obtained by removing a vertex, there are two possibilities:

1. There is some \( x_i \) or \( y_j \) that no longer appears in any vertex of \( \mathcal{F} \). Suppose for specificity that it is some specific \( x_k \). Then all vertices of \( \mathcal{F} \) lie in the product \( F_k \times \Delta^q \), where \( F_k = [x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p] \) is the \( k \)th face of \( \Delta^p \). Thus in the triangulation \(|\Delta^p \times \Delta^q|, \ |\mathcal{F}| \subset |F_k| \times |\Delta^q| \subset \partial(|\Delta^p \times \Delta^q|) \), so we have a \( p + q - 1 \) simplex lying on the boundary of \(|\Delta^p| \times |\Delta^q|\).
2. Each \( x_i \) and \( y_j \) appears in some vertex of \( F \). Suppose \((x_k, y_l)\) is the vertex of \( \eta_{\mu \nu} \) that is removed to create \( F \). Note that this can’t be \((x_0, y_0)\), since the next vertex must be either \((x_0, y_1)\) or \((x_1, y_0)\), and so if we remove \((x_0, y_0)\), then either \(x_0\) or \(y_0\) will not occur in any vertex of \( F \); similarly, the removed vertex cannot be \((x_p, y_q)\). So the removed vertex, \((x_k, y_l)\) must have a predecessor and successor among the vertices of \( \eta_{\mu \nu} \). The next vertex in \( \eta_{\mu \nu} \) is either \((x_{k+1}, y_l)\) or \((x_k, y_{l+1})\), and the preceding vertex is either \((x_{k-1}, y_l)\) or \((x_k, y_{l-1})\). But if the sequence is \((x_{k-1}, y_l), (x_k, y_l), (x_{k+1}, y_l)\), then \( F \) could not have \( x_k \) in a vertex, and similarly there would be a contradiction if the sequence were \((x_k, y_{l-1}), (x_k, y_l), (x_k, y_{l+1})\). So the only two possible subsequences in \( \eta_{\mu \nu} \) are \((x_{k-1}, y_l), (x_k, y_l), (x_{k+1}, y_l)\) or \((x_k, y_{l-1}), (x_k, y_l), (x_{k+1}, y_l)\). Suppose it is the former; the arguments that follow will be equivalent if it is the latter. Then replacing this triple of vertices in \( \eta_{\mu \nu} \) with \((x_{k-1}, y_l), (x_k, y_l), (x_{k+1}, y_l)\) while leaving the other vertices unchanged gives us another \( p + q \) simplex \( \eta_{\mu', \nu'} \) with the same sequence of vertices except for the swap of \((x_k, y_l)\) for \((x_{k-1}, y_{l+1})\). Thus \( \eta_{\mu \nu} \) and \( \eta_{\mu', \nu'} \) share the common face obtained by removing the vertex \((x_k, y_l)\) from \( \eta_{\mu \nu} \) and \((x_{k-1}, y_{l+1})\) from \( \eta_{\mu', \nu'} \). Furthermore, \((x_k, y_l)\) and \((x_{k-1}, y_{l+1})\) are the only possible vertices of \( \Delta^p \times \Delta^q \) that are greater than \((x_{k-1}, y_l)\) and less than \((x_{k+1}, y_l)\) in the partial order, and so these are the only \( \eta \s s \) with \( F \) as a face. Furthermore, we see that the shuffle \((\mu, \nu)\) and the shuffle \((\mu', \nu')\) differ only by a single permutation (rather than having an element of \( \Delta^p \) followed by an element of \( \Delta^q \) in \((\mu, \nu)\), we swap for an element of \( \Delta^q \) followed by an element of \( \Delta^p \) in \((\mu', \nu')\)). So \( \text{sgn}(\mu, \nu) = -\text{sgn}(\mu', \nu') \). But also, if \( F \) is the \( k \)th face of \( \eta_{\mu \nu} \) it is also the \( k \)th face of \( \eta_{\mu', \nu'} \), and so \( F \) occurs with the same sign in \( \partial \eta_{\mu \nu} \) and \( \partial \eta_{\mu', \nu'} \). So altogether \( F \) appears with coefficient 0 in \( \sum \text{sgn}(\mu, \nu)\eta_{\mu \nu} \). As mentioned already, the argument when we have a sequence \((x_k, y_{l-1}), (x_k, y_l), (x_{k+1}, y_l)\) is analogous.

The last statement of the lemma is now clear since the coefficient of \( \sum \text{sgn}(\mu, \nu)\eta_{\mu \nu} \) on each \( p + q \) simplex is \( \pm 1 \).

To finish showing that \( \varepsilon \) is a chain map, it now suffices to show that the simplices of \( \partial |\Delta^p \times \Delta^q| \) are represented with the same coefficients as summands in \( \varepsilon(\partial(\sigma \otimes \tau)) \) as in \( \partial \varepsilon(\sigma \otimes \tau) \). Rather than compute all the details regarding “shuffles of shuffles”, it is useful to observe an implication of the preceding lemma: since \( \sum \text{sgn}(\mu, \nu)\eta_{\mu \nu} \) is a fundamental class on \( |\Delta^p \times \Delta^q| \), its boundary is a fundamental class for the sphere \( \partial |\Delta^p \times \Delta^q| \). Hence, in principle, the sign on any \( p + q - 1 \) simplex of this chain must determine the signs of all the others. Similarly, for each face \( F \) of \( \Delta^p \), the corresponding signed sum of shuffles is a fundamental class for \( F \times \Delta^q \) and so one of its \( p + q - 1 \) simplices determines the signs of all the others, and analogously for each \( \Delta^p \times G \). So it will be sufficient to compare the signs of one \( p + q - 1 \) simplex in each \( |F \times \Delta^q| \) or \( |\Delta^p \times G| \).

First, let \( \sigma_k \) be the \( k \)th face of \( \sigma \), i.e. \( \sigma_k = \sigma \mid F_k \), where \( F_k \) is the \( k \)th face of \( \Delta^p \). Then \( \sigma_k \) occurs with sign \((-1)^k\) in \((\partial \sigma) \otimes \tau \). Then \( \varepsilon((-1)^k\sigma_k \otimes \tau) = (-1)^k \sum \text{sgn}(\xi, \zeta)(\sigma_k \otimes \tau) \circ \delta_{\xi\zeta} \) where the \((\xi, \zeta)\) run over \((p - 1, q)\) shuffles of the sets \( \{x_1, \ldots, \hat{x}_k, \ldots, x_p\} \subset \Delta^p \) and \( \delta_{\xi\zeta} \) is the corresponding linear embedding into \( |F_k \times \Delta^q| \). Since we have seen that it will be sufficient to know the sign on one \( p + q - 1 \) simplex, let’s use the trivial shuffle \((\xi_0, \zeta_0)\) that
yields the sequence \( \{x_0, \ldots, \hat{x}_k, \ldots, x_p, y_0, \ldots, y_q\} \); in terms of vertices of \( \Delta^p \times \Delta^q \), this is the \( p+q-1 \) simplices with vertices \( \{(x_0, y_0), \ldots, (x_p, y_0), (x_p, y_1), \ldots, (x_p, y_q)\} \), omitting the vertex \((x_k, y_0)\). Then \( \text{sgn}(\xi, \zeta) = 1 \), so altogether the corresponding singular \( p+q-1 \) simplex is \((-1)^k(\sigma_k \times \tau) \circ \delta_{\xi_0 \zeta_0} \). The computation for faces \( \Delta^p \times G_i \) is similar, except that there will be an additional sign \((-1)^p\) coming from \( \partial(\sigma \otimes \tau) = (\partial \sigma) \otimes \tau + (-1)^p \sigma \otimes (\partial \tau) \), and the missing vertex pair will be \((x_p, y_i)\), so overall the sign is \((-1)^{p+l}\).

Now let’s turn to the corresponding terms in \( \partial \varepsilon(\sigma \otimes \tau) = \partial \sum \text{sgn}(\mu, \nu)(\sigma \times \tau) \circ \eta_{\mu \nu} \). The \( p+q-1 \) simplex with vertex set \( \{(x_0, y_0), \ldots, (x_k, y_0), \ldots, (x_p, y_0), (x_p, y_1), \ldots, (x_p, y_q)\} \) is the \( k \)th face of the \( \eta_{\mu_0, \nu_0} \), where \((\mu_0, \nu_0)\) is the trivial \((p, q)\) shuffle. This shuffle has sign 1 and since this is the \( k \)th face, there’s a sign \((-1)^k\) for this term in \( \partial \varepsilon(\sigma \otimes \tau) \). This corresponds with the computation in the preceding paragraph. Similarly, the \( p+q-1 \) simplex with vertex set \( \{(x_0, y_0), \ldots, (x_p, y_0), \ldots, (x_p, y_1), \ldots, (x_p, y_q)\} \) is the \((p+l)\)th face of the \( \eta_{\mu_0, \nu_0} \), where \((\mu_0, \nu_0)\) is the trivial \((p, q)\) shuffle. This shuffle has sign 1 and since this is the \((p+l)\)th face, there’s a sign \((-1)^{p+l}\) for this term in \( \partial \varepsilon(\sigma \otimes \tau) \). This also corresponds with the computation in the preceding paragraph.

Finally, we are done proving Proposition 5.10

5.3 K"unneth theorem when one term is a manifold

Now that we have a cross product, in this section we will provide a K"unneth theorem for intersection homology in the case where one factor of the product is an unfiltered manifold \( M \) and the other is a filtered space \( X \). We will not be able to obtain a more general K"unneth theorem (for arbitrary perversities) until we have redefined intersection homology slightly in Section 8, but this version of the K"unneth theorem nonetheless has important applications, including the proof of topological invariance of intersection homology with Goresky-MacPherson perversities (see Section 5.6).

Versions of the K"unneth theorem presented here seem to have been known quite early on; the special case for \( \bar{p} = \bar{m} \), \( X \) a Witt space\(^{60}\) and real coefficient\(^{61}\) is a special case of [43, Section 6.3] of Goresky-MacPherson, while Siegel has a proof in the PL category for Witt spaces with rational coefficients in [95]. A proof for singular intersection homology (and integer coefficients) was provided by King [61]. We provide a different proof, though one that is very consonant with other techniques developed elsewhere in King’s paper.

Theorem 5.28. Suppose \( X \) is a filtered space with perversity \( \bar{p}_X \) and that \( M \) is an \( n \)-dimensional manifold with its trivial filtration. Filter \( M \times X \) with product filtration so that \((M \times X)^i = M \times X^{i-n}\), and define a perversity \( \bar{p} \) on \( M \times X \) whose value on \( M \times S \) is \( \bar{p}(S) \). Then the cross product induces an isomorphism \( H_*(S_*(M) \otimes I^p S_*^{GM}(X)) \xrightarrow{\cong} I^p H_*^{GM}(M \times X) \).

\(^{59}\) Caution: recall that there is a 0th vertex (in this case \((x_0, y_0)\)), so what we call the \((p+l)\)th face removes what is really the \((p+l+1)\)st vertex in the list.

\(^{60}\) See Section 10.4 below.

\(^{61}\) See Section 5.4 below.
Proof. Before starting on the main body of the proof, we observe that, for any $x_0 \in M$, the inclusion of $X$ into $M \times X$ by identifying it with $\{x_0\} \times X$ is a normally nonsingular inclusion. In particular, the codimension of a stratum $S$ in $X$ is the same as the codimension of $M \times S$ in $M \times X$, so $I^pH^*_s GM(M) \times X)$ is isomorphic to the intersection homology group $I^pH^*_s GM(X)$ obtained by thinking of $X$ as a subspace of $M \times X$ and using the inherited filtration and perversity. See Example 4.25 for a full discussion of this scenario. We will use this identification of intersection homology groups without further explicit mention.

Now, to prove the theorem, we will apply the Mayer-Vietoris argument Theorem 5.1. We fix $X$ and define functors from $M$ to $Ab$ as follows: Let $F_*(M) = H_*(S_*(M) \otimes I^pS^* GM(M) \times X)$, let $G_*(M) = I^pH^*_s GM(M \times X)$, and let the natural transformation $\Phi : F_* \to G_*$ be induced by the cross-product.

We first consider the cross product $S_*(\mathbb{R}^n) \otimes I^pS^* GM(X) \to I^pS^* GM(\mathbb{R}^n \times X)$. In this case, we already know from Corollary 4.9 that $I^pH^*_s GM(\mathbb{R}^n \times X) = I^pH^*_s GM(X)$, induced by the inclusion $X = \{0\} \times X \hookrightarrow \mathbb{R}^n \times X$. Furthermore, by the algebraic Künneth Theorem, we also know that $H_i(S_*(\mathbb{R}^n) \otimes I^pS^* GM(X)) = H_0(\mathbb{R}^n) \otimes I^pH^*_i GM(X) = I^pH^*_i GM(X)$. But we can assume $H_0(\mathbb{R}^n)$ is generated by the singular vertex at the origin, $v_0$, and the cross product of $v_0$ with a singular simplex $\sigma \in S_*(X)$ is the same as the composition of $\sigma$ with the inclusion $X = \{0\} \times X \hookrightarrow \mathbb{R}^n \times X$, so these isomorphisms are compatible. More formally speaking, we have a commutative diagram

\[
\begin{array}{ccc}
S_*(\{0\}) \times I^pS^* GM(X) & \xrightarrow{\cdot} & I^pS^* GM(\{0\} \times X) = I^pS^* GM(X) \\
S_*(\mathbb{R}^n) \otimes I^pS^* GM(X) & \xrightarrow{\cdot} & I^pS^* GM(\mathbb{R}^n \times X),
\end{array}
\]

in which the vertical maps are induced by inclusion. By the preceding arguments, the top and the sides induce isomorphisms on homology, and hence so does the bottom. (Technically the argument that the left side induces a homology isomorphism also uses that the chain complexes $S_*$ and $C_*$ and their submodules are torsion-free, and hence flat, so that the algebraic Künneth theorem applies; see [105] Theorem 3.6.3 for the Künneth theorem and [105] Corollary 3.15 and Proposition 3.2.4 for the fact that torsion free implies flat.)

Note that these arguments are invariant up to homeomorphism, and so the above paragraph also applies to subsets of $M$ that are homeomorphic to $\mathbb{R}^n$.

Next suppose that $\{U_\alpha\}$ is a sequence of open subsets of $M$ ordered by inclusion and such that $S_*(U_\alpha) \otimes I^pS^* GM(X) \xrightarrow{\cdot} I^pS^* GM(U_\alpha \times X)$ induces homology isomorphisms for each $\alpha$. Consider the commuting diagrams of the form

\[
\begin{array}{ccc}
H_*(S_*(U_\beta) \otimes I^pS^* GM(X)) & \xrightarrow{\cdot} & I^pH^*_s GM(U_\beta \times X) \\
H_*(S_*(\cup_\alpha U_\alpha) \otimes I^pS^* GM(X)) & \xrightarrow{\cdot} & I^pH^*_s GM(\cup_\alpha U_\alpha \times X),
\end{array}
\]

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where $U_\beta$ is a particular one of the $\{U_\alpha\}$. Suppose $[\xi] \in I^\beta H^*_GM(\cup_\alpha U_\alpha \times X)$, represented
by the cycle $\xi$. Since $\xi$ consists of a finite number of singular simplices, the union of the images of the simplices of $\xi$ is compact and so
must be contained in some $U_\beta \times X$. Hence $[\xi]$ is in the image of $I^\beta H^*_GM(U_\beta \times X)$. But the bottom line of diagram is
an isomorphism for any fixed $\beta$ by assumption, and so it follows that the bottom map must be surjective.

Similarly, suppose that $[\eta] \in H_*(\cup_\alpha U_\alpha \otimes P^x S^*GM(X))$ maps to $0$ in $I^\beta H^*_GM(\cup_\alpha U_\alpha \times X)$. By definition,
$\eta = \sum_j x_j \otimes \xi_j$ for some $x_j \in S_*(\cup_\alpha U_\alpha)$ and $\xi_j \in I^\beta S^*_GM(X)$, so we are assuming $[\sum x_j \times \xi_j] = 0$. Let $\zeta$ be a chain in $I^\beta S^*_GM(\cup_\alpha U_\alpha \times X)$ with $d\zeta = \sum x_j \times \xi_j$. Then,
again by a compactness argument, there is a $\beta$ such that all the $x_j$ are supported in $U_\beta$ and $\zeta$ is supported in $U_\beta \times X$. So then $\sum x_j \times \xi_j$ also represents $0$ as an element of $I^\beta H^*_GM(U_\beta \times X)$. Since the top line is an isomorphism, there must be an element $\mu \in S_*(U_\beta) \otimes I^\beta S^*_GM(X)$ whose boundary
is $\sum_j x_j \otimes \xi_j$. But then this must hold also under the inclusion of $S_*(U_\beta) \otimes I^\beta S^*_GM(X)$ into $S_*(\cup_\alpha U_\alpha) \otimes I^\beta S^*_GM(X)$ and so $[\eta] = 0$.

Altogether, we have now shown that the cross product induces an isomorphism $H_*(\cup_\alpha U_\alpha \otimes I^\beta S^*_GM(X)) \cong I^\beta H^*_GM(\cup_\alpha U_\alpha \times X)$.

Finally, consider the following diagram:

```
0 \rightarrow S_*(U \cap V) \otimes I^\beta S^*_GM(X) \rightarrow S_*(U) \otimes (I^\beta S^*_GM(X)) \oplus (S_*(V) \otimes I^\beta S^*_GM(X)) \rightarrow (S_*(U) + S_*(V)) \otimes I^\beta S^*_GM(X) \rightarrow 0
```

The bottom row is a Mayer-Vietor is short exact sequence, while the top row is obtained by tensoring the short exact Mayer-Vietoris sequence

```
0 \rightarrow S_*(U \cap V) \rightarrow S_*(U) \bigoplus S_*(V) \rightarrow S_*(U) + S_*(V) \rightarrow 0
```

with $I^\beta S^*_GM(X)$ and then summing over degrees (recall that $(A_i \otimes B_i)_j = \oplus_{j+k=1}^i A_j \otimes B_k$). Since each $I^\beta S^*_GM(X)$ is $\mathbb{Z}$-torsion free, it is a flat $\mathbb{Z}$-module [109, Corollary 3.1.5 and Proposition 3.2.4], so tensoring with $I^\beta S^*_GM(X)$ preserves exactness.\(^{62}\) This diagram yields
a map of long exact sequences in homology. By the proof of Theorem 4.41 (or Theorem 4.33 in the PL case), the inclusion $I^\beta S^*_GM(U \times X) + I^\beta S^*_GM(V \times X) \rightarrow I^\beta S^*_GM(U \times X) + I^\beta S^*_GM((U \cup V) \times X)$ induces a homology isomorphism, so the bottom row has the form of a long exact
Mayer-Vietoris sequence. Similarly, by the same arguments, or classically, $S_*(U) + S_*(V) \hookrightarrow S_*(U \cup V)$ induces a homology isomorphism, and it follows from the algebraic Künneth theorem and the five lemma that the map $(S_*(U) + S_*(V)) \otimes I^\beta S^*_GM(X) \rightarrow S_*(U \cup V) \otimes I^\beta S^*_GM(X)$ induced by inclusion is also a homology isomorphism. So, again, substituting
the homology of the latter expression for that of the former in the long exact sequence yields a long exact sequence of Mayer-Vietoris form for the functor $F_*$. We observe that the cross product continues to induce the isomorphism of exact sequences with these substitutions via the diagrams

---

\(^{62}\) Note that this argument also applies to conclude that $I^\beta S^*_GM(X)$ is flat in the PL case.
Proof. Consider the short exact sequence
\[
H_i(S_*(U) \otimes I^p S_*^GM(X)) \oplus H_i([S_*(V) \otimes I^p S_*^GM(X)]) \xrightarrow{\partial_i} H_i((S_*(U) + S_*(V)) \otimes I^p S_*^GM(X)) \xrightarrow{\cong} H_i(S_*(U \cup V) \otimes I^p S_*^GM(X))
\]
and
\[
H_i(S_*(U \cup V) \otimes I^p S_*^GM(X)) \cong H_i([S_*(U) + S_*(V)) \otimes I^p S_*^GM(X)) \xrightarrow{\partial_{i-1}} H_{i-1}(S_*(U \cap V) \otimes I^p S_*^GM(X)) \xrightarrow{\cong} I^p H_{i-1}((U \cap V) \times X),
\]
the commutativity of the righthand square being induced by the functoriality of the long exact sequence construction from our map of Mayer-Vietoris short exact sequences.

Theorem 5.1 now implies this theorem. \qed

Corollary 5.29. Under the assumptions of Theorem 5.28, if \( A \subset X \), then the cross product induces an isomorphism \( H_*(S_*(M) \otimes I^p S_*^GM(X, A)) \cong I^p H_*^GM(M \times X, M \times A) \).

If \( X \) is a PL manifold stratified space, \( A \) is a PL subspace, and \( M \) is a PL manifold, then the same conclusion holds replacing singular chains with PL chains.

Proof. Consider the short exact sequence
\[
0 \longrightarrow I^p S_*^GM(A) \longrightarrow I^p S_*^GM(X) \longrightarrow I^p S_*^GM(X, A) \longrightarrow 0.
\]
As \( S_*(M) \) is torsion-free, and hence flat, tensoring with \( S_*(M) \) preserves exactness, and we obtain a diagram of short exact sequences
\[
0 \longrightarrow S_*(M) \otimes I^p S_*^GM(A) \longrightarrow S_*(M) \otimes I^p S_*^GM(X) \longrightarrow S_*(M) \otimes I^p S_*^GM(X, A) \longrightarrow 0
\]
\[
0 \longrightarrow I^p S_*^GM(M \times A) \longrightarrow I^p S_*^GM(M \times X) \longrightarrow I^p S_*^GM(M \times X, M \times A) \longrightarrow 0.
\]
It is not difficult to verify that this diagram commutes by looking at cross products of generators. The diagram then induces a map of long exact sequences, and the corollary follows from Theorem 5.28 and the five lemma. \qed

5.4 Intersection homology with coefficients; universal coefficient theorems

So far, we have managed to show that intersection homology satisfies many of the key properties of ordinary homology. In this section, we will explore a property whose translation to
intersection homology is more problematic, namely intersection homology with coefficients. Part of the issue is that there are two possible competing definitions of intersection homology with coefficients.

Perhaps the simplest approach to intersection homology with coefficients would be to consider $I^pS^*_{\text{GM}}(X) \otimes G$ for some abelian group $G$. Indeed, one could do this, and then of course the algebraic universal coefficient theorem would give us split short exact sequences of the form

$$0 \to I^pH^G_i(X) \otimes G \to H_i(I^pS^*_M(X) \otimes G) \to I^pH^G_{i-1}(X) \ast G \to 0,$$

where $\ast$ denotes the torsion product $\text{Tor}^1(\cdot, \cdot)$.

However, there is a more intriguing option, which is captured in the following definition:

**Definition 5.30.** Let $G$ be an abelian group and $X$ a filtered space with perversity $\bar{p}$. Define the complex of intersection chains with coefficients $I^pS^*_{\text{GM}}(X; G)$ to be the subcomplex of $S_*(X; G) = S_*(X) \otimes G$ such that $\xi \in I^pS^*_{\text{GM}}(X; G)$ if each simplex of $\xi$ is allowable and each simplex of $\partial \xi$ is allowable. In other words, we simply mirror the definition of the intersection chain complex $I^pS^*_{\text{GM}}(X)$, in terms of allowance, but using $S_*(X; G)$ as our starting point rather than $S_*(X)$.

**Remark 5.31.** More generally, if $M$ is an $R$-module for some commutative ring with unity $R$, then the complex $I^pS^*_{\text{GM}}(X; M)$ defined as above as a subcomplex of $S_*(X; M) = S_*(X) \otimes_Z M$, where the tensor product is defined via the unique ring morphism $\mathbb{Z} \to R$ taking 1 to the unity of $R$, naturally inherits the structure of an $R$-module.

Is there really any difference between these two approaches? Yes! Of course the allowability of a simplex is no different in $I^pS^*_{\text{GM}}(X; G)$ from that in $I^pS^*_{\text{GM}}(X) \otimes G$, but now very interesting things can happen due to the boundary allowability requirement. For example, suppose that $\xi$ is a singular chain in $S_*(X)$ and that $\partial \xi = 2\eta$, for some other chain $\eta$. Suppose every simplex of $\xi$ is allowable, but that $\eta$ contains simplices that are not allowable. In this case, $\xi \notin I^pS^*_{\text{GM}}(X)$, and so $\xi \otimes 1 \notin I^pS^*_{\text{GM}}(X) \otimes \mathbb{Z}_2$. However, $\xi \otimes 1$ is an element of $I^pS^*_{\text{GM}}(X; \mathbb{Z}_2)$: by assumption every simplex of $\xi$ (and hence every simplex of $\xi \otimes 1$) is allowable, and now the boundary vanishes by

$$\partial(\xi \otimes 1) = (\partial \xi) \otimes 1 = 2\eta \otimes 1 = 0$$

and so is also allowable! In fact, $\xi \otimes 1$ is a cycle in $I^pS^*_{\text{GM}}(X; \mathbb{Z}_2)$ and may well represent an intersection homology class.

Similarly, suppose there are chains $y \in S_i(X), x \in S_{i+1}(X)$ that that $x \otimes 1, y \otimes 1 \in I^pS^*_{\text{GM}}(X; G)$ and $\partial(x \otimes 1) = y \otimes 1$. It might nonetheless be possible that $x$ is not allowable as an element of $I^pS^*_{i+1}(X)$; for example, perhaps $G$ is 2-torsion and $\partial x = y + 2z$ for some chain $z \in S_i(X)$ containing non-allowable simplices. In this case, $y \otimes 1$ represents a trivial element of $I^pH^G_{i+1}(X; G)$, but it might not be trivial in $H_i(I^pS^*_{i+1}(X) \otimes G)$.

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63The reader should think for a moment about why there’s no other way to write $\xi \otimes 1$ in a different form that portray it as a legitimate element of $I^pS^*_{\text{GM}}(X) \otimes \mathbb{Z}_2$; it’s not completely obvious and would be more difficult to verify with a more complicated group than $\mathbb{Z}_2$. 

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Let us look at an important concrete example where this latter situation occurs quite explicitly:

**Example 5.32.** Let \( X = X^3 \) be the cone \( c \mathbb{RP}^2 \) with the perversity \( \bar{0} \) that assigns 0 to the cone vertex \( v \). We first use homological tools to compute \( I^0 H^G_M (c \mathbb{RP}^2; \mathbb{Z}_2) \) and \( H_*(I^0 S^G_M (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) \), and then we try to understand the reasons for these computations at the chain level in terms of the issues just discussed.

The cone computation of Theorem 4.12 is easily generalized (see Theorem 5.33, below) to establish that

\[
I^0 H^G_M (c \mathbb{RP}^2; \mathbb{Z}_2) \cong \begin{cases} 
0, & i \geq 2, \\
I^0 H_i (\mathbb{RP}^2; \mathbb{Z}_2), & i < 2,
\end{cases}
\]

so \( I^0 H^G_M (c \mathbb{RP}^2; \mathbb{Z}_2) \cong I^0 H^G_1 (c \mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \), and \( I^0 H^G_M (c \mathbb{RP}^2; \mathbb{Z}_2) = 0 \) otherwise.

On the other hand,

\[
I^0 H^G_M (c \mathbb{RP}^2) \cong \begin{cases} 
0, & i \geq 2, \\
I^0 H_i (\mathbb{RP}^2), & i < 2.
\end{cases}
\]

Since \( \mathbb{RP}^2 \) is an unfiltered manifold, \( I^0 H_* (\mathbb{RP}^2) = H_* (\mathbb{RP}^2) \), and so \( I^0 H^G_M (c \mathbb{RP}^2) \cong \mathbb{Z} \), \( I^0 H^G_1 (c \mathbb{RP}^2) \cong \mathbb{Z}_2 \), and \( I^0 H^G_M (c \mathbb{RP}^2) = 0 \) otherwise.

The algebraic universal coefficient theorem then shows that

\[
\begin{align*}
H_0 & (I^0 S^G_* (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) \cong \mathbb{Z}_2 \\
H_1 & (I^0 S^G_* (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) \cong \mathbb{Z}_2 \\
H_2 & (I^0 S^G_* (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) \cong \mathbb{Z}_2 \\
H_i (I^0 S^G_* (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) & = 0, \quad i > 2.
\end{align*}
\]

This does not agree with the computation of \( I^0 H^G_M (c \mathbb{RP}^2; \mathbb{Z}_2) \).

Homologically, this example is particularly illuminating regarding what is really going on. If we look at the cone formula of Theorem 4.12 (and assume, \( 0 < n - \bar{p}(\{v\}) - 1 \)) we see that there is a sharp cut-off: below a certain dimension the intersection homology of the cone on \( X \) agrees with the intersection homology of \( X \) itself, above that it is 0. But the universal coefficient theorem references not just the homology in a given dimension, but it reaches into a lower dimension to pull out torsion information. As a result, as we have seen in this example, the universal coefficient theorem can then cause nontrivial homology to appear above the cutoff dimension where the homology of a cone should be zero. Since all CS sets are cones locally (or products of cones with Euclidean space), we can expect this local issue to percolate into a failure of the universal coefficient theorem in general.

What is going on at the chain level? Let \( \eta \) be a cycle in \( \mathbb{RP}^2 \) representing the generator of \( H_1 (\mathbb{RP}^2) \cong \mathbb{Z}_2 \). Identifying \( \mathbb{RP}^2 \) with \( \{1/2\} \times \mathbb{RP}^2 \subset c \mathbb{RP}^2 \), then \( \eta \) is also a generator of \( I^0 H_1 (c \mathbb{RP}^2) \), \( I^0 H_1 (c \mathbb{RP}^2; \mathbb{Z}_2) \), and \( H_1 (I^0 S_* (c \mathbb{RP}^2) \otimes \mathbb{Z}_2) \), all of which are isomorphic to \( \mathbb{Z}_2 \). Notice that \( \eta \) does not bound in \( c \mathbb{RP}^2 \) with any coefficients because it certainly does not
bound in \( \mathbb{R}P^2 \), and it cannot bound in \( c\mathbb{R}P^2 \) because if we compute \( 2 - \operatorname{codim}(\{v\}) - \bar{0}(\{v\}) = 2 - 3 - 0 = -1 \), we see that no 2-chain can intersect the cone vertex.

Now, let \( \xi \in S_2(\mathbb{R}P^2) \) be a chain such that \( \partial \xi = 2\eta \). Then \( \xi \) represents a cycle in \( I^pS_2^{GM}(\mathbb{R}P^2; \mathbb{Z}_2) \), but it bounds \( \bar{c}\xi \) since we see from the dimension computation that a singular 3-cycle with a vertex at \( v \) is allowable. But \( \bar{c}\xi \) is not allowable as a chain in \( I^pS_2^{GM}(\mathbb{R}P^2) \), now with \( \mathbb{Z} \) coefficients. This is because \( \partial(\bar{c}\xi) = \xi - \bar{c}\partial\xi = \xi - 2\bar{c}\eta \). But \( \bar{c}\eta \) is not allowable, as we have seen that 2-chains may not intersect the cone vertex. Therefore, \( \xi \) does not bound \( \bar{c}\xi \) in \( I^pS_2^{GM}(\mathbb{R}P^2) \otimes \mathbb{Z}_2 \). Technically, this is not sufficient to demonstrate that \( \xi \) represents a non-trivial element of \( H_2(I^pS_*^{GM}(\mathbb{R}P^2) \otimes \mathbb{Z}_2) \), as we have only shown that \( \xi \) does not bound \( \bar{c}\xi \) and not that it can never bound. However, this example should give some idea of the additional intricacies of working with coefficients.

Since \( H_*(I^pS_*^{GM}(X) \otimes G) \) can be computed from \( I^pH_*^{GM}(X) \) via the universal coefficient theorem, the much more interesting groups are \( I^pH_*(X; G) \), and so we shall focus on them.

As mentioned in the example, the entire argument of the proof of Theorem 4.12 can be copied nearly verbatim to establish the following generalization:

**Theorem 5.33.** If \( X \) is a compact filtered space of formal dimension \( n - 1 \), then

\[
I^pH_i^{GM}(cX; G) \cong \begin{cases} 
0, & i \geq n - \bar{p}(\{v\}) - 1, i \neq 0, \\
G, & i \geq n - \bar{p}(\{v\}), i = 0, \\
G, & i = n - \bar{p}(\{v\}) - 1, i = 0, I^p\pi_0H_0^{GM}(X; G) \neq 0, \\
0, & i = n - \bar{p}(\{v\}) - 1, i = 0, I^p\pi_0H_0^{GM}(X; G) = 0, \\
I^pH_i^{GM}(X; G), & i < n - \bar{p}(\{v\}) - 1.
\end{cases}
\]

Similarly, all of our preceding work generalizes, mostly in the evident ways. Our discussion of simplicial-versus-PL intersection homology, behavior under stratified maps, stratified homotopy invariance, relative intersection homology, the long exact sequences of pairs, subdivisions, Mayer-Vietoris sequences, and excision can be generalized to statements about \( I^pH_i^{GM}(X; G) \) nearly verbatim. The one place where we must be slightly more careful is when considering cross products and the K"unneth theorem of the preceding section because there are places where we assumed we were working with free chain complexes. In particular, we used freeness in Lemma 5.12 to argue that \( I^pS_*^{GM}(X) \otimes I^pS_*^{GM}(Y) \subset S_*(X) \otimes S_*(Y) \), which is necessary to have the cross product defined. We also used freeness multiple times in the proof of Theorem 5.28, including where it is needed to invoke the algebraic K"unneth theorem, which itself does not hold for arbitrary rings. In order to extend to more general coefficients, we will need to put in place some restrictions.

First, notice that if \( R \) is a commutative ring with unity then \( S_*(X; R) \cong S_*(X) \otimes_{\mathbb{Z}} R \) and we can extend the cross product to

\[
\varepsilon : S_*(X; R) \otimes_R S_*(Y; R) \to S_*(X \times Y; R)
\]

by

\[
\varepsilon((x \otimes_R r) \otimes_R (y \otimes_R s)) = (x \times y) \otimes_R rs.
\]
Since $r, s \in R$ live in degree 0,
\[
\partial \varepsilon((x \otimes r) \otimes_R (y \otimes s)) = \partial((x \times y) \otimes_R rs) \\
= (\partial(x \times y)) \otimes \mathbb{Z} rs \\
= ((\partial x) \times y + (-1)^{|x|} x \times (\partial y)) \otimes \mathbb{Z} rs \\
= ((\partial x) \times y) \otimes \mathbb{Z} rs + (-1)^{|x|}(x \times (\partial y)) \otimes \mathbb{Z} rs \\
= \varepsilon(((\partial x) \otimes \mathbb{Z} r) \otimes_R (y \otimes \mathbb{Z} s) + (-1)^{|x|}(x \otimes \mathbb{Z} r) \otimes_R ((\partial y) \otimes \mathbb{Z} s)) \\
= \varepsilon(\partial((x \otimes \mathbb{Z} r) \otimes_R (y \otimes \mathbb{Z} s))),
\]
where in these equations we have used $\varepsilon$ in the sense defined here and $\times$ for the cross product with integer coefficients. This shows that our new, more general, $\varepsilon$ is still a chain map. From here on, we will again use $\varepsilon$ and $\times$ interchangeably to denote the cross product, letting context determine which coefficients are meant. Additionally, when fully in the setting of $R$ coefficients, we will often write $\otimes$ rather than $\otimes_R$.

Now, to extend the cross product to intersection chains with coefficients, we assume that $R$ is a Dedekind domain. When working with coefficient rings in what follows, we will often require them to be Dedekind domains due to the nice homological algebra properties they possess. Recall that a Dedekind domain is an integral domain with the property that every submodule of a projective $R$-module is projective\textsuperscript{64}. In particular, principal ideal domains and fields are Dedekind domains. It is also true that any torsion-free module over a Dedekind domain is flat; in fact, this is true more generally of Prüfer domains\textsuperscript{65} [Proposition 4.20], which could equally well be used below for the arguments where only this property of Dedekind domains is needed.

Since each $S_j(X; R)$ and $\mathfrak{c}_i(X; R)$ are torsion free $R$-modules for any $R$, so will be their respective submodules $I^p S^G_i(X; R)$ and $I^p \mathfrak{c}^G_i(X; R)$, so if $R$ is a Dedekind domain (or, more generally, a Prüfer domain), these $R$-modules will be flat. In fact, as $S_j(X; R)$ is a free $R$-module, $I^p S^G_i(X; R)$ is projective, though in proofs for which we want to run parallel arguments for singular and PL chains, we will focus on the flatness. Therefore, for the purpose of short exact sequences of tensor products, $I^p S^G_i(X; R)$ has the same properties we needed before when we use that $I^p S^G_i(X)$ is flat as an abelian group. In particular, since we have an inclusion $I^p S^G_i(Y; R) \hookrightarrow S_j(Y; R)$, tensoring with the flat module $I^p S^G_i(X; R)$ yields an inclusion $I^p S^G_i(X; R) \otimes_R I^p S^G_j(Y; R) \hookrightarrow I^p S^G_i(X; R) \otimes_R S_j(Y; R)$, and similarly since $S_j(Y; R)$ is flat, tensoring with the inclusion $I^p S^G_i(X; R) \hookrightarrow S_j(Y; R)$ yields an inclusion $I^p S^G_i(X; R) \otimes_R S_j(Y; R) \hookrightarrow I^p S^G_i(X; R) \otimes_R S_j(Y; R)$. Summing over indices, we obtain once again $I^p S^G_i(X; R) \otimes_R I^p S^G_j(Y; R) \subset S_*(X; R) \otimes_R S_*(Y; R)$, which allows us

\textsuperscript{64}This is essentially taken as the definition of a Dedekind domain in Cartan-Eilenberg\textsuperscript{17} Section VII.5 and Theorem I.5.4. Exercise 20 to Section 4 of Chapter VII of\textsuperscript{10} shows that this property can be derived from other defining properties of Dedekind domains. A short literature search reveals that there are a very large number of equivalent definitions for Dedekind domains!

\textsuperscript{65}Prüfer domains satisfy the more general property that submodules of\textsuperscript{finitely-generated} projective modules are projective. A module over a Prüfer domain is torsion free if and only if it is flat\textsuperscript{65} Proposition 4.20].

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to restrict the cross product to intersection chain complexes. The properties of section 5.2.1 follow.

We can now generalize Theorem 5.28 to the following:

**Theorem 5.34.** Suppose $X$ is a filtered space with perversity $\bar{p}_X$ and that $M$ is an $n$-dimensional manifold with its trivial filtration. Filter $M \times X$ with product filtration so that $(M \times X)^i = M \times X^{i-n}$, and define a perversity $\bar{p}$ on $M \times X$ whose value on $M \times S$ is $\bar{p}(S)$. Let $R$ be a Dedekind domain. Then the cross product induces an isomorphism $H_*(S_*(M;R) \otimes_R \check{P} S_*^{GM}(X;R)) \cong \check{I}^p H_*^{GM}(M \times X;R)$.

If $X$ is a PL manifold stratified space and $M$ is a PL manifold, then the same conclusion holds replacing singular chains with PL chains.

*Proof.* The proof is essentially the same as that of Theorem 5.28. □

**Remark 5.35.** The assumption that $R$ be Dedekind is not needed in the proof of Theorem 5.34 to define the cross product for singular chains since $S_*(M;R)$ is free for any manifold, which is sufficient to have $S_*(M;R) \otimes_R \check{P} S_*^{GM}(X;R) \subset S_*(M;R) \otimes_R S_*(X;R)$ for any ring.

**Corollary 5.36.** Under the assumptions of Theorem 5.34, if $A \subset X$, then the cross product induces an isomorphism $H_*(S_*(M;R) \otimes_R \check{P} S_*^{GM}(X,A;R)) \cong \check{I}^p H_*^{GM}(M \times X, M \times A;R)$.

If $X$ is a PL manifold stratified space, $A$ is a PL subspace, and $M$ is a PL manifold, then the same conclusion holds replacing singular chains with PL chains.

The proof is the same as that for Corollary 5.29 using Theorem 5.34 in place of theorem 5.28

### 5.4.1 Universal coefficient theorems

A natural question to ask is under what circumstances might it be true that $\check{I}^p S_*^{GM}(X;G)$ and $\check{I}^p S_*^{GM}(X) \otimes G$ have the same homology groups? This would be a useful property, for then we could use the universal coefficient theorem for computations; at the same time, knowing when this property fails tells us when $\check{I}^p H_*^{GM}(X;G)$ really is fundamentally different from $\check{I}^p H_*^{GM}(X)$. Surprisingly enough, it turns out that the situation of Example 5.32 is essentially the only thing that can go wrong.

Recall that in Example 5.32 we saw a situation where $\check{I}^p H_*^{GM}(cX;G) \not\cong H_*(\check{I}^p S_*^{GM}(cX) \otimes G)$ because the way the universal coefficient theorem blends information from two dimensions contradicts the strict truncation we see in the cone formula of Theorem 5.33. In particular, in the first dimension in which the cone formula tells us that $\check{I}^p H_*^{GM}(cX;G)$ must be 0 (namely dimension $i = n - \bar{p}(\{v\}) - 1$ if $X$ has dimension $n - 1$), the universal coefficient computation reaches down to provide a possibly nontrivial term in $H_0(\check{I}^p S_*^{GM}(cX) \otimes G)$ that comes from the torsion of $\check{I}^p H_{i-1}^{GM}(cX)$. However, if $\check{I}^p H_{i-1}^{GM}(cX) \ast G = 0$, we eliminate this problem. This discussion motivates the following definition, which is a slight generalization of that provided by Goresky and Siegel [16], who first considered this issue.
Definition 5.37. A CS set $X$ is called locally $(\bar{p}, \mathbb{Z}; G)_{\ast}$-torsion free if for each point $x \in X$ and for each distinguished neighborhood $\cong \mathbb{R}^k \times cL$ of $x$, $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L) \ast G = 0$, where $S$ is the stratum of $X$ containing $x$.

More specifically, if the condition holds for all points in a stratum $S \subset X$, we say that $X$ is locally $(\bar{p}; G)_{\ast}$-torsion free along $S$.

Notice that since $\dim(L) + \dim(S) + 1 = n$, the condition can be rewritten as $I^\hat{p}H^{GM}_{\codim(S) - \bar{p}(S) - 2}(L) \ast G = 0$, which more closely approximates the original definition in [66].

If $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L) \ast G = 0$ for all $G$, then we simply say that $X$ is locally $(\bar{p}, \mathbb{Z})_{\ast}$-torsion free. This is equivalent to asking that $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L)$ is flat as a $\mathbb{Z}$-module [64, Theorem XVI.3.11]. In particular, this means that $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L)$ is torsion free by [63, Proposition 4.20].

Remark 5.38. More generally, if $R$ is a Dedekind domain and $M$ is an $R$-module, we can say that a CS set $X$ is locally $(\bar{p}, R; M)_{\ast}$-torsion free if for each point $x \in X$ and for each distinguished neighborhood $\cong \mathbb{R}^k \times cL$ of $x$, $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L; R) \ast_R M = 0$, where $S$ is the stratum of $X$ containing $x$. Again if $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L; R) \ast_R M = 0$ for all $M$, we simply say that $X$ is locally $(\bar{p}, R)_{\ast}$-torsion free, and this is equivalent to asking that $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L; R)$ be flat as an $R$-module by [64, Theorem XVI.3.11]. In particular, this means that $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L)$ is torsion free (as an $R$-module) by [63, Proposition 4.20].

For the remainder of this section, we will concentrate on $\mathbb{Z}$-modules, i.e. abelian groups, but we invite the reader to formulate the appropriate generalizations.

Lemma 5.40 and its corollary, which we will state and prove momentarily, show that the intersection homology of a distinguished neighborhood of a point $x$ depends only on the stratum containing $x$ and not on the specific choice of $x$ or its distinguished neighborhood. Therefore, the condition of the Definition 5.37 can be alternatively stated as requiring only that each stratum contains some point with some link such that $I^\hat{p}H^{GM}_{\dim(L) - \bar{p}(S) - 1}(L) \ast G = 0$.

First, though, we discuss two examples.

Example 5.39. Of course if $X$ is a manifold, trivially stratified, then $X$ is trivially locally $(\bar{p}, \mathbb{Z})_{\ast}$-torsion free for any $\bar{p}$. A more general, and somewhat remarkable, example is that all CS sets are locally $(\bar{t}, \mathbb{Z})_{\ast}$-torsion free for any $G$, where $\bar{t}$ is the top perversity such that $\bar{t}(S) = \codim(S) - 2$ for any singular stratum $S$. To see this, we observe that if $S$ is a stratum and $L$ is a link of a point in $S$, then $\dim(X) = \dim(S) + \dim(L) + 1$, so $\dim(S) = \dim(L) + 1$. Therefore,

$$\dim(L) - \bar{p}(S) - 1 = \dim(L) - (\codim(S) - 2) - 1$$

$$= \dim(L) - (\dim(L) + 1 - 2) - 1$$

$$= 0.$$

But by Example 3.37, $I^\hat{p}H^{GM}_{0}(L) = \mathbb{Z}^m$, where $m$ is the number of path components of $L$. Since $\mathbb{Z}^m$ is free, $I^\hat{p}H^{GM}_{0}(L) \ast G = 0$ for any $G$. This argument also carries over to coefficients in any Dedekind domain $R$.

\footnote{I had never noticed this example until I discovered it quite recently in [18].}
Lemma 5.40. Let $X$ be a CS set and $x \in X_k \subset X$. For $i = 1, 2$, let $N_i \cong \mathbb{R}^k \times cL_i$ be distinguished neighborhood of $x$. Then $I^pH_*^{GM}(L_1) \cong I^pH_*^{GM}(L_2)$ and $I^pH_*^{GM}(N_1) \cong I^pH_*^{GM}(N_2)$.

Proof. First we claim that it is possible to shrink $N_2$ by a stratified homotopy equivalence to a smaller distinguished neighborhood $N_2' \subset N_2$ such that also $N_2' \subset N_1$ and $N_2' \cong N_2$. Suppose $x$ is contained in the stratum $S$. For simplicity of argument, via the given homeomorphisms, let us identify $N_2$ canonically with $\mathbb{R}^k \times cL_2$, and let us assume that $x = (0, v)$, where 0 is the origin of $\mathbb{R}^k$. Then $N_1 \cap N_2 \cap S$ is an open subset of $\mathbb{R}^k$ and so contains a closed disk $D_r$ of some radius $r$ around the origin. $N_2$ is then stratum-preserving homotopy equivalent to $D_r \times cL_2$. Furthermore, since $N_1 \cap (D_r \times cL_2)$ must be a neighborhood of $D_r \times \{v\}$ in $D_r \times cL_2$ and since $D_r$ is compact, by the Tube Lemma (Lemma 26.8), there must be a neighborhood $W$ of $v$ in $cL_2$ such that $D_r \times W \subset N_1 \times (D_r \times cL_2)$. But $v \subset cL_2$ has a fundamental system of neighborhoods of the form

$$c_sL_2 = ([0, s) \times L_2/\sim) \subset cL_2 = ([0, 1) \times L_2/\sim)$$

for $0 < s \leq 1$; this also follows from the Tube Lemma, using the compactness of $L_2$ and the definition of the quotient topology. So for some $s$, $0 < s \leq 1$, there is a cone $c_sL_2 \subset W$. Then $N_2' = D_r \times c_sL_2 \subset N_1 \cap N_2$ is stratified homeomorphic to $\mathbb{R}^k \times cL_2$, and the inclusion into $\mathbb{R}^k \times cL_2$ is a stratified homotopy equivalence (in the cone direction, we can retract along the cone lines).

Next, we notice that $N_i - S \cong \mathbb{R}^k \times (cL_i - \{v_i\})$, which is stratified homotopy equivalent to $L_i$, and so it suffices to show $I^pH_*^{GM}(N_1 - S) \cong I^pH_*^{GM}(N_2 - S)$.

Now, let $V = N_2 - X^k$ and $U = N_1 - X^k$. By the above argument, let us assume that $V \subset U$. But then by repeating the above argument, we can assume there is a stratified homeomorphic copy $U'$ of $U$ inside $V$, and, similarly, a stratified homeomorphic copy $V'$ of $V$ inside $U'$. In other words, we can have a sequence of spaces

$$V' \xrightarrow{f} U' \xrightarrow{g} V \xrightarrow{h} U$$

where $U \cong U'$, $V \cong V'$, and furthermore the inclusions $hg : U' \hookrightarrow U$ and $gf : V' \hookrightarrow V$ are stratified homotopy equivalence, and so induces isomorphisms in intersection homology. Therefore, taking the intersection homology of the sequence (5.4.1) shows that the map $f : I^pH_*^{GM}(V') \rightarrow I^pH_*^{GM}(U')$ must be injective, and the map $h : I^pH_*^{GM}(V) \rightarrow I^pH_*^{GM}(U)$ must be surjective. Putting this together with the isomorphisms from the stratified homotopy equivalence, it follows $h(gf)$ is surjective, while the same map written as $(hg)f$ is injective. Thus, $hgf : I^pH_*^{GM}(V') \rightarrow I^pH_*^{GM}(U)$ must be an isomorphism. Hence, the intersection homology groups of $L_1$ and $L_2$ must be isomorphic.

The last statement now follows because stratified homotopy invariance and the cone formula imply that the intersection homology of a distinguished neighborhood depends only on that of the link. 

\[\square\]

67This is simply a convenience so that we don’t have to say “is homeomorphic to” in each sentence instead of simply “is”. There is no difficulty generalizing the argument back.
Corollary 5.41. Let $X$ be a CS set. Then the intersection homology $I^pH_*^{GM}(L)$ of a link $L$ of a point $x$ in a stratum of $S$ depends only on $S$. In other words, all links for any distinguished neighborhoods of any points in $S$ have isomorphic intersection homology groups.

Proof. Let $S$ be a stratum of $X$, and let $x_0 \in S$. The preceding Lemma shows that all possible links of a given point in $S$ have isomorphic intersection homology. Let $W$ be the set of points of $S$ whose links have intersection homology isomorphic to that of the links of $x_0$. We will show that $W$ is both open and closed as a subset of $S$. Since $S$ is connected, this will imply $W = S$.

Let $x$ be any point in $W$, and let $N \cong \mathbb{R}^k \times cL$ be a distinguished neighborhood of $x$. Then the image under this homeomorphism of all points of the form $(z, v) \subset \mathbb{R}^k \times cL$, with $v$ representing the cone vertex, share this distinguished neighborhood and hence have stratified homeomorphic links. So each such point has a link whose intersection homology is isomorphic to that of a link of $x$, which is in turn isomorphic to the intersection homology of the link of $x_0$. This shows that $W$ must be open.

Next, suppose $y$ is a point in the closure of $W$, and let $N \cong \mathbb{R}^k \times cL$ be a distinguished neighborhood of $y$. The neighborhood $N$ must contain a point $z \in W$. But then $y$ and $z$ share a distinguished neighborhood and hence a link. So the intersection homology of the links of $y$ must agree with that of the links of $z$, which agree with the intersection homology of the links of $x_0$. So $y \in W$, and $W$ must be closed.

We will now use our Mayer-Vietoris argument, Theorem 5.3, to show that $I^pH_*^{GM}(cX; G) \cong H_*(I^pS_*^{GM}(X) \otimes G)$ for locally $(\bar{p}, \mathbb{Z}; G)^{GM}$-torsion free CS sets.

Theorem 5.42. Suppose $X$ is a locally $(\bar{p}, \mathbb{Z}; G)^{GM}$-torsion free CS set. Then $I^pH_*^{GM}(X; G) \cong H_*(I^pS_*^{GM}(X) \otimes G)$.

Corollary 5.43. For any CS set and any field $F$ of characteristic 0, $I^pH_*^{GM}(X; F) \cong I^pH_*^{GM}(X) \otimes \mathbb{Z} F$.

Proof of Corollary. As $F$ is torsion free, any space automatically satisfies the locally torsion free condition with respect to $F$ by [77, Theorem 54.4.c], so the result follows from Theorem 5.42 and the algebraic universal coefficient theorem.

Corollary 5.44. For any CS set and any abelian group $G$, $I^\ell H_*^{GM}(X; G) \cong H_*(I^\ell S_*^{GM}(X) \otimes G)$.

Proof of Corollary. This follows immediately from Theorem 5.42 and Example 5.39.

Proof of Theorem 5.42. We will use the Mayer-Vietoris argument, Theorem 5.3, with $F_*(U) = H_*(I^pS_*^{GM}(U) \otimes G)$ and $G_*(U) = I^pH_*^{GM}(U; G)$. The natural transformation $\Phi : F_*(U) \to G_*(U)$ is induced by the inclusion map $I^pS_*^{GM}(U) \otimes G \to I^pS_*^{GM}(U; G)$; notice that if $\xi$ is allowable in $I^pS_*^{GM}(U)$, then $\xi \otimes g$ will be allowable in $I^pS_*^{GM}(U; G)$, and $I^pS_*^{GM}(U) \otimes G$ is generated by terms of this form.

We must show that the conditions of Theorem 5.3 hold.
Condition (1): Notice that we have a commutative diagram with exact rows

\[
0 \longrightarrow I^\bar{p}S^G_*(U \cap V) \otimes G \longrightarrow (I^\bar{p}S^G_* (U) \oplus I^\bar{p}S^G_* (V)) \otimes G \longrightarrow (I^\bar{p}S^G_* (U) + I^\bar{p}S^G_* (V)) \otimes G \longrightarrow 0
\]

To make sense of the sum terms, we identify \( I^\bar{p}S^G_* (U) + I^\bar{p}S^G_* (V) \subset I^pS^G_* (U \cup V) \) and \( I^\bar{p}S^G_* (U; G) + I^\bar{p}S^G_* (V; G) \subset I^pS^G_* (U \cup V; G) \). The top row is exact by tensoring the short exact sequence of free groups

\[
0 \longrightarrow I^\bar{p}S^G_* (U \cap V) \longrightarrow I^\bar{p}S^G_* (U) \oplus I^\bar{p}S^G_* (V) \longrightarrow I^\bar{p}S^G_* (U) + I^\bar{p}S^G_* (V) \longrightarrow 0
\]

with \( G \), and the bottom sequence is exact by our standard Mayer-Vietoris arguments. The lefthand map is induced by inclusion. Furthermore, by distributivity of tensor products over direct sums, \( (I^\bar{p}S^G_* (U) \oplus I^\bar{p}S^G_* (V)) \otimes G \cong (I^\bar{p}S^G_* (U) \otimes G) \oplus (I^\bar{p}S^G_* (V) \otimes G) \), and so we may interpret the middle vertical map as a direct sum of inclusions, and so the middle map on homology corresponds to \( \Phi \). It is not difficult to check by hand that the diagram commutes using that all groups are subgroups of the corresponding groups of the form \( I^pS^* \) \( S_*(W) \otimes G \). Finally, we have a commutative diagram

\[
(I^\bar{p}S^G_* (U) + I^\bar{p}S^G_* (V)) \otimes G \longrightarrow I^\bar{p}S^G_* (U \cup V) \otimes G
\]

\[
I^\bar{p}S^G_* (U; G) + I^\bar{p}S^G_* (V; G) \longrightarrow I^pS^G_* (U \cup V; G).
\]

Commutativity is again easy to check by viewing all groups as subgroups of \( S_*(U \cup V) \otimes G \). The top induces homology isomorphisms by the proof of Theorem 4.33\footnote{Note that if \( \xi \) is not allowable in \( S_*(W) \), then no multiple of \( \xi \) can be allowable either, and so \( S_*(W) / I^pS^G_* (W) \) is torsion free. Thus \( S_*(W) / I^pS^G_* (W) \otimes G = 0 \).} and the algebraic universal coefficient theorem, and the bottom induces isomorphisms by the analogue of Theorem 4.33 with coefficients. So the resulting long exact Mayer-Vietorises homology sequences are compatible with \( \Phi \).

Condition 2: This property is satisfied for both \( F_* \) and \( G_* \) using Lemma 5.5 and minor modifications of the arguments in the proof of Lemma 5.6

Condition 3: We must show that if \( L \) is a compact filtered space such that \( X \) has an open subset filtered homeomorphic to \( \mathbb{R}^i \times cL \) and \( \Phi : F_*(\mathbb{R}^i \times (cL - \{v\})) \to G_*(\mathbb{R}^i \times (cL - \{v\})) \) is an isomorphism, then so is \( \Phi : F_* (\mathbb{R}^i \times cL) \to G_*(\mathbb{R}^i \times cL) \).

Using the stratified homotopy invariance of both functors \( F_* \) and \( G_* \), this is equivalent to assuming that \( H_*(I^\bar{p}S^G_* (L) \otimes G) \to I^\bar{p}H^G_* (L; G) \) is an isomorphism and needing to
verify that, as a consequence, \( H_*(\ell^p G^{GM}(cL) \otimes G) \rightarrow G_*(U) = I^p H_*^{GM}(cL; G) \) is an isomorphism. By the assumption that \( X \) is locally \((\bar{p}, \mathbb{Z}; G)^{GM}\)-torsion free, we must have \( I^p H_{\dim(L)-\bar{p}(S)-1}(L)_* \ast G = 0 \).

Consider the commutative diagram induced by inclusions

\[
\begin{array}{ccc}
H_i(\ell^p S_*^{GM}(L) \otimes G) & \longrightarrow & I^p H_i^{GM}(L; G) \\
\downarrow & & \downarrow \\
H_i(\ell^p S_*^{GM}(cL) \otimes G) & \longrightarrow & I^p H_i^{GM}(cL; G).
\end{array}
\]

By assumption, the top horizontal map is an isomorphism for all \( i \). Via the cone formula, Theorem \( 5.33 \), the righthand map is an isomorphism for all \( i < \dim(L) - \bar{p}(S) \). Similarly, by Theorem \( 4.12 \), \( I^p H_i^{GM}(L) \rightarrow I^p H_i^{GM}(cL) \) is an isomorphism in the same range, and hence this is also true of the lefthand map using the universal coefficient theorem. Hence the bottom map is also an isomorphism in this range.

For \( i \geq \dim(L) - \bar{p}(S) \), \( i \neq 0 \), by Theorem \( 5.33 \), \( I^p H_i^{GM}(cL; G) = 0 \). But this is also true of \( H_i(\ell^p S_*^{GM}(cL) \otimes G) \) using Theorem \( 4.12 \) and the universal coefficient theorem, and the assumption that \( I^p H_{\dim(L)-\bar{p}(S)-1}(L)_* \ast G = 0 \) (here is where the assumption that \( X \) is locally \((\bar{p}, \mathbb{Z}; G)^{GM}\)-torsion free pays off!).

Finally, if \( i = 0 \geq \dim(L) - \bar{p}(S) \), then \( I^p H_0^{GM}(cL) \cong \mathbb{Z} \) or \( 0 \), so \( H_0(\ell^p S_*^{GM}(cL) \otimes G) \cong G \) or \( 0 \), respectively, by the universal coefficient theorem. Similarly, \( I^p H_0^{GM}(cL; G) \cong G \) or \( 0 \) in the corresponding situations by Theorem \( 5.33 \) note that \( I^p H_0^{GM}(cL; G) = 0 \) if and only if there are no allowable 0-simplices in \( cX \), which is precisely when \( I^p H_0^{GM}(cL) = 0 \). In both non-trivial cases, the elements of the groups can be represented in the form \( \sigma_0 \otimes g \), where \( \sigma_0 \) is any allowable 0-simplex and \( g \in G \), so the bottom map of the diagram is again an isomorphism.

Thus \( \Phi : H_i(\ell^p S_*^{GM}(cL) \otimes G) \rightarrow I^p H_i^{GM}(cL; G) \) is an isomorphism for all \( i \).

**Condition 4** If \( U = \emptyset \), \( F_*(U) = G_*(U) = 0 \), so suppose \( U \subset X \) is an open subset of \( X \) homeomorphic to Euclidean space and contained within a stratum \( S \). Since the images of simplices of \( S_*(U) \) are contained completely in \( U \) and cannot intersect other strata, the allowability condition for an \( i \)-simplex is that

\[
\Delta^i \subset \{ i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i \},
\]

or, in other words, that \( i \leq i - \text{codim}(S) + \bar{p}(S) \). But this is simply the condition that \( \bar{p}(S) \geq \text{codim}(S) \), which is independent of \( i \). So, depending on the value of \( \bar{p}(S) \) and the codimension of \( S \), either all simplices are allowable or none are! If none are, then \( I^p S_*^{GM}(U) \otimes G = 0 = I^p S_*^{GM}(U; G) \), and if all are, \( I^p S_*^{GM}(U) \otimes G = S_*(U) \otimes G = S_*(U; G) = I^p S_*^{GM}(U; G) \). So either way \( \Phi \) is an isomorphism on \( U \).

**Corollary 5.45.** Suppose \( X \) is a locally \((\bar{p}, \mathbb{Z}; G)^{GM}\)-torsion free CS set and that \( A \subset X \) is also a locally \((\bar{p}; G)^{GM}\)-torsion free CS set, in particular if \( A \) is an open subset of \( X \). Then \( I^p H_*^{GM}(X; A; G) \cong H_*(I^p S_*^{GM}(X; A) \otimes G) \).
Proof. As seen in its proof, the isomorphism of Theorem 5.42 is induced by obvious chain map \( I^pS^\ast_GM(X) \otimes G \to I^pS^\ast_GM(X; G) \). So consider the diagram

\[
\begin{array}{cccccc}
0 & \to & I^\bar{p}S^\ast_GM(A) \otimes G & \to & I^\bar{p}S^\ast_GM(X) \otimes G & \to & I^\bar{p}S^\ast_GM(X, A) \otimes G & \to & 0 \\
0 & \to & I^\bar{p}S^\ast_GM(A; G) & \to & I^\bar{p}S^\ast_GM(X; G) & \to & I^\bar{p}S^\ast_GM(X, A; G) & \to & 0.
\end{array}
\]

The top row is obtained by tensoring the short exact sequence for the pair \((X, A)\) with \(G\). Since \(I^\bar{p}S^\ast_GM(X, A)\) is a subgroups (i.e. \(\mathbb{Z}\)-module) \(S^\ast(X, A)\), which is a free \(\mathbb{Z}\)-modules, \(I^\bar{p}S^\ast_GM(X, A)\) is also free as \(\mathbb{Z}\) is a PID \([64, \text{Theorem III.7.1}]\). Therefore, the short exact sequence of the pair \((X, A)\) splits \([77, \text{Corollary 23.2}]\), and so tensoring with \(G\) preserves exactness \([77, \text{Theorem 50.4}]\). The bottom row is exact by the definition of \(I^\bar{p}S^\ast_GM(X, A; G)\) as the quotient \(I^\bar{p}S^\ast_GM(X; G)/I^\bar{p}S^\ast_GM(A; G)\). It is straightforward that the left-hand vertical square commutes, and so the right hand vertical map is induced as the quotient map; commutativity of the right-hand square follows.

The corollary now follows from the ensuing diagram of long exact sequences, Theorem 5.42 and the five lemma. \(\square\)

5.5 Equivalence on PL spaces

In this section we will show that for a PL CS set \(X\), the PL intersection homology groups \(I^\bar{p}S^\ast_GM(X; G)\) are isomorphic to the singular intersection homology groups \(I^\bar{p}H^\ast_GM(X; G)\). This fact is certainly well known via sheaf theory \([43]\). A proof without sheaf theory is suggested in King \([61]\), utilizing King’s Theorem 10. However, on close examination, it was not completely clear to the author precisely what King had in mind for an “ordered PL theory”, at least not without requiring some significant additional work to verify that such a theory would have the needed properties. So here we take a slightly different route (though no doubt this is not very far from what King had in mind).

The idea here is that we would like a nice way to assign a singular chain to a simplicial chain. This is ordinarily done by choosing an ordering of the vertices of a triangulation and using that to decide precisely which singular simplex map to use to represents a simplex in the triangulation. The difficulty comes when we start working with PL chains, which are identified with their subdivisions. Then how do we choose simplices in the subdivision? It is a bit easier to work with barycentric subdivisions, because then we can control in precisely what order we add new vertices. The difficulty then is that while we are free to use just these barycentric subdivision of a triangulation of our space \(X\), we are also going to need to compute PL intersection homology groups of various open subsets of \(X\) in order to employ Mayer-Vietoris sequences. So our first job will be to show that we can compute the intersection homology of these subsets using only simplices coming from the barycentric subdivisions of our triangulation of the entire space. This is the content of Lemma 5.46. Then we will show that the intersection homology groups described this way also possess a
nice map to singular intersection homology. Then in Theorem [5.47] we will use an argument in the spirit of Theorem [5.3] (which is itself based on King’s Theorem 10!) to show that PL and singular intersection homology agree on PL CS sets. It seems reasonable to conjecture that such an isomorphism holds more generally for any PL filtered set, though unfortunately we will not obtain such a result here.

The techniques of this section are not particular to the coefficients, so to keep the notation from getting even more cluttered than necessary, we will use the notation for integers coefficients (meaning that we keep $G$ tacit) in the proofs, though, for completeness we, provide the full notation in the statements of the theorems.

Our first result is applicable to PL filtered spaces in general. Suppose $U \subset X$ is an open subset of a PL filtered space. Let $T$ be a particular triangulation of $X$, and let $T^i$ be the $i$th barycentric subdivision of $T$. The subdivision maps $C^T(X) \to C^{T+1}(X)$ form a direct system, whose limit we denote $C^T(X)$. Clearly we have a chain map, in fact an inclusion, $C^T(X) \to C^i(X)$, and similarly for intersection chains $I^pC^{GM,T}(X) \to I^pC^{GM}(X)$. If we let $I^pC^{GM,T}(U) \subset I^pC^{GM,T}(X)$ be the subcomplex consisting of chains supported in $U$, then we similarly have a monomorphism $\phi : I^pC^{GM,T}(U) \to I^pC^{GM}(U)$. Notice that elements of $I^pC^{GM,T}(U)$ are not defined with respect to any fixed triangulation of $U$, but are rather defined with respect to triangulations of $X$ and have their support in $U$.

**Lemma 5.46.** For a PL filtered space $X$ with open subset $U$, the map $\phi : I^pC^{GM,T}(U;G) \to I^pC^{GM}(U;G)$ defined above induces isomorphisms $\phi : I^pC^{GM,T}(U;G) \to I^pC^{GM}(U;G)$.

**Proof.** The proof utilizes the techniques used to prove Theorem [3.26].

We begin with surjectivity. Let $\xi$ be a chain representing an element of $I^p\gamma^{GM}(U)$. Utilizing the compactness of the support of $\xi$, we can find an iterated barycentric subdivision $T^i$ of $T$ such that every simplex of $T^i$ that intersects $\xi$ is contained in $U$. Furthermore, by definition, there is a triangulation $T_1$ of $X$ such that $\xi$ is represented as a simplicial chain with respect to $T_1$. Let $T^i_1$ be a common subdivision of $T^i$ and $T_1$, and now let $\xi$ be represented as a simplicial chain with respect to $T^i_1$. Since we may assume that $i > 0$, we may assume without loss of generality that $T^i$ is a full triangulation. The proof of Lemma [3.29] then shows how to construct an allowable homology in $I^pC^{GM}(X)$ from $\xi$ to an element of $I^pC^{GM,T}(X)$. However, the details of that argument show that if $\tau$ is a simplex of $\xi$, then there is a prism providing the “homology over $\tau$”, and the image of the prism lies within a single simplex of $T^i$. So, in particular, this homology is contained within $U$ and demonstrates that $\phi$ must be surjective.

Similarly, if $\xi$ is a cycle in $I^pC^{GM,T}(U)$ whose image in $I^pC^{GM}(U)$ bounds an allowable PL chain $\eta$, then we can find a subdivision $T^i$ with respect to which we can represent $\xi$ as a simplicial chain such that every simplex of $T^i$ that intersects $\eta$ is contained in $U$, a $T_1$ with respect to which $\eta$ is simplicial, and a common refinement $T^i_1$. Then Lemma [3.27] provides a chain map $\mu : I^pC^{GM,T_1}(X) \to I^pC^{GM,T}(X)$ which is a left inverse to the subdivision map, so in particular it takes the subdivision of $\xi$ in $T^i_1$ back to the simplicial representative of $\xi$ in $T^i$. Once again, the proof of Lemma [3.27] shows that $\mu$ keeps the image of each simplex of $\eta$ within a single simplex of $T_1$ and so the image of the triangulation of $\eta$ in $T^i_1$ under $\mu$ lives in $I^pC^{GM,T^i}(U)$ and has boundary $\xi$. 

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This completes the lemma.

We make the following observations concerning the intersection homology groups $I^b S_*^{GM,T}(U)$ for a triangulation $T$ of a PL filtered space $X$ of which $U$ is an open subset:

1. By the same arguments used in Section 4.4.1 we have excision $I^b S_*^{GM,T}(U,A) \cong I^b S_*^{GM,T}(U-K,A-K)$ for $K \subset A \subset U$ with the closure of $K$ in $U$ contained in the interior of $A$ in $U$ (which is equal to the interior of $A$ in $X$, since $U$ is open in $X$).

For example, if $[\xi] \in I^b S_*^{GM,T}(U,A)$ is represented by a chain $\xi$ in some subdivision of $T$ that is also contained in $U$, then we can use the same construction as in the proof of Theorem 4.32 to find a further barycentric subdivision of $T$ with respect to which we can split the image of $\xi$ into two $p$-allowable pieces, one contained in $U-K$ and the other contained in $A$. The rest of the argument for excision is then exactly as in the proof of Theorem 4.32. Similarly, minor modifications to the proof of Theorem 4.33 establish that, for two open subsets $U,V \subset X$, there are Mayer-Vietoris sequences

$$
\rightarrow I^b S_*^{GM,T}(U \cap V) \rightarrow I^b S_*^{GM,T}(U) \oplus I^b S_*^{GM,T}(V) \rightarrow I^b S_*^{GM,T}(U \cup V) \rightarrow .
$$

2. Via the standard compactness of chains argument as in Lemma 5.6 if $U_\alpha$ is an increasing sequence of open subsets of $X$, then $\lim \rightarrow I^b S_*^{GM,T}(U_\alpha) \cong I^b S_*^{GM,T}(\cup U_\alpha)$.

Now, continuing to assume we have a triangulation $T$ of $X$, we may choose a total ordering of the vertices of $T$ and use this to construct a chain map $C_*^T(X) \rightarrow S_*^T(X)$ in the usual way (see, e.g. [77]) by using the ordering of the vertices on a (nondegenerate) simplex $[v_0, \ldots, v_i]$ to prescribe a linear map $\Delta^i \rightarrow X$. If $T^1$ is the first barycentric subdivision of $T$, we can partially order the vertices of $T^1$ consistently with the vertices of $T$ by making each barycenter of a $j$ simplex smaller in order than a barycenter of a $k$ simplex if $j < k$. This is enough to prescribe a commutative diagram

$$
\begin{array}{ccc}
C_*^T(X) & \rightarrow & S_*^T(X) \\
\downarrow & & \downarrow \\
C_*^{T^1}(X) & \rightarrow & S_*^T(X),
\end{array}
$$

where the vertical maps are barycentric subdivision operators.

Continuing this process, we obtain a map $C_*^T(X) \rightarrow \mathfrak{S}_*(X)$, where $\mathfrak{S}_*(X)$ is the limit of $S_*^T(X)$ under the barycentric subdivision maps, and similarly we obtain maps $\psi : I^b \mathfrak{S}_*^{GM,T}(X) \rightarrow I^b \mathfrak{S}_*^{GM}(X)$, where $I^b \mathfrak{S}_*^{GM}(X)$ is again a limit under barycentric subdivision. This is well-defined as the barycentric subdivisions of PL and singular intersection chains are allowable by Lemmas 3.23 and 4.35. If $U \subset X$ is open, then since the image of $I^b \mathfrak{S}_*^{GM,T}(U) \subset I^b \mathfrak{S}_*^{GM,T}(X)$ under $\psi$ consists of chains supported in $U$, $\psi$ restricts to a chain map $\psi : I^b \mathfrak{S}_*^{GM,T}(U) \rightarrow I^b \mathfrak{S}_*^{GM}(U)$, where $I^b \mathfrak{S}_*^{GM}(U) \subset I^b \mathfrak{S}_*^{GM}(X)$ is the limit under barycentric subdivisions of
\(I^pS_{\mathcal{G}}^G(U)\). By Lemma 4.36 the barycentric subdivision map \(I^pS_{\mathcal{G}}^G(U) \to I^pS_{\mathcal{G}}^G(U)\) induces the identity on homology, and so by the commutation of direct limits with homology functors, \(H_*(I^p\mathcal{G}_G(U)) \cong I^pH_*^G(U)\).

We now proceed on to the main theorem of this section. The extra generality inherent in not assuming that \(X\) is itself a PL CS set will be utilized below for Corollary 5.50.

**Theorem 5.47.** Let \(X\) be a PL filtered space with triangulation \(T\), and let \(W \subset X\) be an open subset of \(X\) such that \(W\) is a PL CS set. Then the composition \(I^p\mathcal{G}_s^G(W; G) \xrightarrow{\phi^{-1}} I^p\mathcal{G}_s^G(W; G) \xrightarrow{\psi} H_*(I^p\mathcal{G}_s^G(W; G))\) is an isomorphism. In particular, \(I^p\mathcal{G}_s^G(W; G) \cong I^pH_*^G(W; G)\) and, if \(X\) is a PL CS set, then \(I^p\mathcal{G}_s^G(X; G) \cong I^pH_*^G(X; G)\).

**Proof.** The last sentence of the statement of the theorem follows from the preceding statements and our prior observation that \(H_*(I^p\mathcal{G}_s^G(X)) \cong I^pH_*^G(X)\).

The proof will use the Mayer-Vietoris argument of Theorem 5.3 with \(\mathcal{F}_W\) being our domain category. In this case our functors on open sets \(U \subset W\) are \(F_*(U) = I^p\mathcal{G}_s(U)\), \(G_*(U) = H_*(I^p\mathcal{G}_s^G(X))\), and \(\Phi = \psi \phi^{-1}\). Notice that \(\Phi\) is a natural transformation on the category of open subsets of \(W\) using the commutativity of diagrams of the form

\[
\begin{array}{ccc}
I^p\mathcal{G}_s^G(V) & \xrightarrow{\phi} & I^p\mathcal{G}_s^G(V) \\
\downarrow & & \downarrow \\
I^p\mathcal{G}_s^G(U) & \xrightarrow{\phi} & I^p\mathcal{G}_s^G(U)
\end{array}
\]

\[
\begin{array}{ccc}
I^p\mathcal{G}_s^G(V) & \xrightarrow{\psi} & H_*(I^p\mathcal{G}_s^G(V)) \\
\downarrow & & \downarrow \\
I^p\mathcal{G}_s^G(U) & \xrightarrow{\psi} & H_*(I^p\mathcal{G}_s^G(U))
\end{array}
\]

for \(V \subset U\). Similarly, we obtain a natural transformation of Mayer-Vietoris sequences. It is also not difficult to observe, in the usual way as in Lemma 5.6, that if \(\{U_\alpha\}\) is an increasing collection of open subspaces of \(X \in \mathcal{F}\) then the natural maps \(\lim_{\alpha} F_*(U_\alpha) \to F_*(\cup_\alpha U_\alpha)\) and \(\lim_{\alpha} G_*(U_\alpha) \to G_*(\cup_\alpha U_\alpha)\) are isomorphisms. Hence, using Lemma 5.5 conditions (1) and (2) of Theorem 5.3 are satisfied.

Next suppose \(U\) is an open subset of \(W\) that is contained in a singular stratum \(S\) and PL homeomorphic to Euclidean space (or empty). By the same argument as in the proof of Theorem 5.42, either every chain in \(U\) is allowable or none are, depending only on the codimension of \(S\) and \(\bar{p}(S)\). If no chains can be allowable, then all homology groups are 0 and condition (4) of Theorem 5.3 holds trivially. If all chains are allowable, then, using Lemma 5.46, \(I^p\mathcal{G}_s^G(U) \cong I^p\mathcal{G}_s^G(U) \cong \mathcal{G}_s(U)\) and \(H_*(I^p\mathcal{G}_s^G(U)) \cong I^pH_*^G(U) \cong H_*(U)\). Since \(U\) is PL homeomorphic to Euclidean space, these groups are all trivial except for \(\mathcal{G}_0(U) \cong H_0(U) \cong \mathbb{Z}\). In all cases the generator is represented by a single vertex (simplicial or singular), so it follows from the constructions that \(\Phi\) is an isomorphism on \(U\). Thus we have verified hypothesis (4) of Theorem 5.3.

Finally, we need to check condition (3) of Theorem 5.3. Let \(N \cong \mathbb{R}^i \times cL\) be a distin-
guished neighborhood in $W$ such that we have the diagram

$$
\begin{array}{ccc}
I^p\mathcal{S}^\text{GM}_s(R^i \times (cL \setminus \{v\})) & \xrightarrow{\Phi} & H_*(I^p\mathcal{S}^\text{GM}_s(R^i \times (cL \setminus \{v\}))) \\
\downarrow & & \downarrow \\
I^p\mathcal{S}^\text{GM}_s(R^i \times cL) & \xrightarrow{\Phi} & H_*(I^p\mathcal{S}^\text{GM}_s(R^i \times cL)).
\end{array}
$$

Note that, in order for $\Phi$, which uses the triangulation of $X$, to be well defined here, we are tacitly identifying $R^i \times cL$ with an open subset $N$ of $W$, and thus $R^i \times (cL \setminus \{v\})$ is similarly identified as a subset of $R^i \times cL$. We assume that the top line is an isomorphism in all dimensions. The vertical maps can be computed from the cone formula and the Künneth theorem when one factor is a manifold; for these computations, the triangulation of $X$ is not involved, and using the invariance of PL (respectively, singular) intersection homology under PL (respectively, topological) stratified homeomorphisms, we may utilize these earlier results about cones and products. Furthermore, we know that these computations are identical for PL intersection homology and singular intersection homology, to which the terms on the right are isomorphic. In particular, the map on the left is isomorphic to the map induced by inclusion $I^p\mathcal{S}^\text{GM}_s(L) \to I^p\mathcal{S}^\text{GM}_s(cL)$, and the same is true of the singular intersection homology. In the range of dimensions for which the cone formulas tell us that these vertical maps are inclusions, the commutativity of the diagram implies that the bottom map is also an isomorphism. In cases where the cone formula forces the bottom groups to be 0, the bottom map is an isomorphism trivially. Finally, in any of the special cases in which $I^p\mathcal{S}^\text{GM}_0(R^i \times cL) \cong I^p\mathcal{S}^\text{GM}_0(N)$ and $H_0(I^p\mathcal{S}^\text{GM}_s(R^i \times cL)) \cong H_0(I^p\mathcal{S}^\text{GM}_s(N))$ are forced to each be isomorphic to $\mathbb{Z}$, Lemma 5.46 shows that there must be an allowable 0-simplex in $N$ that generates both $I^p\mathcal{S}^\text{GM}_T(N)$ and $I^p\mathcal{S}^\text{GM}_0(N)$, and $\Phi$ then takes this 0-simplex to a singular 0-simplex generating $H_0(I^p\mathcal{S}^\text{GM}_s(N))$. So the bottom map of the diagram is an isomorphism in all dimensions, verifying condition 3 of Theorem 5.3.

This completes our verification of the assumptions of Theorem 5.3 and the conclusion follows from that theorem.

Two relative versions of Theorem 5.47 follow almost immediately:

**Corollary 5.48.** Let $X$ be a PL CS set, and let $W$ be an open subset. Then $I^p\mathcal{S}^\text{GM}_s(X, W; G) \cong I^pH^\text{GM}_s(X, W; G)$.

**Proof.** Consider the following diagram of short exact sequences, where $T$ is a triangulation of $X$:

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as we know that the subdivision operator induces isomorphisms on intersection homology if and only if the associated triangulated as a subcomplex. Once again, we have a diagram of short exact sequences.

This time we assume that we begin with a triangulation $T$ of $X$ such that $X$ is itself a PL CS set in its inherited filtration. Then $I^p S^G_M(X, A; G) \cong I^p H^G_M(X, A; G)$.

**Proof.** This time we assume that we begin with a triangulation $T$ of $X$ such that $A$ is triangulated as a subcomplex. Once again, we have a diagram of short exact sequences

$$0 \longrightarrow I^p \mathcal{E}^G_M(A) \longrightarrow I^p \mathcal{E}^G_M(X) \longrightarrow I^p \mathcal{E}^G_M(X, A) \longrightarrow 0$$

In each case row, the rightmost non-trivial group is defined to be the quotient under the evident inclusion of the leftmost group into the middle group.

This diagram induces a commutative diagram of long exact sequences. By Lemma 5.46, applied to $W$ and $X$ and by the five lemma, we obtain an isomorphism of the top two long exact sequences. It follows then that $\psi \phi^{-1}$ is well defined from the top long exact sequence to the bottom long exact sequence, so now by Theorem 5.47 and the five lemma, $I^p S^G_M(X, W) \cong H^G_*(I^p \mathcal{E}^G_M(X, W))$. But now by the exactness of the direct limit functor,

$$H^G_*(I^p \mathcal{E}^G_M(X, W)) = H^G_*(I^p \mathcal{E}^G_M(X)/I^p \mathcal{E}^G_M(W))$$

$$= H^G_*(\lim \longrightarrow I^p S^G_M(X)/I^p S^G_M(W))$$

$$\cong H^G_*(\lim \longrightarrow (I^p \mathcal{E}^G_M(X)/I^p \mathcal{E}^G_M(W)))$$

$$\cong \lim \longrightarrow H^G_*(I^p S^G_M(X)/I^p S^G_M(W))$$

$$\cong I^p H^G_M(X, W),$$

as we know that the subdivision operator induces isomorphisms on intersection homology (and hence on relative intersection homology by another application of the five lemma). □

**Corollary 5.49.** Let $X$ be a PL CS set with closed PL subset $A$ such that $A$ is itself a PL CS set in its inherited filtration. Then $I^p S^G_M(X, A; G) \cong I^p H^G_M(X, A; G)$.

**Proof.** This time we assume that we begin with a triangulation $T$ of $X$ such that $A$ is triangulated as a subcomplex. Once again, we have a diagram of short exact sequences

$$0 \longrightarrow I^p \mathcal{E}^G_M(A) \longrightarrow I^p \mathcal{E}^G_M(X) \longrightarrow I^p \mathcal{E}^G_M(X, A) \longrightarrow 0$$
where, as before, \( I \bar{p} C^G M, T (A) \) denotes those chains of \( I \bar{p} C^G M, T (X) \) supported in \( X \). Since the barycentric subdivisions of the restriction of \( T \) to \( A \) are compatible with restricting the barycentric subdivisions of \( T \) in all of \( X \) to \( A \), the left and center vertical maps all induce isomorphisms on homology by Lemma 5.46 and Theorem 5.47. The corollary now follows from two applications of the five lemma to the resulting long exact sequences.

**Corollary 5.50.** Suppose \( X \) is a PL \( \partial \)-stratified pseudomanifold. Then the composition \( I \bar{p} \mathcal{S}_*^G M (X; G) \xrightarrow{\phi} I \bar{p} \mathcal{S}_*^G M, T (X; G) \xrightarrow{\psi} H_* (I \bar{p} \mathcal{S}_*^G M (X; G)) \xrightarrow{\cong} I \bar{p} H_*^G M (X; G) \) is an isomorphism.

**Proof.** Let \( X - \partial X \) be the interior of \( X \), and consider the following diagram in which the vertical maps are induced by inclusions:

\[
\begin{array}{ccc}
I \bar{p} \mathcal{S}_*^G M (X - \partial X) & \xrightarrow{\phi} & I \bar{p} \mathcal{S}_*^G M, T (X - \partial X) & \xrightarrow{\psi} & H_* (I \bar{p} \mathcal{S}_*^G M (X - \partial X)) & \cong & I \bar{p} H_*^G M (X - \partial X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I \bar{p} \mathcal{S}_*^G M (X) & \xleftarrow{\phi} & I \bar{p} \mathcal{S}_*^G M, T (X) & \xrightarrow{\psi} & H_* (I \bar{p} \mathcal{S}_*^G M (X)) & \cong & I \bar{p} H_*^G M (X)
\end{array}
\]

Commutativity can be seen at the chain level. The maps \( \phi \) are both isomorphisms by Lemma 5.46 and the composition along the top is an isomorphism by Theorem 5.47. The leftmost and rightmost vertical maps are isomorphisms by stratified homotopy invariance, Corollary 4.9, the homotopy equivalence of the inclusion of \( X - \partial X \) into \( X \) utilizes the collaring of the boundary. It follows that the composition along the bottom of the diagram is an isomorphism, as desired.

While Theorem 5.47 and its corollary provide isomorphisms between singular and PL intersection homology, in practice one prefers simplicial intersection homology as being the most directly computable. While PL techniques were necessary for our proofs in this section given their good functorial properties and the ability to work on open subsets of a larger PL space, our final results, together with our work from Section 3.3.2 allow us to recover a direct isomorphism between simplicial and singular intersection homology.

**Corollary 5.51.** Let \( X \) be a PL CS set of a \( \partial \)-stratified pseudomanifold, and let \( T \) be a full triangulation of \( X \) compatible with the stratification and with an ordering on its vertices. Then the canonical chain map \( I \bar{p} C_*^{G M, T} (X; G) \to I \bar{p} S_*^G M (X; G) \) from simplicial to singular intersection chains determined using the ordering of the vertices induces an isomorphism on intersection homology.

**Proof.** By thinking through the definitions of the maps, we have the following commutative diagram:

---

\(^{69}\)In fact, identifying the collar neighborhood of \( \partial X \) with \([0, 1) \times \partial X\), we see that both \( X \) and \( X - \partial X \) have stratified deformation retraction to \( X - ([0, 1/2) \times \partial X] \).
The map $\phi$ is a quasi-isomorphism (i.e. it induces homology isomorphisms) by Lemma 5.46, while the diagonal map is a quasi-isomorphism by Theorem 3.26. Therefore, the lefthand vertical map is a quasi-isomorphism by commutativity of the diagram. The righthand vertical map is a quasi-isomorphism by our discussion preceding the proof of Theorem 5.47, while $\psi$ is a quasi-isomorphism as a consequence of Theorem 5.47 or Corollary 5.50; the theorem demonstrates that $\phi^{-1}\psi$ is a homology isomorphism, but so is $\phi$, hence $\psi$ is as well, and a similar argument applies using the corollary instead. Thus three of the four sides of the square are quasi-isomorphisms, which implies that the top is, as well. 

5.6 Topological invariance

The definition of the intersection chains uses the stratification of the space to define which chains are allowable. The following fact is therefore quite remarkable: for certain perversities, the intersection homology groups do not depend on the stratification of a CS set, only on the underlying homeomorphism type of the space. We will provide the precise statement below, but, to even begin to make sense of this notion, we cannot work with our arbitrary perversities, which are also defined with reference to the stratification of a space. Instead, we follow the original Goresky-MacPherson definition and use perversities that depend only on the codimension of the strata. Then it makes sense to apply such a perversity to multiple stratifications of the same space (or alternatively, we can think of such a thing as a recipe for how to define a perversity on a space, given a stratification). Thus, throughout this section, we think of a perversity as a function $\bar{p} : \{1, 2, 3, \ldots\} \to \mathbb{Z}$, as perversities are always 0 on the codimension zero strata. Additionally, since we must have a notion of codimension that depends only on the homeomorphism type of the space, we fold the formal dimension of the CS set into the data concerning its homeomorphism type.

Invariance of PL intersection homology under restratification of PL pseudomanifolds (keeping fixed the PL structure) was proven in [42], where Goresky and MacPherson introduced intersection homology. More general topological invariance, i.e. the dependence of intersection homology only on perversity and topological homeomorphism type, was first proven for pseudomanifolds by Goresky and MacPherson in [43] using the techniques of sheaf theory. That proof proceeds by finding axiomatic characterizations for sheaf complexes whose hypercohomology computes intersection homology. They first find such an axiomatic characterization in terms of a stratification, but then they show that this axiomatic characterization is equivalent to other ones that do not rely on the stratification. A good source for this material is [8, Section V], in which the details concerning constructibility
of sheaf complexes are carried out a bit more carefully than in [43] (see, in particular, [8, Remark V.3.16]). The proof of topological invariance of singular intersection homology that we provide here is a modification of that found by King [61] and applies more broadly to CS sets. Note that our CS sets follow the definition of Siebenmann [94] and so our more general than those of King [61], which assume dimensional homogeneity.

5.6.1 The necessity of restricting the perversity, the statement of the theorem, and some corollaries

Let \( X \) be a CS set of a fixed formal dimension, and let \( \bar{p} \) be a perversity \( \bar{p} : \{1, 2, \ldots \} \to \mathbb{Z} \). If we want \( I^\bar{p} H^i_{GM}(X) \) to be a topological invariant, then \( \bar{p} \) still cannot be arbitrary. To see this, consider a basic distinguished neighborhood of the form \( \mathbb{R} \times cL \) for a compact filtered space \( L \) with regular strata. This space is homeomorphic to \( cSL \), the cone on the suspension of \( L \). However, these two descriptions give rise to two different stratifications based on the stratification of \( L \). The natural strata of \( \mathbb{R} \times cL \) are \( \mathbb{R} \times \{v\} \), where \( v \) is the cone vertex, and \( \mathbb{R} \times (0, 1) \times S \), where \( S \) is a stratum of \( L \). For \( cSL \), if we think of \( SL \) as a quotient of \([-1, 1] \times L \) and \( cL \) as a quotient of \([0, 1) \times L \), then there are three types of strata:

1. \( \{w\} \), where \( w \) is the cone point of \( cSL \),
2. \( (0, 1) \times \{v_{-1}\} \) and \( (0, 1) \times \{v_1\} \), where \( v_{-1}, v_1 \) are the vertices of the suspension \( SL \), and
3. \( (0, 1) \times (-1, 1) \times S \) for each stratum \( S \) of \( L \).

Now, suppose \( L \) has dimension \( k - 2 \). By Theorem 4.12 and using the stratified homotopy invariance, we have

\[
I^\bar{p} H^i_{GM}(\mathbb{R} \times cL) \cong \begin{cases} 
0, & i \geq k - 2 - \bar{p}(k - 1), i \neq 0, \\
\mathbb{Z}, & i = 0 \geq k - 2 - \bar{p}(k - 1), \\
I^\bar{p} H_i(L), & i < k - 2 - \bar{p}(k - 1).
\end{cases}
\]

Similarly,

\[
I^\bar{p} H^i_{GM}(cSL) \cong \begin{cases} 
0, & i \geq k - 1 - \bar{p}(k), i \neq 0, \\
\mathbb{Z}, & i = 0 \geq k - 1 - \bar{p}(k), \\
I^\bar{p} H_i(SL), & i < k - 1 - \bar{p}(k),
\end{cases}
\]

and by Theorem 4.43,

\[
I^\bar{p} H^i_{GM}(SL) = \begin{cases} 
I^\bar{p} H^i_{\bar{G}}(L), & i > k - \bar{p}(k - 1) - 2, i \neq 0, \\
0, & i = k - \bar{p}(k - 1) - 2, i \neq 0, \\
I^\bar{p} H^i_{GM}(L), & i < k - \bar{p}(k - 1) - 2,
\end{cases}
\]

\[
\mathbb{Z}, & i = 0 \geq k - \bar{p}(k - 1) - 2.
\]

Now, what can we determine from this? Ignoring for the moment possible complications in low dimensions, we see that \( I^\bar{p} H^i_{GM}(\mathbb{R} \times cL) = 0 \) when \( i \geq k - 2 - \bar{p}(k - 1) \). Furthermore,
\[ I^p H_{k-3-\bar{p}(k-1)}^{GM} (\mathbb{R} \times cL) \cong I^p H_{k-3-\bar{p}(k-1)}^{GM} (L) \]. So at least assuming that \( 0 \leq k - 3 - \bar{p}(k-1) \leq k - 2 \), it would not be hard to rig up an example where \( I^p H_{k-3-\bar{p}(k-1)}^{GM} (\mathbb{R} \times cL) \neq 0 \), or, for that matter, \( I^p H_{k-3}^{GM} (\mathbb{R} \times cL) \neq 0 \) for all \( i \leq k - 3 - \bar{p}(k-1) \). For example, suppose \( L \) is the product of \( k - 2 \) circles. On the other hand, \( I^p H_i^{GM} (cSL) = 0 \) for \( i \geq k - 1 - \bar{p}(k) \). So in order for topological invariance to hold, we should need \( k - 3 - \bar{p}(k-1) < k - 1 - \bar{p}(k) \), or in other words, \( \bar{p}(k) \leq \bar{p}(k-1) + 1 \).

On the other hand, again ignoring low-dimensional issues, we see that \( I^p H_{i}^{GM} (cSL) = 0 \) for \( i \geq k - 1 - \bar{p}(k) \) and also for \( i = k - \bar{p}(k-1) - 2 \), regardless of how \( k - \bar{p}(k-1) - 2 \) compares to \( k - 1 - \bar{p}(k) \). Once again, it is easy to choose \( L \) so that it is non-zero for all other dimensions \( \geq 0 \) and \( < k \). Thus we will run into contradictions if there is a dimension \( j \) such that \( k - \bar{p}(k-1) - 2 < j < k - 1 - \bar{p}(k) \), for then \( I^p H_{j}^{GM} (cSL) \) need not be 0, but \( I^p H_{i}^{GM} (\mathbb{R} \times cL) = 0 \). So to avoid these contradictions we must have \( k - \bar{p}(k-1) - 2 \geq k - 1 - \bar{p}(k) - 1 \), or in other words, \( \bar{p}(k) \geq \bar{p}(k-1) \).

Together, these two arguments show that we must have \( \bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1 \). This is one of the conditions for \( \bar{p} \) to be a GM perversity (see Definition 3.3), and now we see the reason for it. It turns out that this condition is sufficient to obtain topological invariance so long as \( \bar{p}(1) \geq 0 \).

**Theorem 5.52.** Suppose \( X \) is a CS set of formal dimension \( n \) and that \( \bar{p} : \{1, 2, \ldots \} \rightarrow \mathbb{Z} \) is a perversity such that \( \bar{p}(1) \geq 0 \) and \( \bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1 \) for all \( k \geq 2 \). Let \( X^* \) be \( |X| \) with its intrinsic stratification (and the same formal dimension \( n \)). Then the identity map of spaces induces an isomorphism \( I^p H_{*}^{GM} (X; G) \rightarrow I^p H_{*}^{GM} (X^*; G) \). It follows that \( I^p H_{*}^{GM} (X; G) \) is independent (up to isomorphism) of the choice of stratification of \( X \) as a CS set of formal dimension \( n \). In particular, if \( X' \) is another CS set of formal dimension \( n \) that is topologically homeomorphic to \( X \) (not necessarily stratified homeomorphic), then \( I^p H_{*}^{GM} (X; G) \cong I^p H_{*}^{GM} (X'; G) \).

More generally, if \( A \) is an open subset of \( X \) and \( (X, A) \cong (X', A') \), then \( I^p H_{*}^{GM} (X, A; G) \cong I^p H_{*}^{GM} (X', A'; G) \).

**Remark 5.53.** Our statement of the theorem appears somewhat different from King’s in [61], where the only condition seems to be that \( \bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1 \). However, King’s intersection homology groups are first defined with respect to what he calls loose perversities which are defined on the domain \( \{0, 1, 2, \ldots \} \); notice that codimension 0 is included in the domain. Then he defines “perversities” to be those loose perversities for which \( \bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1 \). But notice that if we have a perversity in the sense of King with \( \bar{p}(0) < 0 \), then \( \bar{p}(k) < k \) for all \( k \). So if \( \sigma \) is an allowable \( i \)-simplex in King’s sense, we have \( \sigma^{-1}(S) \subset \{i - \text{codim}(S) + \bar{p}(S) \text{ skeleton of } \Delta^i \} \) for all strata, both regular and singular. But then if \( \text{codim}(S) = k \), we have \( i - \text{codim}(S) + \bar{p}(S) = i - k + \bar{p}(k) < i \) for all \( S \). Which means that no stratum, regular or singular, can intersect the image of the interior of \( \Delta^i \). Which means that no allowable simplices can exists, so all intersection homology groups are trivial. Hence, this case does not quite bear consideration.

On the other hand, suppose we are given a perversity in the sense of King for which \( \bar{p}(0) \geq 0 \). It follows from the condition that \( \bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1 \) for all \( k \geq 2 \) that \( \bar{p}(j) \geq 0 \) for all \( j \). Now, by Remark 3.36, we obtain exactly the same intersection chain.
complexes by changing $\bar{p}(0)$ to 0, which results in one of our perversities, and, in fact, one
that satisfies the hypotheses of the theorem. Hence, for all cases of interest, it suffices to use
our definition of perversity and the assumption $\bar{p}(1) \geq 0$.

*Remark 5.54.* King’s theorem also does not explicitly mention formal dimensions as it is
implicit that each space is meant to be taken with its topological dimension. However, with
the generality of our CS sets, we cannot restrict ourselves to that case even when we only care
about results for spaces whose formal dimensions equal their topological dimensions since
such spaces might have subspaces of lower topological dimension, and we will often need such
subspaces to satisfy the theorem in our inductive arguments. As we have remarked many
times now, for intersection homology of subspaces to make coherent sense, we will assign to
subspaces the formal dimensions of their ambient spaces.

*Remark 5.55.* The relative version of the theorem is stated only for open subsets. This
is because the proof will rely upon intrinsic stratifications, and due to the local nature of
intrinsic stratifications, an open subset of an intrinsically stratified space inherits its own
intrinsic stratification. One could not necessarily expect such nice behavior for arbitrary
subspaces, though in certain other nice situations, for example if $A$ and $A'$ are normally
nonsingular subspaces of $X$ and $X'$, then one could extend the conclusion of the theorem
using stratified homotopy arguments.

*Corollary 5.56.* Let $X$ and $X'$ be two n-dimensional CS set stratifications of the same un-
derlying topological space, say $|X|$. Let $x \in |X|$, and let $N, N'$ be distinguished neighborhoods
of $x$ in $X$ and $X'$, respectively. Suppose that $\bar{p}: \{1, 2, \ldots\} \to \mathbb{Z}$ is a perversity such that
$\bar{p}(1) \geq 0$ and $\bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1)+1$ for $k \geq 2$. Then $I^pH_{GM}^*(N; F) \cong I^pH_{GM}^*(N'; G)$.

This corollary follows immediately from the theorem and Corollary 2.98 will have another
nice application once we have used it to prove Theorem 5.52.

It is interesting to compare this corollary with Lemma 5.40 which provides the same
result for general perversities, but only when $X$ and $X'$ are the same stratification.

It will be useful to have some additional, more specific, variants of this corollary available:

*Lemma 5.57.* Let $X$ and $X'$ be CS sets with no codimension one strata and with $|X| = |X'|$,
and suppose that $\bar{p}: \{1, 2, \ldots\} \to \mathbb{Z}$ is a perversity such that $\bar{p}(1) \geq 0$ and $\bar{p}(k-1) \leq \bar{p}(k) \leq
\bar{p}(k-1)+1$ for $k \geq 2$. Take $x \in |X|$. Let $U$ and $V'$ be distinguished neighborhoods of $x$ in $X$ and $X'$, respectively. Let $U'$
with the stratification it inherits from $X'$, let $V$ denote $V'$ in the stratification inherited from $X$, and let $U^*$ and $V^*$ be $|U|$ and $|V'|$ in
their intrinsic stratifications. Suppose that $|V| \subset |U|$. Then inclusion induces isomorphisms
$I^pH_{GM}^*(V; G) \to I^pH_{GM}^*(U; G), \; I^pH_{GM}^*(V'; G) \to I^pH_{GM}^*(U'; G), \; \text{and} \; I^pH_{GM}^*(V^*; G) \to
I^pH_{GM}^*(U^*; G)$ for any abelian group $G$. Furthermore, the lemma remains true replacing the
various spaces $U, V$, etc. with the deleted neighborhoods $U - \{x\}, V - \{x\}$, etc..

Proof. By shrinking $U$ as in the proof of Lemma 5.40 there is a distinguished neighborhood
$U_1$ of $x$ in $|V|$ such that $U_1$ and $U$ are stratified homotopy equivalent, and then similarly
there is a $V'_1 \subset |U_1|$ with $V'_1$ and $V'$ stratified homotopy equivalent. We can form $U'_1$ and $V
by again letting the primed spaces get there stratification from $X'$ while all those without

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get their stratification from $X$. Similarly, asterisks denote the intrinsic stratification with all intrinsic stratifications inherited from $X^*$. Altogether, we have

$$|V_1| \subset |U_1| \subset |V| \subset |U|,$$

and this leads to the commutative diagram

\[
\begin{array}{cccc}
I^p H^*_G(V_1; G) & \longrightarrow & I^p H^*_G(U_1; G) & \longrightarrow I^p H^*_G(V; G) & \longrightarrow I^p H^*_G(U; G) \\
I^p H^*_G(V^*_1; G) & \longrightarrow & I^p H^*_G(U^*_1; G) & \longrightarrow I^p H^*_G(V^*_1; G) & \longrightarrow I^p H^*_G(U^*_1; G) \\
I^p H^*_G(V^*_1; G) & \longrightarrow & I^p H^*_G(U^*_1; G) & \longrightarrow I^p H^*_G(V^*_1; G) & \longrightarrow I^p H^*_G(U^*_1; G) \\
I^p H^*_G(V'_1; G) & \longrightarrow & I^p H^*_G(U'_1; G) & \longrightarrow I^p H^*_G(V'; G) & \longrightarrow I^p H^*_G(U'; G) \\
\end{array}
\]

The vertical maps are all isomorphism by Theorem 5.52 and its proof; note that if neither $X$ nor $X'$ have codimension one strata, then neither does $X^*$, as the strata of $X^*$ are unions of strata of $X$ (or $X'$).

Now, by stratified homotopy invariance, the composition $I^p H^*_G(U_1; G) \to I^p H^*_G(U; G)$ is an isomorphism, as is the composition $I^p H^*_G(V^*_1; G) \to I^p H^*_G(V^*; G)$. Together with the isomorphisms in the diagram, this is sufficient to conclude that $I^p H^*_G(U^*_1; G) \to I^p H^*_G(U^*; G)$ is surjective and injective, so an isomorphism. It follows that all horizontal arrows in the middle column of the diagram are isomorphisms. Together with our two composite isomorphisms, this implies that $I^p H^*_G(V; G) \to I^p H^*_G(U; G)$ and $I^p H^*_G(V^*_1; G) \to I^p H^*_G(U^*_1; G)$ are isomorphisms, and it follows now that every map in the diagram is an isomorphism.

The proof of the last statement concerning the deleted neighborhoods is identical, noticing that we can choose our stratified homotopy equivalences in the preceding argument to fix $x$. 

For $\partial$-stratified pseudomanifolds, there is also a version for collars, at least if our boundary is compact:

**Lemma 5.58.** Let $\bar{p} : \{1, 2, \ldots \} \to \mathbb{Z}$ be a perversity such that $\bar{p}(1) \geq 0$ and $\bar{p}(k - 1) \leq \bar{p}(k) \leq \bar{p}(k - 1) + 1$ for $k \geq 2$, and let $X$ and $X'$ be $\partial$-stratified pseudomanifolds with no codimension one strata and with $(|X|, |\partial X|) = (|X'|, |\partial X'|)$. Suppose $|\partial X| = |\partial X'|$ is compact. Let $U$ and $V'$ be stratified collar neighborhoods respectively of $\partial X$ in $X$ and of $\partial X'$ in $X'$. Let $U'$ denote $|U|$ with the stratification it inherits from $X'$, and let $V$ denote $V'$ in the stratification inherited from $X$. Suppose that $|V| \subset |U|$. Then inclusion induces isomorphisms $I^p H^*_G(V; G) \to I^p H^*_G(U; G)$ and $I^p H^*_G(V^*; G) \to I^p H^*_G(U^*; G)$.
Note, we do not include statements about $X^*$ in the lemma because we do not have intrinsic stratifications in this setting; see Remark 5.59.

**Proof.** The proof is very analogous to that of Lemma 5.57. Using the Tube Lemma [78, Lemma 26.8], we can find stratified collars $U_1$ and $V'_1$ so that again $|V_1| \subset |U_1| \subset |V| \subset |U|$, the inclusion $U_1 \hookrightarrow U$ is a stratified homotopy equivalence, etc. In fact, if $U \cong [0,1) \times \partial X$, we can assume that $U_1 \cong [0,t) \times \partial X$ for some $0 < t < 1$, and similarly for $V'$ and $V'_1$. Let $\bar{U}_1$, denote $U_1 - \partial X$, and similarly for the other sets. It remains true that the inclusion $\bar{U}_1 \hookrightarrow \bar{U}$ is a stratified homotopy equivalence and analogously for $\bar{V}_1 \hookrightarrow \bar{V}'$. As $\bar{U}, \bar{V}', \bar{U}_1, \bar{V}'_1$ are all CS sets, there are intrinsic stratifications, and repeating the diagram chase of the proof of Lemma 5.57 shows that the inclusions induce isomorphisms $I^\bar{p}H^GM(\bar{V};G) \to I^\bar{p}H^GM(\bar{U};G)$ and $I^\bar{p}H^GM(V';G) \to I^\bar{p}H^GM(\bar{U}';G)$.

Finally we have maps

$$I^\bar{p}H^*_GM(\bar{V};G) \longrightarrow I^\bar{p}H^*_GM(\bar{U};G)$$

$$I^\bar{p}H^*_GM(V;G) \longrightarrow I^\bar{p}H^*_GM(U;G).$$

We have just show that the top map is an isomorphism, but so are the vertical maps by stratified homotopy invariance, noting that $U$ is an open neighborhood of $\partial X$ in $X$ and so, by the Tube Lemma [78, Lemma 26.8], it contains a neighborhood of $\partial X$ stratified homeomorphic to $[0,s) \times \partial X$ (in other words, we may now use the standard argument that, if $X$ is a $\partial$-stratified pseudomanifold, $X$ and $X - \partial X$ are stratified homotopy equivalent). Similarly for $\bar{V}$ and $V'$. The conclusion follows.

**Corollary 5.59.** Let $\bar{p} : \{1,2,\ldots\} \to \mathbb{Z}$ be a perversity such that $\bar{p}(1) \geq 0$ and $\bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1$ for $k \geq 2$, and let $X$ and $X'$ be $\partial$-stratified pseudomanifolds with no codimension one strata and with $(|X|,|\partial X|) = (|X'|,|\partial X'|)$. Suppose $|\partial X| = |\partial X'|$ is compact. Let $V'$ be a stratified collar neighborhood of $\partial X'$, and let $V$ denote $V'$ in the stratification inherited from $X$. Then inclusion induces isomorphisms $I^{\bar{p}}H^*_{GM}(\partial X;G) \to I^{\bar{p}}H^*_{GM}(V;G)$.

**Proof.** The set $V$ is an open neighborhood of $\partial X$, so we can find a collar $U$ of $\partial X$ in $V$ by the Tube Lemma. Now consider the composition $I^{\bar{p}}H^*_{GM}(\partial X;G) \to I^{\bar{p}}H^*_{GM}(U;G) \to I^{\bar{p}}H^*_{GM}(V;G)$. The first map is an isomorphism by stratified homotopy invariance, and the latter is an isomorphism by Lemma 5.59. The result follows.

We can now apply some of these topological invariance results to the issue of a CS set $X$ being locally $(\bar{p},R,M)^{GM}$-torsion. It is fairly evident that the notion of a CS set $X$ being locally $(\bar{p},R,M)^{GM}$-torsion free depends on the stratification of $X$, as the very definition of $\bar{p}$ depends, in general, on $X$: 177
Example 5.60. Consider the space $X = X^5 = \mathbb{R} \times c(\mathbb{R}P^3)$ stratified as $\mathbb{R} \times \{v\} \subset X$. Then $\mathbb{R}P^3$ is a links of each point in the singular stratum $\mathbb{R} \times \{v\}$, and $I^0H_{3-0(S)-1}^{GM}(\mathbb{R}P^3) = H_2(\mathbb{R}P^3) = 0$. So $X$ is locally $(0, \mathbb{Z})^{GM}$-torsion free. But now let’s restratify this space as $Y$ with $\{(0,v)\} \subset \mathbb{R} \times \{v\} \subset Y$, and let $\bar{p}$ be a perversity on $Y$ that remains 0 on the 1-dimensional strata. From the proof of Lemma 2.12, the suspension $S(\mathbb{R}P^3)$ is a link of $(0,v)$, and the cone points of the suspension lie in the 1-dimensional strata, so, by Theorem 4.43

$$I^pH_i^{GM}(S(\mathbb{R}P^3)) = \begin{cases} I^pH_{i-1}^{GM}(\mathbb{R}P^3), & i > 2, \\ 0, & i = 2, \\ I^pH_i^{GM}(\mathbb{R}P^3), & i < 2. \end{cases}$$

In particular, then $I^pH_1^{GM}(S(\mathbb{R}P^3)) \cong H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. So if we take $\bar{p}(\{(0,v)\}) = 2$, we’ll have $I^pH_{\dim(S(\mathbb{R}P^3)) - 2}(S(\mathbb{R}^3)) = \mathbb{Z}_2$, and $Y$ is not locally $(\bar{p}, \mathbb{Z})$-torsion free.

The next result shows, however, that the locally torsion free property is invariant of the stratification in those settings where Theorem 5.52 holds.

Proposition 5.61. Let $\bar{p} : \{1,2,\ldots\} \to \mathbb{Z}$ be a perversity such that $\bar{p}(1) \geq 0$ and $\bar{p}(k-1) \leq \bar{p}(k) \leq \bar{p}(k-1) + 1$ for $k \geq 2$, and let $X$ and $X'$ be CS sets with $|X| = |X'|$. Then $X$ is locally $(\bar{p}, R; M)^{GM}$-torsion free if and only if $X'$ is.

Proof. We will show that $X$ is locally $(\bar{p}, R; M)^{GM}$-torsion free if and only if $X^*$ is, where $X^*$ is $|X|$ with its intrinsic stratification (see Section 2.8). As $X$ and $X'$ have the same intrinsic stratification, the result will follow.

First, assume that $X$ is locally $(\bar{p}, R; M)^{GM}$-torsion free. Recall that every stratum $S$ of $X^*$ is a union of strata of $X$ of dimension $\leq \dim(S)$ (see Section 2.8). So let $S$ be a stratum of $X^*$ of codimension $\ell$, so that the dimension of its link is $\ell - 1$, and let $x$ be a point of $X$ contained in a stratum $T$ of $X$ with $T \subset S$ and $\dim(S) = \dim(T)$; such a stratum $T$ must exist by the definition of a CS set, in particular the compatibility of the filtration with the local conical structure. Let $L$ be a link of $x$ in $X$ and let $L'$ be a link of $x$ in $X^*$. As $\dim(S) = \dim(T)$, we have $\dim(L) = \dim(L') = \ell - 1$. In this case, the locally torsion free condition is a statement about intersection homology in degree $\ell - 1 - \bar{p}(\ell) - 1 = \ell - \bar{p}(\ell) - 2$. By the cone formula, Theorem 5.33, $I^pH_{\ell-\bar{p}(\ell)-2}(cL; R) \cong I^pH_{\ell-\bar{p}(\ell)-2}(L; R)$ and $I^pH_{\ell-\bar{p}(\ell)-2}(cL'; R) \cong I^pH_{\ell-\bar{p}(\ell)-2}(L'; R)$. It follows that if $N$ and $N^*$ are regular neighborhoods of $x$ in $X$ and $X^*$, respectively, we have

$$I^pH_{\ell-\bar{p}(\ell)-2}(L; R) \cong I^pH_{\ell-\bar{p}(\ell)-2}(N; R) \cong I^pH_{\ell-\bar{p}(\ell)-2}(N^*; R) \cong I^pH_{\ell-\bar{p}(\ell)-2}(L'; R),$$

using Corollary 5.56 for the middle isomorphism. As we have assumed that $X$ is locally $(\bar{p}, R; M)^{GM}$-torsion free, the link $L$ of $x$ in $X$ satisfies the required intersection homology torsion condition, and so $L'$ also satisfies the required torsion condition. Since the locally torsion free condition is satisfied for a link at one point in $S$, it is satisfied at all points in $S$ by Corollary 6.28. As $S$ was an arbitrary stratum of $X^*$, it follows that $X^*$ is locally $(\bar{p}, R; M)^{GM}$-torsion free.
Conversely, suppose $X^*$ is locally $(\bar{p}, R; M)_{GM}$-torsion free. Let $x \in X$ be a point with distinguished neighborhood $N \cong \mathbb{R}^k \times cL$. Suppose $\dim(L) = \ell - 1$. As observed in the preceding paragraph, we have $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(L; R) \cong I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(N; R)$. Now, let $N^*$ be a distinguished neighborhood of $x$ in $N^*$. By Corollary 6.28, $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(N; R) \cong I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(N^*; R)$. But $N^* \cong \mathbb{R}^m \times c\mathcal{L}$ for some link $\mathcal{L}$ and some $\mathbb{R}^m$ with $m \geq k$, since the stratification of $X^*$ is coarser than that of $X$. Let $\dim(\mathcal{L}) = d - 1$. By stratified homotopy invariance, $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(N^*; R) \cong I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(c\mathcal{L}; R)$. So, altogether, $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(L; R) \cong I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(c\mathcal{L}; R)$. If $m = k$, then $\dim(\mathcal{L}) = \ell - 1$ as well, and, by the cone formula again, $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(L; R) \cong I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(\mathcal{L}; R)$, which shows that $L$ satisfies the required torsion condition as $\mathcal{L}$ does by assumption. So suppose $m > k$, which implies that $d - 1 = \dim(\mathcal{L}) < \ell - 1$. We will show that $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(c\mathcal{L}; R)$ is equal either to 0, $R$, or to $I^{\bar{p}}H^G_{\ell - \bar{p}(\ell) - 2}(\mathcal{L}; R)$, which will suffice to show the theorem, as we have assumed that $X^*$ is locally $(\bar{p}, R; M)_{GM}$-torsion free and as $R$ is a free.

To prove our claim, it is sufficient, by the cone formula, to verify that $\ell - \bar{p}(\ell) - 2 \geq d - \bar{p}(d) - 2$, i.e. that $\ell - d \geq \bar{p}(\ell) - \bar{p}(d)$. But this is immediate from the conditions on our perversity, which satisfies $\bar{p}(k) \leq \bar{p}(k + 1) \leq \bar{p}(k) + 1$ for all $k \geq 1$.

\subsection{5.6.2 Proof of topological invariance}

We first need one preliminary observation. Let $X$ be a CS set, and recall the intrinsic coarsest stratification $X^*$ constructed in Section 2.8. If $X$ has formal dimension $n$, then we will assume that $X^*$ is also given formal dimension $n$. This allows us to construct a comparison map $I^{\bar{p}}H^G_*(X) \to I^{\bar{p}}H^G_*(X^*)$. Indeed, the finer the stratification of a space, the more difficult it is for a simplex to be allowable, so we have the following lemma:

\begin{lemma}
Suppose that $X$ is a CS set and that $X'$ is a coarsening of $X$, meaning that $X$ and $X'$ have the same underlying topological space but that each stratum of $X'$ is a union of strata of $X$. Suppose $X$ and $X'$ have the same formal dimension, and let $\bar{p}$ be a perversity that depends only on codimension and such that $\bar{p}(k - 1) \leq \bar{p}(k) \leq \bar{p}(k - 1) + 1$ for $k \geq 2$. Then $I^{\bar{p}}S^G_*(X) \subseteq I^{\bar{p}}S^G_*(X')$.
\end{lemma}

\begin{proof}
Suppose that $S$ is a stratum of $X$ of codimension $k$ and that $T$ is the stratum of $X'$ of codimension $j \leq k$ containing $S$. If $\sigma$ is an allowable $i$-simplex with respect to $S$, then $\sigma^{-1}(S) \subseteq \{i - k + \bar{p}(k)\text{ skeleton of }\Delta^i\}$. By the assumption on the perversities, $\bar{p}(k) \leq \bar{p}(j) + (k - j)$, so $i - k + \bar{p}(k) \leq i - k + \bar{p}(j) + (k - j) = i - j + \bar{p}(j)$, so $\sigma$ is also allowable with respect to $T$. Hence there is an inclusion $I^{\bar{p}}S^G_*(X) \subseteq I^{\bar{p}}S^G_*(X')$.
\end{proof}

We will show that if $X^*$ is the intrinsic stratification of $X$, then the inclusion $I^{\bar{p}}S^G_*(X) \subseteq I^{\bar{p}}S^G_*(X^*)$ induces an isomorphism on homology. Then if $X'$ is any other stratification of $X$ (not necessarily a coarsening of $X$), we can show that $I^{\bar{p}}H^G_*(X) \cong I^{\bar{p}}H^G_*(X')$ via the composite

$I^{\bar{p}}H^G_*(X) \cong I^{\bar{p}}H^G_*(X^*) \cong I^{\bar{p}}H^G_*(X')$.

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Similarly, once we show that $I^\beta S^{GM}_*(X) \to I^\beta S^{GM}_*(X^*)$ induces an isomorphism on homology, the claimed relative result also follows as we will have maps of short exact sequences

$$0 \to I^\beta S^{GM}_*(A) \to I^\beta S^{GM}_*(X) \to I^\beta S^{GM}_*(X, A) \to 0$$

The left side of the diagram commutes because the local nature of the definition of the intrinsic stratification implies that the intrinsic stratification of the open set $A$ is the restriction to $A$ of the intrinsic stratification of $X$. The commutativity of the right side then follows from the induced map on quotients. So once we have proven the theorem in the absolute case, the relative case will follow from the five lemma applied to the induced diagrams of long exact sequences.

We are now ready to prove Theorem 5.52 which will proceed by an induction on depth (see Definition 2.28).

**Proof of Theorem 5.52.** The proof is the same for all coefficient systems, so we provide the details only for $\mathbb{Z}$ coefficients.

Continuing to follow King [61], the main body of the proof proceeds by an intertwined set of inductions on the following three statements:

$P(i)$: The comparison map $I^\beta H^{GM}_*(X) \to I^\beta H^{GM}_*(X^*)$ is an isomorphism for all CS sets $X$ of depth $\leq i$,

$Q(i)$: The comparison map $I^\beta H^{GM}_*(X) \to I^\beta H^{GM}_*(X^*)$ is an isomorphism for all CS sets $X$ stratified homeomorphic to $M \times cW$, where $M$ is a manifold and $W$ is a compact filtered space of depth $\leq i$,

$R(i)$: The comparison map $I^\beta H^{GM}_*(X) \to I^\beta H^{GM}_*(X^*)$ is an isomorphism for all CS sets $X$ stratified homeomorphic to $\mathbb{R}^k \times cW$, where $W$ is a compact filtered space of depth $\leq i$.

We will show $P(i) \Rightarrow R(i)$, $R(i) \Rightarrow Q(i)$, and $P(i) \lor Q(i) \Rightarrow P(i + 1)$. So, for a CS set of depth $n$, it will follow from $P(n)$ that $I^\beta H^{GM}_*(X) \cong I^\beta H^{GM}_*(X^*)$. If $X'$ is another $n$-dimensional CS set that is topologically homeomorphic to $X$, say by $h : X' \to X$, then the induced topological homeomorphism (which is the same map on the underlying spaces) $h : (X')^* \to X^*$ must in fact be a *stratified* homeomorphism, as follows from the purely topological character of the definition of the intrinsic stratification. Therefore, we will have $I^\beta H^{GM}_*(X) \cong I^\beta H^{GM}_*(X^*) = I^\beta H^{GM}_*((X')^*) \cong I^\beta H^{GM}_*(X')$.

To get the induction started, we notice that $P(0)$ is trivial, since if $X$ has depth 0, then certainly $X = X^*$. We follow the arguments of King’s closely for the first two parts of the proof. For the third, we follow the basic idea but generalize using Zorn’s lemma so that we do not need to assume second countability of our manifolds. Throughout the proof, we
will need to take intersection homology groups of stratified spaces with the same underlying topological spaces; the reader should be aware that in each case we will let the notation for the space indicate the desired stratification. So, for example, if $\mathbb{R}^k \times cW$ and $\mathbb{R}^m \times cL$ are two topologically homeomorphic spaces, as we shall encounter below, we let $I^p H^G_M(\mathbb{R}^k \times cW)$ be computed with respect to the evident filtration with skeleta of the form $\mathbb{R}^k \times cW^i$, and similarly $I^p H^G_M(\mathbb{R}^m \times cL)$ is to be computed with respect to the filtration with skeleta of the form $\mathbb{R}^m \times cL^i$.

$\mathbf{R}(i) \Rightarrow \mathbf{Q}(i)$: This is the simplest step of the argument. Let $\dim(M) = k$, and consider $M \times cW$, where $W$ is a compact filtered space of depth $\leq i$. By Lemma 2.106, there is a coarsening $Z$ of $cW$ such that $(M \times cW)^* \cong M \times Z$ and $(\mathbb{R}^k \times cW)^* \cong \mathbb{R}^k \times Z$. By assumption, $I^p H^G_M(\mathbb{R}^k \times cW) \to I^p H^G_M(\mathbb{R}^k \times Z)$ is an isomorphism, and using the stratified homotopy invariance of intersection homology, it follows that in fact we must have $I^p H^G_M(cW) \to I^p H^G_M(Z)$. But then it follows that $I^p H^G_M(M \times cW) \to I^p H^G_M(M \times Z)$ is an isomorphism using the Künneth theorem, Theorem 5.28 and naturality of the isomorphisms involved.

$\mathbf{P}(i) \Rightarrow \mathbf{R}(i)$: This is perhaps the most challenging part of the argument.

Consider $\mathbb{R}^k \times cW$, where $W$ is a compact filtered space of depth $\leq i$. We need to show that $I^p H^G_M(\mathbb{R}^k \times cW) \to I^p H^G_M((\mathbb{R}^k \times cW)^*)$ is an isomorphism under the assumption that $I^p H^G_M(X) \to I^p H^G_M(X^*)$ is an isomorphism for all CS sets $X$ of depth $\leq i$.

Let $(\mathbb{R}^k \times cW)^*$ have the stratification of the form $\mathbb{R}^k \times Z$ that exists by Lemma 2.106. Let $w \in Z$ correspond to the cone point of $cW$, and let $y = 0 \times w$. As discussed after the proof of Lemma 2.106, $\mathbb{R}^k \times \{w\}$ is not necessarily a stratum of $\mathbb{R}^k \times Z$, so, in particular, this is not necessarily the stratum containing $y$ in $(\mathbb{R}^k \times cW)^*$. However, since the intrinsic stratification $(\mathbb{R}^k \times cW)^*$ is a CS set, $y$ will nonetheless have some distinguished neighborhood $N \subset (\mathbb{R}^k \times cW)^*$; let us suppose this distinguished neighborhood $N$ is stratified homeomorphic to $\mathbb{R}^m \times cL$ for some $m$ and some compact filtered space $L$. Since $\mathbb{R}^k \times \{w\}$ is a stratum of $\mathbb{R}^k \times cW$, and so all points in it have homeomorphic neighborhoods, the intersection $\mathbb{R}^k \times \{w\}$ with $N$ must be contained in the stratum of $N$ homeomorphic to $\mathbb{R}^m \times \{v\}$, where $v$ is the cone point of $cL$.

Now, up to topological homeomorphism $\mathbb{R}^k \times cW \cong c(S^{k-1} \ast W)$, and $N \cong \mathbb{R}^m \times cL \cong c(S^{m-1} \ast L)$. Since our neighborhood $N$ of $y$ is contained in $\mathbb{R}^k \times cW$, we can conclude from Lemma 2.97 that in fact

$$\mathbb{R}^k \times cW \cong c(S^{k-1} \ast W) \cong c(S^{m-1} \ast L) \cong \mathbb{R}^m \times cL,$$

with each homeomorphism fixing $y$. Let $h : \mathbb{R}^k \times cW \to \mathbb{R}^m \times cL$ be the composite homeomorphism. Since the intrinsic stratification of a CS set is determined locally and purely topologically and since $N$ is an open subset of the intrinsic stratification $(\mathbb{R}^k \times cW)^*$, it follows that the images of the skeleta under $h$ also provide the intrinsic stratification of $\mathbb{R}^k \times cW$. In other words, $h$ provides a stratified homeomorphism $\mathbb{R}^m \times cL \to (\mathbb{R}^k \times cW)^*$. Thus it suffices to show that $I^p H^G_M(\mathbb{R}^k \times cW) \cong I^p H^G_M(\mathbb{R}^m \times cL)$ via the homeomorphisms $h$.

We note that our previous observation that the intersection of $\mathbb{R}^k \times \{w\}$ with $N$ must be contained in the stratum of $N$ homeomorphic to $\mathbb{R}^m \times \{v\}$ now translates into the observation
that \( h(\mathbb{R}^k \times \{w\}) \subset \mathbb{R}^m \times \{v\} \). It will be useful to observe that if \( s = \dim W \) and \( t = \dim L \), then \( k + s = m + t \), so, in particular, \( L \) has dimension \( t = k + s - m \). Since \( m \geq k \), we have \( t \leq s \).

In order to use \( P(i) \), we will need to utilize a space of depth \( \leq i \), and this will be \( \mathbb{R}^k \times cW - \mathbb{R}^k \times \{w\} \cong \mathbb{R}^k \times (cW - \{w\}) \cong \mathbb{R}^{k+1} \times W \). So we must obtain some understanding of how this space interacts with the stratified space \( \mathbb{R}^m \times cL \), where, under the homeomorphism \( h \), it is homeomorphic to the subspace \( \mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\}) \). More generally, we must understand how the two different stratifications interact. We begin by observing that since \( h^{-1}(\mathbb{R}^m \times cL) \) provides the intrinsic stratification of \( \mathbb{R}^k \times cW \), if \( v \) is the vertex of \( cL \), then \( h^{-1}(\mathbb{R}^m \times \{v\}) \) must be a union of strata of \( \mathbb{R}^k \times cW \). Since the skeleta of \( \mathbb{R}^k \times cW \), all have the form \( \mathbb{R}^k \times cW^i \) for some skeleton \( W^i \) of \( W \) (possibly empty), \( h^{-1}(\mathbb{R}^m \times \{v\}) \) must have the form \( \mathbb{R}^k \times cA \) for some closed set \( A \subset W \) (since \( h^{-1}(\mathbb{R}^m \times \{v\}) \) must be closed). Since \( \mathbb{R}^m - \{pt\} \) has the homology of an \( m-1 \) sphere, this must also be true of \( \mathbb{R}^k \times cA - \{0\} \times \{w\} \), which is homotopy equivalent to \( S^{k-1} \ast A \), which is the \( k \)th suspension of \( A \). Since suspension increases the dimension of each reduced homology group by \( 1 \), it follows that \( W^t \) must be a homology \( m-1-k \) sphere.

The preceding observation will now help us compute the intersection homology of \( \mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\}) \). We will utilize the long exact sequence of the pair \((\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\}), \mathbb{R}^m \times cL - \mathbb{R}^m \times \{v\})\). Notice that this makes sense since we have observed that \( h(\mathbb{R}^k \times \{w\}) \subset \mathbb{R}^m \times \{v\} \). If we excise \( h(\mathbb{R}^k \times \{w\}) \times (cL - \{v\}) \), we obtain

\[
I^pH^*_{\text{GM}}(\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\}), \mathbb{R}^m \times cL - \mathbb{R}^m \times \{v\}) \\
\cong I^pH^*_{\text{GM}}((\mathbb{R}^m - h(\mathbb{R}^k \times \{w\})) \times cL, (\mathbb{R}^m - h(\mathbb{R}^k \times \{w\})) \times (cL - \{v\})) \\
\cong I^pH^*_{\text{GM}}((\mathbb{R}^m - h(\mathbb{R}^k \times \{w\})) \times (cL, cL - \{v\})).
\]

But via \( h \), \( \mathbb{R}^m - h(\mathbb{R}^k \times \{w\}) \cong \mathbb{R}^k \times (cA - \{v\}) \cong \mathbb{R}^{k+1} \times A \), so

\[
I^pH^*_{\text{GM}}((\mathbb{R}^m - h(\mathbb{R}^k \times \{w\})) \times (cL, cL - \{v\})) \cong I^pH^*_{\text{GM}}(\mathbb{R}^{k+1} \times A \times (cL, (cL - \{v\}))).
\]

But \( A \) is a homology \( m-1-k \) sphere, so it follows from the Künneth theorem\(^{70}\) Theorem 5.28 that

\[
I^pH^*_{\text{GM}}(\mathbb{R}^{k+1} \times A \times (cL, cL - \{v\})) \cong I^pH^*_{\text{GM}}(cL, cL - \{v\}) \oplus I^pH^*_{\text{GM}}(\mathbb{R}^{k+1} \times A \times (cL, (cL - \{v\}))).
\]

By Theorem 4.28, \( I^pH^*_{\text{GM}}(\mathbb{R}^{k+1} \times A \times (cL, cL - \{v\})) = 0 \) if \( i - m + 1 + k \leq t - p(t + 1) \). So in this range, up to isomorphism of groups, the long exact sequence of the pair \((\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\}), \mathbb{R}^m \times cL - \mathbb{R}^m \times \{v\})\) becomes

\[
\rightarrow I^pH^*_{\text{GM}}(L) \rightarrow I^pH^*_{\text{GM}}(\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\})) \rightarrow I^pH^*_{\text{GM}}(cL, cL - \{v\}) \rightarrow,
\]

using that \( \mathbb{R}^m \times cL - \mathbb{R}^m \times \{v\} \cong \mathbb{R}^m \times (cL - \{v\}) \cong \mathbb{R}^{m+1} \times L \). There is a map of long exact sequences from this one to the sequence of the pair \((cL, cL - \{v\})\) induced by projection of \( \mathbb{R}^m \times cL \) to \( a \times cL \) for some \( a \notin h(\mathbb{R}^k \times \{w\}) \). The five lemma then shows

\(^{70}\mathbb{R}^{k+1} \times A \) is homeomorphic to an open subset of \( \mathbb{R}^m \), so it’s a manifold!
that $I^p H^G_M(\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\})) \cong I^p H^G_M(cL)$ for $i \leq t - \bar{p}(t + 1) + m - 1 - k$, noting that when $j = t - \bar{p}(t + 1) + m - k$ we still have a surjection $I^p H^G_M(cL, cL - \{v\}) \oplus I^p H^G_M(cL, cL - \{v\}) \to I^p H^G_M(cL, cL - \{v\})$.

So, having computed that $I^p H^G_M(\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\})) \cong I^p H^G_M(cL)$ for $i \leq t - \bar{p}(t + 1) + m - 1 - k$, we can use that $\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\})$ provides (up to stratified homeomorphism) the intrinsic stratification of $\mathbb{R}^k \times (cW - \{w\})$ together with $P(i)$ and stratified homotopy equivalence to conclude that

\[I^p H^G_M(W) \cong I^p H^G_M(\mathbb{R}^k \times (cW - \{w\})) \cong I^p H^G_M(\mathbb{R}^m \times cL - h(\mathbb{R}^k \times \{w\})) \cong I^p H^G_M(cL)\]

for $i \leq t - \bar{p}(t + 1) + m - 1 - k$. Note again that the two groups on the left are defined with respect to the stratification of $\mathbb{R}^k \times cW$ while those on the right are defined with respect to the stratification of $\mathbb{R}^m \times cL$.

Now we are almost done; we have compiled all the tools necessary to compare the intersection homology groups $\mathbb{R}^k \times cW$ and its intrinsic coarsening $\mathbb{R}^m \times cL$. If $i \geq s - \bar{p}(s + 1), i \neq 0$, then by the assumption on perversities and since $t \leq s$, we have $\bar{p}(s + 1) - \bar{p}(t + 1) \leq s - t$, so also $i \geq s - \bar{p}(t + 1) + t - s = t - \bar{p}(t + 1)$. So by stratified homotopy invariance and the cone formula, in this range

\[I^p H^G_M(\mathbb{R}^k \times cW) = 0 = I^p H^G_M(\mathbb{R}^m \times cL)\]

If $i < s - \bar{p}(s + 1)$, then $i - m + k + 1 < s - m + k + 1 - \bar{p}(s + 1) \leq s - m + k + 1 - \bar{p}(t + 1) = t + 1 - \bar{p}(t + 1)$. So in this range we have shown $I^p H^G_M(cL) \cong I^p H^G_M(W)$, and by the cone formula, $I^p H^G_M(W) \cong I^p H^G_M(cW)$, induced by inclusion. It follows using stratified homotopy equivalences that also in this range $I^p H^G_M(\mathbb{R}^m \times cL) \cong I^p H^G_M(\mathbb{R}^k \times cW)$.

Finally, suppose $i = 0 \geq s - \bar{p}(s + 1)$, then $s \leq \bar{p}(s + 1) \leq \bar{p}(t + 1) + s - t$, so also $0 \geq t - \bar{p}(t + 1)$. So, by the cone formula, $I^p H^G_M(\mathbb{R}^k \times cW)$ and $I^p H^G_M(\mathbb{R}^m \times cL)$ are each isomorphic to either $\mathbb{Z}$ or $0$ depending on whether or not there is an allowable 0-simplex. If $I^p H^G_M(\mathbb{R}^k \times cW) \cong \mathbb{Z}$, then there is an allowable 0-simplex $\sigma_0$ in $\mathbb{R}^k \times cW$. Since we identify $\mathbb{R}^m \times cL$ as a coarsening of $\mathbb{R}^k \times cW$ and since coarsening preserves allowability by Lemma \[5.62\] $\sigma_0$ is also allowable in $I^p S^G_M(\mathbb{R}^m \times cL)$, and so also $I^p H^G_M(\mathbb{R}^m \times cL) \cong \mathbb{Z}$ and inclusion of complexes induces an isomorphism $I^p H^G_M(\mathbb{R}^k \times cW) \to I^p H^G_M(\mathbb{R}^m \times cL)$. Conversely, suppose there is an allowable 0-simplex $\sigma_0$ in $I^p S^G_M(\mathbb{R}^m \times cL)$ so that $I^p H^G_M(\mathbb{R}^m \times cL) \cong \mathbb{Z}$. Suppose the image of $\sigma_0$ lies in a stratum $T$ of $\mathbb{R}^m \times cL$. Since allowability depends only on the stratum and the perversity, we see that then any singular 0-simplex $\Delta^0 \to T$ is also allowable. Now, since $\mathbb{R}^m \times cL$ is a coarsening of $\mathbb{R}^k \times cW$ via the homeomorphism $h$, it follows that $h^{-1}(T)$ is a union of strata of $\mathbb{R}^k \times cW$, and if $T$ has dimension $j$, at least one of these strata of $\mathbb{R}^k \times cW$, say the stratum $S$, must also have dimension $j$, since $T$ and $h^{-1}(T)$ are $j$-manifolds. But now codim($S$) = codim($T$), so $\bar{p}(S) = \bar{T}$, and we see that any 0-simplex with image in $S$ must be allowable in $\mathbb{R}^k \times cW$. Therefore, $I^p H^G_M(\mathbb{R}^k \times cW) \cong \mathbb{Z}$ and inclusion of complexes induces an isomorphism $I^p H^G_M(\mathbb{R}^k \times cW) \to I^p H^G_M(\mathbb{R}^m \times cL)$. So we have seen that $I^p H^G_M(\mathbb{R}^k \times cW) \to I^p H^G_M(\mathbb{R}^m \times cL)$ is an isomorphism in all cases.

This finishes the proof that $P(i) \implies R(i)$. 

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**P(i) and Q(i) ⇒ P(i + 1):** This final verification is essentially a Mayer-Vietoris argument. However, the setting doesn’t quite fit the situation of Theorem 5.3 so we work through the details.

Suppose \( X \) is a CS set of depth \( i + 1 \). Notice that if \( V \) is an open subset of \( X \), then \( V^* \) is the restriction to \( V \) of the intrinsic stratification of \( X \), hence for any open subsets \( V, Y \subseteq X \), the morphisms \( \mathcal{P}H_{\ast}^{GM}(V) \to \mathcal{P}H_{\ast}^{GM}(V^*) \) and \( \mathcal{P}H_{\ast}^{GM}(Y) \to \mathcal{P}H_{\ast}^{GM}(Y^*) \) are compatible (they form a commutative square with the inclusion maps). Let \( U \) be the largest open subset of \( X \) such that \( \mathcal{P}H_{\ast}^{GM}(U) \to \mathcal{P}H_{\ast}^{GM}(U^*) \) is an isomorphism. Such a \( U \) exists by Zorn’s lemma, since if \( \{U_\alpha\} \) is any increasing sequence of open sets such that \( \mathcal{P}H_{\ast}^{GM}(U_\alpha) \to \mathcal{P}H_{\ast}^{GM}(U_\alpha^*) \) is an isomorphism for each \( \alpha \), then certainly also \( \mathcal{P}H_{\ast}^{GM}(\bigcup U_\alpha) \to \mathcal{P}H_{\ast}^{GM}(\bigcup U_\alpha^*) \). By an inductive argument, this implies that the intersection homology groups of compact stratified pseudomanifolds are finitely generated. We can obtain such a theorem for the intersection homology of CS sets assuming that the local intersection homology groups are finitely generated. By an inductive stratification on open subsets (due to their local character), also so using Mayer-Vietoris sequences, the five lemma, and the compatibility of isomorphisms. So using the long exact Mayer-Vietoris argument, this would contradict the maximality of \( U \). So \( X - X(i + 1) \subset U \).

Now suppose \( x \in X - U \), so in particular \( x \in X^k \). Since \( X \) is a CS set, \( x \) has a neighborhood \( N \) stratified homeomorphic to \( \mathbb{R}^k \times cL \) for some compact filtered \( L \), which must have depth \( \leq i \). Let \( Y = N \cap X(i + 1) \cap U \). Since \( U \) contains all of \( X - X(i + 1) \), \( Y \) has a neighborhood in \( U \) homeomorphic to \( Y \times cL \), and in fact \( N \cap U \cong (Y \times cL) \cup (\mathbb{R}^k \times (cL - \{v\})) \), where \( v \) is the cone vertex of \( cL \). The intersection \((Y \times cL) \cap (\mathbb{R}^k \times (cL - \{v\})) \cong Y \times (cL - \{v\}) \). By \( P(i) \), \( \mathcal{P}H_{\ast}^{GM}(Y \times (cL - \{v\})) \cong \mathcal{P}H_{\ast}^{GM}(Y \times (cL - \{v\})) \). By \( Q(i) \), \( \mathcal{P}H_{\ast}^{GM}(Y \times cL) \cong \mathcal{P}H_{\ast}^{GM}(Y \times cL) \). By using the long exact Mayer-Vietoris argument and the five lemma, also \( \mathcal{P}H_{\ast}^{GM}(N \cap U) \cong \mathcal{P}H_{\ast}^{GM}(N \cap U) \). But now \( \mathcal{P}H_{\ast}^{GM}(U) \cong \mathcal{P}H_{\ast}^{GM}(U^*) \) by assumption and \( \mathcal{P}H_{\ast}^{GM}(N) \cong \mathcal{P}H_{\ast}^{GM}(N^*) \) by \( Q(i) \). So another Mayer-Vietoris and five lemma argument shows that \( \mathcal{P}H_{\ast}^{GM}(N \cup U) \cong \mathcal{P}H_{\ast}^{GM}(N \cup U^*) \).

But this contradicts the maximality of \( U \) and shows that we must have \( U = X \).

### 5.7 Finite generation

It is often useful to know that the homology groups of certain “nice” compact spaces, such as manifolds, are finitely generated. We can obtain such a theorem for the intersection homology of CS sets assuming that the local intersection homology groups are finitely generated. By an inductive argument, this implies that the intersection homology groups of compact stratified pseudomanifolds are finitely generated. Using the results of Section 5.5, it suffices to limit our discussion in this section to singular intersection homology; by Theorem 5.47, the following
results will also hold in the PL setting.

Here is the relevant local definition. Note its similarities to our locally torsion free conditions.

**Definition 5.63.** A CS set $X$ is called locally $(\hat{p}, \mathbb{Z}; G)^{GM}$-finitely-generated if $G$ is a finitely generated abelian group and, for each point $x \in X$, there is a link $L$ of $X$ such that $I^p H_i^{GM}(L; G)$ is finitely generated for each $i$.

**Remark 5.64.** Analogously to Remark 5.38, we can call a CS set $X$ locally $(\hat{p}, R; M)^{GM}$-finitely-generated if $M$ is a finitely generated $R$-module over a Noetherian ring $R$ and, for each point $x \in X$, there is a link $L$ of $X$ such that $I^p H_i^{GM}(L; M)$ is finitely generated as an $R$-module for each $i$.

Arguments completely analogous to those of Lemma 5.40 show that this definition is equivalent to requiring that each $I^p H_i^{GM}(L; G)$ be finitely generated for any link $L$ in $X$.

**Remark 5.65.** The following proposition only requires that every point have a distinguished neighborhood $N \cong \mathbb{R}^k \times cL$ such that each $I^p H_i^{GM}(N; G)$ is finitely generated. Thus, as $I^p H_i^{GM}(N; G) \cong I^p H_i^{GM}(L; G)$ only for some $i$, and is $0$ otherwise, we do not need the full force of Definition 5.63 here. We will, however, need the stronger assumptions below when we discuss general Künneth theorems; see Section 6.4.2.

**Proposition 5.66.** Suppose $X$ is a locally $(\hat{p}, \mathbb{Z}; G)^{GM}$-finitely-generated CS set. Suppose $U \subset W$ are open subsets of $X$, that $\bar{U} \subset W$, and that $\bar{U}$ is compact. Then the image of $I^p H_i^{GM}(U; G)$ in $I^p H_i^{GM}(W; G)$ is finitely generated. In particular, if $X$ is compact, then each $I^p H_i^{GM}(X; G)$ is finitely generated.

**Proof.** We first observe that the last claim follows from the general situation by taking $U = W = X$.

To prove the proposition, we provide the arguments of [8, Theorem V.3.5]. We will perform an induction on $i$. The claim is clearly true when $i < 0$. So we suppose that the statement holds for all $i < k$ and consider dimension $k$.

We assume that $W$ is fixed and let $\mathcal{E}_W$ be the set of open subsets $U \subset W$ such that $\bar{U}$ is compact in $W$. Let $\mathcal{E}_W^k$ be the set of $U \in \mathcal{E}_W$ such that $\text{im}(I^p H_i^{GM}(U; G) \to I^p H_i^{GM}(W; G))$ is finitely generated. We want to show that $\mathcal{E}_W^k = \mathcal{E}_W$ under the induction hypothesis $\mathcal{E}_W^i = \mathcal{E}_W$ for all $i < k$. Suppose that $V \in \mathcal{E}_W^k$ and that $U \subset V$. Then $\text{im}(I^p H_i^{GM}(U; G) \to I^p H_i^{GM}(W; G)) \subset \text{im}(I^p H_i^{GM}(V; G) \to I^p H_i^{GM}(W; G))$. So if $V \in \mathcal{E}_W$, then $U \in \mathcal{E}_W^k$, using that $\mathbb{Z}$ is a Noetherian ring so that any submodule of a finitely generated module over $\mathbb{Z}$ (i.e. of any finitely generated abelian group) is finitely generated; see [64, Chapter X]. Therefore, it suffices to show that every compact set $K \subset W$ has an open neighborhood $V \supset K$ with $V \in \mathcal{E}_W^k$.

As $X$ is a CS set, every point $x \in W$ has a neighborhood $N$ homeomorphic to $\mathbb{R}^j \times cL$ that has compact closure in $W$; if necessary, take a distinguished neighborhood homeomorphic to $\mathbb{R}^j \times cL$ and then let $N$ be a smaller open neighborhood $D_r \times cL$ within that (in the notation of Lemma 5.40) with closure homeomorphic to $\bar{D}_r \times cL$. We know from stratified homotopy equivalence and the cone formula that $I^p H_i^{GM}(N; G) \cong I^p H_i^{GM}(\mathbb{R}^j \times cL; G)$ is either 0 or
isomorphic to $I^pH^G_k(L; G)$ or $G$, all of which we have assumed to be finitely generated. So, again using that $\mathbb{Z}$ is Noetherian, the image of $I^pH^G_k(N; G)$ in $I^pH^G_k(W; G)$ is finitely generated [64, Proposition X.1.1]. Therefore, $N \in \mathcal{E}_W^k$. So now if $K$ is any compact subset of $W$, there are $U_1, \ldots, U_m \in \mathcal{E}_W^k$ that cover $K$. We want to show that there is a single $U \in \mathcal{E}_W^k$ covering $K$, so we will show that we can do an induction to decrease the number of elements of $\mathcal{E}_W^k$ needed to cover $K$.

Suppose we can show that any compact $K' \subset W$ that can be covered by two elements of $\mathcal{E}_W^k$ can be covered by one element of $\mathcal{E}_W^k$ and that $K$ is covered by $U_1, \ldots, U_m \in \mathcal{E}_W^k$. Then $K' = K - \bigcup_{j=1}^{m-2} U_j$ is compact and contained in $U_{m-1} \cup U_m$. So our assumption implies there is a $U''_{m-1} \in \mathcal{E}_W^k$ with $K' \subset U''_{m-1}$, and therefore $K \subset U_1 \cup \cdots \cup U_{m-2} \cup U''_{m-1}$. Thus, we could inductively reduce the number of elements of $\mathcal{E}_W^k$ needed to cover $K$ down to one. So, it remains to show that we can reduce covers of compact subspaces by two elements of $\mathcal{E}_W^k$ to covers by one element.

So let $U_1, U_2 \in \mathcal{E}_W^k$ with $K \subset U_1 \cup U_2$. Let $V_1$ be a neighborhood of $K - U_2$ with $\bar{V}_1 \subset U_1$. As $K - U_2$ is compact and contained in $U_1$, such a $V_1$ exists by Corollary 2.44. Then $K \subset V_1 \cup U_2$, and we can let $V_2$ be an open neighborhood of $K - V_2$ such that $V_2 \subset U_2$ for the same reasons. Then $K \subset V_1 \cup V_2$, and we claim $V_1 \cup V_2 \subset \mathcal{E}_W^k$.

Consider the following diagram:

$$
\begin{array}{c}
I^pH^G_k(V_1; G) \oplus I^pH^G_k(V_2; G) \\
\downarrow \beta \quad \downarrow \gamma \\
I^pH^G_k(U_1; G) \oplus I^pH^G_k(U_2; G) \\
\downarrow \alpha \quad \downarrow \delta \\
I^pH^G_k(U_1 \cup U_2; G) \\
\downarrow \mu \quad \downarrow \nu \\
I^pH^G_k(W; G).
\end{array}
$$

The rows of this diagram are from the Mayer-Vietoris sequences. Our claim is that $\text{im}(\nu \beta)$ is finitely generated. The image of $\mu$ is finitely generated because $U_1, U_2 \in \mathcal{E}_W^k$, and so $\text{im}(\mu \alpha)$ is finitely generated. As $\nu(\text{im}(\beta) \cap \text{im}(\alpha)) \subset \text{im}(\nu \alpha)$, the subgroup $\nu(\text{im}(\beta) \cap \text{im}(\alpha))$ is finitely generated. We also have $\frac{\text{im}(\beta) \cap \text{im}(\alpha)}{\text{im}(\nu \beta)} \cong \text{im}(\delta \beta) \subset \text{im}(\gamma)$. But we claim that $\text{im}(\gamma)$ is finitely generated by the induction hypothesis. Notice that $V_1 \cap V_2 \subset \bar{V}_1 \cap \bar{V}_2 \subset U_1 \cap U_2 \subset \bar{U}_1 \cap \bar{U}_2$. So the closure of the open set $V_1 \cap V_2$ is contained in the open set $U_1 \cap U_2$. Also, as $U_1, U_2 \in \mathcal{E}_W^k$, the closures $\bar{U}_1$ and $\bar{U}_2$ are compact, so $V_1 \cap V_2$ is compact as a closed subset of $\bar{U}_1 \cap \bar{U}_2$. Therefore, $\text{im}(\gamma)$ is finitely generated in dimension $k - 1$ by the induction assumption.

So, now $\nu(\text{im}(\beta) \cap \text{im}(\alpha))$ and $\frac{\text{im}(\beta)}{\text{im}(\beta) \cap \text{im}(\alpha)}$ are finitely generated, and we can consider the
diagram of short exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{im}(\beta) \cap \text{im}(\alpha) & \rightarrow & \text{im}(\beta) & \rightarrow & \text{im}(\beta) \cap \text{im}(\alpha) & \rightarrow & 0 \\
\downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu \\
0 & \rightarrow & \nu(\text{im}(\beta) \cap \text{im}(\alpha)) & \rightarrow & \nu(\text{im}(\beta)) & \rightarrow & \nu(\text{im}(\beta))/\nu(\text{im}(\beta) \cap \text{im}(\alpha)) & \rightarrow & 0.
\end{array}
\]

The commutativity of the left square induces the vertical map on the right, which is well-defined and surjective. Therefore, the bottom right group is finitely generated as a quotient of a finitely generated group. And, lastly, the finite generation of the outer terms in the bottom short exact sequence implies that \(\nu(\text{im}(\beta)) = \text{im}(\nu\beta)\) is finitely generated [64, Proposition X.1.2].

**Corollary 5.67.** If \(G\) is a finitely generated abelian group and \(X\) is a compact recursive CS set, in particular if \(X\) is a compact stratified pseudomanifold, then \(I^pH^\text{GM}_i(X;G)\) is finitely generated for all \(i\).

**Proof.** By Proposition 5.66, \(I^pH^\text{GM}_i(X;G)\) will be finitely generated if the intersection homology groups of the links are all finitely generated. But these must be lower depth compact recursive CS sets, so the result follows by induction on depth. In the base case, depth 0, \(X\) must be a compact manifold, all the links must be empty, and the proof of the proposition applies directly.

**Remark 5.68.** Similar results to those in this section hold by replacing \(G\) with a Noetherian ring \(R\) and considering finite generation as \(R\)-modules. Then analogous assumptions on the \(I^pH^\text{GM}_i(L;R)\) lead to analogous conclusions about \(I^pH^\text{GM}_i(X;R)\). We treat these sorts of coefficients more explicitly in the setting of non-GM intersection homology; see Definition 6.37, Proposition 6.38, and Corollary 6.39.

### 6 Non-GM intersection homology

#### 6.1 Motivation for non-GM intersection homology

In order to proceed on to discuss intersection homology versions of Poincaré duality and a general Künneth theorem, it is first necessary to modify the Goresky-MacPherson intersection homology, which is the theory we have been using thusfar. Our modified PL and singular intersection chain complexes will be denoted simply \(I^pS_*\) and \(I^pC_*\), and the homology groups will be denoted \(I^pH_*\) and \(I^pH_*\). Note that we have dropped the GM from the notation. As we go along, we will explain why the modified definition is necessary and why we consider it to be the “right” definition for intersection homology.

Before providing the definition, we record the following facts that will be verified below:
1. For perversities $\bar{p}$ for which $\bar{p}(S) \leq \text{codim}(S) - 2$ for all singular strata $S$, our new intersection homology groups $I^p H_*$ will be identical to the intersection homology groups $I^p H^{GM}_*$. In particular, this will be true for any GM-perversity on any pseudomanifold without codimension one strata. In fact, we will show in Proposition 6.7 that if $\bar{p}(S) \leq \text{codim}(S) - 2$ then $I^p S_*(X; G) \cong I^p S^{GM}_*(X; G)$.

2. Versions of all our previous theorems will hold for $I^p H_*$ with the exception of the topological invariance. If $\bar{p}$ is a perversity on a CS set $X$ with $\bar{p}(S) > \text{codim}(S) - 2$ for some stratum $S$, it will no longer be true that $I^\bar{p} H_*(X)$ is independent of the stratification of $X$.

3. If $X$ is a closed oriented pseudomanifold, then there is a version of Poincaré duality with field coefficients $F$, extending that developed by Goresky and MacPherson [42, 43]. This can be formulated for $I^\bar{p} H_*(X; F)$ for any perversity $\bar{p}$ but not necessarily for $I^\bar{p} H^{GM}_*(X; F)$ except in those cases where $I^\bar{p} H^{GM}_*(X; F) \cong I^\bar{p} H_*(X; F)$ as in the preceding remark.

To illustrate this last point, let us provide a sample computation. Looking ahead, if we let $F = \mathbb{Q}$, then one consequence of Poincaré duality on a closed oriented $n$-dimensional pseudomanifold $X$ is an isomorphism

$$I^\bar{p} H_i(X; \mathbb{Q}) \cong \text{Hom}(I^{D\bar{p}} H_{n-i}(X; \mathbb{Q}), \mathbb{Q}).$$

Recall that $D\bar{p}$ is defined so that

$$D\bar{p}(S) = \bar{i}(S) - \bar{p}(S) = \text{codim}(S) - 2 - \bar{p}(S).$$

For our example, let $ST$ be the suspensions of the torus $T = S^1 \times S^1$. The only singular strata of $ST$ are the two suspensions points $\{N, S\}$. We let $\bar{p}$ be the perversity such that $\bar{p}(N) = \bar{p}(S) = 2$, which is one less than the codimension of the singular strata, which is 3. Then $D\bar{p}(N) = D\bar{p}(S) = -1$.

Employing our suspension computation of Theorem 4.43, together with the universal coefficient theorem for $\mathbb{Q}$ coefficients, Corollary 5.43, we have the following results.

$$I^\bar{p} H^Gm_i(ST; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & i = 3, \\ \mathbb{Q} \oplus \mathbb{Q}, & i = 2, \\ 0, & i = 1, \\ \mathbb{Q}, & i = 0, \end{cases}$$

$$I^{D\bar{p}} H^Gm_i(ST; \mathbb{Q}) \cong \begin{cases} 0, & i = 3, \\ \mathbb{Q}, & i = 2, \\ \mathbb{Q} \oplus \mathbb{Q}, & i = 1, \\ \mathbb{Q}, & i = 0. \end{cases}$$

Note the asymmetry that prevents Poincaré duality from holding. By contrast, we will see below in Theorem 6.21 that the following formulas hold:

$$I^p H_i(ST; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & i = 3, \\ \mathbb{Q} \oplus \mathbb{Q}, & i = 2, \\ \mathbb{Q}, & i = 1, \\ 0, & i = 0, \end{cases}$$

$$I^{D\bar{p}} H_i(ST; \mathbb{Q}) \cong \begin{cases} 0, & i = 3, \\ \mathbb{Q}, & i = 2, \\ \mathbb{Q} \oplus \mathbb{Q}, & i = 1, \\ \mathbb{Q}, & i = 0, \end{cases}$$

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and the restored symmetry is a reflection of Poincaré duality between $I^pH_*$ and $I^{Db}H_*$. It turns out that the source of the problem with our original definition, and the key to the modification, lies in the cone formula, which we recall here from Theorem 4.12. If $X$ is a compact $n - 1$ dimensional filtered space and we assume that $X$ has regular strata (so that $I^pH^GM_0(X) \neq 0$), then

$$I^pH^GM_i(cX) \cong \begin{cases} 0, & i \geq n - \bar{p}(\{v\}) - 1, i \neq 0, \\ \mathbb{Z}, & i = 0 \geq n - \bar{p}(\{v\}) - 1, \\ I^pH^GM_i(X), & i < n - \bar{p}(\{v\}) - 1. \end{cases}$$

Now, notice that “most” of the cases here follow a fairly simple formula. There is a cut-off dimension at $n - \bar{p}(\{v\}) - 1$. At this dimension and above, the intersection homology of the cone is 0. Below this dimension, we simply recover the intersection homology of $X$. Notice that the smaller the value of $\bar{p}(\{v\})$, the more of $I^pH^GM_i(X)$ we recover; the greater the value of $\bar{p}(\{v\})$, the more of $I^pH^GM_i(X)$ gets killed to 0. The discrepancy from this nice situation arises as $\bar{p}(\{v\})$ gets so big as to be $\geq n - 1$. Following the above pattern, we would expect in this situation that all of the intersection homology groups of the cone should vanish, but rather we maintain the stable situation that $I^pH^GM_0(cX) \cong \mathbb{Z}$ no matter how large $\bar{p}(\{v\})$ gets beyond this point.

Admittedly, it is not immediately clear that this is a “problem”. A first concern is that it conflicts with what happens in sheaf theoretic intersection homology. From that point of view (which, lamentably, we will not discuss here), there is not this bifurcation of cases as perversities grow large. This raises a flag that perhaps something strange is going on here.$^71$

From a chain-theoretic point of view, we can observe that on a non-compact $F$-oriented $n$-manifold, Poincaré duality with field coefficients takes the modified form of a duality between $H_{n-i}(M; F)$ and $H^\infty_i(M; F)$. Here $H^\infty_i(M; F)$ is the locally finite homology of $M$ (also called homology with closed supports or infinite supports). These are the homology groups of the chain complex whose elements are linear combinations of arbitrary numbers of simplices (not just finite numbers of simplices as in ordinary homology) with the only condition being that chains be locally finite, meaning that every point of $M$ must have a neighborhood that intersects the images of only finitely many singular simplices. Now, let $X$ be a compact $n - 1$ dimensional stratified pseudomanifold, and consider $cX$. The appropriate intersection homology duality should have the form $I^pH^\infty_i(cX; F) \cong \text{Hom}(I^{Db}H_{n-i}(cX; F), F)$. While we will not discuss the groups $I^pH^\infty_i(cX; F)$ in detail here, it is not difficult to show that $I^pH^\infty_i(cX; F) \cong I^pH^\infty_i(cX - \{v\}; F)$; see, for example, $^{23}$—the basic argument is that infinite chains provide a method for pushing chains off to infinity. Now, if $\bar{p}(\{v\}) < 0$, no $n$-simplex can intersect $\{v\}$, and so it follows that we must have $I^pH^\infty_{n-i}(cX; F) = 0$. This would require also $I^{Db}H_0(cX; F) = 0$, but this is impossible using $I^{Db}H^GM_0(cX; F)!$ In fact,$^{22}$

$^71$On the other hand, in the original Goresky-MacPherson sheaf formulation there are catastrophes that can occur for negative perversities. However, the author has modified the sheaf formulation to account for that issue also, inspired largely by the modifications to the singular chain definition we discuss here, which itself was inspired by a need to make the singular theory better approximate the sheaf theory.$^{22}$ For an expository account of those developments, we refer the reader to $^{35}$; for the technical details of the modified sheaf theory, see $^{34}$. For the original Goresky-MacPherson sheaf formulation, we again refer to $^{13}$. 

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in this case \( D\bar{p}(\{v\}) = \text{codim}(\{v\}) - 2 - \bar{p}(\{v\}) = n - 2 - \bar{p}(\{v\}) > n - 2 \), so indeed \( 0 \geq n - D\bar{p}(\{v\}) - 1 \) and \( I^{D\bar{p}}H^G_0(cX; F) \cong F \). So if we want Poincaré duality in this situation, we cannot use \( IH^G_0 \).

In order to modify our definition of intersection homology appropriately, it is interesting to observe why \( I^\bar{p}H^G_0(cX) \cong \mathbb{Z} \) even for large values of \( \bar{p}(\{v\}) \). Essentially, it is the same reason that \( H_0(cX) \cong \mathbb{Z} \) even though the rest of the homology of a cone vanishes: if \( \xi \) is an \( i \)-cycle in \( cX \), \( i > 0 \), then the singular cone \( \bar{c}\xi \) provides a null-homology of \( \xi \) — it is a chain whose boundary is \( \xi \). However, suppose \( \sigma \) is a 0-simplex in \( cX - v \). We can still form the cone \( \bar{c}\sigma \), but now \( \partial \bar{c}\sigma = \sigma - \sigma_v \), where \( \sigma_v \) is the 0-simplex whose image is \( v \); we have already observed this phenomenon before when computing the intersection homology of the cone. The result is that \( \bar{c}\sigma \) does not provide a null-homology, but simply a homology from \( \sigma \) to \( \sigma_v \). Therefore every 0-simplex is homologous to the cone point, which is similarly not null-homologous, hence \( H_0(cX) \cong \mathbb{Z} \), due to this special property of 0-cycles compared to cycles of other dimensions.

So how do we modify our definitions to ensure that \( I^\bar{p}H_0(cX) = 0 \) for large enough \( \bar{p}(\{v\}) \)? The idea is to replace \( I^pS^G_0 \) with a chain complex that behaves a bit more like the relative chain groups. In ordinary homology theory, we know that the relative homology group \( H_0(cX, v) = 0 \) because we in some sense declare that all singular chains with image in the cone vertex \( v \) are trivial. Hence once we have established that all singular 0-simplices are homologous to \( \sigma_v \), and that \( \sigma_v \) is trivial by definition, now the 0-dimensional homology will vanish.

Unfortunately, however, for a general \( n \)-dimensional filtered space \( X \), the solution is not as simple as something like replacing \( I^pS^G_0(X) \) with something like \( I^pS^G_0(X, \Sigma X) \cong I^pS^G_0(X) / I^pS^G_0(\Sigma X) \), where \( \Sigma X \) is the singular locus of \( X \) (see Definition 2.13). For one thing, what does this mean? As we observed in our initial discussion of relative intersection homology in Section 4.3, the most natural meaning for \( I^pS^G_0(\Sigma X) \) would be as the subcomplex \( I^pS^G_0(\Sigma X \subset X) \). But if, for example, \( \bar{p}(S) = \text{codim}(S) - 1 \), which is one of the perversities we’re concerned about, then this complex is empty! But this is no good, since quotienting by the trivial group doesn’t change anything.

A solution to this problem was first hinted at in some unpublished lecture notes of MacPherson’s in which the approach, in a subanalytic setting, is to remove the singular set \( \Sigma X \) entirely and work with locally finite subanalytic chains. We will not discuss this approach in detail. It does have the advantage of \( \Sigma X \) becoming something like an “end” of \( X - \Sigma X \) off which boundaries of infinite chains vanish, but, unfortunately, once one begins using infinite chains, it changes the structure of the theory in other ways. This is not all bad, and it fact this is the key to connecting the chain theory and sheaf theory formalisms of intersection homology! However, it is not what we will do here.

Another approach, this time more within the context of finite singular chains, was introduced independently by the author in and by Saralegi in . These two approaches look different at first, but they turn out to be essentially identical (though the approach of...
the author has a slightly broader applicability to local coefficient situations, see [29]. We will work through all the definition in the next section. We will first present the author’s original description, though in somewhat different language; then we will discuss Saralegi’s version of the definition. Since coefficients will play an important role in what follows, we use coefficients in an abelian group $G$ throughout the discussion.

6.2 Definitions of non-GM intersection homology

Let $X$ be an $n$-dimensional filtered space with perversity $\bar{p}$, and let $G$ be an abelian group. Let each $S^p_i(X; G)$ be the subgroup generated by $\bar{p}$-allowable $i$-simplices that are not contained completely in $\Sigma_X$; this differs from $I^pS^G_i(X; G)$ in that we make no assumption concerning the allowability of boundaries. Let $\xi \in S^p_i(X; G)$ and suppose the boundary of $\xi$ (as an element of $S_*(X; G)$) is written $\partial \xi = \sum_{|\sigma| \leq \Sigma_X} c_{\sigma j} \sigma_j + \sum_{|\sigma| \leq \Sigma_X} c_{\sigma j} \sigma_j$ for $c_{\sigma j} \in G$; in other words, we divide the boundary up into terms coming from those simplices whose images are contained in $\Sigma_X$ and those whose images are not. Let $\hat{\partial} \xi = \sum_{|\sigma| \leq \Sigma_X} c_{\sigma j} \sigma_j$; in other words, $\hat{\partial} \xi$ is obtained from $\partial \xi$ by throwing away the simplices contained in $\Sigma_X$. Define $I^pS_i(X; G)$ to consist of those chains in $S^p_i(X; G)$ whose images under $\hat{\partial}$ are contained in $S^p_{i-1}(X; G)$, i.e. such that each simplex of $\hat{\partial} \xi$ is $\bar{p}$-allowable. Let $I^pC_*(X; G)$ be the complex with chain groups $I^pS_i(X; G)$ and with boundary map $\hat{\partial}$.

Remark 6.1. It is not difficult to observe (CHECK!!) that if $\xi = \sum c_k \tau_k$ for simplices $\tau_k$, then $\hat{\partial} \xi = \sum c_k \hat{\partial} \tau_k$, so $\hat{\partial}$ is a homomorphism.

A PL version $I^pC_*(X; G)$ is defined analogously. Note that the compatibility of the PL structure with the filtration implies that if a simplex of some admissible triangulation $\sigma$ is contained in $\Sigma_X$ then so is every subdivision of $\sigma$. Conversely, if $\sigma$ is an arbitrary $i$-simplex in an admissible triangulation and $\sigma'$ is an $i$-simplex in a subdivision of $\sigma$ and $\sigma'$ is contained in $\Sigma_X$, then $\sigma'$ is contained in some skeleton $X^k$ of $X = X^n$ with $k < n$. But since all admissible triangulations are compatible with the filtration by assumption, in fact $\sigma$ must have been contained in $X^k$, which is a subcomplex of any admissible triangulation. Therefore, an $i$-simplex $\sigma$ is contained in $\Sigma_X$ if and only if every $i$-simplex $\sigma'$ in of any subdivision of $\sigma$ is contained in $\Sigma_X$. This says that the group $C^p_i(X; G) \subset C_i(X; G)$ and generated by $\bar{p}$-allowable $i$-simplices that are not contained in $\Sigma_X$ is well-defined, using also that we know by Lemma 3.23 that any $i$-simplex in a subdivision of a $\bar{p}$-allowable $i$-simplex is also $\bar{p}$-allowable. This also shows that $\hat{\partial} \xi$ is well-defined and can be computed using any triangulation of $X$ with respect to which $\xi$ can be defined.

Definition 6.2. Let $I^pH_1(X; G) = H_1(I^pS_1(X; G))$. From here on, we will call these the intersection homology groups, referring to $I^pH^{GM}_i(X; G)$ as GM-intersection homology groups. Similarly, $I^pS_i(X; G) = H_i(I^pC_*(X; G))$ are the PL intersection homology groups, which are defined when $X$ is a PL filtered space.

Remark 6.3. As usual, if $G = \mathbb{Z}$, we write simply $I^pS_i(X)$, $I^pH_i(X)$, $I^pC_*(X)$, and $I^pS_*(X)$.

Lemma 6.4. $I^pS_*(X; G)$ is well-defined.
Proof. We first note that if \( \xi, \eta \in S^p_i(X) \), then so is \( \xi + \eta \). Furthermore, if \( \partial \xi = \sum_{|\sigma| \leq \Sigma_X} c_{\sigma} \sigma + \sum_{|\sigma| \leq \Sigma_X} c_{\sigma} \sigma_j \) and \( \partial \eta = \sum_{|\sigma| \leq \Sigma_X} d_{\sigma_k} \sigma + \sum_{|\sigma| \leq \Sigma_X} d_{\sigma_k} \sigma_k \), then \( \partial \xi = \sum_{|\sigma| \leq \Sigma_X} c_{\sigma} \sigma + \sum_{|\sigma| \leq \Sigma_X} c_{\sigma} \sigma_j \) and \( \partial \eta = \sum_{|\sigma| \leq \Sigma_X} d_{\sigma_k} \sigma + \sum_{|\sigma| \leq \Sigma_X} d_{\sigma_k} \sigma_k \), so certainly \( \partial \xi + \partial \eta = \partial (\xi + \eta) \). It is also evident from the construction that the image of \( \partial \) applied to \( I^p S_i(X; G) \) lies in \( I^p S_{i-1}(X; G) \).

Lastly, we observe that \( \hat{\partial}^2 = 0 \): By definition, \( \partial \xi = \hat{\partial} \xi + \zeta \), where \( \zeta \) consists of simplices in \( \Sigma_X \), so we have \( \hat{\partial} \zeta = \partial \zeta - \zeta \). Analogously then, \( \hat{\partial}(\partial \xi) = \partial(\partial \zeta - \zeta) - \zeta_2 \), where \( \zeta_2 \) contains the simplices of \( \partial(\partial \zeta - \zeta) \) contained in \( \Sigma_X \). But then

\[
\hat{\partial}(\partial \xi) = \partial(\partial \xi - \zeta) - \zeta_2 \\
= \partial \partial \xi - \partial \zeta - \zeta_2 \\
= 0 - \partial \zeta - \zeta_2.
\]

But since all simplices of \( \partial \zeta \) and \( \zeta_2 \) lie in \( \Sigma_X \) and since \( \hat{\partial}(\partial \xi) \) can contain no non-trivial such simplices, we must have \( -\partial \zeta - \zeta_2 = 0 \). \( \square \)

A more formal formulation of \( I^p S_i(X; G) \), closer to the definition of Saralegi, is as follows: Let \( A^p S_i(X; G) \) be the subgroup of \( S_i(X; G) \) generated by the allowable simplices of \( X \) (again with no assumption on boundaries). Then \( I^p S^{CM}_i(X; G) \) can be described as \( A^p S_i(X; G) \cap \partial^{-1}(A^p S_{i-1}(X; G)) \). Instead, we let

\[
I^p S'_i(X; G) = \frac{(A^p S_i(X; G) + S_i(\Sigma_X; G)) \cap \partial^{-1}(A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))}{S_i(\Sigma_X; G)}.
\]

The idea is that we add to the allowable simplices all of the singular simplices living in \( \Sigma_X \) in order to quotient them out, but we do this in such a way as to maintain a chain complex. If we had just used \( A^p S_i(X; G) + S_i(\Sigma_X; G) \), not only would this just be the same as the quotient of \( A^p S_i(X; G) \) by any allowable simplices supported in \( \Sigma_X \), but we could also have some elements whose boundaries contain non-allowable simplices that are not contained in \( \Sigma_X \), so we would not have a chain complex. The definition we have provided for \( I^p S'_i(X; G) \) clearly does not have this problem. Additionally, we note that \( I^p S'_i(X; G) \) is well-defined because \( S_i(\Sigma_X; G) \subset A^p S_i(X; G) + S_i(\Sigma_X; G) \) and also \( S_i(\Sigma_X; G) \subset \partial^{-1}(A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G)) \) as every boundary of an element of \( S_i(\Sigma_X; G) \) lies in \( S_{i-1}(\Sigma_X; G) \).

These definitions in fact yield isomorphic chain complexes:

**Lemma 6.5.** \( I^p S^*_i(X; G) \cong I^p S'_i(X; G) \), and similarly for the PL versions.

*Proof.* As \( I^p S^*_i(X; G) \subset S^p_i(X; G) \subset A^p S^*_i(X; G) \), we will have a canonical inclusion induced homomorphism \( f : I^p S^*_i(X; G) \to I^p S'_i(X; G) \) provided we also have \( I^p S^*_i(X; G) \subset \partial^{-1}(A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G)) \). But indeed, if \( \xi \in I^p S^*_i(X; G) \), then by definition \( \partial \xi = \hat{\partial} \xi + \eta \), where \( |\eta| \subset \Sigma_X \) and \( \hat{\partial} \xi \) must be allowable by definition of \( I^p S^*_i(X; G) \). So indeed \( \partial \xi \in A^p S_{i-1}(X) + S_{i-1}(\Sigma_X; G) \). The inclusion \( f \) is also a chain map since \( \partial f(\xi) \) will be represented by \( \partial \xi = \hat{\partial} \xi + \eta \), which is equivalent in \( I^p S'_i(X; G) \) to \( \hat{\partial} \xi \), because \( |\eta| \subset S_{i-1}(X; G) \). But \( \hat{\partial} \xi \) also represents \( f(\hat{\partial} \xi) \).

Next we observe that \( f \) is injective since by definition \( I^p S^*_i(X; G) \subset S^p_i(X; G) \) and \( S^p_i(X; G) \) consists of simplices whose images do not lie completely in \( \Sigma_X \).
Now, we show that $f$ is surjective. Let $x \in (A^pS_i(X;G)+S_i(\Sigma_X;G)) \cap \partial^{-1}(A^pS_{i-1}(X;G)+S_{i-1}(\Sigma_X;G))$. By definition, $x$ can be written as a sum $\xi + \eta$ with $\xi \in A^pS_i(X;G)$ and $\eta \in S_i(\Sigma_X;G)$. This decomposition is not necessarily unique, but we may assume that $\xi$ contains no simplices contained completely in $\Sigma_X$ (this will make the decomposition unique).

We claim that in fact $\xi$ represents an element of $\hat{I}^pS_i(X;G)$ so that $f(\xi) = x$. We only need to show that $\hat{\partial}\xi \in S^i_{i-1}(X;G)$. But we know that $\partial x = \partial(\xi + \eta) \in A^pS_{i-1}(X;G) + S_{i-1}(\Sigma_X;G)$, hence any simplex of $\partial(\xi + \eta) = \partial\xi + \partial\eta$ that is not contained in $\Sigma_X$ must be allowable. Note that any simplices that $\partial\xi$ and $\partial\eta$ have in common must lie in $\Sigma_X$, so every simplex of $\partial(\xi + \eta)$ that does not lie in $\Sigma_X$ must be part of $\partial\xi$, and this includes all of the simplices of $\hat{\partial}\xi$; thus all simplices of $\hat{\partial}\xi$ are allowable.

The proof of the PL versions is essentially the same, using fixed representatives for the chains.

This lemma demonstrates that the two approaches to $I^pS_\ast(X;G)$ discussed so far are equivalent. The author has some preference for the first definition since it eliminates the ambiguities that arise naturally when dealing with quotient groups. However the second definition certainly has the advantage of providing a clear mathematical formula.

The definition of Saralegi is a version of our second definition. The main difference is that instead of including and then quotienting out all of $S_\ast(\Sigma_X;G)$, instead we only include and quotient out the allowable chains on those strata for which the perversity is “too big”. Following our cone computations above, this is reasonable, since these are the strata for which distinguished neighborhood computations will be affected by the “faulty” cone formula.

We will temporarily denote Saralegi’s chain complex from [89] as \[ I^pS_*''(X;G). \]

To define it, we continue to let $A^pS_i(X;G)$ denote the groups generated by the $\hat{p}$-allowable $i$-simplices of $X$, and we let $X^\hat{p}$ denote the closure of the union of the singular strata $S$ of $X$ such that $\hat{p}(S) > \mathrm{codim}(S) - 2$ (i.e. $D\hat{p}(S) < 0$). Then let $A^qS_i(X^\hat{p};G)$ be generated by the $\hat{q}$-allowable $i$-simplices with support in $X^\hat{p}$. We then define

$$ I^pS_*''(X;G) = \frac{(A^pS_i(X;G) + A^{p+1}S_i(X^\hat{p};G)) \cap \partial^{-1}(A^pS_{i-1}(X;G) + A^{p+1}S_{i-1}(X^\hat{p};G))}{A^{p+1}S_i(X^\hat{p};G) \cap \partial^{-1}A^{p+1}S_{i-1}(X^\hat{p};G)}. $$

We next show that show that $I^pS_*''(X;G)$ is also isomorphic to $I^pS_\ast(X;G)$.

**Lemma 6.6.** $I^pS_*''(X;G) \cong I^pS_\ast(X;G)$, and similarly for the PL versions.

**Proof.** We will construct a chain map $f : I^pS_\ast(X;G) \to I^pS_*''(X;G)$ and show that it is an isomorphism in each fixed degree. The proof is similar in spirit to that of Lemma 6.5 though with some modifications.

To define $f$, let $\xi \in I^pS_i(X;G)$. Then, by definition, $\xi \in A^pS_i(X;G)$. Furthermore, recall that $\hat{\partial}\xi \in I^pS_{i-1}(X;G)$ can be described by taking the boundary of $\xi$ in $S_i(X;G)$ and then throwing away all terms involving simplices with support in $\Sigma_X$. The remaining simplices of $\partial\xi$ must be $\hat{p}$-allowable by definition of $I^pS_\ast(X;G)$. But this means that in $S_{i-1}(X;G)$, $\partial\xi = \hat{\partial}\xi + \eta$, where $\hat{\partial}\xi \in I^pS_{i-1}(X;G)$ and $\eta \in S_{i-1}(\Sigma_X;G)$; as in the previous lemma, we

\[\text{Saralegi’s original notation was } SC^\phi_\ast(X,X_\phi;G), \text{ where } \phi = D\hat{p}.\]
can make the decomposition unique by putting all simplices supported in $\Sigma_X$ into $\eta$. We claim that, in fact, $\eta \in A^{p+1}S_{i-1}(X^p; G)$. If so, it follows that $\xi$ also represents an element of $I^pS^p_i(X; G)$, and we can define $f$ in this manner.

Let $S \subset X_{n-k}$, $k > 0$, be a stratum such that $|\eta| \cap S \neq \emptyset$. For a $j$ chain $\zeta$, define $\dim(\{s| \cap S\})$ to be the maximum over all singular simplices $s$ of $\zeta$ of the minimum $K$ such that $\sigma^{-1}(S)$ is contained in the $K$ skeleton of $\Delta^j$; in other words this is the skeleton that comes into the the allowability computations and $\dim(s| \cap S)$ gives us the “worst case” among the simplices of $\zeta$. Since $\eta$ is part of the boundary of $\xi$, we must have $\dim(|\eta| \cap S) \leq \dim(|\zeta| \cap S) \leq i - k + p(S) = i - 1 - k + p(S) + 1$. Thus, $\eta$ is both contained in $\Sigma_X$ and $p + 1$ allowable. Furthermore, suppose that $S \not\subset X^p$ (continuing to assume $S \subset X_{n-k}$). Then $p(S) \leq k - 2$, so $\dim(|\xi| \cap S) \leq i - k + k - 2 = i - 2$. Thus the interior of no simplex of $\eta$ can lie in $S$. So, the interiors of the simplices of $\eta$ must be in $X^p$, and hence $|\eta| \subset X^p$, since $X^p$ is closed. We have shown $\eta \in A^{p+1}S_{i}(X^p; G)$. It follows that $\xi$ represents an element of $I^pS^p_i(X; G)$.

This assignment taking $\xi$ to an element of $I^pS^p_i(X; G)$ is clearly additive, so this defines our homomorphism $f$. To see that $f$ is a chain map, consider $\xi \in I^pS_{i}(X; G)$. As noted above, we can write $\partial \xi$ in $S_{i-1}(X; G)$ as $\partial \xi = \hat{\partial} \xi + \eta$, so $\partial f(\xi)$ represented by $\hat{\partial} \xi + \eta$ while $f(\partial \xi)$ is represented by $\partial \xi$. But we have also seen that $\eta \in A^{p+1}S_{i-1}(X^p; G)$. It follows immediately that $\partial \eta$ is supported in $X^p$. Additionally, $\partial \eta = \partial (\partial \xi - \hat{\partial} \xi) = \partial \hat{\partial} \xi$, and since $\partial \hat{\partial} \xi$ is $p$-allowable, the argument presented above to show that $\eta \in A^{p+1}S_{i-1}(X^p; G)$ shows similarly that $\partial \eta \in A^{p+1}S_{i-2}(X^p; G)$. So $\eta \in A^{p+1}S_{i-1}(X^p; G) \cap \partial^{-1}A^{p+1}S_{i-2}(X^p; G)$, and therefore $\hat{\partial} \xi + \eta$ and $\hat{\partial} \xi$ represent the same element of $I^pS^p_{i-1}(X; G)$. Thus $f$ is a chain map.

Suppose $\xi \in I^pS_i(X; G)$ and $f(\xi) = 0$. Then $f(\xi) \in A^{p+1}S_i(X^p; G) \cap \partial^{-1}A^{p+1}S_{i-1}(X^p; G) \subset S_i(\Sigma_X; G)$. But this is impossible since $\xi \in I^pS_i(X; G)$ implies that $\xi$ is not supported in $\Sigma_X$. Thus $f$ is injective.

Suppose $z \in I^pS^p_i(X; G)$. We can represent $z$ by $z = x + y$, where $x \in A^pS_i(X; G)$ and $y \in A^{p+1}S^p_i(X^p; G)$. We claim that we also have $\partial y \in A^{p+1}S^p_{i-1}(X^p; G)$. Certainly $\partial y \in S_{i-1}(X^p; G)$, so the issue is just the allowability. We know from the definition of $I^pS^p_i(X; G)$ that $\partial z = \partial x + \partial y \in A^pS_{i-1}(X^p; G) + A^{p+1}S^p_{i-1}(X^p; G)$, so, in particular, $\partial z$ is $p + 1$ allowable. Since $x$ is $p$-allowable, for any stratum $S \subset X_{n-k}$, we have $\dim(|\partial x| \cap S) \leq \dim(|x| \cap S) \leq i - k + p(S) = i - 1 - k + p(S) + 1$, so that $\partial x$ is $p + 1$ allowable. Thus it follows that $\partial y = \partial z - \partial x$ is $p + 1$ allowable. So $\partial y \in A^{p+1}S^p_{i-1}(X^p; G)$. So, $y \in A^{p+1}S^p_i(X^p; G) \cap \partial^{-1}A^{p+1}S^p_{i-1}(X^p; G)$. It follows that $z$ and $x$ represent the same element of $I^pS^p_i(X; G)$. Now, $x \in A^pS_i(X; G)$, and we know that $\partial x = \partial z - \partial y \in A^pS_{i-1}(X^p; G) + A^{p+1}S^p_{i-1}(X^p; G)$, which implies that if we set to zero the coefficients of the simplices of $\partial x$ with support in $\Sigma_X$ (and, in particular, in $X^p$), then what remains will be $p$-allowable. So $x$ fits the description of a chain in $I^pS_i(X; G)$. Thus $f$ is surjective.

The proof of the PL versions is essentially the same, using fixed representatives for the chains.

Saralegi’s proof gives us perhaps the shortest route towards seeing that $I^pS_i(X; G) \cong I^pS^{GM}_i(X; G)$ when $\bar{p}(S) \leq \bar{t}(S)$ for all singular $S$.  

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**Proposition 6.7.** If \( \bar{p}(S) \leq \bar{t}(S) \) for all singular strata of \( X \), then \( I^p S_*(X; G) \cong I^{\bar{p}} S_{G*}^*(X; G) \), and similarly for the PL versions.

**Proof.** If \( \bar{p}(S) \leq \bar{t}(S) \), then \( \bar{p}(S) \leq \text{codim}(S) - 2 \). So there are no strata on which \( D\bar{p}(S) = \text{codim}(S) - 2 - \bar{p}(S) \) is \( < 0 \), and so \( X^{\bar{p}} \) is empty! So \( I^p S_*(X; G) = A^p S_*(X; G) \cap \partial^{-1}(A^p S_{*-1}(X; G)) \), which is precisely \( I^{\bar{p}} S_{G*}^*(X; G) \). The result follows from Lemma 6.6. The PL argument is identical.

**Remark 6.8.** Here is another way to prove Proposition 6.7 this time working with \( I^p S_*(X; G) \) directly. Recall that \( S^p_*(X; G) \) is the complex generated by the allowable simplices. If \( \bar{p} \leq \bar{t} \), then for every \( i \), \( \bar{p}(S) \leq \bar{t}(S) \leq \text{codim}(S) + \bar{p}(S) \leq i - \text{codim}(S) + \bar{p}(S) - 2 = i - 2 \). So in order for an \( i \)-simplex \( \sigma \) to be allowable, \( \sigma^{-1}(S) \) must be contained in the \( i - 2 \) skeleton of \( \Delta^i \) for any singular stratum \( S \), and so \( \sigma^{-1}(\Sigma X) \subset \{ i - 2 \text{ skeleton of } \Delta^i \}. \) But this implies that no \( i - 1 \) dimensional face of \( \sigma \) can be contained completely in \( \Sigma X \). This in turn tells us that \( \hat{\partial} = \partial \) when applied to elements of \( S^p_*(X; G) \). By definition, \( I^p S_*(X; G) \) consists of those elements \( \xi \in S^p_*(X; G) \) such that \( \hat{\partial} \xi \in S^p_*(X; G) \). But in this case, this means precisely that every simplex of \( \xi \) must be allowable and every simplex of \( \partial \xi \) must be allowable. But this is precisely the definition of \( I^{\bar{p}} S_{G*}^*(X; G) \).

So when \( \bar{p}(S) \leq \bar{t}(S) \) for all singular strata of \( X \), we get nothing new. However, when \( \bar{p}(S) > \bar{t}(S) \) for some \( S \), we indeed get something different.

One way to see this immediately is to observe that \( I^p S_*(X; G) \) is not always a topological invariant even if the conditions of Theorem 5.52 hold; i.e. it may depend on the stratification (though of course, by the preceding proposition, it will be a topological invariant when the conditions of Theorem 5.52 and the condition \( \bar{p}(S) \leq \bar{t}(S) \) both hold; notice that this requires that \( X \) have no codimension one strata).

**Example 6.9.** Let \( \bar{0} \) be the 0-perversity. Not that \( \bar{0} \) satisfies the conditions of Theorem 5.52. Let \( \mathbb{R} \) be the real line, unstratified. Then \( I^p S_*(\mathbb{R}) = S_*(\mathbb{R}) \), so \( H_0(S_*(\mathbb{R})) = H_0(\mathbb{R}) \cong \mathbb{Z} \). But now suppose we stratify \( \mathbb{R} \) instead as \( \{ \{ \} \} \subset \mathbb{R} \). Let \( \sigma_x \) be the 0-simplex with image at \( x \in \mathbb{R} \). Let \( I_x \) be the linear 1-simplex with boundary \( \sigma_x - \sigma_0 \). Then \( I_x^{-1}(\{ \} \) is in the 0-skeleton of \( \Delta^1 \), and \( 0 \leq 1 - \text{codim}(\{ \}) + 0(\{ \}) = 1 - 1 + 0 = 0 \), so \( I_x \) is allowable. Furthermore, \( \hat{\partial} I_x = \sigma_x \). So \( I_x \) is a null-homology of \( \sigma_x \) in \( I^p S_*(\mathbb{R}) \), so \( I^p H_0(\mathbb{R}) = 0 \).

Additionally, we expect that \( I^p S_*(X) \) should not equal \( I^{\bar{p}} S^*_G(X) \) in all cases because the entire point of introducing \( I^p S_*(X) \) was to modify the cone formula. Let us verify that we have done that successfully.

**Theorem 6.10.** If \( X \) is a compact filtered space of formal dimension \( n - 1 \), then

\[
I^p H_i(cX; G) \cong \begin{cases} 0, & i \geq n - \bar{p}(\{ \}) - 1, \\ I^p H_i(X; G), & i < n - \bar{p}(\{ \}) - 1. \end{cases}
\]

Furthermore, the isomorphisms of the last case are induced by inclusion. An equivalent conclusion holds for PL intersection homology when \( X \) is a compact PL manifold stratified space.
Proof. The proof is nearly exactly the same as that of Theorems 4.12 and 5.33 except for the special computations that were required in dimension 0. We outline the arguments again, highlighting the necessary modifications.

If \( \xi \) is an \( i \)-cycle in \( I^pS_i(X;G) \) for \( i \geq n - \bar{p}(\{v\}) - 1 \), \( i > 0 \), then, as in the proof of Theorem 4.12 we can check that \(^74\) \( \bar{c}\xi \) is allowable and its boundary is \( \xi \). Recall that \( \xi \in I^pS_i(X;G) \) is an \( i \)-cycle if \( \partial \xi = 0 \), but as a chain in \( S_i(X;G) \), we will have \( \partial \xi = \partial \xi + c \), where \( c \) is contained in \( S_i \).

Then \( \partial(\bar{c}\xi) = \xi - \bar{c}(\partial \xi) = \xi - \bar{c}(\partial \xi + c) \). Since \( c \) is supported in \( S_i \), so is \( \bar{c} \), and we have assumed that \( \partial \xi = 0 \). Therefore, \( \partial(\bar{c}\xi) = \xi \), using that no simplex of \( \xi \) is contained in \( S_i \), and that the definition of \( I^pS_i(X;G) \).

The allowability of the simplices of \( \bar{c}\xi \) follows from the allowability of the simplices of \( \xi \) as in the proof of Theorem 4.12.

But now, remarkably, the argument of the preceding paragraph continues to hold even if \( i = 0 \geq n - \bar{p}(\{v\}) - 1 \), which wasn’t the case in the proof of Theorem 4.12 except for the perversity assumption, if \( \sigma \) is an allowable 0-simplex in \( cX \) not contained in \( S_i \), so that \( \sigma \otimes g \in I^pS_0(cX;G) \), then \( \partial(\sigma \otimes g) = \sigma \otimes g - \sigma_v \otimes g \), where \( \sigma_v \) is the 0-simplex with image in the cone vertex \( v \). But then we have \( \partial(\sigma \otimes g) = \sigma \otimes g \) since \( \sigma_v \) has image in \( S_i \). But then any \( \sigma \otimes g \in I^pS_0(cX;G) \) is null-homologous in \( I^pS_i(X;G) \), and \( I^pH_0(X;G) = 0 \).

Finally, for \( i < n - \bar{p}(\{v\}) - 1 \), \( \bar{c}\xi \) is not allowable, and in fact no allowable simplex can intersect \( \{v\} \) by the arguments in the proof of Theorem 4.12. So \( I^pH_i(cX;G) = I^pH_i(cX - \{v\};G) \). We will see below in Corollary 6.17 that, just as for \( I^pH^GM \), \( I^pH_i \) is a stratified homotopy invariant, and this completes the proof.

\[ \square \]

6.2.1 Relative non-GM intersection homology and the relative cone formula

The relative intersection homology groups are defined just as we defined the relative GM-intersection homology groups. If \( Y \subset X \), we let \( I^pS_*(Y;G) \) be the subcomplex of \( I^pS_*(X;G) \) consisting of chains supported in \( Y \) or, equivalently, the complex defined natively in \( Y \) by using the filtration and perversity inherited from \( X \) (cf. Lemma 4.16). Then we let

\[ I^pS_*(X,Y;G) = I^pS_*(X;G)/I^pS_*(Y;G) \]

and

\[ I^pH_*(X,Y;G) = H_*(I^pS_*(X,Y;G)). \]

Of course it follows that there is a long exact sequence of pairs for intersection homology.

Remark 6.11. It will be useful below to notice that, for each \( i \), \( I^pS_i(X,Y;G) \) is a subcomplex of \( S_i(X,Y;G) \). Indeed, there are evident maps

\[ I^pS_i(X,Y;G) = I^pS_i(X,G)/I^pS_i(Y,G) \rightarrow S_i^p(X,G)/S_i^p(Y,G) \rightarrow S_i(X,G)/S_i(Y,G) \cong S_i(X,Y;G). \]

To see that this composition is injective, we just need to observe that if \( x \in I^pS_i(X,Y;G) \) does not represent 0, then \( x \) has a representative chain contain a simplex that is not supported in \( Y \), and so the image of \( x \) under this sequence of maps cannot be 0.

\[ \text{We let } \bar{c}(\sigma \otimes g) = (\bar{c}\sigma) \otimes g \text{ for } g \in G. \]
The following corollary is an immediate consequence of the long exact sequence of pairs.

**Corollary 6.12.** If $X$ is a compact $n-1$ dimensional filtered space then

$$I^pH_i(cX, cX - \{v\}; G) \cong \begin{cases} I^pH_{i-1}(X; G), & i > n - \bar{p}(\{v\}) - 1, \\ 0, & i \leq n - \bar{p}(\{v\}) - 1. \end{cases}$$

An equivalent conclusion holds for PL intersection homology when $X$ is a compact PL manifold stratified space.

For some of our arguments below, it will be more useful to work with $I^pS_*(X; G)$ in the form $I^pS'_*(X; G)$.

As these complexes are isomorphic, of course it follows that if we define $I^pS'_*(X, A; G)$ to be $I^pS'_*(X; G)/I^pS'_*(A; G)$ then

$$I^pS'_*(X, A; G) = \frac{I^pS'_*(X; G)}{I^pS'_*(A; G)} \cong \frac{I^pS_*(X; G)}{I^pS_*(A; G)} = I^pS_*(X, A; G),$$

but it is also useful to have a better look at the form $I^pS'_*(X, A; G)$ takes from the definitions. Indeed, $S_*(\Sigma_A; G)$ is generated by the singular simplices in $\Sigma_A = A \cap \Sigma_X$, and these are certainly all contained in both $\Sigma_X$ and in the expression in the square brackets. On the other hand, any chain in the left side of the expression is both a singular chain in $\Sigma_X$ and a singular chain in $A$ again from the expression in brackets), so it must be in $S_*(\Sigma_A; G)$. But now by basic group theory, if $C, D, E$ are abelian groups, then

$$\frac{C}{D \cap E} = \frac{C}{D + E} \cdot \frac{D}{D + E} = \frac{C}{D + E}.$$

So

$$I^pS'_*(X, A; G) = \frac{(A^pS_*(X; G) + S_*(\Sigma_X; G)) \cap \partial^{-1}(A^pS_{*+1}(X; G) + S_{*+1}(\Sigma_X; G))}{S_*(\Sigma_X; G) + (A^pS_*(A; G) + S_*(\Sigma_A; G)) \cap \partial^{-1}(A^pS_{*+1}(A; G) + S_{*+1}(\Sigma_A; G))}.$$

(7)
6.3 Properties of $I^pH_*(X; G)$.

We next review the properties we have already established for $I^pH_*^{GM}(X; G)$ and discuss how they carry over for $I^pH_*(X; G)$. In most cases, the proofs of these properties are nearly word-for-word the same, and so we omit them. Instead we focus on necessary modifications to proofs. For completeness, we state all of the relevant major theorems, providing additional justifications as necessary. More detailed proofs of most statements, written directly for $I^pH_*$ can be found in [29].

Recheck all of these!! Especially any claims about PL because might not be able to use freeness!

Proposition 6.13. If $X, Y$ are filtered spaces, $f : X \to Y$ is $(\bar{p}, \bar{q})$-stratified, and $A \subset X$ and $B \subset Y$ with $f(A) \subset B$, then $f$ induces a chain map $f : I^pS_*(X, A; G) \to I^pS_*(Y, B; G)$. If, furthermore, $X, Y$ are PL filtered spaces, $A, B$ are PL subspaces, and $f$ is a PL map that is $(\bar{p}, \bar{q})$-stratified, then $f$ induces a chain map $f : I^p\mathcal{C}_*(X, A; G) \to I^p\mathcal{C}_*(Y, B; G)$ of PL intersection chain complexes. In either case, we obtain corresponding maps of intersection homology groups.

It follows from the GM-intersection homology arguments of Proposition [4.4] that such a stratified map will take allowable chains to allowable chains. However, we must exercise some care to ensure that we still have a chain map in this new setting. Looking at the $I^pS_*(X, A; G)$ form of the definition for non-GM intersection chains shows that it suffices to have $f$ take allowable simplices to allowable simplices and $\Sigma_X$ to $\Sigma_Y$. But this latter condition is also guaranteed by the assumption that $f$ be $(\bar{p}, \bar{q})$-stratified and so preserve codimension of strata.

Remark 6.14. As noted in Remark [4.6], it is sometimes necessary to consider maps between stratified spaces $f : X \to Y$ that satisfy the property of taking each stratum of $X$ into a stratum of $Y$ but such that codimension is not necessarily preserved. As we showed there, the more general condition for taking allowable simplices to allowable simplices is that $\bar{p}(S) - \bar{q}(S') \leq \text{codim}_X(S) - \text{codim}_Y(S')$ for every pair of strata $S, S'$ with $f(S) \subset S'$. However, this condition alone does not guarantee that a chain map is induced on non-GM intersection chains. For this, it is sufficient to also know that $f(\Sigma_X) \subset \Sigma_Y$, which can again be seen by considering the $I^pS'_*(X, A; G)$ form of the definition of non-GM intersection chains. However, even when this condition fails, it is sometimes possible to construct chain maps, as we will see below in Lemma [7.41].

The verification that allowable chains are taken to allowable chains is the same as for the proof of Proposition [4.4]. The assumption that $f$ preserves codimension ensures that simplices are taken by $f$ into the singular strata of $Y$ if and only if they are contained in the singular strata of $X$. Thus $f$ induces a chain map.

Corollary 6.15. If $f : X \to Y$ is a stratified homeomorphism that is also a homeomorphism of pairs $f : (X, A) \to (Y, B)$ and that the perversities $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ correspond, then $I^pH_*(X, A; G) \cong I^pH_*(Y, B; G)$. The corresponding fact holds for PL spaces, PL stratified homeomorphisms, and PL intersection homology.
Proposition 6.16. Suppose $f, g : X \to Y$ are $(\bar{p}, \bar{q})$-stratified maps that are $(\bar{p}, \bar{q})$-stratified homotopic via a $(\bar{p}, \bar{q})$-stratified homotopy taking the pair $(I \times X, I \times A)$ to $(Y, B)$. Then $f$ and $g$ induce chain homotopic maps $I^p S_*(X, A; G) \to I^q S_*(Y, B; G)$ and so $f = g : I^p H_*(X, A; G) \to I^q H_*(Y, B; G)$. The analogous result holds in the PL category.

The proofs of Proposition 4.8 and 4.22 for $IH^{GM}$ followed the standard proofs that homotopic maps topological maps induce chain homotopic chain maps. In particular, we showed that, under the assumptions of that proposition, the prism construction takes intersection chains to intersection chains. Notice that the prisms were all stratum-preserving in the sense that, under the assumptions of that proposition, the prism construction takes intersection chain homotopic chain maps. In particular, we showed that the simplices of $P(\sigma)$ are contained in $\Sigma_Y$. Conversely, if $\sigma$ is not contained in $\Sigma_X$, then none of the simplices of $P(\sigma)$ are contained in $\Sigma_Y$.

So, consider the absolute case $A = \emptyset$, and suppose $\sigma$ is a simplex not contained in the singular locus $\Sigma_X$. We have not only $\partial P(\sigma) = g(\sigma) - f(\sigma) - P(\partial \sigma)$, but $\partial P(\sigma) = g(\sigma) - f(\sigma) - P(\bar{\partial} \sigma)$, as the simplices of $\partial P(\sigma)$ contained in $\Sigma_Y$ are precisely the simplices of $P(\partial \sigma)$, obtained by applying $P$ to simplices in $\partial \sigma$ that are contained in $\Sigma_X$. The map $P$ continues to be a homomorphism (by definition), so if $\xi \in I^p S_*(X; G)$, we have $\hat{\partial} P(\xi) = g(\xi) - f(\xi) - P(\bar{\partial} \xi)$, utilizing Remark 6.1. Again the allowability considerations are demonstrated as before, so the same construction $P$ also provides a chain homotopy in this context. Note that the coefficients $G$ do not alter the argument at all (DOUBLE CHECK!). Similarly, if $A \neq \emptyset$, the prism operator $P$ takes chains in $A$ to chains in $B$, and so it induces a chain homotopy $I^p S_*(X, A; G) \to I^q S_*(Y, B; G)$.

Corollary 6.17. Suppose $f : X \to Y$ is a stratified map with a stratified homotopy between $f$ and $g$ are also maps of pairs $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ and $I \times (Y, B) \to (Y, B)$. Suppose that the values of $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ agree on corresponding strata. Then $f$ induces an isomorphism $I^p H_*(X, A; G) \cong I^q H_*(Y, B; G)$. The analogous result holds in the PL category.

Proposition 6.18. Let $\xi$ be a chain representing an element of $I^p H_*(X, U; G)$. Then $\xi$ is intersection homologous to any singular subdivision $\xi'$, so $\xi$ and $\xi'$ represent the same element of $I^p H_*(X, U; G)$.

We need to exercise a little care here because even if $\sigma$ is not contained in $\Sigma_X$, a singular subdivision $\sigma'$ of $\sigma$ might contain some simplices with images in $\Sigma_X$. Therefore, for this argument it will be best to employ the formulation $I^p S'_*(X, U; G)$ in order for $\xi'$ to make sense as a chain. Since we have only so far studied the absolute group $I^p S'_*(X, U; G)$, we should pause a moment to discuss $I^p S'_*(X, U; G)$.

Recall that

$$I^p S'_*(X; G) = \frac{(A^p S_*(X; G) + S_*(\Sigma_X; G)) \cap \partial^{-1}(A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))}{S_*(\Sigma_X; G)},$$
and $I^p S'_i(X; G) \cong I^p S_*(X; G)$ by Lemma 6.5. There is a map

$$j : \left( A^p S_i(U; G) + S_i(\Sigma_U; G) \right) \cap \partial^{-1} \left( A^p S_{i-1}(U; G) + S_{i-1}(\Sigma_U; G) \right) \to I^p S'_i(X; G)$$

induced by the inclusion $S_i(U; G) \to S_i(X; G)$ (the first group in the displayed equation is a subgroup of $S_i(U; G)$ and the second is the quotient of a subgroup of $S_i(X; G)$ containing the image of the first group). The kernel consists of all elements of $(A^p S_i(U; G) + S_i(\Sigma_U; G)) \cap \partial^{-1} (A^p S_{i-1}(U; G) + S_{i-1}(\Sigma_U; G))$ whose image in $S_i(X; G)$ is contained in $S_i(\Sigma_X; G)$, but this says that the kernel is precisely $S_i(\Sigma_U; G)$; note that $\Sigma_U = \Sigma_X \cap U$. Therefore,

$$I^p S'_i(U; G) = \frac{(A^p S_i(U; G) + S_i(\Sigma_U; G)) \cap \partial^{-1} (A^p S_{i-1}(U; G) + S_{i-1}(\Sigma_U; G))}{S_i(\Sigma_U; G)}$$

is isomorphic to the image of $j$ and so is a subgroup of $I^p S'_i(X; G)$. We let $I^p S'_*(X, U; G)$ be the quotient $I^p S'_i(X; G)/I^p S'_i(U; G)$. Some elementary abstract algebra employing the second and third isomorphism theorems reveals that in fact DOUBLE CHECK

$$I^p S'_i(X; G)/I^p S'_i(U; G) \cong \frac{(A^p S_i(X; G) + S_i(\Sigma_X; G)) \cap \partial^{-1} (A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))}{S_i(\Sigma_X; G) + (A^p S_i(U; G) + S_i(\Sigma_U; G)) \cap \partial^{-1} (A^p S_{i-1}(U; G) + S_{i-1}(\Sigma_U; G))} \cong \frac{(A^p S_i(X; G) + S_i(\Sigma_X; G)) \cap \partial^{-1} (A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))}{(A^p S_i(U; G) + S_i(\Sigma_U; G)) \cap \partial^{-1} (A^p S_{i-1}(U; G) + S_{i-1}(\Sigma_X; G))}.$$

By the Five Lemma and Lemma 6.5, $I^p S'_*(X, U; G) \cong I^p S_*(X, U; G)$.

Now, returning to Proposition P: subdivision00, suppose that $\xi$ is an allowable chain in $I^p S_*(X, U; G)$, which we can identify with the element of $I^p S'_*(X, U; G)$ also represented by $\xi$, so that if $\xi'$ is a singular subdivision of $\xi$, now $\xi'$ makes sense representing an element of the quotient group $I^p S'_*(X; G)$. Just as for $I^p S'_G(X, U; G)$, any subdivision of an allowable simplex is allowable, and it follows that if $\xi'$ is a singular subdivision of $\xi$, then $\xi' \in I^p S'_*(X, U; G)$, using that any simplex in a subdivision of an allowable chain in $\partial \Sigma$ will similarly be allowable and that any subdivision of an allowable singular chain of $\partial \Sigma$ lying in $\Sigma_X$ (or $U$) will also lie in $\Sigma_X$ (or $U$). In other words, subdivisions of chains in $I^p S'_*(X, U; G)$ are still well-defined in $I^p S'_*(X, U; G)$, and if we wanted to, we could use the isomorphisms of Lemma 6.5 to properly define subdivision in $I^p S'_*(X, U; G)$, though we will not need this.

Recall that the prism constructed in the proof of Proposition 6.18 preserves supports in the sense that $\partial D = \xi' - \xi + E$, where $E$ is supported in $|\partial \xi|$. Furthermore, the portion of $E$ “over” the portion of $\partial \xi$ contained in $U$ is contained in $U$, and the portion of $E$ “over” the portion of $\partial \xi$ contained in $\Sigma_X$ is contained in $\Sigma_X$. So, in particular, suppose $\partial \xi = x + y$, where $y$ is contained in $\Sigma_X$ and $x$ consists of allowable chains in $U$; since we assume $\xi$ is a relative cycle, $\partial \xi$ must have this form. Then in fact $\partial D = \xi' - \xi + E_y$, where $E_y$ lives over $y$, and so is contained in $\Sigma_X$ and $E_x$ is a prism over $x$ from $x$ to its subdivision $x'$. Since $x$ consists of allowable simplices, so does $E_x$ by the arguments in the proof of Proposition 6.18 and

$$\partial E_x = \partial D - \partial \xi' + \partial \xi - \partial E_y = -x' - y' + x + y - \partial E_y.$$
Since $E_y$ is contained in $\Sigma_X$, so is $\partial E_y$, as well as $y$ and $y''$. And since $x$ consists of allowable simplices in $U$, so does $x'$. Therefore, $E = E_x + E_y \in (A^pS_i(U; G) + S_i(\Sigma_X; G)) \cap \partial^{-1}(A^pS_{i-1}(U; G) + S_{i-1}(\Sigma_X; G))$, and so $E$ represents 0 in $I^pS'_i(X, U; G)$. Thus $\xi = \xi'$ in $I^pH_i(X, U; G)$.

Using Proposition 6.18, excision and the Mayer-Vietoris sequence follow almost exactly as in the proofs of Theorems 4.40 and 4.41, above. In some sense, the proofs are simpler as any boundary simplex lying in $\Sigma_X$ vanishes and so doesn’t need to be checked to verify any further properties.

**Theorem 6.19.** Let $X$ be a filtered stratified space, and suppose $K \subset U \subset X$ such that $\overline{K} \subset \overline{U}$. Then inclusion induces an isomorphism $I^pH_*(X - K, U - K; G) \cong I^pH_*(X, U; G)$. The equivalent results holds in the PL context.

**Theorem 6.20.** Suppose $X = U \cup V$, where $U, V$ are subspaces such that $X = \overline{U} \cup \overline{V}$. Then there is an exact Mayer-Vietoris sequence

$$\to I^pH_i(U \cap V; G) \to I^pH_i(U; G) \oplus I^pH_i(V; G) \to I^pH_i(U \cup V; G) \to I^pH_{i-1}(U \cap V; G) \to .$$

The equivalent results holds in the PL context. There is also a relative Mayer-Vietoris sequence analogous to that stated in Theorem 4.46.

Applying the Mayer-Vietoris theorem, we obtain the suspension formula for $I^pH_*$:

**Theorem 6.21.** If $X$ is an $n - 1$ dimensional compact filtered space, and $\bar{p}$ is a perversity on $SX$ that takes the same value $p$ on the two suspensions points, then

$$I^pH_i(SX; G) = \begin{cases} 
I^pH_{i-1}(X; G), & i > n - p - 1, \\
0, & i = n - p - 1, \\
I^pH_i(X; G), & i < n - p - 1,
\end{cases}$$

**Proof.** We can write $SX$ as the union of two cones $cX$. The intersection is homeomorphic to $(-1, 1) \times X$, which is stratified homotopy equivalent to $X$. By Theorem 6.10, the inclusion $X \hookrightarrow cX$ induces an intersection homology isomorphism for $i < n - p - 1$. In the Mayer-Vietoris sequence, this becomes a diagonal injection $I^pH_i(X; G) \hookrightarrow I^pH_i(cX; G) \oplus I^pH_i(cX; G) \cong I^pH_i(X; G) \oplus I^pH_i(X; G)$, and so in this range $I^pH_i(SX; G) \cong I^pH_i(X; G)$. For $i \geq n - p - 1$, $I^pH_i(cX; G) = 0$, so we get $I^pH_i(SX; G) \cong I^pH_{i-1}(X; G)$ for $i > n - p - 1$, and $I^pH_{n-p-1}(SX; G) = 0$.

**Remark 6.22.** Notice that this formula is in fact a little cleaner than that of Theorem 4.43 since no reduced homology groups come in.

The following version of Lemma 5.6 has an identical proof to that lemma and will be useful for Mayer-Vietoris arguments.

**Lemma 6.23.** If $X$ is a filtered space with perversity $\bar{p}$ and $\{U_\alpha\}$ is an increasing collection of open subspaces of $X$ then the natural map $f : \lim_{\alpha} I^pH_*(U_\alpha) \to I^pH_*(\cup_\alpha U_\alpha)$ is an isomorphism.
Next we turn to cross products and a version of Theorem \[5.34\]. If \( R \) is a commutative ring with unity and \( M \) is an unfiltered manifold, then since \( I^\bar{p} S_*(X; R) \subset S_*(X; R) \), the classical cross product \( S_*(M; R) \otimes_R S_*(X; R) \to S_*(M \times X; R) \) restricts to a cross product

\[
S_*(M; R) \otimes_R I^\bar{p} S_*(X; R) \to I^\bar{p} S_*(M \times X; R),
\]

where we assume here that \( \bar{p} \) is defined on \( M \times X \) with its product filtration and \( I^\bar{p} S_*(X; R) \) is defined with respect to the inherited filtration and perversity. Furthermore, it continues to be a chain map: Suppose \( x \in S_i(M; R) \) and \( y \in I^\bar{p} S_*(X; R) \); in particular, this implies that no simplex of \( y \) is contained in \( \Sigma_X \). Suppose \( \partial y = \hat{\partial} y + \eta \), where \( \eta \) is contained in \( \Sigma_X \). Then since the cross product is a chain map on ordinary chains, we have

\[
\partial(x \times y) = (\partial x) \times y + (-1)^i x \times (\partial y) = (\partial x) \times y + (-1)^i x \times (\hat{\partial} y + \eta) = (\partial x) \times y + (-1)^i x \times (\hat{\partial} y) + (-1)^i x \times \eta
\]

Since \( \eta \) is supported in \( \Sigma_X \), \( x \times \eta \) is supported in \( M \times \Sigma_X = \Sigma_{M \times X} \); since no simplex of \( y \) or \( \hat{\partial} y \) is contained in \( \Sigma_X \), no simplex of \( \partial x \times y \) or \( x \times (\hat{\partial} y) \) will be contained in \( \Sigma_{M \times X} \). The latter fact can be seen by observing that, in the singular subdivisions of \( \Delta^i \times \Delta^j \) by shuffles used in the construction of the cross product, each \( i + j \) simplex of the triangulation of \( \Delta^i \times \Delta^j \) projects onto both \( \Delta^i \) and \( \Delta^j \) by the standard projections; this is an immediate consequence of the definition of the shuffle product in Section \[5.2\]. Therefore, \( \hat{\partial}(x \times y) = (\partial x) \times y + (-1)^i x \times (\hat{\partial} y) \), showing that the cross product is indeed a chain map \( S_*(M; R) \otimes_R I^\bar{p} S_*(X; R) \to I^\bar{p} S_*(M \times X; R) \).

We also observe that if \( R \) is a Dedekind domain, then \( I^\bar{p} S_*(X; R) \) is a projective \( R \)-module as a submodule of the free module \( S_*(X; R) \), and \( I^\bar{p} \mathcal{C}_*(X; R) \) is a flat module as a submodule of the torsion free module \( \mathcal{C}_*(X; R) \); see [63, Proposition 4.20]. In this case, the properties of the singular cross product established in Section \[5.2.1\] hold for \( IH_* \). We state this as a theorem.

**Theorem 6.24.** If \( R \) is a Dedekind domain, then the properties of the cross product established in Section \[5.2.1\], including naturality, associativity, commutativity, unitality, and stability hold for non-GM singular intersection homology with coefficients in \( R \).

**Theorem 6.25.** Suppose \( X \) is a filtered space with perversity \( \bar{p}_X \) and that \( M \) is an \( n \)-dimensional manifold with its trivial filtration. Filter \( M \times X \) with product filtration so that \((M \times X)^i = M \times X^i\), and define a perversity on \( M \times X \) whose value on \( M \times S \) is \( \bar{p}(S) \). Let \( R \) be a Dedekind domain. Then the cross product induces an isomorphism \( H_*(S_*(M; R) \otimes_R I^\bar{p}_X S_*(X; R)) \cong I^\bar{p} H_*(M \times X; R) \).

The relative version of this theorem analogous to Corollary \[5.29\] also holds.

Just as for Theorems \[5.28\] and \[5.34\], this is an application of the Mayer-Vietoris argument Theorem \[5.1\] with \( F_*(M) = H_*(S_*(M; R) \otimes_R I^\bar{p} S_*(X; R)) \), \( G_*(M) = I^\bar{p} H_*(M \times X; R) \), and natural transformation \( F_* \to G_* \) induced by the cross-product. The relative version follows just as in Corollary \[5.29\].

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Definition 6.26. A CS set $X$ is called \textit{locally $(\bar{p}, \mathbb{Z}; G)$-torsion free} if for each point $x \in X$ and for each distinguished neighborhood $\cong \mathbb{R}^k \times cL$ of $x$, $I^pH_{\dim(L) - \bar{p}(S) - 1}(L) \ast G = 0$, where $S$ is the stratum of $X$ containing $x$.

More specifically, if the condition holds for points in a stratum $S \subset X$, we say that $X$ is \textit{locally $(\bar{p}, \mathbb{Z}; G)$-torsion free along $S$}.

More generally, if $R$ is a Dedekind domain and $M$ is an $R$-module, we say that a CS set $X$ is \textit{locally $(\bar{p}, R; M)$-torsion free} if for each point $x \in X$ and for each distinguished neighborhood $\cong \mathbb{R}^k \times cL$ of $x$, $I^pH_{\dim(L) - \bar{p}(S) - 1}(L; R) \ast_R M = 0$, where $S$ is the stratum of $X$ containing $x$.

If $I^pH_{\dim(L) - \bar{p}(S) - 1}(L; R) \ast_R M = 0$ for all $M$, we simply say that $X$ is \textit{locally $(\bar{p}, R)$-torsion free}. In particular, this means that $I^pH_{\dim(L) - \bar{p}(S) - 1}(L; R)$ is flat as an $R$-module by [64, Theorem XVI.3.11] and so, equivalently, torsion free as an $R$-module by [63, Proposition 4.20].

As for the GM locally torsion free conditions, this condition does not depend on the choice of distinguished neighborhood by the following lemma.

Lemma 6.27. Let $X$ be a CS set and $x \in X_k \subset X$. For $i = 1, 2$, let $N_i \cong \mathbb{R}^k \times cL_i$ be distinguished neighborhood of $x$. Then $I^pH_*(L_1; G) \cong I^pH_*(L_2; G)$.

Corollary 6.28. Let $X$ be a CS set. Then the intersection homology $I^pH_*(L; G)$ of a link $L$ of a point $x$ in a stratum of $S$ depends only on $S$. In other words, all links of points in $S$ have isomorphic intersection homology groups.

The proofs are the same as for Lemma 6.27 and Corollary 6.28.

We then have a universal coefficient theorem proven in the same way as Theorem 5.42.

Theorem 6.29. Suppose $X$ is a locally $(\bar{p}, \mathbb{Z}; G)$-torsion free CS set. Then $I^pH_*(X; G) \cong H_*(I^pS_*(X) \otimes G)$.

In fact, the same proof, using the properties of a Dedekind ring $R$ and the fact that each $I^pS_*(X; R)$ is flat provides a more general version of the theorem:

Theorem 6.30. Suppose $X$ is a locally $(\bar{p}, R; M)$-torsion free CS set for a Dedekind domain $R$ and $R$-module $M$. Then $I^pH_*(X; M) \cong H_*(I^pS_*(X; R) \otimes_R M)$.

Corollary 6.31. For any CS set and any field $F$ of characteristic $0$, $I^pH_*(X; F) \cong I^pH_*(X) \otimes_{\mathbb{Z}} F$.

Corollary 6.32. Suppose $X$ is a locally $(\bar{p}, R; M)$-torsion free CS set for a Dedekind domain $R$ and $R$-module $M$ and that $A \subset X$ is also a locally $(\bar{p}, R; M)$-torsion free CS set, in particular if $A$ is an open subset of $X$. Then $I^pH_*(X, A; M) \cong H_*(I^pS_*(X, A; R) \otimes_R M)$.

For the corollary, at the point in the proof of Corollary 5.45 where we used that $I^pS_{GM}^*(X, A)$ is free as a subgroup of the free group $S_i(X, A)$, here we instead use the analogous observation of Remark 6.11 that each $I^pS_i(X, A; R) \subset S_i(X, A; R)$ to conclude that $I^pS_i(X, A; R)$ is projective as a submodule of a free $R$-module.
Suppose that \( \bar{p} \) is a GM perversity on a CS set \( X \). Then \( \bar{p}(S) \leq \bar{t}(S) \) for all singular strata \( S \) so that \( I^\bar{p}H^\text{GM}_*(X;R) = I^\bar{t}H_*(X;R) \) by Proposition 6.37. Furthermore, if \( X \) has no codimension one strata, then \( X \) satisfies the criteria for topological invariance. In this case, Proposition 5.66 applies directly. We restate the result here, stripped of the GM notation.

**Proposition 6.33.** Let \( \bar{p} \) be a GM perversity, and let \( X \) and \( X' \) be CS sets with no codimension one strata and with \( |X| = |X'| \). Then \( X \) is locally \( (\bar{p},R;M) \)-torsion free if and only if \( X' \) is.

All the material of Section 5.5 also carries over to provide an equivalence between PL and singular intersection homology on PL spaces:

**Theorem 6.34.** Let \( X \) be a PL CS set with triangulation \( T \). Then the composition \( I^\bar{p}\mathcal{S}_*(W;G) \xrightarrow{\phi^{-1}} I^\bar{p}\mathcal{S}_T^*(W;G) \xrightarrow{\psi} H_*(I^\bar{p}\mathcal{S}_*(W;G)) \) is an isomorphism for any open set \( W \subset X \). In particular, \( I^\bar{p}\mathcal{S}_*(X;G) \cong I^\bar{p}H_*(X;G) \).

**Corollary 6.35.** Let \( X \) be a PL CS set, and let \( A \) be an open subset. Then \( I^\bar{p}\mathcal{S}_*(X,A;G) \cong I^\bar{p}H_*(X,A;G) \).

**Corollary 6.36.** Let \( X \) be a PL CS set with closed PL subset \( A \) such that \( A \) is itself a PL CS set in its inherited filtration. Then \( I^\bar{p}\mathcal{S}_*(X,A;G) \cong I^\bar{p}H_*(X,A;G) \).

Similarly, the results of Section 5.7 carry over, using the following definition.

**Definition 6.37.** A CS set \( X \) is called *locally \((\bar{p},R;M)\)-finitely generated* if \( R \) is a Noetherian ring, \( M \) is a finitely generated \( R \)-module, and, for each point \( x \in X \), there is a link \( L \) of \( X \) such that \( I^\bar{p}H_i(L;M) \) is finitely generated as an \( R \)-module for each \( i \). When \( M = R \), we will simply say that \( X \) is *locally \((\bar{p},R)\)-finitely generated* if the CS set \( X \) is locally \((\bar{p},R)\)-finitely generated for all \( \bar{p} \), we will simply say that \( X \) is *locally \( R \)-finitely generated*.

Arguments completely analogous to those of Lemma 5.40 (and Lemma 6.27) show that this definition is equivalent to requiring that each \( I^\bar{p}H_i(L;R) \) be finitely generated for *any* link \( L \) in \( X \).

Then we obtain the following proposition, noting that the proof of Proposition 5.66 goes through with only minor modifications; in particular, there is a small, but insignificant for the purposes of the proposition, change to the local intersection homology computation. We also use that the only properties of abelian groups utilized in the proof of Proposition 5.66 are those arising from them being modules over the Noetherian ring \( \mathbb{Z} \).

**Proposition 6.38.** Let \( R \) be a Noetherian ring, and suppose \( X \) is a locally \((\bar{p},R)\)-finitely generated CS set. Suppose \( U \subset W \) are open subsets of \( X \), that \( \bar{U} \subset W \), and that \( \bar{U} \) is compact. Then the image of \( I^\bar{p}H_*(U;R) \) in \( I^\bar{p}H_*(W;R) \) is finitely generated. In particular, if \( X \) is compact, then each \( I^\bar{p}H_i(X;R) \) is finitely generated.

**Corollary 6.39.** If \( R \) is a Noetherian ring and \( X \) is a compact recursive CS set, in particular if \( X \) is a compact stratified pseudomanifold, then \( I^\bar{p}H_i(X;R) \) is finitely generated for all \( i \).
Of course, as in Section 5.7, we could also replace $R$ with a finitely-generated abelian group $G$ and rewrite these results in terms of finitely-generated intersection homology groups.

To close this section, we make an official statement of a property that we have already seen in use.

**Lemma 6.40.** Suppose $X$ is a filtered space, $A \subset X$, and $R$ is a Dedekind domain. Then each $I^p S_i(X; A; R)$ is a projective $R$-module and each $I^p C_i(X; A; R)$ is a flat $R$-module. If $R$ is a field, then $I^p S_i(X; A; R)$ and $I^p C_i(X; A; R)$ are each free.

**Proof.** By construction, $I^p S_i(X; R) \subset S_i(X; R)$, and the inclusion induces a map $I^p S_i(X; A; R) \to S_i(X; A; R)$. If the image of a chain $\xi \in I^p S_i(X; A; R)$ is 0 in $S_i(X; A; R)$, then $\xi$ must be contained in $A$, but this would force $\xi$ to be 0 in $I^p S_i(X; A; R)$. Therefore, $I^p S_i(X; A; R) \subset S_i(X; A; R)$. But $S_i(X; A; R)$ is a free $R$-module, generated by the singular simplices not contained in $A$. Therefore, as $R$ is Dedekind, $I^p S_i(X; A; R)$ is projective.

For $I^p C_i(X; A; R)$, we cannot make the same argument, as $C_i(X; A; R)$ does not appear to be free in general. However, as previously observed, $I^p C_i(X; A; R)$ is $R$-torsion free, and so, as $R$ is Dedekind, $I^p C_i(X; A; R)$ is flat [63, Proposition 4.20].

The claims when $R$ is a field are immediate, as all vector spaces are free modules. □

### 6.3.1 Non-GM intersection homology and dimensional homogeneity

The trade-off for non-GM intersection homology being the theory that will provide a more general Künneth Theorem and Poincaré duality is that it does not “see” lower-dimensional pieces of a space. More precisely, suppose $X$ is a CS set, and let $X^\bullet$ denote the closure of the union of the regular strata of $X$. In other words, $X^\bullet$ is the union of the strata $S$ of $X$ such that there exists a regular stratum $\mathcal{R}$ with $S \leq \mathcal{R}$. We will show that $I^p H_\ast(X; G) \cong I^p H_\ast(X^\bullet; G)$. In fact, more is true: the intersection homology of the closures of the regular strata do not interact. More precisely, we will see that if $\{\mathcal{R}_\alpha\}$ are the regular strata of $X$, then $I^p H_\ast(X; G) \cong \oplus_\alpha I^p H_\ast(\mathcal{R}_\alpha; G)$ with the isomorphism being the sum of the maps induced by inclusion.

**Example 6.41.** Here are two examples illustrating the concept of $X^\bullet$.

1. Let $X = X^2$ be the one-point union (wedge product) of $S^2$ and $S^1$ with the filtration $\{v\} \subset S^1 \subset X$, where $\{v\}$ is the wedge point. Then $X^\bullet \cong S^2$ with filtration $\{v\} \subset S^2$.

2. If $X$ has formal dimension $n$ but every non-empty stratum of $X$ has dimension $< n$, then $X$ has no regular strata and $X^\bullet = \emptyset$. For example, if $X = S^2$ but is given formal dimension 6, then $X^\bullet = \emptyset$.

If $X$ is a PL filtered space, the fact that $I^p S_\ast(X; G) \cong I^p S_\ast(X^\bullet; G)$ is not difficult to see. In fact, consider the inclusion map $I^p C_\ast(X^\bullet; G) \to I^p C_\ast(X; G)$. Since $X^\bullet$ is a closed union of strata of $X$, it is a subcomplex of any admissible triangulation of $X$ as a filtered space. By definition, no simplex (with respect to some triangulation) $\sigma$ of a chain in $I^p C_\ast(X; G)$ can be contained in $\Sigma_X$, so every such simplex must intersect a regular stratum. But since $X^\bullet$ is a subcomplex, this implies that $\sigma$ is contained in $X^\bullet$. Thus the inclusion
\[ I^p\mathcal{C}_s(X^*; G) \rightarrow I^p\mathcal{C}_s(X; G) \] is also onto, so in fact \( I^p\mathcal{C}_s(X^*; G) = I^p\mathcal{C}_s(X; G) \) and thus \( I^p\mathcal{H}_s(X^*; G) = I^p\mathcal{H}_s(X; G) \). In fact, the same argument demonstrates that every allowable simplex must be contained in the closure of some regular stratum \( R_\alpha \), and thus \( I^p\mathcal{C}_s(X; G) = \bigoplus_\alpha I^p\mathcal{C}_s(R_\alpha; G) \) and \( I^p\mathcal{H}_s(X; G) = \bigoplus_\alpha I^p\mathcal{H}_s(R_\alpha; G) \). We state this result as a lemma:

**Lemma 6.42.** If \( X \) is a PL filtered space and \( X^* \) denotes the closure of the union of the regular strata of \( X \), then \( I^p\mathcal{C}_s(X^*; G) = I^p\mathcal{C}_s(X; G) \) and so \( I^p\mathcal{H}_s(X^*; G) = I^p\mathcal{H}_s(X; G) \). Furthermore, if \( \{R_\alpha\} \) is the collection of regular strata of \( X \), then \( I^p\mathcal{C}_s(X; G) = \bigoplus_\alpha I^p\mathcal{C}_s(R_\alpha; G) \) and \( I^p\mathcal{H}_s(X; G) = \bigoplus_\alpha I^p\mathcal{H}_s(R_\alpha; G) \).

**Example 6.43.** Since the space in item (1) of Example 6.41 can be assumed to be PL, we see that \( I^p\mathcal{H}_s(X; G) \cong I^p\mathcal{H}_s(S^2; G) \), where \( S^2 \) is filtered as above.

In both cases of item (2) of Example 6.41, \( I^p\mathcal{H}_s(X; G) = 0 \).

The corresponding conclusion for singular intersection homology is more difficult to achieve. For one thing, a singular simplex that is not contained in \( \Sigma_X \) might nonetheless have its image intersect \( X - X^* \), for example a singular 1-simplex might have both endpoints in regular strata of \( X \) but pass through strata of \( X - X^* \) in between. In order to draw upon local structure arguments, we must limit ourselves to CS sets, which will let us utilize Mayer-Vietoris arguments to get our desired results.

If \( X \) is a CS set (or, even more generally, a manifold stratified space) with \( n \)-dimensional regular strata, and we let \( X^* \) denote the closure of the union of the regular strata of \( X \), then, by Proposition 2.19, \( X^* \) is itself a manifold stratified space whose strata comprise a subset of the strata of \( X \). It turns out that if \( X \) is a CS set, then \( X^* \) is itself a CS set, which we will show below in Lemma 6.45. Furthermore, by definition, every point of \( X^* \) is contained in the closure of an \( n \)-dimensional stratum, but we can show something a bit more technical: if \( X^* \neq \emptyset \), each distinguished neighborhood in \( X^* \) is \( n \)-dimensional (using cohomological dimension as our dimension theory). This can be interpreted as a dimensional homogeneity property, akin to the sense in which an \( n \)-manifold is dimensionally homogeneous because every point has a neighborhood homeomorphic to \( n \)-dimensional Euclidean space. We will demonstrate this property below in Lemma 6.46. First, these lemmas motivate the following definition:

**Definition 6.44.** If \( X \) is a CS set, let \( X^* \) denote the CS set that is the closure of the union of the regular strata of \( X \). We call \( X^* \) the homogenization of \( X \). If \( X^* = X \), we say that \( X \) is dimensionally homogeneous.

After stating and proving Lemmas 6.45 and 6.46, we will show in Proposition 6.47 that if \( X \) is a CS set with homogenization \( X^* \), then \( I^pH_s(X; G) \cong I^pH_s(X^*; G) \).

**Lemma 6.45.** If \( X \) is a CS set and \( \mathcal{R} \) is any union of regular strata of \( X \), then the closure \( \overline{\mathcal{R}} \) is a CS set. In particular, the homogenization \( X^* \) is a CS set. If \( X \) is a stratified pseudomanifold, then so is \( \mathcal{R} \).

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Proof. The second statement follows from the first letting \( \mathcal{R} \) be the union of all the regular strata of \( X \).

Since \( \mathcal{R} \) is a union of strata of \( X \) by the Frontier Condition for stratified space, \( \mathcal{R} \) is automatically manifold stratified. So we must check that \( \mathcal{R} \) is locally cone-like: Suppose \( x \in \mathcal{R} \) has regular neighborhood \( N \) in \( X \) stratified homeomorphic to \( \mathbb{R}^i \times cL \) by a stratified homeomorphism \( h : N \to \mathbb{R}^i \times cL \). Since \( \mathcal{R} \) is a union of strata of \( X \), \( N \cap \mathcal{R} \) is a union of strata of \( N \) and \( h(N \cap \mathcal{R}) \) must be a union of strata of \( \mathbb{R}^i \times cL \). Since \( x \in \mathcal{R} \), the entire stratum of \( N \) containing \( x \) must be in \( \mathcal{R} \), so \( h(N \cap \mathcal{R}) \) must contain \( \mathbb{R}^i \times \{v\} \); it now follows from the structure of the stratification of \( \mathbb{R}^i \times cL \) that \( h(N \cap \mathcal{R}) \) must have the form \( \mathbb{R}^i \times cL \), where \( \hat{L} \) is a union of strata of \( L \). If we identify \( L \) with \( \{0\} \times \{1/2\} \times L \subset \mathbb{R}^i \times cL \), then \( \hat{L} \) is the intersection of \( L \) with \( h(N \cap \mathcal{R}) \) in \( \mathbb{R}^i \times cL \). Since \( \mathcal{R} \) is closed in \( X \), \( N \cap \mathcal{R} \) is closed in \( N \) and \( h(N \cap \mathcal{R}) \) is closed in \( \mathbb{R}^i \times cL \); therefore, since \( L \) is compact, \( \hat{L} = L \cap h(N \cap \mathcal{R}) \) must also be compact. So \( \mathcal{R} \) is a CS set.

Note that, in general, \( \hat{L} \) is not required to be a CS set, as \( L \) itself is not required to be a CS set. However, suppose now that \( X \) is a stratified pseudomanifold, which implies that \( L \) is a stratified pseudomanifold by Lemma 2.54. Our subspace \( \mathcal{R} \) is certainly the closure of its regular strata, by definition, so we only need to show that the links \( \hat{L} \) are also stratified pseudomanifolds. But a point-set argument shows that \( \mathcal{R} \) is not required to be a CS set, as \( \mathcal{R} \) itself is not required to be a CS set. However, suppose now that \( X \) is a stratified pseudomanifold, which implies that \( L \) is a stratified pseudomanifold by Lemma 2.54. Our subspace \( \mathcal{R} \) is certainly the closure of its regular strata, by definition, so we only need to show that the links \( \hat{L} \) are also stratified pseudomanifolds. But a point-set argument shows that \( \mathcal{R} \) is itself the closure of the union of the regular strata of \( L \) that \( h^{-1} \) takes into \( \mathcal{R} \). So it suffices to show that Lemma 6.45 holds replacing \( X \) with \( L \), which has smaller depth than \( X \). Repeating this argument iteratively, eventually we get to links that have depth 0, i.e. they’re manifolds, and in this case it’s clear that the closure of a union of connected components of a manifold is a manifold.

We now state and prove Lemma 6.46, which shows that if \( X \) is an \( n \)-dimensional CS set and \( X^\bullet \) is non-empty, then every distinguished neighborhood of \( X^\bullet \) has topological dimension \( n \). This justifies our claim that \( X^\bullet \) is dimensionally homogeneous. This lemma is technical and uses some sheaf theory. Except for one later lemma, Lemma 8.8, the argument here will not be needed again and can be safely skipped by those not wishing to think too much about sheaf-theoretic dimension theory.

Lemma 6.46. Suppose \( X \) is an \( n \)-dimensional CS set and that \( \mathcal{R} \) is a non-empty union of regular strata of \( X \). Then every distinguished neighborhood \( N \) in \( \mathcal{R} \) has \( \dim_Z N = n \), where \( \dim_Z E \) is defined as in [13, Definition II.16.6]. In particular, if \( X = \Sigma X \neq \emptyset \), then every distinguished neighborhood \( N \) in \( X^\bullet \) has \( \dim_Z N = n \).

Proof. This argument is a straightforward generalization of that appearing for pseudomani-

folds in [13, Proposition 7.3], which was due to Jim McClure.

We saw in Lemma 2.37 that the stratification of every CS set is locally finite, and so \( N \cong \mathbb{R}^i \times cL \), where \( L \) is a compact filtered space with finitely many strata. We also observe that since any open subset of a CS set is also a CS set, \( U \) is locally compact (every points has a neighborhood homeomorphic to \( D^i \times \bar{c}L \), where \( D^i \) is the closed disk and \( \bar{c}L \) is the closed cone on the compact space \( L \)), as is each skeleton of \( U \), being a closed subset of a locally compact Hausdorff space [13, Corollary 29.3].

Since \( U \subset \mathcal{R} \), \( U \) must intersect a regular stratum of \( X \), and thus \( U \) must contain an open subspace \( M \) homeomorphic to a (non-empty) \( n \)-manifold. By [13, Corollary II.16.28],
dim_\mathcal{Z}(M) = n$, and by [13 Theorem II.16.8], dim_\mathcal{Z}(U) \geq dim_\mathcal{Z}(M) = n (here we use that \( U \) is locally compact, and hence locally paracompact).

Next, we will show that \( \dim_\mathcal{Z}(U) \leq \dim_\mathcal{Z}(M) \leq n \). In fact, we will show that if \( Y \) is any CS set, then the \( i \)-skeleton \( Y^i \) of \( Y \) satisfies \( \dim_\mathcal{Z}(Y^i) \leq i \). As \( U = U^n \), this will suffice. The proof will be by induction on \( i \). In case \( i = 0 \), \( Y^0 \) is a discrete collection of points, and so a 0-manifold, and we can again apply [13 Corollary II.16.28]. We now assume by induction hypothesis that \( \dim_\mathcal{Z}(Y^{i-1}) \leq i - 1 \).

Let \( c \) denote the family of compact supports, and let \( \dim_{c,\mathcal{Z}} \) be as in [13 Definition II.16.3]. Then \( \dim_\mathcal{Z} \) is equal to \( \dim_{c,\mathcal{Z}} \) for any locally compact space by [13 Definition II.16.6], since \( c \) is paracompacting for locally compact spaces (see [13 page 22]) and \( E(c) = Y \) as every point has a compact neighborhood. In particular, by [78 Corollary 29.3], since \( Y^{i-1} \) and \( Y^i \) are closed in \( Y \), they are each locally compact, and since \( Y^i = Y^{i-1} \) is open in \( Y^i \), \( Y^i - Y^{i-1} \) is also locally compact. Thus the equality

\[
\dim_{c,\mathcal{Z}}(Y^i) = \max\{\dim_{c,Y^{i-1},\mathcal{Z}}(Y^{i-1}), \dim_{c,Y^i = Y^{i-1},\mathcal{Z}}(Y^{i-1}, Y^i - Y^{i-1})\}
\]

[13 Exercise II.11], which utilizes that \( c \) is paracompacting on locally-compact spaces, becomes

\[
\dim_\mathcal{Z}(Y^i) = \max\{\dim_\mathcal{Z}(Y^{i-1}), \dim_\mathcal{Z}(Y^i - Y^{i-1})\}.
\]

We have assumed that \( \dim_\mathcal{Z}(Y^{i-1}) \leq i - 1 \) by induction, and, since \( Y^i = Y^{i-1} \) is an \( i \)-manifold, possibly empty, by [13 Corollary II.16.28], \( \dim_\mathcal{Z}(Y^i - Y^{i-1}) \leq i \). In fact, this last inequality will be an equality if \( Y^i - Y^{i-1} \) is non-empty and \( \dim_\mathcal{Z}(Y^i - Y^{i-1}) = -\infty \) if \( Y^i - Y^{i-1} = \emptyset \); technically, \( -\infty \) isn’t allowed as a dimension in [13 Definition II.16.3], but [13 Theorem II.16.4] leaves it as the only reasonable definition when the space is empty. Altogether, this shows that \( \dim_\mathcal{Z}(Y^i) \leq i \), as desired. Thus, we get \( \dim_\mathcal{Z}(U) \leq n \), and so \( \dim_\mathcal{Z}(U) = n \).

We are now ready for the main result of this section:

**Proposition 6.47.** Let \( X \) be a CS set, and let \( \{\mathcal{R}_\alpha\} \) be the regular strata of \( X \). Then the inclusion maps induce isomorphisms

\[
\bigoplus_\alpha \mathcal{I}^*H_\ast(\mathcal{R}_\alpha; G) \rightarrow \mathcal{I}^*H_\ast(X; G)
\]

\[
\downarrow \quad \downarrow
\]

\[
\mathcal{I}^*H_\ast(X^\bullet; G).
\]

The analogous result holds in the PL category.

**Remark 6.48.** As \( \mathcal{I}^*H = \mathcal{I}^*H_{GM}^\ell \) if \( \ell \leq \ell \) by Proposition 6.7, Proposition 6.47 holds for GM intersection homology, as well, under this perversity restriction. However, Proposition 6.47 is not true of GM intersection homology in general. For example, suppose \( X \) has more than one regular stratum and that \( \ell \) is large enough on some stratum to allow a singular 1-simplex to run from one regular stratum to another, say from \( \mathcal{R}_1 \) to \( \mathcal{R}_2 \). Then \( \mathcal{I}^*H_0(\mathcal{R}_1 \cup \mathcal{R}_2) \cong \mathcal{I}^*H_{GM}^0(\mathcal{R}_1) \cong \mathcal{I}^*H_{GM}^0(\mathcal{R}_2) \cong \mathbb{Z} \).

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Proof of Proposition 6.47. The PL claim is an immediate consequence of Lemma 6.42.

For singular intersection homology, we will apply Theorem 5.3 to get the isomorphisms \(\oplus \alpha I^pH_\alpha(R_\alpha; G) \to I^pH_\alpha(X; G)\) and \(I^pH_\alpha(X^\bullet; G) \to I^pH_\alpha(X; G)\). The third isomorphism follows from the clear commutativity of the diagram.

For each open set \(U \subset X\), let \(G_\alpha(U) = I^pH_\alpha(U; G)\), let \(F_\bullet(U) = I^pH_\alpha(U \cap X^\bullet; G)\), and let \(E_\alpha(U) = \oplus \alpha I^pH_\alpha(U \cap R_\alpha; G)\). Let \(\Phi : F_\bullet \to G_\alpha\) be induced by inclusion, and let \(\Psi : E_\alpha \to G_\alpha\) be the sum of the maps induced by inclusion of the \(R_\alpha\). Of course \(G_\bullet\) admits the standard Mayer-Vietoris sequence by Theorem 6.20, and if \(U, V\) are two open sets of \(X\), then \(U \cap X^\bullet\) and \(V \cap X^\bullet\) are two open subsets of \(X^\bullet\) with \((U \cap X^\bullet) \cap (V \cap X^\bullet) = (U \cap V) \cap X^\bullet\) and \((U \cap X^\bullet) \cup (V \cap X^\bullet) = (U \cap V) \cup X^\bullet\), so \(F_\bullet\) admits a Mayer-Vietoris sequence coming from the intersection homology on \(X^\bullet\). Compatibility of the short exact Mayer-Vietoris sequences at the chain level shows that \(\Phi\) induces a map of Mayer-Vietoris sequences. For \(E_\alpha\), we have the direct sum of the Mayer-Vietoris sequences similarly associated to \(U \cap R_\alpha\) and \(V \cap R_\alpha\), and the direct sum of similarly associated maps to the Mayer-Vietoris sequence for \(G_\bullet\). This demonstrates condition (1) of Theorem 5.3.

Condition (2) follows from Lemma 5.5 using Lemma 6.23, applied on \(X\) for \(G_\bullet\) and on \(X^\bullet\) for \(F_\bullet\), noting that an increasing sequence of open subsets \(\{U_\beta\}\) in \(X\) yields and increasing sequence of open subsets \(\{U_\beta \cap X^\bullet\}\) in \(X^\bullet\). Once again, the analogous argument is true for each summand of \(E_\alpha\), and so for the direct sum.

Next we look at condition (3) of Theorem 5.3. If \(U\) is empty, then \(E_\alpha(U) = F_\bullet(U) = G_\alpha(U) = 0\), trivially. If \(U\) is an open subset contained within a regular stratum of \(X\), then \(U\) is contained in a single \(R_\alpha\), say \(R_0\), and \(U \cap R_0 = U \cap X^\bullet = U\), so \(\Phi\) is certainly an isomorphism on such sets, as is \(\Psi\) because all summands will be 0 except for \(I^pH_\alpha(U \cap R_0; G) = I^pH_\alpha(U; G)\). If \(U\) is an open subset of \(X\) contained in any one singular stratum \(S\), then we must have \(S \cap R_\alpha = S \cap X^\bullet = 0\) for all \(\alpha\), for otherwise \(S\) would have to be a stratum of some \(R_\alpha\) (since we have seen that each \(R_\alpha\), and so also \(X^\bullet\), is a union of strata of \(X\)), in which case every neighborhood of every point of \(S\) would have to intersect some regular stratum of \(X\), meaning that \(U\) could not be open. So, in this case, \(U \cap R_\alpha = U \cap X^\bullet = 0\) for all \(\alpha\) and \(E_\alpha(U) = F_\bullet(U) = 0\). But similar, since \(U \subset \Sigma_X\) by assumption, \(I^pS_\alpha(U; G) = 0\), as simplices of \(I^pS_\alpha(X; G) = 0\) cannot be contained in \(\Sigma_X\) by definition, and therefore \(G_\alpha(U) = I^pH_\alpha(U; G) = 0\). Thus \(\Phi\) and \(\Psi\) must again be isomorphisms in this case, trivially.

Finally, consider condition (4). Suppose we have an open subset \(U\) of \(X\) stratified homeomorphic to \(\mathbb{R}^i \times cL\) and suppose that \(\Phi : F_\bullet(\mathbb{R}^i \times (cL - \{v\})) \to G_\alpha(\mathbb{R}^i \times (cL - \{v\}))\) and \(\Psi : E_\alpha(\mathbb{R}^i \times (cL - \{v\})) \to G_\alpha(\mathbb{R}^i \times (cL - \{v\}))\) are isomorphisms (where \(v\) is the cone vertex). By definition, such an open set \(U\) is a distinguished neighborhood of all points of \(U\) contained in the image of \(\mathbb{R}^i \times \{v\}\). Let \(x\) be such a point in \(U\), and first suppose \(x \notin X^\bullet\), so also \(x \notin R_\alpha\) for all \(\alpha\). Then the stratum containing \(x\) is not contained in \(X^\bullet\) (or any \(R_\alpha\)), and, by the partial ordering on the strata of \(X\) and the fact that \(X^\bullet\) (or \(R_\alpha\)) is closed, it follows that none of the strata that intersect \(U\) can be contained in \(X^\bullet\) (or any \(R_\alpha\)), and therefore they do not intersect \(X^\bullet\) (or \(R_\alpha\)). This means that \(E_\alpha(U) = F_\bullet(U) = 0\), as \(U \cap X^\bullet = U \cap R_\alpha = 0\), but also \(G_\alpha(U) = 0\), as \(U \cap X^\bullet = 0\) implies that \(U \subset \Sigma_X\). Thus \(\Phi\) and \(\Psi\) are trivially isomorphisms on \(U\).

Next, suppose that \(x \in X^\bullet\), so the stratum of \(U\) containing the homeomorphic image of
inherited filtrations is the same as the codimension of the cone vertex in all strata inherit their dimensions from the top maps are isomorphisms by assumption. Now, by the arguments in the proof of Lemma \textup{6.45} with the given assumptions \((\mathbb{R}^i \times cL) \cap X^\bullet\) has the form \(\mathbb{R}^i \times \hat{L}\), where \(\hat{L}\) is, roughly speaking, the intersection of \(L\) with \(X^\bullet\). Similarly, each \(\mathcal{R}_\alpha\) intersects \(\mathbb{R}^i \times cL\) as some \(\mathbb{R}^i \times cL_\alpha\). Note that \(\hat{L}_\alpha\) will be empty for those \(\mathcal{R}_\alpha\) whose closures do not contain \(x\); in this case \(cL_\alpha = \{v\}\).

Therefore, employing stratified homotopy invariance, the diagram is isomorphic to the diagram

\[
\oplus \alpha I^p H_*(\mathbb{R}^i \times \{v\}); G) \xrightarrow{\cong} I^p H_*(cL - \{v\}); G) \xrightarrow{\cong} I^p H_*(cL - \{v\}; G)
\]

But now we can employ the cone formula, Theorem \textup{4.12}. Recall that the cut-off dimension of the cone formula depends only on \(\bar{p}(\{v\})\) and the codimension of the cone vertex. Since all strata inherit their dimensions from \(X\), the codimension of the cone vertex in \(cL\) with the inherited filtrations is the same as the codimension of the cone vertex in \(cL\), or in any \(cL_\alpha\), with the inherited filtration. In particular, since the stratum containing \(x\) has dimension \(i\) (by our assumption that the distinguished neighborhood of \(x\) in \(X\) has the form \(\mathbb{R}^i \times cL\), the cone vertex inherits codimension \(n - i\) in \(cL\), \(c\hat{L}\), and each \(cL_\alpha\) which is also consistent with treating \(L\) and \(\hat{L}\) as stand-alone spaces of formal dimension \(n - i - 1\) from which we construct the \(n\)-dimensional spaces \(\mathbb{R}^i \times cL\) and \(\mathbb{R}^i \times c\hat{L}\); cf. Remark \textup{4.19}. Regardless, the cone formula now says that in degrees \(n - i - 1 - \bar{p}(\{v\})\) (using the inherited perversity), \(I^p H_*(cL; G)\), \(I^p H_*(c\hat{L}; G)\), and all of the \(I^p H_*(cL_\alpha; G)\) vanish, so they are trivially isomorphic. We also have trivial isomorphisms \(I^p H_*(c\hat{L} - \{v\}; G) \rightarrow I^p H_*(c\hat{L}_\alpha; G)\) in all degrees if \(\hat{L}_\alpha = \emptyset\); in this case, \(c\hat{L}_\alpha - \{v\} = \emptyset\) and \(cL_\alpha = \{v\}\), but \(I^p H_*(\{v\}; G) = 0\) due to \(v \in \Sigma_X\). Together with the other part of the cone formula and stratified homotopy invariance, we therefore have that all vertical maps are isomorphisms in degrees \(< n - i - 1 - \bar{p}(\{v\})\). Thus again the bottom maps of the diagram are isomorphisms. Therefore, \(\Phi : F_*(U) \rightarrow G_*(U)\) and \(\Psi : E_*(U) \rightarrow G_*(U)\) are isomorphisms in all degrees, and this establishes condition (3) of Theorem \textup{5.3}.

\footnote{\textup{In fact, we can give an even better reason for this statement by showing that the formal dimensions of \(L\) and \(\hat{L}\) must agree, thinking of them as stand-alone spaces out of which we construct respective distinguished}}
We have now verified all conditions of Theorem 5.3, which we can now invoke to conclude that \( \Phi \) and \( \Psi \) are isomorphisms on \( X \).

**Corollary 6.49.** Suppose \( X \) is a CS set with regular strata \( \{ R_\alpha \} \), and let \( x \in \overset{\circ}{X} \). Let \( L, L_\alpha \), and \( L \) be respective links of \( x \) in \( X, R_\alpha \), and \( X \). Furthermore, if \( h : N \to \mathbb{R}^i \times cL \) is a distinguished neighborhood of \( X \), let \( \{ T_\beta \} \) be the intersections of \( L \) with the image under \( h \) of the regular strata \( \{ R_\beta \} \) of \( N \). Then

\[
I^H_*(L; G) \cong I^H_*(\overset{\circ}{L}; G) \cong \bigoplus \beta I^H_*(\overset{\circ}{T}_\beta; G) \cong \bigoplus \alpha I^H_*(L_\alpha; G).
\]

**Proof.** Suppose \( h : N \to \mathbb{R}^i \times cL \) gives a distinguished neighborhood of \( x \) and that \( \overset{\circ}{L} \) and the \( L_\alpha \) are as in Proposition 6.47. Notice that, as a regular stratum \( R_\alpha \) of \( X \) might intersect \( N \) in multiple disjoint regular strata, a single \( L_\alpha \) may contain multiple of the \( T_\beta \). The \( T_\beta \) are the regular strata of \( L \), though as \( X \) is only a CS set, we cannot assume, for example, that they are manifolds. However, a point-set topology exercise does show that \( \mathbb{R}^i \times cT_\beta \cong R_\beta \).

The basic idea for proving this corollary is to observe that all the spaces involved in the claim are stratified homeomorphism equivalent to spaces for which we can apply the preceding proposition.

Using stratified homotopy equivalence, \( I^H_*(L; G) \cong I^H_*(N - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \), where \( v \) is the vertex of \( cL \). By the arguments of Lemma 6.45, \( x \) has a distinguished neighborhood \( \overset{\circ}{N} \) \( \overset{\circ}{X} \) of the form \( h(\mathbb{R}^i \times \overset{\circ}{L}) \), where \( \overset{\circ}{L} \) is a union of strata of \( L \). So we also have \( I^H_*(\overset{\circ}{L}; G) \cong I^H_*(\overset{\circ}{N} - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \). Similarly, \( x \) has a distinguished neighborhood \( \overset{\circ}{N}_\alpha \) in \( R_\alpha \) of the form \( h(\mathbb{R}^i \times c\overset{\circ}{L}_\alpha) \) and \( I^H_*(\overset{\circ}{L}_\alpha; G) \cong I^H_*(\overset{\circ}{N}_\alpha - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \). Finally, the local cone structure also implies that \( \mathbb{R}^i \times cT_\beta \) is stratified homeomorphic to the closure of the single regular stratum \( R_\beta \) of \( N \), which is a CS set by Lemma 6.45, and so \( I^H_*(T_\beta; G) \cong I^H_*(\overset{\circ}{R}_\beta - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \).

We now claim that \( (N - h^{-1}(\mathbb{R}^i \times \{ v \}))^\circ = \overset{\circ}{N} - h^{-1}(\mathbb{R}^i \times \{ v \}) \) and that similarly the closures of the \( R_\alpha \) and \( R_\beta \) in \( N - h^{-1}(\mathbb{R}^i \times \{ v \}) \) correspond precisely to \( \overset{\circ}{N}_\alpha - h^{-1}(\mathbb{R}^i \times \{ v \}) \) and \( \overset{\circ}{R}_\beta - h^{-1}(\mathbb{R}^i \times \{ v \}) \). Assuming these claims, then, by Proposition 6.47 and stratified homotopy invariance, we have

\[
I^H_*(L; G) \cong I^H_*(N - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \cong I^H_*(\overset{\circ}{N} - h^{-1}(\mathbb{R}^i \times \{ v \}); G) \cong I^H_*(\overset{\circ}{L}; G)
\]

neighborhoods of \( X \) and \( \overset{\circ}{X} \); see Remark 4.19.

Suppose \( X \) has formal dimension \( n \). We have assumed that \( x \in \overset{\circ}{X} \), which implies, in particular, that \( \overset{\circ}{X} \neq \emptyset \). Therefore, \( X \) has regular strata, which must be \( n \)-manifolds, and \( x \) is contained in the closure of at least one such regular stratum. The local cone structure then implies that \( L \) must have a stratum, say \( S \) of (formal) dimension \( n - i - 1 \) such that \( \mathbb{R}^i \times (cS - \{ v \}) \) is an \( n \)-dimensional stratum of \( \mathbb{R}^i \times cL \) (in fact \( \mathbb{R}^i \times (cS - \{ v \}) \cong \mathbb{R}^{i+1} \times S \) must be an \( n \)-manifold); again, here we are thinking of \( L \) as a stand-alone space of formal dimension \( n - i - 1 \) from which we construct \( \mathbb{R}^i \times cL \), not as a subspace of \( X \) (see Remark 4.19). Even though \( \mathbb{R}^{i+1} \times S \) is an \( n \)-manifold, we still consider \( S \) as only having formal dimension \( n - i - 1 \) as strange things we do not wish to think about can happen in dimension theory; see [69] Section III.4. But then certainly \( \mathbb{R}^{i+1} \times S \) must also be contained in \( \overset{\circ}{X} \), so we have \( S \subset \overset{\circ}{L} \). Therefore, \( L \) and \( \overset{\circ}{L} \), both have non-empty (formal) \( n - i - 1 \)-dimensional strata, but they cannot have any strata of higher formal dimension or else \( \mathbb{R}^i \times cL \) (and hence \( X \)) could not have formal dimension \( \leq n \). Therefore, \( L \) and \( \overset{\circ}{L} \) must both be filtered spaces of formal dimension \( n - i - 1 \) with regular strata, though these strata need not be manifolds.
and also
\[ I^p H_*(\hat{N} - h^{-1}(\mathbb{R}^i \times \{v\}); G) \cong \oplus_\beta I^p H_*(R_\beta - h^{-1}(\mathbb{R}^i \times \{v\}); G) \cong \oplus_\beta I^p H_*(T_\beta; G). \]

For the part of the statement of the corollary involving \( \oplus_\alpha I^p H_*(\hat{L}_\alpha; G) \), we note that if \( J_\alpha \) is the collection of \( \beta \) with \( R_\beta \subset R_\alpha \), then the CS set \( \hat{N}_\alpha - h^{-1}(\mathbb{R}^i \times \{v\}) \) is the union of the closures in \( N - h^{-1}(\mathbb{R}^i \times \{v\}) \) of its regular strata \( R_\beta \), and so, again invoking Proposition 6.47 and stratified homotopy invariance, we have \( I^p H_*(\hat{L}_\alpha; G) \cong \oplus_{\beta \in J_\alpha} I^p H_*(T_\beta; G) \). So the corollary follows by summing over the \( \alpha \).

It remains to verify the topological claims. We give the argument for \( \hat{N} \), the others being similar. Lemma 6.45 shows that \( \hat{N} = N \cap X^\bullet \). But then every point in \( \hat{N} \) is in the closure of a regular stratum of \( X \), and it must be that every point in \( \hat{N} - h^{-1}(\mathbb{R}^i \times \{v\}) \) is in the closure of a regular stratum of \( N - h^{-1}(\mathbb{R}^i \times \{v\}) \). So \( \hat{N} - h^{-1}(\mathbb{R}^i \times \{v\}) \subset (N - h^{-1}(\mathbb{R}^i \times \{v\}))^\bullet \). Furthermore, if \( x \in N - h^{-1}(\mathbb{R}^i \times \{v\}) \) is in the closure of the regular strata of \( N - h^{-1}(\mathbb{R}^i \times \{v\}) \) (which are the intersections of the regular strata of \( X \) with \( N \)), then \( x \) is in the closure of the regular strata of \( X \), so \( x \in N \cap X^\bullet = \hat{N} \). In other words, \( (N - h^{-1}(\mathbb{R}^i \times \{v\}))^\bullet = \hat{N} - h^{-1}(\mathbb{R}^i \times \{v\}) \).

To conclude this section, we provide one more lemma, which will be useful in the next section.

**Lemma 6.50.** If \( X \) and \( Y \) are CS sets, then the product of the homogenizations \( X^\bullet \times Y^\bullet \) is the homogenization \( (X \times Y)^\bullet \) of \( X \times Y \).

**Proof.** From the definition of the product stratification, the regular strata of \( X \times Y \) are the products of the regular strata of \( X \) and \( Y \). If either \( X \) or \( Y \) has no regular strata, then the corresponding homogenization is empty and so is the homogenization of \( X \times Y \). So assume that all of the spaces have regular strata. If \( x \in X^\bullet \) and \( y \in Y^\bullet \), then every neighborhood \( U \) of \( x \) in \( X \) and every neighborhood \( V \) of \( y \) in \( Y \) intersect regular strata, so the product neighborhood \( U \times V \) intersects a regular stratum of \( X \times Y \). Since such neighborhoods are cofinal among neighborhoods of \( x \times y \), the point \( x \times y \) must be in \( (X \times Y)^\bullet \).

Conversely, if \( x \times y \in (X \times Y)^\bullet \), then every neighborhood of \( x \times y \) intersects some regular strata of \( X \times Y \), so in particular every neighborhood of the form \( U \times V \) has this property, where \( U \) is a neighborhood of \( x \) in \( X \) and \( V \) is a neighborhood of \( y \) in \( Y \). But this means that if \( u \times v \subset U \times V \) is contained in a regular stratum, so are \( u \in X \) and \( v \in V \). Therefore, \( x \) and \( y \) are contained in the closures of regular strata of \( x \) and \( y \), so \( x \times y \in X^\bullet \times Y^\bullet \). \( \square \)

### 6.4 A general Künneth theorem

We next turn toward establishing a more general Künneth theorem that is not limited by the assumption that one factor of the product space must be a trivially-filtered manifold. This will be possible using the non-GM intersection homology groups. Throughout this section we let \( R \) be a Dedekind domain, which includes the possibility that \( R \) is a PID or a field. Recall that this assumption allows us to conclude that any submodule of the torsion-free \( R \)-modules \( I^p S_*(X; R) \) and \( I^p C_*(X; R) \), including the modules themselves, is flat.
If $X, Y$ are filtered spaces, we filter $X \times Y$ as $(X \times Y)^k = \bigcup_{i+j=k} X^i \times Y^j$. Then the strata of $X \times Y$ have the form $S \times T$, where $S \subset X$ and $T \subset Y$ are strata of $X$ and $Y$, because

$$(X \times Y)^k - (X \times Y)^{k-1} = \bigcup_{i+j=k} X^i \times Y^j - \bigcup_{a+b=k-1} X^a \times Y^b$$

$$= \bigcup_{i+j=k} (X^i \times Y^j - ((X^i \times Y^{j-1}) \cup (X^{i-1} \times Y^j)))$$

$$= \bigcup_{i+j=k} (X^i - X^{i-1}) \times (Y^j - Y^{j-1}).$$

Early Künneth theorems focused primarily on attempts to determine for which GM perversities $\bar{p}$ it is true that

$$I^p H_*(X \times Y; R) \cong H_*(I^p S_*(X; R) \otimes_R I^p S_*(Y; R)).$$

Recall that GM perversities are functions of codimension alone, so the formula makes sense. In [43], Goresky and MacPherson provide a sheaf-theoretic proof, based on the work of Cheeger [19] on $L^2$-cohomology, that such a formula holds for $\bar{p} = \bar{m}$, the lower middle perversity, using field coefficients and Witt spaces (see Section 10 below). This result was generalized by Cohen, Goresky, and Ji [20], who, also using sheaf theory but now for coefficients in a principal ideal domain and for arbitrary compact pseudomanifolds, showed that such a formula holds whenever $\bar{p}$ satisfies the condition $\bar{p}(a) + \bar{p}(b) \leq \bar{p}(a + b) \leq \bar{p}(a) + \bar{p}(b) + 1$. They also show that this can be extended to $\bar{p}(a) + \bar{p}(b) \leq \bar{p}(a + b) \leq \bar{p}(a) + \bar{p}(b) + 2$ provided one of $X$ or $Y$ is locally $(\bar{p}, R)$-torsion free (see Definition 6.26).

In [31], the situation was generalized further to consider for what perversities $\bar{p}, \bar{q}, Q$ (not necessarily Goresky-MacPherson perversities) it is true that

$$I^Q H_*(X \times Y; R) \cong H_*(I^p S_*(X; R) \otimes_R I^q S_*(Y; R))$$

for stratified pseudomanifolds $X, Y$. The arguments in [31] were also sheaf-theoretic. We provide here a further generalization to CS sets using singular chains.

We will first briefly discuss cross products in this setting. Then we will use a key example in order to demonstrate what the perversity $Q$ must be in order to obtain isomorphisms

$$H_*(I^p S_*(X; R) \otimes I^q S_*(Y; R)) \rightarrow I^Q H_*(X \times Y; R).$$

Then, we will state the Künneth theorem, Theorem 6.56, and prove it using the computations of the key example and a Mayer-Vietoris argument.

### 6.4.1 Cross products in non-GM intersection homology

Let $X$ and $Y$ be two filtered spaces with respective perversities $\bar{p}$ and $\bar{q}$. Analogously to what we observed in the discussion prior to Theorem 6.25, since $I^p S_*(X; R) \subset S_*(X; R)$ and $I^q S_*(Y; R) \subset S_*(Y; R)$, the cross product restricts to provide a map

$$I^p S_*(X; R) \otimes_R I^q S_*(Y; R) \rightarrow S_*(X \times Y; R).$$

Furthermore, we saw in Lemma 5.12 that the cross product restricts to yield a map $I^p S_*^{GM}(X; R) \otimes I^q S_*^{GM}(Y; R) \rightarrow I^Q S_*^{GM}(X \times Y; R)$ if $Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ for all strata $S \subset X$ and
$S' \subset Y$. For that lemma, we showed that the cross product of a $\bar{p}$-allowable simplex with a $\bar{q}$-allowable simplex is $Q$-allowable, given the assumption on $Q$, and it is straightforward to observe that these allowability considerations continue to hold with coefficients in the ring $R$. So to conclude that the image of the cross product acting on $I^p S_*(X; R) \otimes_R I^q S_*(Y; R)$ lies in $I^Q S_*^{GM}(X \times Y)$, we need only observe that if $\sigma$ is a singular $i$-simplex of $X$ not contained in $\Sigma_X$ and $\tau$ is a singular $j$-simplex of $Y$ not contained in $\Sigma_Y$, then the chain $\sigma \times \tau$ does not contain any simplices contained in $\Sigma_{X \times Y}$. For this, we observe as we did prior to Theorem 6.25 that the shuffle product construction of the triangulation of $\Delta^i \times \Delta^j$ makes it apparent that if $\eta: \Delta^{i+j} \to X \times Y$ is any of the singular $i+j$ simplices of the chain $\sigma \times \tau$, then the images of the composition of $\eta$ with the projections to $X$ or $Y$ will yield the images of $\sigma$ and $\tau$. But since the images of $\sigma$ and $\tau$ contain points that are respectively not contained in $\Sigma_X$ or $\Sigma_Y$, the image of $\eta$ must contained a point in $(X - \Sigma_X) \times (Y - \Sigma_Y) = X \times Y - \Sigma_{X \times Y}$. Therefore, $\eta$ is not contained in $\Sigma_{X \times Y}$.

We must also verify that the cross product

$$I^p S_*(X; R) \otimes_R I^q S_*(Y; R) \to I^Q S_*(X \times Y; R)$$

is a chain map. We know that it is a chain map $S_*(X; R) \otimes_R S_*(Y; R) \to S_*(X \times Y; R)$. Suppose $x \in I^p S_*(X; R)$ and $y \in I^q S_*(Y; R)$ with $\partial x = \hat{\partial} x + \eta_x$ and $\partial y = \hat{\partial} y + \eta_y$.

Then

$$\partial (x \times y) = (\partial x) \times y + (-1)^i x \times (\partial y)$$

$$= (\hat{\partial} x + \eta_x) \times y + (-1)^i x \times (\hat{\partial} y + \eta_y)$$

$$= (\hat{\partial} x) \times y + \eta_x \times y + (-1)^i x \times (\hat{\partial} y) + (-1)^i x \times \eta_y$$

$$= (\hat{\partial} x) \times y + (-1)^i x \times (\hat{\partial} y) + \eta_x \times y + (-1)^i x \times \eta_y.$$

Since $\eta_x$ is contained in $\Sigma_X$ and since $\eta_y$ is contained in $\Sigma_Y$, $\eta_x \times y$ and $x \times \eta_y$ are contained in $\Sigma_{X \times Y}$, and since $x$, $y$, $\hat{\partial} x$, and $\hat{\partial} y$ have no simplices in the respective singular loci, the argument of the preceding paragraph shows that neither do $(\hat{\partial} x) \times y$ or $x \times (\hat{\partial} y)$. Therefore, $\partial (x \times y) = (\partial x) \times y + (-1)^i x \times (\hat{\partial} y)$, which established that the cross product is a chain map as desired.

We state this formally as a lemma:

**Lemma 6.51.** Let $X$ and $Y$ be filtered sets with respective perversities $\bar{p}$ and $\bar{q}$. Suppose $Q$ is a perversity on $X \times Y$ such that $Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ for all strata $S \subset X$ and $S' \subset Y$. Then the singular cross product restricts yield a well-defined chain map $I^p S_*(X; R) \otimes I^q S_*(Y; R) \to I^Q S_*(X \times Y; R)$ and so a homology map

$$H_*(I^p S_*(X; R) \otimes I^q S_*(Y; R)) \to I^Q H_*(X \times Y; R).$$

Now that we have established that the non-GM cross product is well-defined, since it remains induced by the cross product on ordinary singular chains, the properties demonstrated in Section 5.2.1 all carry over to this new setting.
6.4.2 A key example

Next we seek to understand what conditions are necessary and sufficient on the product space perversity $Q$ in order for the map $H_*(I^pS_*(X; R) \otimes I^qS_*(Y; R)) \to I^QH_*(X \times Y; R)$ to be an isomorphism. We will do this by considering the key example of a product of cones $cX \times cY$, under the assumption that the Künneth theorem already holds for product subsets of smaller depth. This situation will then play an important role in the inductive Mayer-Vietoris argument for the general case of the Künneth theorem. We have already seen that we must have $Q(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ in order for the cross product to be well defined. We will see that this condition, in addition to others, is imposed separately by other considerations in our example. The following discussion will be somewhat technically involved, so the reader more interested in the final answers might skip ahead to the statement of Lemma [6.54] and then continue on to later sections.

So to understand what $Q$ may be, we consider $cX \times cY$, where $X, Y$ are (non-empty) compact filtered sets of respective dimensions $n - 1$ and $m - 1$ and $cX \times cY$ is given the product filtration. Given that CS sets are locally products of this form (with additional Euclidean factors that do not influence the intersection homology), this is a good starting place. Topologically, we recall that $cX \times cY \cong c(X \ast Y)$, where $X \ast Y$ is the join of $X$ and $Y$; see Section 2.9. We suppose $cX$ and $cY$ have been endowed with respective perversities $\bar{p}, \bar{q}$.

In order to discover the possibilities for $Q$ so that the cross product will induce a homology isomorphism, we must compare $H_*(I^pS_*(cX; R) \otimes I^qS_*(cY; R))$ with $I^QH_*(cX \times cY; R) \cong I^QH_*(c(X \ast Y); R)$, for which we can use the cone formula. We will assume, as an induction hypothesis, that we already have a Künneth isomorphism for any open subset of $cX \times cY - \{v \times w\}$ of the form $U \times V$, for open $U \subset X$ and open $V \subset Y$. Here $v$ and $w$ are the respective cone points of $cX$ and $cY$, and so any such product subspace has depth less than that of $cX \times cY$. We will discuss this assumption in detail later in the proof of Theorem [6.56], but such an induction hypothesis is reasonable taking as base cases the products where at least one factor is a manifold.

With all of these assumptions, we will see that the cross product induces a homology isomorphism when we have the following conditions on $Q$:

$$\bar{p}(\{v\}) + \bar{q}(\{w\}) \leq Q(\{v \times w\}) \leq \bar{p}(\{v\}) + q(\{w\}) + 1. \quad (9)$$

Furthermore, if the torsion product $I^\bar{p}H_{n-\bar{p}(\{v\})-2}(X; R) \ast I^\bar{q}H_{m-\bar{q}(\{w\})-2}(Y; R) = 0$, then we can weaken this condition to

$$\bar{p}(\{v\}) + \bar{q}(\{w\}) \leq Q(\{v \times w\}) \leq \bar{p}(\{v\}) + q(\{w\}) + 2. \quad (10)$$

We will also see that these conditions are necessary, in general.

Using the cone formula, we can now easily write down a formula for $H_*(I^pS_*(cX; R) \otimes I^qS_*(cY; R))$. In fact, using the algebraic Künneth theorem [55, Theorem V.2.1], we have
\[H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) = \bigoplus_{j+k=i} H_j(I^p S_*(cX; R)) \otimes H_k(I^q S_*(cY; R)) \]
\[\bigoplus_{j+k=i-1} H_j(I^p S_*(cX; R)) \ast H_k(I^q S_*(cY; R)) \]
\[\bigoplus_{j+k=i} I^p H_j(X; R) \otimes I^q H_k(Y; R) \]
\[\bigoplus_{j+k=i-1} I^p H_j(X; R) \ast I^q H_k(Y; R). \]

A useful first observation here is that each tensor product term will be 0 unless simultaneously \( j \leq n - \bar{p}(\{v\}) - 2 \) and \( k \leq m - \bar{q}(\{w\}) - 2 \), so \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \) will have no tensor product terms if \( i > n + m - \bar{p}(\{v\}) - q(\{w\}) - 4 \). Similarly, each torsion product term will be 0 unless simultaneously \( j \leq n - \bar{p}(\{v\}) - 2 \) and \( k \leq m - \bar{q}(\{w\}) - 2 \), so \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \) will have no torsion product terms if \( i - 1 > n + m - \bar{p}(\{v\}) - q(\{w\}) - 4 \), i.e. if \( i > n + m - \bar{p}(\{v\}) - q(\{w\}) - 3 \). So, if \( i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2 \), then \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) = 0 \).

The more interesting computation is the one for \( I^Q H_*(cX \times cY; R) \cong I^Q H_*(c(X \times Y); R) \).

By the cone formula, we will have
\[I^Q H_i(cX \times cY; R) \cong I^Q H_i(c(X \times Y); R) \]
\[\begin{cases} 0, & i \geq n + m - Q(\{v \times w\}) - 1, \\ I^Q H_i(X \times Y; R), & i < n + m - Q(\{v \times w\}) - 1. \end{cases} \]

Given this computation, we can see why it is in general necessary to have
\[\bar{p}(\{v\}) + q(\{w\}) \leq Q(\{v \times w\}) \leq \bar{p}(\{v\}) + q(\{w\}) + 1 \]
in order to have any hope of an isomorphism between \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \) and \( I^Q H_*(cX \times cY; R) \): We have just seen that \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) = 0 \) if \( i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2 \); but it is also not hard to arrange for \( H_{n+m-\bar{p}(\{v\})-q(\{w\})-3} (I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \) to be non-zero. In particular, we have
\[H_{n+m-\bar{p}(\{v\})-q(\{w\})-3} (I^p S_*(cX; R) \otimes I^q S_*(cY; R)) = I^p H_{n-\bar{p}(\{v\})-2} (X; R) \ast I^q H_{m-q(\{w\})-2} (Y; R), \]
which could (generically) be non-zero, for example, by choosing \( X \) and \( Y \) to be manifolds with torsion in their homology modules of appropriate dimensions. For \( i < n + m - \bar{p}(\{v\}) - q(\{w\}) - 3 \), we can obtain non-trivial non-torsion elements in all dimensions by letting \( X \) and \( Y \) be products of circles. On the other hand, we must have \( I^Q H_i(c(X \times Y); R) = 0 \) if \( i \geq \]
\(n + m - Q(\{v \times w\}) - 1\). So we need to have \(n + m - Q(\{v \times w\}) - 1 \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2\), i.e.
\[
Q(\{v \times w\}) \leq \bar{p}(\{v\}) + q(\{w\}) + 1.
\]

If we happen to be in a situation where we know that \(I^p H_{n-\bar{p}(\{v\}) - 2}(X; R) \cong I^q H_{m-q(\{w\}) - 2}(Y; R) = 0\), either by an assumption on the spaces or working with \(I\) being a field, then \(H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R))\) will vanish when \(i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 3\). So in this case we only need \(n + m - Q(\{v \times w\}) - 1 \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 3\), or \(Q(\{v \times w\}) \leq \bar{p}(\{v\}) + q(\{w\}) + 2\).

This provides the upper bounds on \(Q(\{v \times w\})\) we claimed in conditions \([9]\) and \([10]\). We already know that the lower bound \(\bar{p}(\{v\}) + q(\{w\}) \leq Q(\{v \times w\})\) is necessary in order for the cross product to be well defined; see also Remark 6.55. So, together, this establishes the necessity of \([9]\), or \([10]\) if the torsion vanishing condition holds.

**Remark 6.52.** Thinking ahead to the sufficiency of conditions \([9]\) or \([10]\), one might expect that these still offer too much flexibility and that we would need to choose \(Q\) so that the dimension “cutoff” in the cone formula agrees with what we would expect from our computation of \(H_*(I^p S_*(cX; R) \otimes I^q S_*(cY; R))\). However, we will see below that \(I^Q H_*(X \times Y; R)\) turns out to be zero automatically in certain dimensions (reminiscent of the computation of the intersection homology of a suspension), and this provides the additional flexibility.

Now we turn to showing that these conditions on \(Q\) are sufficient to obtain a Künneth isomorphism. We have \(H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) = 0\) for \(i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2\) and \(I^Q H_i(cX \times cY; R) = 0\) for \(i \geq n + m - Q(\{v \times w\}) - 1\). Assuming \(Q(\{v \times w\}) \geq \bar{p}(\{v\}) + q(\{w\})\), we have \(n + m - Q(\{v \times w\}) - 1 \leq n + m - \bar{p}(\{v\}) - q(\{w\}) - 1\). Therefore, for \(i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 1\) both \(H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R))\) and \(I^Q H_i(cX \times cY; R)\) must be trivial, and we can focus on \(i \leq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2\) for the rest of the discussion. Note that when \(Q(\{v \times w\}) = \bar{p}(\{v\}) + q(\{w\})\), this is exactly the range where \(I^Q H_i(cX \times cY; R)\) is not automatically 0. If \(Q(\{v \times w\}) = \bar{p}(\{v\}) + q(\{w\}) + 1\) then \(I^Q H_i(cX \times cY; R)\) is also automatically 0 in dimension \(i = n + m - \bar{p}(\{v\}) - q(\{w\}) - 2\). If \(Q(\{v \times w\}) = \bar{p}(\{v\}) + q(\{w\}) + 2\) then \(I^Q H_i(cX \times cY; R)\) is also automatically 0 in dimension \(i = n + m - \bar{p}(\{v\}) - q(\{w\}) - 3\). Below these dimensions, and under our assumptions about \(Q\), the module \(I^Q H_i(cX \times cY; R)\) is never automatically zero just from the cone formula.

Now, in the dimension range where \(I^Q H_i(cX \times cY; R) \cong I^Q H_i(cX \times Y; R)\) is not automatically 0, it will be isomorphic to \(I^Q H_i(cX \times Y; R) \cong I^Q H_i(cX \times cY - \{v \times w\}; R)\). So, we need to know something about these modules. But, we have
\[
cX \times cY - \{v \times w\} \cong (cX \times (cY - \{w\})) \cup ((cX - \{v\}) \times cY),
\]
while
\[
(cX \times (cY - \{w\})) \cap ((cX - \{v\}) \times cY) \cong (cX - \{v\}) \times (cY - \{w\}).
\]
As each of \(cX \times (cY - \{w\})\), \((cX - \{v\}) \times cY\), and \((cX - \{v\}) \times (cY - \{w\})\) is a product of open subsets of depth less than that of \(cX \times cY\), we can utilize our induction hypothesis that there is a Künneth isomorphism for these products. Furthermore, this decomposition of \(cX \times cY - \{v \times w\}\) allows us to utilize a Mayer-Vietoris sequence. Therefore, we have the following diagram with the long exact Mayer-Vietoris sequence along the right side (coefficients tacit):
Now, consider the terms in the exact sequence. The simplest term is the intersection term, $I^QH_i((cX - \{v\}) \times (cY - \{w\}); R)$, which, by the induction assumption, is isomorphic via the cross product map to

$$H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY - \{w\}; R)) \cong \bigoplus_{j+k=i} H_j(I^pS_s(cX - \{v\}; R)) \otimes H_k(I^qS_s(cY - \{w\}; R))$$

In the Mayer-Vietoris sequence, this maps to the direct sum term $I^QH_i((cX \times (cY - \{w\}); R)) \oplus I^QH_i((cX - \{v\}) \times cY; R)$, which, again using the induction assumptions, is isomorphic to

$$H_i(I^pS_s(cX; R) \otimes I^qS_s(cY - \{w\}; R)) \oplus H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY; R)).$$

Using the algebraic K"unneth theorem and the cone formula, this becomes\footnote{Of course, we could write $I^pH_j(X; R)$ rather than $I^pH_j(cX - \{v\}; R)$ in this formula, but the latter is better suited to the somewhat delicate argument we have coming up.}

$$\bigoplus_{j+k=i} I^pH_j(cX - \{v\}; R) \otimes I^qH_k(cY - \{w\}; R) \bigoplus_{j+k=i-1} I^pH_j(cX - \{v\}; R) \otimes I^qH_k(cY - \{w\}; R) \bigoplus_{k<m-q} I^pH_j(cX - \{v\}; R) \otimes I^qH_k(cY - \{w\}; R).$$
Notice that each summand here also occurs as a summand of \([12]\). In fact, we claim that the maps
\[ H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY - \{w\}; R)) \to H_i(I^p S_*(cX; R) \otimes I^q S_*(cY - \{w\}; R)) \]
\[ H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY - \{w\}; R)) \to H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY; R)) \]
induced by the space inclusions correspond to the obvious projections onto summands. It turns out that this is really not so obvious\(^\text{77}\) even given the nice topological situation, as the splittings guaranteed by the algebraic Künneth theorem are not required to be preserved under morphisms (see \[55\], Section V.2). Luckily, however, this naive expectation does turn out to hold. This will be the upshot of some technical work we will do below in Section 6.4.6, culminating in the following corollary. Although this corollary plays a critical role at several points in the remainder of our key example, its proof goes a bit far afield while we already have a few balls up in the air; hence, we put the proof aside until Section 6.4.6 after we have finished the rest of our discussion of the Künneth theorem.

Here is a statement of Corollary 6.63.

**Corollary 6.53** (Corollary 6.63). Given a Dedekind domain \( R \) and compact filtered sets \( X, Y \), there are splittings of \( H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY - \{w\}; R)) \), \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \), \( H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY; R)) \), and \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \) into direct sums of tensor products \( I^p H_j(cX - \{v\}; R) \otimes I^q H_k(cY - \{w\}; R), j + k = i \), and torsion products \( I^p H_j(cX - \{v\}; R) \otimes I^q H_k(cY - \{w\}; R), j + k = i - 1 \), such that the maps in the diagram

\[ H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY - \{w\}; R)) \]

\[ \xrightarrow{id \otimes \mathfrak{j}} \]

\[ H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \]

\[ \xrightarrow{id \otimes \mathfrak{k}} \]

\[ H_i(I^p S_*(cX - \{v\}; R) \otimes I^q S_*(cY; R)) \]

\[ \xrightarrow{j \otimes \text{id}} \]

\[ H_i(I^p S_*(cX; R) \otimes I^q S_*(cY; R)) \]

induced by the inclusions \( j : cX - \{v\} \hookrightarrow cX \) and \( \mathfrak{k} : cY - \{w\} \hookrightarrow cY \) each restrict on each tensor or torsion product summand either to the 0 map or to an isomorphism with the corresponding summand in the codomain. Furthermore, which of these options is determined in the obvious way; for example, the tensor product summand \( I^p H_j(cX - \{v\}; R) \otimes I^q H_k(cY - \{w\}; R) \) maps to 0 in \( H_i(I^p S_*(cX; R) \otimes I^q S_*(cY - \{w\}; R)) \) when \( j \geq n - \bar{p}(\{v\}) - 1 \) and isomorphically otherwise.

We will assume that we have made these consistent choices in our following work with the Mayer-Vietoris sequence.

So let us examine the Mayer-Vietoris sequence in dimensions \( i \leq n + m - \bar{p}(\{v\}) - q(\{w\}) - 2 \). It will be useful to adopt the following temporary notation: Let \( G_i \) denote one of the

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\(^{77}\)This is an oversight in the proof of the Künneth theorem in \[31\]; however, as we will see here, the result stated there does hold.
summands of \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY - \{w\}; R)) \cong I^QH_i((cX - \{v\}) \times (cY - \{w\}); R) \) in dimension \( i \). So \( G_i \) has the form \( I^pH_j(cX - \{v\}; R) \otimes I^qH_k(cY - \{w\}; R) \), \( j + k = i \), or \( I^pH_j(cX - \{v\}; R) \ast I^qH_k(cY - \{w\}; R) \), \( j + k = i - 1 \). Corollary 6.63 says that the splittings can be chosen so that, under the maps from \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY - \{w\}; R)) \) to \( H_i(I^pS_s(cX; R) \otimes I^qS_s(cY - \{w\}; R)) \) and \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY; R)) \) induced by inclusion, each of the \( G_i \) summands is taken either to 0 or isomorphically to the appropriate corresponding summand.

We next claim that, therefore, \( MV \) restricted to each \( G_i \) is injective if \( i < n + m - \bar{p}(\{v\}) - q(\{w\}) - 2 \), which will imply that, for all \( i \) in this range, the Mayer-Vietoris sequence breaks up into short exact sequences, each in a single dimension. Since the map \( MV \) takes each summand \( G_i \) of (12) either to 0 or to the corresponding term (s) in (13), we need only check that, in this range, each summand \( G_i \) appears in the expression (13). If \( G_i \) is a tensor product term, it does not occur in (13) if and only if \( i = j + k \) with \( j \geq n - \bar{p}(\{v\}) - 1 \) and \( \bar{q}(\{w\}) - \bar{k} \geq m \); in this case, we must have \( i \geq m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 1 \). If \( G_i \) is a torsion product term, it does not occur in (13) if and only if \( i = j + k + 1 \) with \( j \geq n - \bar{p}(\{v\}) - 1 \) and \( \bar{k} \geq m - \bar{q}(\{w\}) - 1 \); in this case, we must have \( i \geq m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 1 \). So if \( i < m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2 \), the summand \( G_i \) does appear in (13), and \( MV \) must be injective when restricted to \( G_i \). But \( G_i \) was an arbitrary summand of (12), so \( MV \) is injective in this range.

So now for \( i < m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2 \), the Mayer-Vietoris sequence breaks up into short exact sequences in each dimension, and there are two possibilities for what happens to each summand \( G_i \) of \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY - \{w\}; R)) \). If both copies of \( G_i \) occur in (13), then, up to isomorphism, \( MV \) restricts on \( G_i \) to the diagonal map \( G_i \rightarrow G_i \oplus G_i \). However, if only one copy of \( G_i \) occurs in the middle Mayer-Vietoris term, then \( MV \) takes \( G_i \) isomorphically to this copy. It follows that \( I^QH_i(cX \times cY - \{v \times w\}; R) \) will be the direct sum of those \( G_i \) for which the first situation occurs, i.e. of those \( G_i \) present in both \( H_i(I^pS_s(cX; R) \otimes I^qS_s(cY - \{w\}; R)) \) and \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY; R)) \).

Let us compute when this happens. If \( G_i \) is a tensor product of groups of dimension \( j \) and \( k \), we will have two copies of \( G_i \) in (13) if both \( j < n - \bar{p}(\{v\}) - 1 \) and \( k < m - \bar{q}(\{w\}) - 1 \). If \( G_i \) is a torsion product of groups of dimension \( j \) and \( k \) (with \( j + k = i - 1 \)), we will again have two copies of \( G_i \) if both \( j < n - \bar{p}(\{v\}) - 1 \) and \( k < m - \bar{q}(\{w\}) - 1 \). So we see that for \( i < m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2 \),

\[
I^QH_i(cX \times cY - \{v \times w\}; R) \cong \bigoplus_{j+k=i} \bigoplus_{j<n-\bar{p}(\{v\})-1 \atop k<m-\bar{q}(\{w\})-1} I^pH_j(cX - \{v\}; R) \otimes I^qH_k(cY - \{w\}; R)
\]

\[\oplus \bigoplus_{j+k=i-1} \bigoplus_{j<n-\bar{p}(\{v\})-1 \atop k<m-\bar{q}(\{w\})-1} I^pH_j(cX - \{v\}; R) \ast I^qH_k(cY - \{w\}; R).
\]

But this is exactly isomorphic to \( H_i(I^pS_s(cX - \{v\}; R) \otimes I^qS_s(cY - \{w\}; R)) \), and we will show below, after cleaning up some other details, that this isomorphism is induced by
the cross product.

Summing up, we have now seen the following, with the assumptions [9] or [10]:

1. If \( i \geq n + m - \bar{p}(\{v\}) - q(\{w\}) - 1 \), then \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) \) and \( I^QH_i(cX \times cY; R) \) are both 0.

2. If \( i \leq n + m - \bar{p}(\{v\}) - q(\{w\}) - 4 \), then \( I^QH_i(cX \times cY; R) \cong I^QH_i(cX \times cY - \{v \times w\}; R) \), and this is isomorphic to \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) \) (we have yet to see that the isomorphism is induced by the cross product).

3. If \( i = n + m - \bar{p}(\{v\}) - q(\{w\}) - 3 \) and \( Q(\{v \times w\}) \leq \bar{p}(\{v\}) + \bar{q}(\{w\}) + 1 \), then again \( I^QH_i(cX \times cY; R) \cong I^QH_i(cX \times cY - \{v \times w\}; R) \), and this is isomorphic to \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) \). If \( Q(\{v \times w\}) = \bar{p}(\{v\}) + \bar{q}(\{w\}) + 2 \), then \( I^QH_i(cX \times cY; R) \cong I^QH_i(cX \times cY; R) \) must be 0 from the cone formula, but, in this dimension \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) \cong I^PH_{n - \bar{p}(\{v\}) - 2}(cX; R) \otimes I^QH_{m - \bar{q}(\{w\}) - 2}(Y; R) \), so we will still have an isomorphism if this torsion term vanishes.

Therefore, it remains only to consider \( i = n + m - \bar{p}(\{v\}) - q(\{w\}) - 2 \). We have already computed that \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) = 0 \) in this dimension. We will show that \( I^QH_i(cX \times cY - \{v \times w\}; R) \) also vanishes, and so we will have isomorphisms in all dimensions whether we choose \( Q(\{v \times w\}) = \bar{p}(\{v\}) + \bar{q}(\{w\}) \) or \( Q(\{v \times w\}) = \bar{p}(\{v\}) + \bar{q}(\{w\}) + 1 \).

So we turn to \( I^QH_{m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2}(cX \times cY - \{v \times w\}; R) \). Since the Mayer-Vietoris map \( MV \) is injective in lower dimensions, this group is the image of the term before it in the Mayer-Vietoris sequence. But if we look at [13], we see that unless \( i = j + k \leq n - \bar{p}(\{v\}) - 2 + m - \bar{q}(\{w\}) = 2 = m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 4 \), it is impossible to have two copies of a tensor product summand \( G_i \) in [13]. But we are assuming \( i = m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2 \), so each copy of \( G_i \) occurs at most once in [13]. Therefore, again using Corollary [6.63], the map \( MV \) takes \( G_i \) in [12] either to 0 or to the lone corresponding summand in [13], and so the summand does not survive into \( I^QH_{m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2}(cX \times cY - \{v \times w\}; R) \). Similarly, if \( G_i \) is a torsion produce summand, it is impossible to have two copies of \( G_i \) in [13] unless \( i - 1 = j + k \leq n - \bar{p}(\{v\}) - 2 + m - \bar{q}(\{w\}) - 2 = m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 4 \). Since this is also not the case, the map of the corresponding \( G_i \) to [13] must again be 0 or an isomorphism onto a lone summand, so there is again no contribution to \( I^QH_{m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2}(cX \times cY - \{v \times w\}; R) \).

Thus \( I^QH_{m + n - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2}(cX \times cY - \{v \times w\}; R) = 0 \).

We have now shown, under the assumption [9] or [10], that the modules \( H_i(I^pS_*(cX; R) \otimes I^qS_*(cY; R)) \) and \( I^QH_i(cX \times cY; R) \cong I^QH_i(cX \times cY; R) \) are isomorphic in all dimensions. Let us confirm that this isomorphism is induced by the cross product. It follows from our Mayer-Vietoris argument that, in the dimension ranges where the modules are not necessarily trivial, every element of \( I^QH_i(cX \times cY - \{v \times w\}; R) \) is in the image of \( I^QH_i((cX - \{v\}) \times cY; R) \) under the inclusion (alternatively, it is also in the image of \( I^QH_i(cX \times (cY - \{w\}); R) \)), and the naturality of the cross product (Lemma [5.16]) provides
a diagram in these dimensions

\[ H_i(\mathcal{I}^p S_*(cX - \{v\}; R) \otimes \mathcal{I}^q S_*(cY; R)) \xrightarrow{\epsilon} H_i(\mathcal{I}^p S_*(cX; R) \otimes \mathcal{I}^q S_*(cY; R)) \]

\[ \cd \quad \cd \]

\[ I^q H_i((cX - \{v\}) \times cY; R) \xrightarrow{\epsilon} I^q H_i(cX \times cY - \{v \times w\}; R) \xrightarrow{\epsilon} I^q H_i(cX \times cY; R), \]

with the surjectivity of the top line again coming from Corollary 6.63. It follows immediately that \( \epsilon : H_i(\mathcal{I}^p S_*(cX; R) \otimes \mathcal{I}^q S_*(cY; R)) \rightarrow I^q H_i(cX \times cY; R) \) is surjective. For injectivity, Corollary 6.63 continues to imply that the top left module has the form \( \oplus G_{i,t} \), where each \( G_{i,t} \) is isomorphic to \( I^p H_j(cX - \{v\}; R) \otimes I^q H_k(cY - \{w\}; R) \) with \( j + k = i \) or \( I^p H_j(cX - \{v\}; R) \otimes I^q H_k(cY - \{w\}; R) \) with \( j + k = i - 1 \), and that the top horizontal map is the projection to those summands with \( j \leq n - \bar{p}(\{v\}) - 2 \) (there are already no summands with \( k > m - q(\{w\}) - 2 \) present). But we know from our arguments with the Mayer-Vietoris sequence that the summands with both \( j \leq n - \bar{p}(\{v\}) - 2 \) and \( k \leq m - q(\{w\}) - 2 \) are the summands that map isomorphically into \( I^q H_i(cX \times cY - \{v \times w\}; R) \cong I^q H_i(cX \times cY; R) \).

Therefore, every non-trivial element of \( H_i(\mathcal{I}^p S_*(cX - \{v\}; R) \otimes \mathcal{I}^q S_*(cY; R)) \) is the image of an element of \( H_i(\mathcal{I}^p S_*(cX - \{v\}; R) \otimes \mathcal{I}^q S_*(cY; R)) \) that does not map to 0 by traveling down then right in the diagram. So the right vertical map is injective.

We can now wrap up the discussion thus far with a formal statement of our conclusions for our key example:

**Lemma 6.54.** Let \( R \) be a Dedekind domain. Let \( X, Y \) be non-empty compact filtered spaces of respective dimensions \( n - 1, m - 1 \) with respective perversities \( \bar{p}, \bar{q} \). Suppose a perversity \( Q \) is chosen on \( cX \times cY \) such that for any open subset of \( cX \times cY - \{v \times w\} \) of the form \( U \times V \) we have an isomorphism induced by the cross product \( H_*(\mathcal{I}^p S_*(U; R) \otimes \mathcal{I}^q S_*(V; R)) \xrightarrow{\sim} I^q H_*(U \times V; R) \). Then the cross product induces an isomorphism \( H_*(\mathcal{I}^p S_*(cX; R) \otimes \mathcal{I}^q S_*(cY; R)) \xrightarrow{\sim} I^q H_*(cX \times cY; R) \) if (and, in general, only if) \( Q(\{v \times w\}) \) is defined on \( cX \times cY \) and \( cX \times cY - \{v \times w\} \). If \( I^p H_{n-\bar{p}(\{v\})-2}(X; R) \otimes I^q H_{m-\bar{q}(\{w\})-2}(Y; R) = 0 \), then we can also use \( Q(\{v \times w\}) = \bar{p}(\{v\}) + \bar{q}(\{w\}) + 2 \).

**Remark 6.55.** As a side note, the Mayer-Vietoris sequence and Corollary 6.63 allow us to see that we need to have \( Q(\{v \times w\}) \geq \bar{p}(\{v\}) + \bar{q}(\{w\}) \) for more reasons than just the well-definedness of the cross product: Indeed, if \( Q(\{v \times w\}) < \bar{p}(\{v\}) + \bar{q}(\{w\}) \), then from the cone formula \( I^q H_{n+m-\bar{p}(\{v\})-\bar{q}(\{w\})-1}(c(X \times Y); R) \cong I^q H_{n+m-\bar{p}(\{v\})-\bar{q}(\{w\})-1}(cX \times cY - \{v \times w\}; R) \).

To study this group, let us consider the map \( MV \) in dimension \( n + m - \bar{p}(\{v\}) - \bar{q}(\{w\}) - 2 \). Suppose that \( I^p H_{n-\bar{p}(\{v\})-1}(X; R) \otimes I^q H_{m-\bar{q}(\{w\})-1}(Y; R) \neq 0 \), which is certainly a possibility (for example, if \( \bar{p} = \bar{q} = 0 \) and \( X, Y \) are oriented stratified pseudomanifolds, this could be the tensor product of the fundamental classes of \( X \) and \( Y \); see Section 8.2 below). This tensor product corresponds to a summand in \( H_i(\mathcal{I}^p S_*(cX - \{v\}; R) \otimes \mathcal{I}^q S_*(cY - \{w\}; R)) \), but we see that its image must vanish in \( H_i(\mathcal{I}^p S_*(cX; R) \otimes \mathcal{I}^q S_*(cY - \{w\}; R)) \oplus H_i(\mathcal{I}^p S_*(cX - \{v\}; R) \otimes \mathcal{I}^q S_*(cY; R)) \) since \( j = n - \bar{p}(\{v\}) - 1 \) and \( k = m - \bar{q}(\{w\}) - 1 \) are both above the cutoff dimensions in the corresponding cone formulas. So \( I^q H_{n+m-\bar{p}(\{v\})-\bar{q}(\{w\})-1}(cX \times cY; R) \) could not be 0, while \( H_{n+m-\bar{p}(\{v\})-\bar{q}(\{w\})-1}(\mathcal{I}^p S_*(cX; R) \otimes \mathcal{I}^q S_*(cY; R)) = 0 \).
6.4.3 The Künneth Theorem

Our key example suggests that to define \( Q \) in general on a product \( X \times Y \) of filtered spaces, we should try to define \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) \) or \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 1 \), where \( S, T \) are singular strata of and \( \bar{p}, \bar{q} \) are perversities on \( X, Y \), respectively. If \( S \) is a regular stratum of \( X \), then the Künneth theorem with a manifold factor, Theorem 5.28, tells us that that we need to be more restrictive and take \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) = \bar{q}(T) \). Similarly, if \( T \) is a regular stratum, we should use \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) = \bar{p}(S) \).

It will turn out that these choices work, and we will obtain the following Künneth theorem:

**Theorem 6.56.** Let \( X, Y \) be CS sets of respective dimensions \( n, m \) and with respective perversities \( \bar{p}, \bar{q} \), and let \( R \) be a Dedekind domain. Let \( Q \) be a perversity defined on \( X \times Y \) such that

1. if \( S \subset X \) is a regular stratum, then \( Q(S \times T) = \bar{q}(T) \), and
2. if \( T \subset Y \) is a regular stratum, then \( Q(S \times T) = \bar{p}(S) \), and
3. if \( S \subset X \) and \( Y \subset T \) are singular strata, \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) \) or \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 1 \).

Then the cross product induces an isomorphism \( H_*(I^pS_*(X; R) \otimes_R I^qS_*(Y; R)) \xrightarrow{\sim} I^Q H_*(X \times Y; R) \).

If for each point \( x \times y \in S \times T \), \( x \) has a distinguished neighborhood in \( X \) of the form \( \mathbb{R}^a \times cL_1 \) and \( y \) has a distinguished neighborhood in \( Y \) of the form \( \mathbb{R}^b \times cL_2 \) such that \( I^pH_{\dim(L_1)}-p(S_{1})-1(L_1; R) \ast I^qH_{\dim(L_2)}-q(T_{1})-1(L_2; R) = 0 \), then condition (3) on \( Q(S \times T) \) may also include the possibility \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 2 \). In particular this is allowable if \( X \) is locally \((\bar{p}, R)\)-torsion free along the singular stratum \( S \) or \( Y \) is locally \((\bar{q}, R)\)-torsion free along the singular stratum \( T \).

**Remark 6.57.** The theorem clearly applies as stated to stratified pseudomanifolds, but it also applies to \( \partial \)-stratified pseudomanifolds: if \( X \) is a \( \partial \)-stratified pseudomanifold, then \( X \) is stratified homotopy equivalent to the stratified pseudomanifold \( X - \partial X \), and similarly for \( Y \). So these interiors are intersection homology substitutes for the full spaces to which the theorem applies.

**Proof of Theorem 6.56.** As observed in Lemma 6.50, \( X^\bullet \times Y^\bullet = (X \times Y)^\bullet \), so we have a commutative diagram

\[
\begin{array}{ccc}
H_*(I^pS_*(X^\bullet; R) \otimes_R I^qS_*(Y^\bullet; R)) & \xrightarrow{\times} & I^Q H_*(X \times Y^\bullet; R) \\
\downarrow & & \downarrow \\
H_*(I^pS_*(X; R) \otimes_R I^qS_*(Y; R)) & \xrightarrow{\times} & I^Q H_*(X \times Y; R)
\end{array}
\]

See Definition 6.26
in which the vertical maps are induced by inclusions. By Proposition 6.47 and the algebraic Künneth Theorem [105, Theorem 3.6.3] (which applies as all intersection chain complexes are submodules of torsion-free modules over a Dedekind domain and so are flat), the vertical maps are isomorphisms. Therefore, it will suffice to show that the top horizontal map is an isomorphism. Equivalently, as the homogenization of a CS set is a CS set by Lemma 6.45, we will assume in the following argument that all spaces are dimensionally homogeneous, and the result in general follows from this diagram. Note that it follows from Corollary 6.49 that our finite generation and torsion free conditions on the links of \( X \) carries over to the links of \( X^\bullet \) and \( Y^\bullet \).

The proof will proceed by an induction on the depth of \( X \times Y \) using Mayer-Vietoris arguments.

If the depth of \( X \times Y \) is 0, then \( X \) and \( Y \) are both manifolds (in fact, since they are dimensionally homogeneous, they are equal to their regular strata), and the theorem reduces to the standard Künneth theorem for ordinary homology.

If the depth of \( X \times Y \) is 1, then one of \( X \) or \( Y \) is a manifold and the theorem reduces to the Künneth theorem with a manifold factor, Theorem 6.25.

So now assume that the depth of \( X \times Y \) is \( K > 1 \) and that we have proven the theorem in all cases where the depth of \( X \times Y \) is \( < K \).

Sill assuming that \( X \times Y \) had depth \( K \), we will first prove the special case of the theorem in which \( Y \) has the form \( Y = \mathbb{R}^j \times c\mathcal{L} \) for some compact filtered \( \mathcal{L} \) (and continues to satisfy the hypotheses of the theorem). We will use Theorem 5.3 with functors defined on the open subsets of \( X \). For \( U \subset X \), we let \( F_\ast(U) = H_\ast(I^pS_\ast(U; R) \otimes_R I^qS_\ast(Y; R)) \) and \( G_\ast(U) = I^qH_\ast(U \times Y; R) \) with the functor \( \Phi \) corresponding to the cross product. Then \( G_\ast \) admits Mayer-Vietoris sequences by Theorem 6.20 and \( F_\ast \) admits Mayer-Vietoris sequences as the long exact sequence of the short exact sequence obtained by tensoring the Mayer-Vietoris short exact sequence for \( I^pS_\ast(\cdot; R) \) for subsets of \( X \) with with \( I^qS_\ast(Y; R) \), which is flat since \( I^qS_\ast(Y; R) \) is torsion free and \( R \) is Dedekind. In fact, as in the proof of Theorem 5.28, \( \Phi \) induces a map of short exact sequences, and hence a map of long exact sequences, even after replacing terms of the form \( H_\ast((I^pS_\ast(U; R) + I^pS_\ast(V; R)) \otimes_R I^qS_\ast(Y; R)) \) with the isomorphic terms \( H_\ast(I^pS_\ast(U \cup V; R) \otimes_R I^qS_\ast(Y; R)) \).

The second condition of Theorem 5.3 follows, as usual, from Lemmas 5.5 and 5.6, in the case of \( F_\ast \) using the algebraic Künneth Theorem and the commutativity of tensor products with direct limits and homology.

The last condition of Theorem 5.3 is trivial if \( U \) is empty. Otherwise, it is a consequence of Theorem 6.25 as the assumption that \( X \) is dimensionally homogeneous ensures that an open subset \( U \) of \( X \) can be contained in a single stratum of \( X \) only if \( U \) is an open subset a regular stratum of \( X \) and so a trivially-filtered manifold of the dimension of \( X \).

Finally, we consider condition (3) of Theorem 5.3. By our induction on depth, the hypothesis that \( \Phi : F_\ast(\mathbb{R}^i \times (cL - \{v\})) \rightarrow G_\ast(\mathbb{R}^i \times (cL - \{v\})) \) be an isomorphism is automatically fulfilled whenever \( \mathbb{R}^i \times cL \) is (homeomorphic to) a distinguished neighborhood of a point in a singular stratum of \( X \). We need to show that \( \Phi : F_\ast(\mathbb{R}^i \times cL) \rightarrow G_\ast(\mathbb{R}^i \times cL) \) is an isomorphism. But this is precisely the content of Lemma 6.54 up to the additional Euclidean space factors, which are not relevant to the computation due to the stratified
homotopy invariance of intersection homology. The hypotheses of the lemma are satisfied due to the induction on depth. And so Theorem 5.3 proves the special case.

Now we can move on to the general case of arbitrary CS sets $X \times Y$ of depth $K$. We again use Theorem 5.3, this time with functors defined on the open subsets of $Y$. For $U \subset Y$, we let $F_*(U) = H_*(\overline{I^pS}_*(X; R) \otimes_R I^qS_*(U; R))$ and $G_*(U) = I^QH_*(X \times U; R)$ with the functor $\Phi$ corresponding to the cross product. By the exact same reasoning as above, we have a commuting diagram of Mayer-Vietoris sequences, the condition on ascending chains of open sets holds, and the last condition of Theorem 5.3 is a consequence of Theorem 6.25, as the dimensional homogeneity assumption on $Y$ again implies that if $U$ is a non-empty open subset of $Y$ contained in a single stratum, then $U$ is a trivially-filtered manifold of the dimension of $Y$. For condition (3) of 5.3, the hypothesis that $\Phi : F_*(R^i \times (cL - \{v\})) \to G_*(R^i \times (cL - \{v\}))$ be an isomorphism (now for $R^i \times cL$ a distinguished neighborhood of a point in a singular stratum of $Y$) is again automatically fulfilled by the induction hypothesis on depth, and we need to show that $\Phi : F_*(R^i \times cL) \to G_*(R^i \times cL)$ is an isomorphism. But this is exactly the special case of the theorem proven above for which the first factor is arbitrary but the second factor is the product of a Euclidean space and a cone. Since an open subset of a space $Z$ has depth less than or equal to that of $Z$, our proof of the special case is allowable here. The theorem now follows from Theorem 5.3.

As a corollary of the proof, we obtain the following version of the theorem with coefficients in a field.

**Corollary 6.58.** Let $X, Y$ be CS sets of respective dimensions $n, m$ and with respective perversities $\overline{p}, \overline{q}$. Let $F$ be a field. Let $Q$ be a perversity defined on $X \times Y$ such that

1. if $S \subset X$ is a regular stratum, then $Q(S \times T) = \overline{q}(T)$, and

2. if $T \subset Y$ is a regular stratum, then $Q(S \times T) = \overline{p}(S)$, and

3. if $S \subset X$ and $Y \subset T$ are singular strata, $Q(S \times T) = \overline{p}(S) + \overline{q}(T)$ or $Q(S \times T) = \overline{p}(S) + \overline{q}(T) + 1$ or $Q(S \times T) = \overline{p}(S) + \overline{q}(T) + 2$.

Then the cross product induces an isomorphism $H_*(I^pS_*(X; F) \otimes_F I^qS_*(Y; F)) \xrightarrow{\sim} I^QH_*(X \times Y; F)$.

**Remark 6.59.** This Künneth Theorem, Theorem 6.56 is not true if we replace $I^H_*$ with $I^H_*^{GM}$. For example, let $X$ and $Y$ each be a cone on a torus. Let $\overline{p}(\{v\}) = 5$ and $\overline{q}(\{w\}) = 0$. Using the cone formula, Theorem 4.12

$$I^pH_*^{GM}(X) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$I^qH_*^{GM}(Y) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$
and so also
\[ H_i(I^\bar{p}S^*_x(X) \otimes I^\bar{q}S^*_y(Y)) \cong \begin{cases} 
\mathbb{Z}, & i = 0, \\
\mathbb{Z} \oplus \mathbb{Z}, & i = 1, \\
0, & \text{otherwise.} 
\end{cases} \]

But if we take \( Q(\{v \times w\}) = \bar{p}(\{v\}) + \bar{q}(\{w\}) = 5 \), then, again by Theorem 4.12, we have
\[ I^QH^*_i(X \times Y; R) \cong \begin{cases} 
\mathbb{Z}, & i = 0, \\
0, & \text{otherwise.} 
\end{cases} \]

Conceivably, there may be some way to modify the theorem so that \( Q \) provides some kind of additional truncation if perversities get “too big” for Theorem 4.12 to apply (which is essentially the problem here) and hence some way to extend the Künneth theorem to \( IH^*_i \), but we will not pursue this here.

### 6.4.4 A relative Künneth theorem

Having established our general Künneth theorem as Theorem 6.56, we now turn to proving a relative version of the theorem. We will not need to work from scratch, instead we will use Theorem 6.56 along with some homological algebra.

For reference purposes, we state the full theorem:

**Theorem 6.60.** Let \( X, Y \) be CS sets of respective dimensions \( n, m \) and with respective perversities \( \bar{p}, \bar{q} \), and let \( R \) be a Dedekind domain. Let \( A \subset X \) and \( B \subset Y \) be open subspace, and let \( Q \) be a perversity defined on \( X \times Y \) such that

1. if \( S \subset X \) is a regular stratum, then \( Q(S \times T) = \bar{q}(T) \), and
2. if \( T \subset Y \) is a regular stratum, then \( Q(S \times T) = \bar{p}(S) \), and
3. if \( S \subset X \) and \( Y \subset T \) are singular strata, \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) \) or \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 1 \).

Then the cross product induces an isomorphism
\[ H_*(I^\bar{p}S_x(X, A; R) \otimes_R I^\bar{q}S_y(Y, B; R)) \xrightarrow{\sim} I^QH_*(X \times Y, (A \times Y) \cup (X \times B); R). \]

If for each point \( x \times y \in S \times T \), \( x \) has a distinguished neighborhood in \( X \) of the form \( \mathbb{R}^a \times cL_1 \) and \( y \) has a distinguished neighborhood in \( Y \) of the form \( \mathbb{R}^b \times cL_2 \) such that
\[ I^\bar{p}H_{\dim(L_1)} - \bar{p}(\{S\}) - 1(L_1; R) * R I^\bar{q}H_{\dim(L_2)} - \bar{q}(\{T\}) - 1(L_2; R) = 0, \]
then condition (3) on \( Q(S \times T) \) may also include the possibility \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 2 \). In particular this is allowable if \( X \) is locally \((\bar{p}, R)\)-torsion free along the singular stratum \( S \) or \( Y \) is locally \((\bar{q}, R)\)-torsion free along the singular stratum \( T \).

---

79It might be possible to prove this theorem in greater generality, but this will be a convenient assumption for us to utilize our previous results.
Proof. Consider the following diagram, in which we leave the $R$ coefficients tacit:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I^pS_*(A) \otimes_R I^qS_*(Y) & \rightarrow & I^pS_*(X) \otimes_R I^qS_*(Y) & \rightarrow & I^pS_*(X, A) \otimes_R I^qS_*(Y) & \rightarrow & 0 \\
\downarrow \times & & \downarrow \times & & \downarrow \times & & \\
0 & \rightarrow & I^qS_*(A \times Y) & \rightarrow & I^qS_*(X \times Y) & \rightarrow & I^qS_*(X \times Y, A \times Y) & \rightarrow & 0.
\end{array}
\]

The top row is the short exact $\bar{p}$-intersection chain sequence of the pair $(X, A)$ tensored over $R$ with $I^qS_*(Y; R)$. Since $I^qS_i(Y; R) \subset S_i(Y; R)$ for all $R$ and $R$ is Dedekind, $I^qS_i(Y; R)$ is flat, and so the sequence remains exact after tensoring. The second row is the short exact $Q$-intersection chain sequence of the pair $(X \times Y, A \times Y)$. The vertical maps are all induced by the chain cross product, and the diagram commutes, as can be seen by working with representative chains. In the resulting diagram of long exact homology sequences, the cross product induces isomorphisms on the absolute homology terms by Theorem 6.56 observing that the links of $A$ and $B$ are all also links of $X$ and $Y$. So

\[
\times : H_*(I^pS_*(X, A; R) \otimes_R I^qS_*(Y; R)) \rightarrow I^qH_*(X \times Y, A \times Y; R)
\]

is also an isomorphism, by the Five Lemma.

Similarly, we now have the diagram, again with coefficients tacit,

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I^pS_*(X, A) \otimes_R I^pS_*(B) & \rightarrow & I^pS_*(X, A) \otimes_R I^pS_*(Y) & \rightarrow & I^pS_*(X, A) \otimes_R I^pS_*(Y, B) & \rightarrow & 0 \\
\downarrow \times & & \downarrow \times & & \downarrow \times & & \\
0 & \rightarrow & I^qS_*(X \times B, A \times B) & \rightarrow & I^qS_*(X \times Y, A \times Y) & \rightarrow & I^qS_*(X \times Y)/(I^qS_*(A \times Y) + I^qS_*(X \times B)) & \rightarrow & 0.
\end{array}
\]

This time, the top row comes from tensoring the short exact $\bar{q}$-intersection chain sequence of the pair $(Y, B)$ with $I^pS_*(X, A; R)$. The sequence stays exact as $I^pS_*(X, A; R)$ is also torsion free, and so flat over a Dedekind domain: if $\xi$ is a chain with $r\xi$ contained in $A$ for some $r \in R$, then $\xi$ must itself be contained in $A$. The bottom row in the diagram is exact via some basic algebra: Certainly the inclusion $X \times B \hookrightarrow X \times Y$ induces an injection $I^QS_*(X \times B, A \times B; R) \rightarrow I^QS_*(X \times Y, A \times Y; R)$. Then

\[
\frac{I^QS_*(X \times Y, A \times Y; R)}{I^QS_*(X \times B, A \times B; R)} \cong \frac{I^QS_*(X \times Y; R)}{I^QS_*(A \times Y; R)} \frac{I^QS_*(A \times Y; R)}{I^QS_*(A \times B; R)}
\]

\[
\cong \frac{I^QS_*(X \times Y; R)}{(I^QS_*(A \times Y; R) + I^QS_*(X \times B; R))/I^QS_*(A \times Y; R)}
\]

\[
\cong \frac{I^QS_*(X \times Y; R)}{(I^QS_*(A \times Y; R) + I^QS_*(X \times B; R))},
\]

using in the last isomorphism the third isomorphism theorem and in the middle isomorphism the second isomorphism theorem, as

\[
I^QS_*(A \times Y; R) \cap I^QS_*(X \times B; R) = I^QS_*(A \times B; R).
\]

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The induced map \( \epsilon \) plays an important role in the next chapter, is that the cross product is a homotopy equivalence.

6.4.5 The cross product is a homotopy equivalence

A remarkable consequence of Theorems 6.56 and 6.60, and one that will play an important role in the next chapter, is that the cross product is a chain homotopy equivalence.

**Theorem 6.61.** Let \( X, Y \) be CS sets of respective dimensions \( n, m \) and with respective perversities \( \bar{p}, \bar{q} \), let \( A \subset X \) and \( B \subset Y \) be open subspaces, let \( R \) be a Dedekind domain, and let \( Q \) be a perversity on \( X \times Y \) satisfying the requirements of Theorem 6.56 (or, equivalently, Theorem 6.60). Then the cross product

\[ \epsilon : I^p S_\ast(X, A; R) \otimes_R I^q S_\ast(Y, B; R) \to I^Q S_\ast(X \times Y, (A \times Y) \cup (X \times B); R) \]

is a chain homotopy equivalence.

**Proof.** We first recall that each \( I^p S_\ast(X, A; R) \), \( I^q S_\ast(Y, B; R) \), and \( I^Q S_\ast(X \times Y, (A \times Y) \cup (X \times B); R) \) is a projective \( R \)-module; see Lemma 6.40. It follows that each \( I^p S_\ast(X, A; R) \otimes I^q S_\ast(Y, B; R) \) is projective. In fact, if \( P \) and \( S \) are projective \( R \)-modules, then there are \( R \)-modules \( U \) and \( V \) such that \( P \oplus U \) and \( S \oplus V \) are each free (recall that a module is free if and
only if it is a direct summand of a free module \([55\text{ Theorem I.4.7}]\). Thus \((P \oplus U) \otimes (S \oplus V)\) is free \([64\text{ Corollary XVI.2.4}]\). But then

\[(P \oplus U) \otimes (S \oplus V) \cong (P \otimes (S \oplus V)) \oplus (U \otimes (S \oplus V)) = (P \otimes S) \oplus (P \otimes V) \oplus (U \otimes S) \oplus (U \times V),\]

so \(P \otimes S\) is a direct summand of a free module.

Since each \(I_{\bar{p}}S_i(X, A; R) \otimes I_{\bar{q}}S_j(Y, B; R)\), so is \(\oplus_{i+j=k} I_{\bar{p}}S_i(X, A; R) \otimes I_{\bar{q}}S_j(Y, B; R)\), and therefore \(I_{\bar{p}}S_*(X, A; R) \otimes I_{\bar{q}}S_*(Y, B; R)\) is a complex of projectives. Under the assumptions of Theorem \([6.60]\), the cross product induces a quasi-isomorphism of these complexes of projectives. This is sufficient for \(\epsilon\) to be a chain homotopy equivalence. This last fact is well known, and we provide a proof as Lemma \([12.7]\) in the Appendix \([12]\). \(\square\)

**6.4.6 Some technical stuff: the proof of Corollary \([6.63]\)**

**Algebra of the algebraic K"unneth theorem.** We need to have a look at the algebraic K"unneth theorem. If \(C_*\) and \(D_*\) are chain complexes of projective modules over the Dedekind domain \(R\), then we know (e.g. \([105\text{ Theorem 3.6.3}]\)) that there is an exact sequence

\[
0 \rightarrow \bigoplus_{i=j+k} H_j(C_*) \otimes H_k(D_*) \rightarrow H_i(C_* \otimes D_*) \rightarrow \bigoplus_{i-1=j+k} H_j(C_*) \ast H_k(D_*) \rightarrow 0.
\]

Furthermore, this sequence splits. However, the splitting is not natural, and this is where we are going to have to take some care.

Given an element \(\sum [x_a] \otimes [y_a] \in H_j(C_*) \otimes H_k(D_*)\), where here \(x_a \in C_j\) and \(y_a \in D_k\) are cycles, the image of this element in \(H_i(C_* \otimes D_*)\) is represented by \(\sum x_a \otimes y_a\). We would like to similarly be able to construct a specific representative in \(H_i(C_* \otimes D_*)\) corresponding to an element of \(H_j(C_*) \ast H_k(D_*)\) under the splitting. In order to do this, we will have to take a closer look at the proof of the algebraic K"unneth theorem. For this, we will begin by following the proof in \([55\text{ Section V.2}]\) with a few minor modifications in the details to pay closer attention to the splitting into summands and under the slightly different assumptions that \(R\) is a Dedekind domain and that \(C_*\) and \(D_*\) are chain complexes of projective \(R\)-modules. The discussion in \([55]\) makes the more general assumption that \(C_*\) and \(D_*\) consist of flat modules but the stronger assumption that \(R\) is a PID; however, the argument there goes through with our assumptions.

We first recall the outline of the proof of the existence of the K"unneth exact sequence. For simplicity of notation, let

\[
\begin{align*}
Z_p & = \ker(\partial : C_p \rightarrow C_{p-1}) \\
B_p & = \text{im}(\partial : C_{p+1} \rightarrow C_p) \\
\bar{Z}_p & = \ker(\partial : D_{p+1} \rightarrow D_{p-1}) \\
\bar{B}_p & = \text{im}(\partial : D_{p+1} \rightarrow D_p).
\end{align*}
\]
As submodules of projective modules over a Dedekind domain, these are each projective modules. We let $Z_*$ be the complex consisting of the modules $Z_p$ and with trivial boundary maps and similarly for $B_*$. We also let $B'_p = B_{p-1}$ so that the map $C_* \to B'_*$ determined by the boundary map is a degree 0 chain map. We then have an exact sequence of degree 0 chain maps

$$0 \longrightarrow Z_* \xrightarrow{i} C_* \xrightarrow{\partial} B'_* \longrightarrow 0. \quad (15)$$

As $D_*$ consists of projective modules,

$$0 \longrightarrow Z_* \otimes D_* \longrightarrow C_* \otimes D_* \longrightarrow B'_* \otimes D_* \longrightarrow 0$$

is exact, yielding a long exact sequence

$$\longrightarrow H_i(Z_* \otimes D_*) \xrightarrow{i \otimes \text{id}} H_i(C_* \otimes D_*) \xrightarrow{\partial \otimes \text{id}} H_i(B'_* \otimes D_*) \longrightarrow .$$

As $B'_*$ has trivial boundary maps, the complex $B'_* \otimes D_*$ has boundary maps $(\text{id} \otimes \partial)$, up to sign. Thus, the complex $B'_* \otimes D_*$ is the direct sum of complexes of the form

$$B'_j \otimes D_{j+k} \xrightarrow{(-1)^j \text{id} \otimes \partial} B'_j \otimes D_k \xrightarrow{(-1)^j \text{id} \otimes \partial} B'_j \otimes D_{k-1} \longrightarrow ,$$

with dimensions shifted so that $B'_j \otimes D_k$ is in dimension $j + k$.

Furthermore, as all modules are projective,

$$\ker(B'_j \otimes D_k \xrightarrow{(-1)^j \text{id} \otimes \partial} B'_j \otimes D_{k-1}) = B'_j \otimes \bar{Z}_k = B_{j-1} \otimes \bar{Z}_k$$

$$\text{im}(B'_j \otimes D_{k+1} \xrightarrow{(-1)^j \text{id} \otimes \partial} B'_j \otimes D_k) = B'_j \otimes \bar{B}_k = B_{j-1} \otimes \bar{B}_k,$$

so that, again using the flatness of $B'_*$ and the right exactness of $\otimes$ and also that the sign of the boundary map does not alter homology computations, we have

$$H_i(B'_j \otimes D_*) \cong B'_j \otimes H_{i-j}(D_*) = B_{j-1} \otimes H_{i-j}(D_*).$$

Therefore, $H_i(B'_* \otimes D_*) \cong \bigoplus_{j+k=i} B'_j \otimes H_k(D_*) = \bigoplus_{j+k=i} B_{j-1} \otimes H_k(D_*)$.

A similar argument shows that $H_i(Z_* \otimes D_*) = \bigoplus_{j+k=i} Z_j \otimes H_k(D_*)$. So we have a long exact sequence

$$\longrightarrow \omega_{i+1} \bigoplus_{j+k=i} Z_j \otimes H_k(D_*) \xrightarrow{i \otimes \text{id}} H_i(C_* \otimes D_*) \xrightarrow{\partial \otimes \text{id}} \bigoplus_{j+k=i-1} B'_{j+1} \otimes H_k(D_*) \xrightarrow{\omega_i} .$$

The connecting map $\omega$ of this exact sequence can be shown (see [55, Section V.2]) to be induced by the inclusions $B_j \otimes H_k(D_*) \to Z_j \otimes H_k(D_*)$, which are equivalent to the maps $\partial C_* \otimes \text{id} : B'_{j+1} \otimes H_k(D_*) \to Z_j \otimes H_k(D_*)$. But, $B'_{j+1} \xrightarrow{\partial C_*} Z_j \to H_j(C_*)$ is a projective resolution of $H_j(C_*)$. So there are, by the definition of the torsion product, exact sequences
0 \longrightarrow H_j(C_s) \ast H_k(D_s) \longrightarrow B'_{j+1} \otimes H_k(D_s) \xrightarrow{\omega_{j,k}} Z_j \otimes H_k(D_s) \longrightarrow H_j(C_s) \otimes H_k(D_s) \longrightarrow 0,

where \( \omega_{j,k} \) is the restriction of \( \omega \) to the given summand \( B_j \otimes H_k(D_s) \). Taking direct sums, we see that \( \text{im}(\partial \otimes \text{id}) = \ker(\omega_i) = \bigoplus_{j+k=i-1} H_j(C_s) \ast H_k(D_s) \) and \( \ker(\partial \otimes \text{id}) = \text{im}(\text{id} \otimes \text{id}) = \text{cok}(\omega_{i+1}) = \bigoplus_{j+k=i} H_j(C_s) \otimes H_k(D_s) \). Hence the Künneth short exact sequence.

For the splitting, as each \( C_j, Z_j, \) and \( B_j = B'_{j+1} \) is projective, we can choose (not necessarily canonically) splittings \( C_j = Z_j \oplus B'_j = Z_j \oplus B_{j-1} \) from the short exact sequence \([15]\), and similarly \( D_k = \tilde{Z}_k \oplus \tilde{B}'_k = \tilde{Z}_k \oplus \tilde{B}_{k-1} \). Therefore, letting \( \phi : C_s \rightarrow Z_s \) and \( \phi : D_s \rightarrow Z_s \) be the splitting maps, we obtain \( \phi \otimes \tilde{\phi} : C_s \otimes D_s \rightarrow Z_s \otimes \tilde{Z}_s \), which induces the splitting of the Künneth exact sequence (again, see \([55, \text{Section V.2}]\)). Notice that, using the distributivity of tensor products over direct sums, we have

\[
C_j \otimes D_k = (Z_j \oplus B'_j) \otimes (Z_k \oplus B'_k) = (Z_j \otimes \tilde{Z}_k) \oplus (Z_j \otimes \tilde{B}'_k) \oplus (B'_j \otimes \tilde{Z}_k) \oplus (B'_j \otimes \tilde{B}'_k),
\]

so that \( \phi \otimes \tilde{\phi} \) is a projection to a summand of each \( C_j \otimes D_k \), and the overall map \( \phi \otimes \tilde{\phi} : C_s \otimes D_s \rightarrow Z_s \otimes \tilde{Z}_s \) is a direct sum of such projections.

Given these choices, we can then recognize \( H_i(C_s \otimes D_s) \) as a direct sum of the image of \( \text{id} \otimes \text{id} : \bigoplus_{i+j+k} H_j(C_s) \otimes H_k(D_s) \rightarrow H_i(C_s \otimes D_s) \) and the kernel of \( \phi \otimes \tilde{\phi} : H_i(C_s \otimes D_s) \rightarrow \bigoplus_{i+j+k} H_j(C_s) \otimes H_k(D_s) \), which is isomorphic to \( \bigoplus_{i-1+j+k} H_j(C_s) \ast H_k(D_s) \). Our next goal is to find representatives in \( C_s \otimes D_s \) of the elements of this kernel. In other words, we want to find representative cycles for the elements of the torsion product summands of \( H_i(C_s \otimes D_s) \).

In general, these are non-canonical, but we will use our choices of splittings.

So, consider once again the projective resolution

\[
B'_{j+1} \xrightarrow{\partial} Z_j \xrightarrow{\eta} H_j(C_s).
\]

We also have the projective resolution

\[
\tilde{B}'_{k+1} \xrightarrow{(-1)^j \partial} \tilde{Z}_k \xrightarrow{\tilde{\eta}} H_k(D_s).
\]

Let us denote the chain complex \( \cdots \rightarrow 0 \rightarrow B'_{j+1} \xrightarrow{\partial} Z_j \rightarrow 0 \rightarrow \cdots \), with \( Z_j \) in degree 0, as \( P'_{j+1} \), and define \( Q^k \) similarly using our second projective resolution. Above, we realized the torsion product module \( H_j(C_s) \ast H_k(D_s) \) as \( \ker(\partial \otimes \text{id}) \), which is \( H_1(P'_{j+1} \otimes H_k(D_s)) \). Recall, however, that \( H_j(C_s) \ast H_k(D_s) \) is also given as \( H_1(P'_{j+1} \otimes Q^k) \) and that the isomorphism between \( H_1(P'_{j+1} \otimes Q^k) \) and \( H_1(P'_{j+1} \otimes H_k(D_s)) \) is induced by \( \text{id} \otimes \tilde{\eta} \). This follows from the proof of \([105, \text{Theorem 2.7.2}]\), concerning “balancing” the Tor and Ext functors.\(^{80}\) In the case at hand, \( (P'_{j+1} \otimes Q^k)_{\text{Tot}} = (B'_{j+1} \otimes \tilde{Z}_k) \oplus (Z_j \otimes \tilde{B}'_{k+1}) \). Therefore, every element of \( H_j(C_s) \ast H_k(D_s) \) can be

\(^{80}\)Caution: in \([105, \text{Section 2.7}]\), Weibel uses the notation \( P \otimes Q \) to denote the double complex and \( \text{Tot}^k(P \otimes Q) \) for the single complex, i.e. \( \text{Tot}^k(P \otimes Q) = \bigoplus_{a+b=k} P_a \otimes Q_b \). Notice also that if we use the given gradings for \( P'_j \) and \( Q^k \) with \( Z_j \) and \( Z_k \) each in degree 0 and \( B'_{j+1} \) and \( B'_{k+1} \) each in degree 231
represented by a cycle $\xi \in (B'_{j+1} \otimes Z_k) \oplus (Z_j \otimes B'_{k+1})$ representing an element $H_1(P_1 \otimes Q_1)$ and the corresponding representative cycle in $H_1(P_1 \otimes H_k(D_*))$ is precisely $(\text{id} \otimes \bar{\eta})(\xi)$. 

Now, consider such a cycle $\xi \in (B'_{j+1} \otimes \bar{Z}_k) \oplus (Z_j \otimes B'_{k+1})$ and the corresponding cycle $\bar{\xi} = (\text{id} \otimes \bar{\eta})(\xi)$ in $P_1 \otimes H_k(D_*)$. By our (non-canonical) choice of splitting we can consider $B'_{j+1}$ as a summand of $C_{j+1}$. Similarly, we can consider $B'_{k+1}$ as a summand of $D_{k+1}$, and the $Z_j$ and $Z_k$ are canonical submodules of $C_j$ and $D_k$. Using these identifications, $\xi$ becomes an element of $(C_{j+1} \otimes D_k) \oplus (C_j \otimes D_{k+1})$. As the boundary maps in $P_j$ and $Q_k$ were chosen to agree with those in $C_*$ and $D_*$ up to the necessary signs for compatibility with the total complex, $\xi$ is a cycle in $C_\ast \otimes D_\ast$.

Now, consider $\text{id} \otimes \bar{\eta} : P_1 \otimes Q_1 \rightarrow P_1 \otimes H_k(D_*)$, which, by definitions acts on such a $\xi$ to give us $\bar{\xi}$, which we can consider to be an element of a torsion summand of $H_1(C_\ast \otimes D_\ast)$ via our initial Künneth theorem constructions. The piece of $\xi$ in the summand $Z_j \otimes B'_{k+1}$ goes to 0; as $Q_1^k$ is a resolution of $H_k(D_*)$, we interpret $\bar{\eta}$ as a chain map to the complex with $H_k(D_*)$ in degree 0 as the only nontrivial term. The piece of $\xi$ in $B'_{j+1} \otimes \bar{Z}_k$, say $\sum b_l \otimes \bar{z}_l$ with $b_l \in B'_{j+1}$ and $\bar{z}_l \in \bar{Z}_k$, gets taken to $\sum b_l \otimes \bar{\eta}(\bar{z}_l) = \sum b_l \otimes \bar{\eta}(\bar{z}_l)$, with $[\bar{z}_l]$ denoting the homology class of $\bar{z}_l$ in $H_k(D_*)$. But now notice that this is exactly the image of $\xi$ also under the map labeled $\partial \otimes \text{id}$ in diagram (16). Notice that the $\partial$ map in that labeling corresponds to $\partial : C_{j+1} \rightarrow B'_{j+1}$, which we can identify as the projection to the boundary summand; restricted to $B'_{j+1}$ considered as a submodule of $C_{j+1}$ via the splitting, this map functions as an identity, hence the different notation in our two discussions. Altogether, this shows that if we consider $\xi$ as representing a cycle in $H_1(C_\ast \otimes D_\ast)$, then the image of $\xi$ under the map $\partial \otimes \text{id}$ of diagram (16) is precisely $\bar{\xi}$.

So what does this show? We have demonstrated that, if $j + k = i - 1$, then $\xi$ represents a homology class in $H_i(C_\ast \otimes D_\ast)$ that maps onto the class represented by $\bar{\xi}$ in $H_j(C_\ast) \ast H_k(D_\ast)$ by the map in the Künneth short exact sequence as we have constructed it. Furthermore, it is clear that $\xi$, as an element of $(B'_{j+1} \otimes \bar{Z}_k) \oplus (Z_j \otimes B'_{k+1})$, cannot map non-trivially under the map $\partial \otimes \text{id}$ of (16) to any other summand $H_a(C_\ast) \ast H_b(D_\ast)$ with $a \neq j$ or $b \neq k$, and $\xi$ is taken to 0 by the splitting map $\phi \otimes \bar{\phi}$. Thus $\xi$ is a cycle in $C_\ast \otimes D_\ast$ that represents precisely an element in the summand $H_j(C_\ast) \ast H_k(D_\ast)$ in the given splitting of $H_i(C_\ast \otimes D_\ast)$. As $\text{id} \otimes \bar{\eta}$ is an isomorphism between the two constructions of $H_j(C_\ast) \ast H_k(D_\ast)$, we see that for any $\bar{\xi} \in H_j(C_\ast) \ast H_k(D_\ast)$, we can construct a corresponding element $[\xi] \in H_i(C_\ast \otimes D_\ast)$ in this way. We will not investigate whether the assignment $\xi \rightarrow \bar{\xi}$ is a homomorphism; such a homomorphism will not be necessary below.

**Intersection homology products with cones.** Now, we wish to consider $j \otimes \text{id} : \mathcal{P}_i^iS_*cX - \{v\}; \mathcal{Y} \otimes \mathcal{P}_i^iS_*\mathcal{Y}; R) \rightarrow \mathcal{P}_i^iS_*cX; R) \otimes \mathcal{P}_i^iS_*\mathcal{Y}; R)$, induce by the inclusion $j : cX - \{v\} \rightarrow cX$. Here $X$ and $\mathcal{Y}$ are filtered spaces, and we suppose that $X$ has dimension $n - 1$ and that $R$ is a Dedekind domain. The naturality of the algebraic Künneth
Theorem gives us a map of short exact sequences (with $R$ coefficients tacit)

\[
\bigoplus_{i=j+k} I^p H_j(cX - \{v\}) \otimes I^q H_k(Y) \hookrightarrow H_i(I^p S_* (cX - \{v\}) \otimes I^q S_* (Y)) \twoheadrightarrow \bigoplus_{i=1+j+k} I^p H_j(cX - \{v\}) \ast I^q H_k(Y)
\]

\[
\bigoplus_{i=j+k} I^p H_j(cX) \otimes I^q H_k(Y) \hookrightarrow H_i(I^p S_* (cX) \otimes I^q S_* (Y)) \twoheadrightarrow \bigoplus_{i=1+j+k} I^p H_j(cX) \ast I^q H_k(Y).
\]

(17)

We also know that the inclusion $cX - \{v\} \hookrightarrow cX$ induces an isomorphism $I^p H_j(cX - \{v\}; R) \to I^p H_j(cX; R)$ for $j < n - \bar{p}(\{v\}) - 1$ and that $I^p H_j(cX; R) = 0$ for $j \geq n - \bar{p}(\{v\}) - 1$. Therefore, naively, we expect that the map $H_i(I^p S_* (cX - \{v\}; R) \otimes I^q S_* (Y; R)) \to H_i(I^p S_* (cX; R) \otimes I^q S_* (Y; R))$ should kill the summands $I^p H_j(cX - \{v\}; R) \otimes I^q H_k(Y; R)$ or $I^p H_j(cX - \{v\}; R) \ast I^q H_k(Y; R)$ with $j \geq n - \bar{p}(\{v\}) - 1$ and take the summands with $j < n - \bar{p}(\{v\}) - 1$ isomorphically to their counterparts in the image.

Unfortunately, the algebraic Künneth exact sequences do not split naturally, in general, and so it is conceivable for there to be unexpected subtleties. For example, one can construct maps $\phi \otimes \psi : C_* \otimes D_* \to C'_* \otimes D'_*$ for which, in the ensuing diagram of Künneth exact sequences, the torsion product summand of $H_i(C'_* \otimes D'_*)$ is 0 and the tensor product summand for $H_i(C_* \otimes D_*)$ is 0, but for which the map $H_i(C_* \otimes D_*) \to H_i(C'_* \otimes D'_*)$ is not trivial; see \[55\].

Therefore, to verify our expectations, we cannot rely solely on algebra but must also utilize the topology involved.

**Lemma 6.62.** Suppose $Y$ is a filtered space with perversity $\bar{q}$, that $X$ is a dimension $n - 1$ filtered space, and that $R$ is a Dedekind domain. Consider $j \otimes \text{id} : I^p S_* (cX - \{v\}; R) \otimes I^q S_* (Y; R) = I^p S_* (cX; R) \otimes I^q S_* (Y; R)$, induced by the inclusion $i : cX - \{v\} \to cX$. Then, in the diagram labeled (17), the splittings of the short exact sequences can be chosen so that each summand $I^p H_j(cX - \{v\}; R) \otimes I^q H_k(Y; R)$ or $I^p H_j(cX - \{v\}; R) \ast I^q H_k(Y; R)$ with $j \geq n - \bar{p}(\{v\}) - 1$ maps to 0 in $H_i(I^p S_* (cX; R) \otimes I^q S_* (Y; R))$ and so that each summand with $j < n - \bar{p}(\{v\}) - 1$ maps to a corresponding summand with identical generators in $H_i(I^p S_* (cX; R) \otimes I^q S_* (Y; R))$.

**Proof.** Recall the following facts from the proof of the cone formula in Theorem 4.12. If $j < n - \bar{p}(\{v\})$ then no allowable simplex can intersect $v$ at all, and so in this range we have $I^p S_j(cX; R) = I^p S_j(cX - \{v\}; R)$. On the other hand, if $j \geq n - \bar{p}(\{v\})$, then simplices may intersect the vertex and, in fact, any cone $\bar{c}(\xi)$ on a chain $\xi \in I^p S_j(cX - \{v\}; R)$ is allowable so long as either $j \geq n - \bar{p}(\{v\})$ or $j = n - \bar{p}(\{v\}) - 1$ with $\partial \xi = 0$ (as in order for $\bar{c}(\xi)$ to be allowable, we also need $\partial(\bar{c}(\xi)) = \xi - \bar{c}(\partial \xi)$ to be allowable - also note that forming the cone adds 1 to the dimension of the chain).

So let us see how this plays out for our map $H_i(I^p S_* (cX - \{v\}; R) \otimes I^q S_* (Y; R)) \to H_i(I^p S_* (cX; R) \otimes I^q S_* (Y; R))$, working one summand at a time.

First, let us consider the tensor product summands with domain $I^p H_j(cX - \{v\}; R) \otimes I^q H_k(Y; R)$. By the naturality of the Künneth exact sequences, the tensor product summand of $H_i(I^p S_* (cX - \{v\}; R) \otimes I^q S_* (Y; R))$ gets taken to the tensor product summand of
$H(I^pS_*(cX; R) \otimes I^qS_*(Y; R))$, so to understand the map on this summand, it is enough to look at each $j \otimes \text{id} : I^pH_j(cX - \{ v \}; R) \otimes I^qH_k(Y; R) \rightarrow I^pH_j(cX; R) \otimes I^qH_k(Y; R)$. But this is straightforward: Each tensor product summand $I^pH_j(cX - \{ v \}; R) \otimes I^qH_k(Y; R)$ is generated by elements of the form $[z] \otimes [y] \in I^pH_j(cX - \{ v \}; R) \otimes I^qH_k(Y; R)$, which map to the corresponding elements of $I^pH_j(cX; R) \otimes I^qH_k(Y; R)$. So if $j \geq n - \bar{p}(\{ v \}) - 1$, the summand maps to 0 and if $j < n - \bar{p}(\{ v \}) - 1$ the summand maps isomorphically to the corresponding tensor product summand $I^pH_j(cX; R) \otimes I^qH_k(Y; R)$. In fact, since, in this latter case, we have $j < n - \bar{p}(\{ v \})$, we are in the range where $I^pS_j(cX; R) = I^pS_j(cX - \{ v \}; R)$ and so the map takes summand generators to the precisely corresponding generators in the image.

Now, we must consider the torsion product summands. This is more subtle, as the lack of natural splitting in the K"unneth exact sequences means that we cannot simply assume that the torsion product summand of $H_I^*(I^pS_*(cX; R) \otimes I^qS_*(Y; R))$ maps only to the torsion product summand of $H_I^*(I^pS_*(cX; R) \otimes I^qS_*(Y; R))$. In our algebraic discussion above, we constructed elements $\xi$ of $I^pS_*(cX - \{ v \}; R) \otimes I^qS_*(Y; R)$ corresponding to elements in torsion product summands of $H_I^*(I^pS_*(cX - \{ v \}; R) \otimes I^qS_*(Y; R))$. In particular, corresponding to each element of the summand $I^pH_j(cX - \{ v \}; R) \otimes I^qH_k(Y; R)$, we can find a representative cycle $\xi$ in $(B_{j+1} \otimes Z_k) \oplus (Z_j \otimes B_{k+1}^*)$, where now we let $B_{j+1}', B_{k+1}', Z_j$, and $Z_k$ be cycle and boundary submodules as in the discussion above, taking $C_* = I^pS_*(cX - \{ v \}; R)$ and $D_* = I^qS_*(Y; R)$. We also assume a choice of splitting so that $B_{j+1}'$ is a summand of $C_{j+1}$ and similarly for $B_{k+1}'$ and $D_{k+1}$. In particular, then, $\xi$ must be of the form

$$\xi = \sum \tilde{b}_t \otimes \tilde{z}_t + \sum z_m \otimes \bar{b}_m$$

with $b_t \in B_{j+1}'$, $\tilde{z}_t \in \bar{Z}_k$, $z_m \in Z_j$ and $\bar{b}_m \in B_{k+1}'$. As $\xi$ is a cycle, we must have

$$0 = \partial \xi = \sum \tilde{b}_t \partial \tilde{z}_t + \sum z_m \partial \bar{b}_m,$$

utilizing that the $z_m$ and $\tilde{z}_t$ are cycles (the $b_t$ and $\bar{b}_m$ map to boundaries but are not themselves necessarily boundaries). Let us see what $\xi$ maps to in homology under $j \otimes \text{id}$.

First, suppose that $j \geq n - \bar{p}(\{ v \}) - 1$. Let

$$\zeta = \sum \tilde{c}(b_t) \otimes \tilde{z}_t + \sum \tilde{c}(z_m) \otimes \bar{b}_m.$$

The chain $\zeta$ is allowable, by our above observations and recalling that the $z_m$ are cycles.
Then we have

\[ \partial \zeta = \partial \left[ \sum_{\ell} (\tilde{c}(b_{\ell}) \otimes \bar{z}_{\ell} + \sum_{m} (\tilde{c}(z_{m}) \otimes \bar{b}_{m}) \right] \\
= \sum_{\ell} (\partial \tilde{c}(b_{\ell}) \otimes \bar{z}_{\ell} + \sum_{m} ((\partial \tilde{c}(z_{m}) \otimes \bar{b}_{m} + (-1)^{j+1}\tilde{c}(z_{m}) \otimes \partial \bar{b}_{m}) \\
= \sum_{\ell} (b_{\ell} - \tilde{c}(\partial b_{\ell}) \otimes \bar{z}_{\ell} + \sum_{m} (z_{m} \otimes \bar{b}_{m} + (-1)^{j+1}\tilde{c}(z_{m}) \otimes \partial \bar{b}_{m}) \\
= \sum_{\ell} b_{\ell} \otimes \bar{z}_{\ell} + \sum_{m} z_{m} \otimes \bar{b}_{m} - \sum_{\ell} \tilde{c}(\partial b_{\ell}) \otimes \bar{z}_{\ell} + (-1)^{j+1}\sum_{m} \tilde{c}(z_{m}) \otimes \partial \bar{b}_{m} \\
= \xi - \left[ \sum_{\ell} \tilde{c}(\partial b_{\ell}) \otimes \bar{z}_{\ell} + (-1)^{j}\sum_{m} (\tilde{c}(z_{m}) \otimes \partial \bar{b}_{m}) \right]. \]

Observe now that \( \xi - \partial \zeta \) looks just like our expression above for \( \partial \xi \) except that the first term in each tensor product now has a cone in the expression. We claim that \( \xi - \partial \zeta = 0 \).

Recall that each of our intersection chain modules is a submodule of an ordinary singular chain module. Therefore, we can write each of the \( b_{\ell} \), etc. as a linear combination of ordinary singular simplices \( \{\sigma_{a}\} \) and \( \{\tau_{b}\} \), which constitute bases of the ordinary singular chain modules \( S_{j}(cX - \{v\}; R) \) and \( S_{k}(Y; R) \). Furthermore, as the intersection chain complexes are projective, \( I^{p}S_{a}(cX - \{v\}; R) \otimes I^{q}S_{b}(Y; R) \) is a submodule of \( S_{j}(cX - \{v\}; R) \otimes S_{k}(Y; R) \), which has basis \( \{\sigma_{a} \otimes \tau_{b}\} \). As \( \partial \xi = 0 \), it follows that if we take \( \sum_{\ell} (\partial b_{\ell}) \otimes \bar{z}_{\ell} + (-1)^{j}\sum_{m} z_{m} \otimes \partial \bar{b}_{m} \) and write out each \( (\partial b_{\ell}, \bar{z}_{\ell}, z_{m}, \partial \bar{b}_{m}) \) in terms of \( \{\sigma_{a}\} \) or \( \{\tau_{b}\} \), then we obtain the 0 element in \( S_{j}(cX - \{v\}; R) \otimes S_{k}(Y; R) \). But now we further recall that \( \tilde{c}(z_{m}) \in I^{p}S_{j+1}(cX; R) \subset S_{j+1}(cX; R) \) is obtained from the expression for \( z_{m} \) in terms of the \( \{\sigma_{a}\} \) by replacing each simplex \( \sigma_{a} \) with the simplex \( \tilde{c}(\sigma_{a}) \), and similarly for \( \tilde{c}(\sigma_{a}) \). Therefore, we have an expression for \( \xi - \partial \zeta \) in terms of \( \{\tilde{c}(\sigma_{a}) \otimes \tau_{b}\} \) that is identical, in terms of coefficients, to the expression for \( \partial \zeta \) in terms of \( \{\sigma_{a} \otimes \tau_{b}\} \). As the assignments \( \sigma_{a} \otimes \tau_{b} \rightarrow \tilde{c}(\sigma_{a}) \otimes \tau_{b} \) determine a linear isomorphism from the free module \( S_{j}(cX - \{v\}; R) \otimes S_{k}(Y; R) \) onto a linear submodule of \( S_{j}(cX; R) \otimes S_{k}(Y; R) \) (the submodule spanned by the generators \( \{\tilde{c}(\sigma_{a}) \otimes \tau_{b}\} \)), it follows from \( \partial \xi = 0 \in S_{j}(cX - \{v\}; R) \otimes S_{k}(Y; R) \) that \( \xi - \partial \zeta = 0 \in S_{j}(cX; R) \otimes S_{k}(Y; R) \)

and so also in the submodule \( I^{p}S_{j}(cX; R) \otimes I^{q}S_{k}(Y; R) \). We conclude that the image of \( \xi \) represents 0 in \( H_{i}(I^{p}S_{a}(cX; R) \otimes I^{q}S_{b}(Y; R)) \).

Thus, for \( j \geq n - \bar{p}(\{v\}) - 1 \), each of the \( I^{p}H_{j}(cX - \{v\}; R) \otimes I^{q}H_{k}(Y; R), j + k = i - 1 \), summands of \( H_{i}(I^{p}S_{a}(cX - \{v\}; R) \otimes I^{q}S_{b}(Y; R)) \) maps to 0 in \( H_{i}(I^{p}S_{a}(cX; R) \otimes I^{q}S_{b}(Y; R)) \).

Next, consider the torsion product summands with \( j < n - \bar{p}(\{v\}) - 1 \). These are also generated by elements of the form of our \( \xi \) and so, in particular, are contained in summands of \( I^{p}S_{a}(cX - \{v\}; R) \otimes I^{q}S_{b}(Y; R) \) of the form

\[ [(I^{p}S_{j+1}(cX - \{v\}; R) \otimes I^{q}S_{k}(Y; R)) \oplus [(I^{p}S_{j}(cX - \{v\}; R) \otimes I^{q}S_{k+1}(Y; R)]. \]

But in this case, \( j + 1 < n - \bar{p}(\{v\}) \), so here we are in the range where \( I^{p}S_{a}(cX - \{v\}; R) = I^{p}S_{a}(cX; R) \). In particular, we can assume that the splittings into kernels and boundaries are identical for the relevant identical \( I^{p}S_{a}(cX - \{v\}; R) \) and \( I^{p}S_{a}(cX; R) \). Thus our construction
above of these particular torsion product summands in $H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(Y; R))$ is identical to the construction in $H_i(I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(Y; R))$, and it follows that each $\xi \in I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(Y; R)$ as constructed above to represent an element of such a summand is taken to the identically corresponding choice of cycle in $I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(Y; R)$.

Altogether then, we can conclude that the map $H_i(I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(Y; R)) \to H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(Y; R))$ is the naive one: it takes any summand $I^\ddot{p}H_j(cX - \{v\}; R) \otimes I^\ddot{q}H_k(Y; R)$ to 0 if $j \geq n - \overline{p}(\{v\}) - 1$, and otherwise it takes it to an identically corresponding summand of $H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(Y; R))$.

**Corollary 6.63.** Given a Dedekind domain $R$ and compact filtered sets $X = X^{n-1}$ and $Y = Y^{m-1}$, there are splittings of $H_i(I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(cY - \{w\}; R))$, $H_i(I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(cY; R))$, and $H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(Y; R))$ into direct sums of tensor products $I^\ddot{p}H_j(cX - \{v\}; R) \otimes I^\ddot{q}H_k(cY - \{w\}; R)$, $j + k = i$, and torsion products $I^\ddot{p}H_j(cX - \{v\}; R) \ast I^\ddot{q}H_k(cY - \{w\}; R)$, $j + k = i - 1$, such that the maps in the diagram

$$H_i(I^\ddot{p}S_*(cX - \{v\}; R) \otimes I^\ddot{q}S_*(cY - \{w\}; R)) \xrightarrow{j \otimes \text{id}} H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(cY - \{w\}; R))$$

induced by the inclusions $j : cX - \{v\} \hookrightarrow cX$ and $\xi : cY - \{w\} \hookrightarrow cY$ each restrict on each tensor or torsion product summand either to the 0 map or to an isomorphism with the corresponding summand in the codomain. Furthermore, which of these options is determined in the obvious way; for example, the tensor product summand $I^\ddot{p}H_j(cX - \{v\}; R) \otimes I^\ddot{q}H_k(cY - \{w\}; R)$ maps to 0 in $H_i(I^\ddot{p}S_*(cX; R) \otimes I^\ddot{q}S_*(cY - \{w\}; R))$ when $j \geq n - \overline{p}(\{v\}) - 1$ and isomorphically otherwise.

**Proof.** The existence of such properties holds independently for each map in the diagram due to Lemma 6.62 However, we must also verify that the choices can be made compatibly.

First, consider the tensor product summands of the expressions in the diagram. The naturality of the Künneth theorem tells us that the tensor product summands due to the Künneth exact sequence always map to tensor product summands, and, in particular, the chain map $j \otimes \text{id}$ induces the corresponding tensor product of homology maps (which we could write $j_* \otimes \text{id}$ to avoid ambiguity), and similarly for the other maps. Therefore, the maps on the tensor product summands behave as expected, and restriction to these summands yields a commutative diagram of the form of the diagram of the corollary.

Now we must consider the torsion product summands. For this, we can assume that we choose once and for all fixed splittings of each $I^\ddot{p}S_j(cX - \{v\}; R)$ and $I^\ddot{q}S_k(cY - \{w\}; R)$ into kernels and boundaries. First, consider the top horizontal map of the diagram of the corollary statement. We showed in Lemma 6.62 that each torsion product summand $I^\ddot{p}H_j(cX -
7 Intersection cohomology and products

7.1 Intersection cohomology

So far, we have focused exclusively on intersection homology. It is time to introduce intersection cohomology. Throughout, we will work with coefficients in a commutative ring with unity \( R \), though for certain results we will need to assume that \( R \) is a Dedekind domain or a field. Recall that we have already seen such coefficient restrictions in Sections 5.4 and 6.4.

We now define cochains and cohomology in the customary way:

**Definition 7.1.** Let \( R \) be a commutative ring with unity, let \( (X, A) \) be a filtered space together \( X \) with a subset \( A \), and let \( \bar{p} \) be a perversity. We define the **intersection cochain complex**

\[
I^\bar{p} S^*(X, A; R) = \text{Hom}_R(I^\bar{p} S^*(X, A; R), R).
\]

We will denote the coboundary operator by \( d \). Following the Koszul sign conventions (see, e.g., [23] pages 151 and 167), if \( \alpha \in I^\bar{p} S^i(X, A; R) \) and \( x \in I^\bar{p} S_{i+1}(X, A; R) \), then \((da)(x) = (-1)^{i+1} \alpha(\partial x)\).
The associated intersection cohomology modules are

\[ I_\bar{p}H^i(X, A; R) = H^i(I_\bar{p}S^*(X, A; R)) = H^i(\text{Hom}_R(I^pS_e(X, A; R), R)). \]

Similarly, if \( X \) is a PL filtered space with PL subspace \( A \), we let \( I_\bar{p}\mathfrak{C}^*(X, A; R) = \text{Hom}_R(I^p\mathfrak{C}_e(X, A; R), R) \) and \( I_\bar{p}\mathfrak{C}^i(X, A; R) = H^i(I_\bar{p}\mathfrak{C}^*(X, A; R)) = H^i(\text{Hom}_R(I^p\mathfrak{C}_e(X, A; R), R)) \).

Notice that, in the notation, both the degree index and the perversity index both shift their subscript/superscript locations. Of course, shifting the degree index is standard between homology and cohomology; the additional shifting of the perversity index is simply meant as an additional visual cue.

**Remark 7.2.** We will not use it here, but one could just as easily define \( I_\bar{p}S^*(X, A; M) = \text{Hom}_R(I^pS_e(X, A; R), M) \) and the corresponding cohomology groups for any \( R \)-module \( M \).

We could also define versions of intersection cohomology based on the complexes \( I^pS^\text{GM}_e(X) \) and \( I^p\mathfrak{C}^\text{GM}_e(X) \), but since our reason for introducing cohomology is to introduce cup products and as these require non-GM intersection cohomology, we will not pursue separately the properties of GM intersection cohomology groups. Recall, though, that when \( \bar{p} \leq \bar{t} \), there is no difference between the GM and non-GM theories, by Proposition 6.7.

**Remark 7.3.** From this point onward, when the ground ring is understood, we will use the notations \( \text{Hom} \) and \( \otimes \), rather than the more cumbersome \( \text{Hom}_R \) and \( \otimes_R \).

When \( R \) is a Dedekind domain, the universal coefficient theorem holds:

**Theorem 7.4.** For a Dedekind domain \( R \) and for every \( i \), there is a noncanonically split natural exact sequence

\[ 0 \to \text{Ext}(I^pH_{i-1}(X, A; R), R) \to I_\bar{p}H^i(X, A; R) \to \text{Hom}(I^pH_i(X, A; R), R) \to 0. \]

Additionally, if \( F \) is a field, then for PL intersection cohomology we have \( I_\bar{p}H^i(X, A; F) \cong \text{Hom}(I^pH_i(X, A; F), F) \).

**Proof.** This is just a special case of the algebraic universal coefficient theorem [105, Section 3.6], for which, for the singular chain case, we only need verify that each \( I^pS_i(X, A; R) \) and \( \partial I^pS_i(X, A; R) \) are projective. But these are respective submodules of the free modules \( S_i(X, A; R) \) and \( S_{i-1}(X; R) \) using the definitions of Section 6.2. Since \( R \) is assumed Dedekind, submodules of projective modules are projective, and the theorem follows.

The difficult with the PL case is that \( \mathfrak{C}_e(X; R) \) is not a free module (or even projective) in any evident way for a general ring. Therefore, even when \( R \) is Dedekind, we cannot conclude that its submodules are necessarily projective, and, therefore, we cannot invoke the needed algebraic results. Of course, if \( R \) is a field, then all modules are free, so in this case we can apply the algebraic theorem, with the Ext term vanishing.

\[ \square \]

Notice that this universal coefficient theorem for cohomology is not inconsistent with the general failure of the intersection homology universal coefficient theorem for homology discussed in Section 5.4.1. There, we introduced \( I^pS_e(X; R) \) not as \( I^pS_e(X) \otimes G \) but as
an “intersection” version of $S_*(X; R)$. Here, by contrast, we define $I_\rho S^*(X; R)$ in terms of $I^p S_*(X; R)$ in the standard way.

Next, we seek to establish that most of the various properties we have established for intersection homology carry over to intersection cohomology in the expected way, sometimes with restrictions, though we reserve discussion of products, including cohomology cross products, to Section 7.2. In order to transport some of these results from homology to cohomology, we will need to utilize two technical results that strengthen what we already know about intersection chains. We also state and prove these results for GM intersection chains, though those cases will not be utilized in our treatment of cohomology. Furthermore, since these are statements about chains, we do not need to worry about rings, so we take as coefficients an abelian group $G$. In particular, we have the following:

**Proposition 7.5.** Let $\mathcal{V}$ be a covering of $X$ such that the interiors of the elements of $\mathcal{V}$ constitute an open covering of $X$, let $A \subset X$, and let $I^p S^*_V(X; A; G) = \sum_{V \in \mathcal{V}} I^p S_*(V, A \cap V; G) \subset I^p S_*(X; A; G)$. Define $I^p S^*_V(X; A; G)$ analogously. Then the inclusions $I^p S^*_V(X, A; G) \hookrightarrow I^p S^*_V(X; A; G)$ and $I^p S_*(X, A; G) \hookrightarrow I^p S_*(X; A; G)$ are chain homotopy equivalences. For PL chains and $A$ a PL subset, the corresponding inclusions $I^p c^*_V(X, A; G) \hookrightarrow I^p c^*_V(X; A; G)$ and $I^p c_*(X, A; G) \hookrightarrow I^p c_*(X; A; G)$ are isomorphisms.

**Lemma 7.6.** Let $A$ be an open subset of the filtered space $X$. Then the maps $I^p S^*_i(A; G) \to I^p S^*_i(X; G)$ and $I^p S_*(A; G) \to I^p S_*(X; G)$ induced by inclusion are split inclusions.

We will prove these technical results at the end of this section, after demonstrating their usefulness. The following statements are the direct analogues of the results we have already established for intersection homology. Most follow almost directly from the intersection homology cases, in which case we will just briefly provide the relevant reasons. Some, however, require more elaboration, which we provide.

**Proposition 7.7.** If $X, Y$ are filtered spaces, $f : X \to Y$ is $(\bar{p}, \bar{q})$-stratified, and $A \subset X$ and $B \subset Y$ with $f(A) \subset B$, then $f$ induces a chain map $f^* : I_\bar{q} S^*(Y, B; R) \to I_\bar{p} S^*(X, A; R)$. If, additionally, $X, Y$ are PL filtered spaces, $A, B$ are PL subspaces, and $f$ is a PL map, then $f$ induces a chain map $f^* : I_\bar{q} c^*(Y, B; R) \to I_\bar{p} c^*(X, A; R)$ of PL intersection chain complexes. In either case, we obtain corresponding maps of intersection cohomology groups.

**Proof.** By Proposition 6.13 these are the conditions that allow the existence of maps of intersection chain complexes, and so the $f^*$ here are just the resulting Hom duals.

**Corollary 7.8.** If $f : X \to Y$ is a stratified homeomorphism that is also a homeomorphism of pairs $f : (X, A) \to (Y, B)$ and the perversities $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ correspond, then $I_\bar{p} H^*(X, A; R) \cong I_\bar{q} H^*(Y, B; R)$. The corresponding fact holds for PL spaces, PL stratified homeomorphisms, and PL intersection cohomology.

**Proof.** As for Corollary 4.7 the maps induce isomorphisms of the intersection chain complexes, and so dually they induce isomorphisms of cochain complexes.
**Proposition 7.9.** Suppose $f, g : X \to Y$ are $(\bar{p}, \bar{q})$-stratified maps that are $(\bar{p}, \bar{q})$-stratified homotopic via a $(\bar{p}, \bar{q})$-stratified homotopy taking the pair $(I \times X, I \times A)$ to $(Y, B)$. Then $f$ and $g$ induce chain homotopic maps $I_\bar{q}S^i(Y, B; R) \to I_\bar{p}S^i(X, A; R)$ and so $f = g : I_\bar{q}H^*(Y, B; R) \to I_\bar{p}H^*(X, A; R)$. The analogous result holds in the PL category.

**Proof.** By Proposition 6.16, $f$ and $g$ induce chain homotopic maps of intersection chain complexes, and so we obtain the corresponding results about cochains by dualizing. 

The following corollary is an immediate consequence of the proposition and the fact that the duals of chain homotopies are (co)chain homotopies.

**Corollary 7.10.** Suppose $f : X \to Y$ is a stratified map with a stratified homotopy inverse $g$ such that $f, g$ are also maps of pairs $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ and the homotopies used to demonstrate that $f$ and $g$ are stratified homotopy inverses are also homotopies of pairs $I \times (X, A) \to (X, A)$ and $I \times (Y, B) \to (Y, B)$. Suppose that the values of $\bar{p}$ on $X$ and $\bar{q}$ on $Y$ agree on corresponding strata. Then $f$ induces an isomorphism $I_\bar{q}H^*(Y, B; R) \cong I_\bar{p}H^*(X, A; R)$. The analogous result holds in the PL category.

**Theorem 7.11.** For a filtered space $X$ and subspace $A$, if either $A$ is an open subset or $R$ is a Dedekind domain, there is a long exact sequence

$$\cdots \to I_\bar{p}H^i(X, A; R) \to I_\bar{p}H^i(X; R) \to I_\bar{p}H^i(A; R) \to I_\bar{p}H^{i+1}(X, A; R) \to \cdots.$$ 

The same is true of PL intersection cohomology using field coefficients.

**Proof.** By Lemma 7.6, if $A$ is open, the short exact sequence of intersection chain complexes of the pair splits in each dimension (the splitting is not necessarily a chain map, but it doesn’t need to be). Similarly, if $R$ is Dedekind, then each $PS_i(X, A; R) \subset S_i(X, A; R)$ is projective as a submodule of a free module, and so the short exact sequence splits in each dimension by [53] Theorem 1.4.7. Therefore, applying the functor $\text{Hom}(\cdot, R)$ preserves exactness to yield the short exact sequence of the pair for intersection cochains. The existence of the long exact sequence of intersection cohomology follows.

In the PL case, field coefficients are necessary to ensure the splittings.

**Theorem 7.12.** Let $X$ be a filtered space, and suppose $K \subset U \subset X$ such that $\bar{K} \subset \bar{U}$. Then inclusion induces an isomorphism $I_\bar{p}H^*(X, U; R) \cong I_\bar{p}H^*(X - K, U - K; R)$. The analogous statement holds for PL chains.

**Proof.** First consider singular chains. We will show that the inclusion $I_\bar{p}S_*(X - K, U - K; R) \to I_\bar{p}S_*(X, U; R)$ is a chain homotopy equivalence. Let $\mathcal{V} = \{X - K, U\}$ be a covering of $X$. Notice that the interiors of the sets of $\mathcal{V}$ continue to be a cover by the assumptions on $\bar{K}$ and $\bar{U}$. By Proposition 7.5, the inclusion $i : PS_\mathcal{V}^*(X; R) \to PS_*(X, R)$ is a chain homotopy equivalence, and as we will see in the proof of the proposition, we have a homotopy inverse $T$ such that $iT$ is a singular subdivision map $T$. In particular, $\hat{T}$ and $T$ preserve supports of simplices. Similarly, the chain homotopies involved also preserve supports. It follows that $i$, $T$, and the chain homotopies descend to maps and chain homotopies relative to $U$, so that $i$
induces a chain homotopy equivalence \( I^\beta S_*^V(X; R)/I^\beta S_*^i(U; R) \rightarrow I^\beta S_*(X, U; R) \). But now, applying the second fundamental theorem of algebra and the basic definitions,

\[
\frac{I^\beta S_*^V(X; R)}{I^\beta S_*^i(U; R)} = \frac{I^\beta S_*^i(X - K; R) + I^\beta S_*^i(U; R)}{I^\beta S_*^i(U; R)}
\]

\[
\cong \frac{I^\beta S_*^i(X - K; R) \cap I^\beta S_*^i(U; R)}{I^\beta S_*^i(U - K; R)}
\]

\[
\cong I^\beta S_*^i(X - K; R)
\]

So \( I^\beta S_*^i(X - K, U - K; R) \) and \( I^\beta S_*^i(X, U; R) \) are chain homotopy equivalent. Thinking through what happens to representative elements, we see that the chain homotopy equivalence is induced by the inclusion map.

The PL case is even more straightforward using the isomorphism given in Proposition 7.5 instead of homotopy equivalences.

\[\text{\textit{Theorem 7.13.}}\quad \text{Suppose } X = U \cup V, \text{ where } U, V \text{ are subspaces such that } X = \hat{U} \cup \hat{V}. \text{ Suppose further that either } U \cap V \text{ is open in both } U \text{ and } V \text{ or that } R \text{ is a Dedekind domain. Then there is an exact Mayer-Vietoris sequence}
\]

\[
\rightarrow I^\beta H^{i-1}(U \cap V; R) \rightarrow I^\beta H^i(U \cup V; R) \rightarrow I^\beta H^i(U; R) \oplus I^\beta H^i(V; R) \rightarrow I^\beta H^i(U \cap V; R) \rightarrow .
\]

The equivalent results holds in the PL context if \( R \) is a field.

\[\text{\textit{Proof.}} \quad \text{We first need to show that the short exact Mayer-Vietoris sequences of chain modules (analogous to that in the proof of Theorem 4.41) split. If } U \cap V \text{ is open in } U \text{ and } V, \text{ then, applying Lemma 7.6, the inclusions } I^\beta S_i(U \cap V; R) \hookrightarrow I^\beta S_i(U; R) \text{ and } I^\beta S_i(U \cap V; R) \hookrightarrow I^\beta S_i(S_i(V; R)) \text{ each split, so } I^\beta S_i(U; R) \cong I^\beta S_i(U \cap V; R) \oplus I^\beta S_i(U, U \cap V; R) \text{ and } I^\beta S_i(V; R) \cong I^\beta S_i(U \cap V; R) \oplus I^\beta S_i(V, U \cap V; R). \text{ Similarly, if } R \text{ is a Dedekind domain, these inclusions each split as } I^\beta S_i(U, U \cap V; R) \subset S_i(U, U \cap V; R) \text{ and } I^\beta S_i(V, U \cap V; R) \subset S_i(V, U \cap V; R) \text{ are projective as submodules of free modules, and so the short exact sequences of the pairs split in each dimension by 55, Theorem 1.4.7}.
\]

Therefore, the middle term of the short exact sequence has the form

\[
I^\beta S_i(U; R) \oplus I^\beta S_i(V; R) \cong I^\beta S_i(U \cap V; R) \oplus I^\beta S_i(U, U \cap V; R) \oplus I^\beta S_i(V, U \cap V; R).
\]

The inclusion map of the short exact sequence takes \( \xi \in I^\beta S_i(U \cap V; R) \) to \((\xi, 0, -\xi, 0)\), so there is a splitting map of the form \((a, b, c, d) \rightarrow a\). Given this splitting of the short exact Mayer-Vietoris sequence of intersection chains, applying \( \text{Hom}(\cdot, R) \) preserves the exactness, and we obtain a short exact sequence of intersection cochain complexes and thus a long exact sequence of cohomology modules. It remains to show that

\[
H^i(\text{Hom}(I^\beta S_*(U; R) + I^\beta S_*(V; R), R)) \cong I^\beta H^i(X; R).
\]

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But, by Proposition 7.5, the corresponding chain inclusion is a chain homotopy equivalence, so the Hom dual is also a chain homotopy equivalence and induces an isomorphism on intersection cohomology.

For PL chains over a field, we use that all short exact sequences of vector spaces split and then apply the PL part of Proposition 7.5.

Applying the material of Section 5.5 provides an equivalence between PL and singular intersection cohomology on PL spaces, though we must again assume field coefficients.

**Theorem 7.14.** Let \( F \) be a field, and let \( X \) be a PL CS set. Then \( I_\bar{p}S^*(X; F) \cong I_\bar{p}H^*(X; F) \).

**Proof.** By the results of Section 5.5, which we have noted in Section 6.3 carry over to non-GM intersection homology, we have a diagram of maps

\[
I_\bar{p}S_*^C(X; F) \xleftarrow{\psi} I_\bar{p}S_*^T(X; F) \xrightarrow{\phi} I_\bar{p}S_*^C(X; F),
\]

in which each map induces isomorphisms on homology. We also have a map \( I_\bar{p}S_*^C(X; F) \to I_\bar{p}S_*^C(X; F) \) that induces an isomorphism on homology. All chain complexes are bounded below and, since we work over a field, free. Therefore, all of the maps involved are in fact chain homotopy equivalences \([77, \text{Theorem 46.2}]\). Dualizing these homotopy equivalences provides homotopy equivalences of cochain complexes and hence the desired intersection cohomology isomorphism.

**Remark 7.15.** It seems reasonable to expect that the restriction that \( R \) be a field in Theorem 7.14 should not be necessary. However, it is not clear how to complete a more general proof, and so not clear that this is true. One method to pursue such a result would be to attempt to strengthen the results of Section 5.5 to make \( \phi \) and \( \psi \) chain homotopy equivalences by constructing the homotopy inverses. An alternate approach would be to attempt to find a way to show that \( I^pC_*^*(X; R) \) is always chain homotopy equivalent to a bounded-below complex of projective modules, say \( A_* \); if \( R \) is Dedekind, then each \( I^pS_*^C(X; R) \) is already projective, as a submodule of the projective module \( S_*^C(X; R) \). We would then have quasi-isomorphisms between \( A_* \) and \( I^pC_*^C(X; R) \), which would induce a chain homotopy equivalence between them by \([105, \text{Theorem 10.4.8}]\). Altogether, then, \( I^pC_*^C(X; R) \) and \( I^pS_*^C(X; R) \) would be chain homotopy equivalent.

Alternatively, a more correct way to proceed might be to work completely in the derived category and so define \( I^pC_*^C(X, A; R) \) instead using derived functors as \( I^pS_*^C(X, A; R) = \text{RHom}_R(I^pS_*^C(X, A; R), R) \) and similarly for \( I^pC_*^C \). This would have advantages at the expense of more sophistication. Since we do not expect the reader to be conversant with derived categories, and since we will not need to work much with PL intersection cohomology, we will not pursue this here.

Finally, we extend the intersection homology results on topological invariance.

**Theorem 7.16.** Suppose \( R \) is a Dedekind domain, \( X \) is a CS set of formal dimension \( n \) with no codimension one strata, and \( \bar{p} : \{2, \ldots \} \to \mathbb{Z} \) is a perversity such that \( \bar{p}(2) = 0 \) and \( \bar{p}(k - 1) \leq \bar{p}(k) \leq \bar{p}(k - 1) + 1 \) for all \( k \geq 3 \), i.e. \( \bar{p} \) is a GM perversity. Then
\[ I_pH^*(X; R) \text{ is independent (up to isomorphism) of the choice of stratification of } X \text{ as a CS set of formal dimension } n. \text{ In particular, if } X' \text{ is another CS set of formal dimension } n \text{ that is topologically homeomorphic to } X \text{ (not necessarily stratified homeomorphic), then } I_pH^*(X; R) \cong I_pH^*(X'; R). \]

More generally, if \( A \) is an open subset of \( X \) and \((X, A) \cong (X', A')\), then \( I_pH^*(X, A; R) \cong I_pH^*(X', A'; R) \)

Proof. The hypotheses on the perversity ensure that \( I_pS_*(X; R) \cong I_pS_*^{GM}(X; R) \) by Proposition 6.7. The condition that there be no codimension one strata is necessary to ensure both that \( \bar{p} \leq \bar{t} \) and that \( \bar{p}(k) \geq 0 \) for all \( k \), which is required for Theorem 5.52 which we can now invoke. Since \( R \) is Dedekind, each \( I_pS_*(X; R) \subset S_*(X; R) \) is projective, so the homotopy equivalences of the proof of Theorem 5.52 are, in fact, homotopy equivalences. This follows again from [105, Theorem 10.4.8], noting that bounded above cochain complexes are equivalent to bounded below chain complexes. The \( \text{Hom}(\cdot, R) \) duals are then also chain homotopy equivalences, yielding isomorphisms on cohomology. The statement for relative intersection cohomology follows via a five lemma argument as in the proof of Theorem 5.52 using the arguments of Theorem 7.11 to obtain the diagram of long exact sequences from the diagram of short exact sequences of chain complexes.

7.1.1 Proofs of technical results

(DOUBLE CHECK THIS - THE RELATIVE VERSIONS ARE NEW!!)

Here we turn to proving Proposition 7.5 and Lemma 7.6, the latter of which will arise as a corollary of the proof of the former. The proofs are somewhat technical and are based upon the arguments in [29] for demonstrating an analogue of Proposition 7.5. Of course the basic subdivision ideas are standard, but some care is required to avoid the ‘standard error’ of creating faces that are not allowable in the boundaries when breaking a subdivided chain into pieces.

Proof of Proposition 7.5. Recall that we assume \( \mathcal{V} \) is a covering of \( X \) such that the associated interiors to the sets in \( \mathcal{V} \) constitute an open covering of \( X \), and we let \( I^pS_*^\mathcal{V}(X, A; G) = \sum_{V \in \mathcal{V}} I^pS_*(V, A \cap V; G) \subset I^pS_*(X, A; G) \). Notice that each \( I^pS_*(V, A \cap V; G) \) really does inject into \( I^pS_*(X, A; G) \): the only chains in the kernel of \( I^pS_*(V; G) \to I^pS_*(X, A; G) \) are those that are supported in \( A \) and \( V \), and those are 0 in \( I^pS_*(V, A \cap V; G) \). Therefore, we can identify the image of \( I^pS_*(V, A \cap V; G) \) as a subgroup of \( I^pS_*(X, A; G) \), and the sum then makes sense. We also observe that \( I^pS_*^\mathcal{V}(X, A; G) \) consists of the elements of \( \xi \in I^pS_*(X, A; G) \) that can be represented as a finite sum of chains \( \xi = \sum_{V \in \mathcal{V}} \xi_V \) with \( \xi_V \in I^pS_*(V; G) \).

The complex \( I^pS_*^{GM, \mathcal{V}}(X, A; G) \) is defined analogously. Proposition 7.5 states that the inclusions \( I^pS_*^{GM, \mathcal{V}}(X, A; G) \hookrightarrow I^pS_*^{GM}(X, A; G) \) and \( I^pS_*^\mathcal{V}(X, A; G) \hookrightarrow I^pS_*(X, A; G) \) are chain homotopy equivalences.

Our method of proof will be to construct a singular subdivision \( T : S_*(X; G) \to S_*(X; G) \) satisfying certain properties that will allow us to show that it induces maps on
$I\bar{p}S^\mathcal{GM}_s(X, A; G)$ and $I\bar{p}S_*^{GM}(X, A; G)$ whose images lie in $I\bar{p}S^\mathcal{GM,V}_s(X, A; G)$ and $I\bar{p}S^V_*(X, A; G)$, respectively. In fact, the required map $I\bar{p}S^\mathcal{GM}_s(X; G) \to I\bar{p}S^\mathcal{GM}_s(X; G)$ will simply be the map induced on the relative groups of the restriction of $T$ to $I\bar{p}S^\mathcal{GM}_s(X; G)$, and since its image will lie in $I\bar{p}S^\mathcal{GM,V}_s(X, A; G)$, we obtain a map $\hat{T} : I\bar{p}S^\mathcal{GM}_s(X, A; G) \to I\bar{p}S^\mathcal{GM,V}_s(X, A; G)$ that we will show is a chain homotopy inverse to the inclusion map.

For $I\bar{p}S_*^\mathcal{A}(X, A; G)$, the entire argument will be more complicated as a consequence of the fact that $I\bar{p}S_*^\mathcal{A}(X, A; G)$ is not a subcomplex of $S_*(X, A; G)$, as its boundary map $\hat{\partial}$ is not compatible with the boundary in $S_*(X, A; G)$, so as to most directly utilize the construction of $T$ on $S_*^\mathcal{A}(X, A; G)$, it is therefore more convenient to use, instead of $I\bar{p}S_*^\mathcal{A}(X, A; G)$, the complex $I\bar{p}S_*^\mathcal{A}(X, A; G)$, our alternative, though isomorphic (see Lemma 6.5), definition of non-$\mathcal{A}$ intersection chains from Section 6.2. Recall from Section 6.2.1 that

$$I\bar{p}S_*^\mathcal{A}(X, A; G) = \frac{(A\bar{p}S_*(X; G) + S_i(\Sigma_X; G)) \cap \partial^{-1}(A\bar{p}S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))}{S_i(\Sigma_X; G) + (A\bar{p}S_i(A; G) + S_i(\Sigma_A; G)) \cap \partial^{-1}(A\bar{p}S_{i-1}(A; G) + S_{i-1}(\Sigma_A; G))},$$

where $A\bar{p}S_i(X; G)$ is the subgroup of $S_i(X; G)$ generated by $\bar{p}$ allowable simplices. Given $T : S_*^\mathcal{A}(X : G) \to S_*^\mathcal{A}(X; G)$, we can restrict $T$ to $(A\bar{p}S_i(X; G) + S_i(\Sigma_X; G)) \cap \partial^{-1}(A\bar{p}S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G))$, and we will see that the image of the restriction also lies in this group. Similarly, $(A\bar{p}S_i(A; G) + S_i(\Sigma_A; G)) \cap \partial^{-1}(A\bar{p}S_{i-1}(A; G) + S_{i-1}(\Sigma_A; G))$ will be taken to itself, and, since subdivision maps preserve supports, $T$ will furthermore take $S_i(\Sigma_X; G)$ to itself. Therefore we obtain an induced map $T' : I\bar{p}S_*^\mathcal{A}(X, A; G) \to I\bar{p}S_*^\mathcal{A}(X, A; G)$. As for the $\mathcal{A}$ theory, we will show that the image of $T'$ is contained in $I\bar{p}S_*^{\mathcal{A,V}}(X, A; G)$, yielding a map $\hat{T} : I\bar{p}S_*^\mathcal{A}(X, A; G) \to I\bar{p}S_*^{\mathcal{A,V}}(X, A; G)$, which we will show is a chain homotopy inverse to the inclusion.

Before moving on with this program, we should briefly discuss what exactly we mean by $I\bar{p}S_*^{\mathcal{A,V}}(X, A; G)$. By Lemma 6.5, $I\bar{p}S_*^\mathcal{A}(X; G) \cong I\bar{p}S'_*^\mathcal{A}(X; G)$ and similarly $I\bar{p}S_*^\mathcal{A}(A; G) \cong I\bar{p}S'_*^\mathcal{A}(A; G)$ and therefore for the relative complexes. Now without the primes, we have that $I\bar{p}S_*^\mathcal{A}(V, A \cap V; G) \to I\bar{p}S_*^\mathcal{A}(X, A; G)$ is an injection because the only chains in the kernel of $I\bar{p}S_*^\mathcal{A}(V; G) \to I\bar{p}S_*^\mathcal{A}(X, A; G)$ are those that are supported in $A$ and $V$, and those are 0 in $I\bar{p}S_*^\mathcal{A}(V, A \cap V; G)$. Therefore, we can identify the image of $I\bar{p}S_*^\mathcal{A}(V, A \cap V; G)$ as a subgroup of $I\bar{p}S_*^\mathcal{A}(X, A; G)$. We then have a diagram

$$I\bar{p}S_*^\mathcal{A}(V, A \cap V; G) \hookrightarrow I\bar{p}S_*^\mathcal{A}(X, A; G)$$

$$\cong \quad \cong$$

$$I\bar{p}S'_*^\mathcal{A}(V, A \cap V; G) \to I\bar{p}S'_*^\mathcal{A}(X, A; G),$$

so it follows that we can also regard each $I\bar{p}S'_*^\mathcal{A}(V, A \cap V; G)$ as a subgroup of $I\bar{p}S'_*^\mathcal{A}(X, A; G)$ (this would be more difficult to show directly. The sum $I\bar{p}S'_*^{\mathcal{A,V}}(X, A; G) \sum_{V \in V} I\bar{p}S'_*^\mathcal{A}(V, A \cap V; G)$ therefore makes sense.

In fact, we then obtain a diagram

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As the composition right then down is injective, it follows that the left vertical map must also be injective. It is also surjective, using the individual isomorphisms $I^p S_*(V; G) \cong I^p S'_*(V; G)$. So the lefthand vertical map is an isomorphism. It follows that the chain complex inclusions represented by the top and bottom horizontal maps of the diagram are isomorphic, and so to show that the top horizontal inclusion is a homotopy equivalence, it suffices to show that the bottom inclusion is.

The construction of $T$, $\hat{T}$, and $\bar{\bar{T}}$ is accomplished by the following lemma:

**Lemma 7.17.** Let $V$ be a covering of $X$ such that $X$ is also covered by the interiors of the elements of $V$. Then there exists a singular subdivision chain map $T : S_*(X; G) \to S_*(X; G)$ that induces chain maps $\hat{T} : I^p S^G_*(X, A; G) \to I^p S^G_*(X, A; G)$ and $\bar{\bar{T}} : I^p S'_*(X, A; G) \to I^p S'_*(X, A; G)$.

The proof of Lemma 7.17 is provided below. First we demonstrate that the existence of such maps is sufficient to obtain the desired homotopy equivalences.

First, consider $I^p S^G_*(X)$. Let $i$ denote the inclusion $I^p S^G_*(X) \hookrightarrow I^p S^G_*(X; G)$. Then $i\hat{T}$ is simply the restriction of the $T : S_*(X; G) \to S_*(X; G)$ of the lemma to a map $I^p S^G_*(X; G) \to I^p S^G_*(X; G)$. By Corollary 4.37 (which applies just as well with coefficients in $G$) since this restriction of $T$ is a singular subdivision map, the induced map $I^p S^G_*(X, A; G) \to I^p S^G_*(X, A; G)$ is chain homotopic to the identity.

Next, let $i : I^p S^G_*(V, A \cap V; G) \hookrightarrow I^p S'_*(X, A; G)$ be the inclusion and consider the map $T' = i\bar{\bar{T}} : I^p S'_*(X, A; G) \to I^p S'_*(X, A; G)$ induced by $T$. We want to show that the chain homotopy $D$ descends to a well-defined map $I^p S^G_*(V, A \cap V; G) \hookrightarrow I^p S'_{i+1}(X, A; G)$. As constructed in Corollary 4.37, $D$ is defined on any simplex of $S_*(X)$, it takes allowable simplices to sums of allowable simplices, and it preserves supports. Thus if $\xi \in A^p S_i(X; G) + S_i(\Sigma_X; G)$, we will have $D(\xi) \in A^p S_{i+1}(X; G) + S_{i+1}(\Sigma_X; G)$, while if also $\partial \xi \in A^p S_{i-1}(X; G) + S_{i-1}(\Sigma_X; G)$, we will have $\partial D(\xi) = T(\xi) - \xi - D(\partial \xi) \in A^p S_i(X; G) + S_i(\Sigma_X; G)$ using that $T$ also preserves allowability and supports. Similarly, $D$ will take $(A^p S_i(A; G) + S_i(\Sigma_A; G)) \cap \partial^{-1}(A^p S_{i-1}(A; G) + S_{i-1}(\Sigma_A; G))$ to itself and $S_i(X; G)$ to $S_{i+1}(X; G)$. Thus, $D$ descends to a chain homotopy $I^p S'_*(X, A; G) \to I^p S'_{i+1}(X, A; G)$ between $T' = i\bar{\bar{T}}$ and the identity.

So we have now shown that $i\hat{T}$ and $i\bar{\bar{T}}$ are chain homotopic to identity maps. Next we consider $\hat{T}i$ and $\bar{\bar{T}}i$.

First, consider $Ti$. Since $i$ is an inclusion map of a subcomplex, $i^{-1}$ is well-defined on its image. We also observe that if $\xi$ is a chain of $I^p S^G_*(X; G)$ supported in some $V$, then the same is true of both $T(\xi)$ and $D(\xi)$. So if $\xi \in I^p S^G_*(X, A; G)$ is represented by $\sum \xi_V$
with $\xi_V \in I^pS^{GM}_s(V; G)$, then $i^{-1}D_i(\xi)$ is represented by $\sum_{V \in \mathcal{V}} i^{-1}D_i(\xi_V)$, which represents an element in $I^pS^{GM,V}_s(X, A; G)$ Now we compute formally

$$
\begin{align*}
\text{id} - \hat{T}i & = i^{-1}(\text{id} - \hat{T}i) \\
& = i^{-1}i - i^{-1}i\hat{T}i \\
& = i^{-1}(1 - i\hat{T})i \\
& = i^{-1}(\partial D + D\partial)i \\
& = i^{-1}(\partial D)i + i^{-1}(D\partial)i \\
& = \partial i^{-1}D_i + i^{-1}D_i\partial.
\end{align*}
$$

So $\hat{T}i$ is chain homotopic to the identity via the chain homotopy $i^{-1}D_i$. Altogether, we have shown that $i$ is a chain homotopy equivalence, completing the proof for GM intersection chains. The argument for showing that $\hat{T}i$ is homotopic to the identity is the same, recognizing that $D$ preserves supports and takes chains $\xi_V \in S_s(V; G)$ representing elements of $I^pS'_s(V, A \cap V; G) \subset I^pS'_s(X, A; G)$ to chains that continue to represent elements in $I^pS'_s(V, A \cap V; G)$. This follows from the same sorts of arguments applied just above to show that $D$ descends to a chain homotopy on $I^pS'_s(X, A; G)$.

This completes the proof for singular chains.

In the PL situation, we can draw the even stronger conclusion that the inclusion map $I^p\mathcal{C}^{GM,V}_s(X, A; G) \hookrightarrow I^p\mathcal{C}^{GM}_s(X, A; G)$ is an isomorphism. Once again this is indeed an injective map because the only chains in the kernel of $I^p\mathcal{C}_s(V; G) \rightarrow I^p\mathcal{C}_s(X, A; G)$ are those that are supported in $A$ and $V$, and those are 0 in $I^p\mathcal{C}_s(V, A \cap V; G)$. So $I^p\mathcal{C}^{GM,V}_s(X, A; G) \subset I^p\mathcal{C}^{GM}_s(X, A; G)$. So it suffices to demonstrate that the inclusion is a surjection. For this, let $[\xi] \in I^p\mathcal{C}^{GM}_s(X, A; G)$, and suppose that we represent $[\xi]$ as a simplicial chain $\xi$ in some triangulation. For any particular simplex $\sigma$ in this triangulation, we can apply $T$, using some orientation-compatible ordering of the vertices to treat $\sigma$ as a singular simplex via the embedding of $\Delta^i$ in $X$ as $\sigma$. It will follow from the construction of $T(\sigma)$ in the proof of Lemma 7.17 below, that the precise choice of vertex ordering does not matter here.

Thus, extrapolating, $T$ can be applied to any representative simplicial chain representing a PL chain. So, just as in the singular case, we can apply $T$ to obtain $T(\xi) = \sum_{V \in \mathcal{V}} \xi_V$, with each $\xi_V$ an element of $I^p\mathcal{C}^{GM}_s(V; G)$ as $T$ subdivides simplices linearly. But in the PL setting, every chain is identified with its subdivisions, so in $I^p\mathcal{C}^{GM}_s(X, A; G)$, we in fact have $[\xi] = \sum_{V \in \mathcal{V}} [\xi_V] = \sum_{V \in \mathcal{V}} [\xi_V]$.

For $I^p\mathcal{C}_s(X, A; G)$, we instead represent an element $[\xi] \in I^p\mathcal{C}'_s(X, A; G)$ by a chain $\xi$ in some triangulation and then similarly treat it as a singular chain and apply $T'$, yielding a well-defined subdivision $T(\xi) \in S_s(X; G)$. Again, since $T$ subdivides simplices linearly and preserves supports, $T(\xi)$ represents a well-defined element of $I^p\mathcal{C}^{GM,V}_s(X, A; G)$ that is equal to $[\xi]$ in $I^p\mathcal{C}'_s(X, A; G)$.

\textbf{Proof of Lemma 7.17} We begin by constructing a singular subdivision chain map $T : S_s(X) \rightarrow S_s(X)$ with image in $S^{GM}_s(X)$, where $S^{GM}_s(X)$ is defined analogously with $I^pS^{GM,V}_s(X; G)$ but for ordinary singular chains and $\mathbb{Z}$-coefficients. In fact, the image of $T$ will lie in
$S^\epsilon_i(X) \subset S^\nu_i(X)$, where $\mathcal{U}$ is the set of interiors of the elements of $\mathcal{V}$. We later discuss the restriction of $T$ to intersection chains.

We first fix a well-ordering on the set $\mathcal{U}$, which is possible using the Well-ordering Theorem; see [78, Section 10]. Suppose that $B \subset X$ is a subset that can be contained in some element $U$ of $\mathcal{U}$. Then let $\psi(B) \in \mathcal{U}$ be the element of $\mathcal{U}$ that is least in the order among elements of $\mathcal{U}$ containing $B$. If $\sigma : \Delta^i \to X$ is a singular simplex such that the image $\sigma(\Delta^i)$ is contained in some $U \in \mathcal{U}$, then define $\psi(\sigma) = \psi(\sigma(\Delta^i))$.

The basic idea of the argument is as follows: we construct $T$ such that, for each $\xi \in S_i(X)$, $T(\xi)$ will be a sum of simplices each of which is supported in some element of $\mathcal{U}$. This alone would allow us to write $T(\xi) = \sum_{U \in \mathcal{U}} T_U(\xi)$ by letting $T_U(\xi)$ be the sum over those simplices $\sigma$ of $T(\xi)$ (with their coefficients) such that $\psi(\sigma) = U$. In other words, if $T(\xi) = \sum n_i \sigma_i$, then $T_U(\xi) = \sum \mathcal{I}_U(\sigma_i) n_i \sigma_i$, where the indicator function $\mathcal{I}_U(\sigma_i)$ is 1 if $\psi(\sigma_i) = U$ and 0 otherwise. Then $T_U(\xi)$ is supported in $U$. Such a $T$ would be sufficient for working with ordinary singular chains (and this is the essence of such arguments in standard texts), but since our ultimate goal is to work with intersection chains, we must be a bit more subtle.

In particular, if $\xi$ is an intersection chain, we require that each of the $\xi_U$ be an intersection chain, which means that we must be careful about the boundaries of the $\xi_U$, which might contain simplices that are not a part of $\partial \xi$. So, as in our arguments concerning excision and Mayer-Vietoris sequences in Section 4.4, we must take some extra care to “shield” bad faces to ensure this doesn’t happen.

For this, we will construct $T$ inductively to satisfy the properties in the following list. After giving this list, we will see why this is sufficient. Then we will see below that we can indeed construct a $T$ with these properties.

1. $T$ is a chain map $S_*(X) \rightarrow S_*(X)$.

2. For each singular simplex $\sigma$, $T(\sigma)$ is a singular subdivision of $\sigma$ as defined in Section 4.4.2. Recall that, roughly, this means that if $\sigma : \Delta^i \to X$, then there is some simplicial subdivision $\hat{\Delta}^i$ of $\Delta^i$ (with ordered vertices) such that $T(\sigma)$ is the sum of the restrictions of $\sigma$ to each of the $i$-simplices of $\hat{\Delta}^i$. Technically speaking, $T(\sigma) = \sum_{j} \sigma_{i j}$, where each $i_j : \Delta^i \to \hat{\Delta}^i$ is a linear homeomorphism of the standard $i$-simplex onto one of the $i$-simplices of $\hat{\Delta}^i$ with the inclusion map determined by the ordering on the vertices.

3. The image of each simplex of $T(\sigma)$ is contained in some element of $\mathcal{U}$.

4. Suppose $\sigma$ is a singular $i$-simplex and $\mu$ is a simplex of the subdivision $\hat{\Delta}^i$ of $\Delta^i$ as in condition [2]. Suppose further that there is some face $\eta$ of $\Delta^i$ such that $\mu \subset \eta$ and $\dim(\mu) = \dim(\eta)$ (in other words, $\mu$ is a top dimensional simplex in the subdivision of some face of $\Delta^i$). Then $\psi(\sigma(\text{St}(\mu, \Delta^i))) = \psi(\sigma(\mu))$, where $\text{St}(\mu, \Delta^i)$ is the closed star of $\mu$ in $\hat{\Delta}^i$, which consists of all (closed) simplices of $\Delta^i$ that have $\mu$ as a face. This condition says that the image under $\sigma$ of every simplex of $\hat{\Delta}^i$ that has $\mu$ as a face has the same minimal containing element of $\mathcal{U}$ as $\mu$ itself does. This is the condition that will create the necessary “shielding”.

**Sublemma 7.18.** There exists a chain map $T$ with these properties.
Next, let us see that if we have a \( T \) with the listed properties and restrict it to \( I^pS^*_{GM}(X) \subset S_*(X) \), then the image lies in \( I^pS^*_{GM}(X) \). By Lemma 4.35, subdivisions preserve allowability, so since \( T \) is a chain map, we certainly have that \( T \) induces a map \( I^pS^*_{GM}(X) \rightarrow I^pS^*_{GM}(X) \subset S_*(X) \). Suppose \( \xi \in I^pS^*_{GM}(X) \) and \( U \in \mathcal{U} \). If \( T(\sigma) = \sum n_j \tau_j \), for a singular simplex \( \sigma \), let \( T_U(\sigma) = \sum I_U(\tau_j)n_j \tau_j \), where \( I_U \) is the indicator function as defined above. Extending by linearity, we then write \( T(\xi) = \sum_{U \in \mathcal{U}} T_U(\xi) \). Although \( \mathcal{U} \) may contain infinite elements, this sum is necessarily finite as \( \xi \) consists of finitely many simplices, and hence so does the subdivision \( T(\xi) \). Furthermore, by construction, \( T_U(\xi) \) must consist of simplices supported in \( U \). It remains to show that each \( T_U(\xi) \) is an intersection chain under the assumption that \( \xi \in I^pS^*_{GM}(X) \).

By Lemma 4.35, all of the \( i \)-simplices of each \( T_U(\xi) \) will be allowable, so we are reduced to consider the allowability of the simplices of \( \partial T_U(\xi) \). Let \( \tau \) be an \( i-1 \) simplex of \( \partial T_U(\xi) \). Each such \( \tau \) can be identified as the restriction of \( \sigma \) to \( \delta \) for some singular simplex \( \sigma : \Delta^i \rightarrow X \) of \( \xi \) and some \( i-1 \) simplex \( \delta \) in the subdivision \( \Delta^i \) of \( \Delta^i \). Technically, \( \tau \) is the composition of \( \sigma \) with the inclusion of \( \Delta^{i-1} \) into \( \Delta^i \) as \( \delta \), but we will abuse the notation slightly. If \( \tau \) is allowable, there is no trouble, so let us assume that \( \tau \) is not allowable.

First, let us consider the possibility that \( \tau \) is an \( i-1 \) simplex that comes from the interior of an \( i \)-simplex \( \sigma \) of \( \xi \). In other words, the \( i-1 \) simplex \( \delta \) in the the subdivision \( \Delta^i \) associated to \( T(\sigma) \) is not contained in \( \partial \Delta^i \). Suppose, furthermore, that for every face \( \eta \) of \( \Delta^i \) it is true that \( \dim(\delta \cap \eta) < \dim(\eta) \). Then by arguments completely analogous to those of Lemma 4.30 and Lemma 4.39, \( \tau \) must in fact be allowable.

Thus, if \( \tau \) is not allowable, either it is an interior face (as in the last paragraph), but such that there is some face \( \eta \) of \( \Delta^i \) for which it is true that \( \dim(\delta \cap \eta) = \dim(\eta) \) or \( \delta \) is contained in the boundary of \( \Delta^i \). But in this latter case, as \( \dim(\delta) = i - 1 \), we again have \( \dim(\delta \cap \eta) = \dim(\eta) \), where \( \eta \) is the \( i-1 \) face of \( \Delta^i \) containing \( \delta \). So, in either case, \( \dim(\delta \cap \eta) = \dim(\eta) \) for some face \( \eta \) of \( \Delta^i \). The intersection \( \delta \cap \eta \) must be a face of \( \delta \) contained in the subdivision of \( \eta \) in \( \Delta^i \) and of the same dimension as \( \eta \), and so \( \delta \cap \eta = \mu \) is a simplex that satisfies the hypotheses of condition (4). Since \( T \) is assumed to satisfy condition (4), it follows that we have \( \psi(\sigma(\mathcal{S}(\mu, \Delta^i))) = \psi(\sigma(\mu)) \). So, in particular, \( \delta \) and any simplex \( \gamma \) of \( \Delta^i \) of which \( \delta \) is a face, both of which contain \( \mu \) as a face, have their images under \( \sigma \) contained in \( \psi(\sigma(\mu)) \). Furthermore, since \( \delta \) and any such \( \gamma \) contain \( \mu \), their images under \( \sigma \) cannot be contained in an element of \( \mathcal{U} \) that is smaller in the ordering because the image under \( \sigma \) of \( \mu \) cannot be. Thus \( \psi(\tau) = \psi(\sigma(\delta)) = \psi(\sigma(\gamma)) = \psi(\sigma(\mu)) \). So, in particular, if \( \gamma \) is an \( i \) simplex of \( \Delta^i \) of which \( \delta \) is a face, then the \( i \)-simplex \( \sigma|_{\gamma} \) is contained in \( T\psi(\tau)(\xi) \) (or is 0 if there is cancellation of the singular simplex \( \sigma|_{\gamma} \) in \( T(\xi) \)).

Now, consider all the ways that \( \tau \) can arise as a face of a simplex of \( T(\xi) \). In all such cases, the discussion of the last paragraph holds, ranging across various possible simplices playing the roles of \( \sigma, \eta, \delta, \mu, \) and \( \gamma \), but in all such cases (except possibly the ones in which \( \tau \) is allowable, in which case there is no problem) the conclusion is that any \( i \)-simplex of \( T(\xi) \) that contains \( \tau \) as a boundary simplex must be included in \( T\psi(\tau)(\xi) \). But since we have assumed \( \tau \) is not allowable, we know it does not occur in \( \partial T(\xi) \). Therefore, there must be cancellations that occur among the boundaries of the \( i \)-simplices of \( T(\xi) \) that contain \( \tau \) as a boundary. Since all such simplices are contained in \( T\psi(\tau)(\xi) \), the coefficients of \( \tau \) must all
cancel in $T_{\psi^\tau}(\xi)$.

Thus we have shown that each $T_{\psi^\tau}(\xi)$ is indeed an intersection chain.

The exact same arguments extend immediately to chains with any coefficients to provide a chain map $I^pS_*^{GM}(X;G) \to I^pS_*^{GMH}(X;G)$. So now consider the relative situation where we want to show that $T$ induces $\tilde{T} : I^pS_*^{GM}(X,A;G) \to I^pS_*^{GMV}(X,A;G)$. Suppose an element of $I^pS_*^{GM}(X,A;G)$ is represented by $\xi + a$ with $\xi \in I^pS_*^{GM}(X,A;G)$ and $a \in I^pS_*^{GM}(X,A;G)$. We know that $T(\xi) = \sum T_U(\xi)$ with $T_U(\xi) \in I^pS_*^{GM}(U;G)$, and similarly we must have $T(a) = \sum T_U(a)$ with $T_U(a) \in I^pS_*^{GM}(A \cap U;G)$ because a subdivision of a chain in $A$ will be in $A$. Therefore, $T(\xi + a) = \sum T_U(\xi + a)$ represents an element in $I^pS_*^{GMV}(X,A;G)$.

If we alter $a$ within $I^pS_*^{GM}(X,A;G)$, we do not change the element that $T(\xi)$ represents in $I^pS_*^{GMV}(X,A;G)$, so $\tilde{T}$ is well defined with the desired properties.

Next, we consider

$$I^pS'_i(X,A;G) = \frac{(A^pS_i(X;G) + S_i(\Sigma_X;G)) \cap \partial^{-1}(A^pS_{i-1}(X;G) + S_{i-1}(\Sigma_X;G))}{S_i(\Sigma_X;G) + (A^pS_i(A;G) + S_i(\Sigma_A;G)) \cap \partial^{-1}(A^pS_{i-1}(A;G) + S_{i-1}(\Sigma_A;G))};$$

recall Section 6.2.1. For convenience of notation, let us denote the “numerator” of the expression by $B$. Since $B \subset S_*(X;G)$, $T$ is defined on elements of $B$ with image in $S_*(X;G)$. Furthermore, the image of $T$ on any chain supported in $\Sigma_X$ will also be supported in $\Sigma_X$, so $T$ will induce a chain map from $I^pS'_i(X;G)$ to itself if we can show that $T$ takes elements of $B$ to elements of $B$. So let $\xi \in B$. Then $\xi$ is a linear combination of allowable simplices and simplices supported in $\Sigma_X$, and the same is true of $\partial \xi$. But the image of $T$ on each allowable simplex is a chain composed of allowable simplices by Lemma 4.35 and the image of $T$ on simplices supported in $\Sigma_X$ is supported in $\Sigma_X$. Since $T$ is a chain map, $\partial T(\xi) = T(\partial \xi)$, which then similarly must be composed of allowable simplices and simplices in $\Sigma_X$. By the same arguments, $T$ must take simplices of $(A^pS_i(A;G) + S_i(\Sigma_A;G)) \cap \partial^{-1}(A^pS_{i-1}(A;G) + S_{i-1}(\Sigma_A;G))$ back into this group and also $S_i(\Sigma_X;G)$ to itself. So $T$ induces a well-defined chain map $T' : I^pS'_i(X,A;G) \to I^pS'_i(X,A;G)$.

Now, we must observe that the image of $T' : I^pS'_i(X,A;G) \to I^pS'_i(X,A;G)$ is contained in $T' : I^pS'_{iU}(X,A;G)$. In fact, if we represent an element of $I^pS'_i(X,A;G)$ by a chain $\xi + a + b$ with $\xi \in B$, $a \in S_i(\Sigma_X;G)$ and $b \in (A^pS_i(A;G) + S_i(\Sigma_A;G)) \cap \partial^{-1}(A^pS_{i-1}(A;G) + S_{i-1}(\Sigma_A;G))$, then, as above, we have $T = \sum T_U(\xi) + T_U(a) + T_U(b)$. Certainly $T_U(a) \in S_i(A \cap U;G)$, and preservation of allowability and supports shows that each $T_U(\xi)$ is contained in $A^pS_i(U;G) + S_i(\Sigma_U;G)$. Next we verify that $\partial T_U(\xi) \in A^pS_{i-1}(U;G) + S_{i-1}(\Sigma_U;G)$. But the preceding arguments for $I^pS_{iU}^{GM}(X;G)$ can be used again verbatim to show here that any simplex of $\partial T_U(\xi)$ not contained in $\Sigma_X$ must be allowable. In particular, notice that if such a simplex $\tau$ is not contained in $\Sigma_X$, no simplex of $T_U(\xi)$ having $\tau$ as a face can be contained in $\Sigma_X$, so there is no disruption to our previous shielding arguments, again using that we know that every simplex of $T(\xi)$ with $\tau$ as a face must be allowable. So $T_U(\xi) \in B$ and the same argument together with preservation of supports shows that $T_U(b) \in (A^pS_i(A;G) + S_i(\Sigma_A;G)) \cap \partial^{-1}(A^pS_{i-1}(A;G) + S_{i-1}(\Sigma_A;G))$. Therefore, each $T_U(\xi + a + b)$ represents an element of $I^pS_{iU}(A \cap U;G)$; in fact, the element is represented by $T_U(\xi)$. Furthermore, varying $a \in S_i(\Sigma_X;G)$ and $b$ in its group do not change this element of $I^pS_{iU}(A \cap U;G)$. This proves our claim that $T$ induces a well-defined map $\tilde{T} : I^pS'_i(X,A;G) \to I^pS'_{iU}(X,A;G)$.

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Proof of Sublemma 7.18. We will construct a subdivision chain map $T : S_*(X) \to S_*(X)$ that satisfies the conditions required. The construction will be inductive over the dimensions of chains. We first define $T : S_0(X) \to S_0(X)$ to be the identity map. Notice that this is consistent with the conditions we need to verify. Next, we assume that $T : S_j(X) \to S_j(X)$ has been constructed for $j < i$ satisfying the required properties. We will show how to extend $T$ to $S_i(X)$.

Let $\sigma : \Delta^i \to X$ be a singular simplex of $S_i(X)$. We will define $T(\sigma)$. In general, $T$ is extended to $S_i(X)$ by linearity. Since $T$ will be a singular subdivision of $\sigma$, we seek to construct an appropriate subdivision $\hat{\Delta}^i$ of $\Delta^i$, and this will determine the subdivision of $\sigma$; see Section 4.4.2 for a general discussion of singular subdivision. By assumption, $T(\partial \sigma)$ has already been defined, so $\partial \hat{\Delta}^i$, which is a subdivision of $\partial \Delta^i$ is already taken as given. We begin by defining a subdivision $\hat{\Delta}^i_1$ of $\Delta^i$ by letting $\hat{\Delta}^i_1$ be the cone on $\partial \hat{\Delta}^i$. In other words, we add the barycenter $v$ of $\Delta^i$ as a vertex, and the $i$-simplices of $\hat{\Delta}^i_1$ will be simplices of the form $\pm[v, w_0, \ldots, w_i-1]$, where $[w_0, \ldots, w_i-1]$ is an $i-1$ simplex of $\partial \hat{\Delta}^i$, and the sign is chosen for consistency with the orientation of $\Delta^i$. We also assume the new barycenter is placed in the order of vertices after the existing vertices of $\partial \hat{\Delta}^i$ (we assume the orderings of the vertices on the $i-1$ face of $\partial \hat{\Delta}^i$ have already been determined by induction).

Next, we perform on $\hat{\Delta}^i_1$ iterated barycentric subdivisions relative to $\partial \hat{\Delta}^i_1$. The process of relative barycentric subdivision is described in detail in [77], Section 16, but here is the basic idea: Recall that barycentric subdivision of a simplicial complex $K$ is performed inductively. The barycentric subdivision of the 0-skeleton of $K$ is always just the 0-skeleton of $K$. Then assuming the barycentric subdivision $K'$ of $K$ has been constructed on the $p-1$ skeleton of $K$, one subdivides each $p$-simplex of $K$ by coning off the barycentric subdivision of its boundary, analogously to our construction of $\hat{\Delta}^i_1$. To obtain ordered simplices in the subdivision, we let each successive barycenter come later in the order than those added at the previous stage of construction. For a relative barycentric subdivision, the difference is that one begins with a subcomplex $L \subset K$ and holds $L$ fixed throughout the procedure: Again the subdivision of the 0-skeleton is just the 0-skeleton itself. Now assume we’ve constructed a relative barycentric subdivision up through the $p-1$ skeleton of $K$ to obtain a $p-1$ dimensional complex $K'$ with $L^{p-1} \subset K'$. Now let $\tau$ be a $p$-simplex of $K$. If $\tau$ is contained in $L$, then $\partial \tau \subset K'$, and we add $\tau$ to $K'$. If $\tau$ is not contained in $L$, then we subdivide $\tau$ by taking the cone on the subdivision of $\partial \tau$ in $K'$ that has already been constructed in the induction. Applying these procedures for all $p$-simplices of $K$, we obtain a $p$-skeleton for $K'$ that contains $L^p$ as a subcomplex. Just as for ordinary barycentric subdivision, relative barycentric subdivision of $K$ relative to $L$ can be repeated iteratively.

We will let $\hat{\Delta}^i$ be such an iterated barycentric subdivision of $\hat{\Delta}^i_1$ relative to $\partial \hat{\Delta}^i_1$, and $T(\sigma)$ will be the singular subdivision of $\sigma$ based on this subdivision of $\Delta^i$. The first two requirements for $T$, that it be a singular subdivision map and a chain map, will thus be satisfied by the construction. To obtain the other conditions, we must ensure that if we perform enough iterations of the relative barycentric subdivision then the other conditions become true. For this, we will prove a modified version of Lemma 16.3 of [77]; our argument (and, for comparison, some of our notation) will be modified versions of those found in [77].

Let $B = \partial \hat{\Delta}^i$, which we assume already constructed by induction. By the inductive
argument, the image of each simplex of $B$ under $\sigma$ is contained in some element of $U$. Furthermore, suppose $\mu$ is a $j$-dimensional simplex of $B$, $j < i$, that is contained in some $j$-dimensional face of $\Delta^i$. By the inductive assumption, if $F$ is any face of $\Delta^i$ containing $\mu$ (as a subset) and if $\tilde{F}$ is the subdivision of $F$ determined by the subdivision $\partial \Delta^i$, then $\psi(\sigma(\tilde{S}t(\mu, \tilde{F}))) = \psi(\sigma(\mu))$. Since this formula holds over all such $F$, we see that in fact $\psi(\sigma(\tilde{S}t(\mu, B))) = \psi(\sigma(\tilde{S}t(\mu, \partial \Delta^i))) = \psi(\sigma(\mu))$. Now let $K = p \ast B$, where $p$ is a vertex, i.e. $K \cong \hat{\Delta}^1$, and let $sd^N(K/B)$ denote the $N$th iterated barycentric subdivision of $K$ relative to the subcomplex $B$. We will show that there is a sufficiently large $N$ such that

- for every $\mu$ in $B$ satisfying the hypotheses of condition [4], $\psi(\sigma(\tilde{S}t(\mu, sd^N(K/B)))) = \psi(\sigma(\mu))$
- for every $i$-simplex $\nu$ of $sd^N(K/B)$, $\sigma(\nu)$ is contained in some element of $U$.

Then if we let $\hat{\Delta}^i = sd^N(K/B)$ and use this to define $T(\sigma)$, we will have satisfied all the conditions we need for $T$.

Here is where we use a slight variation of the method of argument of the proof of [77, Lemma 16.3]. We assume that $B = \partial \hat{\Delta}^i$ lies in some $\mathbb{R}^K \times \{0\} \subset \mathbb{R}^K \times \mathbb{R}$; in fact, since $B$ is the boundary if an $i$-simplex, we can assume that it lies in $\mathbb{R}^i \times \{0\} \subset \mathbb{R}^{i+1}$. Then we let $p = (0, \ldots, 0, 1)$ and form $K = p \ast B$ inside $\mathbb{R}^{i+1}$ in this way; from now on, $K$ will denote this specific complex in $\mathbb{R}^{i+1}$. Notice that $K$ is still PL homeomorphic to our $\hat{\Delta}^i$, and we identify $K$ with $\Delta^i$ so that we may speak of $\sigma : K \to X$. Let $\mu$ be a simplex of $B$ satisfying the hypotheses of condition [4]. We observe that, for any $N$, $\tilde{S}t(\mu, sd^N(K/B)) \subset p \ast \tilde{S}t(\mu, B)$ because any simplex of a relative subdivision having $\mu$ as a face must be contained within a subdivision of a simplex that already has $\mu$ as a face. Consider $\sigma^{-1}(\psi(\mu)) \subset K$, which is open in $K$ and, from our previous observations, contains $\tilde{S}t(\mu, B)$. Since $p \ast \tilde{S}t(\mu, B) - \sigma^{-1}(\psi(\mu))$ is compact but does not intersect $\mathbb{R}^i \times \{0\}$, its projection to $\{0\} \times \mathbb{R}$ has a positive minimum. In particular, there is an $\epsilon_\mu$ such that any simplex contained in $(\mathbb{R}^i \times \{0, \epsilon_\mu\}) \cap (p \ast \tilde{S}t(\mu, B))$ is contained in $\sigma^{-1}(\psi(\mu))$. But now the arguments of Step 1 of the proof of [77, Lemma 16.3] show precisely that, for any given $\epsilon_\mu$, there is a sufficiently large $M_\mu$ such that any simplex of $sd^{M_\mu}(K/B)$ that intersects $\mathbb{R}^i \times \{0\}$ is contained in the strip $\mathbb{R}^i \times \{0, \epsilon_\mu\}$. It follows that in $sd^{M_\mu}(K/B)$, the star of $\mu$ is contained in $\sigma^{-1}(\psi(\mu))$, as desired. Since there are a finite number of such $\mu$ in $B$, it follows that there is an $M = \max_\mu \{M_\mu\}$ such that in $sd^M(K/B)$, the star of any $\mu$ satisfying the hypotheses of [4] is contained in $\sigma^{-1}(\psi(\mu))$.

It remains to show that we can find an $N \geq M$ such that every simplex of $sd^N(K/B)$ is contained in some $\sigma^{-1}(U)$, $U \in U$. By the preceding paragraph, every simplex of $sd^M(K/B)$ that intersects $B$ has this property. Let $Q$ be the union of the simplices of $sd^M(K/B)$ that do not intersect $B$, and let $P$ be the union of the simplices of $sd^N(K/B)$ that do intersect $B$. Then $P$ and $Q$ are finite complexes. As we perform further relative subdivisions, the simplices subdivided from the simplices in $P$ continue to have the desired property, while the simplices in $Q$, since they do not intersect $B$, undergo ordinary iterated barycentric subdivisions. But now we can appeal to the standard arguments: since $Q$ is compact, it has a Lebesgue number [78, Lemma 27.5] with respect to the covering by $\sigma^{-1}(U)$, $U \in U$, and by [77, Theorem 15.4], there is a finitely iterated barycentric subdivision of $Q$ such that...

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the diameters of the simplices of the subdivision are all less than the Lebesgue number. It follows that there is such an $sd^N(K/B)$ as desired.

At last, this completes the proof.

Now we turn to proving Lemma 7.6 which turns out to be a somewhat unexpected corollary of the constructions in the proofs Lemma 7.17 and Sublemma 7.18. Recall that Lemma 7.6 claims that, if $A$ be an open subset of the filtered space $X$, then the maps $I^pS^G_i(A;G) \to I^pS^G_i(X;G)$ and $I^pS_i(A;G) \to I^pS_i(X;G)$ induced by inclusion split.

Proof of Lemma 7.6. Let $U = \{A,X\}$ be an open cover of $X$; in fact, we could let $U$ be any open cover of the form $U = \{A,B\}$. Let us order the cover so that $A < X$. We claim that if we very slightly modify the construction of $\hat{\Delta}$, then the homomorphisms $\hat{T}_A : I^pS^G_i(X;G) \to I^pS^G_i(A;G)$ and $\hat{T}_A : I^pS_i(X;G) \to I^pS_i(A;G)$ provide the splittings. For this, we look carefully at the construction of $T$ in the proof of Sublemma 7.18.

Recall first that $T$ is the identity map in dimension 0, and we continue to take this as the base step in an inductive construction over dimension. Next, if $\sigma$ is an $i$-simplex for $i > 0$, we constructed $T(\sigma)$ using the inductively existing $T(\partial \sigma)$. In fact, the construction of $T(\partial \sigma)$ provides a triangulation $\partial \Delta^i$ of $\partial \Delta^i$. Our next step was to create $\hat{\Delta}^i$ by taking the cone on $\partial \Delta^i$; then we performed iterated barycentric subdivisions relative to $\partial \Delta^i$. The modification we make now is that if $\partial \Delta^i = \partial \Delta$ (in other words if the previous subdivision steps have not actually subdivided any of $\partial \Delta$) and if the image of $\sigma$ is contained in $A$, then we let $\hat{\Delta} = \Delta^i$ and so have $T(\sigma) = \sigma$ for such a $\sigma$. We must verify that our modified $T$ still satisfies the required conditions from Lemma 7.17. But we have only modified $T$ on simplices $\sigma$ contained in $A$, and for these, our new $T$ is still a chain map, is still a subdivision (trivially so), and all the simplices of $T(\sigma)$ map into $A$. Furthermore, the last condition is also satisfied, again because every star of any simplex of $\Delta^i$ in $\Delta^i$ is $\Delta^i$, which is mapped into $A$. For simplices not contained in $A$, our previous procedure is unmodified, and so the required conditions are achieved by the proof of Sublemma 7.18. So this modified $T$ satisfies all the required properties but is constructed so that $T(\sigma) = \sigma$ if the image of $\sigma$ is in $A$.

Now, given our modified $T$, we construct $T_A : S_i(A) \to S_i(A)$ as in the proof of Lemma 7.17. Recall that $T_A(\xi)$ can be thought of as constructing $T(\xi)$ and then throwing away all simplices $\sigma$ such that $\psi(\sigma) \neq A$. But in this case, as $A$ is initial in the ordering on $U$, $\psi(\sigma) = A$ if and only if $\sigma$ is contained in $A$. So $T_A(\xi)$ consists of those simplices of $T(\xi)$ that are contained in $A$. In particular, if $\xi$ is supported in $A$, then we have achieved that $T(\xi) = \xi$. Therefore, $T_A$ is a splitting map for the inclusion $S_i(A) \hookrightarrow S_i(X)$. But now $T_A$ also restricts to a well-defined map between the various relevant intersection chain complexes by the arguments of Lemma 7.17.

81 Here is a sketch of the argument: suppose $K'$ is a subdivision of $K$ relative to $B$ such that any simplex of $K'$ that intersects $B$ is contained in $\mathbb{R}^i \times [0, m]$ for some $m$. There clearly exists such an $m \leq 1$. Let $\delta$ be a simplex of $sd(K'/B)$ that intersect $\mathbb{R}^i \times \{0\}$. Then the vertices of $\delta$ are either vertices of $B$ or barycenters of simplices of $K'$ that intersect $B$. A computation with barycentric coordinates demonstrates that each of these barycenters must be contained in the strip $\mathbb{R}^i \times [0, (\frac{1}{t+1})m]$. Iterating the relative barycentric subdivision $N$ times therefore results in all simplices of the iterated subdivision that intersect $B$ being contained in $\mathbb{R}^i \times [0, (\frac{1}{t+1})^Nm]$, and for a large enough $N$, $(\frac{1}{t+1})^Nm < \epsilon$. 252
7.2 Cup, cap, and cross products

In this section, we introduce and study cup and cap products in intersection homology and cohomology, as well as an intersection cohomology cross product. Broadly, we follow the construction of cup and cap products in [38], though only field coefficients are treated there. Here we work in more general coefficient systems, necessitating more elaborate verifications of properties. We also make an attempt to be more comprehensive in the properties considered.

7.2.1 Philosophy

As for ordinary homology/cohomology theory, the advantage of working with cohomology over homology is that cohomology possesses an internal product. It is well known that singular cohomology always possesses a cup product, while homology only possesses a product in certain special situations, such as when we take the homology of a topological group or H space [23, Section VII.2] or when our space is a manifold, in which case there is an intersection product that is Poincaré dual to the cup product. In fact, a singular cup product can be defined at the level of cochains. Unfortunately, for intersection cohomology, we will not be able to define a cup product quite so broadly. We will mostly need to work at the level of cohomology (not cochains), and even when we do so the cup product will not generally be internal, meaning that the cup product will take a pair of intersection cohomology classes with certain perversities to an intersection cohomology class with a third perversity. This last property is related to the formal structure of Poincaré duality for pseudomanifolds that we will discuss in a later section: in the intersection world, Poincaré duality pairs not just dual dimensions but *dual (or complementary) perversities*.

In fact, the first intersection homology products were the intersection products introduced by Goresky and MacPherson for PL stratified pseudomanifolds in [42]. There, among other results we shall discuss later, it is shown that if $\bar{p}$ and $\bar{q}$ are complementary GM perversities (see Definition 3.5) and $X$ is a closed connected oriented $n$-dimensional PL stratified pseudomanifold, then there is a nonsingular intersection pairing $\llcorner: I^pH^GM_i(X; \mathbb{Q}) \otimes I^qH^{GM}_{n-i}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$. This was the original form of intersection homology Poincaré duality. When $X$ satisfies these properties, our cup product over $\mathbb{Q}$ will be the Poincaré dual of this intersection product in cohomology. More generally, we define our cup product over Dedekind domains on CS sets.

We will begin our discussion with a conceptual review of the cup product in ordinary singular homology. This will help us to see both the limitations and the possibilities for the intersection cohomology cup product.

We begin by recalling perhaps the most familiar version of the cup product. If $\alpha \in S^i(X; R)$ and $\beta \in S^j(X; R)$, and $\sigma \in S_{i+j}(X; R)$, then we often see the cup product $\alpha \smile \beta$ defined by the formula:

$$
(\alpha \smile \beta)(\sigma) = (-1)^{ij} \alpha(\sigma|_{[v_0,\ldots,v_i]}) \beta(\sigma|_{[v_i,\ldots,v_{i+j}]})
$$

Here $\sigma|_{[v_0,\ldots,v_i]}$ and $\sigma|_{[v_i,\ldots,v_{i+j}]}$ are the singular $i$- and $j$-simplices obtained by restricting $\sigma$ to the “front $i$-face” and “back $j$-face” of the standard model $i+j$ simplex $[v_0,\ldots,v_{i+j}]$.

Unfortunately, many of the most readable textbook sources for algebraic topology leave out the sign.
Already from this formula we see what can go wrong for intersection cohomology. Suppose \( \alpha \in I_p S^i(X; R) \) and \( \beta \in I_q S^j(X; R) \). Then we would expect that \( \alpha \smile \beta \) should be an element of \( I_r S^{i+j}(X; R) \) for some appropriate \( r \). Suppose now that \( \xi \in I_r S_{i+j}(X; R) \). If some sort of front/back formula were to hold, we would first expect that \( (\alpha \smile \beta)(\xi) \) should be determined as a linear combination of terms \( (\alpha \smile \beta)(\sigma) \), where \( \sigma \) is a simplex of \( \xi \). Already this is a little problematic, since we know that \( \sigma \) being a simplex of \( \xi \) does not guarantee that \( \sigma \in I^r S_{i+j}(X; R) \). But even if \( \sigma \) is itself allowable as a chain, there is the further difficulty that we should not expect the \( \bar{r} \)-allowability of \( \sigma \) to tell us anything useful about the \( \bar{p} \) and \( \bar{q} \) allowability of its various faces. For example, suppose we would like \( \bar{p}, \bar{q} \), and \( \bar{r} \) to all be GM perversities; the intersection homology Poincaré duality of [42] makes this a not unreasonable request. Then we know that no 0- or 1-simplex that intersects \( \Sigma_X \) can be allowable (see Example 3.37). Therefore, if \( \sigma \) is any singular simplex that maps \([v_0]\) into \( \Sigma_X \), the front 0-face cannot be allowable. Since it is not difficult, in general, to find allowable singular \( i+j \) simplices that map \([v_0]\) into \( \Sigma_X \), we see that here the front face/back face formulation cannot be used to define \( (\alpha \smile \beta)(\sigma) \) for \( \alpha \in I_p S^0(X; R) \).

Luckily, in many ways the oft-used front face/back face formulation of the cup product is not really the beginning of the cup product formula but one of its ends. By backtracking to examine the origins of this formula, we will see that there are other options for defining a cup product.

So where does the cup product formula come from? Another familiar formula (see, e.g., [77], Theorem 61.3) is that if \( \alpha \in S^i(X; R) \) and \( \beta \in S^j(X; R) \), then \( \alpha \smile \beta = d^*(\alpha \times \beta) \), where \( d : X \to X \times X \) is the diagonal map given by \( d(x) = (x, x) \) and \( \alpha \times \beta \) here denotes the cochain cross product \( \times : S^i(X; R) \otimes S^j(X; R) \to S^{i+j}(X \times X; R) \). So how is the cochain cross product defined? First of all, if \( \alpha \in S^i(X; R) = \text{Hom}(S_i(X; R), R) \) and \( \beta \in S^j(X; R) = \text{Hom}(S_j(X; R), R) \) then we naturally obtain an element \( \Theta(\alpha \otimes \beta) \in \text{Hom}(S_i(X; R) \otimes S_j(X; R), R) \). In fact, for any \( \text{Hom}(A, R) \) and \( \text{Hom}(B, R) \) there is a natural map \( \Theta : \text{Hom}(A, R) \otimes \text{Hom}(B, R) \to \text{Hom}(A \otimes B, R) \) defined so that \( \Theta(\alpha \otimes \beta)(x \otimes y) = (-1)^{|\alpha||\beta|} \alpha(x) \beta(y) \). Then \( \alpha \otimes \beta \) is defined to be \( \nu^* \theta \), where \( \nu \) is a chain homotopy inverse to the the Eilenberg-Zilber cross product \( \epsilon : S_*(X; R) \otimes S_*(X; R) \to S_*(X \times X; R) \); see Section 5.2.

That \( S_*(X; R) \otimes S_*(X; R) \) and \( S_*(X \times X; R) \) are chain homotopy equivalent is the content of the Eilenberg-Zilber theorem [97, Theorem 5.3.6] or [77, Theorem 59.2], which is often proven using the acyclic model theorem [97, Theorem 4.2.8] or [77, Theorem 32.1]. That our Eilenberg-Zilber product \( \epsilon \), constructed in detail above, provides such an equivalence in one direction follows from the uniqueness-up-to-homotopy part of the acyclic model theorem [97, Theorem 4.2.8.b], using that \( \epsilon \) induces an isomorphism \( H_0(S_*(X; R) \otimes S_*(X; R)) \to H_0(X \times X; R) \). The reverse chain homotopy equivalence \( \nu : S_*(X \times X; R) \to S_*(X; R) \otimes S_*(X; R) \) is sometimes called an Alexander-Whitney map; in general, such a homotopy inverse of \( \epsilon \)

Of course, provided one only cares about cup products \( H^i(X; R) \times H^j(X; R) \to H^{i+j}(X; R) \) for fixed \( i \) and \( j \), this sign doesn’t really matter up to composition with the group isomorphism \( x \to -x \). However, leaving out the sign ignores the Koszul sign conventions, which play a more important role when working at the level of complexes (as opposed to the level of groups or modules). We will err on the side of caution and attempt to maintain the Koszul conventions. A treatment of cup products that includes the signs can be found in Section VII.8 of Dold [24].
is only defined up to chain homotopy, though this is of course enough to get a well-defined map at the level of cohomology.

Now, we have said that
\[ \alpha \sim \beta = d^*(\alpha \times \beta) = d^*\nu^*\Theta(\alpha \otimes \beta). \]
This means that if \( \sigma \in S_{i+j}(X; R) \), we have
\[
(\alpha \sim \beta)(\sigma) = d^*\nu^*\Theta(\alpha \otimes \beta)(\sigma) \\
= \Theta(\alpha \otimes \beta)(\nu d(\sigma))
\]

Here \( d(\sigma) \in S_{i+j}(X \times X; R) \) and \( \nu d(\sigma) \in S_s(X; R) \otimes S_s(X; R) \). This composition \( \nu d : S_s(X; R) \to S_s(X; R) \otimes S_s(X; R) \) is sufficiently useful that we will below give it its own symbol \( \bar{d} \) and call it the \textit{algebraic diagonal map}. Since \( \nu \) is only defined up to chain homotopy, so is \( \bar{d} \). Suppose that, for some choice of \( \nu \), we write \( \nu d(\sigma) = \sum y_k \otimes z_k \). Then we can compute explicitly
\[
(\alpha \sim \beta)(\sigma) = \Theta(\alpha \otimes \beta)(\nu d(\sigma)) \\
= \Theta(\alpha \otimes \beta)(\sum y_k \otimes z_k) \\
= \sum (-1)^{\beta|y_k|} \alpha(y_k) \beta(z_k). \tag{18}
\]
So what does this have to do with the front face/back face formula from the beginning of our discussion? It turns out [77] Theorem 59.5 that a particular Alexander-Whitney map can be given explicitly by
\[
\nu(\tau) = \sum_{k=0}^{i+j} \pi_1 \circ \tau|_{[v_0, \ldots, v_k]} \otimes \pi_2 \circ \tau|_{[v_k, \ldots, v_{i+j}]},
\]
where \( \tau \) is an \( i + j \) simplex in \( X \times X \) and \( \pi_1, \pi_2 : X \times X \to X \) are the projections to the two factors. Of course if \( \alpha \in \mathcal{S}^i(X; R) \) and \( \beta \in \mathcal{S}^j(X; R) \), then we have
\[
\Theta(\alpha \otimes \beta)(\nu(\tau)) = \Theta(\alpha \otimes \beta)(\sum_{k=0}^{i+j} \pi_1 \circ \tau|_{[v_0, \ldots, v_k]} \otimes \pi_2 \circ \tau|_{[v_k, \ldots, v_{i+j}]}) \\
= \sum_{k=0}^{i+j} (-1)^{\beta|y_k|} \alpha(\pi_1 \circ \tau|_{[v_0, \ldots, v_k]}) \beta(\pi_2 \circ \tau|_{[v_k, \ldots, v_{i+j}]}) \\
= (-1)^{ij} \alpha(\pi_1 \circ \tau|_{[v_0, \ldots, v_i]}) \beta(\pi_2 \circ \tau|_{[v_i, \ldots, v_{i+j}]})
\]
using that \( \alpha \) evaluated to 0 on simplices not of degree \( i \) and similarly for \( \beta \). In case \( \tau = d\sigma \), this becomes
\[
(-1)^{ij} \alpha(\pi_1 \circ d\sigma|_{[v_0, \ldots, v_i]}) \beta(\pi_2 \circ d\sigma|_{[v_i, \ldots, v_{i+j}]}) = (-1)^{ij} \alpha(\sigma|_{[v_0, \ldots, v_i]}) \beta(d\sigma|_{[v_i, \ldots, v_{i+j}]})
\]

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as $\pi_1d = \pi_2d = \text{id}$. Of course, we recognize here the front face/back face cup product formula.

Let us emphasize that it might be more appropriate to say that the front face/back face formulation of the cup product gives us a cup product and not the cup product. The point is that the front/back description of the cup product relies upon a particular choice of Alexander-Whitney map. If we choose another, chain homotopic, Alexander-Whitney map, we will obtain a different cup product formula at the cochain level. A formula of the form of (18) will still apply, but we might not have such nice explicit expressions for $\nu d(\sigma)$. However, changing $\nu$ by a chain homotopy will not change the cup product at the cohomology level, since of course chain homotopic chain maps yield the same (co)homology morphisms. In other words, at the cohomology level, we can view the cup product as a composition.

$$H^i(X; R) \otimes H^j(X; R) \xrightarrow{\Theta} H^{i+j}(\text{Hom}(S_i(X; R) \otimes S_j(X; R), R)) \xrightarrow{\nu^*} H^{i+j}(X \times X; R) \xrightarrow{d^*} H^{i+j}(X; R).$$ (19)

The composition of the first two maps is called the cohomology cross product.

It is perhaps a good time with this composition laid out to remind the reader that this formulation also demonstrates why we have a cup product in cohomology but not always an analogous internal product in homology: in homology we have the homology cross product $\epsilon : H_i(X; R) \otimes H^j(X; R) \to H_{i+j}(X \times X; R)$, but the diagonal map points the wrong way $H_{i+j}(X \times X; R) \xleftarrow{d} H_{i+j}(X; R)$!

Returning to intersection cohomology, we see that the composition (19) is our hope for defining a cup product in intersection cohomology. The maps $\Theta$ and $d$ are canonical to the process of defining a cup product, but there is some flexibility in the use of $\nu^*$. In fact, we see that any chain homotopy inverse to the chain cross product will do. We have seen that we cannot hope in an intersection cohomology analogue to use the particular $\nu$ defined in terms of the front and back faces, but Theorem 6.61 nonetheless promises that our intersection version of the Eilenberg-Zilber map $\epsilon$ is a chain homotopy equivalence, given the proper assumptions, and so there are chain homotopy inverses, which we shall denote IAW for “intersection Alexander-Whitney map”. Although IAW is defined at the chain level only up to chain homotopy, it remains true that any two such IAW maps yield the same maps on cohomology. So, while we lose the precision of having a specific nice Alexander-Whitney map given by a front face/back face formula, the general points of the cup product construction still apply!

Historically, the suggestion by Jim McClure that one could obtain a cup product in intersection cohomology this way led to the author’s work on the Küneth theorem in (which extended previously known intersection homology Künneth theorems from and eventually to the construction of cup products in Friedman and McClure, though in , we worked only with field coefficients and so only needed that $\epsilon$ induces homology isomorphisms, not the full power of $\epsilon$ being a chain homotopy equivalence.

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83 It is not difficult to show that $\Theta$ also makes sense as a map on cohomology $H^i(X; R) \otimes H^j(X; R) \xrightarrow{\Theta} H^{i+j}(\text{Hom}(S_i(X; R) \otimes S_j(X; R), R))$; see Lemma 7.25 below.
Before moving on to intersection homology, we note that the algebraic diagonal $\bar{d}$ is the key not only to the cup product but also to the cap product as well as slant products, which are less well known but sometimes useful. The cap product $H^j(X; R) \otimes H_{i+j}(X; R) \to H_i(X; R)$, $\alpha \otimes \xi \to \alpha \lhd \xi$ is defined by

$$\alpha \lhd \xi = (1 \otimes \alpha)\bar{d}(\xi),$$

so if $\bar{d}\xi = \sum_k y_k \otimes z_k$, then

$$\alpha \lhd \xi = (-1)^{ij}\alpha(z_k)y_k,$$

where again $\alpha(z_k) = 0$ if the dimensions of $\alpha$ and $z_k$ do not agree, which forces the sign in the formula. See [23, Section VII.12].

Although we will not have much use for it, the cohomology slant product is defined similarly as an external product

$$H^j(X; R) \otimes H_{i+j}(Y \times X; R) \to H_i(Y; R), \quad \alpha \times \xi \to \alpha \\cdot \\xi.$$

Here, there is no diagonal map, but rather we utilize the Künneth theorem isomorphism between $H_{i+j}(Y \times X; R)$ and $H_{i+j}(S_*(Y) \otimes S_*(X); R)$. Via this isomorphism, we can associated to $\xi \in H_{i+j}(Y \times X; R)$ a sum $\sum_k y_k \otimes z_k \in H_{i+j}(S_*(Y) \otimes S_*(X); R)$ and then let

$$\alpha \\cdot \\xi = (1 \otimes \alpha)\left(\sum_k y_k \otimes z_k\right) = \sum_k (-1)^{ij}\alpha(z_k)y_k.$$

See [23, Section VII.11].

### 7.2.2 Intersection homology cup, cap, and cross products

The preceding discussion should provide convincing evidence that developing an algebraic diagonal map will be key to defining and utilizing intersection (co)homology analogues of the various products from the classical homology and cohomology theories. We also see that in constructing such an algebraic diagonal, we should consider two steps. We will need an intersection homology version of the diagonal map $d: S_*(X) \to S_*(X \times X)$, and we will need the intersection homology Künneth theorem. We will also want to have available below relative cup and cap products, so we develop the necessary tools in this generality.

#### The Künneth theorem

We have already studied the necessary Künneth theorems. From Theorem [6.61] it follows that if $X$ is a CS set of with open subsets $A$ and $B$, if $\bar{p}$ and $\bar{q}$ are two perversities on $X$, and if $R$ is a Dedekind domain, then the cross product induces a chain homotopy equivalence

$$\epsilon: I^pS_*(X, A; R) \otimes_R I^qS_*(X, B; R) \xrightarrow{\sim} I^QS_*(X \times Y, (A \times X) \cup (X \times B); R)$$

if $Q$ is a perversity satisfying the following conditions:

1. if $S \subset X$ is a regular stratum and $T \subset X$ is any stratum, then $Q(S \times T) = \bar{q}(T)$ and $Q(T \times S) = \bar{p}(T)$, and
2. if $S, T \subset X$ are singular strata, then $Q(S \times T) = \bar{p}(S) + \bar{q}(T)$ or $Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 1$
3. if for each point \( x \times y \in S \times T \), \( x \) has a distinguished neighborhood in \( X \) of the form \( \mathbb{R}^a \times cL_1 \) and \( y \) has a distinguished neighborhood in \( X \) of the form \( \mathbb{R}^b \times cL_2 \) such that \( I^pH_{\dim(L_1)-\bar{p}(\{S\})-1}(L_1; R) \ast_R I^qH_{\dim(L_2)-\bar{q}(\{T\})-1}(L_2; R) = 0 \), then condition (2) on \( Q(S \times T) \) may also include the possibility \( Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 2 \). In particular, this condition holds if \( X \) is locally \((\bar{p}, R)\)-torsion free along the singular stratum \( S \) or locally \((\bar{q}, R)\)-torsion free along the singular stratum \( T \).

Assuming we choose a \( Q \) of the given form, the chain homotopy equivalence \( \epsilon \) is available for the construction of an algebraic diagonal.

The diagonal map. For the geometric diagonal, we will need to have a well-defined map

\[
d : I^pS_*(X, A \cup B; R) \to I^qS_*(X \times X, (A \times X) \cup (X \times B); R)
\]

induced by the topological diagonal map \( d(x) = (x, x) \). Notice that if \( a \in A \), then \( d(a) = (a, a) \in A \times X \), and, if \( b \in B \), then \( d(b) = (b, b) \in X \times B \), so we just need to check for which \( \bar{r} \) and \( Q \) it is true that \( d \) takes allowable chains to allowable chains. Since \( d \) takes \( \Sigma_X \) to \( \Sigma_{X \times X} \), this is sufficient for \( d \) to induce the desired chain map.

Notice that \( d \) is not technically a stratified map in the sense of Definition 4.1, as it does not preserve codimension, but the same principles we utilized to determine allowability in Section 4.1 apply (see Remark 4.6): Let \( \sigma : \Delta^i \to X \) be an \( \bar{r} \)-allowable singular simplex, and consider the composition \( d\sigma \). The image of \( d\sigma \) can only intersect strata of \( X \) of the form \( S \times S \) where \( S \) is a stratum of \( X \). The singular strata of this form are the ones for which \( S \) is a singular stratum of \( S \), and, furthermore, we have \((d\sigma)^{-1}(S \times S) = \sigma^{-1}(S)\). If \( S \) has codimension \( k \) in \( X \), then \( S \times S \) has codimension \( 2k \) in \( X \times X \). So \( d\sigma \) is allowable if and only if \( \sigma^{-1}(S) \) is contained in the \( i - 2k + Q(S \times S) \) skeleton of \( \Delta^i \). The assumption that \( \sigma \) is \( \bar{r} \)-allowable imposes the condition that \( \sigma^{-1}(S) \) be contained in the \( i - k + \bar{r}(S) \) skeleton of \( \Delta^i \). So, \( d \) will take allowable simples to allowable simples if and only if \( i - k + \bar{r}(S) \leq i - 2k + Q(S \times S) \), i.e. if

\[
\bar{r}(S) \leq Q(S \times S) - k \tag{20}
\]

(note, we could have gotten to this point directly by invoking Remark 4.6).

In order for (20) to hold for the largest possible range of \( \bar{r} \), we would like to take the option \( Q(S \times S) = \bar{p}(S) + \bar{q}(S) + 2 \), and we will do so. However, as we have seen, for the K"unneth theorem to apply to such spaces, we must require some extra assumptions about \( X \). Ultimately, this trade-off will be justified by the Poincaré duality theorem REF BELOW (NOTE WHY WE NEED THIS VERSION).

Looking back at the K"unneth theorem again, we see that the most general assumption to make about \( X \) in order to use this \( Q \) would be that, for every pair of strata \( S, T \) of \( X \),

\[
I^pH_{\dim(L_1)-\bar{p}(\{S\})-1}(X; L_1) \ast_R I^qH_{\dim(L_2)-\bar{q}(\{T\})-1}(L_2; R) = 0, \tag{21}
\]

where \( L_1 \) and \( L_2 \) are the corresponding links of \( S \) and \( T \). A useful way to ensure this property would be to assume that, for every pair of strata \( S, T \) of \( X \), either \( X \) is locally \((\bar{p}, R)\)-torsion free along \( S \) or \( X \) is locally \((\bar{q}, R)\)-torsion free along \( T \). However, if \( X \) has some stratum that
S such that X is not locally \((\bar{p}, R)\)-torsion free along S, then such a condition would require that it must be locally \((\bar{q}, R)\)-torsion free along \(T\) for all \(T\), i.e. that \(X\) must be locally \((\bar{q}, R)\)-torsion free. So, the stratum-by-stratum assumption works out to being equivalent to assuming that \(X\) is either locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free.

**Remark 7.19.** These assumptions that \(X\) be some variety of locally torsion free are convenient and will be sufficient for our purposes. However, the reader should bear in mind that the purpose of these assumptions is to make sure that the Künneth theorem holds with \(Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 2\) when \(S\) and \(T\) are both singular strata. Any other assumptions, such as \([21]\), that guarantee this may be substituted into the relevant hypotheses in the various results that follow.

Having decided to work with the option \(Q(S \times T) = \bar{p}(S) + \bar{q}(T) + 2\) when \(S\) and \(T\) are singular strata, we give this perversity a label:

Since we chose a particular \(Q\) to use in the above argument, it will be useful to fix this in the notation with the following definition:

**Definition 7.20.** Let \(\bar{p}\) and \(\bar{q}\) be two perversities on the space \(X\). Let \(Q_{\bar{p},\bar{q}}\) be the perversity such that

1. if \(S \subset X\) is a regular stratum and \(T \subset X\) is any stratum, then \(Q_{\bar{p},\bar{q}}(S \times T) = \bar{q}(T)\) and \(Q_{\bar{p},\bar{q}}(T \times S) = \bar{p}(T)\), and

2. if \(S, T \subset X\) are singular strata, then \(Q_{\bar{p},\bar{q}}(S \times T) = \bar{p}(S) + \bar{q}(T) + 2\).

We will also need to consider below perversities of the form \(Q_{\bar{p},Q_{\bar{q},\bar{r}}}\) and \(Q_{\bar{p},\bar{q},\bar{r}}\). As the reader can verify, these two expressions are equally, and we will abbreviate both of them by \(Q_{\bar{p},\bar{q},\bar{r}}\).

Now, let us return to our discussion of the allowability of the diagonal map. Making the choice \(Q = Q_{\bar{p},\bar{q}}\), we have seen in \([20]\) that the requirement for \(d\) to induce a map of intersection chains is that \(\bar{r}(S) \leq Q_{\bar{p},\bar{q}}(S \times S) - k\) for all singular strata \(S\). Plugging in for \(Q_{\bar{p},\bar{q}}(S \times S)\) when \(S\) is a singular stratum, this condition becomes \(\bar{r}(S) \leq \bar{p}(S) + \bar{q}(S) + 2 - k\). It turns out to be useful to rearrange this formula as follows:

\[
\bar{r}(S) \leq \bar{p}(S) + \bar{q}(S) + 2 - k \iff -\bar{r}(S) \geq -\bar{p}(S) - \bar{q}(S) - 2 + k
\]
\[
\iff k - 2 - \bar{r}(S) \geq k - 2 - \bar{p}(S) + k - 2 - \bar{q}(S)
\]
\[
\iff \bar{t}(S) - \bar{r}(S) \geq \bar{t}(S) - \bar{p}(S) + \bar{t}(S) - \bar{q}(S)
\]
\[
\iff D\bar{r}(S) \geq D\bar{p}(S) + D\bar{q}(S).
\]

Recall that \(\bar{t}\) here is the Goresky-MacPherson top perversity, which is defined by \(\bar{t}(S) = \text{codim}(S) - 2\) for \(S\) a singular stratum, and that the dual perversity \(D\bar{p}\) of a perversity \(\bar{p}\) is defined to be \(D\bar{p}(S) = \bar{t}(S) - \bar{p}(S)\) for \(S\) singular, see Definition 3.5. As we want this inequality to hold for all singular strata \(S\), we can write it more succinctly as \(D\bar{r} \geq D\bar{p} + D\bar{q}\).

\[84\text{Recall that we say that } X \text{ is locally } (\bar{q}, R)\text{-torsion free if it is locally } (\bar{q}, R)\text{-torsion free with respect to all its strata.}\]
In this form, the inequality has a nice symmetry with conditions of the form $\bar{p} + \bar{q} \leq \bar{r}$ that arise when considering intersection pairings, which are dual to the cup product pairing on pseudomanifolds; see [12] and BELOW SOMEWHERE.

The preceding arguments now imply the following lemma.

**Lemma 7.21.** Let $R$ be a Dedekind domain. Suppose that $\bar{p}, \bar{q}, \bar{r}$ are perversities on a CS set $X$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Suppose, furthermore, that $A, B$ are open subsets of $X$ and that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free along each singular stratum. Then the algebraic diagonal $\bar{d}$ is well-defined up to chain homotopy as the composition

$$I^{\bar{r}}S_*(X, A \cup B; R) \xrightarrow{\bar{d}} I^{Q\bar{p}+\bar{q}}S_*(X \times X, (A \times X) \cup (X \times B); R) \xrightarrow{\text{IAW}} I^{\bar{p}}S_*(X, A; R) \otimes I^{\bar{q}}S_*(X, B; R),$$

where IAW is a chain homotopy inverse to the chain cross product $\epsilon$.

**Remark 7.22.** Our assumption that $X$ satisfy certain locally torsion free conditions arose out of our desire to maximize the range of perversities $\bar{r}$ for which the algebraic diagonal, and hence the cup product, is defined, given $\bar{p}$ and $\bar{q}$. If we were unwilling to make such an assumption on $X$, we would instead arrive at the requirement $D\bar{r} \geq D\bar{p} + D\bar{q} + 1$. Certainly we could follow through the remaining discussion in this context instead, but ultimately we will want the additional requirements anyway in order to prove Poincaré duality. The given setting also makes the statements of certain properties cleaner. Thus we leave the more general case for the interested reader.

We also recall again that the torsion free conditions will be satisfied automatically if $R$ is a field.

Given the algebraic diagonal of Lemma 7.21, we can now define cup and cap products completely analogously with the classical setting:

**Definition 7.23.** Let $R$ be a Dedekind domain. Suppose that $\bar{p}, \bar{q}, \bar{r}$ are perversities on a CS set $X$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Suppose, furthermore, that $A, B$ are open subsets of $X$, and that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Then the *intersection cohomology cup product*

$$\lhd: I_{\bar{p}}H^i(X, A; R) \otimes I_{\bar{q}}H^j(X, B; R) \to I_{\bar{r}}H^{i+j}(X, A \cup B; R)$$

is defined by

$$\alpha \lhd \beta = \bar{d}^*\Theta(\alpha \otimes \beta) = \bar{d}^*\text{IAW}^*\Theta(\alpha \otimes \beta).$$

Notice that even though $\bar{d}$ is defined as a chain map only up to chain homotopy, the induced map on cohomology is well defined uniquely.

With the same assumptions, the *intersection cohomology cap product*

$$\rhd: I_{\bar{q}}H^j(X, B; R) \otimes I_{\bar{r}}H_{i+j}(X, A \cup B; R) \to I_{\bar{p}}H_i(X, A; R)$$

is defined by

$$\beta \rhd \xi = \Phi(id \otimes \beta)\bar{d}(\xi),$$

where $\Phi$ is the canonical isomorphism: $I_{\bar{p}}S_*(X, A; R) \otimes R \to I_{\bar{p}}S_*(X, A; R)$.  

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While we are at it, if we return to two CS set pairs \((X, A)\) and \((Y, B)\) and leave out the diagonal map, we can similarly define a cohomology cross product:

**Definition 7.24.** Let \(R\) be a Dedekind domain. Suppose that \(X\) is a CS set with perversity \(\bar{p}\) and \(Y\) is a CS set with perversity \(\bar{q}\). Suppose, furthermore, that \(A \subset X\) and \(B \subset Y\) are open subsets, and that \(X\) is locally \((\bar{p}, R)\)-torsion free or \(Y\) is locally \((\bar{q}, R)\)-torsion free. The intersection cohomology cross product

\[
\times : I_{\bar{p}}H^i(X, A; R) \otimes I_{\bar{q}}H^j(Y, B; R) \to I_{Q_{\bar{p}, \bar{q}}}H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R)
\]

is defined by

\[
\alpha \times \beta = \text{IAW}^*\Theta(\alpha \otimes \beta).
\]

Even though IAW is defined as a chain map only up to chain homotopy, the induced map on cohomology is well-defined uniquely.

It follows immediately from this definition that \(\alpha \simeq \beta = d^*(\alpha \times \beta)\) when \(X = Y\) and \(d : I^pS_s(X, A \cup B; R) \to I^qS_s(X \times Y, (A \times X) \cup (X \times B); R)\) with \(D\bar{p} \geq D\bar{p} + D\bar{q}\).

Even though we have seen that \(d\) and IAW induce well-defined maps on cohomology, there is still a bit of work remaining to verify that our cup, cap, and cross products are well defined. We will verify that \(\Theta\) is indeed a chain map, and we must show that the our formula for the cap product yields a well-defined homology class. We now turn to these verifications. Then, in Section 7.3 we will turn to exploring the properties of these products.

**Well-definedness of the products.** We now verify that our products are well defined at the level of (co)homology. We begin with the purely algebraic map \(\Theta\).

**Lemma 7.25.** Suppose we have chain complexes of \(R\)-modules \(C_*\) and \(D_*\). Then \(\Theta : \text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) \to \text{Hom}(C_* \otimes D_*, R)\) is a chain map, and it induces a well-defined map \(H^*(\text{Hom}(C_*, R)) \otimes H^*(\text{Hom}(D_*, R)) \to H^*(\text{Hom}(C_* \otimes D_*, R))\).

**Proof.** We first check that \(\Theta\) is a chain map. Suppose \(\alpha \in \text{Hom}^i(C_*, R) = \text{Hom}(C_i, R)\) and \(\beta \in \text{Hom}^j(D_*, R) = \text{Hom}(D_j, R)\). Then for any \(x \otimes y \in C_* \otimes D_*\), we have

\[
\Theta(d(\alpha \otimes \beta))(x \otimes y) = \Theta((d\alpha) \otimes \beta + (-1)^i\alpha \otimes d\beta)(x \otimes y)
\]

\[
= (-1)^{i+j}|\alpha|\beta(y) + (-1)^{i+j+1}|\alpha|\alpha(x)((d\beta)(y)),
\]

while

\[
d(\Theta(\alpha \otimes \beta))(x \otimes y) = (-1)^{i+j+1}\Theta(\alpha \otimes \beta)\partial(x \otimes y)
\]

\[
= (-1)^{i+j+1}\Theta(\alpha \otimes \beta)((\partial x) \otimes y + (-1)^{|x|}x \otimes \partial y)
\]

\[
= (-1)^{i+j+1+|x|-1}\alpha(\partial x)\beta(y) + (-1)^{i+j+1+|x|+|y|}\alpha(x)\beta(\partial y)
\]

\[
= (-1)^{i+j+1+|x|}\alpha(\partial x)\beta(y) + (-1)^{i+j+1+|x|+|y|}\alpha(x)\beta(\partial y)
\]

\[
= (-1)^{i+j+1+|x|+|y|+1}((d\alpha)(x))\beta(y) + (-1)^{i+j+1+|x|+|y|+j+1}\alpha((d\beta)(y))
\]

\[
= (-1)^{|x|}(d\alpha)(x)\beta(y) + (-1)^{|x|+j}\alpha(x)((d\beta)(y)).
\]

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So $\Theta(d(\alpha \otimes \beta))$ and $d(\Theta(\alpha \otimes \beta))$ represent the same element of $\text{Hom}(C_* \otimes D_*, R)$. Therefore, $\Theta$ is a chain map.

This is enough to show that $\Theta$ induces a cohomology map $H^*(\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R)) \to H^*(\text{Hom}(C_* \otimes D_*, R))$. It remains to show that we have a well-defined map $H^*(\text{Hom}(C_*, R)) \otimes H^*(\text{Hom}(D_*, R)) \to H^*(\text{Hom}(C_* \otimes D_*, R))$. But this follows just as for the definition of the homology cross product in Remark 5.15.

Remark 7.26. By standard homological algebra, the cohomology map $\Theta$ of the lemma is not necessarily an isomorphism. In fact, unlike the analogous case in homology, it is not even necessarily part of an algebraic Künneth exact sequence without some additional assumptions. For example, $\Theta$ will be an isomorphism if $R$ is a field and $H_i(C_*)$ (or, symmetrically, $H_i(D_*)$) is finitely generated in each dimension [77, Theorem 60.6]. See [23, Proposition VI.12.16] for a more comprehensive statement.

Putting this lemma together with the previous discussion implies the following corollary.

Corollary 7.27. Given the assumptions of Definitions 7.24 and 7.23, the cross product $\times : I_{\bar{p}}H^i(X, A; R) \otimes I_{\bar{q}}H^j(Y, B; R) \to I_{\bar{q}}H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R)$ and the cup product $\cup : I_{\bar{p}}H^i(X, A; R) \otimes I_{\bar{q}}H^j(X, B; R) \to I_{\bar{r}}H^{i+j}(X, A \cup B; R)$ are well defined and independent of the choices of IAW map.

Next, let us see that the cap product is well-defined. This requires some computations. We begin with a useful preliminary lemma.

Lemma 7.28. Given the assumptions of Definition 7.23, suppose $\beta \in I_{\bar{q}}S^j(X, B; R)$ and $\xi \in I_{\bar{r}}S^{i+j}(X, A \cup B; R)$. Then, if we define the cap product $\cap$ on $X$ using any particular fixed choice of IAW map, the following chain-level formula holds:

$$\partial (\beta \cap \xi) = (d\beta) \cap \xi + (-1)^{|\beta|} \beta \cap \partial \xi.$$  \hspace{1cm} (22)

Proof. Suppose that, with our given choice of IAW map, $d(\xi) = \sum_k y_k \otimes z_k$. Since IAW and $d$ are chain maps, $d(\partial \xi) = \partial d(\xi) = \partial (\sum y_k \otimes z_k) = \sum ((\partial y_k) \otimes z_k + (-1)^{|y_k|} y_k \otimes (\partial z_k))$. 

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Now, we compute using the definitions:

\[
(d\beta) \smile \xi = \Phi(id \otimes d\beta)\tilde{d}(\xi) \\
= \Phi(id \otimes d\beta) \sum_k y_k \otimes z_k \\
= \sum_k (-1)^{|(\beta)+1|}y_k(((d\beta)z_k)y_k \\
= \sum_k (-1)^{|(\beta)+1|}y_k+|\beta|+1\beta(z_k)y_k \\
= \Phi(id \otimes \beta) \sum_k (-1)^{|(\beta)+1|}y_k \otimes \partial z_k \\
= (-1)^{|\beta|+1}\Phi(id \otimes \beta) \sum_k (-1)^{|y_k|}y_k \otimes \partial z_k \\
= (-1)^{|\beta|+1}\Phi(id \otimes \beta) \left( \sum_k \partial(y_k \otimes z_k) - \sum_k (\partial y_k) \otimes z_k \right) \\
= (-1)^{|\beta|+1}\Phi(id \otimes \beta)\tilde{d}(\xi) + (-1)^{|\beta|}\Phi(id \otimes \beta) \left( \sum_k (\partial y_k) \otimes z_k \right) \\
= (-1)^{|\beta|+1}\Phi(id \otimes \beta)\tilde{d}(\xi) + (-1)^{|\beta|} \sum_k (-1)^{|\beta|+|y_k|} \beta(z_k) \partial y_k \\
= (-1)^{|\beta|+1}\beta \smile \partial \xi + (-1)^{|\beta|+|\xi|+|\beta|} \sum_k \beta(z_k) \partial y_k. 
\]

In the second to last line, we have used that all terms of the second summand vanish unless \( |z_k| = |\beta| \), in which case \( \partial y_k = |\partial \xi| - |\beta| = |\xi| - 1 - |\beta| \) as \( |\partial y_k| + |z_k| = |\partial \xi| = |\xi| - 1 \). By comparison, and using the same reasoning about degrees,

\[
\partial(\beta \smile \xi) = \partial \Phi(id \otimes \beta)\tilde{d}(\xi) \\
= \partial \Phi(id \otimes \beta) \sum_k y_k \otimes z_k \\
= \partial \sum_k (-1)^{|\beta|(|\xi|+|\beta|)} \beta(z_k)y_k \\
= \sum_k (-1)^{|\beta|(|\xi|+|\beta|)} \beta(z_k) \partial y_k \\
\]

So, altogether, we see that

\[
(d\beta) \smile \xi = (-1)^{|\beta|+1} \beta \smile \partial \xi + \partial(\beta \smile \xi),
\]
or, equivalently,
\[ \partial(\beta \smile \xi) = (d\beta) \smile \xi + (-1)^{\vert \beta \vert} \beta \smile \partial \xi. \]

Now we can show that the cap product induces a well-defined map on (co)homology, independent of choices.

**Lemma 7.29.** Given the assumptions of Definition 7.23, the cap product \( \smile \colon I^qH^j(X, B; R) \otimes I^rH_{i+j}(X, A \cup B; R) \to I^pH_i(X, A; R) \) is well defined and independent of the choice of IAW map.

**Proof.** Suppose \( \beta \in I^qS^j(X, B; R) \) is a cocycle and \( \xi \in I^rS_{i+j}(X, A \cup B; R) \) is a cycle. Let us first verify that \( \beta \smile \xi \) is a cycle for any choice of IAW. We have just seen in Lemma 7.28 that
\[ \partial(\beta \smile \xi) = (d\beta) \smile \xi + (-1)^{\vert \beta \vert} \beta \smile \partial \xi, \]
so if \( d\beta = 0 \) and \( \partial \xi = 0 \), we have \( \partial(\beta \smile \xi) = 0 \).

Next, we must show that altering \( \beta \) and \( \xi \) within their (co)homology classes does not alter \( \beta \smile \xi \). Equivalently, we must show that \( \beta \smile \xi = 0 \) as an intersection homology class if \( \beta \) or \( \xi \) is a boundary. We continue to assume a fixed IAW maps. First, suppose \( d\beta = 0 \) and \( \partial \xi = \partial \zeta \). Then, using equation (22) with \( \zeta \) substituted for \( \xi \), we have
\[ \beta \smile \xi = \beta \smile \partial \zeta \]
\[ = (-1)^{\vert \beta \vert} \partial(\beta \smile \zeta) - (-1)^{\vert \beta \vert}(d\beta) \smile \zeta \]
\[ = (-1)^{\vert \beta \vert} \partial(\beta \smile \zeta). \]

So \( \beta \smile \xi \) is a boundary. Next, suppose \( \beta = d\alpha \) and \( \partial \xi = 0 \). Then, using equation (22) with \( \alpha \) substituted for \( \beta \), we have
\[ \beta \smile \xi = (d\alpha) \smile \xi \]
\[ = \partial(\alpha \smile \xi) - (-1)^{\vert \alpha \vert} \alpha \smile \partial \xi \]
\[ = \partial(\alpha \smile \xi). \]

So, again, \( \beta \smile \xi \) is a boundary.

Summing up our computations thus far, we have seen that for a fixed choice of IAW, the cap product takes a cohomology class and a homology class to a homology class. Next, we must show that altering IAW within its chain homotopy class does not affect the output.

To see this, we observe that if \( \bar{d} \) and \( \bar{d}' \) are two algebraic diagonals based on two different choices of IAW, then we have
\[ \bar{d}(\xi) - \bar{d}'(\xi) = D\partial \xi + \partial D\xi = \partial D\xi, \]
where \( D \) is the chain homotopy between \( \bar{d} \) and \( \bar{d}' \) induced by the chain homotopy between the two choices of IAW. So, changing either IAW within its chain homotopy class results in altering the cycle \( \bar{d}(\xi) \) by a boundary. Notice that a boundary in \( I^pS_*(X, A; R) \otimes I^qS_*(X, B; R) \) has the form \( \partial (\sum u_\ell \otimes v_\ell) = \sum (\partial u_\ell) \otimes v_\ell + (-1)^{\vert u_\ell \vert} u_\ell \otimes (\partial v_\ell) \).
But then we compute

\[
(\Phi(\text{id} \otimes \beta)) \left( \partial \sum_{\ell} u_\ell \otimes v_\ell \right) = (\Phi(\text{id} \otimes \beta)) \left( \sum_{\ell} ((\partial u_\ell) \otimes v_\ell - (-1)^{\|u_\ell\|} u_\ell \otimes \partial(v_\ell)) \right)
\]

\[
= \Phi \left( \sum_{\ell} (-1)^{\|u_\ell\|} (\partial u_\ell) \otimes \beta(v_\ell) + \sum_{\ell} (-1)^{\|u_\ell\|+\|v_\ell\|} u_\ell \otimes \beta(\partial(v_\ell)) \right)
\]

\[
= \sum_{\ell} (-1)^{\|u_\ell\|} \beta(v_\ell) \partial u_\ell + \sum_{\ell} (-1)^{\|u_\ell\|+\|v_\ell\|} \beta(\partial(v_\ell)) u_\ell
\]

\[
= \sum_{\ell} (-1)^{\|u_\ell\|} \beta(v_\ell) \partial u_\ell + \sum_{\ell} (-1)^{\|u_\ell\|+\|v_\ell\|} (-1)^{\|v_\ell\|+1} (d\beta)(v_\ell) u_\ell
\]

\[
= \partial \left( \sum_{\ell} (-1)^{\|u_\ell\|} \beta(v_\ell) u_\ell \right),
\]

using that \( \beta \) is a cocycle. This term is a boundary, so we see that altering \( \bar{d}(\xi) \) by a boundary alters \( \beta \lhd \xi = \Phi(\text{id} \otimes \beta) \bar{d}(\xi) \) by a boundary, and therefore the homology class remains unchanged.

\[\square\]

### 7.3 Properties of cup, cap, and cross products.

In this (lengthy) section, we develop the various properties of cup and cap products in intersection homology and cohomology. Since we do not have a concrete Alexander-Whitney map to work with, but only one defined up to chain homotopy as a chain homotopy inverse of the intersection chain cross product, it is only at the level of homology and cohomology (as opposed to that of chains and cochains) that these properties can be formulated in a way that is independent of these choices. We will derive formulas reminiscent of the familiar ones from ordinary homology and cohomology theory, but statements will require some conditions and the proofs will require some care. The proofs of these properties in the standard textbook treatments generally use either the front face/back face formulas, which allow for very concrete computations at the chain level, or they rely on acyclic model arguments, which are also not available to us as we do not necessarily have acyclic generators of any of our chain complexes. What we rely on instead are the remarkable properties of the Eilenberg-Zilber shuffle product to show that certain diagrams commute on the nose; then we replace the Eilenberg-Zilber cross products with IAW maps going in the opposite directions to obtain homotopy commutative diagrams. These then have to be deployed in the right way in order to obtain the desired properties.

One nice feature of our program is that it demonstrates that acyclic models are unnecessary in the classical literature (which certainly must have been the original point of view when these tools were being developed), though after reading through our contortions below, the reader might well end up grateful for the acyclic model theorem. We also note that, when using acyclic models to obtain homotopy commutative diagrams, many texts tend not to provide all the arguments for the following properties in detail. Therefore, we hope that we are providing a service for those who would like a reference to the standard properties.
even for ordinary (co)homology, to which intersection (co)homology reduces for trivial filtrations. One source that does provide a fairly thorough treatment is Dold’s book [23] (though again relying on acyclic models to produce the initial homotopy commutative diagrams). We will not follow Dold precisely (though we do in many places), but it is a good reference for those wishing to see the treatment for ordinary (co)homology and a good place to find some additional properties that the reader might wish to generalize to the intersection setting.

There is a further point the reader should keep in mind when considering some of the more involved arguments below, especially the reader who will notice that the analogous arguments in [38] are much simpler. The point is that everything would be much easier with coefficients in a field $F$. In that case, for any chain complex $C_*$ over $F$, one has $H^*(\text{Hom}(C_*, F)) \cong \text{Hom}(H_*(C, F); F)$. So, if one wants to check, for example, that $\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha$ on a space $X$, it is only necessary to evaluate each expression in the claimed equality on cycles representing elements of $H_*(X; F)$. If the resulting evaluations are equal for arbitrary such representatives, then the equality is verified. Unfortunately, of course, in general $H^*(\text{Hom}(C_*, R)) \not\cong \text{Hom}(H_*(C, R); R)$ when $R$ is not a field, and, in particular, one cannot distinguish cohomology classes only by evaluating them on cycles — how they act on other chains is relevant. Thus, as already indicated, our main strategy for proving cohomological identities will be to show that two expressions are obtained by applying chain homotopic maps to a single cohomological expression. Since chain homotopic maps induce the same map on cohomology, we obtain identities in the image cohomology. Occasionally, in order for us to carry out this program, it will be necessary to perform some hands-on computations via evaluation of cochains on chains in order to demonstrate equalities of expressions for cochains. However, it is important to note that such expressions will be carried out at the level of cochains (not cohomology) and the evaluations will be applied to chains, not just cycles. Of course, this is an acceptable way to verify equalities at the cochain level.

Our pattern of attack for properties involving the cap product will tend to be a bit more irregular. As cap products involve both homology and cohomology, functoriality will not run in a single direction. Oddly enough, this will not cause a serious problem, and, in fact, proving properties involving cap products will often be easier than proving the analogous properties for cup products. In some sense, this is due to the fact that, in this setting, the data on the homological side really does come in the form of homology classes (i.e. cycles) and not just arbitrary chains.

As we proceed, we will group by topic, rather than by product type. For example, all of the associativity properties are discussed in a single section, as opposed to, say, all of the cup product properties being contained in a single section. As the properties of intersection cup, cap, and cross products are spread out over so many pages of proofs, we have provided a summary in the next subsection, Section 7.3.9.

### 7.3.1 Naturality

The cup, cap, and cross products satisfy naturality conditions with respect to maps of spaces that satisfy enough conditions for all the relevant terms to be well-defined. As we noted in
Remark 5.18 concerning the naturality of the homology cross product, it is possible for the maps of spaces to be identity maps, in which case we obtain statements about naturality with respect to change of perversity; this observation applies just as well in this section.

The key lemmas for our naturality statements are the following:

**Lemma 7.30.** Let $X$ and $Y$ be filtered sets with open subsets $A, B \subset X$ and $C, D \subset Y$. Let $f: X \to Y$ be a map with $f(A) \subset C$ and $f(B) \subset (D)$. Suppose $p, q, r$ are perversities on $X$ with $D^f \geq Dp + Dq$ and $\tilde{u}, \tilde{v}, \tilde{s}$ are perversities on $Y$ with $D\tilde{s} \geq D\tilde{u} + D\tilde{v}$. Suppose that $f$ is $(\tilde{p}, \tilde{u})$-stratified, $(\tilde{q}, \tilde{v})$-stratified, and $(\tilde{r}, \tilde{s})$-stratified. Then the following diagram commutes:

$$
\begin{array}{ccc}
I^*S_*(X, A \cup B; R) & \xrightarrow{d} & I^{Q_\beta \times}S_*(X \times X, (A \times X) \cup (X \times B); R) & \xleftarrow{\epsilon} & I^{\tilde{p}}S_*(X, A; R) \otimes I^{\tilde{s}}S_*(X, B; R) \\
\downarrow f & & \downarrow f \times f & & \downarrow f \otimes f \\
I^*S_*(Y, C \cup D; R) & \xrightarrow{d} & I^{Q_\alpha \times}S_*(Y \times Y, (C \times Y) \cup (Y \times D); R) & \xleftarrow{\epsilon} & I^{\tilde{q}}S_*(Y, C; R) \otimes I^{\tilde{r}}S_*(Y, D; R).
\end{array}
$$

Thus, if $R$ is a Dedekind domain and $X, Y$ are CS sets satisfying the requirements for the K"unneth Theorem (Theorem 6.61) to imply that the $\epsilon$ maps are homotopy equivalences, then \(d \circ f\) is chain homotopic to \((f \otimes f) \circ d\) as maps \(I^*S_*(X, A \cup B; R) \to I^*S_*(Y, C; R) \otimes I^*S_*(Y, D; R)\).

**Proof.** First, we note that it follows from $f$ being $(\tilde{p}, \tilde{u})$-stratified and $(\tilde{q}, \tilde{v})$-stratified that $f \times f$ is $(Q_{\beta \times}, Q_{\alpha \times})$-stratified. The conditions on the perversities also ensure that both diagonal maps $d$ are well defined and that each $\epsilon$ is a chain homotopy equivalence by Theorem 6.61.

The commutativity of the left square of the diagram diagram holds at the level of spaces, as $(f \times f) \circ d(x) = (f, f)(x, x) = (f(x), f(x)) = d \circ f(x)$. The square on the right is a special case of the diagram considered in Lemma 5.16. Therefore, the diagram commutes. As the diagram commutes, the version of the diagram with each $\epsilon$ replaced by an IAW in the opposite direction homotopy commutes, and so $d \circ f$ is chain homotopic to $(f \otimes f) \circ d$.

Since we will use analogous arguments often below, it is worth verifying this sort of claim in detail at least once, which we do here. By Lemma 5.16 we know that $(f \times f) \circ \epsilon = \epsilon(f \otimes f)$ exactly. By applying the appropriate IAW maps to each side, we obtain that IAW$(f \times f) \circ \epsilon \circ$ IAW = IAW$(f \otimes f) \circ$ IAW. But now using that IAW and $\epsilon$ are chain homotopy inverses, IAW$(f \times f) \circ \epsilon \circ$ IAW $\sim$ IAW$(f \times f)$ and IAW$(f \otimes f) \circ \epsilon \circ$ IAW $\sim$ $(f \otimes f) \circ$ IAW. Thus IAW$(f \times f)$ and $(f \otimes f) \circ$ IAW are chain homotopic. Therefore,

\[
\begin{align*}
\circ d f &= \text{IAW}(f \times f) \circ d \\
&= \text{IAW}(f \times f) \circ d \\
&\sim (f \otimes f) \circ \text{IAW} \circ d \\
&= (f \otimes f) \circ d.
\end{align*}
\]
Lemma 7.31. Let $C_*, D_*, C_*', D_*'$ be complexes of $R$-modules, and let $f : C_* \to C_*'$ and $g : D \to D_*'$ be degree 0 chain maps. Then the following diagram commutes:

$$
\begin{array}{ccc}
\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) & \xrightarrow{\Theta} & \text{Hom}(C_\ast \otimes D_*, R) \\
(f \otimes g)^* & & (f \otimes g)^*
\end{array}
$$

Proof. Let $\alpha \in \text{Hom}(C_*, R)$, $\beta \in \text{Hom}(D_*, R)$. To show that the diagram commutes, it suffices to apply both compositions to $\alpha \otimes \beta$ and check how the images act on generators $x \otimes y \in C_* \otimes D_*$. We have

$$
[(f \otimes g)^* \Theta(\alpha \otimes \beta)](x \otimes y) = \Theta(\alpha \otimes \beta)(f \otimes g)(x \otimes y)
$$

$$
= \Theta(\alpha \otimes \beta)(f(x) \otimes g(y))
$$

$$
= (-1)^{[\beta] \cdot [x]} \alpha(f(x)) \beta(g(y))
$$

$$
= (-1)^{[\beta] \cdot [x]} (f^* (\alpha))(x) \cdot (g^* (\beta))(y)
$$

$$
= [\Theta((f^* \alpha) \otimes (g^* \beta))](x \otimes y)
$$

$$
= [\Theta(f^* \otimes g^*)(\alpha \otimes \beta)](x \otimes y).
$$

\qed

Using Lemmas 5.16, 7.30 and 7.31, we can now obtain naturality of the cross and cup products.

Lemma 7.32. Let $R$ be a Dedekind domain, and let $(X, A)$, $(Y, B)$, $(X', A')$ and $(Y', B')$ be pairs of CS sets and open subsets. Let $f : X \to X'$ and $g : Y \to Y'$ be maps with $f(A) \subseteq A'$ and $f(B) \subseteq B'$. Suppose $\bar{p}, \bar{q}, \bar{p}', \bar{q}'$ are respective perversities on $X, Y, X', Y'$. Suppose that $f$ is $(\bar{p}, \bar{p}')$-stratified, that $g$ is $(\bar{q}, \bar{q}')$-stratified, that $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is locally $(\bar{q}, R)$-torsion free, and that $X'$ is locally $(\bar{p}', R)$-torsion free or $Y'$ is locally $(\bar{q}', R)$-torsion free.

Then if $\alpha \in I_{\bar{p}} H^i(X', A'; R)$ and $\beta \in I_{\bar{q}} H^j(Y', B'; R)$, we have

$$(f \times g)^*(\alpha \otimes \beta) = (f^*(\alpha)) \times (g^*(\beta)) \in I_{\bar{p}', \bar{q}'} H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R).$$

Proof. It follows from $f$ being $(\bar{p}, \bar{p}')$-stratified and $g$ being $(\bar{q}, \bar{q}')$-stratified that $f \times g$ is $(\bar{p}, \bar{q}, \bar{p}', \bar{q}')$-stratified. Now, applying the definitions and Lemmas 5.16 and 7.31 we have

$$(f \times g)^*(\alpha \otimes \beta) = (f \times g)^* \text{IAW}^* \Theta(\alpha \otimes \beta)$$

$$= \text{IAW}^* (f \otimes g)^* \Theta(\alpha \otimes \beta)$$

$$= \text{IAW}^* \Theta(f^* \otimes g^*)(\alpha \otimes \beta)$$

$$= \text{IAW}^* \Theta((f^* (\alpha)) \otimes (g^* (\beta)))$$

$$= (f^* (\alpha)) \times (g^* (\beta)).$$

\qed
Lemma 7.33. Let $R$ be a Dedekind domain, and let $X$ and $Y$ be CS sets with open subsets $A, B \subset X$ and $C, D \subset Y$. Let $f : X \to Y$ be a map with $f(A) \subset C$ and $f(B) \subset D$. Suppose $\bar{p}, \bar{q}, \bar{r}$ are perversities on $X$ with $D\bar{r} \geq D\bar{p} + D\bar{q}$ and $\bar{u}, \bar{v}, \bar{s}$ are perversities on $Y$ with $D\bar{s} \geq D\bar{u} + D\bar{v}$. Suppose that $f$ is $(\bar{p}, \bar{u})$-stratified, $(\bar{q}, \bar{v})$-stratified, and $(\bar{r}, \bar{s})$-stratified, that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free, and that $Y$ is locally $(\bar{u}, R)$-torsion free or locally $(\bar{v}, R)$-torsion free.

Then if $\alpha \in I_\bar{a}H^i(Y, C; R)$ and $\beta \in I_\bar{b}H^j(Y, D; R)$, we have

$$f^*(\alpha \sim \beta) = (f^*(\alpha)) \sim (f^*(\beta)) \in I_\bar{r}H^{i+j}(X, A \cup B; R).$$

Proof. The conditions on $f$ and the perversities ensure that all terms in the expression are well-defined.

Now, we compute using Lemmas 7.30 and 7.31:

$$(f^*(\alpha)) \sim (f^*(\beta)) = \tilde{d}^*\Theta((f^*(\alpha)) \otimes (f^*(\beta)))$$

$$= \tilde{d}^*\Theta(f^* \otimes f^*)(\alpha \otimes \beta)$$

$$= \tilde{d}^*(f \otimes f)^*\Theta(\alpha \otimes \beta)$$

by Lemma 7.31

$$= f^*\tilde{d}^*\Theta(\alpha \otimes \beta)$$

by Lemma 7.30

$$= f^*(\alpha \sim \beta).$$

\[\square\]

Now we turn to naturality of the cap product, where the mixed functoriality of the cap product makes the statement of naturality a bit more complex.

Lemma 7.34. Let $R$ be a Dedekind domain, and let $X$ and $Y$ be CS sets with open subsets $A, B \subset X$ and $C, D \subset Y$. Let $f : X \to Y$ be a map with $f(A) \subset C$ and $f(B) \subset D$. Suppose $\bar{p}, \bar{q}, \bar{r}$ are perversities on $X$ with $D\bar{r} \geq D\bar{p} + D\bar{q}$ and $\bar{u}, \bar{v}, \bar{s}$ are perversities on $Y$ with $D\bar{s} \geq D\bar{u} + D\bar{v}$. Suppose that $f$ is $(\bar{p}, \bar{u})$-stratified, $(\bar{q}, \bar{v})$-stratified, and $(\bar{r}, \bar{s})$-stratified. Suppose that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free and that $Y$ is locally $(\bar{u}, R)$-torsion free or locally $(\bar{v}, R)$-torsion free.

Then if $\beta \in I_\bar{b}H^j(Y, D; R)$ and $\xi \in I_\bar{r}H_{r+j}(X, A \cup B; R)$, we have

$$\beta \sim f(\xi) = f(f^*(\beta) \sim \xi) \in I^aH(X, C; R).$$

Proof. Once again, the conditions on $f$ and the perversities ensure that all terms in the expression are well-defined.

We compute

$$\beta \sim f(\xi) = \Phi(1 \otimes \beta)\tilde{d}f(\xi)$$

$$= \Phi(1 \otimes \beta)(f \otimes f)\tilde{d}(\xi)$$

by Lemma 7.30

$$= \Phi(f \otimes f\beta)\tilde{d}(\xi)$$

$$= \Phi(f \otimes f^*\beta)\tilde{d}(\xi)$$

$$= \Phi(f \otimes id)(id \otimes f^*\beta)\tilde{d}(\xi)$$

$$= f\Phi(id \otimes f^*\beta)\tilde{d}(\xi)$$

see below

$$= f(f^*(\beta) \sim \xi).$$

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Here, in the next to last equality, we have used that $\Phi(f \otimes \text{id}) = f \Phi$. This is immediate in generality, as if $x \otimes 1 \in C_* \otimes R$ is a generator for some chain complex $C_*$ and $f : C_* \to D_*$ is some chain map, then

$$\Phi(f \otimes \text{id})(x \otimes 1) = \Phi(f(x) \otimes 1) = f(x) = f(\Phi(x \otimes 1)).$$

\[\square\]

Remark 7.35. Recall from Examples 4.2 and 4.3 that inclusions of open subsets and normally nonsingular inclusions are what we might call $(\bar{p}, \bar{p})$-stratified maps, where the first perversity is the restricted perversity on the subset. Therefore, the naturality statements of this section apply to such inclusion maps. This is a nice fact that we will use below in proving Poincaré duality for stratified pseudomanifolds.

Before moving on, we pause to observe that our naturality results also demonstrate that our products are, up to canonical isomorphisms, independent of stratification under the hypotheses necessary for Theorems 5.52 and 7.16 to show that our intersection homology and cohomology modules are likewise stratification independent. In particular, suppose that $X$ and $X'$ are two CS set stratifications of the same underlying space $|X|$, neither of which possesses codimension one strata. Let $X^*$ denote $|X|$ with its intrinsic stratification. Furthermore, suppose that $\bar{p}$, $\bar{q}$, and $\bar{r}$ are GM perversities; then these perversities depend only on the codimension of strata and so are defined on $X$, $X'$, and $X^*$. By Proposition 6.33 $X$ is locally $(\bar{p}, R)$-torsion free if and only if $X'$ is if and only if $X^*$ is and similarly for $\bar{q}$. Then our naturality results for the cup product together with Theorems 5.52 and 7.16, show that we have a commutative diagram

$$
\begin{array}{ccc}
I_{\bar{p}}H^i(X, A; R) \otimes I_{\bar{q}}H^j(X, B; R) & \sim & I_{\bar{r}}H^{i+j}(X, A \cup B; R) \\
\downarrow \cong & & \downarrow \cong \\
I_{\bar{p}}H^i(X^*, A^*; R) \otimes I_{\bar{q}}H^j(X^*, B^*; R) & \sim & I_{\bar{r}}H^{i+j}(X^*, A^* \cup B^*; R) \\
\downarrow \cong & & \downarrow \cong \\
I_{\bar{p}}H^i(X', A'; R) \otimes I_{\bar{q}}H^j(X', B'; R) & \sim & I_{\bar{r}}H^{i+j}(X', A' \cup B'; R).
\end{array}
$$

We leave the reader to formulate the appropriate analogues for the other products and state our conclusion as follows:

**Theorem 7.36.** Up to canonical isomorphisms, the cup, cap, and cross products are independent of CS set stratification assuming that no stratification has codimension one strata and all perversities are GM perversities.
7.3.2 Commutativity

Next we turn to the (graded) commutativity of cup and cross products. These properties are based on the following lemmas:

**Lemma 7.37.** Let $R$ be a Dedekind domain. Suppose that $\bar{p}, \bar{q}, \bar{r}$ are perversities on a CS set $X$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$ and that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Let $A, B \subset X$ be open subsets.

Then the following diagram commutes up to chain homotopy:

$$
\begin{array}{c}
I^\bar{r}S_*(X, A \cup B; R) \xrightarrow{d} I^\bar{p}S_*(X, A; R) \otimes I^\bar{q}S_*(X, B; R) \\
\downarrow \tau \downarrow \\
I^\bar{q}S_*(X, B; R) \otimes I^\bar{p}S_*(X, A; R).
\end{array}
$$

Here $\tau$ is the standard (signed!) interchange map of tensor product factors and the two algebraic diagonals are defined with respect to the appropriate ordering of perversities (tacit in the notation).

**Proof.** The following diagram is commutative:

$$
\begin{array}{c}
I^\bar{r}S_*(X, A \cup B; R) \xrightarrow{d} I^{Q_{\bar{p}, \bar{q}}}S_*(X \times X; (A \times X) \cup (X \times B), R) \xleftarrow{\epsilon} I^{Q_{\bar{q}, \bar{p}}}S_*(X, A; R) \otimes I^{Q_{\bar{p}, \bar{q}}}S_*(X, B; R) \\
\downarrow \tau \downarrow \\
I^{Q_{\bar{q}, \bar{p}}}S_*(X \times X, (B \times X) \cup (X \times A); R) \xleftarrow{\epsilon} I^\bar{p}S_*(X, B; R) \otimes I^\bar{q}S_*(X, A; R).
\end{array}
$$

Here the lefthand vertical arrow is induced by the topological map $t(x, y) = (y, x)$, which clearly satisfies $td = d$ and takes $Q_{\bar{p}, \bar{q}}$-allowable chains to $Q_{\bar{q}, \bar{p}}$ allowable chains. The commutativity of the square follows from Lemma 5.20.

It now follows that replacing the maps $\epsilon$ with the homotopy inverses IAW (going in the opposite directions) will yield a diagram that is homotopy commutative. Commutativity of the diagram in the statement of the lemma now follows from the definition $d = \text{IAW}d$. \qed

We also need to know that $\Theta$ satisfies an interchange property. This property is purely algebraic:

**Lemma 7.38.** Suppose $C_*, D_*$ are chain complexes of $R$-modules. The following diagram commutes:

$$
\begin{array}{c}
\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) \xrightarrow{\Theta} \text{Hom}(C_* \otimes D_*, R) \\
\tau \downarrow \quad \quad \quad \quad \tau^* \downarrow \\
\text{Hom}(D_*, R) \otimes \text{Hom}(C_*, R) \xrightarrow{\Theta} \text{Hom}(D_* \otimes C_*, R).
\end{array}
$$
Here both maps labeled $\tau$ are the maps that interchange factors, with appropriate signs.

Proof. The $\tau$ on the left is analogous to that of the preceding lemma, and so is a chain map by Lemma 5.20. The map on the right is technically the Hom dual of the chain map $\tau : D_* \otimes C_* \rightarrow C_* \otimes D_*$. We proceed by direct computation. If $y \otimes x$ is a generator of $D_* \otimes C_*$ and $\alpha, \beta$ are respective generators of $\text{Hom}(C_*, R)$ and $\text{Hom}(D_*, R)$, then

$$
(\tau^* \Theta(\alpha \otimes \beta))(y \otimes x) = \Theta(\alpha \otimes \beta)\tau(y \otimes x)
= \Theta(\alpha \otimes \beta)(-1)^{|x||y|} x \otimes y
= (-1)^{|x||y|+|\beta||x|} \alpha(x) \beta(y),
$$

while

$$
\Theta(\tau(\alpha \otimes \beta))(y \otimes x) = (-1)^{|\alpha||\beta|} \Theta(\beta \otimes \alpha)(y \otimes x)
= (-1)^{|\alpha||\beta|+|\alpha||y|} \alpha(x) \beta(y).
$$

Now, both expressions will be 0 unless $|y| = |\beta|$ and $|x| = |\alpha|$, and so both expressions are $\alpha(x) \beta(y)$. \qed

Corollary 7.39. Let $R$ be a Dedekind domain. Suppose that $X, Y$ are CS sets with respective perversities $\bar{p}, \bar{q}$. Let $A \subset X$ and $B \subset Y$ be open subsets, and suppose that $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is locally $(\bar{q}, R)$-torsion free.

Let $\alpha \in I_{\bar{p}}H^i(X, A; R)$ and $\beta \in I_{\bar{q}}H^j(Y, B; R)$. Then $t^*(\alpha \times \beta) = (-1)^{ij} \beta \times \alpha \in I_{Q_{\bar{p}, \bar{q}}}H^{i+j}(Y \times X, (B \times X) \cup (Y \times A); R)$.

Proof. We compute

$$
t^*(\alpha \times \beta) = t^*\text{IAW}^* \Theta(\alpha \otimes \beta)
= \text{IAW}^* \tau^* \Theta(\alpha \otimes \beta)
= \text{IAW}^* \Theta \tau(\alpha \otimes \beta)
= (-1)^{ij} \text{IAW}^* \Theta(\beta \otimes \alpha)
= (-1)^{ij} \beta \times \alpha.
$$

In the second line, we have used that the strict commutativity of Lemma 5.20 becomes commutativity up to chain homotopy if we replace each map $\epsilon$ by a map IAW going in the opposite direction, and this implies that $t^*\text{IAW}^* = (\text{IAW}t)^* = (\tau \text{IAW})^* = \text{IAW}^* \tau^*$ as a map on cohomology. \qed

Corollary 7.40. Let $R$ be a Dedekind domain. Suppose that $\bar{p}, \bar{q}, \bar{r}$ are perversities on a CS set $X$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$ and that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Let $A, B \subset X$ be open subsets, and let $\alpha \in I_{\bar{p}}H^i(X, A; R)$ and $\beta \in I_{\bar{q}}H^j(X, B; R)$. Then $\alpha \sim \beta = (-1)^{ij} \beta \sim \alpha \in I_{\bar{r}}H^{i+j}(X, A \cup B; R)$. 272
Lemma 7.41. Let \( \alpha = 1 \) play a privileged role in the unital property, and hence in intersection cohomology theory.

\( i = 1 \)

The unital property of cup products, requires consideration of projection maps an algebraic diagonal of the form \( \bar{\alpha} \).

Based on our requirements for the algebraic diagonal, we see that this will require having \( \bar{\alpha} \) and together with the last paragraph, this forces the top perversity \( \bar{t} \) to play a privileged role in the unital property, and hence in intersection cohomology theory.

\( \bar{d} \) is counital in the homotopy category, with appropriate conditions, and which leads to the unital property of cup products, requires consideration of projection maps \( p_i : X \times X \to Y, i = 1, 2 \), which project the product to its factors, i.e. \( p_1(x,y) = x \) and \( p_2(x,y) = y \). For these maps to induce maps on intersection homology, we will see in the next lemma that we need to have \( \bar{q} \leq \bar{t} \), and together with the last paragraph, this forces the top perversity \( \bar{t} \) to play a privileged role in the unital property, and hence in intersection cohomology theory.

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The unitality of the standard cup product is the fact that \( 1 \sim \alpha = \alpha \sim 1 = \alpha \), where \( \alpha \in H^*(X;R) \) and \( 1 \in H^0(X;R) \) is the element that evaluates each (positively-oriented) singular 0-simplex to 1. Similarly, the unital property for cross products is that, if \( 1 \in H^0(pt,R) \), then \( \alpha \times 1 = 1 \times \alpha = \alpha \) in \( H^*(X;R) \).

To have a unit for the cup product in intersection cohomology, we first need to see when it can be true that we have a cup product \( I_\beta H^*(X;R) \otimes I_q H^0(X;R) \to I_\beta H^*(X;R) \).

Based on our requirements for the algebraic diagonal, we see that this will require having \( D\bar{p} \geq D\bar{p} + D\bar{q} \), i.e. that \( \bar{D}\bar{q} \leq \bar{t} \) that \( \bar{q} \geq \bar{t} \). So the condition \( \bar{q} \geq \bar{t} \) is necessary to have an algebraic diagonal of the form \( \bar{d} : I_\bar{p} H_\bar{p}(X;R) \to H_\bar{\alpha}(I_\bar{\beta} S_\bar{\alpha}(X;R) \otimes I_\bar{\beta} S_\bar{\alpha}(X;R)) \).

On the other hand, the proof of Lemma 7.45, below, which demonstrates that \( \bar{d} \) is counital in the homotopy category, with appropriate conditions, and which leads to the unital property of cup products, requires consideration of projection maps \( p_i : X \times X \to Y, i = 1, 2 \), which project the product to its factors, i.e. \( p_1(x,y) = x \) and \( p_2(x,y) = y \). For these maps to induce maps on intersection homology, we will see in the next lemma that we need to have \( \bar{q} \leq \bar{t} \), and together with the last paragraph, this forces the top perversity \( \bar{t} \) to play a privileged role in the unital property, and hence in intersection cohomology theory.

Lemma 7.41. Let \( X,Y \) be filtered spaces with respective perversities \( \bar{p}, \bar{q} \). The projection map \( p_1 : X \times Y \to X \) induces a well-defined chain map \( p_1 : I_{\bar{p},\bar{q}} S_{\bar{p}}(X \times Y;R) \to I_{\bar{p}} S_{\bar{p}}(X;R) \) if \( \bar{q} \leq \bar{t} \). Similarly, the projection map \( p_2 : X \times Y \to Y \) induces a well-defined chain map \( p_2 : I_{\bar{p},\bar{q}} S_{\bar{p}}(X \times Y;R) \to I_{\bar{q}} S_{\bar{q}}(Y;R) \) if \( \bar{p} \leq \bar{t} \). If these perversity requirements are not satisfied, then such maps do not exist in general.

Proof. We will demonstrate the lemma for \( p_1 \), the proof for \( p_2 \) being equivalent.

First, let us see when \( p_1 \) preserves allowability. By the computation of Remark 4.6 (see also Remark 6.14), to show that \( p_1 \) takes allowable simplices to allowable simplices, we only need to check that \( Q_{\bar{p},\bar{q}}(S \times T) \leq \bar{p}(S) \leq \text{codim}_{X\times Y}(S \times T) = \text{codim}_X(S) + \text{codim}_Y(T) \) for each singular stratum \( S \) of \( X \) and arbitrary stratum \( T \) of \( Y \). Notice that \( \text{codim}_{X\times Y}(S \times T) = \text{codim}_X(S) + \text{codim}_Y(T) \), so we need \( Q_{\bar{p},\bar{q}}(S \times T) \leq \bar{p}(S) \leq \text{codim}_Y(T) \). If \( T \) is a regular stratum, \( Q_{\bar{p},\bar{q}}(S \times T) \leq \bar{p}(S) \), and the inequality is satisfied. If \( T \) is singular, \( Q_{\bar{p},\bar{q}}(S \times T) \leq \bar{p}(S) + \bar{q}(T) = \bar{q}(T) + \bar{p}(S) = \bar{q}(T) + 2 \), so the condition becomes \( \bar{q}(T) + 2 \leq \text{codim}_X(T) \),
or $\bar{q}(T) \leq \text{codim}_X(T) - 2 = \bar{t}(T)$. So, if $\bar{q} \leq \bar{t}$, then $p_1$ takes allowable simplices to allowable simplices.

Now, as observed in Remark 6.14, preserving allowability is not in itself sufficient to guarantee a chain map of non-GM intersection chains. Furthermore, if $Y$ possesses a singular stratum $S$ and $\mathcal{R}$ is a regular stratum of $X$ then $p_1$ takes $\mathcal{R} \times S$ to $\mathcal{R}$, so it is not the case that $p_1(\Sigma_{X \times Y}) \subset \Sigma_X$. Therefore, the conclusions of Remark 6.14 are not sufficient to demonstrate that $p_1$ induces a chain map, even when $\bar{q} \leq \bar{t}$. It turns out that $p_1$ does induce a chain map with these assumptions, but we still need a bit more work.

The critical observation is that if $\mathcal{R}$ is a regular stratum of $X$, then $p_1^{-1}(\mathcal{R})$ consists of strata of the form $\mathcal{R} \times S$, and, by the definition of $Q_{p,q}$, we have $Q_{p,q}(\mathcal{R} \times S) = q(S)$. For an $i$-simplex in $X \times Y$ to be $Q_{p,q}$ allowable with respect to $\mathcal{R} \times S$, we must have that $\sigma^{-1}(\mathcal{R} \times S)$ is contained in the $i - \text{codim}_{X \times Y}(\mathcal{R} \times S) + Q_{p,q}(\mathcal{R} \times S)$ skeleton of $\Delta^i$. But $\text{codim}_X(\mathcal{R}) = 0$ and $Q_{p,q}(\mathcal{R} \times S) = q(S)$. If we assume that $\bar{q} \leq \bar{t}$, as we have seen is also necessary for $p_1$ to preserve allowability, then we obtain

$$i - \text{codim}_{X \times Y}(\mathcal{R} \times S) + Q_{p,q}(\mathcal{R} \times S) = i - \text{codim}_Y(S) + q(S)$$

$$\leq i - \text{codim}_Y(S) + \bar{t}(S)$$

$$= i - \text{codim}_Y(S) + \text{codim}_Y(S) - 2$$

$$= i - 2.$$

It follows that if $\xi \in I^{Q_{p,q}}S_i(X \times Y; \mathcal{R})$, then any simplex of $\partial \xi$ that is contained completely in $\Sigma_{X \times Y}$ must in fact be contained in $p_1^{-1}(\Sigma_X)$, as the preceding argument shows that the interior of an $i - 1$ face of an allowable simplex of $\xi$ cannot intersect any singular stratum of the form $\mathcal{R} \times S$. So if $\tau$ is such a face that is contained in $\Sigma_{X \times Y}$, its interior must be contained in $p_1^{-1}(\Sigma_X)$, but this is a closed set, so all of it must be contained in $p_1^{-1}(\Sigma_X)$.

So now consider $\xi \in I^{Q_{p,q}}S_i(X \times Y; R)$. Let $\partial_1 \xi = \partial \xi - \hat{\partial} \xi$, where here we let $\partial$ denote the boundary of $\xi$ as a chain in $S_1(X \times Y; R)$. Recall that $\hat{\partial} \xi$ consists, by definition, of the simplices of $\partial \xi$ (with their coefficients) that do not have image in $\Sigma_{X \times Y}$. Thus $\partial_1 \xi$ comprises those simplices that are contained in $\Sigma_{X \times Y}$, and, by the preceding paragraph, such simplices are actually contained in $p_1^{-1}(\Sigma_X)$. Since $p_1$ induces a chain map on ordinary chains, $\partial p_1(\xi) = p_1(\partial \xi) = p_1(\hat{\partial} \xi) + p_1(\partial_1 \xi)$. Since each simplex of $\partial_1 \xi$ is contained in $p_1^{-1}(\Sigma_X)$, each simplex of $p_1(\partial_1 \xi)$ is contained in $\Sigma_X$. On the other hand, since no simplex of $\hat{\partial} \xi$ is contained in $\Sigma_{X \times Y}$, it follows that no image of a simplex of $p_1(\hat{\partial} \xi)$ is contained in $\Sigma_X$. So, from the definition, we must have $\hat{\partial} p_1(\xi) = p_1(\hat{\partial} \xi)$, showing that $p_1$ is a chain map $p_1 : I^{Q_{p,q}}S_*(X \times Y; R) \rightarrow I^pS_*(X; R)$ when $\bar{q} \leq \bar{t}$.

Lastly, suppose $\bar{q} \leq \bar{t}$. Let $S$ be a stratum of $Y$ such that $\bar{q}(S) > \text{codim}_Y(S) - 2$, and let $U$ be a regular stratum of $Y$ with $S$ in its closure. Let $\sigma : \Delta^i \rightarrow Y$ be a simplex with the image of one $i - 1$ face in $S$ and with the rest of $\Delta^i$ mapping into $U$. Also, let $x_0$ be a point in a regular stratum $\mathcal{R}$ of $X$ and let $\eta : \Delta^i \rightarrow X$ be the unique map with image $x_0$. Then the map $(\eta, \sigma) : \Delta^i \rightarrow X \times Y$ with $(\eta, \sigma)(z) = (\eta(z), \sigma(z))$ is a singular simplex with image in $p_1^{-1}(\mathcal{R})$. Furthermore, $(\eta, \sigma)$ is allowable, as the only singular stratum it intersects is $\mathcal{R} \times S$ and $i - \text{codim}_{X \times Y}(\mathcal{R} \times S) + Q_{p,q}(\mathcal{R} \times S) \geq i - 1$, using our above calculations, but replacing $\bar{q}(S) \leq \bar{t}(S)$ with $\bar{q}(S) > \bar{t}(S)$. In fact, by the same computation, $(\eta, \sigma)$ is allowable as a
Lemma 7.44. Let \( p_1(\eta, \sigma) = \eta \) is allowable with boundary given by the full boundary. In other words, \( p_1 \) is not a chain map of intersection chain complexes in this example. This shows that the condition that \( \hat{q} \leq \hat{\ell} \) is necessary provided \( Y \) has any singular strata.

\[ \square \]

**Corollary 7.42.** Let \( X, Y \) be filtered spaces with respective perversities \( \bar{p}, \bar{q} \) and respective subspaces \( A, B \). The map \( p_1 \) induces a well-defined chain map \( p_1 : I^{\bar{p} \cdot \bar{q}} S_*(X \times Y, A \times Y; R) \to I^\bar{p} S_*(X, A; R) \) if \( \bar{q} \leq \bar{\ell} \). Similarly, the map \( p_2 \) induces a well-defined chain map \( p_2 : I^{\bar{p} \cdot \bar{q}} S_*(X \times Y, X \times B; R) \to I^\bar{p} S_*(Y, B; R) \) if \( \bar{p} \leq \bar{\ell} \). If the conditions on perversities are not satisfied, then such maps do not exist in general.

**Proof.** An equivalent argument to that in the proof of Lemma 7.41 demonstrates that \( p_1 \) induces a well-defined chain map \( I^{\bar{p} \cdot \bar{q}} S_*(A \times Y; R) \to I^\bar{p} S_*(A; R) \). Therefore, \( p_1 : I^{\bar{p} \cdot \bar{q}} S_*(X \times Y; R) \to I^\bar{p} S_*(X; R) \) induces a well-defined map on the quotient complexes \( p_1 : I^{\bar{p} \cdot \bar{q}} S_*(X \times Y, A \times Y; R) \to I^\bar{p} S_*(X, A; R) \). The argument for \( p_2 \) is identical.

The next lemma is similar in spirit to the last, though simpler. It will also be used below in the proof of Lemma 7.45.

**Lemma 7.43.** Let \( X \) be a filtered set with perversity \( \bar{q} \), and let \( p : X \to \text{pt} \) be the map from \( X \) to a point. Then \( p \) induces a well-defined chain map \( p : I^\bar{q} S_*(X; R) \to S_*(\text{pt}; R) \) if \( \bar{q} \leq \bar{\ell} \). If \( \bar{q} \not\leq \bar{\ell} \), then this is not true in general.

**Proof.** If \( \bar{q} \leq \bar{\ell} \), then \( I^\bar{q} S_*(X; R) = I^\bar{q} S_*^{\text{GM}}(X; R) \) by Proposition 6.7. All simplices are allowable in \( S_*(\text{pt}; R) \), so \( p \) takes allowable chains to allowable chains, and since \( I^\bar{q} S_*^{\text{GM}}(X; R) \subset S_*(X; R) \), the desired \( p : I^\bar{q} S_*(X; R) \to S_*(\text{pt}; R) \) is simply the restriction of the chain map \( p : S_*(X; R) \to S_*(\text{pt}; R) \). Therefore, \( p \) is a well-defined chain map when \( \bar{q} \leq \bar{\ell} \).

If \( \bar{q} \not\leq \bar{\ell} \) and \( X \) has a singular stratum, then \( p \) will not necessarily be a chain map. In particular, let \( \sigma \) be as in the example at the end of the proof of Lemma 7.41. Then \( \partial p(\sigma) = \partial p(\sigma) = p(\partial \sigma) \neq p(\partial \sigma) \).

\[ \square \]

We will also need the following purely algebraic lemma:

**Lemma 7.44.** Suppose \( C_* \) and \( D_* \) are chain complexes of \( R \)-modules with \( C_i = D_i = 0 \) for \( i < 0 \) and that \( D_* \) has a degree 0 augmentation map \( a : D_* \to R \). In other words, \( a(x) = 0 \) if \( |x| \neq 0 \) and \( a(x) = 0 \) if \( x \in D_1 \). Let \( \alpha \in \text{Hom}(C_*, R) \). Then

\[
(id \otimes a)^{\Phi^*}(\alpha) = \Theta(\alpha \otimes a) \in \text{Hom}(C_* \otimes D_*, R),
\]

where \( \Phi : C_* \otimes R \to C_* \) is the standard isomorphism.

**Proof.** It suffices to verify that both expressions in the claimed equality act identically on elements of \( C_* \otimes D_* \). So let \( x \otimes y \) be a generator of \( C_* \otimes D_* \). Then

\[
((id \otimes a)^{\Phi^*}(\alpha))(x \otimes y) = a(\Phi(id \otimes a)(x \otimes y)) = a(\Phi(x \otimes a(y))) = a(a(y)x) = a(y)\alpha(x) = \Theta(\alpha \otimes a)(x \otimes y),
\]

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using that $|a| = 0$.  

The discussion preceding Lemma 7.41 as well as the use of Lemma 7.41 in the following argument, demonstrates that we need $\bar{q} = \bar{t}$ in the next lemma in order to have a counital property (up to homotopy) for the algebraic diagonal. This condition also allows us to dispense with explicit mention of the torsion free conditions as every CS set is locally torsion free with respect to $\bar{t}$ by Example 5.39.

**Lemma 7.45.** Let $R$ be a Dedekind domain. Suppose that $\bar{p}$ is a perversity on a CS set $X$ and that $A \subset X$ is an open subset. Then the compositions

$$I^\bar{p}S_*(X, A; R) \xrightarrow{\bar{d}} I^\bar{q}S_*(X; R) \otimes I^\bar{p}S_*(X, A; R) \xrightarrow{\text{id} \otimes \text{id}} I^\bar{p}S_*(X, A; R)$$

and

$$I^\bar{p}S_*(X, A; R) \xrightarrow{\bar{d}} I^\bar{p}S_*(X, A; R) \otimes I^\bar{t}S_*(X; R) \xrightarrow{\text{id} \otimes \text{id}} I^\bar{p}S_*(S, A; R) \otimes R \xrightarrow{\Phi} I^\bar{p}S_*(X, A; R),$$

in which $a$ is the augmentation map, are each homotopic to the identity map $I^\bar{p}S_*(X, A; R) \rightarrow I^\bar{p}S_*(X, A; R)$.

**Proof.** We will demonstrate the claim regarding the first composition. The second argument is equivalent.

As $D\bar{t} = 0$, $D\bar{p} \geq D\bar{t} + D\bar{p}$, and, as just noted, every space is locally $(\bar{t}, R)$-torsion free by Example 5.39. Thus the algebraic diagonal is defined.

Now, consider the diagram

$$I^{Q_{\bar{t}}}S_*(X \times X, A \times X; R) \xleftarrow{\epsilon} I^\bar{p}S_*(X, A; R) \otimes I^\bar{t}S_*(X; R)$$

$$\xrightarrow{p_1} I^\bar{p}S_*(X, A; R) \xleftarrow{\epsilon} I^\bar{p}S_*(X, A; R) \otimes S_*(pt; R).$$

The map $p_1$ is here induced by the projection to the first factor $X \times X \rightarrow X$, which is well defined by Corollary 7.42 and $p$ is induced by the unique map $X \rightarrow pt$ and is well-defined by Lemma 7.43. The bottom cross product is that which occurs in the version of the Künneth theorem for which one factor is a manifold. For fixed indices, each of the relevant chain modules is a submodule of either $S_k(X; R)$ or $S_k(X, A; R)$ for some $k$, and the maps are induced, at the level of modules (i.e. ignoring boundary maps), by the corresponding maps for the ordinary chain modules. Thus, to show that this diagram commutes, it suffices to see that the diagram

$$S_{i+j}(X \times X, A \times X) \xleftarrow{\epsilon} S_i(X, A; R) \otimes S_j(X; R)$$

$$\xrightarrow{p_1} S_i(X, A; R) \xleftarrow{\epsilon} S_i(X, A; R) \otimes S_j(pt; R).$$
commutes. Notice that in the bottom left term, we identify \( X \) with \( X \times \text{pt.} \)

If \( \sigma \) is an \( i \)-simplex representing an element of \( S_i(X, A; R) \) and \( \tau : \Delta^j \to X \) is a \( j \)-simplex, then \( \epsilon(\sigma \otimes \tau) \) is defined by applying \( \sigma \times \tau \) to the singular triangulation of \( \Delta^i \times \Delta^j \) determined by the Eilenberg-Zilber shuffle process. So, proceeding left then down, we obtain the chain in \( S_{i+j}(X, A; R) \) that comes from applying \( p_1(\sigma \times \tau) \) as a chain map to this singular triangulation. But is \((x, y) \in \Delta^i \times \Delta^j\) we have

\[
\epsilon(\sigma) = p_1(\sigma),
\]

so \( p_1(\sigma \times \tau) = \sigma \). If we let \( \pi_1 : \Delta^i \times \Delta^j \to \Delta^i \) be the projection to the first factor, we similarly have \( \sigma \pi_1(x, y) = \sigma(x) \), so \( p_1(\sigma \times \tau) = \sigma \). On the other hand, if \( \pi : \Delta^j \to \text{pt} \) is the unique map to a point, \( \epsilon(\text{id} \otimes p) = \epsilon(p) \) is given by applying \( \sigma \times p(\tau) = \sigma \times \pi \) to the Eilenberg-Zilber singular triangulation of \( \Delta^i \times \Delta^j \). But \( \sigma \pi_1 \) and \( \sigma \times \pi \) agree up to identifying \( X \) with \( X \times \text{pt} \), demonstrating the commutativity.

It follows that if we replace each \( \epsilon \) in the diagram by a homotopy inverse \( \text{IAW} \), we obtain a homotopy commutative diagram

\[
\begin{array}{ccc}
I^{Q_{p,t}}S_*(X \times X, A \times X; R) & \xrightarrow{\text{IAW}} & I^pS_*(X, A; R) \otimes I^lS_*(X; R) \\
| \ & \ & | \\
p_1 & \xrightarrow{\text{id} \otimes p} & \text{id} \\
I^pS_*(X, A; R) & \xrightarrow{\text{IAW}} & I^pS_*(X, A; R) \otimes S_*(\text{pt}; R).
\end{array}
\]

Next, in the special case we are considering here, it is possible to write down an explicit homotopy inverse for the bottom map \( \text{IAW} : I^pS_*(X, A; R) \xrightarrow{\text{IAW}} I^pS_*(X, A; R) \otimes S_*(\text{pt}; R) \). In fact, we claim that the map \( \nu : \xi \to \xi \otimes v_0 \), where \( v_0 \) is the unique singular 0 simplex generating \( S_*(\text{pt}; R) \) is such a homotopy inverse. For this, we note that it follows from the definitions that \( \epsilon \nu = \text{id} \) as chain maps (again identifying \( X \times \text{pt} \) with \( X \)). But this is enough to imply that \( \nu \) is a chain homotopy inverse to \( \epsilon \) by the following argument: Let \( g \) be a homotopy inverse to \( \epsilon \) so that \( g \epsilon \) and \( \epsilon g \) are each homotopic to the appropriate identity; such a \( g \) exists by Theorem 6.61 using that \( S_*(\text{pt}; R) = I^qS_*(\text{pt}; R) \) for any \( q \) and that the ensuing \( Q_{p,q} \) agrees with \( \bar{p} \) on \( X \times \text{pt} = X \) as \( \text{pt} \) has only a regular stratum. Then, as \( g \epsilon \) is homotopic to the identity, \( g \epsilon \nu \) is homotopic to \( \nu \), but \( g \epsilon \nu = g \). So \( \nu \) is homotopic to \( g \) and so is a chain homotopy inverse for \( \epsilon \).

Now consider the larger diagram
\[ I^p S_\ast(X, A; R) \xrightarrow{\mathfrak{d}} I^{q_p, \mathfrak{t}} S_\ast(X \times X, A \times X; R) \xrightarrow{\text{IAW}} I^p S_\ast(X, A; R) \otimes I^\mathfrak{t} S_\ast(X; R) \]

We have already seen that the square commutes; here we are letting \( \nu \) be the map constructed in the preceding paragraph.

Commutativity of the upper left triangle is straightforward as \( p_1 \mathfrak{d} = \text{id}_X \). For the commutativity of the bottom right triangle, we need only observe that \( \Phi(\text{id} \otimes a)\nu(\xi) = \Phi(\text{id} \otimes a)(\xi \otimes v_0) = \Phi(\xi \otimes 1) = \xi \), as \( a \) has degree 0 and \( a(v_0) = 1 \). Therefore, each component of the diagram homotopy commutes, and using that \( \nu \) is a homotopy equivalence (and so its arrow can be reversed), the composition counterclockwise around the outside of the diagram is homotopic to the identity. But, noting that the composition \( I^\mathfrak{t} S_\ast(X; R) \to S_\ast(\text{pt}; R) \xrightarrow{\text{id}} R \) factors the direct augmentation \( I^\mathfrak{t} S_\ast(X; R) \xrightarrow{\text{id}} R \), the path around the outside of the diagram is precisely \( \Phi(\text{id} \otimes a)\mathfrak{d} \).

**Corollary 7.46.** Let \( R \) be a Dedekind domain. Suppose that \( \bar{p} \) is a perversity on a CS set \( X \) and that \( A \subset X \) is an open subset. Let \( \alpha \in I^p H^\ast(X, A; R) \), and let \( 1 \in I^1 H^0(X; R) \) be represented by the cocycle that evaluates to 1 on each singular 0 simplex of \( I^1 S_0(X; R) \). Then \( 1 \sim \alpha = \alpha \sim 1 = \alpha \).

**Proof.** We begin by observing that 1 really is a cocycle. For this, we need only show that if \( \xi \) is a 1-chain in \( I^1 S_1(X; R) \), then \( 1(\partial \xi) = 0 \). By Proposition \( 6.7 \), \( I^1 S_\ast(X; R) = I^1 S_\ast^\text{GM}(X; R) \), so \( \partial \xi \) is the usual boundary of \( \xi \), treating \( \xi \) as an element of \( S_1(X; R) \). Since 1 is a cocycle in \( S^0(X; R) \), it follows that \( 1(\partial \xi) = 0 \).

Now to verify the given property, we observe that

\[
\alpha = \text{id}^* \alpha = (\Phi(\text{id} \otimes a)\mathfrak{d})^* \alpha \quad \text{by Lemma 7.45}
\]

\[
= \mathfrak{d}^* (\text{id} \otimes a)^* \Phi^* (\alpha).
\]

But by Lemma \( 7.44 \) \( (\text{id} \otimes a)^* \Phi^* (\alpha) = \Theta(\alpha \otimes 1) \) at the chain level, using that 1 acts as an augmentation map on \( I^1 S_0(X; R) \). Therefore, \( \alpha = \mathfrak{d}^* \Theta(\alpha \otimes 1) = \alpha \sim 1 \). 

A unital property for cross products can be proven by reworking some of the pieces from Lemma \( 7.45 \).
Corollary 7.47. Let $R$ be a Dedekind domain. Suppose that $\bar{p}$ is a perversity on a CS set $X$ and that $A \subset X$ is an open subset. Let $\alpha \in I^p H^i(X; R)$, and let $1 \in H^0(\text{pt}; R)$ be represented by the cocycle that evaluates to 1 on the singular 0 simplex of $S_0(\text{pt}; R)$. Then $1 \times \alpha = \alpha \times 1 = \alpha$ in $I^p H^i(\text{pt} \times X, \text{pt} \times A; R) = I^p H^i(X \times \text{pt}, A \times \text{pt}; R) = I^p H^i(X; A; R)$.

Proof. We will provide the argument for $\alpha \times 1$, the argument for $1 \times \alpha$ being equivalent.

By definition, $\alpha \times 1 = IAW^{\ast} \Theta(\alpha \otimes 1)$, and by Lemma 7.44 $\Theta(\alpha \otimes 1) = (\text{id} \otimes a)^\ast \Phi^\ast(\alpha)$, again using that the cocycle 1 acts as an augmentation on $S(\text{pt}; R)$. Thus, $\alpha \times 1 = IAW^\ast(\text{id} \otimes a)^\ast \Phi^\ast(\alpha)$. But, we saw in the final paragraph of the proof of Lemma 7.45 that $\Phi(\text{id} \otimes a) IAW$ is the identity, up to identifying $1 \times 1$ with $X$, for a specific choice of IAW that we there labeled $\nu$. Hence $IAW^\ast(\text{id} \otimes a)^\ast \Phi^\ast(\alpha) = \alpha$ at the level of cohomology.

Corollary 7.48. Let $R$ be a Dedekind domain. Suppose that $\bar{p}$ is a perversity on a CS set $X$ and that $A \subset X$ is an open subset. Let $Y$ be a CS set with perversity $\bar{q} \leq \bar{t}$. Suppose that $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is locally $(\bar{q}, R)$-torsion free. Let $\alpha \in I^p H^i(X; A; R)$, and let $1_Y \in I^p H^0(Y; R)$ be represented by the cocycle that evaluates to 1 on each singular 0 simplexes of $I^p S_0(Y; R)$. Then $\alpha \times 1_Y = p_1^\ast(\alpha) \in I^p_\bar{q} H^x(X \times Y, A \times Y; R)$, where $p_1 : X \times Y \to X$ is the projection.

Proof. By Lemma 7.42 $p_1^\ast$ is well-defined. Let $p : Y \to \text{pt}$ be the unique map, and observe that by Lemma 7.43 $p$ also induces a well-defined chain map $p : I^p S_0(\text{pt}; R) \to S_0(\text{pt}; R)$. We observe that if $1_{\text{pt}}$ denotes the element of $S_0(\text{pt}; R)$ that takes the unique 0-simplex to 1, then $p^\ast(1_{\text{pt}}) = 1_Y$. Thus $\alpha \times 1_Y = \alpha \times (p^\ast(1_{\text{pt}})) = (\text{id} \times p)^\ast(\alpha \times 1_{\text{pt}})$ by Lemma 7.32 technically, $p$ and $\text{id} \times p = p_1$ are not stratified maps in that they do not preserve codimension, but we know that they induces maps on intersection chains by Lemmas 7.42 and 7.43 and, given this, the arguments of Lemma 7.32 and Lemma 5.16 continue to apply. But $\alpha \times 1_{\text{pt}} = \alpha$ by Corollary 7.47 and $\text{id} \times p = p_1$ as maps (identifying $X \times \text{pt}$ with $X$, as usual). Therefore, $\alpha \times 1_Y = (\text{id} \times p)^\ast(\alpha \times 1_{\text{pt}}) = p_1^\ast(\alpha)$.

Lemma 7.45 can also be used to show that the cap product corresponds to evaluation in the appropriate setting. Once again, the role played by the top perversity obviates the need for restrictions on $X$.

Lemma 7.49. Let $R$ be a Dedekind domain, and suppose that $\bar{p}$ is a perversity on a CS set $X$ and that $A \subset X$ is an open subset. Let $\alpha \in I^p H^i(X; A; R)$ and $\xi \in I^p H_i(X; A; R)$. Then $a(\alpha \otimes \xi) = a(\xi) \in R$, where $a : I^p H_0(X; R) \to R$ is the augmentation map.

Proof. First, observe that we have a well-defined cap product $I^p H^i(X; A; R) \otimes I^p H_i(X; A; R) \to I^p H_0(X; R)$ as $Dp \geq Dp + D\bar{t} = Dp + 0 = Dp$.

We next claim that if $\alpha \in I^p S_i(X; A; R)$, $a : I^p S_\ast(X; R) \to R$ is the augmentation map, and $\Phi : R \otimes I^p S_\ast(X; R) \to I^p S_\ast(X; R)$ is the canonical isomorphism, then $a\Phi(\text{id} \otimes \alpha) = a\Phi(\alpha \otimes \text{id}) \in \text{Hom}(I^p S_\ast(X; R) \otimes I^p S_\ast(X, A; R), R)$. To check this, let $x \otimes y \in I^p S_\ast(X; R) \otimes
\( I^\beta S_s(X, A; R) \) be a generator. Then we have

\[
(\alpha \Phi(\text{id} \otimes \alpha))(x \otimes y) = (-1)^{|x|} \alpha \Phi(x \otimes \alpha(y)) \\
= (-1)^{|x|} \alpha(x) \alpha(y) \\
= (-1)^{|x|} \alpha(y) \alpha(x) \\
= \alpha(y) \alpha(x),
\]

where the last equality comes from the observation that \( \alpha(x) = 0 \) unless \( |x| = 0 \). Meanwhile,

\[
(\alpha \Phi(\alpha \otimes \text{id}))(x \otimes y) = (\alpha \Phi)(\alpha(x) \otimes y) \\
= \alpha(\alpha(x)y) \\
= \alpha(x)\alpha(y).
\]

So, indeed, \( a\Phi(\text{id} \otimes \alpha) = \alpha \Phi(\alpha \otimes \text{id}) \).

Now, suppose \( \alpha \in I^\beta H^s(X, A; R) \) and \( \xi \in I^\beta H_s(X, A; R) \), and consider the evaluation \( \alpha(\xi) \). Since evaluation is well defined at the level of (co)homology, we know that the element \( \alpha(\xi) \in R \) is independent of the choice of intersection chain representing \( \xi \). By Lemma 7.45 we can therefore replace \( \xi \) in this computation with \( \Phi(\alpha \otimes \text{id})\tilde{d}(\xi) \), where \( \tilde{d} \) is defined with respect to some specific choice of IAW. Therefore, \( \alpha(\xi) = \alpha(\Phi(\alpha \otimes \text{id})\tilde{d}(\xi)) \). But we have just seen that \( \alpha \Phi(\alpha \otimes \text{id}) = \alpha \Phi(\text{id} \otimes \alpha) \), so \( \alpha(\xi) \) becomes equal to \( (\alpha \Phi(\text{id} \otimes \alpha))(\tilde{d}(\xi)) = \alpha((\Phi(\text{id} \otimes \alpha))\tilde{d}(\xi)) \), which is precisely \( \alpha(\alpha \otimes \xi) \).

**Remark 7.50.** There is an observation to be made concerning the proof of Lemma 7.49 that will be useful later in Section 8.5.3 though its utility isn’t likely to appear so useful now. Let us continue to assume that \( \alpha \in I^\beta S^s(X, A; R) \) and \( \xi \in I^\beta S_s(X, A; R) \) but that they are not necessarily a cocycles and a cycle. We also continue to assume we have made a fixed choice of IAW with which to define the cap product. As Lemma 7.45 is stated at the chain level, it tells us that \( \xi - \Phi(\alpha \otimes \text{id})\tilde{d}(\xi) = D\partial \xi + \partial D\xi \), where \( D \) is a chain homotopy guaranteed by Lemma 7.45. The argument in the proof continues to imply that \( \alpha(\Phi(\alpha \otimes \text{id})\tilde{d}(\xi)) = \alpha(\alpha \otimes \xi) \), so, applying \( \alpha \) to the entire expression yields

\[
\alpha(\xi) = \alpha(\alpha \otimes \xi) + \alpha(D\partial \xi + \partial D\xi).
\]

See Section 8.5.3 for our application of this formula.

Of course, there is also a nice formula for evaluation of the cohomology cross product on the homology cross product:

**Lemma 7.51.** Let \( R \) be a Dedekind domain. Suppose that \( X, Y \) are CS sets with respective perversities \( \bar{p}, \bar{q} \). Let \( A \subseteq X \) and \( B \subseteq Y \) be open subsets, and suppose that \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free. Let \( \alpha \in I^{\bar{p}} H^s(X, A; R) \), \( \beta \in I^{\bar{q}} H^s(Y, B; R) \), \( \xi \in I^{\bar{p}} H_i(X, A; R) \) and \( \eta \in I^{\bar{q}} H_j(Y, B; R) \). Then \( (\alpha \times \beta)(\xi \times \eta) = (-1)^{bi}\alpha(\xi)\beta(\eta) \).
Proof. Given the assumptions, $\alpha \times \beta$ is well-defined in $I_{Q_{\beta\delta}}H^{a+b}(X \times Y, (A \times Y) \cup (X \times B); R)$ and $\xi \times \eta \in I_{Q_{\beta\delta}}H_{i+j}(X \times Y, (A \times Y) \cup (X \times B); R)$. Now we have

\[
(\alpha \times \beta)(\xi \times \eta) = IAW^*\Theta(\alpha \otimes \beta)\epsilon(\xi \otimes \eta)
= \Theta(\alpha \otimes \beta)IAW\epsilon(\xi \otimes \eta)
= \Theta(\alpha \otimes \beta)(\xi \otimes \eta)
= (-1)^{bi}\alpha(\xi)\beta(\eta).
\]

In the third line, we have used that IAW and $\epsilon$ are chain homotopy inverses, so that $IAW\epsilon(\xi \otimes \eta) = \xi \otimes \eta \in H_{i+j}(I^pS_*(X, A; R) \otimes I^qS_*(Y, B; R))$. Since $\Theta(\alpha \otimes \beta) \in H^{a+b}(\text{Hom}(I^pS_*(X, A; R) \otimes I^qS_*(Y, B; R), R))$ and evaluation is well-defined from cohomology to homology, these equalities make sense.

7.3.4 Associativity

In this section, we turn to the associativity properties of the various products. One nuisance here is that in order to take the cup (or cross) product of three intersection cohomology classes, associated in some order, we will need to make iterate use of the Künneth theorem. For example, we will need to consider compositions of the form

\[
I^pS_*(X; R) \otimes I^qS_*(Y; R) \otimes I^rS_*(Z; R) \xrightarrow{\epsilon \otimes \text{id}} I^{p+q}S_*(X \times Y; R) \otimes I^rS_*(Z; R) \rightarrow I^{p+q+r}S_*(X \times Y \times Z; R),
\]

and we will need these to be chain homotopy equivalences. We know by Theorem \[\ref{6.61}\] that in order for $I^pS_*(X; R) \otimes I^qS_*(Y; R) \rightarrow I^{p+q}S_*(X \times Y; R)$ to be a chain homotopy equivalence it is sufficient to have either $X$ be locally $(\tilde{p}, R)$-torsion free or $Y$ be locally $(\tilde{q}, R)$-torsion free. However, to employ the Künneth theorem again to a chain homotopy equivalence $I^{p+q}S_*(X \times Y; R) \otimes I^rS_*(Z; R) \rightarrow I^{p+q+r}S_*(X \times Y \times Z; R)$, we then need either $Z$ to be locally $(\tilde{r}, R)$-torsion free or $X \times Y$ to be locally $(Q_{\beta\delta}; R)$-torsion free.

It turns out that asking for a product to be locally $(Q_{\beta\delta}; R)$-torsion free is equivalent to asking for the factors to be respectively locally $(\tilde{p}, R)$-torsion free and locally $(\tilde{q}, R)$-torsion free, as the following lemma demonstrates.

**Lemma 7.52.** Suppose $X$ and $Y$ are CS sets. Then $X \times Y$ is locally $(Q_{\beta\delta}; R)$-torsion free if and only if $X$ is locally $(\tilde{p}, R)$-torsion free and $Y$ is a locally $(\tilde{q}, R)$-torsion free.

**Proof.** Let $S \times T$ be a stratum of $X \times Y$, let $K$ be the link of $S$ in $X$ and let $L$ be the link of $T$ in $Y$. Then the join $K \ast L$ is the link of $S \times T$ in $X \times Y$. If $S$ is a regular stratum of $X$, then $K = \emptyset$, and similarly for $T$ and $L$. We will consider the join with the empty set to be the identity construction, e.g. $K \ast \emptyset = K$.

By definition, $X \times Y$ is locally $(Q_{\beta\delta}; R)$-torsion free if for any such $S \times T$ the torsion product of $I^{Q_{\beta\delta}}H_{\dim(K \ast L) - Q_{\beta\delta}(S \times T) - 1}(K \ast L; R)$ with any $R$-module vanishes, or, equivalently by \[63\] Theorem XVI.3.11, if this intersection homology module is flat. Recall that this

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expression makes sense as we can restrict the perversity $Q_{\bar{p}, \bar{q}}$ on $X \times Y$ to the link $K \ast L$, which we can consider embedded in $X \times Y$. The locally torsion free conditions for $X$ and $Y$ can be similarly restated.

If $K = \emptyset$, i.e. $S$ is a regular stratum, then $Q_{\bar{p}, \bar{q}}(S \times T) = \bar{q}(T)$ and $Q_{\bar{p}, \bar{q}}$ restricts to $\bar{q}$ on $L$, so the condition that $I^{Q_{\bar{p}, \bar{q}}} H_{\dim(K \ast L) - Q_{\bar{p}, \bar{q}} - 1}(K \ast L; R)$ be flat reduces to $I^{\bar{q}} H_{\dim(L) - \bar{q}(T) - 1}(L; R)$ being flat, which is the condition for $Y$ to be locally $(\bar{q}, R)$-torsion free along $T$. Therefore, we see that if $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-torsion free, it follows that $Y$ is locally $(\bar{q}, R)$-torsion free. The analogous argument holds when $L = \emptyset$. Thus if $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-torsion free, we must also have that $X$ is locally $(\bar{p}, R)$-torsion free and $Y$ is a locally $(\bar{q}, R)$-torsion free. Conversely, if $X$ is locally $(\bar{p}, R)$-torsion free and $Y$ is a locally $(\bar{q}, R)$-torsion free, then $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-torsion free along strata $S \times T$ for which one of $S$ or $T$ is regular. It remains to show that if $X$ and $Y$ are each locally torsion free then $X \times Y$ is locally torsion free along products of singular strata.

So, now, suppose that $X$ and $Y$ are both locally torsion free and that neither $K$ nor $L$ is empty. To best mesh with our earlier computations, suppose $\dim(K) = m - 1$ and $\dim(L) = n - 1$. Then

$$
\dim(K \ast L) - Q_{\bar{p}, \bar{q}}(S \times T) - 1 = m + n - 1 - (\bar{p}(S) + \bar{q}(T) + 2) - 1
= m + n - \bar{p}(S) - \bar{q}(T) - 4.
$$

Now let us apply equation [14], which holds by our assumption that we use $Q_{\bar{p}, \bar{q}}$ and that $X$ and $Y$ are appropriately locally torsion free. In our case, deleting terms that must vanish, we obtain\(^{85}\)

$$
I^{Q_{\bar{p}, \bar{q}}} H_{m + n - \bar{p}(S) - \bar{q}(T) - 4}(K \ast L; R) = 
(I^{\bar{p}} H_{m - \bar{p}(S) - 2}(K; R) \otimes_R I^{\bar{q}} H_{n - \bar{q}(T) - 2}(L; R))
\oplus (I^{\bar{p}} H_{m - \bar{p}(S) - 3}(K; R) *_R I^{\bar{q}} H_{n - \bar{q}(T) - 2}(L; R))
\oplus (I^{\bar{p}} H_{m - \bar{p}(S) - 2}(K; R) *_R I^{\bar{q}} H_{n - \bar{q}(T) - 3}(L; R)).
$$

Recalling that $\dim(K) = m - 1$ and $\dim(L) = n - 1$, we see that the locally torsion free conditions on $X$ and $Y$ imply that $I^{\bar{p}} H_{m - \bar{p}(S) - 2}(K; R)$ and $I^{\bar{q}} H_{n - \bar{q}(T) - 2}(L; R)$ are both flat. It follows that the two torsion product terms vanish and that the tensor product term is flat as a tensor product of flat modules. Therefore, we obtain that $I^{Q_{\bar{p}, \bar{q}}} H_{m + n - \bar{p}(S) - \bar{q}(T) - 4}(K \ast L; R)$ is flat, as desired. \(\square\)

According to Lemma [7.52] asking that $X \times Y$ be locally $(Q_{\bar{p}, \bar{q}}, R)$-torsion free is somewhat demanding! Therefore, in order to apply the K"unneth theorem twice in the composition [23], we need to assume

1. $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is locally $(\bar{q}, R)$-torsion free, and

2. $(X$ is locally $(\bar{p}, R)$-torsion free and $Y$ is locally $(\bar{q}, R)$-torsion free) or $Z$ is locally $(\bar{r}, R)$-torsion free.

\(^{85}\)In this expression $K \ast L$ denotes the join of spaces and we use $*_R$ for the torsion product over $R$. 282
Together, these conditions are equivalent to asking that at least two of the following three statements be true:

1. $X$ is locally $(\bar{p}, R)$-torsion free,
2. $Y$ is locally $(\bar{q}, R)$-torsion free,
3. $Z$ is locally $(\bar{r}, R)$-torsion free.

Symmetrically, dealing with the alternatively associated composition of products, first forming $Y \times Z$ and then $X \times Y \times Z$, leads us to the same requirements. To repeat this requirement more succinctly in the following statements, we will simply say that “two out of three of $X$, $Y$, and $Z$ are locally torsion free” when the assigned perversities for the spaces are known, or, when working with a single space $X$, we can say that “$X$ is locally torsion free with respect to two out of the three perversities $\bar{p}$, $\bar{q}$, $\bar{r}$”.

Analogously to Lemma 7.52, we have the following for finite generation. We will not be needing this result in this section, but as the proof is so similar, this is a convenient place for it.

**Lemma 7.53.** Suppose $X$ and $Y$ are CS sets and that $R$ is a Noetherian ring. If $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-finitely generated then $X$ is locally $(\bar{p}, R)$-finitely generated and $Y$ is a locally $(\bar{q}, R)$-finitely generated. The converse holds if we also assume that either $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is a locally $(\bar{q}, R)$-torsion free.

**Proof.** Let $S \times T$ be a stratum of $X \times Y$, let $K$ be the link of $S$ in $X$ and let $L$ be the link of $T$ in $Y$. Then the join $K * L$ is the link of $S \times T$ in $X \times Y$. If $S$ is a regular stratum of $X$, then $K = \emptyset$, and similarly for $T$ and $L$. We will consider the join with the empty set to be the identity construction, e.g. $K \ast \emptyset = K$.

By definition, $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-finitely generated if, for any such $S \times T$ and each $i$, the modules $I^{Q_{\bar{p}, \bar{q}}}H_i(K \ast L; R)$ are finitely generated. Recall that this expression makes sense as we can restrict the perversity $Q_{\bar{p}, \bar{q}}$ on $X \times Y$ to the link $K \ast L$, which we can consider embedded in $X \times Y$.

If $K = \emptyset$, i.e. $S$ is a regular stratum, then $Q_{\bar{p}, \bar{q}}(S \times T) = \bar{q}(T)$ and $Q_{\bar{p}, \bar{q}}$ restricts to $\bar{q}$ on $L$, so the condition that $I^{Q_{\bar{p}, \bar{q}}}H_i(K \ast L; R)$ be finitely generated reduces to $I^{\bar{q}}H_i(L; R)$ being finitely generated, which is the condition for $Y$ to be locally $(\bar{q}, R)$-finitely generated along $T$. Therefore, we see that if $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-finitely generated, it follows that $Y$ is locally $(\bar{q}, R)$-finitely generated. The analogous argument holds when $L = \emptyset$. Thus if $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-finitely generated, we must also have that $X$ is locally $(\bar{p}, R)$-finitely generated and $Y$ is a locally $(\bar{q}, R)$-finitely generated. Conversely, if $X$ is locally $(\bar{p}, R)$-finitely generated and $Y$ is a locally $(\bar{q}, R)$-finitely generated, then $X \times Y$ is locally $(Q_{\bar{p}, \bar{q}}, R)$-finitely generated along strata $S \times T$ for which one of $S$ or $T$ is regular. It remains to show that if $X$ and $Y$ are each locally finitely generated then $X \times Y$ is locally finitely generated along products of singular strata.

So, now, suppose that $X$ and $Y$ are both locally finitely generated and that neither $K$ nor $L$ is empty. We need for the intersection homology modules $I^{Q_{\bar{p}, \bar{q}}}H_i(K \ast L; R)$ to be
finitely generated. But in our discussion in Section 6.4.2, we saw that \( I^{Q_{\phi^g}}H_i(K \ast L; R) \cong I^{Q_{\phi^g}}H_i(cK \times cL - \{v \times w\}; R) \), where \( v, w \) are the respective cone vertices. Furthermore, \( cK \times cL - \{v \times w\} \) is the union of \( (cK - \{v\}) \times cL \) and \( cK \times (cL - \{w\}) \), which have intersection \( (cK - \{v\}) \times (cL - \{w\}) \), and there is a Mayer-Vietoris sequence involving these spaces. Using the assumption that either \( X \) or \( Y \) is locally torsion free, the same is true of all the factors of these product spaces, which are open subsets of \( X \) or \( Y \), and so the Künneth theorem, Theorem 6.56, applies to \( I^{Q_{\phi^g}}H_i((cK - \{v\}) \times cL; R) \), \( I^{Q_{\phi^g}}H_i(cK \times (cL - \{w\}); R) \), and \( I^{Q_{\phi^g}}H_i((cK - \{v\}) \times (cL - \{w\}); R) \). Therefore, using also the cone formula and stratified homotopy invariance, each of these modules is a finite direct sum of tensor and torsion products of modules of the form \( I^{\bar{p}}H_j(K; R) \) or \( I^{\bar{q}}H_k(L; R) \), each of which is finitely generated by hypothesis. So the terms on either side of \( I^{Q_{\phi^g}}H_i(K \ast L; R) \) in the Mayer-Vietoris sequence are finitely generated, and so, as \( R \) is Noetherian, \( I^{Q_{\phi^g}}H_i(K \ast L; R) \) must also be finitely generated, using the basic properties of Noetherian modules; see [64, Section X.1]. \( \square \)

We can now turn to the associativity properties of intersection homology products. We begin with the following lemma:

**Lemma 7.54.** Let \( R \) be a Dedekind domain. Suppose that \( \bar{\rho}, \bar{\sigma}, \bar{\tau}, \bar{s} \) are perversities on a CS set \( X \) such that \( D\bar{s} \geq D\bar{\rho} + D\bar{\sigma} + D\bar{\tau} \). Let \( A, B, C \subset X \) be open subsets, and suppose \( X \) is locally torsion free with respect to two out of the three perversities \( \bar{\rho}, \bar{\sigma}, \bar{\tau} \). Let \( \bar{u} = D(D\bar{\rho} + D\bar{\sigma}) \), and let \( \bar{v} = D(D\bar{\sigma} + D\bar{\tau}) \).

Then the following diagram commutes up to chain homotopy:

\[
\begin{array}{c}
I^\bar{u}S_*(X, A \cup B \cup C; R) \xrightarrow{\bar{d}} I^\bar{u}S_*(X, A \cup B; R) \otimes I^\bar{u}S_*(X, C; R) \\
\downarrow d \quad \quad \quad \quad \quad \downarrow d \otimes id \\
I^\bar{u}S_*(X, A; R) \otimes I^\bar{u}S_*(X, B \cup C; R) \xrightarrow{id \otimes \bar{d}} I^\bar{u}S_*(X, A; R) \otimes I^\bar{u}S_*(X, B; R) \otimes I^\bar{u}S_*(X, C; R).
\end{array}
\]

**Proof.** The proof will utilize the following diagram in which the \( R \) coefficients are implicit:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
I^\rho S_*(X, A \cup B \cup C) \\
\downarrow d \\
P^\rho S_*(X \times X, (A \times X) \cup (B \times C)) \\
\downarrow \text{id} \otimes d \\
P^\rho S_*(X, A \cup B \cup C) \\
\downarrow d \otimes \text{id} \\
P^\rho S_*(X, A) \otimes P^\rho S_*(X, B \cup C) \\
\downarrow \text{id} \otimes \text{id} \\
P^\rho S_*(X, A) \otimes P^\rho S_*(X, B) \otimes P^\rho S_*(X, C).
\end{array}
\end{array}
\end{array}
\]

The maps \( \text{id} \otimes d \) and \( d \otimes \text{id} \) are defined on spaces by \((\text{id} \otimes d)(x, y) = (x, y, y)\) and \((d \otimes \text{id})(x, y) = (x, x, y)\). We will show just below in Lemma 7.55 that these induce well-defined maps of intersection chains with the perversities as given. We do note now that since \( DD\bar{u} = \bar{a} \) for any perversity \( \bar{a} \), we have \( D\bar{u} = D\bar{\rho} + D\bar{\sigma} \) and \( D\bar{v} = D\bar{\sigma} + D\bar{\tau} \). Therefore, as \( D\bar{s} \geq D\bar{\rho} + D\bar{\sigma} + D\bar{\tau} \), we also have \( D\bar{s} \geq D\bar{u} + D\bar{\tau} \) and \( D\bar{s} \geq D\bar{\rho} + D\bar{\sigma} \), so both of the maps \( d \) in the upper left of the diagram are well-defined, as well as the the maps \( d \otimes \text{id} \) and \( \text{id} \otimes d \).
We will show that the squares in the diagram commute. Since each cross-product is a chain homotopy equivalence, this implies that if we replace each $\epsilon$ with a map $I_{A,W}$ going in the opposite direction, we obtain a diagram in which each square is homotopy commutative. Notice that the top and left of the diagram become algebraic diagonals $d$, while the right and bottom of the diagram take the form $\bar{d} \otimes \text{id}$ and $\text{id} \otimes \bar{d}$, assuming each symbol $\bar{d}$ is interpreted with respect to the correct $R$-modules.

Since we know that all maps in the diagram are well-defined chain maps, assuming for now Lemma 7.55, it suffices to verify commutativity at the level of $R$-modules (as opposed to chain complexes). But we know that each $R$-module here is a submodule of the analogous ordinary singular chain $R$-modules, which are all free, generated by the singular simplices. So it suffices to verify commutativity on singular simplices.

The upper left square is induced by maps of spaces, and since $(d \times \text{id})d(x) = (x, x, x) = (\text{id} \times d)d(x)$, this square commutes already at the space level. For the right upper square, consider a generator $\sigma \otimes \tau \in S_i(X; R) \otimes S_j(X; R)$. The map left then down yields the singular chain corresponding to applying $(d \otimes \text{id})(\sigma \times \tau) = (d\sigma) \times \tau$ to the singular triangulation of $\Delta^i \times \Delta^j$ arising from the Eilenberg-Zilber shuffle procedure. On the other hand, the right hand vertical map of the square takes $\sigma \otimes \tau$ to $(d\sigma) \otimes \tau$, and then the bottom horizontal map takes this to $(d\sigma) \times \tau$ applied to the singular triangulation of $\Delta^i \times \Delta^j$. So the two ways around the square agree. The argument for the lower left square is equivalent. This leaves the commutativity of the bottom right square, which is associativity of the chain cross product, Lemma 7.58.

Lemma 7.55. If $D\bar{v} \geq D\bar{q} + D\bar{r}$ and $D\bar{u} \geq D\bar{p} + D\bar{q}$, then the following two maps are well defined:

\[
I_{Q,M}^i S_i(X \times X, (A \times X) \cup (X \times (B \cup C)); R) \xrightarrow{id \times d} I_{Q,M}^i S_i(X \times X, (A \times X) \cup (X \times B \times X) \cup (X \times X \times C); R)
\]
\[
I_{Q,M}^i S_i(X \times X, ((A \cup B) \times X) \cup (X \times C); R) \xrightarrow{d \times \text{id}} I_{Q,M}^i S_i(X \times X, (A \times X \times C); R).
\]

Proof. Again, we will limit ourselves to the first case, the other being equivalent. Clearly $(id \times d)((A \times X) \cup (X \times (B \cup C))) \subset (A \times X \times X) \cup (X \times B \times X) \cup (X \times X \times C)$ and $id \times d$ takes $\Sigma_{X \times X}$ to $\Sigma_{X \times X \times X}$, so it suffices to check that $id \times d$ takes allowable chains to allowable chains. If $S \times T$ is a stratum of $X \times X$, then $(id \times d)(S \times T) = S \times T \times T$, so it suffices to consider allowability with respect to strata of this form. Suppose $\sigma \in S_i(X \times X, (A \times X) \cup (X \times (B \cup C)); R)$ is a $Q_{\bar{p},\bar{q},\bar{r}}$-allowable simplex. Then $(id \times d)\sigma$ will be $Q_{\bar{p},\bar{q},\bar{r}}$-allowable if, for all $S \times T$, the preimage of $(id \times d)\sigma)^{-1}(S \times T \times T) = \sigma^{-1}(S \times T)$ is contained in the $\bar{r} - \text{codim}_{X \times X} (S \times T \times T) + Q_{\bar{p},\bar{q},\bar{r}}(S \times T \times T)$ skeleton of $\Delta^i$. We can simplify $\text{codim}_{X \times X} (S \times T \times T) = \text{codim}_{X \times X} (S \times T) + \text{codim}_{X} (T)$. By the assumption that $\sigma$ is allowable, we already know that $\sigma^{-1}(S \times T)$ is contained in the $i - \text{codim}_{X \times X} (S \times T) + Q_{\bar{p},\bar{q}}(S \times T)$ skeleton of $\Delta^i$. So it suffices to show that $Q_{\bar{p},\bar{q}}(S \times T) \leq Q_{\bar{p},\bar{q},\bar{r}}(S \times T \times T) - \text{codim}_{X} (T)$.

As we do not need to check allowability with respect to regular strata, there are three cases to consider:

- $S$ and $T$ are singular. In this case, we need to check that

\[
\bar{p}(S) + \bar{v}(T) + 2 \leq \bar{p}(S) + \bar{q}(T) + \bar{r}(T) + 4 - \text{codim}_X(T),
\]
which is equivalent to $\bar{v}(T) \leq \bar{q}(T) + \bar{r}(T) + 2 - \codim X(T)$. This is equivalent to $D\bar{v} \geq D\bar{q} + D\bar{r}$ by the computation just before Definition 7.20.

- **$S$ is regular and $T$ is singular.** In this case, we need to check that

$$\bar{v}(T) \leq Q_{\bar{q},\bar{r}}(T \times T) - \codim X(T) = \bar{q}(T) + \bar{r}(T) + 2 - \codim X(T),$$

which is again equivalent to $D\bar{v} \geq D\bar{q} + D\bar{r}$ by the computation just before Definition 7.20.

- **$S$ is singular and $T$ is regular.** In this case, we need to check that

$$\bar{p}(S) \leq Q_{\bar{p},\bar{q},\bar{r}}(S \times T \times T) - \codim X(T) = \bar{p}(S),$$

noting that if $T$ is regular then $\codim X(T) = 0$. Of course, this is true.

$\square$

We again need a purely algebraic lemma, as well:

**Lemma 7.56.** Let $C_*, D_*, E_*$ be chain complexes of $R$-modules. Then the following diagram commutes:

$$\begin{array}{c}
\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) \otimes \text{Hom}(E_*, R) \\
\downarrow \Theta \otimes \text{id} \quad \downarrow \Theta \\
\text{Hom}(C_* \otimes D_*, R) \otimes \text{Hom}(E_*, R) \\
\text{id} \otimes \Theta \\
\downarrow \Theta \\
\text{Hom}(C_* \otimes D_* \otimes E_*, R) \\
\end{array}$$

**Proof.** Let $\alpha \in \text{Hom}(C_*, R)$, $\beta \in \text{Hom}(D_*, R)$, and $\gamma \in \text{Hom}(E_*, R)$. To verify the commutativity, it suffices to check the evaluation of $\alpha \otimes \beta \otimes \gamma$ on a generator $x \otimes y \otimes z$ of $C_* \otimes D_* \otimes E_*$. We have:

$$\Theta(\Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma)(x \otimes y \otimes z) = \Theta(\Theta(\alpha \otimes \beta) \otimes \gamma)(x \otimes y \otimes z)$$

$$= (-1)^{|\gamma||x| + |y|}[\Theta(\Theta(\alpha \otimes \beta))(x \otimes y)]\gamma(z)$$

$$= (-1)^{|\gamma||x| + |y| + |\beta||x|} \alpha(x)\beta(y)\gamma(z).$$

On the other hand,

$$\Theta(\text{id} \otimes \Theta)(\alpha \otimes \beta \otimes \gamma)(x \otimes y \otimes z) = \Theta(\alpha \otimes \Theta(\beta \otimes \gamma))(x \otimes y \otimes z)$$

$$= (-1)^{|[\beta] + |\gamma||x|} \alpha(x)\Theta(\beta \otimes \gamma)(y \otimes z)$$

$$= (-1)^{|[\beta] + |\gamma||x| + |\beta||y|} \alpha(x)\beta(y)\gamma(z).$$

The two expressions are equal, completing the proof. $\square$
Lemma 7.57 (Associativity). Let $R$ be a Dedekind domain. Suppose that $p, q, r, s$ are perversities on a CS set $X$ such that $D s \geq D p + D q + D r$. Let $A, B, C \subset X$ be open subsets, and suppose $X$ is locally torsion free with respect to two out of the three perversities $p$, $q$, $r$. Let $\alpha \in I_p H^i(X, A; R)$, $\beta \in I_q H^j(Y, B; R)$, and $\gamma \in I_r H^k(Z, C; R)$. Then $(\alpha \smile \beta) \smile \gamma$ and $\alpha \smile (\beta \smile \gamma)$ are well defined and equal elements of $I_s H^{i+j+k}(X, A \cup B \cup C; R)$.

Proof. Representing $\alpha$, $\beta$, and $\gamma$ by cocycles and making specific choices of IAW maps, by definition we have

$$(\alpha \smile \beta) \smile \gamma = \dd^* \Theta((\alpha \smile \beta) \otimes \gamma) = \dd^* \Theta(\dd^* \Theta(\alpha \otimes \beta) \otimes \gamma) = \dd^* \Theta(\dd^* \Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma).$$

Similarly,

$$\alpha \smile (\beta \smile \gamma) = \dd^* \Theta(\alpha \otimes (\beta \smile \gamma)) = \dd^* \Theta(\alpha \otimes (\dd^* \Theta(\beta \otimes \gamma))) = \dd^* \Theta(\text{id} \otimes \dd^* \Theta)(\alpha \otimes \beta \otimes \gamma).$$

Notice here that, at the cochain level, these expressions depend on the choices of IAW maps but that they are well-defined expressions independent of these choices upon passing to cohomology as all IAW maps are well defined up to chain homotopy. Now we compute at the level of cohomology:

$$(\alpha \smile \beta) \smile \gamma = (\dd^* \Theta(\dd^* \Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma) = \dd^* \Theta(\dd^* \otimes \text{id})(\Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma) = \dd^* \Theta(\dd^* \otimes \text{id})(\Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma) = \dd^* \Theta(\text{id} \otimes \dd^* \Theta)(\alpha \otimes \beta \otimes \gamma) = \alpha \smile (\beta \smile \gamma).$$

Similarly, the cross product is associative:

Lemma 7.58 (Associativity). Let $R$ be a Dedekind domain. Suppose that $p, q, r$ are perversities on CS sets $X, Y, Z$. Let $A \subset X$, $B \subset Y$, and $C \subset Z$ be open subsets. Suppose two out of three of $X$, $Y$, and $Z$ are locally torsion free. Let $\alpha \in I_p H^i(X, A; R)$, $\beta \in I_q H^j(Y, B; R)$, and $\gamma \in I_r H^k(Z, C; R)$. Then $(\alpha \times \beta) \times \gamma$ and $\alpha \times (\beta \times \gamma)$ are well defined and equal elements of $I_{p, q, r} H^{i+j+k}(X \times Y \times Z, (A \times Y \times Z) \cup (X \times B \times Z) \cup (X \times Y \times C); R)$. 

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Proof. From the conditions on the spaces, both iterated cross products are well defined. We choose fixed representative IAW maps and compute at the level of cohomology

\[(\alpha \times \beta) \times \gamma = \text{IAW}^*\Theta((\alpha \times \beta) \otimes \gamma)\]

\[= \text{IAW}^*\Theta(\text{IAW}^*\Theta(\alpha \otimes \beta) \otimes \gamma)\]

\[= \text{IAW}^*\Theta(\text{IAW}^*\Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma)\]

\[= \text{IAW}^*(\text{IAW} \otimes \text{id})(\Theta \otimes \text{id})(\alpha \otimes \beta \otimes \gamma)\]

\[= \text{IAW}^*(\text{id} \otimes \text{IAW})^*\Theta(\alpha \otimes \beta \otimes \gamma)\]

by Lemma 7.31

\[= \text{IAW}^*(\text{id} \otimes \text{IAW})^*\Theta(\alpha \otimes \beta \otimes \gamma)\]

by Lemma 7.56

\[= \text{IAW}^*\Theta(\alpha \otimes \beta \otimes \gamma)\]

\[= \text{IAW}^*\Theta(\alpha \otimes (\beta \times \gamma))\]

\[= \alpha \times (\beta \times \gamma).\]

Here, the seventh equality follows from the commutativity of Lemma 7.58 using the Künneth theorem, Theorem 6.61 to replace each \(\epsilon\) arrow with the corresponding IAW arrow in the opposite direction. The resulting diagram homotopy commutes, and so \(\text{IAW}^*(\text{IAW} \otimes \text{id})^* = \text{IAW}^*(\text{id} \otimes \text{IAW})^*\) as a map on cohomology.

\[\square\]

The cap product version of associativity has the following form:

**Lemma 7.59 (Associativity).** Let \(R\) be a Dedekind domain. Suppose that \(\bar{p}, \bar{q}, \bar{r}, \bar{s}\) are perversities on a CS set \(X\) such that \(D\bar{s} \geq D\bar{p} + D\bar{q} + D\bar{r}\). Let \(A, B, C \subset X\) be open subsets, and suppose \(X\) is locally torsion free with respect to two out of the three perversities \(\bar{p}, \bar{q}, \bar{r}\). Let \(\alpha \in I^qH^j(X, B; R), \beta \in I^rH^k(X, C; R), \) and \(\xi \in I^sH_{i+j+k}(X, A \cup B \cup C; R).\) Then \((\alpha \smile \beta) \smile \xi\) and \(\alpha \smile (\beta \smile \xi)\) are well defined and equal elements of \(I^pH_i(X, A; R)\).

**Proof.** To see that both expressions are well defined, let \(\bar{u} = D(D\bar{p} + D\bar{q})\) and \(\bar{v} = D(D\bar{q} + D\bar{r})\). Then \(\alpha \smile \beta\) is well-defined in \(I_\delta H^{j+k}(X, B \cup C; R)\), and, since we then have \(D\bar{s} \geq D\bar{p} + D\bar{q} + D\bar{r} = D\bar{p} + D\bar{r}\), the cap product \((\alpha \smile \beta) \smile \xi\) is well-defined in \(I^pH_i(X, A; R)\). Similarly, since \(D\bar{s} \geq D\bar{p} + D\bar{q} + D\bar{r} = D\bar{u} + D\bar{r}\), we have that \(\beta \smile \xi \in I^pH_i(X, A \cup B; R)\) is well defined, and then \(\alpha \smile (\beta \smile \xi)\) is well-defined in \(I^pH_i(X, A; R)\). To demonstrate the equality, let us again assume that we have chosen fixed chain maps IAW and compute both expressions for given elements \(\alpha \in I^qS^j(X, B; R), \beta \in I^rS^k(X, C; R),\) and \(\xi \in I^sS_{i+j+k}(X, A \cup B \cup C; R).\)

\[(\alpha \smile \beta) \smile x = \Phi(\text{id} \otimes (\alpha \smile \beta))\tilde{d}(x)\]

\[= \Phi(\text{id} \otimes \tilde{d}^\gamma \Theta(\alpha \otimes \beta))\tilde{d}(x)\]

\[= \Phi(\text{id} \otimes \Theta(\alpha \otimes \beta))\tilde{d}(x)\]

\[= \Phi(\text{id} \otimes \Theta(\alpha \otimes \beta))((\text{id} \otimes \tilde{d})\tilde{d})(x).\]
The other computation is a bit more complicated. For it, we will want to assume we have fixed IAW maps and that \( \bar{d}(x) = \sum_{\ell} y_{\ell} \otimes z_{\ell} \in I^q \ast S_*(X, A \cup B; R) \otimes I^q \ast S_*(X, C; R) \) and that for each \( y_{\ell} \), \( \bar{d}(y_{\ell}) = \sum_a u_{\ell a} \otimes v_{\ell a} \in I^q \ast S_*(X; A; R) \otimes I^q \ast S_*(X, B; R) \).

\[
\alpha \hookrightarrow (\beta \hookrightarrow x) = \Phi(\text{id} \otimes \alpha) \bar{d}(\beta \hookrightarrow x) \\
= \Phi(\text{id} \otimes \alpha) \bar{d}(\Phi(\text{id} \otimes \beta)) \bar{d}(x) \\
= (-1)^{|y_{\ell}|} \Phi(\text{id} \otimes \alpha) \bar{d}(\Phi(\text{id} \otimes \beta)) \left( \sum_{\ell} y_{\ell} \otimes z_{\ell} \right) \\
= (-1)^{|y_{\ell}|} \Phi(\text{id} \otimes \alpha) \bar{d} \sum_{\ell} \beta(z_{\ell}) y_{\ell} \\
= (-1)^{|y_{\ell}|} \Phi(\text{id} \otimes \alpha) \sum_{\ell} \beta(z_{\ell}) \bar{d}(y_{\ell}) \\
= (-1)^{|y_{\ell}|} \Phi(\text{id} \otimes \alpha) \sum_{\ell} \beta(z_{\ell}) \sum_a u_{\ell a} \otimes v_{\ell a} \\
= (-1)^{|y_{\ell}| + j|u_{\ell a}|} \sum_{\ell} \sum_a \alpha(v_{\ell a}) \beta(z_{\ell}) u_{\ell a} \\
= (-1)^{|y_{\ell}| + j|u_{\ell a}| + k|v_{\ell a}|} \Phi \left( \sum_{\ell,a} u_{\ell a} \otimes \Theta(\alpha \otimes \beta)(v_{\ell a} \otimes z_{\ell}) \right) \\
= (-1)^{|y_{\ell}| + j|u_{\ell a}| + k|v_{\ell a}| + (j+k)|u_{\ell a}|} \Phi(\text{id} \otimes \Theta(\alpha \otimes \beta)) \sum_{\ell,a} u_{\ell a} \otimes v_{\ell a} \otimes z_{\ell} \\
= \Phi(\text{id} \otimes \Theta(\alpha \otimes \beta))(\bar{d} \otimes \text{id}) \bar{d}(x).
\]

For the signs in the last line, we notice that these expressions vanish unless \( j = |\alpha| = |v_{\ell a}| \) and \( k = |\beta| = |z_{\ell}| \), which leaves \( |u_{\ell a}| = i \) in the nonvanishing terms. Therefore,

\[
k|y_{\ell}| + j|u_{\ell a}| + k|v_{\ell a}| + (j+k)|u_{\ell a}| = k(i+j) + ij + jk + (j+k)i \\
= ik + jk + ij + jk + ik,
\]

which is even.

So, now suppose that \( x \in I^q H_{i+j+k}(X, A \cup B \cup C; R) \). Since we know by Lemma \ref{7.54} that \((\bar{d} \otimes \text{id}) \bar{d}\) and \((\text{id} \otimes \bar{d}) \bar{d}\) are chain homotopic chain maps, we know that \((\bar{d} \otimes \text{id}) \bar{d}(x) = (\text{id} \otimes \bar{d}) \bar{d}(x) \in H_{i+j+k}(I^q \ast S_*(X, A; R) \otimes I^q \ast S_*(X, B; R) \otimes I^q \ast S_*(X, C; R))\). Similarly, we know that if \( \alpha \in I_q H^j(X, B; R) \) and \( \beta \in I^q H^k(X, C; R) \), then \( \Theta(\alpha \otimes \beta) \) is well-defined in \( H^*(\text{Hom}(I^q \ast S_*(X, B; R) \otimes I^q \ast S_*(X, C; R), R)) \). From here, the verification that

\[
\Phi(\text{id} \otimes \Theta(\alpha \otimes \beta))(\bar{d} \otimes \text{id}) \bar{d}(x) = \Phi(\text{id} \otimes \Theta(\alpha \otimes \beta))(\text{id} \otimes \bar{d}) \bar{d}(x)
\]

follows exactly as in the proof of Lemma \ref{7.29} where we showed that the cap product is independent of the choice of algebraic diagonal map up to chain homotopy. \( \square \)
7.3.5 Stability

We now turn to what Dold\cite{Dold} refers to as “stability” properties of products. These are the properties that involve the connecting morphisms $\partial_*$ and $d^*$ in the long exact homology sequences. Here we run into some additional difficulties because we do not have the fact from ordinary homology, which Dold achieves via acyclic model arguments, that the IAW maps are natural as chain maps with respect to maps of spaces; this naturality property plays a subtle role in the arguments of\cite{Dold}. We do have such a fact for the chain cross product $\epsilon$ (see Lemma 5.16), but we cannot assume that this naturality continues to hold when we replace each $\epsilon$ with a chain homotopy inverse IAW. Therefore, our arguments will have to be more elaborate than those of Dold. We begin with the stability of cap products, which requires some big diagrams but fewer new techniques.

Lemma 7.60. Let $R$ be a Dedekind domain. Suppose $X$ is a CS set with perversities $\bar{p}, \bar{q}, \bar{r}$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let $A, B \subset X$ be open subsets with $i : B \hookrightarrow X$ the inclusion map, and suppose that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Suppose $\alpha \in I_{\bar{q}}H^j(B; R)$ and $\xi \in I_{\bar{r}}H^{i+j}(X, A \cup B; R)$. Then

$$(d^*(\alpha)) \mapsto \xi = (-1)^{j+1}i(\alpha \mapsto e^{-1}\partial_*(\xi)) \in I_{\bar{p}}H_{i-1}(X, A; R),$$

where we interpret $\partial_*(\xi)$ as landing in $I_{\bar{r}}H^i(A \cup B, A; R)$ and $e : I_{\bar{r}}H_{i+j-1}(B, A \cap B; R) \rightarrow I_{\bar{r}}H_{i+j-1}(A \cup B, A; R)$ is the excision isomorphism.

In other words, the following diagram commutes:\footnote{Recall that we treat $\partial_*$ as a degree $-1$ map.}

$$
\begin{array}{c}
I_{\bar{q}}H^j(B; R) \otimes I_{\bar{r}}H_{i+j}(X, A \cup B; R) \\
\downarrow -\text{id} \otimes \partial_* \\
I_{\bar{q}}H^j(B; R) \otimes I_{\bar{r}}H_{i+j-1}(A \cup B, A; R) \\
\end{array}
\begin{array}{c}
\xrightarrow{d^* \otimes \text{id}} \\
\xrightarrow{\text{id} \otimes e} \\
\xrightarrow{\text{id} \otimes e} \\
\end{array}
\begin{array}{c}
I_{\bar{q}}H^j(B; R) \otimes I_{\bar{r}}H_{i+j-1}(B, A \cap B; R) \\
\xrightarrow{i} \\
I_{\bar{p}}H_{i-1}(B, A \cap B; R).
\end{array}
$$
Proof. The proof will eventually utilize the following diagram, with \( R \) coefficients tacit:

\[
\begin{array}{ccc}
H_{i+j}(I^pS_\ast(X, A) \otimes I^qS_\ast(X, B)) & \xrightarrow{\epsilon} & I^{p+q}H_{i+j}(X \times X, (A \times X) \cup (X \times B)) \xrightarrow{d} I^pH_{i+j}(X, A \cup B) \\
\partial_i \downarrow & & \downarrow \partial_s \\
I^{p+q}H_{i+j-1}((A \times X) \cup (X \times B), A \times X) & \xrightarrow{d} & I^pH_{i+j-1}(A \cup B, A) \\
\cong e' & & \cong e \\
H_{i+j-1}(I^pS_\ast(X, A) \otimes I^qS_\ast(B)) & \xrightarrow{\epsilon} & I^{p+q}H_{i+j-1}(X \times B, A \cap B) \xrightarrow{d} I^pH_{i+j-1}(B, A \cap B) \\
\end{array}
\]

Here, each of the diagonal maps is also composed with the evident inclusion. The unlabeled maps are also the evident inclusions or projections. The map labeled \( \partial_s \) on the left is meant to be the boundary map in the long exact sequence associated to the short exact sequence

\[
0 \rightarrow I^pS_\ast(X, A) \otimes I^qS_\ast(B; R) \rightarrow I^pS_\ast(X, A; R) \otimes I^qS_\ast(X; R) \rightarrow I^pS_\ast(X, A; R) \otimes I^qS_\ast(X, B; R) \rightarrow 0.
\]

This is obtained by tensoring the exact sequence of \( \bar{q} \) intersection chains of the pair \((X, B)\) with the complex \( I^pS_\ast(X, A; R)\). The short sequence remains exact after tensoring as \( I^pS_\ast(X, A; R) \) is projective and so flat.

Now, every element of \( I^pS_\ast(X, A; R) \otimes I^qS_\ast(X, B; R) \) can be represented by a chain in \( I^pS_\ast(X; R) \otimes I^qS_\ast(X; R) \), and if \( x \) is such a chain that is a cycle in \( I^pS_\ast(X, A; R) \otimes I^qS_\ast(X, B; R) \), and so represents a homology class \( \xi \), then, by the standard zig-zag construction, \( \partial_x \xi \) is represented by \( \partial x \), representing an element of \( I^pS_\ast(X, A; R) \otimes I^qS_\ast(B; R) \). Therefore, \( \epsilon \partial_x(\xi) \in I^{p+q}H_{i-1+j}(X \times B, A \times B; R) \) is represented by \( \epsilon(\partial x) \), which also represents the image of this class in \( I^{p+q}H_{i+j-1}((A \times X) \cup (X \times B), A \times X; R) \). But since \( \epsilon \) is a chain map, \( \epsilon(\partial x) = \partial \epsilon(x) \), which represents \( \partial_x \epsilon(\xi) \) in \( I^{p+q}H_{i+j-1}((A \times X) \cup (X \times B); R) \) and so also its image in \( I^{p+q}H_{i+j-1}((A \times X) \cup (X \times B), A \times X; R) \). So the upper left rectangle commutes.

The square on the bottom left commutes by the non-GM version of Lemma \([5.16]\). The triangle and the two bottom squares on the right commute at the space level because diagonal maps are natural, i.e. if \( f : Z \rightarrow W \) is any map of spaces, then \( d_W f = (f \times f) d_Z \). For the upper square on the right, let \( x \) be a chain in \( I^pS_{i+j}(X; R) \) representing an element \( \xi \in I^pH_{i+j}(X, A \cup B; R) \) and recall that \( \partial_s \) can be represented by taking the boundary of \( x \). So \( d_{A \cup B} \partial_s(\xi) \) is represented by including \( d_{A \cup B}(\partial x) \) into \( (A \times X) \cup (X \times B) \subset X \times X \). On the other hand, \( \partial_s d(\xi) \) is represented by \( \partial d_X(x) = d_X \partial x \), as \( d \) is a chain map. But
again using the naturality of $d$, these are the same chain. Therefore, both images of $\xi$ are represented by the same chain.

Let us also verify that the two maps labeled $e$ and $e'$, which are meant to indicate excision, really are isomorphisms. Since we have assumed $A$ and $B$ to be open subsets of $X$, $\{A, B\}$ is an open cover of $A \cup B$. Consider now the maps that factor $e$:

\[
\frac{I^p S_s(B; R)}{I^p S_s(A \cap B; R)} \to \frac{I^p S_s(A; R) + I^p S_s(B; R)}{I^p S_s(A; R)} \to \frac{I^p S_s(A \cup B; R)}{I^p S_s(A; R)}.
\]

The first map is an isomorphism by the second isomorphism theorem, noting that $I^p S_s(A \cap B; R) \cap I^p S_s(B; R) = I^p S_s(A \cap B; R)$, while the second map induces a homology isomorphism by the non-GM version of the arguments in the proof of Theorem 4.46. The map $e'$ is similarly a homology isomorphism by the same arguments, replacing $A$ with $A \times X$ and $B$ with $X \times B$.

Next we recall how $d^*$ works. Suppose that $\alpha$ represents an element of $I_q H^j(B; R)$. Then the zig-zag construction of $d^*$ shows that $d^* \alpha$ is represented by $d\bar{\alpha}$, where $\bar{\alpha} \in I_q S^j(X; R)$ restricts to $\alpha$ over $B$.

Now, suppose that $\xi \in I^p H_{i+j}(X, A \cup B; R)$. Then $d(\xi)$ is obtained by going left across the top row of the diagram, using that $\epsilon$ is an isomorphism by the K"unneth theorem. Suppose we choose chain maps IAW and that $\bar{\xi}$ is then represented by $\sum_k y_k \otimes z_k \in I^p S_s(X; R) \otimes I^p S_s(X; R)$, noting that every element of $I^p S_s(X, A; R) \otimes I^p S_s(X, B; R)$ has such representatives. By definition, we then have

\[
(d^* \alpha) \smile \xi = \Phi(\text{id} \otimes d^* \alpha)(d\bar{\xi})
\]

\[
= \Phi(\text{id} \otimes d\bar{\alpha}) \left( \sum_k y_k \otimes z_k \right)
\]

\[
= \sum_k (-1)^{j+1} |y_k| y_k d\bar{\alpha}(z_k)
\]

\[
= \sum_k (-1)^{j+1} |y_k| |y_k| y_k \bar{\alpha}(\partial z_k)
\]

\[
= \Phi(\text{id} \otimes \bar{\alpha}) \left( \sum_k (-1)^{j+1} |y_k| |y_k| + j+1 |y_k| y_k \otimes \partial z_k \right)
\]

\[
= \Phi(\text{id} \otimes \bar{\alpha}) \left( \sum_k (-1)^{|y_k|+j+1} y_k \otimes \partial z_k \right)
\]

\[
= (-1)^{j+1} \Phi(\text{id} \otimes \bar{\alpha}) \left( \sum_k (-1)^{|y_k|} y_k \otimes \partial z_k \right)
\]

which we know must be a cycle in $I^p S_{i-1}(X, A; R)$.

Notice that the expression here $\sum_k (-1)^{|y_k|} y_k \otimes \partial z_k$ is a piece of $\partial(\sum_k y_k \otimes z_k) = \sum ((\partial y_k) \otimes z_k + (-1)^{|y_k|} y_k \otimes \partial z_k)$. Suppose we replace $\sum_k (-1)^{|y_k|} y_k \otimes \partial z_k$ in the last line with $\partial(\sum_k y_k \otimes z_k)$. Then we would have
Let Lemma 7.61.

Thus we have that \((\text{some cancellation of terms}), we can assume each \((\partial y_k) \otimes z_k\) thus \((\partial y_k) \otimes z_k\) which agrees with the above computation (see the fourth line) except for the terms of the form \((\partial y_k) \otimes z_k\). but these terms are boundaries in \(I^pS_\ast(X; R)\), and so also in \(I^pS_\ast(X, A; R)\), and thus \((-1)^{j+1} \Phi(id \otimes \alpha) \partial(\sum_k y_k \otimes z_k)\) also represents \((d' \alpha) \triangleright \xi\). Furthermore, since we know that \(\partial(\sum_k y_k \otimes z_k)\) must represent an element of \(I^{p}S_\ast(X, A; R) \otimes I^qS_\ast(B; R)\) (perhaps after some cancellation of terms), we can assume each \(z_k\) is supported in \(B\) so that \(\alpha(z_k) = \alpha(z_k)\).

Thus we have that \((d' \alpha) \triangleright \xi\) is also represented by \((-1)^{j+1} \Phi(id \otimes \alpha) \partial(\sum_k y_k \otimes z_k)\), thinking of \(\partial(\sum_k y_k \otimes z_k)\) as a cycle in \(I^pS_\ast(X, A; R) \otimes I^qS_\ast(B; R)\).

Next, consider \(i \Phi(\alpha \triangleright e^{-1} \partial_s(\xi)) = i(\Phi(id \otimes \alpha)de^{-1} \partial_s(\xi)).\) If \(\eta\) is a chain representing \(de^{-1} \partial_s(\xi)\) as an element of \(H_{i+j-1}(I^pS_\ast(B, A \cup B; R) \otimes I^qS_\ast(B; R))\), then this same chain also represents \((i \otimes id)(\eta)\) in \(H_{i+j-1}(I^pS_\ast(X, A; R) \otimes I^qS_\ast(B; R))\), and \(i \Phi(\alpha \triangleright e^{-1} \partial_s(\xi))\) is represented by \(\Phi(id \otimes \alpha)(i \otimes id)(\eta)\). But by the commutativity of the diagram, \((i \otimes id)(\eta)\) is homologous to \(\partial(\sum_k y_k \otimes z_k)\) in \(I^pS_\ast(X, A; R) \otimes I^qS_\ast(B; R)\), and so \(i \Phi(\alpha \triangleright e^{-1} \partial_s(\xi))\) is represented by \(\Phi(id \otimes \alpha) \partial(\sum_k y_k \otimes z_k)\). The claim of the lemma now follows.

**Lemma 7.61.** Let \(R\) be a Dedekind domain. Suppose \(X\) is a CS set with perversities \(\bar{p}, \bar{q}, \bar{r}\) such that \(D\bar{r} \geq D\bar{p} + D\bar{q}\). Let \(A, B \subset X\) be open subsets with \(i : A \hookrightarrow X\) the inclusion map, and suppose that \(X\) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free. Suppose \(\alpha \in I^qH^j(X, B; R)\) and \(\xi \in I^pH_{i+j}(X, A \cup B; R)\). Then

\[
\partial_s(\alpha \triangleright \xi) = (-1)^j (i^\ast(\alpha)) \triangleright (e^{-1} \partial_s(\xi)) \in I^pH_{i-1}(A; R),
\]

where we interpret \(\partial_s(\xi)\) as landing in \(I^pH_{i+j-1}(A \cup B; R)\) and \(e : I^pH_{i+j-1}(A, A \cap B; R) \rightarrow I^pH_{i+j-1}(A \cup B; R)\) is the excision isomorphism.

In other words, the following diagram commutes:

\[
\begin{array}{ccc}
I^qH^j(X, B; R) \otimes I^pH_{i+j}(X, A \cup B; R) & \xrightarrow{\sim} & I^pH_i(X, A; R) \\
\| \downarrow \ & \ & \uparrow \partial_s \downarrow \\
I^qH^j(A, A \cap B; R) \otimes I^pH_{i+j-1}(A \cup B; R) & \overset{i^\ast \otimes \partial_s}{\xrightarrow{\cong}} & I^qH^j(A, A \cap B; R) \otimes I^pH_{i+j-1}(A, A \cap B) \\
\end{array}
\]

\[\overset{87}\text{Recall that we treat} \ \partial_s \text{ as a degree} -1 \text{ map.}\]
Proof. Consider the following diagram, with \( R \) coefficients tacit:

\[
\begin{array}{c}
H_{i+j}(I^pS_*(X,A) \otimes I^qS_*(X,B)) \xrightarrow{\epsilon} I^{q+p}H_{i+j}(X \times X, (A \times X) \cup (X \times B)) \xrightarrow{d} I^pH_{i+j}(X, A \cup B) \\
\begin{array}{c}
\partial_* \\
\partial_\ast
\end{array}
\end{array}
\]

\[
\begin{array}{c}
I^{q+p}H_{i+j-1}((A \times X) \cup (X \times B), X \times B) \xrightarrow{d} I^pH_{i+j-1}(A \cup B, B) \\
\cong e' \\
\cong e
\end{array}
\]

\[
\begin{array}{c}
H_{i+j-1}(I^pS_*(A) \otimes I^qS_*(X,B)) \xrightarrow{\epsilon} I^{q+p}H_{i+j}(A \times X, A \times B) \xrightarrow{d} I^pH_{i+j}(A, A \cap B) \\
\begin{array}{c}
\text{id} \otimes i \\
\text{id} \times i
\end{array}
\end{array}
\]

\[
H_{i+j-1}(I^pS_*(A) \otimes I^qS_*(A,A \cap B)) \xrightarrow{\epsilon} I^{q+p}H_{i+j-1}(A \times A, A \times (A \cap B)).
\]

This is the same as the diagram in Lemma 7.60 except that the roles of \((X, A)\) and \((X, B)\) have been reversed. In particular, the \( \partial_\ast \) on the left is now the boundary map in the long exact sequence associated to the short exact sequence

\[
0 \rightarrow I^pS_*(A; R) \otimes I^qS_*(X, B; R) \rightarrow I^pS_*(X; R) \otimes I^qS_*(X, B; R) \rightarrow I^pS_*(X, A; R) \otimes I^qS_*(X, B; R) \rightarrow 0.
\]

The arguments for commutativity, however, remain the same.

Suppose that \( \xi \in I^pH_{i+j}(X, A \cup B; R) \) and that we have chosen fixed IAW maps. Then \( \overline{d}(\xi) \) is obtained by going left across the top row of the diagram, using that \( \epsilon \) is an isomorphism by the Künneth theorem. Suppose \( d(\xi) \) is represented by \( \sum_k y_k \otimes z_k \in I^pS_*(X; R) \otimes I^qS_*(X; R) \). By definition, \( \alpha \sim \xi \) is represented by

\[
\alpha \sim \xi = \Phi(id \otimes \alpha)\overline{d}(\xi)
\]

\[
= \Phi(id \otimes \alpha) \left( \sum_k y_k \otimes z_k \right)
\]

\[
= \sum_k (-1)^{j|y_k|} \alpha(z_k)y_k,
\]

which is an element of \( I^pS_*(X; R) \) representing a cycle in \( I^pS_*(X, A; R) \). Therefore, by the
Part of the problem is that the maps ∂ are some algebra. The stability formulas for the cup and cross products are a little trickier.

zig-zag construction, ∂∗(α ∼ ξ) is represented by

\[
\partial^* (\alpha \sim \xi) = \partial \left( \sum_k (-1)^{|y_k|} \alpha(z_k)y_k \right)
\]

\[
= \sum_k (-1)^{|y_k|} \alpha(z_k) \partial y_k
\]

\[
= \Phi(id \otimes \alpha) \sum (-1)^{|y_k|+j(|y_k|-1)}(\partial y_k) \otimes z_k
\]

\[
= (-1)^j \Phi(id \otimes \alpha) \sum (\partial y_k) \otimes z_k,
\]

which must be a cycle in \(I^pS_*(A;R) \otimes I^qS_*(X,B;R)\).

As in the proof of Lemma 7.60, we recognize that \(\sum (\partial y_k) \otimes z_k\) is a piece of \(\partial(\sum y_k \otimes z_k) = \sum ((\partial y_k) \otimes z_k + (-1)^{|y_k|}y_k \otimes (\partial z_k))\). But since \(\alpha \in I^qS_*(X,B;R)\) is a cocycle, for any term of the form \(y_k \otimes (\partial z_k)\), we have

\[
\Phi(id \otimes \alpha)(y_k \otimes (\partial z_k)) = \pm y_k \otimes (\alpha(\partial z_k)) = \pm y_k \otimes ((d\alpha)(z_k)) = 0.
\]

Thus we have

\[
\partial^* (\alpha \sim \xi) = (-1)^j \Phi(id \otimes \alpha) \sum (\partial y_k) \otimes z_k
\]

\[
= (-1)^j \Phi(id \otimes \alpha) \left( \sum ((\partial y_k) \otimes z_k + (-1)^{|y_k|}y_k \otimes (\partial z_k)) \right)
\]

\[
= (-1)^j \Phi(id \otimes \alpha) \partial \left( \sum_k y_k \otimes z_k \right).
\]

Next, consider \((i^*\alpha) \sim e^{-1}\partial^* (\xi) = (\Phi(id \otimes i^*\alpha)\tilde{d}e^{-1}\partial^* (\xi))\). If \(\eta\) is a chain representing \(\tilde{d}e^{-1}\partial^*(\xi)\) as an element of \(H_{i+j-1}(I^pS_*(A;R) \otimes I^qS_*(A,A \cap B;R))\), then this same chain also represents an element \((id \otimes i)(\eta)\) of \(H_{i+j-1}(I^pS_*(A;R) \otimes I^qS_*(X,B;R))\), and \((i^*\alpha) \sim e^{-1}\partial^* (\xi)\) is represented by \(\Phi(id \otimes i^*\alpha)(\eta) = \Phi(id \otimes \alpha)(id \otimes i)(\eta)\). But by the commutativity of the diagram, \((id \otimes i)(\eta)\) is homologous to \(\partial(\sum_k y_k \otimes z_k)\), and so \(\Phi(id \otimes i^*\alpha)(\eta)\) and \(\Phi(id \otimes \alpha)\partial(\sum y_k \otimes z_k)\) represent the same homology class. The claim of the lemma now follows.

Some algebra. The stability formulas for the cup and cross products are a little trickier. Part of the problem is that the maps \(\partial^*\) and \(d^*\), in their usual constructions, aren’t described via chain maps, which makes it difficult to apply our previous results as tools. To get around this difficulty, we utilize the algebraic mapping cone construction, which lets us replace quotient complexes by algebraic mapping cones (up to chain homotopy equivalence) and the connecting morphisms in long exact homology sequences by morphisms induced by chain maps. Having done this, we will be able to establish our desired stability formulas for cup and cross products, but first we must review the necessary algebra.
Shifts. It is useful to be able to reindex chain complexes. Most often, one sees this done for cohomologically indexed complexes, in which case, if \( D \) is such a chain complex (of \( R \)-modules) with (co)boundary map \( d_D \), the shifted complex \( D[k] \) is defined such that \((D[k])^i = D^{k+i}\) and \(d_{D[k]} = (-1)^k d_D\); see [III Section III.3]. For homological indexing, using the standard bijection between cohomologically indexed complexes and homologically indexed complexes such that \( C_i = C^{-i} \), we see that if \( C \) is a homologically indexed complex, then we should have

\[
(C[k])_i = (C[k]^*)_i = C^{k-i} = C_{i-k}.
\]

In other words, given \( C \), we should let \( C[k] \) be the chain complex with \((C[k])_i = C_{i-k} \) and \( \partial_{C[k]} = (-1)^k \partial_C \). For our purposes, we will only ever need to consider \( k = 1 \), and so we make that specialization in the rest of this section.

Taking \( k = 1 \) and \( C \) a chain complex (of \( R \)-modules), we obtain \( C[1] \) with \((C[1])_i = C_{i-1} \) and \( \partial_{C[1]} = -\partial_C \). Let us define \( s : C[1] \to C \) so that it takes \( C[1] \), identically to the corresponding module \( C_{i-1} \). Then from the definition of the boundary map on \( C[1] \), we see that \( s \partial_{C[1]} = -\partial_C s \), which is consistent with \( s \) being a (homological) degree \(-1\) chain map.

Unfortunately, it is easy to get confused when attempting to consider \( C_{i-1} \) and \((C[1])_i \) as two separate entities, especially when working with individual elements. Indeed, it is very tempting to write things like \( s(x) = x \), which is right and wrong: right because \( C_{i-1} \) and \((C[1])_i \) are identical modules, but wrong because they live in different chain complexes.

In an attempt to mitigate the confusion, if \( x \) is an element of \( C_{i-1} \), we will write \( \bar{x} \) for the corresponding element of \((C[1])_i \), i.e. \( s(\bar{x}) = x \). Of course we could also write \( s^{-1}(x) \) instead of \( \bar{x} \), but it is convenient to have both notations available.

Additionally, the assignment \( C \to C[1] \) is functorial. Suppose \( f : C \to D \) is a chain map of chain complexes. Then we can define \( f[1] : C[1] \to D[1] \) so that if \( \bar{x} \in C[1] \) then \( f[1](\bar{x}) = f(x) \). Alternatively, \( f[1] \) is defined so that the following diagram commutes, which can be done as \( s \) is an isomorphism:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\uparrow{s} & & \uparrow{s} \\
C[1] & \xrightarrow{f[1]} & D[1]
\end{array}
\]

It also follows that \( f[1] \) is a degree 0 chain map. It is then clear that \((\cdot)[1] \) is a functor.

We will also need later the observation that if \( C \) and \( D \) are two chain complexes then there is a canonical (degree 0) isomorphism \( t : C[1] \otimes D \cong (C \otimes D)[1] \). Indeed, we can realize \( t \) as the composition

\[
C[1] \otimes D \xrightarrow{s \otimes \text{id}} C \otimes D \xrightarrow{s^{-1}} (C \otimes D)[1].
\]

This takes a generator \( \bar{x} \otimes y \) to \( \bar{x} \otimes y \).
**Algebraic mapping cones.** Suppose $f : C_* \hookrightarrow D_*$ is a chain map of chain complexes. We let $E^f_*$ (or simply $E_*$ if there’s no ambiguity) denote the algebraic mapping cone of $f : C_* \to D_*$. Recall [40, Section III.3] that this means that $E_i = D_i \oplus C_{i-1} = D_i \oplus C[1]_i$ and $\partial(x, y) = (f(y) + \partial x, -\partial y)$. This is a chain complex, as

$$\partial(\partial(x, y)) = \partial(f(y) + \partial x, -\partial y) = (-f(\partial y) + \partial f(y) + \partial(\partial x), \partial(\partial y)) = 0.$$ 

This construction mimics algebraically the chain complex one obtains from a topological mapping cone; the shift can be thought of as being due to taking the cone on the domain space, and so increasing the dimension by one. Lemma [7.62] below, should provide a more technically convincing version of this claim. We should also note that there are alternative conventions for the algebraic mapping cone construction; see, for example, [105, Section 1.5].

We observe that there is a short exact sequence of chain complexes

$$0 \to D_* \xrightarrow{e} E_* \xrightarrow{b} C[1]_* \to 0$$

with $e(x) = (x, 0)$ and $b(x, y) = \bar{y}$, where $\bar{y}$ uses our notation for shifted elements from just above. It is immediate to verify that $e$ and $b$ are both chain maps. Notice, however, that it is not true that $E_* = D_* \oplus C[1]_*$ as chain complexes, since the boundary map of $E_*$ is not a direct sum of the boundary maps of the summands.

The mapping cone construction is also functorial in the following sense: suppose we have a commutative diagram of chain maps

$$C_* \xrightarrow{g} C'_* \xrightarrow{h} D_* \xrightarrow{f} D'_*.$$ 

Then there is an induced map $k : E^f_* \to E^{f'}_*$ with $k(x, y) = (h(x), g(y))$. We check that this is a chain map:

$$k\partial(x, y) = k(f(y) + \partial x, -\partial y) = (hf(y) + h(\partial x), -g(\partial y) = (f'g(y) + \partial h(x), -\partial g(y)) = \partial(h(x), g(y)) = \partial k(x, y).$$

We also obtain a commutative diagram

$$0 \to D_* \xrightarrow{e} E_* \xrightarrow{b} C[1]_* \to 0$$

$$0 \to D'_* \xrightarrow{e} E'_* \xrightarrow{b} C'[1]_* \to 0.$$
We leave the easy verification of commutativity to the reader.

Now, suppose that \( i : C_* \to D_* \) is an inclusion and consider \( E^i_* \). In this case, we will simply write \( \partial(x, y) = (f(y) + \partial x, -\partial y) = (y + \partial x, -\partial y) \), as elements of \( C_* \) can also be considered elements of \( D_* \). In this setting, there are useful interactions between the long exact homology and cohomology sequences associated with \( (24) \) and the usual long exact homology sequences of the pair \( (D_*, C_*) \).

**Lemma 7.62.** Suppose that \( i : C_* \to D_* \) is an inclusion of chain complexes with algebraic mapping cone \( E_* \). There is a diagram of long exact homology sequences,

\[
\begin{array}{ccccccccc}
& & H_{i+1}(C[1]_*) & \partial_* & H_i(D_*) & \epsilon & H_i(E_*) & b & H_i(C[1]_*) & \\
& & \downarrow{s} & 1 & = & 1 & q & -1 & \downarrow{s} \\
& & H_i(C_*) & \downarrow{i} & H_i(D_*) & \downarrow{p} & H_i(D_*/C_*) & \partial_* & H_{i-1}(C_*) & \\
\end{array}
\]

which commutes up to the signs indicated in each square. Here the top sequence is the long exact sequence associated to the short exact sequence \( (24) \) and the bottom sequences is the long exact sequence associated to the short exact sequence

\[
0 \to C_* \xrightarrow{i} D_* \xrightarrow{p} D_*/C_* \to 0 \quad (25)
\]

and the bottom sequence. In particular, \( q : H_*(E) \to H_*(D_*/C_*) \) is an isomorphism.

**Proof.** We first observe that, via \( s \), the cycles of \( C[1]_* \) are taken bijectively to the cycles of \( C_{i-1} \) and the boundaries of \( C[1]_* \) are taken bijectively to the boundaries of \( C_{i-1} \), so \( H_i(C[1]_*) = H_{i-1}(C) \). The vertical maps \( s \) in the diagram involving \( C_* \) represent simply this canonical isomorphism.

Next, let us define \( q : E_* \to D_*/C_* \). If \((x, y) \in E_* \), we let \( q(x, y) = x \), where we let \( x \) also represent the class of \( x \) in \( D_*/C_* \). This is evidently a homomorphism, and we have \( q(\partial(x, y)) = q((y + \partial x, -\partial y)) = y + \partial x \), which represents the same class as \( \partial x \) in \( D_*/C_* \). So \( q \) is a chain map. We can also see immediately from the definitions that \( q \epsilon = p \), so the middle square in the diagram commutes.

Next, let us check that the other squares commute. Each of these includes one map that is the boundary map of a long exact sequence. Let us first suppose that \((x, y) \in E_* \) represents a cycle in \( H_i(E_*) \). Then \( q(x, y) \) is represented by \( x \), and by the standard zig-zag definition of the boundary map in a long exact homology sequence, \( \partial q(x, y) \) is represented by a cycle in \( C_{i-1} \) that maps to \( \partial x \) under the injection \( i \). On the other hand \( b(x, y) = \bar{y} \), so \( s \bar{y} = x \). But we have stipulated that \((x, y) \) is a cycle so that \((0, 0) = \partial(x, y) = (y + \partial x, -\partial y) \), and it follows that \( \partial x = -y \). So the square commutes up to a sign of \(-1 \).

Now, suppose \( \bar{y} \in C[1]_{i+1} \) is a cycle representing a homology class. Again we use the standard zig-zag construction to compute \( \partial s(\bar{y}) \). We notice that \((0, y) \in E_{i+1} \) satisfies
Suppose \( i : C_\ast \to D_\ast \) is a chain map of chain complexes of projective \( R \)-modules. There is a diagram of long exact cohomology sequences,

\[
\begin{array}{ccccccccc}
& & & H^{i+1}(\text{Hom}(C[1], R)) & \xrightarrow{d^*} & H^i(\text{Hom}(D_\ast, R)) & \xrightarrow{\epsilon^*} & H^i(\text{Hom}(E_\ast, R)) & \xrightarrow{b^*} & H^i(\text{Hom}(C[1], R)) & \\
& & & \downarrow{s^*} & & \downarrow{-1} & & \downarrow{1} & & \downarrow{q^*} & & \downarrow{1} & & \downarrow{s^*} & \\
& & & H^i(\text{Hom}(C_\ast, R)) & \xrightarrow{i^*} & H^i(\text{Hom}(D_\ast, R)) & \xrightarrow{\beta^*} & H^i(\text{Hom}(D_\ast/C_\ast, R)) & \xrightarrow{d^*} & H^{i-1}(\text{Hom}(C_\ast, R)) & \\
\end{array}
\]

which commutes up to the signs indicated in each square. Here the top sequence is the long exact sequence associated to the short exact sequence

\[
0 \quad \text{Hom}(C_\ast, R) \quad \xrightarrow{i^*} \quad \text{Hom}(D_\ast, R) \quad \xrightarrow{\epsilon^*} \quad \text{Hom}(D_\ast/C_\ast, R) \quad \xrightarrow{b^*} \quad 0
\]

and the bottom sequence is the long exact sequence associated to the dual short exact sequence to (24).

In particular, \( q^* : H^*(\text{Hom}(D_\ast/C_\ast, R)) \to H^*(\text{Hom}(E_\ast, R)) \) is an isomorphism.

Proof. First notice that the assumption that all modules be projective implies that the Hom duals of the short exact sequences (24) and (25) remain exact and so generate long exact cohomology sequences.

Next, recall that if \( \alpha \in \text{Hom}(C_i, R) \) and \( \bar{x} \in C[1]_{i+1} \), we have \( s^*(\alpha)(\bar{x}) = (-1)^i\alpha(s(\bar{x})) = (-1)^i\alpha(x) \); the sign is due to the Koszul convention as \( s \) is a degree \(-1\) map and \( \alpha \) is a degree \(-i\) map. Since \( s \) is a (degree \(-1\) chain map that is the identity on modules up to indexing, the same is true of \( s^* \) up to sign. In particular, \( s^* \) takes cocycles in \( \text{Hom}(C_i, R) = \text{Hom}^i(C_\ast, R) \) bijectively to cocycles of \( \text{Hom}^{i+1}(C[1], R) \) and coboundaries in \( \text{Hom}(C_i, R) = \text{Hom}^i(C_\ast, R) \) bijectively to coboundaries of \( \text{Hom}^{i+1}(C[1], R) \), so \( H^{i+1}(\text{Hom}(C[1], R)) \cong H^i(\text{Hom}(C_\ast, R)) \). The vertical maps \( s^* \) in the diagram involving \( C_\ast \) represent simply this canonical isomorphism.

We saw in the proof of Lemma 7.62 that \( qe = p \), so \( e^*q^* = p^* \), so the middle square in the diagram commutes.

Next, let us check that the other squares commute. Each of these includes one map that is the coboundary map of a long exact sequence.

\[\text{b}(0, y) = \bar{y}, \text{ and then consider } \partial(0, y) = (y, -\partial y) = (y, 0) = \epsilon(y). \text{ Therefore, } y \text{ represents } \partial_s(\bar{y}), \text{ and of course } is(\bar{y}) = y, \text{ so this shows that the left square of the diagram commutes.} \]

Finally, since the vertical maps of the diagram involving \( C_\ast \) and \( D_\ast \) are isomorphisms, the map \( q \) must also induce a homology isomorphism by the five lemma. Technically, we do not have a strictly commutative diagram, but if we choose a fixed degree \( i \) and change the signs of the two vertical maps to the right of \( q : H_i(E_\ast) \to H_i(D_\ast/C_\ast) \), then we obtain a strictly commutative diagram in a large enough vicinity of this particular \( q \) to apply the five lemma and conclude that this \( q \) is an isomorphism. But of course the same argument works for any \( i \).

Lemma 7.63. Suppose \( i : C_\ast \to D_\ast \) is a chain map of chain complexes of projective \( R \)-modules.
First, suppose $\alpha \in \text{Hom}^i(D_*, R)$ is a cocycle representing an element of $H^i(\text{Hom}(D_*, R))$. By the zig-zag construction, $d^*\alpha \in H^{i+1}(\text{Hom}(C[1]_*, R))$ is represented by choosing a cochain $\bar{\alpha} \in \text{Hom}(E_i, R)$ that restricts to $\alpha$ on $D_*$, taking $d\bar{\alpha}$, and then restriction $d\bar{\alpha}$ to $\text{Hom}^{i+1}(C[1]_*, R)$. We are free to choose $\bar{\alpha}$ to be $\bar{\alpha} = (\alpha, 0) \in \text{Hom}(E_i, R) = \text{Hom}(D_i \oplus C_{i-1}, R) = \text{Hom}(D_i, R) \oplus \text{Hom}(C_{i-1}, R)$. So then $d(\bar{\alpha})$ acts on a chain $(0, y) \in D_{i+1} \oplus C_i$ by

$$
(d\bar{\alpha})(0, y) = (-1)^{i+1}\bar{\alpha}\partial(0, y)
= (-1)^{i+1}(\alpha, 0)(y, -\partial y)
= (-1)^{i+1}\alpha(y).
$$

On the other hand, we have

$$
\mathbf{s}^*i^*(\bar{\alpha})(\bar{y}) = (-1)^i\bar{\alpha}(\bar{y})
= (-1)^i\alpha(i(y))
= (-1)^i\alpha(y).
$$

So the left square commutes up to the sign $-1$.

Now, suppose $\alpha \in \text{Hom}^{i-1}(C_*, R) = \text{Hom}(C_{i-1}, R)$ represents an element of $H^{i-1}(\text{Hom}(C_*, R))$. Then $d^*\alpha$ is represented by choosing a cochain $\bar{\alpha} \in \text{Hom}(D_{i-1}, R)$ that restricts to $\alpha$ on $C_{i-1}$ and then taking $d\bar{\alpha}$ and restricting it to act on $D_*/C_*$.

If $(x, y) \in E_i$, we have

$$
q^*d\bar{\alpha}(x, y) = (d\bar{\alpha})q(x, y)
= (-1)^i\bar{\alpha}(\partial q(x, y))
= (-1)^i\bar{\alpha}(\partial x).
$$

On the other hand, noting that $b$ is a degree 0 chain map,

$$
b^*\mathbf{s}^*\alpha(x, y) = (\mathbf{s}^*\alpha)(b(x, y))
= (\mathbf{s}^*\alpha)(\bar{y})
= (-1)^{i-1}\alpha(\mathbf{s}(\bar{y}))
= (-1)^{i-1}\alpha(y).
$$

But now consider $(\bar{\alpha}, 0) \in \text{Hom}(D_{i-1}, R) \oplus \text{Hom}(C_{i-2}, R) = \text{Hom}(D_{i-1} \oplus C_{i-2}, R) = \text{Hom}(E_{i-1}, R)$. So we can compute

$$
d(\bar{\alpha}, 0)(x, y) = (-1)^i(\bar{\alpha}, 0)\partial(x, y)
= (-1)^i(\bar{\alpha}, 0)(y + \partial x, -\partial y)
= (-1)^i(\bar{\alpha}(y) + \bar{\alpha}(\partial x))
= (-1)^i\bar{\alpha}(y) + (-1)^i\bar{\alpha}(\partial x)
= (-1)^i\alpha(\bar{y}) + q^*d\bar{\alpha}(x, y)
= -b^*\mathbf{s}^*\alpha(x, y) + q^*d\bar{\alpha}(x, y)
\text{ by } (26)
= -b^*\mathbf{s}^*\alpha(x, y) + q^*d\bar{\alpha}(x, y)
\text{ by } (27).
$$

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In the fifth line, we have also used that $\alpha(y) = \tilde{\alpha}(y)$, as $y \in C_*$. So $b^*s^*\alpha$ and $q^*d\tilde{\alpha}$ represent the same cohomology class.

Finally, since the vertical maps of the diagram involving $C_*$ and $D_*$ are isomorphisms, the map $q^*$ must also induce a cohomology isomorphism by the five lemma. Technically, we do not have a strictly commutative diagram, but for any fixed $i$, we can change the signs of nearby vertical maps to obtain a strictly commutative diagram in a large enough vicinity of this particular $q^*$ to apply the five lemma and conclude that this $q^*$ is an isomorphism. \qed

**Back to stability.** Now we can return to establishing stability formulas for cup and cross products. Of course, these come in pairs corresponding, for example, to $(d^*\alpha) \times \beta$ or $\alpha \times (d^*\beta)$. We only prove one result for each such pair, as the other can be obtained using the commutative properties.

Various excision isomorphisms come into play in stating and proving the stability formulas. We denote these generically with an $e$, specifying in more detail where relevant.

**Lemma 7.64.** Suppose that $\bar{p}, \bar{q}$ are perversities on filtered sets $X, Y$. Let $A \subset X$ and $B \subset Y$ be open subsets. Let $F_*$ denote the algebraic mapping cone of the inclusion $i : I^pS_* (A; R) \to I^pS_* (X; R)$, let $E_*$ be the algebraic mapping cone of the map $i \times id : I^{Q\bar{p}, \bar{q}}S_* (A \times Y, A \times B; R) \to I^{Q\bar{p}, \bar{q}}S_* (X \times Y, X \times B; R)$, and let $G_*$ be the algebraic mapping cone of the inclusion $i \times id : I^{Q\bar{p}, \bar{q}}S_* ((A \times Y) \cup (X \times B), X \times B; R) \to I^{Q\bar{p}, \bar{q}}S_* (X \times Y, X \times B; R)$. The following diagram (with implicit $R$ coefficients) commutes:

\[
\begin{array}{ccc}
I^{Q\bar{p}, \bar{q}}S_* (X \times Y, (A \times Y) \cup (X \times B)) & = & I^{Q\bar{p}, \bar{q}}S_* (X \times Y, (A \times Y) \cup (X \times B)) \\
\downarrow & & \downarrow \\
I^{Q\bar{p}, \bar{q}}S_* ((A \times Y) \cup (X \times B), X \times B) & \cong & I^{Q\bar{p}, \bar{q}}S_* (X \times Y, (A \times Y) \cup (X \times B)) \\
\downarrow & & \downarrow \\
I^{Q\bar{p}, \bar{q}}S_* ((A \times Y) \cup (X \times B), X \times B) & \cong & I^{Q\bar{p}, \bar{q}}S_* (X \times Y, (A \times Y) \cup (X \times B)) \\
\end{array}
\]

**Proof.** Let us first verify that all the maps make sense. The unlabeled maps are induced by the obvious inclusions and/or quotients. The upper left vertical map is an isomorphism by the functorial properties.

The map $E_* \to G_*$ makes sense by the functoriality of the algebraic mapping cone construction, and we define the map labeled $q'$ to act on $(x, y) \in E_i = I^{Q\bar{p}, \bar{q}}S_i (X \times Y, X \times B; R) \oplus I^{Q\bar{p}, \bar{q}}S_{i-1} (A \times Y, A \times B; R)$ by taking $(x, y)$ to the class of $x$ in $I^{Q\bar{p}, \bar{q}}S_i (X \times Y, (A \times Y) \cup (X \times B); R)$. This is a chain map because $q'(x, y) = q'(y + \partial dx, \partial y) = y + \partial x$. But $y$ here is represented by a chain supported in $A \times Y$, and so $y + \partial x$ and $\partial x$ represent the same element in $I^{Q\bar{p}, \bar{q}}S_* (X \times Y, (A \times Y) \cup (X \times B))$. It also follows readily from this definition that the upper left rectangle commutes.
For the map labeled \( \epsilon \), we let this be the composition of the canonical isomorphism \( t \) (see page 296), and the shifted cross product \( \epsilon[1] \).

Similarly, for the mapped labeled \((\epsilon, \epsilon)\), recall that as modules, we have

\[
E_i = I^{q \& q} S_i(X \times Y, X \times B; R) \oplus I^{q \& q} S_{i-1}(A \times Y, A \times B; R)
\]

and

\[
F_j = I^p S_j(X; R) \oplus I^p S_{j-1}(A; R).
\]

Thus \((F_\ast \otimes I^q S_\ast(Y, B; R))\), is the direct sum over \( j + k = i \) of terms of the form

\[
(I^p S_j(X; R) \oplus I^p S_{j-1}(A; R)) \otimes I^q S_k(Y, B; R)
= (I^p S_j(X; R) \otimes I^q S_k(Y, B; R)) \oplus (I^p S(A; R)_{j-1} \otimes I^q S_k(Y, B; R)).
\]

Such modules are generated by elements of the form \((x, a) \otimes y = (x \otimes y, a \otimes y)\), and we let \((\epsilon, \epsilon)\) act in the obvious way by \((\epsilon, \epsilon)(x \otimes y, a \otimes y) = (\epsilon(x \otimes y), \epsilon(a \otimes y))\). Let us check that this gives us a chain map, keeping \(|x| = j\) and \(|y| = k\), so that \(|a| = j - 1\) and \(|(x, a)| = j:

\[
(\epsilon, \epsilon)(\partial((x, a) \otimes y)) = (\epsilon, \epsilon)(\partial(x, a) \otimes y + (-1)^j(x, a) \otimes \partial y)
= (\epsilon, \epsilon)((\partial x + i(a), -\partial a) \otimes y + (-1)^j(x, a) \otimes \partial y)
= (\epsilon, \epsilon)((\partial x \otimes y + i(a) \otimes y, -(\partial a) \otimes y) + (-1)^j(x \otimes \partial y, a \otimes \partial y))
= (\epsilon, \epsilon)((\partial x \otimes y + i(a) \otimes y + (\partial a) \otimes y) - (\partial a) \otimes y + (-1)^j a \otimes \partial y))
= (\epsilon, \epsilon)(\partial(x \otimes y) + i(a) \otimes y, -\partial(a \otimes y))
= (\epsilon(\partial(x \otimes y) + i(a) \otimes y), -\epsilon(\partial(a \otimes y)))
= (\partial(\epsilon(x \otimes y)) + (i \times \text{id})\epsilon(a \otimes y), -\partial(\epsilon(a \otimes y))
= \partial(\epsilon(x \otimes y), \epsilon(a \otimes y)).
\]

Here, in the next to last line, we use that \( \epsilon \) is a chain map and that it is naturale by Lemma 5.16.

For commutativity of the top right square, we compute using representatives

\[
q(\epsilon, \epsilon)((x, a) \otimes y) = q(\epsilon, \epsilon)(x \otimes y, a \otimes y)
= q(\epsilon(x \otimes y), \epsilon(a \otimes y))
= \epsilon(x \otimes y)
= \epsilon(q(x, a) \otimes y)
= \epsilon(q \otimes \text{id})(x, a) \otimes y).
\]
Similarly, for the bottom right square, we have

\[
\begin{align*}
\mathbf{b}(\epsilon, \epsilon)((x, a) \otimes y) &= \mathbf{b}(\epsilon, \epsilon)(x \otimes y, a \otimes y) \\
&= \mathbf{b}(\epsilon(x \otimes y), \epsilon(a \otimes y)) \\
&= \epsilon(a \otimes y) \\
&= \epsilon[1]a \otimes y \\
&= \epsilon[1](\bar{a} \otimes y) \\
&= \epsilon(\mathbf{b}(x, a) \otimes y) \\
&= \epsilon(\mathbf{b} \otimes \text{id})(x, a) \otimes y).
\end{align*}
\]

The commutativity of the bottom left square is a consequence of the naturality properties of the mapping cone construction.

\begin{proof}
This top portion of this diagram is obtained from the diagram of Lemma \[7.64\] by replacing some maps (or composition of maps) with their homotopy inverses. In particular, by Lemma \[7.62\], the \(\mathbf{q}\) maps are quasi-isomorphisms of chain complexes of projective modules, so they are homotopy equivalences by Lemma \[12.7\] (noting that it is sufficient in the argument for the complexes simply to both be bounded below). It follows that \(\mathbf{q} \otimes \text{id}\) is also a homotopy equivalence. We know that \(\epsilon\) is a quasi-isomorphism by the Künneth theorem, and \(\text{IAW}\) is our label for its inverse. It follows that the composition in the middle horizontal row of the diagram of Lemma \[7.63\] is also a homotopy equivalence. We also use \(\text{IAW}\) as the chain homotopy inverse for \(\bar{\epsilon} = \epsilon[1]t\), which is a composition of an isomorphism and a shifted homotopy equivalence.
\end{proof}

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The inclusion-induced map \( I_{p,q}^* S_* (A \times Y, A \times B) \to I_{p,q}^* S_* ((A \times Y) \cup (X \times B), X \times B) \) is also a quasi-isomorphism, and so a homotopy equivalence as all modules are projective. We can see this noting that the map factors as

\[
\frac{I_{p,q}^* S_* (A \times Y; R)}{I_{p,q}^* S_* (A \times B; R)} = \frac{I_{p,q}^* S_* ((A \times Y) \cap (X \times B); R)}{I_{p,q}^* S_* (X \times B; R)} \to \frac{I_{p,q}^* S_* ((A \times Y) \cup (X \times B); R)}{I_{p,q}^* S_* (X \times B; R)}
\]

The first map in the second line is an isomorphism by the second isomorphism theorem, noting that \( I_{p,q}^* S_* (A \times Y; R) \cap I_{p,q}^* S_* (X \times B; R) = I_{p,q}^* S_* (A \times B; R) \), while the second map induces a homology isomorphism by the non-GM version of the arguments in the proof of Theorem 4.46.

Therefore, appropriately reversing the arrows in the diagram of Lemma 7.64 results in a homotopy commutative diagram as labeled, except for the bottom row, which we have added on. The bottom left square commutes by the functoriality of shifting. The bottom right square is the following square with its horizontal maps inverted up to homotopy:

\[
\begin{array}{ccc}
I_{p,q}^* S_* (A \times Y, A \times B)[1] & \overset{\epsilon}{\longrightarrow} & I^p S_* (A)[1] \otimes I^q S_* (Y, B) \\
\downarrow s & & \downarrow s \otimes \text{id} \\
I_{p,q}^* S_* (A \times Y, A \times B) & \overset{\epsilon}{\longrightarrow} & I^p S_* (A) \otimes I^q S_* (Y, B).
\end{array}
\]

This commutes because

\[
se(x \otimes y) = se[1]t(x \otimes y) = se[1]x \otimes y = s(\epsilon(x \otimes y)) = \epsilon(x \otimes y) = \epsilon(s(x) \otimes y) = \epsilon(s \otimes \text{id})(x \otimes y).
\]

\[
\square
\]

**Lemma 7.66.** Let \( R \) be a Dedekind domain. Suppose \( X \) and \( Y \) are CS sets with respective perversies \( \bar{p} \) and \( \bar{q} \) and open subspaces \( A \subset X \) and \( B \subset Y \). Suppose that \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free. Let \( \alpha \in I_{\bar{p}} H^i(A; R) \) and \( \beta \in I_{\bar{q}} H^j(Y, B; R) \). Then

\[
(d^*(\alpha)) \times \beta = d^*(e^{-1})^*(\alpha \times \beta) \in I_{p,q} H^{i+j+1}(X \times Y, (A \times Y) \cup (X \times B); R),
\]

where \( e : I_{p,q} H_* (A \times Y, A \times B; R) \to I_{p,q} H_* ((A \times Y) \cup (X \times B), X \times B; R) \) is an excision isomorphism and where we interpret the right hand \( d^* \) as a map \( I_{p,q} H^{i+j}(A \times Y) \cup (X \times B), X \times B; R) \to I_{p,q} H^{i+j+1}(X \times Y, (A \times Y) \cup (X \times B); R) \).
Proof. By Lemma 7.63, $d^*(\alpha) = (q^*)^{-1}b^*s^*\alpha$. As observed in Corollary 7.65, since all of our modules are projectives, the quasi-isomorphism $q$ (see Lemma 7.62) is in fact chain homotopy equivalence, so we can replace $(q^*)^{-1}$ with $(q^{-1})^*$, which is well defined up to chain homotopy. This lets us write $d^*(\alpha) = (q^{-1})^*b^*s^*\alpha = (sbq^{-1})^*\alpha$. So, from the definition of the cross product, we have

$$(d^*\alpha) \times \beta = IAW^*(d^*(\alpha) \otimes \beta)$$

$$= IAW^*((sbq^{-1})^*(\alpha) \otimes \beta)$$

$$= IAW^*((sbq^{-1}) \otimes \text{id})^*(\alpha \otimes \beta)$$

$$= IAW^*((sbq^{-1}) \otimes \text{id})^*\Theta(\alpha \otimes \beta)$$

by Lemma 7.31

$$= (e^{-1}sbq^{-1})^*IAW^*(\alpha \otimes \beta)$$

by Corollary 7.65

$$= d^*(e^{-1})^*(\alpha \times \beta)$$

by Lemma 7.63.

\[ \square \]

Lemma 7.67. Let $R$ be a Dedekind domain. Suppose $X$ is a CS set with open subsets $A$ and $B$ and that $i : A \to X$ is the inclusion map. Suppose $\bar{p}, \bar{q}, \bar{r}$ are perversities on $X$ with $D_{\bar{r}} \geq D_{\bar{p}} + D_{\bar{q}}$, that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Let $\alpha \in I_{\bar{p}}H^i(A; R)$ and $\beta \in I_{\bar{q}}H^j(X, B; R)$. Then

$$(d^*(\alpha)) \sim \beta = d^*(e^{-1})^*(\alpha \sim i^*(\beta)) \in I_{\bar{r}}H^{i+j+1}(X, A \cup B; R),$$

where $e : I_{\bar{r}}H_*(A, A \cap B; R) \to I_{\bar{r}}H_*(A \cup B, B; R)$ is induced by inclusion and where we interpret the right hand $d^*$ as a map $I_{\bar{r}}H^{i+j}(A \cup B, B; R) \to I_{\bar{r}}H^{i+j+1}(X, A \cup B; R)$.

Proof. Consider the following diagram with tacit coefficients. Here $D_*$ is the algebraic mapping cone of the inclusion $I_{\bar{r}}S_*(A \cup B, B; R) \to I_{\bar{r}}S_*(X, B; R)$, and, as in Lemma 7.64 but with $X = Y$, $G_*$ is the algebraic mapping cone of the inclusion $I^{Q_{\bar{p}, \bar{q}}}S_*((A \times X) \cup (X \times B), X \times B; R) \to I^{Q_{\bar{p}, \bar{q}}}S_*((X \times X), X \times B; R)$:
The bottom quadrilateral commutes by the naturality of the chain cross product, Lemma 5.16. The triangle and the square with the $e$ maps commute at the level of maps of pairs of spaces. The isomorphisms in the top square come from the third isomorphism theorem, and this square also commutes by looking at representative elements. So let us consider the squares involving the mapping cones. By the naturality properties of the mapping cone construction, the map labeled $(d, d)$ is a chain map and the square involving the $b$ maps
commutes. Furthermore, if \((x, y) \in D_i = I^\ell S_i(X, B; R) \oplus I^\ell S_{i-1}(A \cup B, B; R)\), we have
\[ dq(x, y) = d(x) = q(d(x), d(y)) = q(d, d)(x, y), \]
so the second square from the top commutes.

Therefore, the full diagram commutes. Furthermore, \(q\) is again a quasi-isomorphism by Lemma 7.62 and it is a homotopy equivalence since we work with all projective modules. Also, our new \(e\) on the left of the diagram is a quasi-isomorphism, and hence a homotopy equivalence of projective modules, since we can factor it as
\[
\frac{I^\ell S_*(A; R)}{I^\ell S_*(A \cap B; R)} \to \frac{I^\ell S_*(A; R) + I^\ell S_*(B; R)}{I^\ell S_*(B; R)} \to \frac{I^\ell S_*(A \cup B; R)}{I^\ell S_*(B; R)}.
\]
The first map is an isomorphism by the second isomorphism theorem, noting that \(I^\ell S_*(A) \cap I^\ell S_*(B) = I^\ell S_*(A \cap B; R)\), while the second map induces a homology isomorphism by the non-GM version of the arguments in the proof of Theorem 4.46. Therefore, we can invert homotopy equivalences and adjoin to part of the diagram of Corollary 7.65 to get the following diagram that commutes up to homotopy:

Now we can compute. By Lemma 7.63 \(d^*(\alpha) = (q^*)^{-1}b^s s^*\alpha\). As observed in Corollary 7.65 since all of our modules are projectives, the quasi-isomorphisms \(q\) (see Lemma 7.62) are in fact chain homotopy equivalences, so we can replace \((q^*)^{-1}\) with \((q^{-1})^*\), which is well defined up to homotopy. This lets us write \(d^*(\alpha) = (q^{-1})^*b^s s^*\alpha = (sbq^{-1})^*\alpha\). So, from
the definition of the cup product, we have

\[
(d^*(\alpha)) \sim \beta = d^*IAW^*\Theta(d^*(\alpha) \otimes \beta) \\
= d^*IAW^*\Theta(((s\phi_q)^{-1})^*(\alpha)) \otimes \beta) \\
= d^*IAW^*\Theta((s\phi_q^{-1})^* \otimes \text{id})(\alpha \otimes \beta) \\
= d^*IAW^*((s\phi_q^{-1}) \otimes \text{id})^*\Theta(\alpha \otimes \beta) \quad \text{by Lemma 7.31} \\
= (e^{-1}s\phi_q^{-1})^*d^*IAW^*(\text{id} \otimes i)^*\Theta(\alpha \otimes \beta) \quad \text{by the diagram} \\
= d^*(e^{-1})^*d^*IAW^*(\text{id} \otimes i^*)^*\Theta(\alpha \otimes \beta) \quad \text{by Lemma 7.63} \\
= d^*(e^{-1})^*d^*IAW^*\Theta((\text{id} \otimes i^*)^*(\alpha \otimes \beta)) \quad \text{by Lemma 7.31} \\
= d^*(e^{-1})^*(\alpha \sim i^*(\beta)).
\]

\[
\square
\]

7.3.6 Criss-crosses

There are a variety of relations involving combinations of the cup, cap, and cross products, some of which we have already seen. We discuss the rest here. Perhaps the simplest is the following, which comes right from the definitions:

Lemma 7.68. Let \( R \) be a Dedekind domain. Suppose that \( \bar{p}, \bar{q}, \bar{r} \) are perversities on a CS set \( X \) such that \( D\bar{r} \geq D\bar{p} + D\bar{q} \). Suppose, furthermore, that \( A, B \) are open subsets of \( X \), and that \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free. If \( \alpha \in I_{\bar{p}}H^i(X, A; R) \) and \( \beta \in I_{\bar{q}}H^j(X, B; R) \), then \( d^*(\alpha \times \beta) = \alpha \sim \beta \in I_{\bar{r}}H^{i+j}(X, A \cup B; R) \).

The preceding lemma shows that the cup product can be obtained from the cross product. The next lemma demonstrates that the cross product is really a cup product.

Lemma 7.69. Let \( R \) be a Dedekind domain. Suppose that \( X \) is a CS set with perversity \( \bar{p}, \bar{q}, \bar{r} \) and \( A \subset X, B \subset Y \) are open sets. Furthermore, suppose that \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free. Let \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \) be the projection maps and \( p_1^* : I_{\bar{p}}H^*(X, A; R) \to I_{Q_{\bar{p},\bar{q}}}H^*(X \times Y, A \times Y; R) \) and \( p_2^* : I_{\bar{q}}H^*(Y, B; R) \to I_{Q_{\bar{p},\bar{q}}}H^*(X \times Y, X \times B; R) \), where \( \bar{p} \) and \( \bar{q} \) are the respective top perversities on \( X \) and \( Y \). Then if \( \alpha \in I_{\bar{p}}H^1(X, A; R) \) and \( \beta \in I_{\bar{q}}H^1(X, B; R) \), we have

\[
(p_1^*(\alpha)) \sim (p_2^*(\beta)) = \alpha \times \beta \in I_{Q_{\bar{p},\bar{q}}}H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R).
\]

Proof. The maps \( p_1^* \) and \( p_2^* \) are well-defined with respective images in \( I_{Q_{\bar{p},\bar{q}}}H^*(X \times Y, A \times Y; R) \) and \( I_{Q_{\bar{p},\bar{q}}}H^*(X \times Y, X \times B; R) \) by Corollary 7.42. Furthermore, if \( X \) is locally \((\bar{p}, R)\)-torsion free, then \( X \times Y \) is locally \((Q_{\bar{p},\bar{q}})\)-torsion free by Lemma 7.52 and Example 5.39 (which says that every space is locally torsion free with respect to the top perversity). Similarly, if \( Y \) is locally \((\bar{q}, R)\)-torsion free, then \( X \times Y \) is locally \((Q_{\bar{p},\bar{q}})\)-torsion free. Therefore, by the assumption that \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, the product \( X \times Y \) is locally \((Q_{\bar{p},\bar{q}})\)-torsion free or locally \((Q_{\bar{p},\bar{q}})\)-torsion free. Therefore, the cup product of \( p_1^*(\alpha) \) and \( p_2^*(\beta) \) is well defined with image in \( I_{Q_{\bar{p},\bar{q}}}H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R) \).
if $DQ_{p,q} \geq DQ_{t_x,q} + DQ_{p,i_y}$. Let us verify that this works out. The required expression is equivalent to

$$\bar{t}_{X \times Y} - Q_{p,q} \geq \bar{t}_{X \times Y} - Q_{t_x,q} + \bar{t}_{X \times Y} - Q_{p,i_y},$$

which becomes

$$Q_{p,i_y} + Q_{t_x,q} - Q_{p,q} \geq \bar{t}_{X \times Y}.$$  

If $R \subset X$ is a regular stratum and $T \subset Y$ is singular, the left side evaluates on $R \times T$ to

$$\bar{t}_{Y}(T) + \bar{q}(T) - \bar{q}(T) = \text{codim}_Y(T) - 2,$$

as does the right side, using that $\text{codim}_{X \times Y}(S \times T) = \text{codim}_X(S) + \text{codim}_Y(T)$ for any strata $S \subset X$ and $T \subset Y$, while, if $R$ is regular, $\text{codim}_X(R) = 0$. A similar computation shows that the two sides agree using a regular stratum from $Y$. If $S \subset X$ and $T \subset Y$ are both singular strata, then the left side becomes

$$\bar{p}(S) + \bar{t}_{Y}(T) + 2 + \bar{t}_{X}(S) + \bar{q}(T) + 2 - \bar{p}(S) - \bar{q}(T) - 2 = \bar{t}_{Y}(T) + \bar{t}_{X}(S) + 2 = \text{codim}_X(S) - 2 + \text{codim}_Y(T) - 2 + 2 = \text{codim}_{X \times Y}(S \times T) - 2 = \bar{t}_{X \times Y}(S \times T).$$

Incidentally, these computations demonstrate that we would not obtain the required inequality if we took perversities less than $\bar{t}_X$ or $\bar{t}_Y$ in their place, but we also cannot take larger perversities due to the restrictions of Corollary [7.42]. Thus the choice of top perversities in these arguments is forced.

Now we must see that the expressions are equal. We have

$$(p^*_1(\alpha)) \sim (p^*_2(\beta)) = \d^*\Theta((p^*_1(\alpha) \otimes (p^*_2(\beta)))$$

$$= \d^*\Theta(p^*_1 \otimes p^*_2)(\alpha \otimes \beta)$$

$$= \d^*(p_1 \otimes p_2)^*\Theta(\alpha \otimes \beta)$$

by Lemma [7.31]

$$= \d^*(p_1 \otimes p_2)^*\text{IAW}^*\Theta(\alpha \otimes \beta)$$

$$= \text{IAW}^*\Theta(\alpha \otimes \beta)$$

by Lemma [5.16]

$$= \alpha \times \beta.$$  

The next to last line uses the fact that, at the level of spaces, $(p_1 \times p_2)d : X \times Y \to X \times Y$ is the identity map. Indeed, $(p_1 \times p_2)d(x,y) = (p_1 \times p_2)((x,y),(x,y)) = (p_1(x,y),p_2(x,y)) = (x,y).$  

We would next like a statement of the form

$$(\alpha \times \gamma) \sim (\beta \times \delta) = (-1)^{|\gamma| \cdot |\beta|}(\alpha \sim \beta) \times (\gamma \sim \delta).$$

For ordinary cohomology, the proof is very simple, utilizing the ordinary cohomology version of Lemma [7.69]. It runs like this:

$$(\alpha \times \gamma) \sim (\beta \times \delta) = (p_1^*(\alpha) \sim p_2^*(\gamma)) \sim (p_1^*(\beta) \sim p_2^*(\delta))$$

$$= (-1)^{|\gamma| \cdot |\beta|}p_1^*(\alpha) \sim p_1^*(\beta) \sim p_2^*(\gamma) \sim p_2^*(\delta)$$

$$= (-1)^{|\gamma| \cdot |\beta|}p_1^*(\alpha \sim \beta) \sim p_2^*(\gamma \sim \delta)$$

$$= (-1)^{|\gamma| \cdot |\beta|}(\alpha \sim \beta) \times (\gamma \sim \delta).$$

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However, in order to apply this argument here, of course we need to make a number of assumptions on the spaces involved; in particular, we need to make sure all the products in these formulas are defined. Rather than look into the above approach directly, we instead develop some more general technical lemmas that we will be able to apply both to the interchange of cup and cross products and to the interchange of cap and cross products. Thus we take a slightly different route than, e.g., Dold [23]. The official statements of these two “interchange” properties can be found in Lemmas 7.12 and 7.13.

**Lemma 7.70.** Let $R$ be a Dedekind domain. Suppose that $X$ is a filtered space with perversities $\bar{p}, \bar{q}, \bar{r}$ and that $Y$ is a filtered space with perversities $\bar{u}, \bar{v}, \bar{s}$. Suppose $D\bar{r} \geq D\bar{p} + D\bar{q}$ and $D\bar{s} \geq D\bar{u} + D\bar{v}$. Let $A, B \subset X$ and $C, D \subset Y$. Then the following diagrams commute (coefficients tacit):

\[
\begin{array}{ccc}
P^{\bar{p}} \otimes S_{(X \times Y; (A \cup B) \times Y) \cup (X \times (C \cup D))} & \xrightarrow{d} & P^{\bar{q}} \otimes S_{(X \times Y \times X \times Y; ((A \times Y) \cup (X \times C)) \times (X \times Y)) \cup ((X \times Y) \times ((B \times Y) \cup (X \times D)))} \\
\downarrow \epsilon & & \downarrow \epsilon \\
P^{\bar{r}} \otimes S_{(X \times Y \times X \times Y; ((A \times X) \cup (X \times B)) \times (Y \times Y)) \cup ((X \times X) \times ((C \times Y) \cup (Y \times D)))} & \xrightarrow{\text{id} \times \text{id} \times \text{id}} & P^{\bar{q}} \otimes S_{(X \times X; A \times X) \cup (X \times B)) \otimes P^{\bar{r}} \otimes S_{(Y \times Y; (C \times Y) \cup (Y \times D))}
\end{array}
\]

\[
\begin{array}{ccc}
P^{\bar{p}} \otimes S_{(X \times Y; (A \cup B) \times Y) \cup (X \times (C \cup D))} & \xrightarrow{\text{id} \times \text{id} \times \text{id}} & P^{\bar{q}} \otimes S_{(X \times Y \times X \times Y; ((A \times Y) \cup (X \times C)) \times (X \times Y)) \cup ((X \times Y) \times ((B \times Y) \cup (X \times D)))} \\
\downarrow \epsilon & & \downarrow \epsilon \\
P^{\bar{r}} \otimes S_{(X \times Y \times X \times Y; ((A \times X) \cup (X \times B)) \times (Y \times Y)) \cup ((X \times X) \times ((C \times Y) \cup (Y \times D)))} & \xrightarrow{\epsilon \otimes \epsilon} & P^{\bar{r}} \otimes S_{(X \times X; A \times X) \cup (X \times B)) \otimes P^{\bar{r}} \otimes S_{(Y \times Y; (C \times Y) \cup (Y \times D))}
\end{array}
\]

**Proof.** We first show that $D\bar{r} \geq D\bar{p} + D\bar{q}$ and $D\bar{s} \geq D\bar{u} + D\bar{v}$ together imply that $D\bar{Q}_{\bar{r}, \bar{s}} \geq D\bar{Q}_{\bar{p}, \bar{q}} + D\bar{Q}_{\bar{q}, \bar{v}}$ so that the diagonal map at the top of the first diagram is well defined. For this, we recall that $D\bar{r} \geq D\bar{p} + D\bar{q}$ is equivalent to $\bar{t}_X - \bar{r} \geq \bar{t}_X - \bar{p} + \bar{t}_X - \bar{q}$, i.e. to

$$\bar{p} + \bar{q} - \bar{r} \geq \bar{t}_X,$$

and similarly for the analogous inequalities. So, in particular, we must show that

$$Q_{\bar{p}, \bar{q}} + Q_{\bar{q}, \bar{v}} - Q_{\bar{r}, \bar{s}} \geq \bar{t}_X \times Y.$$

Let $S \subset X$ and $T \subset Y$ be strata. The inequality holds trivially if $S$ and $T$ are both regular strata. Suppose $T$ is regular and $S$ is singular. Then, plugging $S \times T$ into $Q_{\bar{p}, \bar{q}} + Q_{\bar{q}, \bar{v}} - Q_{\bar{r}, \bar{s}}$, we get

$$\bar{p}(S) + \bar{q}(S) - \bar{r}(S),$$

which, by hypothesis, is

$$\geq \bar{t}_X(S) = \text{codim}_X(S) - 2 = \text{codim}_{X \times Y}(S \times T) - 2 = \bar{t}_{X \times Y}(S \times T).$$

\[\text{Notice that the diagrams fit together to make one large diagram, but it wouldn't fit on the page!}\]
A similarly computation verifies the claim if $S$ is regular. Now, suppose $S$ and $T$ are both singular. Then, plugging $S \times T$ into $Q_{p,\bar{a}} + Q_{q,\bar{v}} - Q_{r,\bar{s}}$ gives

$$Q_{p,\bar{a}}(S \times T) + Q_{q,\bar{v}}(S \times T) - Q_{r,\bar{s}}(S \times T)$$

$$= \bar{p}(S) + \bar{u}(T) + 2 + \bar{q}(S) + \bar{v}(T) + 2 - \bar{r}(S) - \bar{s}(T) - 2$$

$$= \bar{p}(S) + \bar{q}(S) - \bar{r}(S) + \bar{u}(T) + \bar{v}(T) - \bar{r}(S) - \bar{r}(T) + 2$$

$$\geq \bar{t}_X(S) + \bar{t}_Y(T) + 2$$

$$= \operatorname{codim}_X(S) - 2 + \operatorname{codim}_Y(T) - 2 + 2$$

$$= \operatorname{codim}_X(S) + \operatorname{codim}_Y(T) - 2$$

$$= \operatorname{codim}_{X \times Y}(S \times T) - 2$$

$$= \bar{t}_{X \times Y}(S \times T),$$

as desired.

Next, we notice that

$$(A \times Y) \cup (X \times C) \cup (B \times Y) \cup (X \times D) = (A \times Y) \cup (B \times Y) \cup (X \times C) \cup (X \times D)$$

$$= ((A \cup B) \times Y) \cup (X \times (C \cup D))$$

so that the subspaces are correct for the top diagonal map.

We also need to show that the map labeled id $\times t \times id$ makes sense and is an isomorphism. At the space level, this map interchanges the second and third coordinates. We can rewrite

$$(((A \times X) \cup (X \times B)) \times (Y \times Y)) \cup ((X \times X) \times ((C \times Y) \cup (Y \times D)))$$

as

$$(A \times X \times Y \times Y) \cup (X \times B \times Y \times Y) \cup (X \times X \times C \times Y) \cup (X \times X \times Y \times D).$$

Applying id $\times t \times id$, which is a homeomorphism, we get the space

$$(A \times Y \times X \times Y) \cup (X \times Y \times B \times Y) \cup (X \times C \times X \times Y) \cup (X \times Y \times X \times D),$$

and this is equal to

$$(((A \times Y) \cup (X \times C)) \times (X \times Y)) \cup ((X \times Y) \times ((B \times Y) \cup (X \times D))).$$

So id $\times t \times id$ is a homeomorphism of pairs of spaces.

Next, we need to check that id $\times t \times id$ is well-defined as a map of intersection chains. In particular, we have to check the perversities of corresponding strata agree, at which point it becomes clear that allowable chains are taken to allowable chains. In other words, if $S, T$ are strata of $X$ and $U, V$ are strata of $Y$, we need to know that

$$Q_{Q_{p,\bar{a},q,\bar{v}}} \times U \times T \times V = Q_{Q_{p,\bar{a},q,\bar{v}}} \times T \times U \times V.$$

Surprisingly enough, this is true! We compute the relevant values in the following table depending on whether each of $S, T, U, V$ is regular (denoted with an $r$) or singular (denoted with an $s$).
with an $s$). The last column shows the common value of the perversity evaluations. The other two columns are meant to indicate the intermediate steps in each case, eliminating terms that evaluate directly to 0.

$$
\begin{array}{|c|c|c|c|c|}
\hline
S & T & U & V & Q_{D,0}(S \times T \times U \times V) = Q_{D,0}(S \times U \times T \times V) \quad \text{common value} \\
\hline
r & r & r & r & 0 \\
\hline
s & r & r & r & Q_{D,0}(S \times T) \\
\hline
r & s & r & r & Q_{D,0}(S \times T) \\
\hline
r & r & s & r & Q_{D,0}(U \times V) \\
\hline
r & r & r & s & Q_{D,0}(U \times V) \\
\hline
s & r & r & r & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
s & s & r & r & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
s & r & s & r & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
s & r & r & s & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
s & s & s & r & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
s & s & s & s & Q_{D,0}(S \times T) + Q_{D,0}(U \times V) + 2 \\
\hline
\end{array}
$$

Now that we know all the arrows in the diagram make sense, we need to verify the commutativity. As in our previous arguments in this section (for example, in the proof of Lemma 7.45), it now suffices to verify commutativity of the analogous diagram of ordinary singular chain complexes. So, suppose $σ$ is a singular $a$-simplex of $X$ and $τ$ is a singular $b$-simplex of $Y$. So $σ \otimes τ$ represents an element of $S_0(X, A \cup B; R) \otimes S_0(Y, C \cup D; R)$. Then $ε(σ \otimes τ)$ is the singular chain obtained by applying the product map $σ \times τ : Δ^a \times Δ^b \to X \times Y$ to the Eilenberg-Zilber shuffle triangulation of $Δ^a \times Δ^b$, and $de(σ \otimes τ)$ applies $(σ \times τ, σ \otimes τ)$ to the Eilenberg-Zilber shuffle triangulation of $Δ^a \times Δ^b$. Note that here we write $σ \times τ : Δ^a \times Δ^b \to X \times Y$ and $(σ \times τ, σ \otimes τ) : Δ^a \times Δ^b \to (X \times Y) \times (X \times Y)$. In other words, each point $(x, y) \in Δ^a \times Δ^b$ gets taken to $(σ(x), τ(y), σ(x), τ(y)) \in X \times Y \times X \times Y$.

On the other hand, $(d \otimes d)(σ \otimes τ) = d(σ) \otimes d(τ) = (σ, σ) \otimes (τ, τ)$. Then $ε(d \otimes d)(σ \otimes τ)$ applies $(σ, σ) \times (τ, τ)$ to the Eilenberg-Zilber shuffle singular triangulation of $Δ^a \times Δ^b$, and so each point $(x, y) \in Δ^a \times Δ^b$ gets taken to $(σ(x), σ(x), τ(y), τ(y)) \in X \times Y \times X \times Y$. Including the map $id \times t \times id$ therefore results in the top diagram commuting.

Next, suppose $σ, τ, \nu, \omega$ are singular simplices such that $σ \otimes τ \otimes \nu \otimes ω$ represents an element of $S_0(X, A; R) \otimes S_0(X, B; R) \otimes S_0(Y, C; R) \otimes S_0(Y, D; R)$. Then

$$(id \otimes τ \otimes id)(σ \otimes τ \otimes \nu \otimes ω) = (-1)^{bc}σ \otimes \nu \otimes τ \otimes ω.$$

So we see that, up to our sign $(-1)^{bc}$, the chain

$$ε(ε(ε) \otimes ε)(id \otimes τ \otimes id)(σ \otimes τ \otimes \nu \otimes ω)$$

is obtained by applying $σ \otimes τ \otimes τ \otimes ω$ to a singular triangulation of $Δ^a \times Δ^c \times Δ^b \times Δ^d$ obtained by taking the Eilenberg-Zilber shuffle triangulations of $Δ^a \times Δ^c$ and $Δ^b \times Δ^d$ and then the
Eilenberg-Zilber shuffle triangulations of the products of the simplices in the triangulations of $\Delta^a \times \Delta^c$ and $\Delta^b \times \Delta^d$.

On the other hand,

$$(id \times t \times id)\epsilon(\epsilon \otimes \epsilon),$$

up to sign, is similarly defined by applying $(id \times t \times id)(\sigma \times \tau \times \nu \times \omega)$ to a singular triangulation of $\Delta^a \times \Delta^b \times \Delta^c \times \Delta^d$ obtained by taking the Eilenberg-Zilber shuffle triangulations of $\Delta^a \times \Delta^b$ and $\Delta^c \times \Delta^d$ and then the Eilenberg-Zilber shuffle triangulations of the products of the simplices in the triangulations of $\Delta^a \times \Delta^b$ and $\Delta^c \times \Delta^d$. But we have seen in the proofs of Lemma 5.20 that, ignoring orientations, the Eilenberg-Zilber triangulation process is strictly associative and commutative, implying that if we apply $id \times t \times id$ to our triangulation of $\Delta^a \times \Delta^b \times \Delta^c \times \Delta^d$ in this paragraph, we obtain our triangulation of $\Delta^a \times \Delta^b \times \Delta^c \times \Delta^d$ from the preceding paragraph. However, since the orientation of the singular subdivision chain depends on the orientation of the underlying space, the signs of the chains in the singular triangulation will differ by a factor of $(-1)^{bc}$, accounting for the interchange of $\Delta^b$ with $\Delta^c$. Now since

$$(id \times t \times \sigma)(\sigma \times \tau \times \nu \times \omega) = (\sigma \times \nu \times \tau \times \omega)(id \times t \times \sigma),$$

where the $t$ on the left is the one in the diagram and the one on the right acts on $\Delta^b \times \Delta^c$, we see that, also accounting for the sign $(-1)^{bc}$ that comes from $\tau$, the rectangle on the right side of the diagram commutes exactly. Note: it might be helpful to compare this argument with the proof of Lemma 5.20.

We will also need the following algebraic lemma:

**Lemma 7.71.** For chain complexes of $R$-modules $C_*, D_*, E_*, F_*$, the following diagram commutes:

![Diagram](image)

**Proof.** Suppose $\alpha \in \text{Hom}(C_i, R)$, $\beta \in \text{Hom}(D_j, R)$, $\gamma \in \text{Hom}(E_k, R)$, $\delta \in \text{Hom}(F_\ell, R)$, $x \in C_i$, $y \in D_j$, $z \in E_k$, and $w \in F_\ell$. Then

$$[(id \otimes \tau \otimes id)^* \Theta(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta)](x \otimes z \otimes y \otimes w)$$

$$= (-1)^{jk}[\Theta(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta)](x \otimes y \otimes z \otimes w)$$

$$= (-1)^{jk}[\Theta(\Theta(\alpha \otimes \beta) \otimes \Theta(\gamma \otimes \delta))](x \otimes y \otimes z \otimes w)$$

$$= (-1)^{jk+(k+l)(l+j)}(\Theta(\alpha \otimes \beta)(x \otimes y))(\Theta(\gamma \otimes \delta)(z \otimes w))$$

$$= (-1)^{jk+(k+l)(l+j)+ij+k\ell}(\alpha(x) \beta(y) \gamma(z) \delta(w))$$

$$= (-1)^{jk+i\ell+j\ell+ij+k\ell}(\alpha(x) \beta(y) \gamma(z) \delta(w))$$

$$= (-1)^{ik+i\ell+jk+j\ell+ij+k\ell}(\alpha(x) \beta(y) \gamma(z) \delta(w)).$$

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On the other hand,

\[
\left[\Theta(\Theta \otimes \Theta)(\text{id} \otimes \tau \otimes \text{id})(\alpha \otimes \beta \otimes \gamma \otimes \delta)\right](x \otimes z \otimes y \otimes w) \\
= (-1)^j k [\Theta(\Theta)(\alpha \otimes \gamma \otimes \beta \otimes \delta)](x \otimes z \otimes y \otimes w) \\
= (-1)^{j k} [\Theta(\Theta)(\alpha \otimes \gamma)](x \otimes z \otimes y \otimes w) \\
= (-1)^{j k} [\Theta(\Theta)(\alpha \otimes \gamma)](x \otimes z \otimes y \otimes w) \\
= (-1)^{j k + (j + \ell)(i + k)}(\Theta(\alpha \otimes \gamma))(y \otimes w)) \\
= (-1)^{j k + (j + \ell)(i + k) + \ell}(\Theta(\alpha \otimes \gamma))(y \otimes w) \\
= (-1)^{j k + j + j + k + \ell}(\alpha(x)\gamma(z)\beta(y)\delta(w)) \\
= (-1)^{j j + k + \ell + k + j}(\alpha(x)\gamma(z)\beta(y)\delta(w)).
\]

These two expressions are equal. With any other combination of degrees (i.e. if \(\alpha\) and \(x\) do not have corresponding degrees), then each expression is 0. \(\square\)

**Lemma 7.72.** Let \(R\) be a Dedekind domain. Suppose that \(X\) is a CS set with perversities \(\bar{p}, \bar{q}, \bar{r}\) with \(D^{\bar{r}} \geq D\bar{p} + D\bar{q}\) and that \(Y\) is a CS set with perversities \(\bar{u}, \bar{v}, \bar{s}\) with \(D\bar{s} \geq D\bar{u} + D\bar{v}\). Let \(A, B \subset X\) be open subsets and \(C, D \subset Y\) be open subsets. Let \(\alpha \in I_{\bar{p}}H^i(X, A; R), \beta \in I_{\bar{q}}H^i(X, B; R), \gamma \in I_{\bar{u}}H^k(Y, C; R), \delta \in I_{\bar{v}}H^\ell(Y, D; R)\). Then

\[(\alpha \otimes \beta) \times (\gamma \otimes \delta) = (-1)^{j k}(\alpha \otimes \gamma) \cup (\beta \otimes \delta) \in I_{\bar{q}, \bar{s}}H^{i+j+k+\ell}(X \times Y, ((A \cup B) \times Y) \cup (X \times (C \cup D)); R)\]

if the following conditions are satisfied\(^{89}\)

- \(X\) is locally (\(\bar{p}, R\))-torsion free or locally (\(\bar{q}, R\))-torsion free
- \(Y\) is locally (\(\bar{u}, R\))-torsion free or locally (\(\bar{v}, R\))-torsion free
- \(X\) is locally (\(\bar{p}, R\))-torsion free or \(Y\) is locally (\(\bar{u}, R\))-torsion free
- \(X\) is locally (\(\bar{q}, R\))-torsion free or \(Y\) is locally (\(\bar{v}, R\))-torsion free.
- \(X \times Y\) is locally (\(Q_{\bar{p}, \bar{u}}, R\))-torsion free or locally (\(Q_{\bar{q}, \bar{v}}, R\))-torsion free
- \(X \times X\) is locally (\(Q_{\bar{p}, \bar{q}}, R\))-torsion free or \(Y \times Y\) is locally (\(Q_{\bar{u}, \bar{v}}, R\))-torsion free
- \(X\) is locally (\(\bar{r}, R\))-torsion free or \(Y\) is locally (\(\bar{s}, R\))-torsion free

**Proof.** The assumptions on perversities guarantees that all of the terms in the two expressions are well defined and that the assumptions of Lemma 7.70 are satisfied\(^{90}\). Furthermore, the torsion free conditions each ensure that one of the \(\epsilon\) maps in the diagrams of Lemma 7.70 is a homotopy equivalence by the K¨unneth Theorem, Theorem 6.61. So, if we reverse the

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\(^{89}\)These conditions are not all independent; see, in particular, Lemma 7.52. For example, omitting the last condition, the others together are equivalent to requiring local torsion freeness of the appropriate spaces with respect to at least three out of four of \(\bar{p}, \bar{q}, \bar{u}, \bar{v}\).

\(^{90}\)Recall that we showed in the proof of Lemma 7.70 that the other assumptions imply \(DQ_{\bar{r}, \bar{s}} \geq DQ_{\bar{p}, \bar{u}} + DQ_{\bar{q}, \bar{v}}\).
arrows of those diagrams, replacing each isomorphism with its inverse and each $\epsilon$ with an IAW, we obtain a diagram that is homotopy commutative.

Now, we compute\footnote{Notice that, abusing notation, $\text{id}^\ast = \text{id}$, $\tau^{-1} = \tau$ and $\tau^* = \tau$. These expressions make perfect sense if each symbol is interpreted with the correct domain and codomain.}

$$(\alpha \sim \beta) \times (\gamma \sim \delta) = \text{IAW}^*\Theta((\alpha \sim \beta) \otimes (\gamma \sim \delta)) = \text{IAW}^*\Theta(d^\ast\Theta(\alpha \otimes \beta) \otimes d^\ast\Theta(\gamma \otimes \delta)) = \text{IAW}^*\Theta(d^\ast \otimes d^\ast)(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta) = \text{IAW}^*(d \otimes d)^\ast\Theta(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta) = \text{IAW}^*(\text{IAW}d \otimes \text{IAW}d)^\ast\Theta(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta) = [(\text{IAW} \otimes \text{IAW})(d \otimes d)]\text{IAW}^*\Theta(\Theta \otimes \Theta)(\alpha \otimes \beta \otimes \gamma \otimes \delta)$$

by Lemma \footnote{Using Lemma \ref{lem:7.52}, these conditions are together equivalent to requiring local torsion freeness of the appropriate spaces with respect to at least three out of four of $\bar{p}, \bar{q}, \bar{v}, \bar{v}$.} \ref{lem:7.31}.

Next we look at the interaction of the cap and cross products. Because we do not need the Künneth Theorem to form the cross product of chains (as opposed to cochains), there are fewer requirements needed on the spaces in this lemma than in the preceding one.

**Lemma 7.73.** Let $R$ be a Dedekind domain. Suppose that $X$ is a CS set with perversities $\bar{p}$, $\bar{q}$, $\bar{r}$ with $D^\ell \geq D\bar{p} + D\bar{q}$ and that $Y$ is a CS set with perversities $\bar{s}$, $\bar{t}$, $\bar{v}$ with $D\bar{s} \geq D\bar{u} + D\bar{v}$. Let $A, B \subset X$ and $C, D \subset Y$ be open subsets.

Let $\alpha \in I_{\bar{q}}H^1(X, B; R)$, $x \in I^\ell H_{i+j}(X, A \cup B; R)$, $\beta \in I_{\bar{s}}H^\ell(Y, D; R)$, and $y \in I^\ell H_{k+\ell}(Y, C \cup D; R)$. Then

$$(\alpha \otimes \beta) \sim (x \times y) = (-1)^{i+j}(\alpha \sim x) \times (\beta \sim y) \in I^{Q_{\bar{p},\bar{q}}}_\ast H_{i+k}(X \times Y, (A \times Y) \cup (X \times C); R)$$

if the following conditions are satisfied\footnote{Using Lemma \ref{lem:7.52}, these conditions are together equivalent to requiring local torsion freeness of the appropriate spaces with respect to at least three out of four of $\bar{p}, \bar{q}, \bar{u}, \bar{v}$.}:

- $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free
• $Y$ is locally $(\bar{u}, R)$-torsion free or locally $(\bar{v}, R)$-torsion free
• $X$ is locally $(\bar{q}, R)$-torsion free or $Y$ is locally $(\bar{v}, R)$-torsion free
• $X \times Y$ is locally $(Q_p, \bar{r}, R)$-torsion free or locally $(Q_q, \bar{v}, R)$-torsion free.

Proof. The assumptions on perversities guarantees that all of the terms in the two expressions are well-defined. Furthermore, the torsion free conditions ensure that all of the horizontal $\epsilon$ maps in the diagrams of Lemma 7.70 are homotopy equivalences by the Künneth Theorem, Theorem 6.61. So, if we reverse these horizontal arrows, replacing each such $\epsilon$ with an IAW, we obtain a diagram that is homotopy commutative. Notice that IAW $\otimes$ IAW is indeed a chain homotopy inverse of $\epsilon \otimes \epsilon$ since the tensor products of chain homotopic maps are chain homotopic and therefore IAW$\epsilon$ $\otimes$ IAW$\epsilon$ and $\epsilon$IAW $\otimes$ $\epsilon$IAW are each homotopic to an appropriate id $\otimes$ id.

Now we begin to compute using the definitions. Let us make specific choices of IAW maps and suppose that $\bar{d}(x) = \sum_a u_a \otimes v_a \in I^0 S_*(X, A; R) \otimes I^q S_*(X, B; R)$ and that $\bar{d}(y) = \sum_b w_b \otimes z_b \in I^0 S_*(Y, C; R) \otimes I^v S_*(Y, D; R)$.

93If $f, f': C_* \to D_*$ are chain homotopic and $g, g': E_* \to F_*$ are chain homotopic, then $f \otimes \text{id}_{E_*}$ and $f' \otimes \text{id}_{E_*}$ are chain homotopic by Lemma IV.3.4, and similarly $\text{id}_{D_*} \otimes g$ is chain homotopic to $\text{id}_{D_*} \otimes g'$. Therefore, $f \otimes g = (\text{id}_{D_*} \otimes g)(f \otimes \text{id}_{E_*})$ is chain homotopic to $f' \otimes g' = (\text{id}_{D_*} \otimes g')(f' \otimes \text{id}_{E_*})$ using that the compositions of homotopic maps are homotopic.

94Suppose $f, f': B_* \to C_*$ are chain maps with $f - f' = D\partial + \partial D$ and $g, g': A_* \to B_*$ are chain maps with $g - g' = E\partial + \partial E$. Then

$$f \circ g - f' \circ g' = f \circ g - f \circ g' + f \circ g' - f' \circ g'$$
$$= f \circ (g - g') + (f - f') \circ g$$
$$= f \circ (E\partial + \partial E) + (D\partial + \partial D) \circ g$$
$$= fE\partial + f\partial E + D\partial g + \partial Dg$$
$$= fE\partial + \partial fE + Dg\partial + \partial Dg$$
$$= (fE + Dg)\partial + \partial (fE + Dg).$$

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Then we have
\[
(\alpha \smile x) \times (\beta \smile y) = [\Psi(id \otimes \alpha) \bar{d}(x)] \times [\Psi(id \otimes \beta) \bar{d}(y)]
\]
\[
= \left[ \Psi(id \otimes \alpha) \left( \sum_a u_a \otimes v_a \right) \right] \times \left[ \Psi(id \otimes \beta) \left( \sum_b w_b \otimes z_b \right) \right]
\]
\[
= \left[ \left( \sum_a (-1)^{ij} u_a \alpha(v_a) \right) \right] \times \left[ \left( \sum_b (-1)^{kl} w_b \beta(z_b) \right) \right]
\]
\[
= \sum_{a,b} (-1)^{ij+kl} \alpha(v_a) \beta(z_b) u_a \times w_b
\]
\[
= \sum_{a,b} (-1)^{ij+kl+j\ell+\ell(k+j+i+k)} \Theta(\alpha \otimes \beta)(v_a \otimes z_b) u_a \times w_b
\]
\[
= \sum_{a,b} (-1)^{ij+kl+j\ell+\ell(k+j+i+k)} \Psi(id \otimes (\alpha \times \beta))((u_a \times w_b) \otimes (v_a \times z_b))
\]
\[
= (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) \sum_{a,b} ((u_a \times w_b) \otimes (v_a \times z_b))
\]
\[
= (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) \sum_{a,b} (\epsilon(u_a \times w_b) \otimes \epsilon(v_a \times z_b))
\]
\[
= (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) \sum_{a,b} (\epsilon \otimes \epsilon)(u_a \otimes w_b \otimes v_a \otimes z_b)
\]
\[
= (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) (\epsilon \otimes \epsilon)(id \otimes \tau \otimes id)(\bar{d}(x) \otimes \bar{d}(y)).
\]

Now, we observe using the commutativity of the diagrams in Lemma \[7.70\] and the homotopy commutativity that ensues if we replace the horizontal \(\epsilon\) maps in those diagrams with IAW maps in the opposite directions, that \((\epsilon \otimes \epsilon)(id \otimes \tau \otimes id)\bar{d}(x) \otimes \bar{d}(y))\) represents the same homology class as \(\bar{d}\epsilon(x \otimes y) = \bar{d}(x \times y)\). So, using our argument from the proof of Lemma \[7.29\] where we showed that the cap product is well-defined, we can replace \((\epsilon \otimes \epsilon)(id \otimes \tau \otimes id)\bar{d}(x) \otimes \bar{d}(y))\) with the homologous \(\bar{d}(x \times y)\) without altering the homology class of the full expression. Therefore, we have
\[
(\alpha \smile x) \times (\beta \smile y) = (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) (\epsilon \otimes \epsilon)(id \otimes \tau \otimes id)(\bar{d}(x) \otimes \bar{d}(y))
\]
\[
= (-1)^{j+it+jk} \Psi(id \otimes (\alpha \times \beta)) \bar{d}(x \times y)
\]
\[
= (-1)^{j(i+j)}(\alpha \times \beta) \smile (x \times y).
\]
7.3.7 Locality

There is another desirable property of the traditional cap product in ordinary homology that we will need to approximate. If we form the singular chain cap product \( \alpha \smile \xi \) using the front face/back face formula, then every simplex of \( \alpha \smile \xi \) will be a face of a simplex of \( \xi \). This shows at the level of chains that the support of \( \alpha \smile \xi \) will be a subset of the support of \( \xi \) itself, which one can imagine is a useful property. For example, it is utilized in the proof that Poincaré duality isomorphisms are compatible with Mayer-Vietoris sequences in Hatcher [53, Lemma 3.36]. We will need a version of this fact for intersection homology below REF!! However, there is no reason to supposed in the intersection world that the support of \( \alpha \smile \xi \) will be contained in the support of \( \xi \), and we will not be able to recreate this property exactly. However, we can show that acting on an intersection chain by a cap product “doesn’t move it too far” in a sense made precise in the following lemma. The arguments in this section are based closely on those in [38, 39].

**Lemma 7.74.** Let \( R \) be a Dedekind domain. Suppose \( X \) is a CS set with perversities \( \bar{p}, \bar{q}, \bar{r} \) such that \( D\bar{r} \geq D\bar{p} + D\bar{q} \) and that \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free. Let \( A, B \) be open subsets of \( X \). Let \( U \) be an open covering of \( X \). Then the image of

\[
\tilde{d} : I^\bar{p}H_\ast(X, A \cup B; R) \to H_\ast(I^\bar{p}S_\ast(X, A; R) \otimes I^\bar{q}S_\ast(X, B; R))
\]

is contained in the image of \( \kappa : H_\ast(\sum_{U \in \mathcal{U}} I^\bar{p}S_\ast(U, U \cap A; R) \otimes I^\bar{q}S_\ast(U, U \cap B; R)) \to H_\ast(I^\bar{p}S_\ast(X, A; R) \otimes I^\bar{q}S_\ast(X, B; R)) \), where \( \kappa \) is induced by inclusions.

Notice that \( \sum_{U \in \mathcal{U}} I^\bar{p}S_\ast(U, U \cap A; R) \otimes I^\bar{q}S_\ast(U, U \cap B; R) \) makes sense as each \( I^\bar{p}S_\ast(U, U \cap A; R) \otimes I^\bar{q}S_\ast(X, B; R) \) is a submodule of \( I^\bar{p}S_\ast(X, A; R) \otimes I^\bar{q}S_\ast(X, B; R) \), using that the intersection chain modules are all projective by Lemma 6.40.

The point of the lemma is that if we apply \( \epsilon \) to an element of \( \sum_{U \in \mathcal{U}} I^\bar{p}S_\ast(U; R) \otimes I^\bar{q}S_\ast(U; R) \), then the image will live in \( \bigcup_{U \in \mathcal{U}} U \times X \), which, depending on \( \mathcal{U} \), can be considered a small neighborhood of the diagonal \( \mathbf{d}(\xi) \subset X \times X \). This doesn’t really ensure that \( \mathbf{d}(\xi) \) consists of chains “near \( \mathbf{d}(\xi) \)” but it does mean that \( \mathbf{d}(x) \) is a sum of tensor products of chains that are near each other, which will be sufficient for REF!! BELOW!!

**Proof.** Consider the following diagram, in which the unlabeled map and the maps labeled \( \lambda \) and \( \kappa \) are induced by inclusions and \( \mu \) is induced by the chain cross product \( \epsilon \).

\[
\begin{array}{c}
I^\bar{p}S_\ast(X, A \cup B; R) \xrightarrow{\mathbf{d}} I^\bar{p}S_\ast(X \times X, (A \times X) \cup (X \times B); R) \xrightarrow{\epsilon} I^\bar{p}S_\ast(X, A; R) \otimes I^\bar{q}S_\ast(X, B; R) \\
\sum_{U \in \mathcal{U}} I^\bar{p}S_\ast(U \times U, ((A \cap U) \times U) \cup (U \times (B \cap U)); R) \xrightarrow{\lambda} \sum_{U \in \mathcal{U}} I^\bar{p}S_\ast(U, A \cup U; R) \otimes I^\bar{q}S_\ast(U, B \cap U; R)
\end{array}
\]
The diagram commutes. In particular, the triangle commutes because the underlying map of spaces commutes, while the rectangle commutes using Lemma 5.16.

We know that $\epsilon$ induces a homology isomorphism by the Künneth theorem, Theorem 6.56. We will show that $\lambda$ and $\mu$ induce homology isomorphisms, as well. The lemma then follows by applying $H_*$ to the diagram and reversing the arrows of the quasi-isomorphisms. For $\lambda$, we observe that

$$(U \times U) \cap [(A \times X) \cup (X \times B)] = [(U \times U) \cap (A \times X)] \cup [(U \times U) \cap (X \times B)]$$

$$= [(U \cap A) \times (U \cap X)] \cup [(U \cap X) \times (U \cap B)]$$

$$= [(U \cap A) \times U] \cup [U \times (U \cap B)].$$

So $\lambda$ is an isomorphism on homology by Proposition 7.5.

The proof that $\mu$ is a quasi-isomorphism is presented as Lemma 7.77, below.

We now work toward proving Lemma 7.77 in order to finish the proof of Lemma 7.74.

**Lemma 7.75.** Let $X$ be a filtered space, let $B$ be an open subset, and let $G$ be an abelian group. Suppose $\{U_j\}_{j=1}^k$ is a finite collection of open subsets of $X$. Then

$$I^pS_*(U_k, B \cap U_k; G) \cap \sum_{j=1}^{k-1} I^pS_*(U_j, B \cap U_j; G) = \sum_{j=1}^{k-1} I^pS_*(U_j \cap U_k, B \cap U_j \cap U_k; G)$$

as subgroups of $I^pS_*(X, B; G)$.

Notice that the analogous lemma would be straightforward for ordinary singular chains, using a basis represented by singular simplices. However, it is not completely obvious in the intersection world where we do not have the complete freedom in how to break chains apart into pieces. For example, it is certainly possible that there might be chains $x_1 \in I^pS_*(U_1; G)$ and $x_2 \in I^pS_*(U_2; G)$ such that $x_1 + x_2$ is supported in $U_3$ but such that $x_1$ is not supported in $U_1 \cap U_3$. If $x_1$ and $x_2$ were ordinary chains, that would mean that they share some simplices that cancel in $x_1 + x_2$, and we could throw away these simplices to be left with $y_1, y_2$ such that $y_1 + y_2 = x_1 + x_2$ but with $y_j$ supported in $U_j \cap U_3$. As usual, we cannot throw away simplices so cavalierly in the setting of intersection chains, so more argument is needed. Luckily, the groundwork has already been laid within our earlier discussion of excision.

**Proof of Lemma 7.75.** First, observe that the expressions in the lemma make sense, as inclusion induces injections $I^pS_*(U_j, B \cap U_j; G) \hookrightarrow I^pS_*(X, B; G)$ for each $j$: for example, the only chains in the kernel of $I^pS_*(U_j; G) \rightarrow I^pS_*(X, B; G)$ are those that are supported in $B$ and $U_j$, and those are 0 in $I^pS_*(U_j, B \cap U_j; G)$. Therefore, we can identify the image of $I^pS_*(U_j, B \cap U_j; G)$ as a subgroup of $I^pS_*(X, B; G)$, and the sum on the left then makes sense. The same argument holds for the terms involving further intersections.

Next, notice that we only need to prove the lemma for groups, i.e. that $I^pS_*(U_k, B \cap U_k; G) \cap \sum_{j=1}^{k-1} I^pS_*(U_j, B \cap U_j; G) = \sum_{j=1}^{k-1} I^pS_*(U_j \cap U_k, B \cap U_j \cap U_k; G)$ for each $i$, as each side of the equality takes care of itself as a chain complex. It is also obvious that
The proof is essentially the same as that for Lemma 7.75. First, recall from that proof that each \( I^pS_i(U_j, A \cap U_j; G) \) \( \to \) \( I^pS_i(X, A; G) \) is an injection and similarly for the analogous terms. But we know that each of the individual groups involved in the expressions represent the same element in \( I^pS_i(U_j, A \cap U_j; G) \). In particular, abusing notation, this means that there is an intersection chain \( \xi \in I^pS_i(U_j; G) \) representing an element of \( I^pS_i(U_k, B \cap U_k; G) \) and intersection chains \( \xi_j \in I^pS_i(U_j; G) \) representing elements of \( I^pS_i(U_j, B \cap U_j; G) \) such that \( \xi \) and \( \sum_j \xi_j \) represent the same element in \( I^pS_i(X, B; G) \). We want to show that this element can also be represented by a chain in \( \sum_j I^pS_i(U_j \cap U_k; G) \).

Consider now Lemma 7.6 which says that if \( A \) is an open subset of \( X \) then the map \( i : I^pS_i(A; G) \to I^pS_i(X; G) \) induced by space inclusion is a split inclusion. In our case, let \( A = U_k \) and let \( P : I^pS_i(X; G) \to I^pS_i(U_k; G) \) be the splitting such that \( P \) = id. Of course all the (non-relative) groups we have been considering are subgroups of \( I^pS_i(X; G) \), so we can restrict \( P \) to any of these subgroups. In the proof of Lemma 7.6, the map \( P \) is constructed by taking subdivisions of chains and then throwing away some of the subdivision. As a result, for any chain \( \zeta \), the support of \( P(\zeta) \) is contained in the support of \( \zeta \). So \( P \) takes chains in \( B \) to chains in \( B \) and is therefore well-defined \( P : I^pS_i(X, B; G) \to I^pS_i(U_k, B; G) \). Now, since \( \xi \) and \( \sum_j \xi_j \) represent the same element in \( I^pS_i(X, B; G) \), so do \( P(\xi) \) and \( P(\sum_j \xi_j) = \sum_j P(\xi_j) \).

As \( \xi \in I^pS_i(U_k; G) \), we have, in fact, \( P(\xi) = \xi \). By the properties of \( P \), each \( P(\xi_j) \in I^pS_i(U_k; G) \), but as \( \xi_j \in I^pS_i(U_j; G) \), \( P(\xi_j) \) will also be supported in \( U_j \) by preservation of supports. Therefore \( P(\xi_j) \in I^pS_i(U_k; G) \cap I^pS_i(U_j; G) = I^pS_i(U_k \cap U_j; G) \). So \( P(\xi_j) = \xi_j \) and \( P(\sum_j \xi_j) = \sum_j P(\xi_j) \in \sum_{j=1}^{k-1} I^pS_i(U_j \cap U_k; G) \) represent the same element in \( I^pS_i(X, B; G) \).

Therefore, we have shown that every element \( \xi \) of \( I^pS_i(U_k, B \cap U_k; G) \cap \sum_{j=1}^{k-1} I^pS_i(U_j, B \cap U_j; G) \) can be represented by an element in \( \sum_{j=1}^{k-1} I^pS_i(U_j \cap U_k, B \cap U_j \cap U_k; G) \), and this completes the proof.

**Lemma 7.76.** Let \( X \) be a filtered space, let \( A \) and \( B \) be open subsets, and let \( R \) be a Dedekind domain. Suppose \( \{U_j\}_{j=1}^k \) is a finite collection of open subsets of \( X \). Then

\[
I^pS_i(U_k, A \cap U_k; R) \otimes I^qS_i(U_k, U_k \cap B; R) \cap \sum_{j=1}^{k-1} [I^pS_i(U_j, A \cap U_j; R) \otimes I^qS_i(U_j, U_j \cap B; R)]
\]

\[
= \sum_{j=1}^{k-1} I^pS_i(U_j \cap U_k, A \cap U_j \cap U_k; R) \otimes I^qS_i(U_j \cap U_k, U_j \cap U_k \cap B; R)
\]

as submodules of \( I^pS_i(X, A; R) \otimes I^qS_i(X, B; R) \).

**Proof.** The proof is essentially the same as that for Lemma 7.75. First, recall from that proof that each \( I^pS_i(U_j, A \cap U_j; G) \) \( \to \) \( I^pS_i(X, A; G) \) is an injection and similarly for the analogous terms. But we know that each of the individual groups involved in the expressions are projective \( R \)-modules by Lemma 6.40 and so also each \( I^pS_i(U_j, A \cap U_j; R) \otimes I^qS_i(U_j, U_j \cap B; R) \) injects into \( I^pS_i(X, A; R) \otimes I^qS_i(X, B; R) \) and so can be considered a submodule, and similarly for the \( I^pS_i(U_j \cap U_k, A \cap U_j \cap U_k; R) \otimes I^qS_i(U_j \cap U_k, U_j \cap U_k \cap B; R) \). It also follows that the inclusion \( \supseteq \) holds for the expression in the statement of the lemma.

The elements of \( I^pS_i(U_k, A \cap U_k; R) \otimes I^qS_i(U_k, U_k \cap B; R) \) can be represented by chains of the form \( \sum_\ell \xi_\ell \otimes \xi'_\ell \), where \( \xi_\ell \in I^pS_i(U_k; R) \) and \( \xi'_\ell \in I^qS_i(U_k; R) \). Similarly, elements of
$I^pS_i(U_j, A \cap U_j; R) \otimes I^qS_i(U_j, U_j \cap B; R)$ for $1 \leq j \leq k - 1$ can be represented by chains
$\sum_{\ell} \eta_{j\ell} \otimes \eta'_{j\ell}$ with $\eta_{j\ell} \in I^pS_i(U_j; R)$ and $\eta'_{j\ell} \in I^qS_i(U_j; R)$. Technically, the indexing sets for
the $\ell$ should depend on $j$, but by including some 0 terms, we can assume that the indexing sets for the $\ell$ are all the same. So, suppose we have an element from the lefthand side of the expression in the statement of the lemma. Then there are choices of $\xi_{\ell}, \eta_{j\ell}, \eta'_{j\ell}$ so that both
$\sum_{\ell} \xi_{\ell} \otimes \xi'_{\ell}$ and $\sum_{\ell} \eta_{j\ell} \otimes \eta'_{j\ell}$ both represent this element in $I^pS_i(X, A; R) \otimes I^qS_i(X, B; R)$.

Consider again the splitting map $P : I^pS_j(X; R) \to I^pS_j(U_k; R)$ of Lemma 7.75 that exists due to Lemma 7.76. Similarly, we have a splitting map $P' : I^pS_i(X; R) \to I^qS_i(U_k; R)$. As
$P, P'$ preserve supports, we have $P : I^pS_i(X, A; R) \to I^pS_i(X, A; R)$ and $P' : I^qS_i(X, B; R) \to
I^qS_i(X, B; R)$, so we can apply $P \otimes P'$ to our chains $\sum_{\ell} \xi_{\ell} \otimes \xi'_{\ell}$ and $\sum_{\ell} \eta_{j\ell} \otimes \eta'_{j\ell}$, and their images will represent the same element in $I^pS_i(X, A; R) \otimes I^qS_i(X, B; R)$. As each $\xi_{\ell}$ and $\xi'_{\ell}$ is already supported in $U_k$, $(P \otimes P')\left(\sum_{\ell} \xi_{\ell} \otimes \xi'_{\ell}\right) = \sum_{\ell} P(\xi_{\ell}) \otimes P'(\xi'_{\ell}) = \sum_{\ell} \xi_{\ell} \otimes \xi'_{\ell}$, so
$P \otimes P'$ acts by the identity on these elements. On the other hand, $(P \otimes P')\left(\sum_{\ell} \eta_{j\ell} \otimes \eta'_{j\ell}\right) = \sum_{\ell} P(\eta_{j\ell}) \otimes P'(\eta'_{j\ell})$. By the properties of $P$ and $P'$, as we saw in Lemma 7.75, each $P(\eta_{j\ell})$ or $P'(\eta'_{j\ell})$ will be contained in $I^pS_i(U_k \cap U_j; R)$ or $I^qS_i(U_k \cap U_j; R)$, so
$\sum_{\ell} P(\eta_{j\ell}) \otimes P'(\eta'_{j\ell}) \in \sum_{\ell} I^pS_i(U_j \cap U_k; R) \otimes I^qS_i(U_j \cap U_k; R)$.

Therefore, we have shown that every element of
$I^pS_i(U_k, A \cap U_k; R) \otimes I^qS_i(U_k, U_k \cap B; R) \cap \sum_{j=1}^{k-1} \left[I^pS_i(U_j, A \cap U_j; R) \otimes I^qS_i(U_j, U_j \cap B; R)\right]$

in $I^pS_i(U_k, A \cap U_k; R) \otimes I^qS_i(U_k, U_k \cap B; R)$ can be represented by an element in
$\sum_{j=1}^{k-1} I^pS_i(U_j \cap U_k, A \cap U_j \cap U_k; R) \otimes I^qS_i(U_j \cap U_k, U_j \cap U_k \cap B; R)$,

and this completes the proof. \qed

Lemma 7.77. Let $R$ be a Dedekind domain. Suppose $X$ is a CS set with perversities $\bar{p}, \bar{q}, \bar{r}$ such that $D\bar{r} \geq D\bar{p} + D\bar{q}$ and that $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free. Let $A, B$ be open subsets of $X$. Let $\mathcal{V}$ be a collection of open subsets of $X$.

Then the map

$\mu : H_*\left(\sum_{\mathcal{V}} I^pS_*(V, A \cap V; R) \otimes I^qS_*(V, B \cap V; R)\right) \to H_*\left(\sum_{\mathcal{V}} I^{p+q}S_*(V \times X; ((A \cap V) \times V) \cup (V \times (B \cap V)); R)\right)$

induced by the chain cross product $\epsilon$ is an isomorphism.

Proof. We will proceed by induction on the size of $\mathcal{V}$. If $|\mathcal{V}| = 1$, the result follows by the Künneth Theorem, Theorem 6.56. So now suppose that $|\mathcal{V}| = k > 1$, i.e. that $\mathcal{V} = \{V_i\}_{i=1}^k$, $k > 1$. Let $\mathcal{V}' = \{V_i\}_{i=1}^{k-1}$, and let $\mathcal{V}'' = \{V_i \cap V_k\}_{i=1}^{k-1}$.
Now we consider the following diagram of short exact sequences with coefficients tacit. Here we use the notation \((Y, Z) \times (C, D)\) to represent \((Y \times C, (Y \times D) \cup (Z \times C))\).

\[
\begin{array}{ccccccccc}
0 & \to & C_* \cap D_* & \to & C_* \oplus D_* & \to & C_* + D_* & \to & 0 \\
\sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r, A \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i ((V_i \cap V_r \cap V_r) \times (V_i \cap V_r \cap V_r)) \\
\sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i ((V_i \cap V_r \cap V_r) \times (V_i \cap V_r \cap V_r)) \\
\sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i (V_i \cap V_r \cap V_r) & \hspace{1cm} & \sum_{i=1}^{k-1} I^q S_i ((V_i \cap V_r \cap V_r) \times (V_i \cap V_r \cap V_r)) \\
0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

Although this looks horrendous, we claim that each vertical sequence has the standard Mayer-Vietoris form

\[
0 \longrightarrow C_* \cap D_* \longrightarrow C_* \oplus D_* \longrightarrow C_* + D_* \longrightarrow 0
\]

and so is exact. In fact, it is clear that the transitions from the middle terms to the bottom terms have the form \(B \oplus C \rightarrow B + C\), and and it follows from Lemma 7.76 that, in each vertical sequence, the top nontrivial term is the intersection of the two summands of the middle term, as desired.

The map of short exact sequences now induces a map of long exact homology sequences. By the induction hypothesis, and by the fact that the homology map induced on a direct sum is a direct sum of homology maps, we have homology isomorphisms on every two out of three terms, so the five lemma completes the proof for \(|V| = k\).

By induction, the lemma is now proven for any finite collection \(V\).

Suppose now that \(V\) is not necessarily finite, and suppose

\[
\xi \in H_i \left( \sum_{V} I^{p,q} S_*(V \times V_r, ((A \cap V_r) \times V_r) \cup (V \times (B \cap V_r)); R) \right).
\]

By definition, every such element can be represented as a finite sum \(\xi = \sum_{j=1}^{k} \xi_j\), with

\[
\xi_j \in I^{p,q} S_*(V_j \times V_j, ((A \cap V_j) \times V_j) \cup (V_j \times (B \cap V_j)); R)
\]

for \(\{V_j\}_{j=1}^{k} \subset V\). Now consider the diagram

\[
H_i \left( \sum_{j=1}^{k} I^p S_*(V_j, A \cap V_j; R) \oplus I^p S_*(V_j, B \cap V_j; R) \right) \overset{\mu}{\longrightarrow} H_i \left( \sum_{j=1}^{k} I^{p+q} S_*(V_j \times V_j, ((A \cap V_j) \times V_j) \cup (V_j \times (B \cap V_j)); R) \right)
\]

\[
H_i \left( \sum_{V} I^p S_*(V, A \cap V; R) \oplus I^p S_*(V, B \cap V; R) \right) \overset{\mu}{\longrightarrow} H_i \left( \sum_{V} I^{p+q} S_*(V \times V_r, ((A \cap V_r) \times V_r) \cup (V \times (B \cap V_r)); R) \right).
\]

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We have just seen that $\xi$ is in the image of the righthand vertical map, and the $\mu$ in the top line is an isomorphism by the arguments above. Therefore, $\xi$ is also in the image of the bottom $\mu$, and the bottom $\mu$ is surjective.

Next, assume that $\zeta \in H_i \left( \bigoplus V I^p S_*(V, A \cap V; R) \otimes I^q S_*(V, B \cap V; R) \right)$ with $\mu(\zeta) = 0$. Once again, by definition, we can represent $\zeta$ by a chain contained in a finite sum of terms, say over the sets $\{W_j\}_{j=1}^k \subset V$. As $\mu(\zeta) = 0$, this means that there must be an element $Z \in \bigoplus V I^{p,q} S_*(V \times V; (A \cap V) \times V) \cup (V \times (B \cap V); R)$ whose boundary is $\mu(\zeta)$, and again there must be some $\{U_m\}_{m=1}^k \subset V$ such that

$$
\eta \in \bigoplus_{m=1}^k I^{p,q} S_*(U_m \times U_m; (A \cap U_m) \times U_m) \cup (U_m \times (B \cap U_m); R).
$$

The collection $\{W_j\}_{j=1}^k \cup \{U_m\}_{m=1}^k \subset V$ is finite, so let us relabel this as $\{V_j\}_{j=1}^k$ and again consider a diagram of the form just above. We start with $\zeta$ in the chain module in the upper left corner, representing a homology class. By our assumptions, the composition down then right yields a trivial homology class and $Z$ with $\partial Z = \mu(\zeta)$ is contained in the chain complex in the upper right. Therefore, $\mu(\zeta)$ represents the 0 homology class already in the upper right. But as the upper $\mu$ is an isomorphism on homology, $\zeta$ must already represent the trivial homology class in the upper left, and therefore in the bottom left. This shows that the bottom $\mu$ is injective. Altogether now, the bottom $\mu$ is an isomorphism. 

7.3.8 The cohomology Künneth theorem

As one final property of intersection (co)homology products, we can prove a cohomology Künneth theorem.

**Theorem 7.78.** Let $R$ be a Dedekind domain. Suppose that $X$ is a CS set with perversity $p$ and $Y$ is a CS set with perversity $q$. Suppose that $A \subset X$ and $B \subset Y$ are open subsets, and that $X$ is locally $(p, R)$-torsion free or $Y$ is locally $(q, R)$-torsion free. Furthermore, suppose that either $I^p H_i(X, A; R)$ is finitely generated for each $i$ or $I^q H_j(Y, B; R)$ is finitely generated for each $j$. Then there is a natural exact sequence

$$
0 \to \bigoplus_{i+j=k} I^p H^i(X, A; R) \otimes I^q H^j(Y, B; R) \to I^{p,q} H^k(X \times Y, (A \times Y) \cup (X \times B); R) \to \bigoplus_{i+j=k+1} I^p H^i(X, A; R) \ast I^q H^j(Y, B; R) \to 0
$$

that splits (non-naturally).

In particular, if $X$ is a manifold $X = M$ (trivially stratified) and the finite generation hypotheses are satisfied, we have an exact sequence

$$
0 \to \bigoplus_{i+j=k} H^i(M, A; R) \otimes I^q H^j(Y, B; R) \to I^{p,q} H^k(M \times Y, (A \times Y) \cup (M \times B); R) \to \bigoplus_{i+j=k+1} H^i(M, A; R) \ast I^q H^j(Y, B; R) \to 0.
$$
Before the proof, we provide a standard important example:

**Example 7.79.** By taking \((X, A) = (\mathbb{R}^k, \mathbb{R}^k - \{0\})\) and recalling \(H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R) \cong R\) and \(H^i(\mathbb{R}^k, \mathbb{k} - \{0\}; R) = 0\) for \(i \neq k\), we see that the cohomology cross product produces an isomorphism

\[
I_q H^j(Y, B; R) \cong R \otimes I_q H^j(Y, B; R) \\
\cong H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R) \otimes I_q H^j(Y, B; R) \\
\xrightarrow{\sim} I_q H^{j+k}(\mathbb{R}^k \times Y, (\mathbb{R}^k - \{0\}) \times Y) \cup (\mathbb{R}^k \times B); R).
\]

**Proof of Theorem 7.78.** The last statement, for which \(X\) is a manifold, comes by the observation that, for any perversity \(\bar{p}\), intersection cohomology is just ordinary cohomology on a trivially stratified space, while, by definition, the product perversity value \(Q_{\bar{p}, \tilde{q}}(M \times T)\) is simply \(\bar{q}(T)\) for any singular stratum \(T \subset Y\). By abuse of notation, it is our habit to denote this perversity on \(M \times Y\) also by \(\tilde{q}\).

For the exact sequence, we follow the standard procedure that can be found, for example, in [77 Section 60] or [23 Proposition VI.12.16]. These sources, however, only consider \(R\) a PID, so we will indicate the proof and the necessary generalizations over a Dedekind domain. The basic ideas is to treat \(I_p S^*(X, A; R)\) and \(I_q S^j(Y, B; R)\) as if they were homologically indexed and then apply the standard algebraic Künneth theorem to obtain the short exact top row in a diagram of the form

\[
\bigoplus_{i+j=k} I_p H^i(X, A; R) \otimes I_q H^j(Y, B; R) \xrightarrow{\sim} H^k(I_p S^*(X, A; R) \otimes I_q S^j(Y, B; R)) \\
\xrightarrow{\text{IAW}^*} I_{Q_{\tilde{p}, \tilde{q}}} H^k(X \times Y, (A \times Y) \cup (X \times B); R)
\]

(28)

The first horizontal map composed with the vertical maps is the intersection cohomology cross product by definition, and replacing the middle term of the short exact sequence with its claimed isomorphic image \(I_{Q_{\tilde{p}, \tilde{q}}} H^k(X \times Y, (A \times Y) \cup (X \times B); R)\) will provide the claimed sequence of the theorem.

We already know that IAW is a chain homotopy equivalence by definition, and so IAW* is an isomorphism. What remains is to verify that \(I_p S^*(X, A; R)\) and \(I_q S^j(Y, B; R)\) satisfy the necessary conditions for the algebraic Künneth theorem to hold and that the top vertical map is an isomorphism.

Suppose we have chain homotopy equivalences \(f : C_* \rightarrow I^p S_*(X, A; R)\) and \(g : D_* \rightarrow I^q S_*(Y, B; R)\) such that \(C_*\) and \(D_*\) are bounded below complexes of projectives and such that each \(C_i\) is finitely generated if each \(I^p H_i(X, A; R)\) is finitely generated and each \(D_i\) is
finely generator if each $I^gH_i(Y, B; R)$ is finitely generated. For simplicity, we will assume that it is $C_*$ that satisfies the finiteness condition. Lemma 12.9 in Appendix 12 shows that such chain homotopy equivalences can be constructed. Then, we have a diagram

$$
\begin{array}{ccc}
\bigoplus_{i+j=k} H^i(\text{Hom}(C_*, R)) & \xrightarrow{\theta} & H^i(\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R)) \\
\oplus_{i+j=k} I_pH^i(X, A; R) & \xrightarrow{\theta} & H^i(I_pS^*(X, A; R) \otimes I_qS^*(Y, B; R); R) \\
\end{array}
$$

The vertical maps here are all isomorphisms as $f$ and $g$, and so also their duals and tensor products, are all homotopy equivalences. More specifically, in the middle we are using that $f^* \otimes g^*$ as a chain map is a chain homotopy equivalence and so induces an isomorphism in (co)homology; on the left, $f^*$ and $g^*$ are (co)homology maps that are isomorphisms and so their tensor product is an isomorphism by the functoriality of $\otimes$, and similarly on the right by the functoriality of the torsion product. The top line here is split short exact by the standard algebraic Künneth theorem\footnote{Curiously, in \cite[Theorem 3.6.3]{Weibel}, Weibel only discusses the (non-natural) splitting of the Künneth exact sequence when the base ring is $\mathbb{Z}$. Hilton and Stammbach \cite[Theorem V.2.1]{HiltonStammbach} give a complete proof of the algebraic Künneth theorem, including the splitting, when $R$ is a PID. The proof given in \cite{HiltonStammbach} works over a Dedekind domain replacing “free” with “projective” throughout the argument.}, up to the switch between homological and cohomological indexing. To cite this theorem, we need to know that each $\text{Hom}(C_i, R)$ and $d(\text{Hom}(C_i, R))$ is flat. We will show that $\text{Hom}(C_i, R)$ is projective, which will suffice as a submodule of a projective module over a Dedekind domain is projective: As $C_i$ is finitely generated, there is a finitely generated free $R$-module $F_i$ with $F_i \to C_i$ surjective and with kernel $K_i \subset F_i$. As $C_i$ is projective, $F_i \cong K_i \oplus C_i$, and as $F_i$ is finitely generated free, $F_i = R^{m_i}$ for some $m_i$. So $\text{Hom}(F_i, R) \cong \text{Hom}(R^{m_i}, R) \cong (\text{Hom}(R, R))^{m_i}$ is free, and, furthermore, $\text{Hom}(F_i, R) \cong \text{Hom}(K_i \oplus C_i, R) \cong \text{Hom}(K_i, R) \oplus \text{Hom}(C_i, R)$. Hence $\text{Hom}(C_i, R)$ is a direct summand of a free module and is therefore projective.

Next, we want to show that we have a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{i+j=k} H^i(\text{Hom}(C_*, R)) \otimes H^j(\text{Hom}(D_*, R)) & \xrightarrow{\theta} & H^i(\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R)) \\
\oplus_{i+j=k} I_pH^i(X, A; R) \otimes I_qH^j(Y, B; R) & \xrightarrow{\theta} & H^i(I_pS^*(X, A; R) \otimes I_qS^*(Y, B; R); R) \\
\end{array}
$$

Notice that the first part of this diagram is compatible with the first part of Diagram (29), while the bottom line consists of the lefthand horizontal map and the first vertical map in Diagram (28). So, if we show that this diagram commutes and that the maps labeled as such are isomorphisms, it will follow that we have achieved a diagram of the form of (28),
in which we know that the row is exact, that the top vertical map is an isomorphism, and that the composition right then one step down is $\Theta$, so that the full map right then all the way down is the cohomology cross product. This will prove the theorem.

The commutativity of the square on the right follows from Lemma 7.31. The commutativity on the left is trivial as, following Remark 5.15, if $\alpha \in I_\mu S^i(X, A; R)$ and $\beta \in I_\nu S^j(Y, B; R)$ are cocycles representing cohomology classes, the map labeled $\theta$ simply takes $\alpha \otimes \beta$, as a tensor product of cohomology classes, to the cohomology class represented by $\alpha \otimes \beta$. The commutativity then comes by seeing what happens to such cochain representatives. We have already observed that the two leftmost vertical maps are isomorphisms. The rightmost vertical map is similar an isomorphism as the tensor products and duals of chain homotopy equivalences are chain homotopy equivalences.

Lastly, we need to see that $H^k(\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R)) \rightarrow (\text{Hom}(C_* \otimes D_*, R)$ is an isomorphism, which will imply that the bottom right horizontal map of Diagram (30) is an isomorphism. In fact, $\text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) \rightarrow \text{Hom}(C_* \otimes D_*, R)$ is an isomorphism with the given hypotheses. This is shown just below as Lemma 7.80.

**Lemma 7.80.** Let $R$ be a Dedekind domain. Suppose $C_*$ and $D_*$ are bounded-below complexes of projective $R$-modules and that one of $C_*$ or $D_*$ consists entirely of finitely-generated modules. Then $\Theta : \text{Hom}(C_*, R) \otimes \text{Hom}(D_*, R) \rightarrow \text{Hom}(C_* \otimes D_*, R)$ is an isomorphism.

**Proof.** For any fixed total degree $k$, we have $\text{Hom}^k(C_* \otimes D_*, R) = \text{Hom}(\oplus_{i+j=k} C_i \otimes D_j, R) \cong \oplus_{i+j=k} \text{Hom}(C_i \otimes D_j, R)$, as there are a finite number of non-zero summands by the bounded-below conditions. For any fixed $i$ and $j$, if $f \in \text{Hom}(C_i, R)$ and $g \in \text{Hom}(D_j, R)$, then $\Theta(f \otimes g)$ acts trivially on element of $C_{i'} \otimes D_{j'}$ unless $i = i'$ and $j = j'$. As every generator of $C_i \otimes D_j$ has such a form, we see that $\Theta$ preserves the summation structure, so it suffices to prove that, if $A$ and $B$ are fixed projective $R$-modules one of which is finitely generated, then $\Theta : \text{Hom}(A, R) \otimes \text{Hom}(B, R) \rightarrow \text{Hom}(A \otimes B, R)$ is an isomorphism.

The proof is standard when $A$ is free and finitely generated (see, e.g., [97, Lemma 5.5.6]): In this case, $A \cong R^m$ for some $m$, so $\text{Hom}(R^m, R) \otimes \text{Hom}(B, R) \cong R^m \otimes \text{Hom}(B, R) \cong \bigoplus_{i=1}^m \text{Hom}(B, R)$ and, similarly, $\text{Hom}(R^m \otimes B, R) \cong \text{Hom}(\bigoplus_{i=1}^m B, R) \cong \bigoplus_{i=1}^m \text{Hom}(B, R)$. So the modules are abstractly isomorphic. Furthermore, $\Theta$ again preserves the splittings: If $\{a_i\}_{i=1}^m$ is a basis for $R^m$ and $\{a_i^*\}_{i=1}^m$ is the dual basis (i.e. $a_i^*(a_j) = \delta_{i,j}$), then we see that $\Theta(a_i^* \otimes \beta)$ (for any $\beta \in \text{Hom}(B, R)$) acts trivially on the summands $\langle a_j \rangle \otimes B$ for $i \neq j$. So, $\Theta$ splits into maps $\text{Hom}(B, R) \cong \text{Hom}(R, R) \otimes \text{Hom}(B, R) \rightarrow \text{Hom}(R \otimes B, R) \cong \text{Hom}(B, R)$, which are easily isomorphisms.

Suppose now $A$ is finitely-generated projective and that $A \oplus A'$ is a free modules. We can find such a finitely generated $A'$. To see this, we observe that, as $A$ is finitely generated, there is a finitely generated free $R$-module $F$ with $F \rightarrow A$ surjective and with kernel $A' \subset F$. As Dedekind domains are Noetherian [10, Theorem VII.2.2.1], $F$ and $A'$ are thus Noetherian (see [64 Section X.1]), and so $A'$ is finitely generated. The proof for free modules says that $\Theta$ induces an isomorphism from

$$\text{Hom}(A \oplus A', R) \otimes \text{Hom}(B, R) \cong (\text{Hom}(A, R) \otimes \text{Hom}(B, R)) \oplus (\text{Hom}(A', R) \otimes \text{Hom}(B, R))$$
Given the large number of properties of cup, cross, and cap products we have now seen and given how spread out through the proofs the statements have been, we here present for the benefit of the reader a quick summary of these properties and the required conditions necessary for the properties to hold. We include also the properties of the homology cross product from Section 5.2.1, which hold for non-GM intersection homology by Theorem 6.24. Throughout, $X$, $Y$, etc. are CS sets (unless noted otherwise), $A$, $B$, etc. are open subspaces (unless noted otherwise), and $R$ is a Dedekind domain. The perversity $\mathfrak{t}_X$ is the top perversity on the space $X$.

1. Associativity

- **Conditions:** All spaces only filtered; $P(S \times S') \geq \bar{p}(S) + \bar{q}(S')$ on $X \times Y$, $Q(S' \times S'') \geq \bar{q}(S') + \bar{r}(S'')$ on $Y \times Z$, $T(S \times S' \times S'') \geq P(S \times S') + \bar{r}(S'')$ and $T(S \times S' \times S'') \geq \bar{p}(S) + Q(S' \times S'')$ on $X \times Y \times Z$, $x \in I^qS_+(X, A; R)$, $y \in I^qS_+(Y, B; R)$, $z \in I^qS_+(Z, C; R)$

  Property:

  $$(x \times y) \times z = x \times (y \times z) \in I^T S_+(X \times Y \times Z, (A \times Y \times Z) \cup (X \times B \times Z) \cup (X \times Y \times C); R)$$

  **Location:** Lemma 6.19

- **Conditions:** two out of three of $X$, $Y$, and $Z$ are locally torsion free, $\alpha \in I_\mathfrak{p}H^i(X, A; R)$, $\beta \in I_\mathfrak{q}H^j(Y, B; R)$, and $\gamma \in I_\mathfrak{r}H^k(Z, C; R)$

  Property:

  $$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$$

  in $I_{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}}H^{i+j+k}(X \times Y \times X, (A \times Y \times X) \cup (X \times B \times Z) \cup (X \times Y \times C); R)$

  **Location:** Lemma 7.58

- **Conditions:** $D \bar{s} \geq D \bar{p} + D \bar{q} + D \bar{r}$ on $X$, $X$ is locally torsion free with respect to two out of three of $\bar{p}, \bar{q}, \bar{r}$, $\alpha \in I_\mathfrak{p}H^i(X, A; R)$, $\beta \in I_\mathfrak{q}H^j(X, B; R)$, and $\gamma \in I_\mathfrak{r}H^k(X, C; R)$

  $$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$$

  in $I_{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}}H^{i+j+k}(X \times Y \times X, (A \times Y \times X) \cup (X \times B \times Z) \cup (X \times Y \times C); R)$
Property:
\[(\alpha \sim \beta) \sim \gamma = \alpha \sim (\beta \sim \gamma) \in I_s H^{i+j+k}(X, A \cup B \cup C; R)\]

Location: Lemma 7.57

2. Commutativity

- **Conditions:** All spaces only filtered; \(P(S \times S') \geq \bar{p}(S) + \bar{q}(S')\) on \(X \times Y\), \(Q(S' \times S) = P(S \times S')\) on \(Y \times X\), \(t : X \times Y \to Y \times X\) such that \(t(x, y) = (y, x)\), \(x \in I^p S_\ast(X, A; R), y \in I^q S_\ast(Y, B; R)\)

Property: \(t(x \times y) = (-1)^{|x||y|} y \times x \in I^q S_\ast(Y, B; R) \otimes I^p S_\ast(X, A; R)\)

Location: Lemma 5.20

- **Conditions:** \(X\) is locally \((\bar{p}, R)\)-torsion free or \(Y\) is locally \((\bar{q}, R)\)-torsion free, \(t : X \times Y \to Y \times X\) such that \(t(x, y) = (y, x)\), \(\alpha \in I^p H^i(X, A; R)\) and \(\beta \in I^q H^j(Y, B; R)\)

Property:
\[t^* (\alpha \times \beta) = (-1)^{ij} \beta \times \alpha \in I^q S_\ast(Y, B; R) \otimes I^p S_\ast(X, A; R)\]

Location: Corollary 7.39

- **Conditions:** \(X\) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free, \(D\bar{r} \geq D\bar{p} + D\bar{q}\), \(\alpha \in I^p H^i(X, A; R)\) and \(\beta \in I^q H^j(X, B; R)\)

Property: \(\alpha \sim \beta = (-1)^{ij} \beta \sim \alpha \in I^p H^{i+j}(X, A \cup B; R)\)

Location: Corollary 7.40

3. Unital properties

- **Conditions:** \(X\) filtered, \(\sigma_0 : \Delta^0 \to pt\) the unique singular 0 simplex in \(S_0(pt; R)\) (with coefficient 1 \(\in R\)), \(\xi \in I^p S_i(X, A; R)\)

Property:
\[\sigma_0 \times \xi = \xi \times \sigma_0 = \xi \in I^p S_i(pt \times X, pt \times A; R) = I^p S_i(X \times pt, A \times pt; R) = I^p S_i(X, A; R).\]

Location: Lemma 5.21

- **Conditions:** \(\alpha \in I^p H^i(X, A; R), 1 \in I^p H^0(X; R)\), (Note: no torsion free condition is necessary because all CS sets are locally \((\bar{t}, R)\)-torsion free)

Property:
\[1 \sim \alpha = \alpha \sim 1 = \alpha\]

Location: Corollary 7.46

- **Conditions:** \(\alpha \in I^p H^i(X, A; R), 1 \in H^0(pt; R)\)
Property:

\[ 1 \times \alpha = \alpha \times 1 = \alpha \in I_\beta H^i(pt \times X, pt \times A; R) = I_\beta H^i(X \times pt, A \times pt; R) = I_\beta H^i(X, A; R) \]

Location: Corollary \[7.47\]

- **Conditions:** \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, \( \bar{q} \leq \bar{t}_Y \), \( \alpha \in I_\beta H^i(X, A; R) \), \( 1 \in I_{\bar{q}} H^0(Y; R) \), \( p_1 : X \times Y \to X \) the projection

Property:

\[ \alpha \times 1_Y = p_1^* \alpha \in I_{\bar{q}, \bar{q}_Y} H^*(X \times Y, A \times Y; R) \]

Location: Corollary \[7.48\]

4. Evaluations

- **Conditions:** \( \alpha \in I_\beta H^i(X, A; R) \), \( \xi \in I^\beta H_i(X, A; R) \), \( a : I^\beta H_0(X; R) \to R \) the augmentation map, (Note: no torsion free condition is necessary, see Lemma \[7.49\])

Property:

\[ a(\alpha \smallint \xi) = \alpha(\xi) \in R \]

Location: Lemma \[7.49\]

- **Conditions:** \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, \( \alpha \in I_\beta H^a(X, A; R) \), \( \beta \in I_{\bar{q}} H^b(Y, B; R) \), \( \xi \in I^\beta H_i(X, A; R) \) and \( \eta \in I^{\bar{q}} H_j(Y, B; R) \)

Property:

\[ (\alpha \times \beta)(\xi \times \eta) = (-1)^{bi} \alpha(\xi)\beta(\eta) \]

Location: Lemma \[7.51\]

5. Stability

- **Conditions:** All spaces filtered, \( Q(S \times S') \geq \bar{p}(S) + \bar{q}(S') \) on \( X \times Y \), \( \xi \in I^\beta H_i(X, A; R) \), \( \eta \in I^{\bar{q}} H_j(Y, B; R) \)

Property:

\[ (\partial_\ast \xi) \times \eta = \partial_\ast(\xi \times \eta) \in I^Q H_{i+j-1}((A \times Y) \cup (X \times B), X \times B; R) \]

Location: Lemma \[5.23\]

- **Conditions:** All spaces filtered, \( Q(S \times S') \geq \bar{p}(S) + \bar{q}(S') \) on \( X \times Y \), \( \xi \in I^\beta H_i(X, A; R) \), \( \eta \in I^{\bar{q}} H_j(Y, B; R) \)

Property:

\[ \partial_\ast(\xi) \times \eta + (-1)^{i+1} \xi \times \partial_\ast(\eta) = \partial_\ast(\xi \times \eta) \in I^Q H_{i+j-1}((A \times Y) \cup (X \times B), A \times B; R) \]

Location: Lemma \[5.24\]
• **Conditions:** $D\bar{r} \geq D\bar{p} + D\bar{q}$ on $X$, $i : B \hookrightarrow X$ the inclusion map, $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free, $\alpha \in I_qH^1(B; R)$ and $\xi \in I^pH_{i+j}(X, A \cup B; R)$, $\partial_*(\xi) \in I^pH_{i+j-1}(A \cup B, A; R)$, and $e : I^pH_{i+j-1}(B, A \cap B; R) \to I^pH_{i+j-1}(A \cup B, A; R)$ is the excision isomorphism

**Property:**

$$(d^*(\alpha)) \otimes \epsilon = (-1)^{j+1}i(\alpha \otimes e^{-1}\partial_*(\xi)) \in I^pH_{i-1}(X, A; R),$$

**Location:** Lemma 7.60

• **Conditions:** $D\bar{r} \geq D\bar{p} + D\bar{q}$ on $X$, $i : A \hookrightarrow X$ the inclusion map, $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free, $\alpha \in I_qH^1(X, B; R)$ and $\xi \in I^pH_{i+j}(X, A \cup B; R)$, $\partial_*(\xi) \in I^pH_{i+j-1}(A \cup B, B; R)$ and $e : I^pH_{i+j-1}(A, A \cap B; R) \to I^pH_{i+j-1}(A \cup B, B; R)$ is the excision isomorphism

**Property:**

$$\partial_*(\alpha \otimes \xi) = (-1)^j(i^*(\alpha)) \otimes (e^{-1}\partial_*(\xi)) \in I^pH_{i-1}(A; R)$$

**Location:** Lemma 7.61

• **Conditions:** $X$ is locally $(\bar{p}, R)$-torsion free or $Y$ is locally $(\bar{q}, R)$-torsion free, $\alpha \in I_\bar{p}H^i(A; R)$ and $\beta \in I_qH^j(Y; R)$, $e : I^\bar{p}H_*^i(A \times Y; A \times B; R) \to I^\bar{p}H_*^j((A \times Y) \backslash (X \times B), X \times B; R)$ is an excision isomorphism, $d^* : I_{\bar{p}, \bar{q}}H^{i+j}(A \times Y) \cup (X \times B), X \times B; R) \to I^\bar{p}H^{i+j+1}(X \times Y, (A \times Y) \cup (X \times B); R)$

**Property:**

$$(d^*\alpha) \times \beta = d^*(e^{-1})^*(\alpha \times \beta) \in I_{\bar{p}, \bar{q}}H^{i+j+1}(X \times Y, (A \times Y) \cup (X \times B); R)$$

**Location:** Lemma 7.66

• **Conditions:** $i : A \to X$ is the inclusion map, $D\bar{r} \geq D\bar{p} + D\bar{q}$ on $X$, $X$ is locally $(\bar{p}, R)$-torsion free or locally $(\bar{q}, R)$-torsion free, $\alpha \in I^pH^1(A; R)$ and $\beta \in I^pH^2(X, B; R)$, $e : I_pH_*^i(A, A \cap B; R) \to I_pH_*^j(A \cup B, B; R)$ is an excision isomorphism, $d^* : I^pH^{i+j}(A \cup B, B; R) \to I^pH^{i+j+1}(X, A \cup B; R)$

**Property:**

$$(d^*\alpha) \otimes \beta = d^*(e^{-1})^*(\alpha \otimes i^*(\beta)) \in I^pH^{i+j+1}(X, A \cup B; R)$$

**Location:** Lemma 7.67

6. Combinations - properties that involve multiple types of products

• **Conditions:** $D\bar{s} \geq D\bar{p} + D\bar{q} + D\bar{r}$ on $X$, $X$ is locally torsion free with respect to two out of the three perversities $\bar{p}$, $\bar{q}$, $\bar{r}$, $\alpha \in I^pH^1(X, B; R)$, $\beta \in I^pH^k(X, C; R)$, and $\xi \in I^pH_{i+j+k}(X, A \cup B \cup C; R)$

**Property:**

$$(\alpha \otimes \beta) \otimes \xi = \alpha \otimes (\beta \otimes \xi) \in I^pH_i(X, A; R)$$

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Location: Lemma [7.59]

- **Conditions:** \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \), \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free, \( \alpha \in I_\bar{p}H^i(X, A; R) \) and \( \beta \in I_\bar{q}H^j(X, B; R) \)

**Property:**
\[
d^*(\alpha \times \beta) = \alpha \rhd \beta \in I_\bar{p}H^{i+j}(X, A \cup B; R)
\]

Location: Lemma [7.68]

- **Conditions:** \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, 
  \( p_1 : X \times Y \rightarrow X \) and \( p_2 : X \times Y \rightarrow Y \) the projection maps, and \( p_1^* : I_\bar{p}H^*(X, A; R) \rightarrow I_{\bar{q}_{i',\bar{r}}^\bar{p}}H^*(X \times Y, A \times Y; R) \) and \( p_2^* : I_\bar{q}H^*(Y, B; R) \rightarrow I_{\bar{p}_{i',\bar{r}}^\bar{q}}H^*(X \times Y, X \times B; R) \), \( \alpha \in I_\bar{p}H^i(X, A; R) \) and \( \beta \in I_\bar{q}H^j(X, B; R) \)

**Property:**
\[
(p_1^*(\alpha)) \rhd (p_2^*(\beta)) = \alpha \times \beta \in I_{\bar{q}_{i',\bar{r}}^\bar{p}}H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R)
\]

Location: Lemma [7.69]

- **Conditions:** \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \), \( D\bar{s} \geq D\bar{u} + D\bar{v} \) on \( Y \), \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free, \( Y \) is locally \((\bar{u}, R)\)-torsion free or locally \((\bar{v}, R)\)-torsion free, \( X \) is locally \((\bar{s}, R)\)-torsion free, \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, \( X \times Y \) is locally \((Q_\beta, R)\)-torsion free or locally \((Q_\alpha, R)\)-torsion free, \( X \times X \) is locally \((Q_{\bar{p},\bar{q}}, R)\)-torsion free or \( Y \times Y \) is locally \((Q_{\bar{q},\bar{p}}, R)\)-torsion free, \( \alpha \in I_\bar{p}H^i(X, A; R) \), \( \beta \in I_\bar{q}H^j(X, B; R) \), \( \gamma \in I_\bar{u}H^k(Y, C; R) \), and \( \delta \in I_\bar{v}H^\ell(Y, D; R) \),

**Property:**
\[
(\alpha \rhd \beta) \times (\gamma \rhd \delta) = (-1)^{jk}(\alpha \times \gamma) \rhd (\beta \times \delta)
\]

in \( I_{\bar{q}_{i',\bar{r}}^\bar{p}}H^{i+j+k+\ell}(X \times Y, ((A \cup B) \times Y) \cup (X \times (C \cup D)); R) \)

Location: Lemma [7.72]

- **Conditions:** \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \), \( D\bar{s} \geq D\bar{u} + D\bar{v} \) on \( Y \), \( D_{\bar{q},\bar{s}} \geq D_{\bar{q}_{i',\bar{r}}^\bar{p}} + D_{\bar{q}_{i',\bar{r}}^\bar{q}} \), 
  \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free, \( Y \) is locally \((\bar{u}, R)\)-torsion free or locally \((\bar{v}, R)\)-torsion free, \( X \) is locally \((\bar{s}, R)\)-torsion free, \( X \times Y \) is locally \((Q_{\bar{p},\bar{u}}, R)\)-torsion free or \( Y \times Y \) is locally \((Q_{\bar{q},\bar{v}}, R)\)-torsion free or locally \((Q_{\bar{q}_{i',\bar{r}}^\bar{p},\bar{v}}, R)\)-torsion free, \( \alpha \in I_\bar{p}H^i(X, B; R) \), \( x \in I^\ell H_{i+j}(X, A \cup B; R) \), 
  \( \beta \in I_\bar{q}H^j(X, D; R) \), and \( y \in I^\delta H_{k+\ell}(X, C \cup D; R) \)

**Property:**
\[
(\alpha \times \beta) \rhd (x \times y) = (-1)^{\ell(i+j)}(\alpha \rhd x) \times (\beta \rhd y) \in I_{\bar{q}_{i',\bar{r}}^\bar{p}}H_{i+k}(X \times Y, (A \times Y) \cup (X \times C); R)
\]

Location: Lemma [7.73]

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96 Altogether, these torsion free conditions are equivalent to requiring local torsion freeness of the appropriate spaces with respect to at least three out of four of \( \bar{p}, \bar{q}, \bar{u}, \bar{v} \).
7. Naturality

- **Conditions**: All spaces only filtered; \( f : X \to X' \) is \((\bar{p}, \bar{p}')\)-stratified, \( g : Y \to Y' \) is \((\bar{q}, \bar{q}')\)-stratified, and \( f \times g : X \times Y \to X' \times Y' \) is \((P, Q)\)-stratified; \( P(S \times T) \geq \bar{p}(S) + \bar{q}(T), Q(S' \times T') \geq \bar{p}'(S') + \bar{q}'(T'); \xi \in I^p S_*(X, A; R) \) and \( \eta \in I^q S_*(Y, B; R) \)

**Property:**

\[ f(\xi) \times g(\eta) = (f \times g)(\xi \times \eta) \in I^Q S_*(X' \times Y', (A' \times Y') \cup (X' \times B'); R) \]

**Location**: Lemma 5.16

- **Conditions**: \( f : (X, A) \to (X', A') \) is \((\bar{p}, \bar{p}')\)-stratified, \( g : (Y, B) \to (Y', B') \) is \((\bar{q}, \bar{q}')\)-stratified, \( X \) is locally \((\bar{p}, R)\)-torsion free or \( Y \) is locally \((\bar{q}, R)\)-torsion free, \( X' \) is locally \((\bar{p}', R)\)-torsion free or \( Y' \) is locally \((\bar{q}', R)\)-torsion free, \( \alpha \in I_{\bar{p}} H^i(X', A'; R) \) and \( \beta \in I_{\bar{q}} H^j(Y', B'; R) \)

**Property:**

\[ (f \times g)^*(\alpha \times \beta) = (f^*(\alpha)) \times (g^*(\beta)) \in I_{\bar{p}, \bar{q}} H^{i+j}(X \times Y, (A \times Y) \cup (X \times B); R). \]

**Location**: Lemma 7.32

- **Conditions**: \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \); \( D\bar{s} \geq D\bar{u} + D\bar{v} \) on \( Y \); \( f : (X; A, B) \to (Y; C, D) \) is \((\bar{p}, \bar{u})\)-stratified, \((\bar{q}, \bar{v})\)-stratified, and \((\bar{r}, \bar{s})\)-stratified; \( X \) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free; \( Y \) is locally \((\bar{u}, R)\)-torsion free or locally \((\bar{v}, R)\)-torsion free; \( \alpha \in I_{\bar{q}} H^i(Y, C; R) \) and \( \beta \in I_{\bar{q}} H^j(Y, D; R) \)

**Property:**

\[ f^*(\alpha \smile \beta) = (f^*(\alpha)) \smile (f^*(\beta)) \in I_{\bar{p}} H^{i+j}(X, A \cup B; R) \]

**Location**: Lemma 7.33

- **Conditions**: \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \); \( D\bar{s} \geq D\bar{u} + D\bar{v} \) on \( Y \); \( f : (X; A, B) \to (Y; C, D) \) is \((\bar{p}, \bar{u})\)-stratified, \((\bar{q}, \bar{v})\)-stratified, and \((\bar{r}, \bar{s})\)-stratified; \( \beta \in I_{\bar{q}} H^j(Y, D; R) \) and \( \xi \in I^p H_{i+j}(X, A \cup B; R) \)

**Property:**

\[ \beta \smile f(\xi) = f(f^*(\beta) \smile \xi) \in I^q H_i(Y, C; R). \]

**Location**: Lemma 7.34

8. Locality

**Conditions**: \( D\bar{r} \geq D\bar{p} + D\bar{q} \) on \( X \); \( X \) locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free, \( \mathcal{U} \) a covering of \( X \); \( \kappa : H_*(\sum_{U \in \mathcal{U}} I^p S_*(U, U \cap A; R) \otimes I^p S_*(U, U \cap B; R)) \to H_*(I^p S_*(X, A; R) \otimes I^p S_*(X, B; R)) \) induced by inclusions

**Property:**

\[ \text{im}(\bar{d} : I^p H_*(X, A \cup B; R) \to H_*(I^p S_*(X, A; R) \otimes I^p S_*(X, B; R))) \subset \text{im}(\kappa) \]
Despite this drawback, $\partial$-strata may be particular, we would like to be able to work with to have intersection (co)homology products on spaces that are not technically CS sets. In turn, to provide our IAW maps, which give our algebraic diagonal maps, which allow us the version stated in Theorem 6.61 which provides a homotopy equivalence. This is used, open subsets. This has largely been so that we can invoke the Künneth theorem, usually cross products under the assumptions that all spaces are CS sets and that all subsets are

Throughout this chapter, we have developed the definitions and properties of cap, cup, and $\partial$-manifold, not manifolds, and so they fail to satisfy the qualifying definition. In general, suppose $(\bar{p}, R)$-torsion free or $Y$ locally $(\bar{q}, R)$-torsion free, all $I^p H_i(X, A; R)$ finitely generated or all $I^q H_j(Y, B; R)$ finitely generated

**Property:** There is a natural exact sequence

$$0 \to \bigoplus_{i+j=k} I_\bar{p}^i H^i(X, A; R) \otimes I_\bar{q}^j H^j(Y, B; R) \xrightarrow{\epsilon} I_{I_\bar{p} \bar{q}} H^k(X \times Y, (A \times Y) \cup (X \times B); R)$$

$$\to \bigoplus_{i+j=k+1} I_\bar{p}^i H^i(X, A; R) \ast I_\bar{q}^j H^j(Y, B; R) \to 0$$

that splits (non-naturally)

**Location:** Theorem 7.78

### 7.3.10 Products on non-CS spaces

Throughout this chapter, we have developed the definitions and properties of cap, cup, and cross products under the assumptions that all spaces are CS sets and that all subsets are open subsets. This has largely been so that we can invoke the Künneth theorem, usually the version stated in Theorem 6.61 which provides a homotopy equivalence. This is used, in turn, to provide our IAW maps, which give our algebraic diagonal maps, which allow us to define the products. However, there are certain circumstances in which we would like to have intersection (co)homology products on spaces that are not technically CS sets. In particular, we would like to be able to work with $\partial$-stratified pseudomanifolds, but their strata may be $\partial$-manifold, not manifolds, and so they fail to satisfy the qualifying definition. Despite this drawback, $\partial$-stratified pseudomanifolds are stratified homotopy equivalent to their interiors. More generally, if $N$ is an open stratified collar neighborhood of $\partial X$ for a $\partial$-stratified pseudomanifold $X$, then the pair $(X, \partial X)$ is stratified homotopy equivalent to the pair $(X - \partial X, N - \partial X)$. These homotopy equivalences provide an alternative route to defining intersection (co)homology products.

In general, suppose $(X, A)$ and $(Y, B)$ are any filtered space pairs but that we also have have CS sets $X'$ and $Y'$ with respective open subsets $A'$ and $B'$, a $(\bar{p}, \bar{q})$-stratified map $f : (X, A) \to (X', A')$, and a $(\bar{q}, \bar{q}')$ stratified map $g : (Y, B) \to (Y', B')$, such that the induced maps $f_* : I^p H_*(X, A; R) \to I^\bar{p} H_*(X', A'; R)$, $g_* : I^q H_*(Y, B; R) \to I^\bar{q} H_*(Y', B'; R)$, and $(f \times g)_* : I^{p+q} H_*(X \times Y, (A \times Y) \cup (X \times B); R) \to I^{\bar{p}+\bar{q}} H_*(X' \times Y', (A' \times Y') \cup (X' \times B'); R)$ are all isomorphisms. Then we can consider the following diagram:

$$I^p S_*(X, A; R) \otimes_R I^q S_*(Y, B; R) \xrightarrow{\epsilon} I^{p+q} S_*(X \times Y, (A \times Y) \cup (X \times B); R)$$

$$f \otimes g$$

$$I^\bar{p} S_*(X', A'; R) \otimes_R I^\bar{q} S_*(Y', B'; R) \xrightarrow{\epsilon} I^{\bar{p}+\bar{q}} S_*(X' \times Y', (A' \times Y') \cup (X' \times B'); R)$$

$$f \times g$$
This diagram commutes by the naturality of the cross product, Lemma 5.16. As $f$, $g$, and $f \times g$ induce intersection homology isomorphisms between chain complexes of projective modules over a Dedekind domain, the vertical maps in the diagram are chain homotopy equivalences (see Lemmas 6.40 and 12.7 and Corollary 12.5). Suppose also that the CS sets $X'$ and $Y'$ satisfy the locally torsion-free conditions needed for the bottom map to be a chain homotopy equivalence by Theorem 6.61. Then three sides of our square are homotopy equivalences, so the cross product on the top must be a homotopy equivalence as well.

In this setting, we therefore have an inverse IAW map

$$\text{IAW} : I^{p,q}S_*(X \times Y, (A \times Y) \cup (X \times B); R) \to I^pS_*(X, A; R) \otimes_R I^qS_*(Y, B; R)$$

and this allows for the construction of an algebraic diagonal and cap, cup, and cross products just as in the previous sections. In fact, a careful reading demonstrates that all of our previous properties apply with the possible exceptions of the stability properties in Section 7.3.5 and Section 7.3.7. The issue with Section 7.3.5 is that there are certain excision maps involved in the statements and proofs that do rely on our subsets being open subsets, or at least on $\{A, B\}$ being an excisive couple for $A \cup B$ as well as some product pairs being excisive couples. Thus, recovering the results of Section 7.3.5 require some extra conditions. One sufficient condition would be the assumption that either $A$ or $B$ is empty, in which case all the excision maps in the lemmas and their proofs becomes identity maps, independent of whether the remaining non-empty subset is open. In fact, for Lemmas 7.60 and 7.61 it is sufficient to assume that $A \cap B = \emptyset$. We will discuss another possible condition below in Lemma 7.84.

The results of Section 7.3.7 are more difficult to reacquire as, for example, Lemma 7.77 involves not just the global cross product but the cross products over small subsets. No doubt a version of the results of this section could be recovered with enough careful hypotheses, but, as we will not need such results below, we will not pursue this.

Of course, these arguments work just as well with maps going from the CS sets to the filtered sets.

For later use, we summarize this discussion as a (somewhat inelegant) theorem.

**Theorem 7.81.** Let $(X, A)$ and $(Y, B)$ be pairs of filtered spaces with respective perversities $\bar{p}$ and $\bar{q}$. Let $R$ be a Dedekind domain. Suppose we have CS sets $X'$ and $Y'$ with respective open subsets $A'$ and $B'$, a $(\bar{p}, \bar{p}')$-stratified map $f : (X, A) \to (X', A')$, and a $(\bar{q}, \bar{q}')$-stratified map $g : (Y, B) \to (Y', B')$ such that the induced maps $f_* : I^pH_*(X, A; R) \to I^pH_*(X', A'; R)$, $g_* : I^qH_*(Y, B; R) \to I^qH_*(Y', B'; R)$, and $(f \times g)_* : I^{p,q}H_*(X \times Y, (A \times Y) \cup (X \times B); R) \to I^{p,q}H_*(X' \times Y', (A' \times Y') \cup (X' \times B'); R)$ are all isomorphisms. Suppose further that $X'$ is locally $(\bar{p}', R)$-torsion free or $Y'$ is locally $(\bar{q}', R)$-torsion free. Then the cross product $I^pS_*(X, A; R) \otimes_R I^qS_*(Y, B; R) \xrightarrow{\sim} I^{p,q}S_*(X \times Y, (A \times Y) \cup (X \times B); R)$ is a chain homotopy equivalence. Its inverse can therefore be used to construct, as appropriate, cup, cap, or cross products satisfying the established properties, except perhaps those of Sections 7.3.3 and Section 7.3.7. Those of Section 7.3.3 can be recovered with suitable additional hypotheses; one such suitable hypothesis is that $A$ or $B$ be empty.

Analogous results hold with the directions of the maps $f$ and $g$ reversed.
As our primary non-CS sets of interest will be $\partial$-stratified pseudomanifolds, we would now like to establish some conditions that will be sufficient for Theorem 7.81 to apply and that will be relevant later when discussing Lefschetz duality.

**Proposition 7.82.** Suppose that $X'$ and $Y'$ are CS sets with respective open subsets $A'$ and $B'$. Suppose $f : (X, A) \hookrightarrow (X', A')$ and $g : (Y, B) \hookrightarrow (Y', B')$ are inclusions such that $X$ is a stratified deformation retract\[97\] of $X'$, $Y$ is a stratified deformation retract of $Y'$, $A$ is a stratified deformation retract of $A'$, and $B$ is a stratified deformation retract of $B'$. Then the inclusions induce isomorphisms\[98\] $f : I^p H_*(X, A; R) \to I^p H_*(X', A'; R)$, $g : I^q H_*(Y, B; R) \to I^q H_*(Y', B'; R)$, and $f \times g : I^{q+p} H_*(X \times Y, (A \times Y) \cup (X \times B); R) \to I^{q+p} H_*(X' \times Y', (A' \times Y') \cup (X' \times B'); R)$.

**Proof.** The statements about $f$ and $g$ are routine using the stratified homotopy invariance of intersection homology (applied to the inclusions $X \hookrightarrow X'$, $Y \hookrightarrow Y'$, $A \hookrightarrow A'$, and $B \hookrightarrow B'$), the long exact sequences of the pairs, and the Five Lemma. For $f \times g$, the product of the stratified deformation retractions gives a stratified deformation retraction of $X' \times Y'$ to $X \times Y$, so, again via the stratified homotopy invariance, the exact sequences of the pairs, and the Five Lemma, it suffices to show that we have an isomorphism $I^{q+p} H_*(((A \times Y) \cup (X \times B); R) \to I^{q+p} H_*(((A' \times Y') \cup (X' \times B'); R)$.

For this, we utilize the following diagram (coefficients and perversities tacit):

\[
\begin{array}{ccc}
I^* H_*(A \times B) & \oplus I^* H_*(A \times Y) & I^* H_*(X \times B) \\
\downarrow a & \downarrow b & \downarrow c \\
I^* H_*(A \times B') & I^* H_*(A \times Y) \oplus I^* H_*(X \times B) & I^* H_*(X \times B') \\
\downarrow d & \downarrow e & \downarrow f \\
I^* H_*(A' \times B') & I^* H_*(A' \times Y) \oplus I^* H_*(X' \times B) & I^* H_*(X' \times B')
\end{array}
\]

The bottom row is the long exact Mayer-Vietoris sequence of the pair \{A' \times Y', X' \times B'\}, each of which is an open subset of $X' \times Y'$. The middle row is the Mayer-Vietoris sequence of the pair \{(A \times Y') \cup (A' \times B), (A \times B') \cup (X' \times B)\}. Each of these sets is open in their union (A \times Y') \cup (X' \times B), so this is a valid Mayer-Vietoris sequence. The modules in the top row do not automatically fit into a Mayer-Vietoris sequence, as the pair \{A \times Y, X \times B\} is not necessarily excisive (at least not evidently so). We will show that the vertical arrows, each of which is induced by spatial inclusion, are all isomorphisms; in particular, then, the composition map $fe$ is an isomorphism, which is what we need to show.

First, as $A'$ stratified deformation retracts to $A$ and $B'$ stratified deformation retracts to $B$, the product $A' \times B'$ has a stratified deformation retraction to $A \times B$, so the composition $ca$ is an isomorphism. Similarly, starting with $(A \times B') \cup (A' \times B)$, we can first use the stratified

---

\[97\] We assume that the homotopies in our deformation retractions hold the subspace fixed; some authors call these strong deformation retractions, though the definition used here agrees with that in [78, 53].

\[98\] As the maps are inclusions, we assume the perversities on the subspaces are induced from the larger spaces and use the un-primed notation.
deformation retraction of \(A \times B'\) to \(A \times B\) to stratified deformation retract \((A \times B') \cup (A' \times B)\) to \(A' \times B\) (holding \(A' \times B\) fixed), and then we stratified deformation retract this to \(A \times B\). So \(a\) is an isomorphism, and it follows that \(c\) is an isomorphism. In the middle column, we have product stratified deformation retractions of \(A' \times Y'\) to \(A \times Y\) and of \(X' \times B'\) to \(X \times B\), so \(db\) is an isomorphism. As \(B \subset Y\), the stratified deformation retraction of \(Y'\) to \(Y\) can be used to stratified deformation retract \((A \times Y') \cup (A' \times B)\) to \((A \times Y) \cup (A' \times B)\), which then stratified deformation retracts to \((A \times Y) \cup (A \times B) = A \times Y\). Analogously, \((A \times B') \cup (X' \times B)\) stratified deformation retracts to \(X \times B\). So the middle vertical maps are isomorphisms.

It follows now from the Five Lemma that \(f\) is an isomorphism. Lastly, again using that \(B \subset Y\) and \(A \subset X\), holding \(X' \times B\) fixed, \((A \times Y') \cup (X' \times B)\) stratified deformation retracts to \((A \times Y) \cup (X' \times B)\); then holding \(A \times Y\) fixed, \((A \times Y) \cup (X' \times B)\) stratified deformation retracts to \((A \times Y) \cup (X \times B)\). So \(e\) is an isomorphism. At last we conclude that \(fe\) is an isomorphism.

\[\square\]

**Corollary 7.83.** Let \(R\) be a Dedekind domain, and let \(X\) be an \(n\)-dimensional \(\partial\)-stratified pseudomanifold with perversities \(\bar{p}, \bar{q}, \bar{r}\) such that \(D\bar{r} \geq D\bar{p} + D\bar{q}\) and \(X\) is locally \((\bar{p}, R)\)-torsion free or locally \((\bar{q}, R)\)-torsion free. Then there are well-defined cup products

\[I_{\bar{p}}H^{i}(X, \partial X; R) \otimes I_{\bar{q}}H^{j}(X, R) \rightarrow I_{\bar{p}}H^{i+j}(X, \partial X; R)\]

\[I_{\bar{p}}H^{i}(X; R) \otimes I_{\bar{q}}H^{j}(X, \partial X; R) \rightarrow I_{\bar{p}}H^{i+j}(X, \partial X; R)\]

\[I_{\bar{p}}H^{i}(X, \partial X; R) \otimes I_{\bar{q}}H^{j}(X, \partial X; R) \rightarrow I_{\bar{p}}H^{i+j}(X, \partial X; R)\]

and cap products

\[I_{\bar{q}}H^{j}(X; R) \otimes I_{\bar{p}}H_{i+j}(X, \partial X; R) \rightarrow I_{\bar{p}}H_{i}(X, \partial X; R)\]

\[I_{\bar{q}}H^{j}(X, \partial X; R) \otimes I_{\bar{p}}H_{i+j}(X, \partial X; R) \rightarrow I_{\bar{p}}H_{i}(X; R)\]

\[I_{\bar{q}}H^{j}(X, \partial X; R) \otimes I_{\bar{p}}H_{i+j}(X, \partial X; R) \rightarrow I_{\bar{p}}H_{i}(X, \partial X; R)\].

**Proof.** Let \(N_1\) be a stratified open collar of \(\partial X\) in \(X\), which is guaranteed to exist from the definition of a \(\partial\)-stratified pseudomanifold. Let \(N_2 \cong [0, 1) \times \partial X\) be an external collar, which we can glue on to \(X\) to form the stratified pseudomanifold \(X' = X \cup_{\partial X} N_2\). Then \(X'\) has a stratified deformation retraction to \(X\) by retracting the \(N_2\). Similarly, \(N' = N_1 \cup_{\partial X} N_2\) has a stratified deformation retraction to \(\partial X\). Note that both \(X'\) and \(Y'\) are stratified pseudomanifolds, and so CS sets, and \(N'\) is an open subset of \(X'\). The corollary now follows from Proposition 7.82 and Theorem 7.81 by considering the maps \((X, \partial X) \rightarrow (X', N')\) and \((X, \emptyset) \rightarrow (X', \emptyset)\) for the appropriate cases.

\[\square\]

To close this section, we highlight one scenario in which Lemmas 7.60 and 7.61 will remain true given pairs \((X, A)\) and \((Y, B)\) that satisfy the hypotheses of Theorem 7.81. By that theorem, an IAW map, and so all the necessary products, exist, so the only parts of the proofs where we need to be careful are in the existence of the claimed excision isomorphisms. For Lemma 7.60, these are the maps \(e : I^pH_*(B, A \cap B; R) \rightarrow I^pH_*(A \cup B, A; R)\) and \(e' : I^pH_*(X \times B, A \times B; R) \rightarrow I^pH_*((A \times X) \cup (X \times B), A \times X; R)\). The maps for Lemma
are equivalent with the roles of $A$ and $B$ reversed. So the following lemma can be useful in adapting Lemmas 7.60 and 7.61 for non-CS spaces. We will apply them later in our proof of Corollary 8.48.

**Lemma 7.84.** Suppose $X$ is a filtered space with closed subspaces $A, B \subset X$. Let $\bar{r}$ be any perversity.

1. If $A \cap B$ has an open neighborhood $U$ in $A$ such that $A \cap B$ is a stratified deformation retract of $U$, then the map induced by inclusion $e : I^*H_*(B, A \cap B; R) \to I^*H_*(A \cup B, A; R)$ is an isomorphism.

2. If $B$ has an open neighborhood $N$ in $X$ such that $B$ is a stratified deformation retract of $N$ then the map induced by inclusion $e' : I^*H_*(X \times B, A \times B; R) \to I^*H_*(((A \times X) \cup (X \times B), A \times X; R)$ is an isomorphism.

**Proof.** The proof of the first statement is very elementary. Note that $A - U$ is closed in $A$, which is closed in $X$, so $A - U$ is closed in $X$ and hence in $A \cup B$. Furthermore, $A - U$ is contained in the $(A \cup B) - B$, which is a subset of $A$ and open in $A \cup B$. So $A - U$ is closed and contained in the interior of $A$ in $A \cup B$; thus excision applies we have an excision isomorphism $I^*H_*(U \cup B, U; R) \to I^*H_*(A \cup B, A; R)$ with excised set $A - U$. Next, the inclusion $I^*H_*(B, A \cap B; R) \to I^*H_*(U \cup B, U; R)$ is an isomorphism using the Five Lemma and that the inclusions $A \cap B \hookrightarrow U$ and $B \hookrightarrow U \cup B$ are stratified homotopy equivalences. As $e$ is the composition of these inclusions, it induces an intersection homology isomorphism.

The proof of the second statement is similar. In this case, we have an excision isomorphism $I^*H_*(((A \times N) \cup (X \times B), A \times N; R) \to I^*H_*((A \times X) \cup (X \times B), A \times X; R)$ that excises $A \times (X - N)$, which is closed and contained in $A \times (X - B)$, an open subset of $A \times X$. Then the stratified deformation retraction of $N$ to $B$ induces stratified deformation retractions from $A \times N$ to $A \times B$ and from $(A \times N) \cup (X \times B)$ to $(A \times B) \cup (X \times B) = X \times B$. So, again employing the Five Lemma, we have an isomorphism $I^*H_*(X \times B, A \times B; R) \to I^*H_*((A \times N) \cup (X \times B), A \times N; R)$. Also once again, $e'$ is the composition of these inclusions, so it induces an intersection homology isomorphism.

### 7.4 Intersection cohomology with compact supports

Cohomology with compact supports plays an important role in ordinary homology/cohomology theory, particularly in the statement and proof of the Poincaré duality theorem, and the same remains true for intersection homology/cohomology. In this section we define intersection cohomology with compact supports and also discuss how, with proper assumptions, the cap product can be viewed as a map from compactly-supported intersection cohomology to intersection homology and, moreover, one with certain compatibilities with the Mayer-Vietoris sequence.

Recall (from, e.g., [53, Section 3.3]) that, for ordinary cohomology, the cohomology groups with compact supports of a space $X$ with coefficients in a group $G$ can be defined as

$$H^i_c(X; G) = \lim_{\to} H^i(X, X - K; G),$$
where here $K$ is a compact subset of $X$ and the limit is taken over the directed set of all such compact subsets. In fact, we define $K < L$ in the directed set if and only if $K \subset L$. Then, clearly, $K < K$, $K < L$ and $L < J$ implies $K < J$, and, given compact $K$ and $L$, both are subsets of the compact set $K \cup L$. Thus the collection of compact subsets of $X$ does satisfy the definition to be a directed set (see, e.g., [77 Section 73]). Furthermore, if $K < L$, then $X - L \subset X - K$, so the restriction $H^i(X, X - K; G) \to H^i(X, X - L; G)$ is well defined for each such pair. We can think of the limit as being made up of cocycles that annihilate any chains in $X$ sufficiently “close to infinity”.

The idea for intersection cohomology is identical:

**Definition 7.85.** Let $X$ be a filtered space with perversity $\overline{p}$, and let $R$ be a commutative ring with unity. The **intersection cohomology groups with compact supports**, $I_{\overline{p}}H^i_c(X; R)$, are defined to be $\lim_{\to} I_{\overline{p}}H^i(X, X - K; R)$, where the limit is over compact subsets of $X$.

**Example 7.86.** Let $X$ be a compact filtered space. We claim $I_{\overline{p}}H^i_c(cX; R) = I_{\overline{p}}H^i(cX, cX - \{v\}; R)$. Indeed, recall that we have $cX = ([0, 1] \times X)/\sim$, and, for $0 < r < 1$, let $\overline{c}_rX$ be the subspace $([0, r] \times X)/\sim$. Then the compact sets $\overline{c}_rX$ are cofinal among the compact subsets of $cX$. In this case, this means that every compact subset of $cX$ is contained within one of the $\overline{c}_rX$. Recall that, for computing direct limits, it is sufficient to restrict to any cofinal directed subset [77 Lemma 73.1]. But then if $r > s$, the restriction $I_{\overline{p}}H^i(cX, cX - \overline{c}_rX; R) \to I_{\overline{p}}H^i(cX, cX - \overline{c}_sX; R)$ is an isomorphism via stratified homotopy equivalence. Therefore, $I_{\overline{p}}H^i_c(cX; R)$ is isomorphic to any of these $I_{\overline{p}}H^i(cX, cX - \overline{c}_rX; R)$, and they are all isomorphic to $I_{\overline{p}}H^i(cX, cX - \{v\}; R)$, again using stratified homotopy equivalence.

The functoriality of cohomology with compact supports runs opposite to the standard functoriality of cohomology and requires some additional assumptions. For example, suppose $U$ is an open subsets of $X$. Then, for each compact $K \subset U$, excision guarantees that restriction induces an isomorphism $I_{\overline{p}}H^*(X, X - K; R) \to I_{\overline{p}}H^*(U, U - K; R)$. These isomorphisms commute with restriction maps induced by including $K$ into a larger compact $L \subset U$, and so we can form the composition

$$I_{\overline{p}}H^*_c(U; R) = \lim_{\to} I_{\overline{p}}H^*(U, U - K; R) \cong \lim_{\to} I_{\overline{p}}H^*(X, X - K; R) \to \lim_{\to} I_{\overline{p}}H^*(X, X - K; R) = I_{\overline{p}}H^*_c(X; R).$$

Here, the last map is obtained by observing that each $I_{\overline{p}}H^*(X, X - K; R)$ for $K \subset U$ has an evident map to $\lim_{\to} I_{\overline{p}}H^*(X, X - K; R)$, and these are all compatible under restrictions induced by inclusions $K \hookrightarrow L \subset U$ with $L$ compact. So the map exists by the universal property of direct limits.

The following nearly immediate observation is useful:

---

99 Unfortunately, we have two different “c”s here: the $c$ for compact supports and the $c$ for the cone construction. However, context should make clear which is meant in each case.

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Lemma 7.87. Suppose $U \subset X$ is an open subset and that $L \subset U$ is compact. Then there is a commutative diagram

$$
\begin{align*}
I_\bar{p}H^*(U, U - L; R) & \cong I_\bar{p}H^*(X, X - L; R) \\
\downarrow & \downarrow \\
I_\bar{p}H_c^*(U; R) & \rightarrow I_\bar{p}H_c^*(X; R).
\end{align*}
$$

Here the horizontal maps are induced by inclusion of $U$ into $X$, the bottom map being that defined just above. The vertical maps are those taking an element of a directed system to its image in the direct limit.

Proof. The top horizontal map in the diagram is an isomorphism by excision. The commutativity of the diagram of the lemma follows from the commutativity of the following diagram.

$$
\begin{align*}
I_\bar{p}H^*(U, U - L; R) & \cong I_\bar{p}H^*(X, X - L; R) \\
\downarrow & \downarrow \\
\lim_{K \subset U} I_\bar{p}H^*(U, U - K; R) & \cong \lim_{K \subset U} I_\bar{p}H^*(X, X - K; R) \rightarrow \lim_{K \subset X} I_\bar{p}H^*(X, X - K; R) \\
\downarrow & \downarrow \\
I_\bar{p}H_c^*(U; R) & \rightarrow I_\bar{p}H_c^*(X; R).
\end{align*}
$$

Here the bottom rectangle simply recounts our definition of the map $I_\bar{p}H_c^*(U; R) \rightarrow I_\bar{p}H_c^*(X; R)$ induced by inclusion. The upper left square commutes already at the level of cochains, noting for ease of observation that the maps $I_\bar{p}C^*(U, U - L; R) \rightarrow \lim_{K \subset U} I_\bar{p}C^*(U, U - K; R)$ and $I_\bar{p}C^*(X, X - L; R) \rightarrow \lim_{K \subset X} I_\bar{p}C^*(X, X - K; R)$ are injective. Then we can apply the cohomology functor, which commutes with direct limits. Similarly, the triangle is a commutative triangle of injections at the cochain level.

We also have the following Mayer-Vietoris sequence for cohomology with compact supports (compare [53, Lemma 3.36]):

Lemma 7.88. Let $X$ be a CS set\footnote{We will use that $X$ is a CS set, not just a filtered space, in order to employ Corollary 2.43 in the argument below. From the proof of that corollary, it would be sufficient to assume that $X$ is locally compact Hausdorff.} with perversity $\bar{p}$, and let $R$ be a commutative ring with unity. Suppose $X = U \cup V$ for $U, V$ open subsets. Then there is an exact Mayer-Vietoris sequence

$$
\begin{align*}
I_\bar{p}H_c^i(U \cap V; R) \rightarrow I_\bar{p}H_c^i(U; R) \oplus I_\bar{p}H_c^i(V; R) \rightarrow I_\bar{p}H_c^i(X; R) \rightarrow .
\end{align*}
$$
Proof. Suppose $K \subset U$ and $L \subset V$ are compact sets. Then, by Theorem 7.13 we have the exact Mayer-Vietoris sequence

$$
\begin{array}{c}
\rightarrow I_pH^i(X, X - (K \cap L); R) \rightarrow I_pH^i(X, X - K; R) \oplus I_pH^i(X, X - L; R) \rightarrow I_pH^i(X, X - (K \cup L); R) \rightarrow.
\end{array}
$$

Note that $(X - K) \cap (X - L) = X - (K \cup L)$ and $(X - K) \cup (X - L) = X - (K \cap L)$. Via excision isomorphisms, this diagram has the form

$$
\begin{array}{c}
\rightarrow I_pH^i(U \cap V, U \cap V - (K \cap L); R) \rightarrow I_pH^i(U, U - K; R) \oplus I_pH^i(V, V - L; R) \rightarrow I_pH^i(X, X - (K \cup L); R) \rightarrow.
\end{array}
$$

Now, we take the direct limits with respect to the directed system of pairs $(K, L)$ such that $K$ is compact in $U$ and $L$ is compact in $V$, letting $(K, L) < (K', L')$ if $K \subset K'$ and $L \subset L'$. Let’s call this directed set $\gamma$. Taking direct limits with respect to $\gamma$ preserves exactness (e.g. [55, Proposition 5.33]). Let us verify that we obtain the correct cohomology groups with compact supports.

By applying functoriality and the preservation of exactness to split exact sequences, direct limits distribute over direct sums, so

$$
\lim_\gamma (I_pH^i(U, U - K; R) \oplus I_pH^i(V, V - L; R))
\cong (\lim_\gamma I_pH^i(U, U - K; R)) \oplus (\lim_\gamma I_pH^i(V, V - L; R))
\cong I_pH^i(U; R) \oplus I_pH^i(V; R).
$$

For $\lim_\gamma I_pH^i(U \cap V, U \cap V - (K \cap L); R)$, we obviously have compatible maps from each $I_pH^i(U \cap V, U \cap V - (K \cap L); R)$ to $I_pH^i(U \cap V; R) = \lim_\gamma I_pH^i(U \cap V, U \cap V - J; R)$, with $J$ running over the compact subsets of $U \cap V$, so there is a map $\phi : \lim_\gamma I_pH^i(U \cap V, U \cap V - (K \cap L); R) \to I_pH^i(U \cap V; R)$. If an element of $I_pH^i(U \cap V; R)$ is represented by $\alpha \in I_pH^i(U \cap V, U \cap V - J; R)$, then we can let $K = L = J$ and obtain a commutative diagram

$$
\begin{array}{cc}
I_pH^i(U \cap V, U \cap V - (J \cap J); R) \rightarrow & \lim_\gamma I_pH^i(U \cap V, U \cap V - (K \cap L); R) \\
\downarrow \quad & \quad \downarrow \phi \\
I_pH^i(U \cap V, U \cap V - J; R) & \rightarrow \quad I_pH^i(U \cap V; R),
\end{array}
$$

so $\phi$ is surjective. Similarly, if we have an element $[\alpha] \in \lim_\gamma I_pH^i(U \cap V, U \cap V - (K \cap L); R)$ represented by some particular $\alpha \in I_pH^i(U \cap V, U \cap V - (K \cap L); R)$ for some specific $K, L$ and if $\phi([\alpha]) = 0$, then the image of $\alpha$ in some $I_pH^i(U \cap V, U \cap V - J; R)$ with $K \cap L \subset J$ must be 0. But then the image of $\alpha$ in $I_pH^i(U \cap V, U \cap V - (J \cap J); R)$ must be 0, so $[\alpha] = 0$.

The argument for $\lim_\gamma I_pH^i(X, X - (K \cup L); R)$ is similar in spirit but requires some minor additional work. Again we have and evident map $\phi : \lim_\gamma I_pH^i(X, X - (K \cup L); R) \to I_pH^i(X; R) = \lim_\gamma I_pH^i(X, X - J; R)$, now with $J$ running over the compact subsets of $X$. 

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Suppose \([\alpha] \in I_pH^i_c(X; R)\) represented by \(\alpha \in I_pH^i(X, X - J; R)\). By arguments similar to those of the last paragraph, it suffices for surjectivity of \(\phi\) to show that we can write \(J\) as \(J = K \cup L\) with \(K \subset U\) and \(L \subset V\). For this, let us consider the disjoint closed sets \(J - (J \cap V)\) and \(X - U\). The subspace \(J - (J \cap V)\) is compact, so we can find disjoint open sets \(W_1, W_2\) with \(J - (J \cap V) \subset W_1\) and \(X - U \subset W_2\) by Corollary 2.43. Now, let \(J_i\) be the closure of \(J \cap W_i\). As \(W_1\) and \(W_2\) are disjoint, \(J_1 \cap W_2 = \emptyset\), which implies that \(J_1 \cap (X - U) = \emptyset\), so \(J_1 \subset U\). Also \(J_1\) is a closed subset of \(J\) and so is compact. Let \(J_2 = J - (J \cap W_1) = J \cap (X - W_1)\), which is also compact as a closed subset of \(J\). Furthermore, \(J \cap W_1\) contains \(J - (J \cap V)\), i.e. all points of \(J\) outside of \(V\) are contained in \(J \cap W_1\), so all the points left in \(J_2\) must be contained in \(V\). Finally, every point of \(J\) is in either \(J \cap W_1\), and so in \(J_1\), or in \(J - (J \cap W_1)\), and so in \(J_2\). Therefore, \(J = J_1 \cup J_2\) with \(J_1 \subset U\) and \(J_2 \subset V\). This provides the surjectivity of \(\phi\).

Finally, we consider injectivity of the \(\phi\) of the preceding paragraph. Now we suppose we have \([\alpha] \in \varprojlim_{\gamma} I_pH^i(X, X - (K \cup L); R)\) with \(\phi([\alpha]) = 0\). Let \(\alpha \in I_pH^i(X, X - (K \cup L); R)\) represent \([\alpha]\). As \(\phi([\alpha]) = 0\), there is some compact \(J \subset X\) with \(K \cup L \subset J\) and the image of \(\alpha\) in \(I_pH^i(X, X - J; R)\) trivial. As in the preceding paragraph, let us choose compact \(J_1\) and \(J_2\) such that \(J = J_1 \cup J_2\) and \(J_1 \subset U\), \(J_2 \subset V\). We do not know that either \(K \subset J_1\) or \(L \subset J_2\). However, we do have \(K \cup J_1 \subset U\) and \(L \cup J_2 \subset V\), so we can consider the image of \(\alpha\) in \(I_pH^i(X, X - ((K \cup J_1) \cup (L \cup J_2)); R)\). Then we have \(K \cup L \subset J \subset (K \cup J_1) \cup (L \cup J_2)\), so the image of \(\alpha\) in this latter group factors through its image in \(I_pH^i(X, X - J; R)\) and thus must be trivial. Therefore, \([\alpha] = 0\) in \(\varprojlim_{\gamma} I_pH^i(X, X - (K \cup L); R)\).

Let us turn to cap products. Suppose \(X\) is a locally \((\vec{q}; R)\)-torsion free CS set with perversities \(D\vec{r} \geq D\vec{p} + D\vec{q}\) and that \(R\) is a Dedekind domain so that we have a well-defined intersection (co)homology cap product. Suppose that for every compact \(K \subset X\) we have a class \(\xi_K \in I^\ell H^i_{i+j}(X, X - K; R)\) such that, if \(K \subset L\) and \(i_{K,L} : (X, X - L) \hookrightarrow (X, X - K)\) is the inclusion, then \(i_{K,L*}(\xi_L) = \xi_K\). In other words, the collection \(\{\xi_K\}\) represents an element of the inverse limit \(\varprojlim_{\gamma} I^\ell H^i_{i+j}(X, X - K; R)\). Again, the relevant direct system is that of compact subsets of \(K\), but now the corresponding modules and maps have the form \(I^\ell H^i_{i+j}(X, X - L; R) \rightarrow I^\ell H^i_{i+j}(X, X - K; R)\) for \(K \subset L\).

Let \(\alpha \in I_qH^i_c(X; R)\). By the definition of the direct limit, \(\alpha\) is represented as the image in the limit of some \(\alpha_K \in I_qH^i(X, X - K; R)\), and we can consider the cap product \(\alpha_K \cap \xi_K \in P^pH_i(X, R)\). Suppose now \(K \subset L\). We have \(\alpha_L = i_{K,L*}^{*}\alpha_K \in I_qH^j(X, X - L; R)\), which also represents \(\alpha\) in \(I_qH^j_c(X; R)\). Furthermore, using the naturality property of the cap product, Lemma 7.34 in the second line below, we have the following computation:

\[
\begin{align*}
\alpha_K \cap \xi_K &= \alpha_K \cap i_{K,L*}(\xi_L) \\
&= i_{K,L*}(i_{K,L}^{*}(\alpha_K) \cap \xi_L) \\
&= i_{K,L*}(\alpha_L \cap \xi_L).
\end{align*}
\]

In this case, the \(i_{K,L*}\) in the last line is the inclusion of the pair \((X, \emptyset) \rightarrow (X, \emptyset)\), as we can see from the usage in Lemma 7.34, so we have \(\alpha_K \cap \xi_K = i_{K,L*}(\alpha_L \cap \xi_L) = \alpha_L \cap \xi_L\).

Altogether then, this demonstrates the following lemma:
Lemma 7.89. Suppose $X$ is a locally $(\bar{p}; R)$-torsion free CS set with perversities $D\bar{p} \geq D\bar{p} + D\bar{q}$ and that $R$ is a Dedekind domain. Then the cap product induces a well-defined map

$$\sim: I^qH^i_c(X; R) \otimes \lim_{\xi \in U, V} I^rH_{i+j}(X, X - K; R) \to I^pH_i(X; R).$$

In our treatment of Poincaré duality, below, the element of $\lim_{\xi \in U, V} I^rH_{i+j}(X, X - K; R)$ will correspond to the fundamental class of an $R$-oriented stratified pseudomanifold $X$, and so, up to some sign issues we will discuss later, this will be the cap product that induces Poincaré duality.

In the remainder of this section, we will prove the following lemma, which is critical to the proof of Poincaré duality. We base our treatment on Proposition 6.7 and Lemma 6.8 of [38], which itself is based on Hatcher [53, Lemma 3.36], though with some modification necessitated by the lack of exact control on supports of intersection chains under cap products (see Section 7.3.7). As observed even in Hatcher, the proof is surprisingly non-trivial.

Lemma 7.90. Let $X$ be a locally $(\bar{p}; R)$-torsion free CS set with perversities $D\bar{p} \geq D\bar{p} + D\bar{q}$, let $U, V \subset X$ be two open subsets with $U \cup V = X$, let $R$ be a Dedekind domain, and let $\xi \in \lim_{\xi \in U, V} I^rH_{i+j}(X, X - K; R)$. If $W \subset X$ is open, let $\xi^W$ denote the image of $\xi$ under the natural map $\lim_{\xi \in U, V} I^rH_{i+j}(X, X - K; R) \to \lim_{\xi \in W} I^rH_{i+j}(W, W - K; R)$. Let $D^W : I^qH^i_c(W; R) \to I^pH_i(W; R)$ be given by $D^W(\alpha) = \alpha \sim \xi^W$. Then the following diagram of Mayer-Vietoris sequences commutes up to signs:

$$\begin{array}{ccc}
I^qH^i_c(U \cap V; R) & \to & I^qH^i_c(U; R) \oplus I^qH^i_c(V; R) \to I^qH^i_c(X; R) \\
\downarrow D^{U \cap V} & & \downarrow D^U \oplus -D^V \\
I^pH_i(U \cap V; R) & \to & I^pH_i(U; R) \oplus I^pH_i(V; R) \to I^pH_i(X; R)
\end{array}$$

Proof. First, we observe that the $\xi^W$ make sense for any open $W \subset X$. In fact, let $\xi \in \lim_{\xi \in U, V} I^rH_{i+j}(X, X - K; R)$. Then $\xi$ determines elements $\xi_K \in I^rH_{i+j}(X, X - K; R)$ for every $K \subset X$ with the property that if $L \subset X$ then the image of $\xi_K$ in $I^rH_{i+j}(X, X - K; R)$ under inclusion is $\xi_L$; conversely, any such compatible collection of $\xi_K$ determines $\xi$. Now, suppose we restrict our attention to the $K$ with $K \subset W$. By excision, $I^rH_{i+j}(X, X - K; R) \cong I^rH_{i+j}(W, W - K; R)$, and for all $L \subset K$, we have commutative diagrams

$$I^rH_{i+j}(X, X - K; R) \to I^rH_{i+j}(X, X - L; R)$$

$$I^rH_{i+j}(W, W - K; R) \to I^rH_{i+j}(W, W - L; R).$$

\[\text{We also include some necessary details that were overlooked in [38] (and are also not treated in full detail in the printed version of [53], though see the online errata).}\]
Thus $\xi \in \lim \leftarrow I^p H_{i+j}(X, X - K; R)$ determines compatible $\xi^I_K \in I^p H_{i+j}(W, W - K; R)$ and so an element $\xi^W_K \in \lim \leftarrow I^p H_{i+j}(W, W - K; R)$. So we do have a well-defined map $\lim \leftarrow I^p H_{i+j}(X, X - K; R) \to \lim \leftarrow I^p H_{i+j}(W, W - K; R)$. It is also worth noting that for any fixed compact $L \subset W$ we have commutative diagrams of the form

$$\lim \leftarrow I^p H_{i+j}(X, X - K; R) \to \lim \leftarrow I^p H_{i+j}(W, W - K; R) \to I^p H_{i+j}(X, X - L; R) \sim I^p H_{i+j}(W, W - L; R).$$

We also have such diagrams replacing $X$ with any open $W' \subset X$ with $W \subset W'$. We let $\xi^I_K$ denote the image of $\xi$ in $I^p H_{i+j}(W, W - K; R)$, and we let $D^W_K : I^q H^j(W, W - K; R) \to I^p H_i(W; R)$ be given by $D^W_K(\alpha) = \alpha \cap \xi^W_K$.

Now, to demonstrate the commutativity-up-to-sign of the diagram, we will demonstrate the commutativity of the three subdiagrams corresponding to the three squares (up to index shift) in the diagram of the lemma.

**First square.** Suppose $K \subset U$ and $L \subset V$ are compact. We first consider the diagram

$$I^q H^j(X, X - K \cap L; R) \longrightarrow I^q H^j(X, X - K; R) \oplus I^q H^j(X, X - L; R)$$

$$\downarrow \downarrow \downarrow \downarrow$$

$$I^q H^j(U \cap V, U \cap V - K \cap L; R) \sim I^q H^j(U, U - K; R) \oplus I^q H^j(V, V - L; R)$$

$$\downarrow D^U_{K \cap L} \quad \quad \downarrow D^V_{K \cap L} \quad \quad D^V_K \oplus -D^V_L$$

$$I^p H_i(U \cap V; R) \longrightarrow I^p H_i(U; R) \oplus I^p H_i(V; R).$$

The the direct limit over the pairs $(K, L)$ of the top half of this diagram determines the map $I^q H^j(U \cap V; R) \to I^q H^j(U; R) \oplus I^q H^j(V; R)$ in the Mayer-Vietoris sequence, as we saw in the proof of Lemma 7.88. Furthermore, the lower vertical maps are precisely the cap products that correspond to the cap product of Lemma 7.89. So, once we show this diagram commutes, taking the limit of such diagrams over pairs $(K, L)$ will provide the commutativity of the first square of Lemma 7.90.

For commutativity, we can work with the two summands on the right independently,
with analogous arguments. So consider

\[ I_qH^j(X, X - K \cap L; R) \rightarrow I_qH^j(X, X - K; R) \]

\[ I_qH^j(U \cap V, U \cap V - K \cap L; R) \leftarrow I_qH^j(U, U - K \cap L; R) \rightarrow I_qH^j(U, U - K; R) \]

\[ D_{U \cap V}^{K \cap L} \]

\[ D_{K \cap L}^U \]

\[ D_U^V \]

\[ I^pH_i(U \cap V; R) \rightarrow I^pH_i(U; R). \]

The maps in the upper half of the diagram are all induced by inclusion and commute at the space level. The map pointing to the left is an isomorphism by excision; the excised subspace is the closed subset \( U - U \cap V \) of \( U \), which is contained in the open subset \( U - K \cap L \) of \( U \). The map \( D_{U \cap V}^{K \cap L} \) on the left is the cap product with the image \( \xi_{U \cap V}^{K \cap L} \) of \( \xi \) in \( I^pH_{i+1}(U \cap V, U \cap V - K \cap L; R) \), while the \( D_{K \cap L}^U \) on the right is the cap product with the image \( \xi_{K \cap L}^U \) of \( \xi \) in \( I^pH_{i+1}(U, U - K \cap L; R) \). If we let \( i : (U \cap V; \emptyset, U \cap V - K \cap L) \rightarrow (U; \emptyset, U - K \cap L) \) denote the inclusion of the triple, then \( i(\xi_{U \cap V}^{K \cap L}) = \xi_{U \cap V}^{K \cap L} \). So the bottom left rectangle commutes by the naturality of the cap product, Lemma 7.34 (compare the argument for Lemma 7.89).

Similarly, if we let \( j : (U; \emptyset, U - K) \rightarrow (U; \emptyset, U - K \cap L) \) denote another inclusion, and \( \xi_K^U \) the image of \( \xi \) in \( I^pH_{i+1}(U, U - K; R) \), then \( j(\xi_K^U) = \xi_K^U \), and the bottom right triangle also commutes by Lemma 7.34.

Finally, notice that the analogous square involving \( V \) acquires a sign in \(-D_L^V\) to counter the negative sign of the second Mayer-Vietoris inclusion \( I^pH_i(U \cap V; R) \rightarrow I^pH_i(V; R) \).

This establishes the commutativity of the first square in the diagram of the lemma.

**Second square.** Next, we consider the diagram

\[ I_qH^j(X, X - K; R) \oplus I_qH^j(X, X - L; R) \rightarrow I_qH^j(X, X - K \cup L; R) \]

\[ I_qH^j(U, U - K; R) \oplus I_qH^j(V, V - L; R) \]

\[ D_{K \cup L} \]

\[ D_{K \cup L} \]

\[ D_{K \cup L} \]

\[ D_{K \cup L} \]

\[ D_{K \cup L} \]

\[ I^pH_i(U; R) \oplus I^pH_i(V; R) \rightarrow I^pH_i(X; R). \]

Once again, in the limit, the top part of the diagram corresponds to the map of the Mayer-Vietoris sequence of Lemma 7.88 this time as the inverse of the upper left vertical isomorphism composed with the upper horizontal map. Also once again, to show that this diagram commutes, we can consider the summands separately. So consider
The lower left $D_U^L$ is the cap product with $\xi^U_K$, the diagonal $D_K$ is the cap product with $\xi_K$, and if $i : (U; \emptyset, U - K) \to (X; \emptyset, X - K)$ is the inclusion, then $i(\xi^U_K) = \xi_K$. So the left triangle commutes by naturality of the cap product, Lemma 7.34.

Similarly, if $j : (X; \emptyset, X - K \cup L) \to (X; \emptyset, X - K)$ is the inclusion, then $j(\xi_{K \cup L}) = \xi_K$, and the right triangle commutes by Lemma 7.34.

In the analogous version of the diagram for $V$, the sign on $-D_L^V$ counteracts the negative sign from $I_q^j(X, X - L; R) \to I_q^j(X, X - K \cup L; R)$ that the inclusion map acquires in the definition of the Mayer-Vietoris cohomology sequence (dualized from the homology sequence).

This demonstrates the commutativity of the second square.

**Third square.** This is the “hidden square” in the diagram that involves the boundary maps of the Mayer-Vietoris sequence. Here it is revealed:

$$
I_q^j(X; R) \xrightarrow{d^*} I_q^{j+1}(U \cap V; R)
$$

$$
\downarrow D^X \quad \downarrow D^{U \cap V}
$$

$$
I^pH_i(X; R) \xrightarrow{\partial_*} I^pH_{i-1}(U \cap V; R).
$$
Commutativity of this diagram will follow from that of the diagram

\[
\begin{array}{ccc}
I_qH^j(X, X - K \cup L; R) & \xrightarrow{d^*} & I_qH^{j+1}(X, X - K \cap L; R) \\
\downarrow & & \downarrow \\
D_{K\cup L} & \xrightarrow{\cong} & I_qH^{j+1}(U \cap V, U \cap V - K \cap L; R) \\
\downarrow & & \\
D_{K\cap L}^{U \cap V} & \xrightarrow{\partial_*} & I^pH_{i-1}(U \cap V; R).
\end{array}
\]  

(31)

Once again, the upper three maps compose to the map of the preceding diagram under the direct limits, and this is the map in the Mayer-Vietoris sequence of Lemma \ref{lem:MayerVietoris}.

The proof utilizes a concrete realization of \( d(\xi_{K\cup L}) \in H_{i+j}(I^pS_*(X; R) \otimes I^qS_*(X, X - K \cup L; R)) \). In fact, let’s take \( \xi_{K\cup L} \in I^pH_{i+j}(X, X - K \cup L; R) \). By Lemma \ref{lem:Realization}, \( d(\xi_{K\cup L}) \) can be realized as the image under inclusion of an element of \( H_*(\sum_{W \in \mathcal{W}} I^pS_*(W, W \cap A; R) \otimes I^qS_*(W, W \cap B; R)) \) where \( \mathcal{W} \) is an open covering of \( X \) and \( A, B \) are open subsets of \( X \), in this case with \( A = \emptyset \) and \( B = X - K \cup L \). We will use the following specific covering of \( X \): Let \( W_1 = U - U \cap L, W_2 = U \cap V, \) and \( W_3 = V - V \cap K \). As \( L \subset V \) and \( K \subset U \), \( W_2 \) contains all the points of \( U \) that are removed to form \( W_1 \) and all the points of \( V \) that are removed to form \( W_3 \), so \( \mathcal{W} = \{W_1, W_2, W_3\} \) is a covering of \( X \). Then

\[
W_1 \cap B = (U - U \cap L) \cap (X - K \cup L) = U - U \cap (K \cup L),
\]

\[
W_2 \cap B = (U \cap V) \cap (X - K \cup L) = U \cap V - (U \cap V) \cap (K \cup L),
\]

and

\[
W_3 \cap B = (V - V \cap K) \cap (X - K \cup L) = V - V \cap (K \cup L).
\]

To simplify notation, we will abbreviate \( U - U \cap (K \cup L) \) as \( U - K \cup L \), and similarly for the others.

Thus, applying Lemma \ref{lem:Realization}, \( d(\xi_{K\cup L}) \) can be represented by a cycle in

\[
(I^pS_*(U - L; R) \otimes I^qS_*(U - L, U - K \cup L; R)) \\
\oplus (I^pS_*(U \cap V; R) \otimes I^qS_*(U \cap V, U \cap V - K \cup L; R)) \\
\oplus (I^pS_*(V - K; R) \otimes I^qS_*(V - K, V - K \cup L; R)).
\]

Following \ref{lem:Realization}, let us therefore represent \( d(\xi_{K\cup L}) \) by a chain \( \eta = \eta_{U-L} + \eta_{U\cap V} + \eta_{V-K} \) with \( \eta_{U-L} \in I^pS_*(U - L; R) \otimes I^qS_*(U - L; R), \eta_{U\cap V} \in I^pS_*(U \cap V; R) \otimes I^qS_*(U \cap V; R), \) and \( \eta_{V-K} \in I^pS_*(V - K; R) \otimes I^qS_*(V - K; R) \).

\begin{footnotesize}
\textsuperscript{102}The idea here is that, e.g., \( \eta_{U-L} \in I^pS_*(U - L; R) \otimes I^qS_*(U - L; R) \) is a precise choice of element representing an element of \( I^pS_*(U - L; R) \otimes I^qS_*(U - L, U - K \cup L; R) \).
\end{footnotesize}
The argument of the proof of Lemma [7.74] is natural with respect to an inclusion \( B \to B' \), so taking \( B' = X - K \cap L \), so as \( \xi_{K \cap L} \) is the image of \( \xi_K \cup L \) under the inclusion \((X; \emptyset, X - K \cup \emptycup) \to (X; \emptyset, X - K \cap L)\), we obtain that the image of \( \partial(\xi_{K \cap L}) \) in \( H_{i+1}(IqS_*(X; R) \otimes IqS_*(X, X - K \cap L; R)) \) is also represented by \( \eta_{U-V} + \eta_{U \cap V} + \eta_{V - K} \). However, observe that 
\( (U - L) \cap (X - K \cap L) = U - L \) and \( (V - K) \cap (X - K \cap L) = V - K \), so 
\[
(IqS_* (U - L; R) \otimes IqS_*(U - L, U - K \cap L; R)) \\
+ (IqS_* (U \cap V; R) \otimes IqS_*(U \cap V, U \cap V - K \cap L; R)) \\
+ (IqS_* (V - K; R) \otimes IqS_*(V - K, V - K \cap L; R)) \\
= IqS_* (U \cap V; R) \otimes IqS_*(U \cap V, U \cap V - K \cap L; R),
\]
and so \( \partial(\xi_{K \cap L}) \) can be represented simply by \( \eta_{U \cap V} \).

Now, let \( \alpha \in IqH^j(X, X - K \cap L; R) = IqH^j(X, (X - K) \cap (X - L); R) \). Let us find a cochain representing \( d^*(\alpha) \in IqH^{j+1}(X, X - K \cap L; R) = IqH^{j+1}(X, (X - K) \cap (X - L); R) \). Treating \( \alpha \) as a cochain, \( d^*(\alpha) \) is determined by the output of a zig-zag chase in the Mayer-Vietoris short exact sequence 
\[
0 \to IqS^*(X, (X - K) + (X - L); R) \xrightarrow{\partial} IqS^*(X, X - K; R) \oplus IqS^*(X, X - L; R) \\
\to IqS^*(X, (X - K) \cap (X - L); R) \to 0.
\]
Here \( IqS^*(X, (X - K) + (X - L); R) = \text{Hom} \left( \frac{IqS_*(X; R)}{IqS_*(X - K; R) + IqS_*(X - L; R)}, R \right) \). So these are intersection cochains that vanish on intersection chains in \( IqS_*(X - K; R) \) or \( IqS_*(X - L; R) \). Of course, we already know, as usual for Mayer-Vietoris sequences, that the cohomology modules of this cochain complex are isomorphic to \( IqH^*(X, (X - K) \cup (X - L); R) \), which is the identification we always tacitly use in Mayer-Vietoris cohomology sequences. The zig-zag argument tells us that \( \alpha \in IqS^j(X, (X - K) \cap (X - L); R) \) must be the image of some \( \alpha_K \oplus \alpha_L \in IqS^j(X, X - K; R) \oplus IqS^j(X, X - L; R) \); in fact, pulling back by the Mayer-Vietoris inclusion map gives us \( \alpha = \alpha_K - \alpha_L \). Then we take \( d(\alpha_K \oplus \alpha_L) = d\alpha_K \oplus d\alpha_L \), and, \( \alpha \) being a cocycle, \( d\alpha_K \oplus d\alpha_L \) is in the image of the map labeled \( \partial \). In fact, the map \( \partial \) is the diagonal (up to restrictions), as we can verify from \( \partial \) being the dual of the map that adds two chains (up to inclusions). Therefore, \( d^*(\alpha) \) is represented by \( d\alpha_K \). Well, almost. Remember that \( d\alpha_K \in IqH^{j+1}(X, (X - K) + (X - L); R) \), while we want an element of \( IqH^{j+1}(X, (X - K) \cup (X - L); R) = IqH^{j+1}(X, X - K \cap L; R) \). The isomorphism between these modules is induced by the dual of the inclusion \( \mathfrak{t}: IqS^*(X, (X - K) + (X - L); R) \to IqS^*(X, X - K \cap L; R) \). So let \( \beta \in IqS^{j+1}(X, X - K \cap L; R) \) be a cochain such that \( \mathfrak{t}^*(\beta) = d\alpha_K \) in \( IqH^{j+1}(X, (X - K) + (X - L); R) \), say by \( \mathfrak{t}^*(\beta) - d\alpha_K = d\theta \). The \( \beta \) represents \( d^*(\alpha) \).

So the composition right then down in diagram (31) takes the class of \( \alpha \) to the class of \( \beta \), then restricts it to act on chains in \( U \cap V \), and finally forms \( \beta \sim \xi_{U \cap V}^{\xi_{K \cap L}} \). Note, here and in what follows, for simplicity of notation we will leave certain inclusion- and restriction-induced maps tacit; so, for example, the \( \beta \) in \( \beta \sim \xi_{U \cap V}^{\xi_{K \cap L}} \) is really the restriction of \( \beta \) to \( U \cap V \). This is not unreasonable, as both \( \beta \) and its restriction act the same way on chains; the context should be clear throughout, so this should not cause too much confusion.
Now, using our above observation that \( \bar{d}(\xi_{K \cap L}) \) can be represented by \( \eta_{U \cap V} \in I^p S_*(U \cap V; R) \otimes I^q S_*(U \cap V; R) \), we obtain that the image of the composition right then down diagram (31) is represented by \( \Phi(\text{id} \otimes \beta) \eta_{U \cap V} \). In fact, recall \( \eta_{U \cap V} \) represents a chain in \( I^p S_*(U \cap V; R) \otimes I^q S_*(U \cap V, U \cap V - K \cup L; R) \), and so also chains in \( I^p S_*(U \cap V; R) \otimes I^q S_*(U \cap V, U \cap V - K) \) and \( I^p S_*(U \cap V; R) \otimes I^q S_*(U \cap V, U \cap V - K \cap L; R) \) via inclusion. Therefore, invoking naturality, \( \Phi(\text{id} \otimes \beta) \eta_{U \cap V} \) and \( \Phi(\text{id} \otimes \xi^*(\beta)) \eta_{U \cap V} \) represent exactly the same intersection chain in \( U \cap V \), so we can just as well use \( \Phi(\text{id} \otimes \xi^*(\beta)) \eta_{U \cap V} \) as our representative for the composition in the diagram.

We next want to get back to an expression involving \( d\alpha_K \), rather than \( \beta \). For this, we recall that \( \eta = \eta_{U-L} + \eta_{U \cap V} + \eta_{V-K} \) represents a cycle in \( I^p S_*(X; R) \otimes I^q S_*(X, X-K \cup L; R) \). So

\[
\partial \eta = \partial \eta_{U-L} + \partial \eta_{U \cap V} + \partial \eta_{V-K} \in I^p S_*(X; R) \otimes I^q S_*(X - K \cup L; R).
\]

But we also have

\[
\eta_{U-L} \in I^p S_*(U - L; R) \otimes I^q S_*(U - L; R) \subset I^p S_*(X; R) \otimes I^q S_*(X - L; R)
\]
\[
\eta_{V-K} \in I^p S_*(V - K; R) \otimes I^q S_*(V - K; R) \subset I^p S_*(X; R) \otimes I^q S_*(X - K; R).
\]

Therefore, \( \partial \eta_{U \cap V} = \partial \eta - \partial \eta_{U-L} - \partial \eta_{V-K} \) is a chain in \( I^p S_*(X; R) \otimes (I^q S_*(X - K; R) + I^q S_*(X - L; R)) \). In fact, as \( \partial \eta_{U \cap V} \) is supported in \( U \cap V \), it is a chain, therefore, in \( I^p S_*(U \cap V; R) \otimes (I^q S_*(U \cap V - K; R) + I^q S_*(U \cap V - L; R)) \).

Now, recall that \( \xi^*(\beta) - d\alpha_K = d\theta \in I^q S^j(U \cap V, (X - K) + (X - L); R) \), and this relation remains under the restriction to \( I^q S^j(U \cap V, (U \cap V - K) + (U \cap V - L); R) \). Suppose
\[ \eta_{U \cap V} = \sum y_k \otimes z_k \in \mathcal{P}^n S_\ast(U \cap V; R) \otimes \mathcal{P}^m S_\ast(U \cap V; R). \]

Then we can compute

\[
(id \otimes \ell^\ast(\beta))\eta_{U \cap V} = (id \otimes (d\alpha_K + d\theta))\beta_{U \cap V} = (id \otimes d\alpha_K)\beta_{U \cap V} + (id \otimes d\theta)\beta_{U \cap V} = (id \otimes d\alpha_K)\beta_{U \cap V} + (id \otimes d\theta) \sum_k y_k \otimes z_k = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{(j+1)(i-1)} \sum_k y_k \otimes (d\theta)(z_k) = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{(j+1)(i-1)} \sum_k y_k \otimes (-1)^{j+1}\theta(\partial z_k) = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{(j+1)(i-1)+j+1} \sum_k y_k \otimes \partial z_k = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i+j} \sum_k y_k \otimes \partial z_k = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i+j+(i-1)} \sum_k (-1)^{|y_k|} y_k \otimes \partial z_k = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i-1} \sum_k (\partial(y_k \otimes z_k) - (\partial y_k) \otimes z_k) = (id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i-1} \sum_k (\partial y_k) \otimes z_k.
\]

Let us explain all this. In the fourth line, we have used that \((d\theta)(z_k) = 0\) unless \(|z_k| = |d\theta| = j + 1\), in which case the corresponding \(y_k\) has \(|y_k| = i - 1\). The next few lines are just computations and simplifications. In the fourth line from the bottom, we again use that all the terms on the right are trivial unless \(|y_k| = i - 1\), and this allows us to include the \((-1)^{|y_k|}\) in all terms balanced off by \((-1)^{i-1}\) outside the sum. In the last line, we have used that \(\theta\) kills elements of \(\mathcal{P}^n S_\ast(X - K; R) + \mathcal{P}^m S_\ast(X - L; R)\), and that we have seen that these are all that occur in the second tensor factors of \(\partial \eta_{U \cap V}\).

Now, applying \(\Phi\) to both sides of this computation, we get

\[
\Phi(id \otimes \ell^\ast(\beta))\eta_{U \cap V} = \Phi(id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i} \Phi(id \otimes \theta) \sum_k (\partial y_k) \otimes z_k = \Phi(id \otimes d\alpha_K)\beta_{U \cap V} + (-1)^{i+1} \sum_k \theta(z_k) \partial y_k.
\]

Therefore, \(\Phi(id \otimes \ell^\ast(\beta))\eta_{U \cap V}\) and \(\Phi(id \otimes d\alpha_K)\beta_{U \cap V}\) are homologous in \(\mathcal{P}^n S_\ast(U \cap V; R)\). So, the composition right then down in the diagram is represented by \(\Phi(id \otimes d\alpha_K)\eta_{U \cap V}\).

Now that we’ve gotten \(d\alpha_K\) back in the picture, we need to massage this just a bit more to fit with what we’ll get going around the diagram the other way. Continue to suppose
\[ \eta_{U \cap V} = \sum y_k \otimes z_k. \] We notice that
\[ \Phi(\text{id} \otimes \alpha_K) \partial \eta_{U \cap V} = \Phi(\text{id} \otimes \alpha_K) \partial \sum y_k \otimes z_k \]
\[ = \Phi(\text{id} \otimes \alpha_K) \sum ((\partial y_k) \otimes z_k + (-1)^{|y_k|} y_k \otimes \partial z_k) \]
\[ = \sum (-1)^{j|y_k|} \alpha_K(z_k) \partial y_k + \sum (-1)^{i-1+j(i-1)} \alpha_K(\partial z_k) y_k \]
\[ = \partial \left( \sum (-1)^{j|y_k|} \alpha_K(z_k) y_k \right) + \sum (-1)^{i-1+j(i-1)+j+1} \left( d\alpha_K(z_k) \right) y_k. \]
\[ = \sum (-1)^{j(i-1)+j} \partial \Phi(\text{id} \otimes \alpha_K)(y_k \otimes z_k) + \sum (-1)^{i+j(i-1)+j+1(i-1)} \Phi(\text{id} \otimes d\alpha_K) y_k \otimes z_k. \]
\[ = (-1)^j \partial \Phi(\text{id} \otimes \alpha_K) \eta_{U \cap V} + (-1)^{j+1} \Phi(\text{id} \otimes d\alpha_K) \eta_{U \cap V}. \]

Here, we have again used that \( \eta_{U \cap V} = \sum y_k \otimes z_k \) is an \( i+j \) chain, that \( \alpha \) is a \( j \)-cochain, and that the expressions above will vanish unless a cochain acts on a chain of the same degree. So, we see that up to signs, \( \Phi(\text{id} \otimes \alpha_K) \partial \eta_{U \cap V} \) and \( \Phi(\text{id} \otimes d\alpha_K) \eta_{U \cap V} \) together bound; therefore, \( \Phi(\text{id} \otimes \alpha_K) \partial \eta_{U \cap V} \) also represents the composition right then down in diagram (31), up to sign. At last, this is the final form that we want for this element.

Next, we consider the other way around the diagram (31). We first take the cap product of \( \alpha \) with \( \xi_{K \cup L} \), which is
\[ \alpha \smile \xi_{K \cup L} = \Phi(\text{id} \otimes \alpha) \hat{d}(\xi_{K \cup L}) \]
\[ = \Phi(\text{id} \otimes \alpha)(\eta_{U-L} + \eta_{U \cap V} + \eta_{V-K}) \]
\[ = \Phi(\text{id} \otimes \alpha) \eta_{U-L} + \Phi(\text{id} \otimes \alpha) \eta_{U \cap V} + \Phi(\text{id} \otimes \alpha) \eta_{V-K}. \]

The first of these chains is supported in \( U \) while the other two are supported in \( V \). Therefore, the zig-zag construction of the map \( \partial_s \) in the homology Mayer-Vietoris sequence can proceed by pulling our chain representative for \( \alpha \smile \xi_{K \cup L} \) back to
\[ \Phi(\text{id} \otimes \alpha) \eta_{U-L} \oplus (\Phi(\text{id} \otimes \alpha) \eta_{U \cap V} + \Phi(\text{id} \otimes \alpha) \eta_{V-K}) \in I^p S_i(U; R) \oplus I^p S_i(V; R), \]
then taking its boundary under \( \partial \oplus \partial \) and then finally arrive at a preimage in \( I^p S_i(U \cap V; R) \) under the map \((i_U, -i_V)\), with the \( i \) denoting the inclusion maps. In this case, the preimage is represented by \( \partial(\Phi(\text{id} \otimes \alpha) \eta_{U-L}) \). Now, using computations identical to those in (32), but with \( \alpha \) in place of \( \alpha_K \) and \( \eta_{U-L} \) in place of \( \eta_{U \cap V} \), we have that, up to signs,
\[ \partial(\Phi(\text{id} \otimes \alpha) \eta_{U-L}) = \pm \Phi(\text{id} \otimes \alpha) \partial \eta_{U-L} \pm \Phi(\text{id} \otimes d\alpha) \eta_{U-L}. \]

But \( \alpha \) is a cocycle, so this becomes \( \partial(\Phi(\text{id} \otimes \alpha) \eta_{U-L}) = \pm \Phi(\text{id} \otimes \alpha) \partial \eta_{U-L} \). Now, recall that \( \eta_{U-L} \), and so also its boundary, is in \( I^p S_s(U-L; R) \otimes I^p S_s(U-L; R) \), and that \( \alpha = \alpha_K - \alpha_L \),

103 These signs disagree with Hatcher [53, Lemma 3.36] because Hatcher’s version of the cap product has the chain on the left and the cochain on the right.
as above. But \( \alpha_L \) kills chains outside of \( L \), so \((\text{id} \otimes \alpha)L\partial U - L = (\text{id} \otimes \alpha)L\partial U - L\). Next, once again, as \( \overline{d}(\xi_{K \cup L}) \) is a cycle in \( I^pS_\ast(X; R) \otimes P^qS_\ast(X, X - K \cup L; R) \), we have that \( \partial \eta \), which represents \( \overline{d}(\xi_{K \cup L}) \), is in \( I^pS_\ast(X; R) \otimes P^qS_\ast(X, X - K \cup L; R) \). Now, apply \((\text{id} \otimes \alpha_K)\) to the expression \( \partial \eta = \partial \eta_{U - L} + \partial \eta_{U \cap V} + \partial \eta_{V - K} \) and observe that \((\text{id} \otimes \alpha_K)\) must kill \( \partial \eta_{V - K} \) and \( \partial \eta \), which are made of chains supported outside of \( K \) (in the second tensor factor). Therefore,

\[
\Phi(\text{id} \otimes \alpha_K)\partial \eta_{U - L} = -\Phi(\text{id} \otimes \alpha_K)\partial \eta_{U \cap V}.
\]

So our representative for the image of \( \alpha \) down then right in diagram (31) is, up to sign, \( \Phi(\text{id} \otimes \alpha_K)\partial \eta_{U \cap V} \). And, again up to sign, this is the same expression we obtained earlier for running right then down in diagram (31).

This completes the third square and so the proof of Lemma 7.90.

8 Poincaré duality

8.1 PL motivation using intersection products

This section needs to be written. Please see [42] for now.

8.2 Orientations and fundamental classes

In this section, we continue to head toward an intersection homology version of Poincaré duality by constructing orientations and fundamental classes for stratified pseudomanifolds. Notice that we are here restricting ourselves from the larger generality of CS sets down to spaces with a bit more structure. Stratified pseudomanifolds are required to be recursive CS sets and to have a dense union of regular strata. This latter condition is necessary to have something that is dimensionally homogeneous, which we need in order for all points to be able to carry anything like an orientation in the proper degree. Such a restriction is not completely necessary for Poincaré duality as, by Proposition 6.47, non-GM intersection homology does not detect strata outside the homogenization of a CS set. But this is also an argument that we might as well restrict our attention to the homogeneous CS sets. The idea for using recursive CS sets is that orientation properties are local and so we will need to demonstrate the proper homological properties on distinguished neighborhoods, and these properties are completely controlled by the links, via stratified homotopy invariance and the cone formula. One alternative would be to simply make the necessary homological assumptions about the links, which could perhaps be done. However, a pleasant feature of stratified pseudomanifolds is that their links are also stratified pseudomanifolds by Lemma 2.54, and so the local homological properties will exist inductively via the fundamental class of the link.

The arguments in this section are based on those of [38, Section 5], which are themselves based on the arguments for manifolds in Section 3.3 of Hatcher [53]. The primary difference

\footnote{Recall Definition 6.44 in Section 6.3.1}
from [33] is that there we first developed the results for normal stratified pseudomanifolds and then made additional arguments to obtain them for arbitrary pseudomanifolds using the properties of normalization maps. Here, we take a more direct route, treating arbitrary stratified pseudomanifolds throughout.

WARNING: In this section, we require some elementary sheaf theory at a variety of points. While we attempt to provide an overview of the relevant notions where necessary, the reader should be aware that not all of this section will be self-contained given our development thusfar. Good references for most of what we need can be found in the first few chapters of Swan [100] or Bredon [13], each of which develops sheaf theory considerably more than we will need here.

8.2.1 Orientation and fundamental classes of manifolds

Let us first briefly review the principal notions concerning orientation and fundamental classes for manifolds. One good reference, and the one we will mostly follow in our treatment below for pseudomanifolds, can be found in [53, Section 3.3].

Recall that, on an \( n \)-dimensional manifold \( M \) and for any coefficient ring \( R \), we have an orientation bundle \( O \) with stalks \( O_x = H_n(M, M - \{x\}; R) \cong R \). At any moment, we will work with a fixed base ring, so we will omit it from the notation for the orientation bundle. The bundle structure is determined by noting that every point in \( M \) has an open ball neighborhood \( B \) such that \( H_n(M, M - B; R) \cong R \) and for any two \( x, y \in B \), we have canonical isomorphisms induced by inclusion \( H_n(M, M - \{x\}; R) \cong H_n(M, M - B; R) \cong H_n(M, M - \{y\}; R) \). Hence every point of \( M \) has a neighborhood on which we have a canonical identification between fibers, and this determines the bundle \( O \). The manifold \( M \) is \( R \)-orientable if \( O \) has a global section \( o \) that restricts to a generator of \( R \) over each point; this is equivalent to assuming that the bundle \( O \) is trivial. In particular, every manifold is \( \mathbb{Z}_2 \)-orientable. An \( R \)-orientation is a choice of such a global section.

Of course, there are also standard homological results about manifolds that are closely intertwined with their orientation properties. The following lemma, which is a restatement of [53 Lemma 3.27], lays the principal cornerstone:

Lemma 8.1. Let \( M \) be an \( n \)-dimensional manifold and \( K \subset M \) a compact set. Then:

1. \( H_i(M, M - K; R) = 0 \) for \( i > n \), and a class in \( H_n(M, M - K; R) \) is zero if and only if its image in \( H_n(M, M - x; R) \) is zero of all \( x \in K \).

2. Given a section \( s \) of \( O \) over \( M \), there is a unique class \( \gamma_K \in H_n(M, M - K; R) \) whose image in \( H_n(M, M - \{x\}; R) \) is \( s(x) \) for any \( x \in K \). In particular, if \( M \) is \( R \)-oriented, there is a unique class \( \Gamma_K \in H_n(M, M - K; R) \) whose image in \( H_n(M, M - \{x\}; R) \) is \( o(x) \) for any \( x \in K \).

This lemma leads to following theorem, which is two thirds of [53 Theorem 3.26]:

Theorem 8.2. Let \( M \) be a closed connected \( n \)-manifold. Then:

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1. If $M$ is $R$-orientable, then $H_n(M;R) \to H_n(M, M - \{x\}; R) \cong R$ is an isomorphism for all $x \in M$.

2. $H_i(M; R) = 0$ for $i > n$.

The last statement of the theorem follows immediately from the first statement of the lemma, taking $K = M$. The first statement of the theorem follows from the second statement of the lemma as follows: As $M$ is connected and $R$-oriented, the orientation bundle $\mathcal{O}$ is the trivial bundle with stalk $R$, and its module of sections $\Gamma(M, \mathcal{O})$ is therefore isomorphic to $R$ via the evaluation map that takes $s \in \Gamma(M, \mathcal{O})$ to $s(x) \in \mathcal{O}_x = H_n(M, M - \{x\}; R) \cong R$ for any $x \in M$. In particular, $o$ is a generator of $\Gamma(M, \mathcal{O}) \cong R$. Now, any element of $\xi \in H_n(M; R)$ determines a global section $s_\xi$ of $\mathcal{O}$ by letting $s_\xi(x)$ be the image of $\xi$ in $H_n(M, M - \{x\}; R)$. The map $H_n(M; R) \to \Gamma(M, \mathcal{O})$ so described is both surjective and injective by the second statement of the lemma, taking $K = M$. Thus we have isomorphisms $H_n(M; R) \cong \Gamma(M, \mathcal{O}) \cong H_n(M, M - \{x\}; R) \cong R$ for all $x \in M$.

**Definition 8.3.** We call the element of $H_n(M; R)$ corresponding to a given $R$-orientation $o$ of $M$ the fundamental class $\Gamma_M \in H_n(M; R)$.

The proof of Lemma 8.1 requires some work. We will not discuss this here, but rather we refer again to [53]. Additionally, a proof for $R$-orientable manifolds will follow as a special case of our more general development for stratified pseudomanifolds in the following sections.

### 8.2.2 Orientation of pseudomanifolds

Now, we turn to pseudomanifolds. In fact, the basic definitions concerning orientation are applicable more generally to CS sets; we will only need to restrict to pseudomanifolds when considering more precise homological issues, so we will work in the greater generality until then.

**Definition 8.4.** Let $X$ be an $n$-dimensional CS set. We say that $X$ is $R$-orientable if the $n$-manifold $X - X^{n-1} = X - \Sigma_X$ is $R$-orientable, and we say $X$ is $R$-oriented if $X - X^{n-1} = X - \Sigma_X$.

Before launching into the (intersection) homological properties related to orientability on a stratified space, we will discuss the relationship between orientations of different stratifications of a single space. We begin with a simple observation.

**Lemma 8.5.** Suppose $X$ is an $n$-dimensional CS set and that $X'$ is an $n$-dimensional CS set with the same underlying space $|X|$ and with a finer stratification, i.e. each stratum of $X'$ is contained in a stratum of $X$. Then, if $X$ is $R$-orientable, so is $X'$, and any $R$-orientation of $X$ determines a unique $R$-orientation of $X'$.

**Proof.** The assumption that the stratification of $X'$ is finer than that of $X$ implies immediately that $X' - \Sigma_{X'} \subset X - \Sigma_X$. So if the orientation bundle over $X - \Sigma_X$ is trivial, it restricts to a trivial bundle over $X' - \Sigma_{X'}$, and any choice of global section of generators similarly restricts. \qed

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Unfortunately, the converse to Lemma 8.5 is not true in general, as the following examples demonstrate.

**Example 8.6.** Let $M$ be the unstratified open Mobius band, and let $M'$ be $M$ restratified as as $\emptyset \subset \Sigma \subset M$, where $\Sigma$ is an arc running widthwise across $M$. In other words, let $M$ be formed from $[0,1] \times (0,1)$ by identifying $\{0\} \times (0,1)$ with $\{1\} \times (0,1)$ by $(0,t) \sim (1,1-t)$, and let $\Sigma$ be the image of $\{0,1\} \times (0,1)$. Of course $M$ is not $\mathbb{Z}$-orientable, but $M' - \Sigma_{M'} = M - \Sigma$ is homeomorphic to the open disk, so it is $\mathbb{Z}$-orientable.

Example 8.6 shows that it is possible to have one stratification of a space be $R$-orientable while another is not. The next example shows that, even if both stratifications yield $R$-orientable CS sets, an $R$-orientation on a finer stratification does not necessarily determine an $R$-orientation on a coarser stratification.

**Example 8.7.** Let $\mathbb{R}'$ consist of the space $\mathbb{R}$ stratified by $\{0\} \subset \mathbb{R}$. Then $\mathbb{R}'$ has two regular strata, corresponding to the positive and negative real numbers, and we can orient these submanifolds, and hence $\mathbb{R}'$, in a way that is not compatible with a single orientation on all of the CS set $\mathbb{R}$.

The trouble in both of these examples is caused by the addition of a stratum of codimension one. It turns out that Lemma 8.5 does have a converse if we forbid the addition of “new” codimension one strata. The two following lemmas utilize some sheaf-theoretic notions of dimension theory that go even a bit further beyond the elementary treatment of sheaves that we will need more seriously below. The reader unacquainted with such notions should still be able to follow the general idea of the proof, though the reader willing to believe the result can safely skip the argument, which won’t be needed again later.

**Lemma 8.8.** Suppose $X$ is an $n$-dimensional CS set and that $X'$ is an $n$-dimensional CS set with the same underlying space $|X|$ and with a finer stratification, i.e. each stratum of $X'$ is contained in a stratum of $X$. Suppose further that any codimension one stratum of $X'$ is contained in a codimension one stratum of $X$. Then if $X'$ is $R$-orientable, so is $X$, and any $R$-orientation of $X'$ determines a unique $R$-orientation of $X$.

**Proof.** Let $M = X - \Sigma_X$ and $M' = X' - \Sigma_{X'}$. The spaces $M$ and $M'$ are $n$-dimensional manifolds, and as $X'$ is stratified more finely than $X$, we have $M' \subset M$. We claim that $M'$ is an open dense subset of $M$ such that $M - M'$ has codimension at least 2, utilizing the sheaf-theoretic $\dim_\mathbb{Z}$ as our notion of dimension (see Lemma 6.46 above, and [13] Section II.16) for a full treatment). This will imply the lemma using a result about bundle theory that we provide below. It is clear that $M'$ is open in $M$ as $M$ and $M'$ are both open subsets of the underlying space $|X|$. It is also immediate that $M'$ is dense in $M$, as if $x \in M$ then any neighborhood of $x$ must intersect an $n$-dimensional stratum of any restratification of $X$. For the issue of codimension, as $M$ is an $n$-manifold, $\dim_\mathbb{Z}(M) = n$ by [13] Corollary II.16.28. To see that the complement of $M'$ in $M$ has dimension $\leq n - 2$, suppose $x \in M - M'$. Then $x$ must be contained in a singular stratum of $X'$, and we claim it is not a codimension one singular stratum. But, by assumption, if $x$ is contained in a codimension one stratum of $X'$ then it is contained in a codimension one stratum of $X$, contradicting $x \in M$. Therefore, $M - M'$ is an open subset of the $n - 2$ skeleton of $X'$. We showed above in the proof of
Lemma 6.46 that the $i$-skeleton of a CS set has $\mathbb{Z}$-dimension $\leq i$, and it follows that any open subset of such a skeleton also has $\mathbb{Z}$-dimension $\leq i$ by [13, Theorem II.16.8]. Thus $\dim_{\mathbb{Z}}(M - M') \leq n - 2$. So we have shown that $M'$ is an open dense subset of $M$ such that $M - M'$ has codimension $\geq 2$.

The lemma now follows from a basic result about bundle theory, which we present as Lemma 8.9 below. In our setting, Lemma 8.9 says that an isomorphism of bundles over $M'$ must extend uniquely over all of $M$. In particular, suppose $\mathcal{O}_M$ is the $R$-orientation bundle over $M$ and that $\mathfrak{R}_M$ is the trivial bundle over $M$ with stalk $R$. If $X'$ is $R$-orientable, there is an isomorphism $\mathfrak{R}_M|_{M'} \rightarrow \mathcal{O}_M|_{M'} = \mathcal{O}_{M'}$, and an $R$-orientation corresponds to a specific choice of isomorphism (given a fixed identification of $\mathfrak{R}_M$ with $M \times R$, the $R$-orientation is the image section under the bundle isomorphism of the section of $\mathfrak{R}_M|_{M'}$ that takes each $x \in M'$ to $1 \in R$). Lemma 8.9 guarantees a unique extension of this isomorphism uniquely to all of $M$, demonstrating that $M$ is orientable and extending uniquely the chosen $R$-orientation.

The following lemma is basic to bundle theory, though the techniques required for the proof are a bit beyond the required background of most this book. We’ll provide the proof from [8, Lemma V.4.11], though, once again, the reader willing to believe the result can safely skip the argument, which won’t be needed again later.

**Lemma 8.9.** Suppose $M$ is an $n$-dimensional manifold and that $U$ is a dense open subset of $M$ whose complement has codimension at least 2. Then if $\mathcal{E}$ and $\mathcal{F}$ are bundles of coefficients on $M$ and $\phi : \mathcal{E}|_U \rightarrow \mathcal{F}|_U$ is a bundle morphism, then there exists a unique bundle morphism $\psi : \mathcal{E} \rightarrow \mathcal{F}$ that extends $\phi$. Furthermore, if $\phi$ is an isomorphism then so is $\psi$.

**Proof.** Without loss of generality, we can assume $M$ is path connected; otherwise we can argue on each path component separately. It follows that $U$ must also be path connected. The basic idea is that, because $M - U$ has codimension $\geq 2$, any path can be altered by a homotopy to avoid $M - U$. If we were working entirely with smooth objects, this would follow from general position arguments, but as our objects are purely topological, this is not completely straightforward. We will use the fact that, if $Y$ is a locally compact Hausdorff space and $W$ is an open subspace, there is a long exact sequence

$$
\cdots \rightarrow \mathbb{H}^i_c(W; \mathbb{Z}_2) \rightarrow \mathbb{H}^i_c(Y; \mathbb{Z}_2) \rightarrow \mathbb{H}^i_c(Y - W; \mathbb{Z}_2) \rightarrow \cdots,
$$

where $\mathbb{H}^i$ denotes sheaf cohomology. The existence of such a long exact sequence is not so evident using the singular cohomology definition of $H^i_c$, but it follows for $\mathbb{H}^i_c$ from basic sheaf cohomology theory [105, 13, Section II.10.3]. Now, $\dim_{\mathbb{Z}}(M - U) \leq n - 2$ by assumption, and as $M - U$ is locally compact, it is locally paracompact, so, from [13, Proposition II.16.15], we also have $\dim_{\mathbb{Z}_2}(M - U) \leq n - 2$. By [13, Definition II.16.6], $\dim_{\mathbb{Z}_2}(M - U) = \dim_{c, \mathbb{Z}_2}(M - U)$,

105Note that the exact sequence in Bredon is stated for any paracompactifying family of supports $\Phi$. In our case, $\Phi = c$, the family of compact supports, and the restrictions of $\Phi|W$ and $\Phi|Y - W$ are also $c$, as follows directly from [13, Definition II.6.3]. In each case, $c$ is paracompactifying by the observations of [13, page 22], as all of the spaces here are locally compact. Note that open and closed subspaces of locally compact Hausdorff spaces are locally compact [78, Corollary 29.3].
and so we have \( H^i_c(M - U; \mathbb{Z}_2) = 0 \) for \( i = n - 1 \) and \( i = n \) by \cite{13} Theorem II.16.4. Thus, from the exact sequence, \( H^n_c(U; \mathbb{Z}_2) \cong H^n_c(M; \mathbb{Z}_2) \). Finally, by Poincaré duality \cite{13} Example IV.2.9], as \( U \) and \( M \) are \( n \)-manifolds and as the \( \mathbb{Z}_2 \)-orientation sheaf of any space is constant, we have \( H^n_c(U; \mathbb{Z}_2) \cong H_0(U; \mathbb{Z}_2) \) and \( H^n_c(M; \mathbb{Z}_2) \cong H_0(M; \mathbb{Z}_2) \). So, if \( M \) is path connected, \( H_0(U; \mathbb{Z}_2) \cong H_0(M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \), and \( U \) is also path connected.

The same argument works locally to show that if \( B \) is an open ball neighborhood of a point in \( M \), then \( U \cap B \) is path connected and dense in \( B \). This observation is enough to imply that, for any basepoint \( x_0 \in U \), \( i_* : \pi_1(U, x_0) \to \pi_1(M, x_0) \) is surjective, where \( i \) is the inclusion \( U \hookrightarrow M \). Indeed, this is a basic exercise (that we leave for the reader) in locally modifying paths by homotopies within small balls.

Now, as \( M \) is path connected, the category of bundles on \( M \) whose stalks are finitely generated \( R \)-modules is equivalent to the category of finitely generated \( \pi_1(M, x_0) \)-modules (see \cite{53} Section 3H), and similarly for \( U \). Furthermore, restriction of a bundle to \( U \) corresponds to the change of scalars induced by \( i_* : \pi_1(U, x_0) \to \pi_1(M, x_0) \). Recall that this means that if \( E \) is a \( \pi_1(M, x_0) \)-module and \( z \in E \), then we obtain a corresponding module, say \( E_U \), with the same elements as \( E \) (though we will denote the version of \( z \) in \( E_U \) as \( z_U \)) and with action of \( \gamma \in \pi_1(U, x_0) \) on \( E_U \) given by \( \gamma z_U = (i_*(\gamma)z)_U \).

The hypothesis of the lemma is equivalent to assuming that we have two \( \pi_1(U, x_0) \)-modules, say \( E, F \), and a \( \pi_1(U, x_0) \)-morphism \( \phi : E_U \to F_U \), where \( E_U \) and \( F_U \) denote \( E \) and \( F \) as modules after the restriction of scalars. We must show that there is a unique morphism \( \psi : E \to F \) that induces \( \phi \). But \( E = E_U \) and \( F = F_U \) as groups, so \( \phi \) certainly provides a function \( \psi \) determined by \( (\psi(z))_U = \phi(z_U) \) for \( z \in E \). We must show that \( \psi \) is a morphism of \( \pi_1(M, x_0) \)-modules by showint that \( \psi(mz) = m\psi(z) \) for any \( m \in \pi_1(M, x_0) \).

As \( i_* \) is surjective, we can choose \( \bar{m} \in \pi_1(U, x_0) \) such that \( i_*(\bar{m}) = m \). Then, using that \( \phi \) is a map of \( \pi_1(U, x_0) \)-modules, we have

\[
\begin{align*}
(\psi(mz))_U &= \phi((mz)_U) \\
&= \phi((i_*(\bar{m})z)_U) \\
&= \phi(\bar{m}z_U) \\
&= \bar{m}\phi(z_U) \\
&= \bar{m}(\psi(z))_U \\
&= (m\psi(z))_U.
\end{align*}
\]

As the assignment \( z \to z_U \) is bijective, we thus see that \( \psi \) is a \( \pi_1(M, x_0) \)-module morphism, as desired. It is also clearly the unique such morphism compatible with \( \phi \), and it is bijective if and only if \( \phi \) is bijective. This completes the proof.

\[ \qed \]

**Corollary 8.10.** Suppose \( X \) is a CS set all of whose codimension one strata are contained in the codimension one strata of \( X^* \), the intrinsic stratification of \( X \) (recall Definition 2.102); in particular, this will be the case if \( X \) has no codimension one strata. Then any \( R \)-orientation of \( X \) determines a unique \( R \)-orientation for any CS set stratification \( X' \) of the underlying space \( |X| \).

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Proof. Recall that $X^*$ is a coarsening of any CS set stratification of $|X|$; see Remark \[2.103\]. Therefore, by Lemma \[8.8\] and the assumptions, an $R$-orientation of $X$ determines a unique $R$-orientation of $X^*$, which determines a unique $R$-orientation on any other stratification $X'$ of $|X|$ by Lemma \[8.5\].

Remark 8.11. As just observed in the proof of Corollary \[8.10\], it follows from Lemma \[8.5\] that any $R$-orientation of $X^*$ determines an $R$-orientation on any CS set stratification of $|X|$. As the CS set stratification $X^*$ is intrinsic to the underlying space, this provides a sense in which the notion of $R$-orientability is stratification independent, assuming that $X^*$ is $R$-orientable. However, Example \[8.6\] just above, demonstrates that we might still have $R$-orientability of $X$ even when $X^*$ is not $R$-orientable.

8.2.3 Homological properties of orientable pseudomanifolds

In this section, we look at intersection homology versions of Lemma \[8.1\] and Theorem \[8.2\]; in particular we construct intersection homology fundamental classes for oriented stratified pseudomanifolds. An immediate question is what perversity we should use. This will turn out to be immaterial, so long as $\bar{p} \geq 0$, i.e. if $\bar{p}(S) \geq 0$ for all singular strata $S$. In this section, we will proceed simultaneously with all such perversities, and in the next we will show that, in fact, the top dimension intersection homology groups behave identically with respect to any of these perversities, so there is no real distinction. In later sections, we will work with $0$, which is initial among all these perversities and so provides a canonical fundamental class. It is the cap product with this 0-perversity fundamental class that takes $\bar{p}$-allowable intersection cochains to $D\bar{p}$-allowable intersection chains, demonstrating again why the notion of dual perversities is so relevant. On the other hand, if $\bar{p}(S) < 0$ for some singular stratum $S$, then we will see below, as a consequence of Proposition \[8.21\], that it is not possible for $X$ to have a (global) fundamental class.

The orientation sheaf. We begin with a preliminary observation that, if $X$ is an $n$-dimensional CS set and $\bar{p}$ is a perversity satisfying $\bar{p} \geq 0$, the assignment $x \mapsto I^\bar{p}H_n(X, X - \{x\}; R)$ will no longer be locally constant, and, in fact, at singular points of $X$ we do not necessarily have $I^\bar{p}H_n(X, X - \{x\}; R) \cong R$. Therefore, we cannot talk about an orientation bundle over all of $X$, though we do still have such a bundle over $X - \Sigma X$. The appropriate object on all of $X$ is a sheaf. Sheaves generalize bundles of coefficients by allowing the possibility of different modules over different points. We will only need a bare minimum of material about sheaves, but since we do not assume the reader is necessarily familiar with any sheaf theory, we provide the basic idea. A readable elementary account from the following point of view can be found in the early chapters of Swan \[100\].

There are actually multiple equivalent definitions of sheaves, but for our purposes the simplest is the following: like a bundle of coefficients, a sheaf $S$ of $R$-modules over a space $Y$ is a space $S$ together with a local homeomorphism $\pi : S \rightarrow Y$, meaning that for each $z \in S$, $\pi$ takes some neighborhood $V$ of $z$ in $S$ homeomorphically onto a neighborhood of $\pi(z)$ in $Y$. Additionally, for each $x \in Y$, $\pi^{-1}(x)$, which is also denoted $S_x$ and called the stalk of the sheaf at $x$, must be an $R$-module (with the discrete topology). It is not required that the various
$\mathcal{S}_x$ be isomorphic to each other. Finally, there is also a requirement that algebraic operations should be continuous; in other words, the map $\{(y, z) \in \mathcal{S} \times \mathcal{S} \mid \pi(y) = \pi(z) \in V\} \to \mathcal{S}$
given by $(y, z) \to y + z$ must be continuous, and an analogous statement holds for scalar multiplication. A bundle of coefficients with fiber module $F$ is simply a sheaf for which each point in $Y$ has a neighborhood $V$ on which $\pi^{-1}(V) \cong V \times F$ with $\pi$ corresponding to the projection onto $V$. In general, however, while each stalk of a sheaf $\mathcal{S}$ must inherit the discrete topology as a subspace of $\mathcal{S}$, the overall topology of $\mathcal{S}$ might be quite complicated and is very often non-Hausdorff.

In our particular case, the orientation sheaf $\mathcal{O}^p$ on $X$ will have stalks $\mathcal{O}^p_x = I^pH_n(X, X - \{x\}; R)$, and the topology on $\mathcal{O}^p$ is given so that if $U \subset X$ is any open subset of $X$ and $\xi \in I^pH_n(X, X - U; R)$ is any homology class, then the union of images of $\xi$ in $I^pH_n(X, X - \{x\}; R)$, as $x$ runs over all points of $U$, is an open subset of $\mathcal{O}^p$. One can then check that the set of subsets of $\mathcal{O}^p$ does yield a topology such that the projection $\pi : \mathcal{O}^p \to X$ is a local homeomorphism; see [13, Example I.1.11]. We call $\mathcal{O}^p$ the $R$-orientation sheaf on $X$ with perversity $p$.

Remark 8.12. A useful consequence of the local homeomorphism property of the sheaf map $\pi : \mathcal{S} \to Y$ is that if $s$ and $t$ are any two sections of $\mathcal{S}$ defined on an open set $U \subset Y$ and if $s(y) = t(y)$ for some $y \in U$, then $\{z \in U \mid s(z) = t(z)\}$ is an open subset of $U$; see [100 Section II.2.1]. This takes a bit of getting used to for those of us who generally work with Hausdorff topologies!

In particular, for our sheaf $\mathcal{O}^p$ over a CS set $X$, this has the following consequence: Suppose that $[\xi] \in I^pH_n(X, X - \{x\}; R)$ represents a value in the stalk of $\mathcal{O}^p$ at $x$. Representing $[\xi]$ as a specific chain $\xi$, let $U = X - |\partial \xi|$. Then $\xi$ also represents an element $[\xi'] \in I^pH_n(X, X - U; R) = I^pH_n(X, |\partial \xi|; R)$. We can now define a section $\tilde{s}_\xi$ of $\mathcal{O}^p$ over $U$ by letting $\tilde{s}_\xi(z), z \in U$, equal the element of $I^pH_n(X, X - \{z\}; R)$ represented by the chain $\xi$. Clearly, $\tilde{s}_\xi(x) = [\xi'] \in I^pH_n(X, X - \{x\}; R)$. So $\tilde{s}_\xi$ is a section whose value at $x$ is the element of $\mathcal{O}^p_x$ that we started with. Now, suppose that $t$ is any other section of $\mathcal{O}^p$ defined in a neighborhood of $x$ and such that $t(x) = s_\xi(x)$. Due to our general observation about sections, $t$ and $\tilde{s}_\xi$ will take equal values in some open neighborhood of $x$, i.e. $t = s_\xi$ on some neighborhood of $x$ (though not necessarily on all of $U$). This argument shows that every section of $\mathcal{S}$ defined on an open set $U$ is determined locally (i.e. in some neighborhood $V_x$ of each point $x \in U$) as the set of images in the $I^pH_n(X, X - \{z\}; R), z \in V_x$, of some homology class in some $I^pH_n(X, X - V; R)$. This will be a useful observation below.

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106For the reader either familiar with some sheaf theory or who wants to compare our rough description here with the details in [13], our sheaf $\mathcal{O}^p$ is really the sheafification of the presheaf $U \to I^pH_n(X, X - U; R)$. Technically, the stalk of this presheaf at $x \in X$ is $\lim_{\overrightarrow{\delta} U \subseteq U} I^pH_n(X, X - U; R)$, but we are free to replace this direct limit with one over a cofinal system of distinguished neighborhoods of $x$. But we can choose such a sequence of distinguished neighborhoods, $\cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots$, so that the inclusion-induced maps $I^pH_n(X, X - N_i; R) \to I^pH_n(X, X - N_{i+1}; R) \to I^pH_n(X, X - \{x\}; R)$ are all isomorphisms by stratified homotopy invariance. So the direct system over these distinguished neighborhoods is constant, and the direct limit is isomorphic to $I^pH_n(X, X - \{x\}; R)$.
**Homological theorems.** Now, we turn to a version of Lemma 8.1 in the stratified case, which will be Lemma 8.15 below. The proof of Lemma 8.15, which concerns global intersection homology classes, in particular fundamental classes, will be somewhat intertwined with another lemma, Lemma 8.13, which is concerned with properties of the orientation sheaf, and so, in particular, with local intersection homology. We will perform an induction on depth that requires Lemma 8.15 at depth \( d \) to prove Lemma 8.13 at depth \( d \), which is needed for Lemma 8.15 at depth \( d \), and so on. Thus, we will state both results together, as well as the theorem to which they lead, which will be our stratified version of Theorem 8.2, and then we will move on to the proofs.

We remark that, even though Lemma 8.1 has some parts that do not require orientability, our main interest here is in orientable pseudomanifolds, and so we will make that assumption throughout in order to avoid making our web of intertwined arguments any more complicated than it already is. Such an assumption will also be useful in our induction steps. Additionally, while our construction of fundamental classes will require perversities \( \bar{p} \) with \( \bar{p} \geq 0 \), we will also prove some results concerning intersection homology in degrees \( i > \dim(X) \) for arbitrary perversities. Throughout the proofs in this section, we will use \( \bar{q} \) to denote an arbitrary perversity and \( \bar{p} \) to denote a perversity with \( \bar{p} \geq 0 \).

Here is our lemma concerning properties of the orientation sheaf \( \mathcal{O}^\bar{p} \):

**Lemma 8.13.** Let \( R \) be a Dedekind domain, and let \( X \) be an \( R \)-oriented \( n \)-dimensional stratified pseudomanifold. Let \( \bar{p} \) be a perversity with \( \bar{p} \geq 0 \). Then the following statements hold:

1. For all \( x \in X \) and all \( i > n \), \( \text{IH}_i(X, X - \{x\}; R) = 0 \) for any perversity \( \bar{q} \).
2. Any section defined over \( X - \Sigma_X \) of the sheaf \( \mathcal{O}^\bar{p} \) extends uniquely to a section of \( \mathcal{O}^\bar{p} \) on all of \( X \). In particular, there is a unique global section \( \sigma^\bar{p} \) of the sheaf \( \mathcal{O}^\bar{p} \) that restricts to the given \( R \)-orientation on \( X - \Sigma_X \), and, if a section \( s \) of \( \mathcal{O}^\bar{p} \) is such that \( s(x) = 0 \) for all \( x \in X - \Sigma_X \), then \( s(x) = 0 \) for all \( x \in X \).

**Definition 8.14.** We call the section \( \sigma^\bar{p} \) of \( \mathcal{O}^\bar{p} \) the orientation section of \( X \) with respect to the perversity \( \bar{p} \).

Now we have our lemma concerning fundamental classes:

**Lemma 8.15.** Let \( R \) be a Dedekind domain, and let \( X \) be an \( R \)-oriented \( n \)-dimensional stratified pseudomanifold. Let \( \bar{p} \) be a perversity with \( \bar{p} \geq 0 \). Let \( K \subset X \) be a compact subset. Then:

1. \( \text{IH}_i(X, X - K; R) = 0 \) for \( i > n \) and for any perversity \( \bar{q} \).
2. Given a section \( s \) of \( \mathcal{O}^\bar{p} \) over \( X \), there is a unique class \( \gamma \in \text{IH}_n(X, X - K; R) \) whose image in \( \text{IH}_n(X, X - \{x\}; R) \) is \( s(x) \), for any \( x \in K \). In particular, if \( \sigma^\bar{p} \) is an \( R \)-orientation section for \( X \), there is a unique class \( \Gamma^\bar{p}_K \in \text{IH}_n(X, X - K; R) \) whose image in \( \text{IH}_n(X, X - \{x\}; R) \) is \( \sigma^\bar{p}(x) \), for any \( x \in K \), and, if \( \gamma \in \text{IH}_n(X, X - K; R) \), then \( \gamma \) restricts to 0 in \( \text{IH}_n(X, X - \{x\}; R) \) for all \( x \in K \) if and only if \( \gamma = 0 \).
Definition 8.16. We call the class $\Gamma_K$ of Lemma 8.15 the fundamental class of $X$ over $K$ with respect to the chosen $R$-orientation.

These results lead to the following important theorem:

Theorem 8.17. Let $R$ be a Dedekind domain, and let $X$ be a compact $R$-oriented $n$-dimensional stratified pseudomanifold with perversity $\bar{p} \geq 0$. Then:

1. $I^pH_i(X; R) = 0$ for $i > n$ and for any perversity $\bar{q}$.

2. There is a unique class $\Gamma^p_X \in I^pH_n(X; R)$ whose image in $I^pH_n(X, X - \{x\}; R)$, for any $x$, corresponds to the image of the orientation section $\sigma^p(x)$.

3. If $\{x_j\}_{j=1}^m$ is a collection of points of $X$, one in each regular stratum, then $I^pH_n(X; R) \cong \oplus_j I^pH_n(X, X - \{x_j\}; R) \cong R^m$ via the map that takes an element of $I^pH_n(X; R)$ to the direct sum of its images in the $I^pH_n(X, X - \{x_j\}; R)$.

Definition 8.18. We call the class $\Gamma^p_X$ of Lemma 8.15 the fundamental class of $X$ with respect to the chosen $R$-orientation and perversity $\bar{p}$. If $\bar{p} = 0$, we write simply $\Gamma_X$ and call $\Gamma_X$ the fundamental class of $X$ with respect to the chosen $R$-orientation.

Proof of Theorem 8.17, assuming Lemmas 8.13 and 8.15. The first two statements follow immediately from Lemma 8.15 by taking $K = X$ and by using Lemma 8.13 to guarantee the existence of the orientation section $\sigma^p$.

For the last statement, we know by Proposition 6.47 that, if $\{R_j\}$ are the regular strata of $X$, then $I^pH_n(X; R) \cong \oplus_j I^pH_n(\overline{R}_j; R)$. Let $\Gamma^p_{X,j}$ denote the component of $\Gamma^p_X$ in $I^pH_n(\overline{R}_j; R)$. As $\Gamma^p_{X,j}$ is supported in $\overline{R}_j$, and as the image of $\Gamma^p_{X,j}$ in $I^pH_n(\overline{R}_j; R)$ is the generator $\sigma^p(x_j)$, it follows that the image of $\Gamma^p_{X,j}$ in $I^pH_n(X, X - \{x_j\}; R)$ is also $\sigma^p(x_j)$, while the image of $\Gamma^p_{X,j}$ in $I^pH_n(\overline{R}_j; R)$ is $0$ for $k \neq j$. Therefore, the map $I^pH_n(X, X - \{x_j\}; R)$ is surjective, with $\Gamma^p_{X,j}$ mapping onto the element of $R^m$ that is a generator in the $j$th slot and $0$ in the other slots.

For injectivity, we observe that, for any $\xi \in I^pH_n(X; R)$, there is a section $s_X$ of $O^p$ given by letting $s_X(x)$ be the image of $\xi$ in $I^pH_n(X, X - \{x\}; R)$. Suppose the image of some $\xi$ is $0$ in each $I^pH_n(X, X - \{x\}; R)$ so that $s_X(x_j) = 0$ for each $j$. As $O^p$ is constant on each regular stratum, due to $X$ being $R$-oriented by assumption, we must have $s_X(x) = 0$ for all $x \in X - \Sigma_X$. By item (2) of Lemma 8.13, $s_X$ is thus the 0 section on all of $X$. So, by item (2) of Lemma 8.15, $\xi$ must be in $I^pH_n(X; R)$. This demonstrates that our map $I^pH_n(X; R) \to \oplus_j I^pH_n(X, X - \{x_j\}; R)$ is injective and so an isomorphism.

We now turn to proving the two lemmas, Lemma 8.13 and Lemma 8.15. As noted above, the proofs are inductively intertwined. They also utilize that the lemmas, at depth $d$, together prove Theorem 8.17 at depth $d$. Both lemmas and the theorem are true for stratified pseudomanifolds of depth 0, i.e. manifolds, by Theorem 3.26 and Lemma 3.27. We now turn to showing that Lemma 8.13 holds for an $X$ of depth $d > 0$ under the assumption that Lemmas 8.13 and 8.15 and Theorem 8.17 hold on all stratified pseudomanifolds of depth $< d$.  

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Proof of Lemma 8.13, given the induction assumptions. We begin with the first statement of the proposition, noting that it is immediate when \( x \in X - \Sigma_X \), as, in this case, \( x \) has a Euclidean neighborhood. So, by excision, \( I^q H_i(X, X - \{x\}; R) \cong I^q H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \); of course, this is isomorphic to \( R \) if \( i = n \) and 0 otherwise.

Next, suppose \( x \) has a distinguished neighborhood of the form \( \mathbb{R}^k \times cL \) with \( L \neq \emptyset \) a compact stratified pseudomanifold of dimension \( n - k - 1 \). By excision, \( I^q H_i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{x\}; R) \cong \). This pair is homeomorphic to the product of the two pairs of spaces \( (\mathbb{R}^k, \mathbb{R}^k - \{0\}) \) and \( (cL, cL - \{v\}) \), using the convention \( (A, B) \times (C, D) = (A \times C, (A \times D) \cup (B \times C)) \). By the K"unneth theorem with one term being a manifold, Theorem 6.24 we have

\[
I^q H_i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{x\}; R) \cong H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R) \otimes I^q H_{i-k}(cL, cL - \{v\}; R) \\
\cong I^q H_{i-k}(cL, cL - \{v\}; R).
\]

We now apply the cone formula, Corollary 6.12, by which \( I^q H_{i-k}(cL, cL - \{v\}; R) \cong I^q H_{i-k-1}(L; R) \) if \( i - k > n - k - q(\{v\}) - 1 \), i.e. if \( i > n - q(\{v\}) - 1 \), and is 0 otherwise.

If \( L \) is \( R \)-oriented, or even \( R \)-orientable, \( I^q H_j(L; R) = 0 \) for \( j > n - k - 1 \) by the induction assumption and Theorem 8.17, which would imply the first statement of the lemma, independent of which case of the cone formula applies for our particular choice of \( j \). To see that \( L \) must be \( R \)-orientable if \( X \) is, we note that any orientation on \( X - \Sigma_X \) restricts to an \( R \)-orientation on the union of regular strata of \( \mathbb{R}^k \times cL \). But, by the local structure for CS sets, this union of regular strata must be isomorphic to \( \mathbb{R}^k \times (0, 1) \times (L - \Sigma_L) \). We now invoke the fact that a product manifold is orientable if and only if its manifold factors are orientable. So \( L - \Sigma_L \) is \( R \)-orientable and thus \( L \) is \( R \)-orientable, which is enough to draw the necessary conclusion from Theorem 8.17. This completes the proof of the first statement of the lemma.

We turn to the second statement of the lemma. For this case, we care about \( i = n \), and we are now in the setting where we have assumed that \( \bar{p} \geq 0 \). It is thus true that \( n > n - \bar{p}(\{v\}) - 1 \), and therefore, \( I^p H_i(X, X - \{x\}; R) \cong I^p H_{i-k-1}(L; R) \) in this case, so \( I^p H_i(X, X - \{x\}; R) \) is not automatically 0.

The second statement of the second section of the lemma follows directly from the first and from the fact that every sheaf has a zero section [13 page 4]. We must show that any section \( s \) of \( O^p \) over \( X - \Sigma_X \) extends uniquely to all of \( X \). So, let \( x \in \Sigma_X \); we must

\footnote{This is the first exercise in Section VI.7 of Bredon’s [12]. Here’s a sketch of the proof: Suppose \( M_1^{n_1} \times M_2^{n_2} \) is a product of manifolds. Letting \( B_i \) be a local Euclidean ball in \( M_i \), the local isomorphisms \( H_{m_i}(M_1, M_1 - B_1; R) \otimes H_{m_2}(M_2, M_2 - B_2; R) \cong H_{m_1+m_2}(M_1 \times M_2, M_1 \times M_2 - B_1 \times B_2; R) \) induce an isomorphism of bundles (locally constant sheaves) \( O_1 \otimes O_2 \rightarrow O_X \), where \( O_i \) is the orientation bundle of \( M_i \) and \( O_X \) is the orientation bundle of the product. The “total tensor product” \( O_1 \otimes O_2 \) has fiber \( O_{1,x_1} \otimes O_{2,x_2} \) at \((x_1, x_2) \in M_1 \times M_2 \); it can be formally defined as the tensor product over \( M_1 \times M_2 \) of the pullback bundles \( \pi^*_1 O_1 \otimes \pi^*_2 O_2 \), where \( \pi_i : M_1 \times M_2 \rightarrow M_i \) is the projection. If \( M_1 \times M_2 \) is \( R \)-orientable, \( O_X \) is the constant bundle and has a global orientation section \( o_X \). Consider now the subspace \( \{x_1\} \times M_2 \) for some fixed \( x_1 \in M_1 \). The restriction to this subspace of \( O_1 \otimes O_2 \) has the constant bundle with fiber \( R \) in the first factor, and so the restriction is isomorphic to \( O_2 \), living on a copy of \( M_2 \). But, as \( O_X \) was constant, this restricted bundle is also constant, so \( O_2 \) is constant. Therefore, \( M_2 \) is \( R \)-orientable. The argument for \( M_1 \) is identical. Conversely, if \( o_i \) is a global \( R \)-orientation section in \( O_i \), then \( o_1 \otimes o_2 \), which takes values \( o_1(x_1) \otimes o_2(x_2) \) at \((x_1, x_2) \), is an \( R \)-orientation section over \( M_1 \times M_2 \).}
define $s(x)$ for each $x$ and show that, overall, we obtain a well-defined global section. We choose a distinguished neighborhood $N$ of $x$, which we identify with $\mathbb{R}^k \times cL$ via a stratified homeomorphism. We can assume that $x \in N$ has the coordinates $(0, v) \in \mathbb{R}^k \times cL$.

We must assign to $x$ an element of $I^pH_n(X, X - \{x\}; R) \cong I^pH_n(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R)$. Within $N$, we have a smaller compact neighborhood of $x$ of the form $\bar{B}_r \times \bar{c}_a$, where $\bar{B}_r$ is closed Euclidean ball of radius $r$ in $\mathbb{R}^k$ and $\bar{c}_aL = [0, s] \times L/\sim$, for $0 < s < 1$, is a smaller closed cone on $L$ within $cL = [0, 1] \times L/\sim$. Let $\bar{B}_r \times \bar{c}_aL$ be denoted by $\bar{N}'$. We can also let $N' = B_s \times c_1L$ be the interior of $\bar{N}'$. Suppose that $\{\mathcal{L}_\alpha\}$ is the collection of the regular strata of $L$, which is finite as $L$ is compact, and that $\mathcal{R}_\alpha = \mathbb{R}^k \times (0, 1) \times \mathcal{L}_\alpha$ are the corresponding regular strata of $N$. For each $\alpha$, let $x_\alpha$ be some point in $\mathcal{R}_\alpha \cap N'$. We claim that there are isomorphisms

$$I^pH_n(X, X - \{x\}; R) \cong I^pH_n(X, X - \bar{N}'; R) \cong \oplus_\alpha I^pH_n(X, X - \{x_\alpha\}; R) \cong R^m.$$ 

Here, the map to the left is induced by inclusion, and the map to the right is the direct sum of maps induced by inclusions. The claim will be sufficiently useful later that we separate it out as a lemma in its own right, Lemma 8.19, which we prove just below. For now, we finish the proof of Lemma 8.13 assuming this claim.

By (8.2.3), there is a unique $\xi \in I^pH_n(X, X - \bar{N}'; R)$ that restricts to the given $s(x_\alpha)$ in each $I^pH_n(X, X - \{x_\alpha\}; R)$. In fact, as we vary through all $z \in N'$, the image of $\xi$ in $I^pH_n(X, X - \{z\}; R)$, $z \in N'$, determines a section $s_\xi$ of $\mathcal{O}^p$ over $N'$. As $\mathcal{O}^p$ is constant over the regular strata of $X$ and as there is a representative $x_\alpha$ in each regular stratum of $N'$, $s_\xi$ must agree with $s$ at all regular stratum points of $N'$. Thus $s_\xi$ extends $s$ to a section over $(X - \Sigma X) \cup N'$. For uniqueness, suppose $t$ is any other section defined on a neighborhood of $x$ that extends $s$ over a neighborhood of $x$. By Remark 8.12, there must be some open neighborhood $V$ of $x$ and some $\zeta \in I^pH_n(X, X - V; R)$ such that, for each $z \in V$, the section value $t(z)$ is the image of $\zeta$ in $I^pH_n(X, X - \{z\}; R)$. But now let $N_1$ be an even smaller distinguished neighborhood of $x$ inside $V \cap N'$ and with its own smaller $\bar{N}_1$. Then $\xi$ and $\zeta$ both represent elements of $I^pH_n(X, X - \bar{N}_1; R)$, and they each have the same images at all points in regular strata of $N_1'$. Therefore, applying Lemma 8.19 (or, equivalently, equation (8.2.3)) again, this time utilizing $N_1$ and $N_1'$ as the neighborhoods in the lemma, we see that $\xi$ and $\zeta$ must represent the same element of $I^pH_n(X, X - \{x\}; R)$. Therefore, $s_\xi(x) = t(x)$. This shows that there is only one possible value at $\mathcal{O}_x^p = I^pH_n(X, X - \{x\}; R)$ that extends $s$ from the regular strata to some neighborhood of $x$.

Finally, to obtain a global section that extends $s$ to all of $X$, we have seen that for every $x \in X$ there is a section $s_x$ defined on a neighborhood $U_x$ of $x$ and such that $s_x$ agrees with $s$ at all points in regular strata of $U_x$. If $U_x$ and $U_y$ are two such neighborhoods, then the uniqueness of the extensions, proven in the preceding paragraph, implies that $s_x = s_y$ on the overlap $U_x \cap U_y$: for any point $z \in U_x \cap U_y$, the sections $s_x$ and $s_y$ are both defined in neighborhoods of $z$ and, by assumption, extend $s$ from the regular strata to a neighborhood of $z$. So, by the result of the preceding paragraph, $s_x(z) = s_y(z)$. It is now another fundamental property of sheaf theory that, given such local sections that agree on overlaps, they can be patched together to provide a global section; see [13, Section I.1].

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As we have seen, the following lemma was needed in the proof of Lemma 8.13, but we will use it several more times in this section. For Lemma 8.19, we will use as slightly more general $N'$, which will be needed below.

**Lemma 8.19.** Let $R$ be a Dedekind domain, and let $X$ be an $R$-oriented $n$-dimensional stratified pseudomanifold. Suppose $x \in \Sigma_X$ has a distinguished neighborhood $N \cong \mathbb{R}^k \times cL$. Suppose $x$ is contained in a compact subset of $\mathbb{R}^k \times cL$ of the form $N' = C \times \tilde{c}_aL \subset \mathbb{R}^k \times cL$, where $C$ is a compact convex subset of $\mathbb{R}^k$ and $\tilde{c}_aL = [0,s] \times L/\sim$, for $0 < s < 1$, is a smaller closed cone on $L$ within $cL = [0,1] \times L/\sim$. Let $\{\mathcal{R}_\alpha\}_{\alpha=1}^m$ denote the regular strata of $\tilde{N}'$, which are bijective with the regular strata of $L$, and let $\{x_\alpha\}_{\alpha=1}^m$ be any collection of points with $x_\alpha \in \mathcal{R}_\alpha$. Then, for any $\bar{p}$ with $\bar{p} \geq 0$, the inclusion maps induce isomorphisms

$$I^\bar{p}H_n(X, X - \{x\}; R) \xleftarrow{\cong} I^\bar{p}H_n(X, X - \tilde{N}'; R) \xrightarrow{\cong} \oplus_\alpha I^\bar{p}H_n(X, X - \{x_\alpha\}; R) \cong R^m.$$

Therefore, if $\xi \in I^\bar{p}H_n(X, X - \tilde{N}'; R)$ and $\mathfrak{s}_\xi$ is the section of $\mathcal{O}^\bar{p}$ over $\tilde{N}'$ such that, for $z \in \tilde{N}'$, $\mathfrak{s}_\xi(z)$ is the image of $\xi$ in $I^\bar{p}H_n(X, X - \{z\}; R)$, then the value $\mathfrak{s}_\xi(x)$ is completely determined by the collection of values $\{\mathfrak{s}_\xi(x_\alpha)\}$, and vice versa.

**Proof.** For the proof, we continue to assume we are within the overall inductive scenario of this section, so that Lemmas 8.13 and 8.15 and Theorem 8.17 are all available for spaces of depth less than that of $X$, which we assume has depth $d$. We observe quickly that the “therefore” statement of the lemma follows directly from the existence of the stated isomorphisms of the lemma. Further, before getting into the details, we should explain that the space $\tilde{N}'$ is not a stratified pseudomanifold, so when we speak of its regular strata, we mean its intersection with the regular strata of $N$. In this case, the regular strata of $\tilde{N}'$ have the form $C \times (0,s) \times \mathcal{L}_\alpha$, where the $\mathcal{L}_\alpha$ are the regular strata of $L$.

The map to the left in the diagram of the lemma is an isomorphism via stratified homotopy invariance. For the map to the right, which is the direct sum of the maps induced by the inclusions, we consider that following diagram

$$I^\bar{p}H_{n-k-1}(L; R) \xleftarrow{\cong} \oplus I^\bar{p}H_{n-k-1}(\mathcal{L}_\alpha; R) \xrightarrow{\phi} \oplus I^\bar{p}H_n(\mathbb{R}^k \times c\mathcal{L}_\alpha, \mathbb{R}^k \times c\mathcal{L}_\alpha - \{x_\alpha\}; R) \xrightarrow{\cong} \oplus_\alpha I^\bar{p}H_n(N, N - \{x_\alpha\}; R) \quad (33)$$

The bottom map is isomorphic to the map $I^\bar{p}H_n(X, X - \tilde{N}'; R) \xrightarrow{\cong} \oplus_\alpha I^\bar{p}H_n(X, X - \{x_\alpha\}; R)$ of the claim by excision, so it suffices to show that this diagram commutes and that the other maps of the diagram are all isomorphisms.

The lefthand vertical map is the inverse of the isomorphism $I^\bar{p}H_{n-k}(cL, cL - \tilde{c}_aL; R) \xrightarrow{\cong} I^\bar{p}H_{n-k}(cL, cL - \{v\}; R) \xrightarrow{\partial} I^\bar{p}H_{n-k-1}(cL - \{v\}; R) \cong I^\bar{p}H_{n-k-1}(L; R)$, which is an isomorphism from the cone formula and stratified homotopy invariance, composed with the Künneth isomorphism.

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\[ I^p H_{n-k}(cL, cL - \tilde{c}_s L; R) \cong R \otimes I^p H_{n-k}(cL, cL - \tilde{c}_s L; R) \]
\[ \cong H_k(\mathbb{R}^k, \mathbb{R}^k - C; R) \otimes I^p H_{n-k}(cL, cL - \tilde{c}_s L; R) \]
\[ \cong I^p H_{n-k}(\mathbb{R}^k \times cL; \mathbb{R}^k \times cL - \tilde{N'}; R). \]

Tracing through these isomorphisms, the lefthand vertical map takes an element \( \zeta \in I^p H_{n-k}(L; R) \) to an element of \( I^p H_n(N, N - \tilde{N'}; R) \cong I^p H_n(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - C - \tilde{c}_s L; R) \) that can be represented by the product chain \( \eta \times Z \), where \( \eta \) is a generator of \( H_k(\mathbb{R}^k, \mathbb{R}^k - C; R) \cong R \) and where \( Z \) is an allowable chain in \( cL \) whose boundary is a copy of \( \zeta \) that we can suppose lives in a copy of \( L \) at the coordinates \( \{0\} \times \{s'\} \times L \subset \mathbb{R}^k \times cL \) with \( s < s' < 1 \). For convenience below, we will now describe an explicit such \( Z \). For this, let us write each \( x_\alpha \) as \( x_\alpha = (y_\alpha, t_\alpha, z_\alpha) \) with \( z_\alpha \in L_\alpha, t_\alpha \in (0, 1) \) and \( y_\alpha \in \mathbb{R}^k \). So this provides coordinates for \( x_\alpha \) in the distinguished neighborhood \( N \cong \mathbb{R}^k \times cL \).

Let us choose a 1-simplex generator of \( H_1((0, 1), (0, 1) - \{t_\alpha\}; R) \) given by the linear homeomorphism \( e : [0, 1] \to [s'', s'] \subset (0, 1) \) with \( e(0) = s'' \) and \( e(1) = s' \). Here \( s' \) is as above, but we choose \( s'' \) such that \( s'' < t_\alpha \) for all \( \alpha \). We propose to take \( Z = \tilde{c}_s \zeta + e \times \tilde{c}_t, \) where \( \tilde{c}_s \zeta \) is the singular cone on the copy of \( \zeta \) located at cone coordinate \( s'' \) (see the construction of Example 3.38) and where \( e \times \zeta \) is the standard cross product. Then \( \partial(\tilde{c}_s \zeta + e \times \zeta) = \zeta'' + \zeta_t - \zeta'' = \zeta_t, \) where \( \zeta_t \) denotes \( \zeta \), thought of as living in a copy of \( L \) at cone coordinate \( t \). This is as desired.

Returning now to Diagram (33), we let the righthand vertical map be a direct sum of excision inclusions, and so it is an isomorphism. The map to the left in the diagram, also induced by inclusions, is an isomorphism by Proposition 6.47. For the map labeled \( \phi \), if \( \zeta_\alpha \in I^p H_{n-k-1}(\tilde{L}_\alpha; R) \), and if we continue to think of \( L \) as embedded in \( N \) at the coordinates \( \{0\} \times \{s'\} \times L \subset \mathbb{R}^k \times cL \) with \( s < s' < 1 \), then we let this map take \( \zeta_\alpha \) to \( \eta \times Z_\alpha \in I^p H_n(\mathbb{R}^k \times cL_\alpha, \mathbb{R}^k \times cL_\alpha - \{x_\alpha\}; R) \), where \( Z_\alpha \) is defined analogously to the earlier \( Z \) and \( \eta \) is some generator of \( H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R) \). We observe that, for support reasons, this chain is automatically 0 in each \( I^p H_n(\mathbb{R}^k \times cL_\beta, \mathbb{R}^k \times cL_\beta - \{x_\beta\}; R) \) with \( \beta \neq \alpha \). With this definition, it follows from our constructions that the diagram commutes. What remains then to proving our claim is showing that \( \phi \) is an isomorphism.

Consider next the diagram

\[ \begin{array}{ccc}
\oplus I^p H_{n-k-1}(\tilde{L}_\alpha; R) & \xrightarrow{\phi} & \oplus I^p H_n(\mathbb{R}^k \times cL_\alpha, \mathbb{R}^k \times cL_\alpha - \{x_\alpha\}; R) \\
\cong & & \cong \\
\downarrow & & \downarrow \\
\oplus I^p H_{n-k-1}(\tilde{L}_\alpha - \{z_\alpha\}; R) & \cong & \oplus I^p H_n(\mathbb{R}^k \times (0, 1) \times \tilde{L}_\alpha, \mathbb{R}^k \times (0, 1) \times \tilde{L}_\alpha - \{x_\alpha\}; R)
\end{array} \]

The lefthand vertical map, induced by inclusions, is an isomorphism as each \( \tilde{L}_\alpha \) is a stratified pseudomanifold (by Lemma 6.45) with only one regular stratum and of depth \( < d \), so Theorem 8.17 applies via our induction assumptions. The righthand vertical map is a
direct sum of excision isomorphisms. The bottom map comes from the Künneth theorem with one term a manifold, Theorem 6.25. We let it be the direct sum of the isomorphisms

\[ I^pH_{n-k-1}(\bar{\mathcal{L}}_\alpha, \bar{\mathcal{L}}_\alpha - \{z_\alpha\}; R) \]

\[ \cong R \otimes I^pH_{n-k-1}(\bar{\mathcal{L}}_\alpha, \bar{\mathcal{L}}_\alpha - \{z_\alpha\}; R) \]

\[ \cong H_k(\mathbb{R}^k, \mathbb{R}^k - \{y_\alpha\}; R) \otimes H_1((0, 1), (0, 1) - \{t_\alpha\}; R) \otimes I^pH_{n-k-1}(\bar{\mathcal{L}}_\alpha, \bar{\mathcal{L}}_\alpha - \{z_\alpha\}; R) \]

\[ \cong I^pH_n(\mathbb{R}^k \times (0, 1) \times \bar{\mathcal{L}}_\alpha, \mathbb{R}^k \times (0, 1) \times \bar{\mathcal{L}}_\alpha - \{x_\alpha\}; R) \]

defined so that if \( \zeta_\alpha \in I^pH_{n-k-1}(\bar{\mathcal{L}}_\alpha, \bar{\mathcal{L}}_\alpha - \{z_\alpha\}; R) \), then this map takes \( \zeta_\alpha \) to \( \eta \times e \times Z_\alpha \), where \( \eta \) and \( e \) are as above. It remains to see that this last diagram commutes. But, by our constructions, each \( \phi(\zeta_\alpha) \) is represented by \( \eta \times Z_\alpha = \eta \times (\bar{e}_s \cdot \zeta_\alpha + e \times \zeta_\alpha) \), which is equal to \( \eta \times e \times \zeta_\alpha \) in \( I^pH_n(\mathbb{R}^k \times c\mathcal{L}_\alpha, \mathbb{R}^k \times c\mathcal{L}_\alpha - \{x_\alpha\}; R) \), as \( \bar{e}_s \cdot \zeta_\alpha \) is supported in \( \mathbb{R}^k \times c\mathcal{L}_\alpha - \{x_\alpha\} \). This proves the commutativity and so finishes the proof of the lemma.

Remark 8.20. The computations in the proof of Lemma 8.19 demonstrate one of the problems if we allow \( \tilde{p}(S) < 0 \) for some singular stratum \( S \). In particular, if \( x \in S \) and \( S \) has dimension \( k \), the computations in the proof shows that \( I^pH_i(X, X - \{x\}; R) \cong I^pH_{i-k}(cL, cL - \{v\}; R) \), which, by the cone formula, Corollary 6.12, is 0 if \( i - k \leq n - k - \tilde{p}(\{v\}) - 1 \), i.e. if \( i \leq n - \tilde{p}(\{v\}) \). But if \( \tilde{p}(\{v\}) < 0 \), this scenario will include \( i = n \). This still results in a unique extension of sections of \( \mathcal{O}^p \) to \( S \), as any section will be forced to be 0 at each point on \( S \). However, the isomorphism on the right in Lemma 8.19 will no longer hold, and this would cause some of the arguments we will see in the proof of Lemma 8.15 below, to fall apart.

Proof of Lemma 8.15, given the induction assumptions. We will broadly follow the pattern of the proof of [53] Lemma 3.27. It consists of a few separate steps.

Mayer-Vietoris reduction step. As a first step, suppose \( K, L \) are compact subsets of \( X \) such that the claims of Lemma 8.15 hold with respect to \( K, L, \) and \( K \cap L \). We will show that they then hold for \( K \cup L \). Consider the following portion of the long exact Mayer-Vietoris sequence

\[ I^qH_{n+1}(X, X - (K \cup L); R) \rightarrow I^qH_n(X, X - (K \cup L); R) \]

\[ \xrightarrow{\Phi} I^qH_n(X, X - K; R) \oplus I^qH_n(X, X - L; R) \rightarrow I^qH_n(X, X - (K \cup L); R). \]

Given the assumptions on \( K, L, \) and \( K \cap L \), we have \( I^qH_i(X, X - K; R) = I^qH_i(X, X - L; R) = I^qH_i(X, X - (K \cap L); R) = 0 \) for \( i > n \). It thus follows from the portion of the sequence further to the left that \( I^qH_i(X, X - (K \cup L); R) = 0 \) for \( i > n \). We also have that the map labeled \( \Phi \) is injective.

Continuing to assume the lemma on \( K, L, \) and \( K \cap L \) and now assuming \( \tilde{p} \geq 0 \) (and replacing \( \tilde{q} \) with \( \tilde{p} \) in our Mayer-Vietoris sequence), suppose that \( \xi \in I^pH_n(X, X - (K \cup L); R) \) maps to 0 in \( I^pH_n(X, X - \{x\}; R) \) for all \( x \in K \cup L \). Then the image of \( \xi \) under \( \Phi \) in the summand \( I^pH_n(X, X - K; R) \) must map to 0 in each \( I^pH_n(X, X - \{x\}; R) \) for \( x \in K \), and analogously for the \( L \) summand. By the assumption, this implies that \( \Phi(\xi) = 0 \), so \( \xi = 0 \). This implies that any two elements \( \xi, \xi' \in I^pH_n(X, X - (K \cup L); R) \) that map to the same
section of $\mathcal{O}^p$ over $K \cup L$ must be equal, as $\xi - \xi' = 0$ by this argument. This provides the uniqueness part of the second statement of the lemma.

Finally, suppose we are given a section $s$ of $\mathcal{O}^p$, and suppose $\gamma_K \in I^pH_n(X, X - K; R)$ and $\gamma_L \in I^pH_n(X, X - L; R)$ restrict to this section at points of $K$ or $L$, respectively. Then $\Psi(\gamma_K, -\gamma_L)$, which is represented by $\gamma_K - \gamma_L$, restricts to $0 \in I^pH_n(X, X - \{x\}; R)$ for each $x \in K \cap L$. But, again assuming the lemma holds for $K \cap L$, this means that $\Psi(\gamma_K, -\gamma_L) = 0$, and, from the exact sequence, there exists a $\gamma_{K \cup L} \in I^pH_n(X, X - (K \cup L); R)$, with $\Phi(\gamma) = (\gamma_K, -\gamma_L)$. As $\Phi(\xi)$ is represented by $(\xi, -\xi)$ in general, we see that $\gamma_{K \cup L}$ must restrict to the values $s(x)$ at each $I^pH_n(X, X - \{x\}; R)$ for $x \in K \cup L$.

*Distinguished neighborhood reduction step.* Suppose $K$ is any compact subset of $X$. For each $x \in X$, there is a distinguished neighborhood $N_x$ of $x$ stratified homeomorphic to $\mathbb{R}^k \times cL$, with $k$ and $L$ depending on $x$. Within this neighborhood, $x$ has smaller compact neighborhoods, say of the form $N'_x \cong B_r \times \bar{c}_sL$, where $B_r$ is a closed ball in $\mathbb{R}^k$ and $\bar{c}_sL$ is the compact subcone of $cL$ up to radius $s$; let $N''_x \cong B_r \times c_sL$ be the interior of $N'_x$. As $K$ is compact, it can be covered by a finite number of the $N'_x$, and therefore $K$ is the union of a finite number of the compact spaces $N''_x \cap K$, with each of these compact spaces being contained in the corresponding distinguished neighborhood $N_x$.

Suppose we can prove the lemma for any such compact set that is contained in some distinguished neighborhood. For the purposes of this step, let us call such compact sets *distinguished compact sets*. Then, we claim that, by induction and the Mayer-Vietoris discussion above, the lemma will hold for any finite union of such distinguished compact sets, and hence it will hold for $K$. To verify the claim, let $\{J_\alpha\}$ be any collection of distinguished compact sets. We have assumed the lemma holds for each $J_\alpha$. Suppose, as induction hypothesis, that it holds for any union of $\ell - 1$ such sets, and let $\{J_1, \ldots, J_\ell\}$ be a collection of $\ell$ such sets, $\ell > 1$. Then $J_\ell \cap (\bigcup_{i=1}^{\ell-1} J_i) = \bigcup_{i=1}^{\ell-1} (J_\ell \cap J_i)$. As each $J_\ell \cap J_i$ is a distinguished compact set and as $\bigcup_{i=1}^{\ell-1} (J_\ell \cap J_i)$ is a union of $\ell - 1$ such sets, the lemma holds for $J_\ell \cap (\bigcup_{i=1}^{\ell-1})$ by the induction hypothesis. It also holds for $J_\ell$ and $\bigcup_{i=1}^{\ell-1} J_i$ by our assumptions, so it holds for $\bigcup_{i=1}^{\ell-1} J_i$ by the Mayer-Vietoris reduction step.

Therefore, as we have shown that $K$ is the union of finitely many distinguished compact sets, the lemma will hold for $K$, provided that we show that the lemma holds for any single distinguished compact set.

*Proof for PM-convex sets.* We have shown that it now suffices to prove the lemma for any distinguished compact set in $X$, where a distinguished compact set is a compact set contained within a distinguished neighborhood of some point in $x$. If $K$ is such a distinguished compact set in $X$ contained in the distinguished neighborhood $N$, then we have $P^pH_n(X, X - K; R) \cong I^pH_n(N, N - K; R)$, and both the orientation bundle $\mathcal{O}^p$ of $X$ and the hypothesized section $s$ restrict to an orientation bundle and section over $N$. As the requirements of the lemma for $K$ only concern the values of $s$ at points in $K$, it therefore suffices to prove the lemma with $N$ in place of $X$. In the remaining steps, we are therefore free to assume that $X = \mathbb{R}^k \times cL$. We can also assume that $L \neq \emptyset$, otherwise we know the claimed result holds from the manifold case [53].

Following [33], we next prove the lemma for the case where $K$ is a PM-convex set in $X = \mathbb{R}^k \times cL$. These are defined to be subsets $K$ of $\mathbb{R}^k \times cL$ of either of the following forms:
1. \( K = C \times \bar{c}_s L \), where \( C \subset \mathbb{R}^k \) is a compact convex set and, as usual, \( \bar{c}_s L = [0, s] \times L / \sim \) is a closed subcone of \( cL \),

2. \( K = C \times [r, s] \times D \), where \( C \subset \mathbb{R}^k \) is a compact convex set, \( [r, s] \) is an interval “along the cone line” with \( 0 < r \leq s < 1 \), and \( D \subset L \) is a compact subset.

As the set-theoretic intersection of products is the product of the intersections and as the intersection of two compact convex subsets of \( \mathbb{R}^k \) is a compact convex subset, we see that the intersection of two PM-convex sets is also a PM-convex set. We also observe that every point in \( X \) has a PM-convex neighborhood. In fact, by some basic point-set topology, given any \( x \in U \subset X \) with \( U \) open, there is a PM convex neighborhood of \( x \) contained in \( U \).

We now prove Lemma 8.15 in the case where \( K \) is a PM-convex set.

First, suppose that \( K \) is of the second type, \( K = C \times [r, s] \times D \). Then \( K \) does not intersect \( \mathbb{R}^k \times \{v\} \subset \mathbb{R}^k \times cL = X \), so, by excision, \( I^qH_*(X, X - K; R) \cong I^qH_q(X - (\mathbb{R}^k \times \{v\}), X - (K \cup (\mathbb{R}^k \times \{v\})); R) \). As \( X - (\mathbb{R}^k \times \{v\}) \) must have depth \( d \), where \( d \) continues to denote the depth of \( X \), the lemma must hold on \( X - (\mathbb{R}^k \times \{v\}) \) by our induction assumptions. So, for \( i > n \),

\[
I^qH_i(X, X - K; R) \cong I^qH_i(X - (\mathbb{R}^k \times \{v\}), X - (K \cup (\mathbb{R}^k \times \{v\})); R) = 0.
\]

Also, for \( \bar{p} \geq 0 \), any section \( s \) of \( O^\bar{p} \) over \( X \) restricts to a section over \( X - (\mathbb{R}^k \times \{v\}) \). By induction, there is an element of \( I^pH_n(X - (\mathbb{R}^k \times \{v\}), X - (K \cup (\mathbb{R}^k \times \{v\})); R) \) that is compatible with this section over \( K \), and the excision isomorphism therefore yields such an element of \( I^pH_n(X, X - K; R) \).

Next, suppose \( K \) is of the first type, \( K = C \times \bar{c}_s L \). Notice that this is precisely the form of the sets \( N' \) of Lemma 8.19. If \( y \in C \), then \( I^qH_i(X, X - K; R) \cong I^qH_i(X, X - \{(y, v)\}; R) \) by stratified homotopy invariance. But we already know from our computations in the proof of Lemma 8.13 that, under the induction assumptions, \( I^qH_i(X, X - \{(y, v)\}; R) = 0 \) for \( i > n \).

For \( i = n \) and \( \bar{p} \geq 0 \), we know by Lemma 8.19 that, given a collection of points \( \{x_\alpha\} \) consisting of one point in each regular stratum of \( N'' \), there is a unique element \( \xi \in I^pH_n(X, X - \bar{N}'; R) \) whose image at each \( x_\alpha \) is \( s(x_\alpha) \). We must verify that the image of \( \xi \) in the other \( I^pH_n(X, X - \{z\}; R) \), \( z \in \bar{N}' \), is \( s(z) \); this will provide the unique element of \( I^pH_n(X, X - K; R) \) promised by the lemma. For this, it is simpler to work with a slightly larger open set that contains \( \bar{N}' \). So let \( B_\tau \) be an open ball in \( \mathbb{R}^k \) containing \( C \), let \( s < s'' < 1 \), and let \( N'' = B_\tau \times \bar{c}_s L \). Then \( \bar{N}' \subset N'' \), and the inclusion \( I^pH_n(X, X - N''; R) \to I^pH_n(X, X - \bar{N}'; R) \) is an isomorphism by stratified homotopy invariance. Let \( \xi'' \in I^pH_n(X, X - N''; R) \) be the unique element that maps to \( \xi \in I^pH_n(X, X - \bar{N}'; R) \). Then \( \xi'' \) determines a section \( \bar{s}_{\xi''} \) over \( N'' \), and again \( \bar{s}_{\xi''}(x_\alpha) = s(x_\alpha) \); in fact, \( \bar{s}_{\xi''}(z) \) must agree with the image of \( \xi \) in \( I^pH_n(X, X - \{z\}; R) \) for each \( z \in \bar{N}' \). As \( N'' \) is a stratified pseudomanifold with orientation sheaf obtained by restricting \( O^\bar{p} \) from \( X \), the restriction of \( \bar{s} \) to \( N'' \) and the section \( \bar{s}_{\xi''} \) must each be constant over each regular stratum of \( N'' \). Thus, the two sections must agree at all points of regular strata of \( N'' \), as they agree over one point in each regular stratum (each regular stratum \( B_\tau \times (0, s'') \times L_\alpha \)).
of $N''$ contains a regular stratum $C \times (0, s] \times \mathcal{L}_\alpha$ of $\tilde{N}'$, the $\mathcal{L}_\alpha$ being the regular strata of $L$, and so each regular stratum of $N''$ contains one of the $x_\alpha$). But, again using that $N''$ is a stratified pseudomanifold with orientation bundle restricted from that on $X$, Lemma 8.13 states that the restriction of $s$ over the regular strata extends uniquely to a section over all of $N'$. As the restriction of $s$ to $N''$ and $s_{\alpha''}$ are both defined on all of $N''$ and are equal on the regular strata, this uniqueness means that $s(z) = s_{\alpha''}(z)$ for all $z \in N''$. Thus, the image of $\xi''$ in each $I^pH_n(X, X - \{z\}; R)$, $z \in N''$, is $s(z)$, so the image of $\xi$ must then be $s(z)$ for each $z \in N'$, as desired.

**Proof for arbitrary $K \subset \mathbb{R}^k \times cL$.** Finally, suppose $K \subset X = \mathbb{R}^k \times cL$ is an arbitrary compact set. We will first verify the second statement of the lemma.

Let $s$ be a section of $O^0$ on $X$. Any compact $K$ must be contained in some PM-convex set, say $Q$. Let $\gamma_Q \in I^pH_n(X, X - Q; R)$ be the unique element guaranteed by the previous step such that the image of $\gamma_Q$ in $I^pH_n(X, X - \{x\}; R)$ is $s(x)$ for each $x \in Q$. Then the image $\gamma_K \in I^pH_n(X, X - K; R)$ of $\gamma_Q$ must have image $s(x) \in I^pH_n(X, X - \{x\}; R)$ for each $x \in K$. We must show that $\gamma_K$ is the unique such element of $I^pH_n(X, X - K; R)$.

Suppose $\gamma' \in I^pH_n(X, X - K; R)$ also has image $s(x) \in I^pH_n(X, X - \{x\}; R)$ for each $x \in K$. We will show that $\gamma_K - \gamma_K' = 0$. Let $\xi$ be a relative cycle representing $\gamma_K - \gamma_K' \in I^pH_n(X, X - K; R)$. Let $|\partial \xi|$ be the support of $\partial \xi$. Then $|\partial \xi| \cap K = \emptyset$, and $\xi$ determines a section $s_\xi$ of $O^0$ over $U = X - |\partial \xi|$ that must be 0 at each $x \in K$. Now, choose any $x \in K$. By Remark [8.12], as $s_\xi(x) = 0$, the section $s_\xi$ must be identically 0 in an open neighborhood $V_x$ of $x$. Let $\bar{A}_x$ be a PM-convex neighborhood of $x$ in $V_x$, and let $A_x$ be the interior of $\bar{A}_x$. As we let $x$ run through all the elements of $K$, the open subsets $A_x$ cover $K$. Since $K$ is compact, there is a finite subcollection $\{A_{x_j}\}_{j=1}^m$ that covers $K$, and the corresponding union $P = \cup_{j=1}^m \bar{A}_{x_j}$ is a compact subset of $U$ that contains $K$. By construction, the image of $\xi$ in $I^pH_n(X, X - \{z\}; R)$ is 0 for each $z \in P$; in other words, the images of $\xi$ agree with the zero section over $P$. But $P$ is a finite union of PM-convex sets, and we have proven the lemma for PM-convex sets. So, by the Mayer-Vietoris step and the induction argument in the distinguished neighborhood reduction step, the lemma holds for $P$. Therefore, $0 \in I^pH_n(X, X - P; R)$ is the unique element that agrees with the zero section at each point of $P$, and so $\xi = 0 \in I^pH_n(X, X - P; R)$. But then $\xi$ must also equal 0 under the inclusion-induced map $I^pH_n(X, X - P; R) \to I^pH_n(X, X - K; R)$, which is what we needed to show.

Lastly, we need to see that $I^qH_i(X, X - K; R) = 0$ for $i > n$ and any $\bar{q}$. Suppose $\xi \in I^qH_i(X, X - K; R)$ for $i > n$. Once again, let $U = X - |\partial \xi|$, and, for each $x \in K$, let $\bar{A}_x$ be a PM convex neighborhood of $x$ in $U$, although this time we impose no additional conditions on $\bar{A}_x$. By the same argument as in the proceeding paragraph, there is a union $P$ of a finite number of the $\bar{A}_x$ with $K \subset P$. But now, for each $\bar{A}_x$, $I^qH_i(X, X - \bar{A}_x; R) = 0$, because we have already proven Lemma 8.15 for PM-convex sets. Also, once again, the Mayer-Vietoris reduction step, together with the induction argument in the distinguished neighborhood reduction step, now shows that $I^qH_i(X, X - P; R) = 0$. So $\xi$ must represent 0 in $I^qH_i(X, X - P; R)$, and so also 0 under the inclusion-induced map $I^qH_n(X, X - P; R) \to I^qH_n(X, X - K; R)$. But $\xi$ was an arbitrary element of $I^qH_i(X, X - K; R)$, so $I^qH_i(X, X - K; R) = 0$. \qed
So, just to catch up, we have now seen that assuming Lemma 8.13, Lemma 8.15, and Theorem 8.17 for depths < d implies both lemmas and the theorem for depth d. So, by induction, our proof of these results is complete.

### 8.2.4 Lack of global fundamental classes for subzero perversities

We are now in a position to investigate a bit further what happens if $\bar{p}$ is a perversity on $X$ with $\bar{p}(S) < 0$ for some singular stratum $S$. The following proposition shows that $I^{\bar{p}}H_*(X - S; R) \rightarrow I^{\bar{p}}H_*(X; R)$ is an isomorphism, so putting negative perversities on some strata is equivalent to leaving such strata out of the space altogether. Among other consequences, this shows that Theorem 8.17 cannot possibly hold for $X$, as $X - S$ will be non-compact (it also might not be a stratified pseudomanifold). In fact, given this isomorphism, any global fundamental class $\Gamma^\bar{p}_X \in I^pH_*(X; R)$ can be represented by a chain in $X - S$, and this chain must have compact support. The chain’s image in $I^pH_n(X, X - \{x\}; R)$ has to be 0 for any $x$ outside this support. But, $S$ must have a neighborhood that does not intersect the support of the chain and, using that $X - \Sigma_X$ is dense, there are therefore points of $X - \Sigma_X$ not contained in the support of the chain. This contradicts $\Gamma^\bar{p}_X$ being a fundamental class.

**Proposition 8.21.** Let $R$ be a Dedekind domain, and let $X$ be a stratified pseudomanifold with perversity $\bar{p}$. Suppose $S \subset X$ is a singular stratum with $\bar{p}(S) < 0$. Then inclusion induces an isomorphism $I^{\bar{p}}H_*(X - S; R) \rightarrow I^pH_*(X; R)$.

**Proof.** We will use a Mayer-Vietoris argument, and so we will invoke Theorem 5.3. In this case, for any open $U \subset X$, let $F_*(U) = I^{\bar{p}}H_*(U - (U \cap S); R)$ and $G_*(U) = I^pH_*(U; R)$, and let $\Phi : F_* \rightarrow G_*$ be induced by inclusion. We must check the conditions of Theorem 5.3.

The functors $F_*$ and $G_*$ admit Mayer-Vietoris sequences by Theorem 6.20, and $\Phi$ induces a map between them. The direct limit condition follows from Lemmas 5.5 and 6.23.

Skipping to the fourth condition, if $U$ is empty or contained in a stratum other than $S$, then $U - (U \cap S) = U$, so $\Phi$ is the identity map for this $U$. If $U \subset S$ is an open subset homeomorphic to Euclidean space and if $\sigma$ is an $i$-simplex in $U$, then $\sigma$ takes all of $\Delta^i$ into $S$, and the $\bar{p}$-allowability condition for $\sigma$ becomes $i \leq i - \text{codim}(S) + \bar{p}(S)$, i.e. $\text{codim}(S) \leq \bar{p}(S)$. But this is impossible if $\bar{p}(S) < 0$, so no simplices are allowable and $I^{\bar{p}}H_*(U) = 0 = I^pH_*(U - (U \cap S))$, as $U - (U \cap S)$ is the empty set.

For the remaining condition, we must show that if $\Phi : F_*(\mathbb{R}^k \times (cL - \{v\})) \rightarrow G_*(\mathbb{R}^k \times (cL - \{v\}))$ is an isomorphism, then so is $\Phi : F_*(\mathbb{R}^k \times cL) \rightarrow G_*(\mathbb{R}^k \times cL)$. Here, $\mathbb{R}^k \times cL$ is a distinguished neighborhood of a point in $X$. Let us make the inductive assumption that we have verified the proposition already for stratified pseudomanifolds of depth less than that of $X$. The proposition is trivial when the depth of $X$ is 0, as in this case $S$ must be empty. Therefore, if we can verify the condition for our $X$ under the inductive assumption, the proposition will be proven on $X$ by Theorem 5.3. As $X$ was arbitrary of its depth, the entire proposition will follow by induction.

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So, consider now the diagram:

\[
\begin{array}{ccc}
F_\ast(\mathbb{R}^k \times (cL - \{v\})) & \xrightarrow{\Phi} & G_\ast(\mathbb{R}^k \times (cL - \{v\})) \\
\downarrow & & \downarrow \\
F_\ast(\mathbb{R}^k \times cL) & \xrightarrow{\Phi} & G_\ast(\mathbb{R}^k \times cL). 
\end{array}
\]

We are assuming the top horizontal map is an isomorphism. The vertical arrows are induced by inclusion. There are two cases to consider, depending upon whether \(x \notin S\). First, suppose \(x \notin S\). If \(x\) is not contained in the closure of \(S\), then the distinguished neighborhood cannot intersect \(S\), so the bottom map in this diagram is trivially an isomorphism. Suppose \(x\) is in the closure of \(S\). This means that \(L\) has a stratum that is the intersection of \(L\) with \(S\), thinking of some copy of \(L\) as being embedded in \(X\) by the distinguished neighborhood homeomorphism. Therefore, \(L \cap S\) is a union of strata of \(L\), on each of which \(\bar{p} < 0\), and the top horizontal arrow is an isomorphism by induction and stratified homotopy invariance. The vertical arrow on the right is basically (up to stratified homotopy equivalence) the inclusion of a link into its cone, and so, by the cone formula, Theorem 6.10, the codomain is 0 for \(\ast \geq \dim(L) - \bar{p}(\{v\})\) and the map is an isomorphism otherwise. The vertical map on the left is similarly, up to stratified homotopy equivalence, the inclusion of \(L - (L \cap S)\) into \(c(L - (L \cap S))\). Our cone formula as stated in Theorem 6.10 doesn’t precisely apply because \(L - (L \cap S)\) is not compact. But our main concern with having compact links was always to avoid weird topologies on cones, which can happen in the quotient topology if the space being coned is not compact. In this case, we can avoid this problem by letting \(c(L - (L \cap S))\) have the subspace topology from \(cL\), and with this assumption the argument of Theorem 6.10 goes through without any trouble. So, for \(\ast \geq \dim(L) - \bar{p}(\{v\})\), both modules on the bottom line of the diagram are trivially, and otherwise the top and sides of the diagram are isomorphisms, implying that the bottom map is also.

Next, suppose \(x \in S\). In this case, the diagram reduces to

\[
\begin{array}{ccc}
I^pH_\ast(\mathbb{R}^k \times (cL - \{v\}); R) & \xrightarrow{\Phi} & I^pH_\ast(\mathbb{R}^k \times (cL - \{v\}); R) \\
\downarrow & & \downarrow \\
I^pH_\ast(\mathbb{R}^k \times (cL - \{v\}); R) & \xrightarrow{\Phi} & I^pH_\ast(\mathbb{R}^k \times cL; R), 
\end{array}
\]

so this case reduces to demonstrating that the inclusion-induced \(I^pH_\ast(\mathbb{R}^k \times (cL - \{v\}); R) \rightarrow I^pH_\ast(\mathbb{R}^k \times cL; R)\) is an isomorphism in all dimensions. By the cone formula (and stratified homotopy invariance), this is true for \(\ast < \dim(L) - \bar{p}(\{v\})\), and \(I^pH_\ast(\mathbb{R}^k \times cL; R) = 0\) for \(\ast \geq \dim(L) - \bar{p}(\{v\})\), so we must show that \(I^pH_\ast(\mathbb{R}^k \times (cL - \{v\}); R) \cong I^pH_\ast(L; R)\) is also 0 in this range. But, by assumption, \(\bar{p}(\{v\}) < 0\), so the dimension range \(\ast \geq \dim(L) - \bar{p}(\{v\})\) only includes dimension that are greater than \(\dim(L)\). But, as \(L\) is a stratified pseudomanifold, \(I^pH_\ast(L; R) = 0\) for \(\ast > \dim(L)\) (and for any perversity) by Theorem 8.17.

\(\square\)
8.2.5 Invariance of fundamental classes

In this section, we will show that the fundamental classes we constructed in the previous section are invariants in two different sense. First, we will show that they are essentially independent of the choice of perversity \( \bar{p} \) such that \( \bar{p} \geq 0 \). The precise meaning of this claim can be found in the statements of Proposition 8.22 and Corollary 8.23. Then we will go on to see in what sense our fundamental classes are invariant of the choice of stratification.

Fundamental classes under change of perversity. We begin with change of perversity.

Proposition 8.22. Let \( R \) be a Dedekind domain, and let \( X \) be an \( R \)-oriented \( n \)-dimensional stratified pseudomanifold. Suppose \( \bar{p}, \bar{q} \) are perversities on \( X \) with \( 0 \leq \bar{p} \leq \bar{q} \). Then:

1. The natural morphism of sheaves \( \mathcal{O}^{\bar{p}} \to \mathcal{O}^{\bar{q}} \) is an isomorphism. In particular, every \( \mathcal{O}^{\bar{p}} \) is isomorphic to \( \mathcal{O}^{\bar{0}} \), which we can simply denote \( \mathcal{O} \) and call the \( R \)-orientation sheaf over \( X \).

2. If \( X \) is compact, the evident map \( \tau_{\bar{p}, \bar{q}} : I^{\bar{p}}H_n(X; R) \to I^{\bar{q}}H_n(X; R) \) is an isomorphism. Furthermore, given a section \( s \) of \( \mathcal{O} \) over \( X \), if \( \gamma^{\bar{p}} \in I^{\bar{p}}H_n(X; R) \) and \( \gamma^{\bar{q}} \in I^{\bar{q}}H_n(X; R) \) are, respectively, the classes that restrict to \( s(x) \) in \( I^{\bar{p}}H_n(X, X - \{x\}; R) \cong I^{\bar{q}}H_n(X, X - \{x\}; R) \) for each \( x \in X \), then \( \tau_{\bar{p}, \bar{q}}(\gamma^{\bar{p}}) = \gamma^{\bar{q}} \). In particular, if \( s = \sigma^{\bar{p}} = \sigma^{\bar{q}} \), then \( \tau_{\bar{p}, \bar{q}}(\Gamma_X^{\bar{p}}) = \Gamma_X^{\bar{q}} \).

3. For \( X \) not necessarily compact and any compact \( K \subset X \) and any section \( s \) of \( \mathcal{O} \) over \( X \), let \( \gamma^{\bar{p}}_K \in I^{\bar{p}}H_n(X, X - K; R) \) and \( \gamma^{\bar{q}}_K \in I^{\bar{q}}H_n(X, X - K; R) \) be, respectively, the classes that restrict to \( s(x) \) in \( I^{\bar{p}}H_n(X, X - \{x\}; R) \cong I^{\bar{q}}H_n(X, X - \{x\}; R) \) for each \( x \in K \). Then, if \( \tau_{\bar{p}, \bar{q}} : I^{\bar{p}}H_n(X, X - K; R) \to I^{\bar{q}}H_n(X, X - K; R) \) is the evident map, \( \tau_{\bar{p}, \bar{q}}(\gamma^{\bar{p}}_K) = \gamma^{\bar{q}}_K \). In particular, if \( s = \sigma^{\bar{p}} = \sigma^{\bar{q}} \), then \( \tau_{\bar{p}, \bar{q}}(\Gamma^{\bar{p}}_K) = \Gamma^{\bar{q}}_K \).

As a corollary, we have the following statement about global fundamental classes on compact stratified pseudomanifolds. This corollary demonstrates the sense in which fundamental classes do not depend on the choice of perversity, provided we don’t allow perversities with negative values.

Corollary 8.23. Let \( R \) be a Dedekind domain, and let \( X \) be a compact \( R \)-oriented \( n \)-dimensional stratified pseudomanifold. Then, for any \( \bar{p} \) and \( \bar{q} \) that are both \( \geq 0 \), the diagram

\[
I^{\bar{p}}H_n(X; R) \leftarrow I^{\bar{q}}H_n(X; R) \to I^{\bar{q}}H_n(X; R)
\]

consists of isomorphisms, and the composition left to right takes \( \Gamma^{\bar{p}}_X \) to \( \Gamma^{\bar{q}}_X \).

Proof of Corollary 8.23. The corollary follows directly from the second statement of the proposition taking \( K = X \) and using \( 0 \) as an intermediary. \( \square \)

Definition 8.24. It follows from Proposition 8.22 that every \( \Gamma^{\bar{p}}_K, \bar{p} \geq 0 \), is the image of \( \Gamma^0_K \), which we will simply denote \( \Gamma_K \) and call the fundamental class of \( X \) over \( K \) with respect to the \( R \)-orientation. If \( X \) is compact, then we call \( \Gamma_X \) the fundamental class of \( X \) with respect to the \( R \)-orientation.
Proof of Proposition 8.22. We will induct on the depth $d$ of $X$. If $d = 0$, $X$ is a manifold and all intersection homology groups reduce to ordinary homology, making the statements trivial. So, suppose now that we have proven the proposition up through depth $d - 1$ and that $X$ has depth $d$.

For the first statement, assume $0 \leq \bar{p} \leq \bar{q}$ so that we have maps $\tau_{\bar{p},\bar{q}}: I^pH_s(X, A; R) \rightarrow I^qH_s(X, A; R)$ for any subset $A$. In particular, if $x \in X$ is any point and $U$ is any neighborhood of $x$, we have maps $\tau_{\bar{p},\bar{q}}: I^pH_n(X, X - U; R) \rightarrow I^qH_n(X, X - U; R)$. Such maps induce a morphism of sheaves $\mathcal{O}^p \rightarrow \mathcal{O}^q$; see [13] Section I.2. To show that this morphism is an isomorphism, it suffices to demonstrate that the induced map of stalks $I^pH_n(X, X - \{x\}; R) \rightarrow I^qH_n(X, X - \{x\}; R)$ is an isomorphism for all $x$. If $x$ is contained in a regular stratum of $X$, this map is isomorphic to the identity map $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R)$ by excision. So suppose suppose $x$ has a distinguished neighborhood $\mathbb{R}^k \times cL, L \neq \emptyset$. Then, applying excision, the naturality of the Künneth theorem, the naturality of the boundary map of the long exact sequence of the pair, the isomorphism of the cone formula Corollary 6.12, and stratified homotopy invariance, this map reduces to the map $I^pH_{n-k-1}(L; R) \rightarrow I^qH_{n-k-1}(L; R)$. By the arguments of the proof of Lemma 8.13 $L$ is orientable, so this map is an isomorphism by the induction assumption, as $L$ must have lower depth than $X$.

For the third statement, consider the commutative diagram

$$
\begin{array}{ccc}
I^pH_n(X, X - K; R) & \longrightarrow & I^pH_n(X, X - \{x\}; R) \\
\tau_{\bar{p},\bar{q}} & & \tau_{\bar{p},\bar{q}} \\
I^qH_n(X, X - K; R) & \longrightarrow & I^qH_n(X, X - \{x\}; R).
\end{array}
$$

We have just demonstrated the righthand vertical isomorphism, which is the stalk-wise realization of the isomorphism $\mathcal{O}^p \rightarrow \mathcal{O}^q$. So if $\gamma^p_K \in I^pH_n(X, X - K; R)$ maps to $s(x)$ in $I^pH_n(X, X - \{x\}; R)$, it follows from the commutativity of the diagram that $\tau_{\bar{p},\bar{q}}(\gamma^p_K)$ maps to the corresponding section value at $x$ of the version of $s$ in $\mathcal{O}^q$. As this holds for all $x \in K$, it follows from the uniqueness part of Lemma 8.15 that $\tau_{\bar{p},\bar{q}}(\gamma^p_K)$ must be $\gamma^q_K$. Of course, this argument also holds when $X$ is compact, demonstrating part of the second statement of the lemma.

It remains to show that, if $X$ is compact, then $\tau_{\bar{p},\bar{q}}: I^pH_n(X; R) \rightarrow I^qH_n(X; R)$ is an isomorphism. For this, let $\{x_j\}_{j=1}^m$ be a collection of points of $X$, one in each regular stratum. Using Theorem 8.17 and excision and identifying neighborhoods of the $x_j$ as Euclidean neighborhoods, we have a diagram

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108We are using here that the collection of maps $I^pH_n(X, X - U; R) \rightarrow I^qH_n(X, X - U; R)$ constitute a map of presheaves, which induces a map of sheaves.
The horizontal maps are isomorphisms by Theorem 8.17 and excision, and the righthand vertical map is an identity map. It follows that \( \tau_{p,q} : I^pH_n(X;R) \to I^qH_n(X;R) \) is an isomorphism.

Remark 8.25. Looking at the statements of Proposition 8.22 it is tempting to expect that \( \tau_{p,q} : I^pH_n(X, X-K; R) \to I^qH_n(X, X-K; R) \) should be an isomorphism. This seems likely, though it also does not seem to follow directly from the kinds of arguments we have been employing in this section. The problem is that our techniques have been geared to working with classes of the form \( \gamma^p \in I^pH_n(X, X-K; R) \) that are determined by a global section of \( O \) on \( X \). And, in fact, we have shown that, given such a section, we can construct a uniquely corresponding \( \gamma^q \) and that its image in \( I^qH_n(X, X-K; R) \) must be the corresponding \( \gamma^q \).

But if we start with an arbitrary \( \xi \in I^qH_n(X, X-K; R) \), this certainly determines a section of \( O \) over \( K \), but not a global section, and this deprives us of our technique for attempting to construct a \( \bar{p} \)-allowable chain that might also represent \( \xi \). It is tempting to look back at the proof of Lemma 8.15 to see if we really need \( s \) to be a global section, but the construction of \( \gamma_Q \) in the “arbitrary K” step of the proof shows that it might indeed be possible, at least using this particular proof, that we need \( s \) to be specified on a larger set than just \( K \).

We also note that it is not always possible to extend a local section \( s \) of \( O \) to a global section. For example, let \( K = \{ x,y \} \) be two points in \( \mathbb{R}^n \), and let \( s \) take two different values in \( H_n(\mathbb{R}^n, \mathbb{R}^n - \{ x \}; R) \cong R \) and \( H_n(\mathbb{R}^n, \mathbb{R}^n - \{ y \}; R) \cong R \).

Nonetheless, Proposition 8.22 is sufficient to tell us that, for any given global section of \( O \), all of the corresponding unique elements \( \gamma^p \) are the images of a single unique \( \gamma^0 \), and in this sense the fundamental class does not depend meaningfully on the perversity.

**Fundamental classes under change of stratification.** Next we consider how fundamental classes behave under change of stratification. We first consider two stratifications that are related by coarsening/refinement. In particular, we suppose \( X' \) is a refinement of \( X \). Of course, two different stratifications of a single underlying space must each be equipped with their own perversity. In general, we could consider perversities \( \bar{p} \) and \( \bar{p}' \) on \( X \) and \( X' \), respectively, such that the set-theoretic identity map \( X' \to X \) is \((\bar{p}', \bar{p})\)-stratified. However, given the “independence of perversity” results just above, it is simpler, and no great loss of generality, to consider the zero perversities, which we can both label as \( \bar{0} \), on both \( X \) and \( X' \). In this case the set-theoretic identity map \( X' \to X \) satisfies the conditions of Definition 4.1 and so induces maps on intersection homology.

Notationally, we let \(|X| = |X'| = |X^*|\) denote the common underlying topological space, while \( X^* \), as always, is the intrinsic CS set stratification. As for our other statements.
concerning fundamental classes, there will be a compact subset, which we denote \( K \subset |X| \).
Of course, \( K \) can inherit different filtrations depending on whether we think of it as contained in \( X \), \( X' \), or \( X^* \), but for simplicity, and as our main interest will be in removing \( K \) to consider spaces such as \( X - K \) or \( X' - K \), we will abuse notation a bit and just use \( K \), rather than \( |K| \), throughout.

**Proposition 8.26.** Let \( R \) be a Dedekind domain, let \( X \) be an \( n \)-dimensional stratified pseudomanifold, and let \( X' \) be an \( n \)-dimensional stratified pseudomanifold refining the stratification of \( X \). Suppose \( X \) and \( X' \) are compatibly \( R \)-oriented in the sense that the \( R \)-orientation on \( X \) induces that on \( X' \). Let \( K \) be a compact subset of \( |X| \), and let \( \Gamma_K \in I^0 H_n(X, X - K; R) \) and \( \Gamma'_K \in I^0 H_n(X', X' - K; R) \) be the fundamental classes over \( K \) given the orientation. Let \( \phi : I^0 H_n(X', X' - K; R) \to I^0 H_n(X, X - K; R) \) be induced by the identity map of the underlying space \( |X| = |X'| \). Then \( \phi(\Gamma'_K) = \Gamma_K \).

**Proof.** Let \( \sigma \) and \( \sigma' \) be the orientation sections on \( X \) and \( X' \), respectively. By the uniqueness clause of Lemma 8.15, it suffices to show that \( \phi(\Gamma'_K) \), which is still represented by the chain \( \Gamma'_K \), restricts to \( \sigma(x) \in I^0 H_n(X, X - \{x\}; R) \) for each \( x \in K \).

So, suppose \( x \in K \), and let \( U' \) be a distinguished neighborhood of \( x \) in the space \( |X' - \partial \gamma'| \) in the stratification \( X' \). As \( \Gamma'_K \) has image \( \sigma'(x) \in I^0 H_n(X', X' - \{x\}; R) \) by assumption, it follows from Lemma 8.19 that there is a smaller distinguished neighborhood \( V' \) of \( x \) in \( U' \) such that at every \( z \) in the regular strata of \( V' \), \( \Gamma'_K \) restricts to \( \sigma'(z) \in I^0 H_n(X, X - \{z\}; R) \). Now, let \( W \) be a distinguished neighborhood of \( x \) in the space \( |V'| \) in the stratification \( X \). As the regular strata of \( V' \) are dense in \( V' \), every regular stratum of \( W \) contains a point that is also in a regular stratum of \( V' \). As \( |W| \subset |V'| \), at any such point \( y \) we continue to have that \( \Gamma'_K \) represents \( \sigma'(y) \in I^0 H_n(X', X' - \{y\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R) \). But, on the regular strata of \( X' \), which are subsets of the regular strata of \( X \), the orientation bundle of \( X' - \Sigma_{X'} \) is the restriction of the orientation bundle over \( X - \Sigma_X \), so it follows that \( \phi(\Gamma'_K) \) also has image \( \sigma(y) \in I^0 H_n(X, X - \{y\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R) \) for any such \( y \). In other words, each regular stratum of \( W \) has a point \( y \) at which \( \phi(\Gamma'_K) \) represents \( \sigma(y) \). But the orientation bundle is constant on the regular strata, so, in fact, \( \phi(\Gamma'_K) \) represents \( \sigma(y) \) at any point \( y \) in a regular stratum of \( W \). It now follows again from Lemma 8.19 that \( \phi(\Gamma'_K) \) must represent \( \sigma(x) \) in \( I^0 H_n(X, X - \{x\}; R) \).

**Remark 8.27.** Notice that no claim is ever made in the proof of Proposition 8.26 that \( I^0 H_n(X, X - \{x\}; R) \cong I^0 H_n(X', X' - \{x\}; R) \). Indeed, by Lemma 8.19, this will not be true if the distinguished neighborhoods of \( x \) in \( X \) and \( X' \) have different numbers of regular strata. As a simple example, if \( X = \mathbb{R} \) with the trivial stratification, then \( I^0 H_1(\mathbb{R}, \mathbb{R} - \{0\}; R) \cong H_1(\mathbb{R}, \mathbb{R} - \{0\}; R) \cong R \), but if \( X' \) is stratified as \( \{0\} \subset \mathbb{R} \), then, by Lemma 8.19, \( I^0 H_1(X', X' - \{0\}; R) \cong R \oplus R \).

Unfortunately, unlike Corollary 8.10 Proposition 8.26 does not lead us directly to a way to compare fundamental classes for arbitrary pseudomanifold stratifications of a single underlying space \( |X| \). The problem is that, for two arbitrary pseudomanifold stratifications

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109We avoid using the notation \( N' \) from Lemma 8.19 to avoid a clash with the “prime” notation we are using here.
of $|X|$, the only way to compare them might be via the intrinsic stratification $X^*$. However, our previous work only guarantees that $X^*$ will be a CS set, even if $X$ is itself a stratified pseudomanifold, and our preceding work on fundamental classes requires a pseudomanifold stratification.

A workaround for this difficulty would be to utilize isomorphisms

$$I^0 H_n(X, X - K; R) \xrightarrow{\sim} I^0 H_n(X^*, X^* - K; R) \xleftarrow{\sim} I^0 H_n(X', X' - K; R).$$

We have such isomorphisms when the topological invariance of intersection homology, Theorem 5.52 applies for the perversity 0; however, we recall that topological invariance requires certain restrictions. First of all, Theorem 5.52 requires GM intersection homology, as non-GM intersection homology is not topologically invariant in general. However, we know that if $\bar{p} \leq \bar{t}$, then $I^0 H_\ast \cong I^0 H^\ast_{GM}$ by Proposition 6.7. There is also a condition on the perversity that is required for Theorem 5.52. The zero perversity $\bar{0}$, which is the perversity we want to use, does satisfy the condition of Theorem 5.52. However, it is not true that $\bar{0}(S) \leq \bar{t}(S)$ if $S$ has codimension one, in which case we would not necessarily have $I^0 H_\ast \cong I^0 H^\ast_{GM}$. Therefore, to utilize topological invariance, we must assume that our stratifications do not include codimension one strata, in which case it will be true that $\bar{0}(S) \leq \bar{t}(S)$ for all existing singular strata and so $I^0 H_\ast \cong I^0 H^\ast_{GM}$ and Theorem 5.52 applies.

**Proposition 8.28.** Let $R$ be a Dedekind domain, and let $X$ and $X'$ be any two $n$-dimensional stratified pseudomanifolds with the same underlying space $|X|$ and without codimension one strata. Suppose $X$ and $X'$ are compatibly $R$-oriented in the sense of Corollary 8.10. Let $K$ be a compact subset of $|X|$, and let $\Gamma_K \in I^0 H_n(X, X - K; R)$ and $\Gamma'_K \in I^0 H_n(X', X' - K; R)$ be the fundamental classes over $K$ given the $R$-orientation. Then $\Gamma_K$ and $\Gamma'_K$ correspond under the canonical isomorphisms

$$I^0 H_n(X, X - K; R) \xrightarrow{\sim} I^0 H_n(X^*, X^* - K; R) \xleftarrow{\sim} I^0 H_n(X', X' - K; R).$$

In particular, if $|X|$ is compact then the fundamental classes $\Gamma_X$ and $\Gamma_{X'}$ correspond in this manner.

**Proof.** The idea of the proof is essentially the same as that of Proposition 8.26, though we must make some modifications. Here, we cannot use the uniqueness clause of Lemma 8.15 for $X^*$, but the topological invariance provides the workaround. Let’s suppose we start with $\Gamma_K$. We will show that the image of $\Gamma_K$ under the isomorphisms is $\Gamma'_K$. For this, we can invoke the uniqueness of Lemma 8.15 for $X'$.

As topological invariance implies that $I^0 H_n(X, A; R) \cong I^0 H_n(X^*, A^*; R) \cong I^0 H_n(X', A'; R)$ for any open $|A| \subset |X|$, we see that $X$, $X'$, and $X^*$ all share a common orientation sheaf (up to canonical isomorphisms) and so the compatible $R$-orientations on these spaces come with a common orientation orientation section $\sigma$, which restricts to the given compatible manifold orientations on the regular strata of each stratification.

Now, let’s take a chain representing $\Gamma_K$. As $K$ is compact, we can find disjoint open sets $U$ and $U'$ with $K \subset U$ and $|\partial U| \subset U'$ by Corollary 2.43. Then the chain $\Gamma_K$ also represents an element of $I^0 H_n(X, X - |U|; R)$ that maps onto $\Gamma_K \in I^0 H_n(X, X - K; R)$. 375
Arguing as in the proof of Proposition 8.26, Lemma 8.19 implies that, for each \( x \in K \), there is a neighborhood \( V \) of \( x \) with \( |V| \subset U \) on which \( \Gamma_K \) represents \( \sigma(z) \in I^0H_n(X, X - \{z\}; R) \) for each \( z \) in each regular stratum of \( V \). Letting \( W \) be a distinguished neighborhood of \( x \) with \( |W| \subset |V| \), and letting \( y \in W \) be a point contained simultaneously in regular strata of \( X \) and \( X' \) (and so also of \( X^* \) as \( X^* \) coarsens \( X \) and \( X' \)), we have a diagram

\[
I^0H_n(X, X - \{|U|\}; R) \cong I^0H_n(X^*, X^* - \{|U|\}; R) \cong I^0H_n(X', X' - \{|U|\}; R) \\
\downarrow \quad \downarrow \quad \downarrow \\
I^0H_n(X, X - \{|W|\}; R) \cong I^0H_n(X^*, X^* - \{|W|\}; R) \cong I^0H_n(X', X' - \{|W|\}; R) \\
\downarrow \quad \downarrow \quad \downarrow \\
I^0H_n(X, X - \{y\}; R) \cong I^0H_n(X^*, X^* - \{y\}; R) \cong I^0H_n(X', X' - \{y\}; R).
\]

The isomorphisms of the diagram come from Theorem 5.52. So, suppose we let \( \gamma' \) denote the image of the element of \( I^0H_n(X, X - \{|U|\}; R) \) represented by the chain \( \Gamma_K \) under the maps across the top row, so \( \gamma' \in I^0H_n(X', X' - \{|U|\}; R) \). By our choice of \( W \), \( \Gamma_K \) represents \( \sigma(y) \in I^0H_n(X, X - \{y\}; R) \), and so it follows from the diagram and the compatibility of the orientation sections (especially at points of regular strata) that \( \gamma' \) also represents \( \sigma(y) \in I^0H_n(X', X' - \{y\}; R) \). But \( y \) was an arbitrary point in a regular stratum of both \( X \) and \( X' \) in \( |W| \). As the unions of regular strata in \( X \) and \( X' \) are dense in \( X \) and \( X' \), respectively, every regular stratum of \( W \) contains a point that is also in a regular stratum of \( X \). So every regular stratum of \( W \subset X' \) has a point at which \( \gamma' \) represents \( \sigma(y) \) in \( I^0H_n(X', X' - \{y\}; R) \), and it follows from Lemma 8.19 that \( \gamma' \) must map to \( \sigma(x) \) in \( I^0H_n(X', X' - \{x\}; R) \). But \( x \) was arbitrary in \( K \), so the chain \( \gamma' \) must represent \( \Gamma'_K \in I^0H_n(X', X' - K; R) \). Finally, we have a diagram

\[
I^0H_n(X, X - \{|U|\}; R) \cong I^0H_n(X^*, X^* - \{|U|\}; R) \cong I^0H_n(X', X' - \{|U|\}; R) \\
\downarrow \quad \downarrow \quad \downarrow \\
I^0H_n(X, X - K; R) \cong I^0H_n(X^*, X^* - K; R) \cong I^0H_n(X', X' - K; R),
\]

and as the chain \( \Gamma_K \) in \( I^0H_n(X, X - \{|U|\}; R) \) maps to the fundamental class \( \Gamma_K \in I^0H_n(X, X - K; R) \) and to \( \gamma' \in I^0H_n(X', X' - \{|U|\}; R) \), and as \( \gamma' \) maps to \( \Gamma'_K \in I^0H_n(X', X' - K; R) \), it follows that the composition along the bottom of the diagram takes \( \Gamma_K \) to \( \Gamma'_K \).

**Remark 8.29.** It follows from Proposition 8.28 that if \( X \) is a compact \( n \)-dimensional \( R \)-oriented stratified pseudomanifold without codimension one strata, then the fundamental
class $\Gamma_X$ is a topological invariant in the following sense: Suppose that $Y$ is another compact $R$-oriented stratified pseudomanifold without codimension one strata and that $f : |X| \to |Y|$ is a topological homeomorphism, i.e. that it is a homeomorphism of the underlying spaces without regard to the stratifications. Then $X$ induces an image stratification, say $Y'$, on $Y$, and an image $R$-orientation on $Y'$ (via the pointwise isomorphisms $I^0H_n(X, X - \{x\}; R) \cong I^0H_n(Y', Y' - \{f(x)\}; R)$ induced by the stratified homeomorphism $X \to Y'$). Suppose that $f$ is orientation-preserving in the sense that the image $R$-orientation is compatible with the given $R$-orientation on $Y$ in the sense of Corollary 8.10. Then it must also be the case, applying Proposition 8.28, that $f(\Gamma_X) \in I^0H_n(Y'; R)$ corresponds to $\Gamma_Y$ under the canonical isomorphisms $I^0H_n(Y'; R) \xrightarrow{\cong} I^0H_n(Y^*; R) \xrightarrow{\cong} I^0H_n(Y; R)$.

This is essentially the content of [38 Corollary 5.23], although there it is mistakenly asserted that $f(\Gamma_X) \in I^0H_n(Y; R)$.

### 8.2.6 Products

We include here one result on product orientations that we shall need below.

Recall that if $M_1, M_2$ are $R$-oriented manifolds of dimensions $n_1, n_2$, then $M_1 \times M_2$ has a natural orientation. In fact, suppose $\mathcal{O}_1, \mathcal{O}_2$ are the orientation bundles over $M_1, M_2$, respectively, and that $\mathcal{O}$ is the orientation bundle for $M_1 \times M_2$. Then the isomorphisms $H_{n_1}(M_1, M_1 - \{x_1\}; R) \otimes H_{n_2}(M_2, M_2 - \{x_2\}; R) \xrightarrow{\cong} H_{n_1+n_2}(M_1 \times M_2, M_1 \times M_2 - \{(x_1, x_2)\}; R)$, and their analogues replacing $x_1$ and $x_2$ with small Euclidean disk neighborhoods, provide an isomorphism $\mathcal{O} \cong \mathcal{O}_1 \otimes \mathcal{O}_2$. Furthermore, if $\mathcal{O}_1, \mathcal{O}_2$ are global sections of the respective orientation bundles $\mathcal{O}_1, \mathcal{O}_2$, then at every points $(x_1, x_2) \in M_1 \times M_2$ we have $\mathcal{O}_1(x_1) \otimes \mathcal{O}_2(x_2) \in H_{n_1}(M_1, M_1 - \{x_1\}; R) \otimes H_{n_2}(M_2, M_2 - \{x_2\}; R) \xrightarrow{\cong} H_{n_1+n_2}(M_1 \times M_2, M_1 \times M_2 - \{(x_1, x_2)\}; R) \cong R$. So we can think of $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2$ as a global section of the product orientation bundle $\mathcal{O}$ of $M_1 \times M_2$. Locally, if $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ are represented by chains $\xi_1 \in H_{n_1}(M_1, M_1 - \{x_1\}; R)$ and $\xi_2 \in H_{n_2}(M_2, M_2 - \{x_2\}; R)$, then $\mathcal{O}((x_1, x_2))$ is represented by $\xi_1 \times \xi_2 \in H_{n_1+n_2}(M_1 \times M_2, M_1 \times M_2 - \{(x_1, x_2)\}; R)$.

In the case of pseudomanifolds, we therefore have the following result:

**Proposition 8.30.** Let $R$ be a Dedekind domain. Suppose $X$ is an $R$-oriented $n$-dimensional stratified pseudomanifold and that $M$ is an $R$-oriented $m$-dimensional manifold. Let $K_1 \subset M$ and $K_2 \subset X$ be compact subsets. Then $\Gamma_{K_1} \times \Gamma_{K_2} \in I^0H_{n+m}(M \times X, (M \times X) - (K_1 \times K_2); R)$ is the fundamental class of $M \times X$ over $K_1 \times K_2$ with respect to the product orientation on $M \times X$.

**Proof.** The proof is similar in spirit to the other proofs in this section, so we will be a little sketchy in the details here.

Suppose $(x_1, x_2) \in K_1 \times K_2$. We must show that, for any such $(x_1, x_2)$, the chain $\Gamma_{K_1} \times \Gamma_{K_2}$ represents $\mathcal{O}((x_1, x_2)) \in I^0H_{n+m}(M \times X, (M \times X) - \{(x_1, x_2)\}; R)$. But suppose $U_1, U_2$ are distinguished neighborhoods of $x_1$ and $x_2$ in $M$ and $X$, respectively. We can assume that $U_1 \subset M - \partial \Gamma_{K_1}$ and $U_2 \subset X - \partial \Gamma_{K_2}$. Then, we know via Lemma 8.19 that, assuming $U_1$ and $U_2$ are sufficiently small, $\Gamma_{K_1}$ represents $\mathcal{O}_1(z_1)$ for each $z_1 \in U_1$ and that $\Gamma_{K_2}$ represents $\mathcal{O}(z_2)$ for each $z_2$ contained in a regular stratum of $U_2$. But this implies that $\Gamma_{K_1} \times \Gamma_{K_2}$ must
represent \( o((z_1, z_2)) \) \( \in I^0H_{n+m}(M \times X, (M \times X) - \{(z_1, z_2)\}; R) \). So, by applying Lemma 8.19 again to a small enough distinguished neighborhood of \((x_1, x_2)\) in \( M \times X \), \( \Gamma_{K_1} \times \Gamma_{K_2} \) must indeed represent \( o((z_1, z_2)) \).

Remark 8.31. Proposition 8.30 focuses on the case where one term is a manifold, as that is the result we will need later. A version of this proposition in which both terms are pseudomanifolds would be a bit more involved as \( Q_{0,0} \) is not, in general, the zero perversity on the product space. This is mitigated by the perversity invariance we proved in Proposition 8.22, but we will not go into the details.

### 8.3 Poincaré duality

At last, in this section, we come to the raison d’être for intersection homology: Poincaré duality on stratified pseudomanifolds.

In previous sections, we have seen that if \( X \) is a compact \( R \)-oriented stratified pseudomanifold of dimension \( n \), then it admits a fundamental class \( \Gamma_X \in I^0H_n(X; R) \) that is consistent with the \( R \)-orientation in the sense that, for any \( x \in X \), the image of \( \Gamma_X \) in \( I^pH_n(X, X - \{x\}; R) \) takes the value of the orientation section \( o(x) \). The cap product with this class induces a map

\[
\sim \Gamma_X : I_pH^i(X; R) \to I^{Dp}H_{n-i}(X; R)
\]

if \( X \) is locally \((\bar{p}; R)\)-torsion free.\footnote{The requirement for the existence of this cap product is actually that \( X \) be either locally \((\bar{p}; R)\)-torsion free or locally \((D\bar{p}; R)\)-torsion free, but it will follow as a consequence of Poincaré duality that an \( R \)-oriented stratified pseudomanifold is locally \((\bar{p}; R)\)-torsion free if and only if it is locally \((D\bar{p}; R)\)-torsion free. See Corollary 8.36.} Note that

\[
D\bar{p} + D(D\bar{p}) = D\bar{p} + \bar{p} = \bar{t} - \bar{p} + \bar{p} = \bar{t},
\]

so indeed,

\[
D\bar{0} = \bar{t} \geq D\bar{p} + D(D\bar{p}),
\]

which is the requirement for this cap product to be well defined in Definition 7.23. The intersection (co)homology version of Poincaré duality states that this map is an isomorphism. Of course, if \( X \) is an unstratified manifold, and hence automatically locally torsion free, this reduces to precisely the statement of classical Poincaré duality.

More generally, if \( X \) is any \( R \)-oriented locally \((\bar{p}; R)\)-torsion free \( n \)-dimensional stratified pseudomanifold, we have a Poincaré duality isomorphism \( \mathcal{D} : I_\bar{p}H^i(X; R) \to I^{D\bar{p}}H_{n-i}(X; R) \), where \( X \) is no longer assumed to be compact. For this, recall that if \( X \) is an \( R \)-oriented stratified pseudomanifold of dimension \( n \) (not necessarily compact) and if \( K \subset X \) is a compact subset, we have shown in the preceding section that we have a fundamental class \( \Gamma_K \in I^0H_n(X, X - K; R) \) that is consistent with the \( R \)-orientation in the sense that, for any \( x \in K \), the image of \( \Gamma_K \) in \( I^pH_n(X, X - \{x\}; R) \) takes the value of the orientation section \( o(x) \). Suppose now that \( K' \subset X \) is another compact subset with \( K \subset K' \). If
\( \Gamma' \in I^0H_n(X, X - K'; R) \) is the corresponding fundamental class, then the image of \( \Gamma' \) in \( I^0H_n(X, X - K; R) \) must be \( \Gamma_K \): the map between homology groups is induced by inclusion so the image of \( \Gamma_K' \) and its image in \( I^0H_n(X, X - K; R) \) are represented by the same chain. But, by definition, this chain represents \( \sigma(x) \) for each \( x \in K' \), and so, in particular, at each \( x \in K \). It follows that the chain therefore represents \( \Gamma_K \) by the uniqueness of Lemma 8.15. This argument shows that the collection \( \{ \Gamma_K \} \), as \( K \) varies over the compact subsets of \( X \), constitutes an element of \( \lim_{\leftarrow} I^0H_n(X, X - K; R) \). By Lemma 7.89, the cap product induces a map

\[
\sim: I^0H_c^i(X; R) \otimes \lim_{\leftarrow} I^0H_n(X, X - K; R) \to I^{D^\beta}H_n-i(X; R).
\]

Fixing the element \( \Gamma \in \lim_{\leftarrow} I^0H_n(X, X - K; R) \) corresponding to the collection \( \{ \Gamma_K \} \), we define \( \mathcal{D} \) to be the map

\[
I^0H_c^i(X; R) \to I^{D^\beta}H_n-i(X; R)
\]

\[
\alpha \mapsto (-1)^{in} \alpha \sim \Gamma.
\]

In other words, if \( \alpha \in I^0H_c^i(X; R) \), then

\[
\mathcal{D}(\xi) = (-1)^{in} \alpha \sim \Gamma.
\]

When \( X \) is compact, the module \( I^0H_n(X; R) \) is initial among the \( I^0H_n(X, X - K; R) \), so \( \lim_{\leftarrow} I^0H_n(X, X - K; R) \cong I^0H_n(X; R) \), and, in this case, \( \mathcal{D} \) reduces, up to sign, to the standard cap product with \( \Gamma \).

**Remark 8.32.** Wait a minute — where did that sign come from? Notice that we have defined \( \mathcal{D}(\alpha) = (-1)^{|\alpha|}n \cdot \sim \Gamma \) and not simply as \( \alpha \sim \Gamma \). This deserves some explanation. The issue is that there are many circumstances in geometric topology where it is desirable to think of the Poincaré duality map as a chain map from cochains to chains. Of course, in our treatment here, we have generally only pursued cap products as operators on homology and cohomology elements, but, in classical algebraic topology, if \( \xi \in S_k(X) \) is a fixed chain and we have chosen a fixed definition of the cap product then \( \sim \xi \) induces a function of chain complexes \( S^*(X) \to S_{k-*}(X) \). In particular, if \( M \) is a compact manifold, then one would like the Poincaré duality map determined by some sort of cap product with a fundamental class to provide a map \( S^*(M) \to S_{k-*}(M) \) in the appropriate category of (co)chain complexes. This is important, for example, in surgery theory.

The problem is that \( \sim \Gamma \) is not a chain map because it does not obey the proper sign conventions. To explain, first let us recall that if we have a map of (homologically indexed) chain complexes \( f: C_* \to D_* \) that raises degrees by \( k \), i.e. \( f \) restricts to homomorphisms \( f_i: C_i \to D_{i+k} \), then \( f \) is considered a chain map of degree \( k \) if \( \partial f = (-1)^k f \partial \); see [23, Section VI.10]. Most of the chain maps we have considers so far have been degree 0 chain maps, and so the sign is invisible. Next, recall that we can consider cohomologically indexed complexes to be equivalent to homologically indexed complexes via the identification \( C^* = C_{-*} \). Unfortunately, this clashes with the standard topological use whereby, say, \( S_*(X) \) and \( S^*(X) \)

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111 As we have discussed, there is some flexibility in the definition of the cap product, but the ambiguities can be removed by passing to the appropriate homotopy category of (co)chain complexes.
are not reindexings of the same complex, but rather different complexes; so, for clarity in
the remainder of this remark, we replace \( S^*(X) \) with \( \text{Hom}^*(S_*(X); R) = \text{Hom}_{-*}(S_*(X); R) \).
This last identification allows us to treat cochain complexes as homologically indexed.

Now, fixing an element \( \xi \in S_k(X) \), the function \( \alpha \to \alpha \sim \xi \) provides homomorphisms
\( \text{Hom}_{-*}(S_*(X); R) \to S_{k-*}(M) \), and so raises the (homological) degree by \( k \). However, by
Lemma \( 7.28 \), fixing a particular cap product at the chain level yields
\[
\partial(\alpha \sim \xi) = (d\alpha) \sim \xi + (-1)^{|\alpha|} \alpha \sim \partial \xi.
\]
If \( \partial \xi = 0 \), we obtain \( \partial(\alpha \sim \xi) = (d\alpha) \sim \xi \), so \( \alpha \to \alpha \sim \xi \) is not signed as a chain map.

Continuing to assume \( \xi \) is a cycle, consider now the map \( \mathcal{D}_\xi \) such that \( \mathcal{D}_\xi(\alpha) = (-1)^{|\alpha||\xi|} \alpha \sim \xi \). Then, we have
\[
\mathcal{D}_\xi(d\alpha) = (-1)^{|\alpha|+1+|\xi|}(d\alpha) \sim \xi \\
= (-1)^{|\alpha|+1+|\xi|}\partial(\alpha \sim \xi) \\
= (-1)^{|\xi|}\partial((-1)^{|\alpha||\xi|} \alpha \sim \xi) \\
= (-1)^{|\xi|}\partial(\mathcal{D}_\xi(\alpha)).
\]
Therefore, \( \mathcal{D}_\xi \) is a chain map. So, in particular, if \( M \) is a compact oriented manifold and \( \Gamma \)
is the fundamental class, then \( \mathcal{D} = \mathcal{D}_\Gamma \) is a chain map from cochains to chain, as desired.
Similarly, if \( X \) is a compact oriented stratified pseudomanifold, \( \mathcal{D} = \mathcal{D}_\Gamma \) is a chain map,
assuming we have fixed a particular choice\(^{112} \) of \( \bar{d} \) so that we can speak of the cap product
at the chain level.

Of course these signs are not critical for obtaining our Poincaré duality isomorphisms
in each fixed degree, but it allows us to stay consistent with the necessary properties of
Poincaré duality as a map in other sources. In particular, this is consistent with \([38, 39]\). For more about this sign convention, see \([33\text{ Section 4.1}].

Before proceeding on to the proof of Poincaré duality, we next present an example that
demonstrates the necessity of the torsion free condition.

Example 8.33. Let \( X \) be the suspension of \( \mathbb{R}P^3 \), i.e. \( X = S\mathbb{R}P^3 \) with the standard suspension
stratification, assuming \( \mathbb{R}P^3 \) is stratified trivially. So \( X \) has two singular points corresponding
to the suspension points. As \( \mathbb{R}P^3 \) is \( \mathbb{Z} \)-orientable, so is \( X - \Sigma X \cong (-1,1) \times \mathbb{R}P^3 \). Let
us choose a perversity \( \bar{p} \) on \( X \) so that its value on each suspension point is 1. As the codimension
of the suspension points is 4, the value of \( D\bar{p} \) on these points is then \( 4 - 2 - 1 = 1 \), so \( D\bar{p} = \bar{p} \) in this situation. We also have \( \bar{p} \leq \bar{t} \), so we even have \( I^pH_*(X) \cong I^pH^*_M(X) \).
Applying the suspension computation of Theorem \( 4.43 \), we have
\[
I^pH_i(X) \cong \begin{cases} 
I^p\tilde{H}_{i-1}(\mathbb{R}P^3), & i > 2, \\
0, & i = 2, \\
I^pH_i(\mathbb{R}P^3), & i < 2.
\end{cases}
\]
So \( I^pH_4(X) \cong I^pH_0(X) \cong \mathbb{Z} \) and \( I^pH_1(X) \cong \mathbb{Z}_2 \), and the other groups are 0.

\(^{112}\)In the manifold case, we can assume that we are using the traditional Alexander-Whitney diagonal.

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Similarly, as $X$ is compact, we have

$$I\bar{p}H_4^i(X) = I\bar{p}H_i(X) \cong \text{Hom}(I\bar{p}H_1(X); \mathbb{Z}) \oplus \text{Ext}(I\bar{p}H_{i-1}(X); \mathbb{Z}),$$

using Theorem 7.4. So, applying our above computation for $I\bar{p}H_4(X)$, we obtain the following.

<table>
<thead>
<tr>
<th>$I\bar{p}H_0(X)$</th>
<th>$I\bar{p}H_1(X)$</th>
<th>$I\bar{p}H_2(X)$</th>
<th>$I\bar{p}H_3(X)$</th>
<th>$I\bar{p}H_4(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cong \mathbb{Z}$</td>
<td>$\cong \mathbb{Z}_2$</td>
<td>$\cong 0$</td>
<td>$\cong 0$</td>
<td>$\cong \mathbb{Z}$</td>
</tr>
</tbody>
</table>

Comparing across the rows, we see that Poincaré duality fails, at least in the torsion subgroups. We do observe, however, that if we replaced our $\mathbb{Z}$ coefficients with, say, coefficients in $\mathbb{Q}$, then similar computations would result in identical answers except with $\mathbb{Z}$ terms replaced with $\mathbb{Q}$ terms and $\mathbb{Z}_2$ terms replaced with 0. Then we have

<table>
<thead>
<tr>
<th>$I\bar{p}H_0(X; \mathbb{Q})$</th>
<th>$I\bar{p}H_1(X; \mathbb{Q})$</th>
<th>$I\bar{p}H_2(X; \mathbb{Q})$</th>
<th>$I\bar{p}H_3(X; \mathbb{Q})$</th>
<th>$I\bar{p}H_4(X; \mathbb{Q})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cong \mathbb{Q}$</td>
<td>$\cong 0$</td>
<td>$\cong 0$</td>
<td>$\cong 0$</td>
<td>$\cong \mathbb{Q}$</td>
</tr>
</tbody>
</table>

In this case, we see that the corresponding groups are isomorphic, although we have not yet shown that this isomorphism is via the cap product with a fundamental class.

We now turn to our proof of Poincaré duality.

**Theorem 8.34** (Poincaré duality).\footnote{Or, equivalently, $X$ can be locally $(D\bar{p}; R)$-torsion free; see Footnote 110 and Corollary 8.36} Suppose $R$ is a Dedekind domain, and let $X$ be an $n$-dimensional $R$-oriented locally $(\bar{p}; R)$-torsion free stratified pseudomanifold. Then the duality map

$$D : I\bar{p}H^i_c(X; R) \to I\bar{p}D_{n-i}(X; R)$$

is an isomorphism. In particular, if $X$ is compact, then the cap product

$$\Gamma_X : I\bar{p}H^i(X; R) \to I\bar{p}D_{n-i}(X; R)$$

is an isomorphism.
Proof. Although we have been careful to define our duality map \( D \) to account for the signs needed for \( D \) to be a chain map when thought of at the chain/cochain level, we can safely ignore these signs and work with the cap product in fixed degrees for the purposes of proving the theorem.

The proof will be by induction on depth. The base case is that for which \( X \) is an unstratified manifold, in which case this is classical Poincaré duality, for which see [53, Theorem 3.35]. So, assume that the theorem is proven for stratified pseudomanifolds of depth \( < d \), and suppose \( X \) has depth \( d \). We will make a Mayer-Vietoris argument by applying Theorem 5.3. If \( U \subset X \) is an open set, let \( F_*(U) = I^D \phi \Lambda(U; R) \), let \( G_*(U) = I^D \phi H_{n-*}(U; R) \), and let \( \Phi = D^U \). Here \( D^U \) is defined as in Lemma 7.90 given the element, say \( \Gamma \in \lim I^D H_n(X, X-K; R) \) represented by the \( \{\Gamma_K\} \), there is an image element \( \Gamma^U \in \lim I^D H_n(U, U-K; R) \), where now the \( K \) range over compact subsets of \( U \). This image element is obtained by the composition of the canonical map \( \lim_{K \subset X} I^D H_n(X, X-K; R) \to \lim_{K \subset U} I^D H_n(X, X-K; R) \) with the isomorphism \( \lim_{K \subset U} I^D H_n(X, X-K; R) \cong I^D H_n(U, U-K; R) \) determined by excision isomorphisms for each \( K \). The map \( D^U \) is the signed cap product, in the sense of Lemma [7.89] with \( \Gamma^U \). However, if \( K \subset U \), then the isomorphism \( I^D H_n(X, X-K; R) \cong I^D H_n(U, U-K; R) \), induced by inclusion, takes \( \Gamma_K \in I^D H_n(X, X-K; R) \) to the corresponding fundamental class in \( I^D H_n(U, U-K; R) \) for the orientation induced on \( U \) from \( X \). Therefore, \( D^U \) is also the Poincaré duality map on \( U \) determined by restricting the orientation from \( X \).

We must verify that the conditions of Theorem 5.3 are satisfied.

The functors \( F_* \) and \( G_* \) admit Mayer-Vietoris sequences with \( \Phi \) inducing a map between them by Lemma 7.90. Incidentally, the commutativity of the diagram of Lemma 7.90 appropriately restricted to each summand in the middle term, demonstrates that \( \Phi \) is indeed a natural transformation. If \( U \) is empty, \( \Phi \) is trivially an isomorphism, and if \( U \) is an open subset of \( X \) contained in a single stratum and homeomorphic to Euclidean space, then \( U \) must be contained in a regular stratum, as \( X \) is a stratified pseudomanifold, and in this case \( \Phi \) is again an isomorphism by Poincaré duality for manifolds.

For the limit condition in Theorem 5.3, we aim to employ Lemma 5.5 for which we need to verify that the maps \( \lim_{\alpha} F_*(U_\alpha) \to F_*(\cup U_\alpha) \) and \( \lim_{\alpha} G_*(U_\alpha) \to G_*(\cup U_\alpha) \) are isomorphisms. For \( G_* \), this is the content of Lemma 6.23. We need to check that \( F_* \) also has this property. As this could potentially be a useful fact on its own, we state and prove this as a separate lemma, Lemma 8.35, following the remainder of the proof.

Finally, we must show that if \( U \cong \mathbb{R}^k \times cL^{n-k-1} \) is a distinguished neighborhood in \( X \) and \( \Phi : F_*(\mathbb{R}^k \times (cL - \{v\})) \to G_*(\mathbb{R}^k \times (cL - \{v\})) \) is an isomorphism, then so is \( \Phi : F_*(\mathbb{R}^k \times cL) \to G_*(\mathbb{R}^k \times cL) \). As has often been the case in our Mayer-Vietoris arguments, we will show directly that \( \Phi : F_*(\mathbb{R}^k \times cL) \to G_*(\mathbb{R}^k \times cL) \) is always an isomorphism, relying on our induction assumptions more so than the supposition that \( \Phi \) is an isomorphism \( F_*(\mathbb{R}^k \times (cL - \{v\})) \to G_*(\mathbb{R}^k \times (cL - \{v\})) \).

There are essentially two cases to consider, the first being the trivial case: By stratified homotopy invariance and the cone formula, Theorem 6.10

\[
G_i(\mathbb{R}^k \times cL) = I^D \phi H_{n-i}(\mathbb{R}^k \times cL; R) \cong I^D \phi H_{n-i}(cL; R) = 0
\]

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if \( n - i \geq n - k - D\bar{p}(v) - 1 \), i.e. if \( i \leq k + D\bar{p}(\{v\}) + 1 \). But now

\[
D\bar{p}(\{v\}) = \overline{\ell}(\{v\}) - \bar{p}(\{v\}) = \text{codim}(\{v\}) - 2 - \bar{p}(\{v\}) = n - k - 2 - \bar{p}(\{v\}).
\]

So, altogether, \( G_i(\mathbb{R}^k \times cL) = 0 \) if \( i \leq n - \bar{p}(\{v\}) - 1 \).

Next, consider \( F_i(\mathbb{R}^k \times cL) = I^L_h(\mathbb{R}^k \times cL; R) \). To compute the compactly supported intersection cohomology, we can choose a cofinal collection of compact subsets of the form \( K_{r,s} = B_r \times \tilde{c}_sL \), where \( B_r \) is the closed ball of radius \( r \) in \( \mathbb{R}^k \) and \( \tilde{c}_sL \) is our closed subcone out to \( s \) in the cone coordinate. But by stratified homotopy invariance, the direct system \( I^L_h(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - K_{r,s}; R) \) is constant, with all terms being isomorphic to \( I^L_h(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0,v)\}; R) \). Therefore, applying the universal coefficient theorem (Theorem 7.4) and the Künneth Theorem (Theorem 6.25), we have

\[
F_i(\mathbb{R}^k \times cL) = I^L_h(\mathbb{R}^k \times cL; R) \\
\cong I^L_h(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0,v)\}; R) \\
\cong \text{Hom}(I^L_h(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0,v)\}; R), R) \\
\oplus \text{Ext}(I^L_h(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0,v)\}; R), R) \\
\cong \text{Hom}(I^L_h(\mathbb{R}^k \times cL; R), R) \oplus \text{Ext}(I^L_h(cL, cL - \{v\}; R), R). \]

By the relative cone formula (Corollary 6.12), \( I^L_h(cL, cL - \{v\}; R) = 0 \) for \( j \leq n - k - \bar{p}(\{v\}) - 1 \). So, the first summand of \( F_i(\mathbb{R}^k \times cL) = 0 \) when \( i - k \leq n - k - \bar{p}(\{v\}) - 1 \), i.e. when \( i \leq n - \bar{p}(\{v\}) - 1 \). Similarly, the second summand vanishes when \( i - k - 1 \leq n - k - \bar{p}(\{v\}) - 1 \) i.e. when \( i \leq n - \bar{p}(\{v\}) \). Therefore, \( F_i(\mathbb{R}^k \times cL) = 0 \) for \( i \leq n - \bar{p}(\{v\}) - 1 \), so that \( F_i(\mathbb{R}^k \times cL) = 0 = G_i(\mathbb{R}^k \times cL) \) for \( i \leq n - \bar{p}(\{v\}) - 1 \), and this must be induced by \( \Phi \) as there is a unique map between trivial modules.

Next, we must consider \( i \geq n - \bar{p}(\{v\}) \). We first compute abstractly the relevant modules in this range. Our above computations with the cone formula show that we are in the range where we have isomorphisms

\[
I^{D\bar{p}}_i H_{n-i}(L; R) \cong I^{D\bar{p}}_i H_{n-i}(cL - \{v\}; R) \\
\cong I^{D\bar{p}}_i H_{n-i}(cL; R).
\]

For cohomology, the analogous fact we will require is that, for these same degrees, we have isomorphisms \( I^{L}_i H_{i-k-1}(L; R) \cong I^{L}_i H_{i-k-1}(cL - \{v\}; R) \) and \( I^{L}_i H_{i}(cL; R) \). Here \( I : L \hookrightarrow cL - \{v\} \) is the inclusion of \( L \) into the cone at some cone radius; it is a stratified homotopy equivalence. For the second isomorphism, we consider again that \( I^{L}_i H_i(cL; R) \cong \text{Hom}(I^{L}_i H_j(cL; R), R) \oplus \text{Ext}(I^{L}_i H_{j-1}(cL; R), R) \) while \( I^{L}_i H_{a}(cL; R) = 0 \) for \( a \geq n - k - \bar{p}(\{v\}) - 1 \). So \( I^{L}_i H_i(cL; R) = 0 \) for \( j > n - k - \bar{p}(\{v\}) - 1 \). If \( j = n - k - \bar{p}(\{v\}) - 1 \), then \( I^{L}_i H_i(cL; R) \cong \text{Ext}(I^{L}_i H_{n-k-\bar{p}(\{v\})-2}(cL; R), R) \cong \text{Ext}(I^{L}_i H_{n-k-\bar{p}(\{v\})-2}(L; R), R) \), using the cone formula. But we have assumed that \( X \) is locally \((\bar{p}; R)\)-torsion free, which means
that $I^pH_{n-k-p(v)}(L; R)$ is $R$-torsion free. Therefore, we have $I^pH^j(cL; R) = 0$ for $j \geq n - k - p(v) - 1$, and so, by the long exact sequence of the pair, $I^pH^j(cL - \{v\}; R) \to I^pH^{j+1}(cL, cL - \{v\}; R)$ for $j \geq n - k - p(v) - 1$. Taking $j = i - k - 1$, this provides the isomorphisms $I^pH^{i-k-1}(L; R) \cong I^pH^{i-k-1}(cL - \{v\}; R) \to I^pH^{i-k}(cL, cL - \{v\}; R)$ for $i - k - 1 \geq n - k - p(v) - 1$, i.e. for $i \geq n - p(v)$, as desired. Notice the role that the locally torsion free condition plays in this argument!

These observations allow us to move on to the following diagram:

\[
\begin{array}{ccc}
I^pH^{i-k-1}(L; R) & \xrightarrow{\sim} & I^pH_{n-i}(L; R) \\
\downarrow \cong & & \downarrow \cong \\
I^pH^{i-k-1}(cL - \{v\}; R) & \xrightarrow{\sim} & I^pH_{n-i}(cL - \{v\}; R) \\
\downarrow d^* \cong & & \downarrow d^* \cong \\
I^pH^{i-k}(cL, cL - \{v\}; R) & \xrightarrow{\sim} & I^pH_{n-i}(cL; R) \\
\downarrow \eta^* \cong & & \downarrow \eta^* \cong \\
I^pH^i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R) & \xrightarrow{(\eta \times \tilde{c}(\Gamma_L))} & I^pH_{n-i}(\mathbb{R}^k \times cL; R).
\end{array}
\]

If we can show that this diagram commutes up to sign and that all the maps so labeled are isomorphisms, it will follow that all of the maps are isomorphisms, and so, in particular, the map at the bottom, $I^pH^i(\mathbb{R}^k \times cL - \{(0, v)\}; R) \xrightarrow{\sim(\eta \times \tilde{c}(\Gamma_L))} I^pH_{n-i}(\mathbb{R}^k \times cL; R)$, is an isomorphism. If, furthermore, we can show that this bottom horizontal map is the cap product with the fundamental class of $\mathbb{R}^k \times cL$ over $\{(0, v)\}$, this will suffice to prove the theorem, as $I^pH^i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R)$ is isomorphic to the direct limit $\lim_{\rightarrow} I^pH^i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - K; R)$. More precisely, taking $K = B_r \times \bar{c}_s L$ and $K' = B_{r'} \times \bar{c}_s' L$ with $r < r'$ and $s < s' < 1$, this last claim follows from the commutativity of the diagrams of the form

\[
\begin{array}{ccc}
I^pH^i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R) & \xrightarrow{\sim} & I^pH^i(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - K; R) \\
\downarrow \Gamma_{K'} & & \downarrow \Gamma_{K'} \\
I^pH_{n-i}(\mathbb{R}^k \times cL; R) & = & I^pH_{n-i}(\mathbb{R}^k \times cL; R).
\end{array}
\]

The horizontal maps are isomorphisms by stratified homotopy invariance. Here, the class $\Gamma_{K'}$ is the fundamental class of $\mathbb{R}^k \times cL$ over $K'$, but, by the uniqueness statement of Lemma 8.15, the same chain represents also the fundamental classes over $K$ and $\{(0, v)\}$, hence the notation. Commutativity follows from naturality of the cap product, Lemma 7.34. Putting
such diagrams together for a cofinal system consisting of $K$ of the form $\bar{B}_r \times \bar{c}_r L$ provides the claimed isomorphism of direct limits, assuming we’ve shown the map on the left is an isomorphism.

Turning to our claims about Diagram (34), the top horizontal map $I^*_p H^{1-k-1}(L; R) \to I^Dp H_{n-1}(L; R)$ is an isomorphism by our induction assumption. For this, we observe that $L$ is a stratified pseudomanifold by Lemma 2.54, and it is $R$-orientable as a link of the $R$-orientable $X$ by the arguments in the proof of Lemma 8.13. Additionally, $L$ is locally $(\bar{p}; R)$-torsion free because, by Remark 2.55, its links are all also links of $X$, which is locally $(\bar{p}; R)$-torsion free, and the locally torsion free vanishing condition can be stated in terms of the dimensions of the links themselves, without reference to the ambient space. Therefore, as $L$ has lower depth than $X$, the induction hypothesis applies. We can here choose a particular orientation of $L$ such that, if we give $\mathbb{R}^k$ and $(0, 1)$ their standard orientations, then the product orientation on $\mathbb{R}^k \times (0, 1) \times L$ agrees with the given orientation inherited from $\mathbb{R}^k \times cL \subset X$.

The map $I$ is the inclusion $I: L \hookrightarrow cL - \{v\}$ of $L$ into the cone at some fixed cone coordinate, so the two top vertical maps of Diagram (34) are isomorphisms by stratified homotopy invariance. Furthermore, $I$ is a normally nonsingular inclusion, so the top square commutes by Lemma 7.34 and Remark 7.35.

For the center square of Diagram (34), the vertical maps are isomorphisms by the discussion just above the diagram. We let $\bar{c}(\Gamma_L)$ be the class of the singular cone on the chain $\bar{c}(\Gamma_L)$ in particular, this means that $\bar{c}(\Gamma_L)$ maps to $I(\Gamma_L)$ under the isomorphism $\partial_*: I^0 H_{n-1} I(cL, cL - \{v\}; R) \to I^0 H_{n-k-1}(cL - \{v\}; R)$; this is an isomorphism by the relative cone formula, Corollary 6.12. For the commutativity of the this square, up to sign, we can apply Lemma 7.60. Comparing that lemma to our setting, the $X$ of the lemma is $cL$, the subspace $B$ is $cL - \{v\}$, and $A = \emptyset$. The chain $\xi$ of the lemma is our $\bar{c}(\Gamma_L) \in I^0 H_{n-k-1}(cL - \{v\}; R)$, and, as $A = \emptyset$, the map $e$ in the lemma is the identity map. With these identifications, the lemma applies to demonstrate that this square commutes up to sign.

We turn to the bottom square of Diagram (34). Here, we let $\eta \in H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R)$ be the generator consistent with the standard orientation, which we can assume to be represented by a single embedded $k$-simplex, and we let $\eta^* \in H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; R)$ be the dual cohomology class such that $\eta^*(\eta) = 1$. The map labeled $\eta^* \times$ is the cohomology cross product with $\eta^*$, which in this case is an isomorphism, employing the cohomology Künneth theorem, Theorem 7.78 and Corollary 7.79. The class $\eta \times \bar{c}(\Gamma_L)$ represents the fundamental class $\Gamma_{\{(0, v)\}} \in I^0 H_0(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R)$. This follows from the uniqueness statements of Lemmas 8.13 and 8.15 and from the arguments in the proof of Lemma 8.19, which show that a chain of this form is consistent with representing $\sigma((0, v))$, so long as it represents $\sigma(z)$ at a sufficient collection of nearby points in regular strata. This is the case by our choice of the orientation of $L$ and hence our choice of $\Gamma_L$.

---

More precisely, in Lemma 8.19 we used chains of the form $\eta \times Z$ with $Z = \bar{c}_p I(\Gamma_L) + e \times I(\Gamma_L)$, for appropriate choices of $e$ and $s''$. But in $I^0 H_0(\mathbb{R}^k \times cL, \mathbb{R}^k \times cL - \{(0, v)\}; R)$, the chain $\eta \times e \times \Gamma_L$ vanishes. We are also free to manipulate $s''$ and $e$ so that the points $z$ we care about all lie in the interior of the support of $\eta \times e \times I(\Gamma_L)$.
For commutativity of the bottom square of Diagram (34), we apply Lemma 7.73. As \( \mathbb{R}^k \) is a manifold, all conditions of Lemma 7.73 on \( \mathbb{R}^k \) are satisfied. For \( cL \), all links of \( cL \) are links of \( X \) and so \( cL \) is locally \((\bar{p}; R)\)-torsion free by the discussion just above. We also need \( \mathbb{R}^k \times cL \) to be locally torsion free with respect to the product perversity; but the links of \( \mathbb{R}^k \times cL \) are just the links of \( cL \) and the product perversity is just \( \bar{p} \) again (in its product version on \( \mathbb{R}^k \times cL \)). So \( \mathbb{R}^k \times cL \) being locally torsion free with respect to the product perversity is equivalent to \( cL \) being locally \((\bar{p}; R)\)-torsion free, which holds. Therefore, we can apply Lemma 7.73. If \( \alpha \in I_{\bar{p}}H^{i-k}(cL, cL - \{v\}; R) \), we thus have
\[
(\eta^* \times \alpha) \sim (\eta \times \overline{d}(\Gamma_L)) = \pm (\eta^* \sim (\alpha \sim \overline{d}(\Gamma_L))
\]
As \( \eta^* \) is dual to \( \eta \), the homology class \( \eta^* \sim \eta \) can be represented by a single 0-simplex, say \( \sigma_y \), with image \( y \in \mathbb{R}^k \). So \( (\eta^* \times \alpha) \sim (\eta \times \overline{d}(\Gamma_L)) = \pm \sigma_y \times (\alpha \sim \overline{d}(\Gamma_L)) \). But now if the vertical map on the right of the bottom square of the Diagram (34) is the inclusion of \( cL \) as \( \{y\} \times cL \), then \( \sigma_y \times (\alpha \sim \overline{d}(\Gamma_L)) \) represents the image of \( \alpha \sim \overline{d}(\Gamma_L) \) under this inclusion by Theorem 6.24 (compare Lemma 5.21). Thus the diagram commutes.

This completes the proof of the theorem.

\[\square\]

**Lemma 8.35.** Suppose \( R \) is any commutative ring with unity. If \( X \) is a filtered space with perversity \( \bar{p} \) and \( \{U_{\alpha}\} \) is an increasing collection of open subspaces of \( X \) then the natural map \( f : \lim_{\alpha} I_{\bar{p}}H^*_c(U_{\alpha}; R) \to I_{\bar{p}}H^*_c(\cup_{\alpha} U_{\alpha}; R) \) is an isomorphism.

**Proof.** First, recall that for any open \( U \subset V \subset X \), we do have maps \( I_{\bar{p}}H^*_c(U; R) \to I_{\bar{p}}H^*_c(V; R) \). These are discussed in Section 7.4. Additionally, if \( U \subset W \subset V \), then
\[
I_{\bar{p}}H^*_c(U; R) \to I_{\bar{p}}H^*_c(W; R) \\
\downarrow \\
I_{\bar{p}}H^*_c(V; R)
\]
commutes. It follows that our map \( f \) is defined.

Now, suppose \( a \in I_{\bar{p}}H^*_c(\cup_{\alpha} U_{\alpha}; R) \). For convenience, denote \( \cup_{\alpha} U_{\alpha} \) by \( \mathcal{U} \). By definition, \( a \) is represented by an element \( a_K \in I_{\bar{p}}H^*(\mathcal{U}, \mathcal{U} - K; R) \), for some compact \( K \subset \mathcal{U} \). As \( K \) is compact and the collection \( \{U_{\alpha}\} \) is increasing, there is some \( U_{\alpha} \), say \( U_0 \), with \( K \subset U_0 \). By Lemma 7.87 we have a commutative diagram
\[
I_{\bar{p}}H^*(U_0, U_0 - K; R) \leftarrow I_{\bar{p}}H^*(\mathcal{U}, \mathcal{U} - K; R) \\
\downarrow \\
I_{\bar{p}}H^*_c(U_0; R) \to I_{\bar{p}}H^*_c(\mathcal{U}; R).
\]
It follows that \( a \in I_{\bar{p}}H^*_c(\mathcal{U}; R) \) must be in the image of \( I_{\bar{p}}H^*_c(U_0; R) \). Therefore, \( a \) is in the image of \( \lim_{\alpha} I_{\bar{p}}H^*_c(U_{\alpha}; R) \). So \( f \) is surjective.

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Next, suppose that \( a \in \lim_{\alpha} I_\rho H^*_c(U_\alpha; R) \) and \( f(a) = 0 \in I_\rho H^*_c(\mathcal{U}; R) \). By definition, \( a \) is represented by an element \( a_0 \in I_\rho H^*_c(U_0; R) \) for some \( U_0 \), and, furthermore, \( a_0 \) must be represented by some \( a_0^K \in I_\rho H^*(U_0, U_0 - K; R) \) for some compact \( K \subset U_0 \). The excision isomorphism \( I_\rho H^*(U_0, U_0 - K; \mathcal{U}; \mathcal{U} - K; R) \sim I_\rho H^*(\mathcal{U} - K; R) \) then takes \( a_0^K \) to an element \( a_0^{K,\mathcal{U}} \) that represents \( f(a) \). But if \( f(a) = 0 \), this implies that there is some compact \( K' \) with \( K \subset K' \) such that \( a_0^{K,\mathcal{U}} \) represents 0 in \( I_\rho H^*(\mathcal{U} - K'; R) \). As \( K' \) is compact, there must be some \( U_1 \) with \( K \subset K' \subset U_1 \) and \( U_0 \subset U_1 \), and we have a commutative diagram

\[
\begin{array}{ccc}
I_\rho H^*_c(U_0; R) & \hookrightarrow & I_\rho H^*_c(U_0, U_0 - K; R) \\
& \downarrow \mathcal{U} & \downarrow \mathcal{U} \\
I_\rho H^*_c(U_1; R) & \hookrightarrow & I_\rho H^*_c(U_1, U_1 - K'; R) \\
& \downarrow \mathcal{U} & \downarrow \mathcal{U} \\
& \vdots & \vdots \\
& I_\rho H^*_c(U_1; R) & \hookrightarrow I_\rho H^*_c(U_1, U_1 - K'; R) \\
\end{array}
\]

with all the maps labeled as isomorphisms being excision maps. The stop left square commutes by Lemma 7.87, the remaining squares commute more evidently. This diagram shows that the image of \( a_0 \) is trivial in \( I_\rho H^*_c(U_1; R) \) as running around the outside of the diagram takes \( a_0 \) through the image of \( a_0^{K,\mathcal{U}} \) in \( I_\rho H^*(\mathcal{U} - K'; R) \). This implies that \( a = 0 \).

\[\square\]

### 8.3.1 Duality of torsion free conditions

As an immediate consequence of Poincaré duality, we prove the fact alluded to in Footnote 110 that an oriented stratified pseudomanifold is locally \((\bar{p}; R)\)-torsion free if and only if it is locally \((D\bar{p}; R)\)-torsion free.

**Corollary 8.36.** Suppose \( R \) is a Dedekind domain and that \( X \) is an \( n \)-dimensional \( R \)-oriented stratified pseudomanifold (or \( \partial \)-stratified pseudomanifold). Then \( X \) is locally \((\bar{p}; R)\)-torsion free if and only if \( X \) is locally \((D\bar{p}; R)\)-torsion free.

**Proof.** Let \( L \) be a link of a point \( x \) contained in the singular stratum \( S \); suppose \( \dim(L) = \ell \). Then \( L \) is a stratified pseudomanifold by Lemma 2.54, and it is \( R \)-orientable as a link of the \( R \)-orientable \( X \) by the arguments in the proof of Lemma 8.13. By Remark 2.55, the links of \( L \) are links of \( X \), and the definition of locally torsion free (Definition 6.20) shows that only dimensions (and not codimensions) are involved in the locally torsion free condition. Therefore, if \( X \) is locally \((\bar{q}; R)\)-torsion free for some perversity \( \bar{q} \), then so is \( L \).

Suppose now that \( X \) and so \( L \), is locally \((\bar{p}; R)\)-torsion free. Then the Poincaré duality theorem, Theorem 8.34, applies to \( L \). So, noting that \( L \) is compact by definition, recalling that \( D(D\bar{p}) = \bar{p} \), and applying the Universal Coefficient Theorem, Theorem 7.4, we have

\[
I^{D\bar{p}} H_{\ell - D\bar{p}(S) - 1}(L; R) \cong I_{D(D\bar{p})} H^{D\bar{p}(S) + 1}(L; R)
\cong \text{Hom}(I^\bar{p} H_{D\bar{p}(S) + 1}(L; R), R) \oplus \text{Ext}(I^\bar{p} H_{D\bar{p}(S)}(L; R), R).
\]

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Next, using that \( \dim(X) = \ell + \dim(S) + 1 \), so that \( \operatorname{codim}_X(S) = \ell + 1 \), we compute
\[
D\bar{p}(S) = \operatorname{codim}_X(S) - 2 - \bar{p}(S) \\
= \ell + 1 - 2 - \bar{p}(S) \\
= \ell - \bar{p}(S) - 1.
\]

By assumption, \( I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R) \) is a flat \( R \)-module (see Definition \ref{flat_module}); it is also finitely generated by Corollary \ref{finitely_generated_modules}, using that Dedekind domains are Noetherian \cite[Theorem VII.2.2.1]{10}. But finitely-generated flat modules over Noetherian rings are projective \cite[Theorem 4.38]{63}, so \( \operatorname{Ext}(I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R), R) = 0 \). Thus our formula for \( I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R) \) reduces to \( I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R) \cong \operatorname{Hom}(I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R), R) \). Furthermore, for any \( R \)-module \( A \), the module \( \operatorname{Hom}(A, R) \) is torsion free. This is an elementary fact: if \( f \in \operatorname{Hom}(A, R) \) and \( f \neq 0 \), then there is some \( x \in A \) such that \( f(x) \neq 0 \). But then if \( r \in R \) with \( r \neq 0 \), we have \( (rf)(x) = r \cdot f(x) \neq 0 \), as \( R \) is an integral domain. So \( f \neq 0 \) implies \( rf \neq 0 \). Thus \( I^{\bar{p}} \mathcal{H}_{\ell-\bar{p}(S)-1}(L; R) \) is \( R \)-torsion free as desired. This concludes the proof that locally \((\bar{p}; R)\)-torsion free implies locally \((D\bar{p}; R)\)-torsion free.

For the other direction, if \( X \) is locally \((D\bar{p}; R)\)-torsion free, we can use \( D\bar{p} \) in place of \( \bar{p} \) in the part of the corollary already proven to conclude that \( X \) is locally \((D(D\bar{p}); R)\)-torsion free. But \( D(D\bar{p}) = \bar{p} \), completing the proof.

## 8.3.2 Topological invariance of Poincaré duality

We demonstrated in Proposition \ref{orientation_invariant} that the orientation class of a stratified pseudomanifold is invariant under appropriate changes of stratification. We also know that intersection homology, in general, is invariant of the stratification assuming that \( X \) has no codimension one strata and that \( \bar{p} \) is a GM perversity; this follows from Theorem \ref{intersection_invariant} and Proposition \ref{intersection_invariant_result}, noting that the two hypotheses of these results are incompatible if codimensions one strata are present. We have also demonstrated that, with equivalent hypotheses, our (co)homological products are stratification independent; see Theorem \ref{cohomological_products}. Putting these results together, we see that the Poincaré duality isomorphism is independent of the stratification in the following sense:

**Theorem 8.37.** Suppose \( R \) is a Dedekind domain and \( \bar{p} \) is a GM perversity. Let \( X \) and \( X' \) be two \( n \)-dimensional compatibly \( R \)-oriented compact stratified pseudomanifold stratifications with no codimension one strata of the same underlying space \( |X| \). Suppose \( X \) (or, equivalently, \( X' \)) is locally \((\bar{p}; R)\)-torsion free. Then there are canonical isomorphisms...
Technically, we haven’t defined the $\mathcal{D}$ in the middle row, as $X^*$ is not guaranteed to be a stratified pseudomanifold, only a CS set. However, we do have a stand-in for a fundamental class in $X^*$, as guaranteed by Proposition 8.28, so we can interpret the middle row $\mathcal{D}$ as the signed cap product with that class.

**Remark 8.38.** As noted in Remark 8.29, it follows from such invariance results that Poincaré duality is topologically invariant in the following broader sense: Suppose $X$ and $Y$ are compact $n$-dimensional $R$-oriented stratified pseudomanifold without codimension one strata, and suppose that $f : |X| \to |Y|$ is a topological homeomorphism, i.e. that it is a homeomorphism of the underlying spaces without regard to the stratifications. Then $X$ induces an image stratification, say $Y'$, on $Y$, and an image $R$-orientation on $Y'$. Suppose that $f$ is orientation preserving in the sense the image $R$-orientation is compatible with the given $R$-orientation on $Y$ in the sense of Corollary 8.10. Then employing Remark 8.29, Theorem 8.37 and naturality, we arrive at a canonical diagram of isomorphisms of the following form:

$$I_pH^i(X; R) \xrightarrow{\mathcal{D}} I^{Dp}H_{n-i}(X; R)$$

$$I_pH^i(X^*; R) \xrightarrow{\mathcal{D}} I^{Dp}H_{n-i}(X^*; R)$$

$$I_pH^i(Y'; R) \xrightarrow{\mathcal{D}} I^{Dp}H_{n-i}(Y'; R)$$

$$I_pH^i(Y; R) \xrightarrow{\mathcal{D}} I^{Dp}H_{n-i}(Y; R).$$

### 8.4 Lefschetz duality

In this section, we will extend our duality results to compact orientable $\partial$-stratified pseudomanifolds. The reader might want to look back at Section 2.3 for the definitions and details.
concerning these spaces. There appear to be more general versions of manifold duality that do not require compactness (for example, see [33 Section 3.3, Exercise 25]), but it does not seem to be as straightforward, for example, to construct the relevant fundamental class without compactness. With $X$ compact, and using that $\partial X$ must have a stratified collar in $X$ by Definition 2.75, it is relatively straightforward to derive Lefschetz duality as a consequence of Poincaré duality.

### 8.4.1 Orientations and fundamental classes

We first consider orientations and fundamental classes for $\partial$-stratified pseudomanifolds.

**Definition 8.39.** Let $X$ be an $n$-dimensional $\partial$-stratified pseudomanifold. We say that $X$ is $R$-orientable if and only if the stratified pseudomanifold $X - \partial X$ is $R$-orientable. Equivalently, by Definition 8.4, $X$ is $R$-orientable if and only if the manifold $(X - \Sigma_X) - \partial(X - \Sigma_X)$ is $R$-orientable. An $R$-orientation of $X$ is an $R$-orientation of $X - \partial X$.

**Lemma 8.40.** If $X$ is an $R$-orientable $\partial$-stratified pseudomanifold, then so is $\partial X$.

**Proof.** By definition, if $X$ is an $R$-orientable $\partial$-stratified pseudomanifold then $(X - \Sigma_X) - \partial(X - \Sigma_X)$ is $R$-orientable. But $(X - \Sigma_X)$ is a $\partial$-manifold. Therefore, its boundary $\partial(X - \Sigma_X)$ is $R$-orientable by classic manifold theory [23 Proposition VIII.2.19]. Lastly, we observe that $\partial(X - \Sigma_X) = \partial X - \Sigma_{\partial X}$, so $\partial X$ is $R$-orientable.

So, by definition, if the $\partial$-stratified pseudomanifold $X$ is $R$-oriented, then so is the stratified pseudomanifold $X - \partial X$. It follows from Lemma 8.13 and the rest of Section 8.2.3 that there is an orientation sheaf $\mathcal{O}^\partial$ over $X - \partial X$ and a unique global section $\sigma^\partial$ determined by the orientation. We then have the following analogue of Theorem 8.17.

**Theorem 8.41.** Let $R$ be a Dedekind domain, and let $X$ be a compact $R$-oriented $n$-dimensional $\partial$-stratified pseudomanifold with perversity $\bar{p} \geq 0$. Then:

1. $I^{\bar{p}}H_i(X; R) = I^{\bar{p}}H_i(X, \partial X; R) = 0$ for $i > n$ and for any perversity $\bar{q}$.

2. There is a unique class $\Gamma^\partial_X \in I^{\bar{p}}H_n(X, \partial X; R)$ such that, for any $x \in X - \partial X$, the image of $\Gamma^\partial_X$ under the composition induced by inclusion and excision, $I^{\bar{p}}H_n(X, \partial X; R) \to I^{\bar{p}}H_n(X, X - \{x\}; R) \cong I^{\bar{p}}H_n(X - \partial X, (X - \partial X) - \{x\}; R)$, corresponds to the image of the orientation section $\sigma^\partial(x)$.

3. If $\{x_j\}_{j=1}^m$ is a collection of points of $X - \partial X$, one in each regular stratum, then $I^{\bar{p}}H_n(X, \partial X; R) \cong \oplus I^{\bar{p}}H_n(X, X - \{x_j\}; R) \cong R^m$ via the map that takes an element of $I^{\bar{p}}H_n(X, \partial X; R)$ to the direct sum of its images in the $I^{\bar{p}}H_n(X, X - \{x_j\}; R)$.

**Proof.** Recall that, by Definition 2.75, $\partial X$ has a collar neighborhood $N$ in $X$ stratified homeomorphic to $[0, 1) \times \partial X$. Then the inclusion $\partial X \hookrightarrow N$ is a stratified homotopy equivalence, so it follows from the long exact sequences and the Five Lemma that $I^{\bar{p}}H_*(X, N; R) \cong I^{\bar{p}}H_*(X, \partial X; R)$. 

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The collar $N$ is stratified homotopy equivalent to $\partial X$, which is $R$-orientable by Lemma 8.40. So $I^qH_i(N; R) \cong I^qH_i(\partial X; R) = 0$ for $i > n - 1$ by Theorem 8.17. Furthermore, by excision, $I^qH_i(X, N; R) \cong I^qH_i(X - \partial X, N - \partial X; R)$. As $X - \partial X$ is an $R$-orientable stratified pseudomanifold and as $X - N = (X - \partial X) - (N - \partial X)$ is compact, $I^qH_i(X, N; R) = 0$ for $i > n$ by Lemma 8.15. Thus $I^qH_i(X, \partial X; R) = 0$ for $i > n$. Together with $I^qH_i(\partial X; R) = 0$ for $i > n - 1$, the long exact sequence of the pair $(X, \partial X)$ shows that $I^qH_i(X, \partial X; R) = 0$ for $i > n$.

Now, let $\bar{p} \geq 0$, and let $K = X - N$, which is compact as $X$ is compact. Note that $(X - \partial X) - K = N - \partial X$. Let $\Gamma^p_K \in I^pH_n(X - \partial X, N - \partial X; R)$ be the fundamental class of $X - \partial X$ over $K$, as guaranteed by Lemma 8.15. By excision and homotopy equivalence, $I^pH_n(X - \partial X, N - \partial X; R) \cong I^pH_n(X, N; R) \cong I^pH_n(X, \partial X; R)$. We let $\Gamma^p_X$ be the image of $\Gamma^p_K$ under these isomorphisms. We will show that $\Gamma^p_X$ has the desired properties and that it is independent of the choice of $N$.

If $\gamma \in I^pS_n(X, \partial X; R)$ is a chain representing $\Gamma^p_X$, then $\Gamma^p_K$ can be represented by a subdivision of $\gamma$, minus some simplices contained in $N$. Therefore, if $x \in K = X - N$, then $\Gamma^p_X$ has the desired image in $I^pH_n(X - \partial X, (X - \partial X) - \{x\}; R)$ by the properties of $\Gamma^p_K$ and an easy diagram chase (diagram omitted).

Now, suppose that $N'$ is another collar neighborhood of $\partial X$ within $N$, and let $K' = X - N'$. Then we have a commutative diagram

$$
\begin{array}{ccc}
I^pH_n(X - \partial X, N - \partial X; R) & \cong & I^pH_n(X, N; R) \\
& \cong & I^pH_n(X, \partial X; R) \\
\end{array}
$$

By the uniqueness properties of Lemma 8.15, the fundamental class $\Gamma^p_{K'} \in I^pH(X - \partial X, N' - \partial X; R)$ must map to $\Gamma^p_K \in I^pH(X - \partial X, N - \partial X; R)$. It therefore follows from the diagram that $N$ and $N'$ both yield the same $\Gamma^p_K$. If now $N''$ is any other collar of $\partial X$ (so not necessarily contained in $N$), then there is a collar $N'''$ of $\partial X$ in $N \cap N''$; as $\partial X$ is compact, this follows from the Tube Lemma [28, Theorem 26.8]. Using the preceding argument twice, we see that the corresponding $\Gamma^p_K$, $\Gamma^p_{K''}$, and $\Gamma^p_{K'''}$ all map to the same $\Gamma^p_X$. So $\Gamma^p_X$ is independent of the choice of collar. As every $x \in X - \partial X$ lies outside of some collar of $\partial X$, it follows that $\Gamma^p_X$ restricts as desired for every $x \in X - \partial X$. Uniqueness of $\Gamma^p$ follows from the uniqueness of the $\Gamma^p_K$.

For the last part of the theorem, we may suppose $\partial X \neq \emptyset$, or the result follows immediately from Theorem 8.17. Let $N$ be a collar of $\partial X$ in the complement of $\cup_{j=1}^m \{x_j\}$. Let $X^+ = X \cup_{\partial X} c(\partial X)$, and let $N^+ = N \cup_{\partial X} c(\partial X)$. Notice that $N^+ \cong c(\partial X)$. We stratify $X^+$ so that if $v$ is the cone vertex of $c(\partial X)$, then $X^+ - \{v\}$ is stratified homeomorphic to $X - \partial X$, and we let $\{v\}$ be a 0-dimensional stratum. We also observe that the regular strata of $X^+$ are the regular strata of $X^+ - \{v\}$, and so are bijectively paired with (in fact, homeomorphic to) the regular strata of $X - \partial X$. In particular, the set $\{x_j\}_{j=1}^m$ contains one point in each regular stratum of $X^+$.
Let \( \bar{p}^+ \) be a perversity on \( X^+ \) that agrees with \( p \) on \( X^+ - \{v\} \) and such that \( \bar{p}^+(\{v\}) \geq n \). Then \( H_*(c(\partial X); R) = 0 \) by the cone formula, Theorem 6.10. We have a commutative diagram

\[
\begin{array}{c}
\xymatrix{
I^pH_n(X, \partial X; R) \ar[r]^{\sim} & I^pH_n(X^+ - \{v\}, N^+ - \{v\}; R) \ar[r]^{\sim} & I^pH_n(X^+; R) \\
\oplus_j I^pH_n(X - \{x_j\}; R) \ar[r]^{\sim} & \oplus_j I^pH_n(X^+ - \{x_j\}; R) \ar[r]^{\sim} & \oplus_j I^pH_n(X^+; R)
}\end{array}
\]

The isomorphisms in the top row are due, respectively, to stratified homotopy equivalence, excision, and the long exact sequence of the pair (using \( I^pH_*(c(\partial X); R) = 0 \)). The isomorphism in the bottom row is also excision. The commutativity comes from the commutativity of the space maps. The vertical map on the far right is an isomorphism by Theorem 8.17 so the map on the left is also an isomorphism, as desired.

**Remark** 8.42. If \( X \) has no codimension one strata and \( \bar{p} \) is a GM perversity, then by employing Lemma 5.58 and Corollary 5.59 we need not even assume in the proof of part (2) of Theorem 8.41 that our collars were formed in the stratification \( X \). They might just as well be collars from another stratification \( X' \), restratified to inherit the stratification from \( X \). This observation will be handy below in proving the topological invariance of fundamental classes of \( \partial \)-stratified pseudomanifolds, Theorem 8.44.

By Lemma 8.40, if \( X \) is an \( R \)-orientable \( \partial \)-stratified pseudomanifold, then so is \( \partial X \). We will next show that, just like for manifolds, the boundaries of fundamental classes of compact oriented \( \partial \)-stratified pseudomanifolds are fundamental classes on the boundaries for the induced orientations. For this, we should first be more clear than we were in the proof of Lemma 8.40 about which orientation on the boundary is determined by the orientation on the interior.

In the proof of Lemma 8.40 we cited Dold’s [23, Proposition VIII.2.19] for the proof there that the boundary of an orientable manifold is orientable. To see which orientation the boundary gets, here is Dold’s construction (though we continue to refer to Dold for proofs): Let \( M \) be an \( R \)-oriented \( n \)-dimensional \( \partial \)-manifold, let \( O_M \) be the \( R \)-orientation bundle on \( M - \partial M \), and let \( O_{\partial M} \) be the \( R \)-orientation bundle on \( \partial M \). Let \( V \) be an open subset of \( V \). Abusing notation somewhat (and also differing from Dold’s), let \( \partial V = V \cap (M - \partial M) \) and let \( V^\circ = V \cap (M - \partial M) \). Then an element of \( \gamma \in H_n(M, M - V^\circ; R) \) determines a section \( s \) of \( O_M \) over \( V^\circ \) by letting \( s(x) \) be the image of \( \gamma \) in \( H_n(M, M - \{x\}; R) \) for \( x \in V^\circ \). Similarly, if \( \eta \in H_{n-1}(M - V^\circ, M - V; R) \), then \( \eta \) determines a section \( t \) of \( O_{\partial M} \) over \( \partial V \) by letting \( t(y) \) be the image of \( \eta \) in \( H_{n-1}(M - V^\circ, (M - V^\circ) - \{y\}; R) = H_{n-1}(\partial M, \partial M - \{y\}; R) \) for \( y \in \partial V \). Here, the isomorphism is an excision isomorphism; note that the excised set is \( M - (\partial M \cup V) \), whose closure in \( M - V^\circ \) is contained in \( (M - V^\circ) - \{y\} \), which is open in
Proposition 8.43. Suppose $X$ is a compact $R$-oriented $n$-dimensional $\partial$-stratified pseudo-manifold, and let $\Gamma_X$ be the fundamental class of $X$ with respect to the given $R$-orientation. Then $\partial_* : I^0H_n(X, \partial X ; R) \to I^0H_{n-1}(\partial X ; R)$ takes $\Gamma_X$ to $\Gamma_{\partial X}$, the fundamental class of $\partial X$ with respect to the induced orientation.

Proof. We follow the proof of [38, Proposition 7.9].

Given the element $\partial_* \Gamma_X \in I^0H_{n-1}(\partial X ; R)$, it generates a global section of the orientation sheaf $O_{\partial X}$ by its images in $I^0H_{n-1}(\partial X, \partial X - \{y\}; R)$ as $y$ runs over $\partial X$. By Lemma 8.13, if this section agrees with the orientation section on $\partial X - \Sigma_{\partial X}$ (for the induced orientation), then it agrees with the orientation section on all of $\partial X$, and then, by Theorem 8.17, $\partial_* \Gamma_X$ must be the appropriate fundamental class.

So let $y \in \partial X - \Sigma_{\partial X}$, and let $V$ be an open neighborhood of $y$ in $X - \Sigma_X$ such that $\bar{V} \cap \Sigma_X = \emptyset$. It is always possible to choose such a $V$ by first choosing a Euclidean half-space neighborhood $W$ of $y$ in the $\partial$-manifold $X - \Sigma_X$ and then choosing a sufficiently smaller Euclidean half-space neighborhood of $y$ in $W$. Continuing our use of the notation introduced just above, we have the following commutative diagram:

$$
\begin{array}{cccccc}
I^0H_n(X, \partial X ; R) & \longrightarrow & I^0H_n(X, X - V^0 ; R) & \cong & H_n(X - \Sigma_X, (X - \Sigma_X) - V^0 ; R) \\
\downarrow \partial_* & & \downarrow \partial_* & & \downarrow \partial_* \\
I^0H_{n-1}(\partial X ; R) & \longrightarrow & I^0H_{n-1}(X - V^0 ; R) & \longrightarrow & H_{n-1}((X - \Sigma_X) - V^0 ; R) \\
& & \downarrow \partial_* & & \downarrow \partial_* \\
& & & \cong & H_{n-1}((X - \Sigma_X) - V^0, (X - \Sigma_X) - V ; R). \\
\end{array}
$$
The two leftward maps labeled as isomorphisms are excision maps with $\Sigma_X$ being excised (which is allowable by our assumption about $V$). Note also that $I^0H_*$ is equivalent to $H_*$ on $X - \Sigma_X$. The composition on the right is precisely the boundary map of the triple from Diagram (35), taking $M = X - \Sigma_X$. By putting that diagram together with this one, we see that if the local images of $\Gamma_X$ provide the orientation section over $X - (\Sigma_X \cup \partial X)$, which is the case by Theorem $8.41$, then $\partial, \Gamma_X$ maps to the induced orientation fiber over $y$ (in fact to the section of the induced orientation over all of $\partial V$).

\textbf{Topological invariance.} As we did in Section $8.2.5$ for pseudomanifolds, we can also discuss the invariance of the fundamental classes when working with pseudomanifolds with boundary. The treatment over varying perversities is essentially equivalent to our work in Proposition $8.22$ and its corollary, so we will not run through all the arguments again. By contrast, invariance of stratification is a bit trickier because, as noted in Remark $2.119$, we have not constructed a natural (intrinsic) stratification for pseudomanifolds with boundary. Nonetheless, we can state the following analogue of Proposition $8.28$.

\textbf{Proposition 8.44.} Let $R$ be a Dedekind domain, and let $X_1$ and $X_2$ be any two compact $n$-dimensional $\partial$-stratified pseudomanifolds with the same underlying space pairs $(|X_1|, \partial X_1) = (|X_2|, \partial X_2)$ and without codimension one strata. Suppose $X_1$ and $X_2$ are compatibly $R$-oriented in the sense of Corollary $8.10$ (applied to $|X| - |\partial X|$). Let $\Gamma_1 \in I^0H_n(X_1, \partial X_1; R)$ and $\Gamma_2 \in I^0H_n(X_2, \partial X_2; R)$ be the fundamental classes over the $R$-orientation. Then there is a canonical isomorphism

$$I^0H_n(X_1, \partial X_1; R) \cong I^0H_n(X_2, \partial X_2; R),$$

and it takes $\Gamma_1$ to $\Gamma_2$.

\textbf{Remark 8.45.} Analogously to Remark $8.29$, it follows from Proposition $8.44$ that if $X$ is a compact $n$-dimensional $R$-oriented $\partial$-stratified pseudomanifold without codimension one strata, then the fundamental class $\Gamma_X$ is a topological invariant of the pair $(|X|, |\partial X|)$ in the following sense: Suppose that $Y$ is another compact $R$-oriented $\partial$-stratified pseudomanifold without codimension one strata and that $f : (|X|, |\partial X|) \to (|Y|, |\partial Y|)$ is a topological homeomorphism, i.e. that it is a homeomorphism of the underlying spaces without regard to the stratifications. Then $X$ induces an image stratification, say $Y'$, on $Y$, and an image $R$-orientation on $Y'$ (via the pointwise isomorphisms $I^0H_n(X, X - \{x\}; R) \cong I^0H_n(Y', Y' - \{f(x)\}; R)$ for $x \in |X| - |\partial X|$ induced by the stratified homeomorphism $X \to Y'$). Suppose that $f$ is orientation preserving in the sense the image $R$-orientation is compatible with the given $R$-orientation on $Y$ in the sense of Corollary $8.10$ applied to $|Y| - |\partial Y|$. Then it must also be the case, applying Proposition $8.44$, that $f(\Gamma_X) \in I^0H_n(Y'; R)$ corresponds to $\Gamma_Y$ under the canonical isomorphisms $I^0H_n(Y', \partial Y'; R) \cong I^0H_n(Y, \partial Y; R)$.

\textbf{Proof of Proposition 8.44} We must first construct an isomorphism $I^0H_n(X_1, \partial X_1; R) \cong I^0H_n(X_2, \partial X_2; R)$ that can be considered canonical. Without an intrinsic coarsest stratification available (see the discussion preceding the statement of the proposition), we cannot
directly apply the methods of Proposition 8.28 instead, we combine those methods with some of our other techniques from this section.

For each \( i = 1, 2 \), let \( \hat{X}_i = X_i \cup_{\partial X_i} \tilde{c}(\partial X_i) \), and let \( \hat{X}^* \) denote \( |\hat{X}_1| \cong |\hat{X}_2| \) with its intrinsic stratification. If \( v \in \hat{X} \) is the added cone point, let \( \hat{N}^* \) be a distinguished neighborhood of \( v \) in \( \hat{X}^* \). Let \( N_i \) be a stratified collar of \( \partial X_i \) in \( X_i \), and let \( \hat{N}_i = N_i \cup_{\partial X_i} \tilde{c}(\partial X_i) \). Given that the \( \hat{N}_i \) are homeomorphic to (stratified!) cone neighborhoods of \( v \), we can assume, up to stratified homeomorphisms, that \( |\hat{N}_i| \subset |\hat{N}^*| \) for each \( i \) by shrinking along the cone lines. We will leave the necessary homeomorphisms tacit and simply assume that the inclusions hold (the following arguments can be easily modified to include the homeomorphisms). Finally, let \( \hat{N}_i^* \) denote \( \hat{N}_i \) with its intrinsic stratification, which is inherited from \( \hat{X}^* \).

Now, consider the following diagram of isomorphisms, coefficients tacit:

\[
\begin{align*}
\bar{I}^0H_n(X_1, \partial X_1) & \xrightarrow{\cong} \bar{I}^0H_n(\hat{X}_1, \hat{N}_1) \xrightarrow{\cong} \bar{I}^0H_n(\hat{X}_1^*, \hat{N}_1^*) \xrightarrow{\cong} \bar{I}^0H_n(X^*, \hat{N}^*) \\
& \xleftarrow{\cong} \bar{I}^0H_n(X_2, \hat{N}_2) \xleftarrow{\cong} \bar{I}^0H_n(\hat{X}_2^*, \hat{N}_2^*) \xleftarrow{\cong} \bar{I}^0H_n(X_2, \partial X_2).
\end{align*}
\]

The far right and left maps are isomorphisms by stratified homotopy invariance and excision. The next inner arrows are isomorphisms by Theorem 5.52 (and Proposition 6.7, by which \( \bar{I}^0H^{GM}_n(X; R) \cong \bar{I}^0H_n(X; R) \), as we have no codimension one strata). The innermost arrows are isomorphisms by Lemma 5.57.

So this gives an isomorphism \( \phi : \bar{I}^0H_n(X_1, \partial X_1) \cong \bar{I}^0H_n(X_2, \partial X_2) \) that looks pretty noncanonical. However, our only choices were of \( N_1, N_2, \) and \( \hat{N}^* \). Suppose we had chosen instead \( \hat{N}_1', N_2, \) and \( \hat{N}^* \). Then we can find a distinguished neighborhood \( \hat{N}'' \) of \( v \) in \( \hat{X}^* \) with \( |\hat{N}''| \subset |\hat{N}^*| \cap |\hat{N}^*| \) and stratified collars \( N_i'' \) with \( |N_i''| \subset |N_i| \cap |N_i| \). Furthermore, again by employing some stratified stretching homeomorphisms around \( v \) in the stratifications \( X_i \), we can continue to assume that \( |\hat{N}''| \) contains each \( |\hat{N}_i''| \).
Now we have a big diagram

\[
\begin{array}{c}
I^0 H_n(X_1, \partial X_1) \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(\hat{X}_1, \hat{N}_1) & \quad I^0 H_n(\hat{X}_1, \hat{N}_1'') & \quad I^0 H_n(\hat{X}_1, \hat{N}_1') \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(\hat{X}^*, \hat{N}_1^*) & \quad I^0 H_n(\hat{X}^*, \hat{N}_1''^*) & \quad I^0 H_n(\hat{X}^*, \hat{N}_1'^*') \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(X^*, \hat{N}_1^*) & \quad I^0 H_n(X^*, \hat{N}_1''^*) & \quad I^0 H_n(X^*, \hat{N}_1'^*') \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(\hat{X}^*, \hat{N}_2^*) & \quad I^0 H_n(\hat{X}^*, \hat{N}_2''^*) & \quad I^0 H_n(\hat{X}^*, \hat{N}_2'^*') \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(\hat{X}_2, \hat{N}_2) & \quad I^0 H_n(\hat{X}_2, \hat{N}_2'') & \quad I^0 H_n(\hat{X}_2, \hat{N}_2') \\
\downarrow \cong \downarrow \cong \\
I^0 H_n(X_2, \partial X_2).
\end{array}
\]

All the arrows so marked are isomorphisms by our previous discussion, and it follows that all arrows are isomorphisms. This shows that our isomorphism \( \phi \) is independent of choices.

Finally, we must say why \( \phi \) takes \( \Gamma_1 \) to \( \Gamma_2 \). As neither \( X_i \) has any codimension one strata, by assumption, each regular stratum of \( X_i - \partial X_i \) is contained as a dense subset of one of the regular strata of \( |X_i - \partial X_i|^* \). In fact, it follows from the argument in the proof of Lemma 8.8 that each \( X_i - (\Sigma X_i \cup \partial X_i) \) is an open submanifold of \( |X_i - \partial X_i|^* - \Sigma|X_i - \partial X_i|^* \) that is dense and such that the difference of these sets has codimension at least two. Furthermore, by the argument provided in the proof of Lemma 8.9, the intersection of each regular stratum of \( |X_i - \partial X_i|^* - \Sigma|X_i - \partial X_i|^* \) with each \( X_i - (\Sigma X_i \cup \partial X_i) \) is path connected. Therefore, we have a bijection between the regular strata from \( X^* \) and the regular strata from \( X_i \), with each regular stratum of \( |X_i - \partial X_i|^* - \Sigma|X_i - \partial X_i|^* \) containing a unique regular stratum of each \( X_i - (\Sigma X_i \cup \partial X_i) \) as a dense, path-connected open set. 

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Thus, we can find a collection of points \( \{x_j\}_{j=1}^m \) such that

1. each \( x_j \) is contained in regular strata of \( X_1 - \partial X_1 \) and \( X_2 - \partial X_2 \) (and therefore also a regular stratum of \( (|X_i - \partial X_i|)^* \)),
2. each regular stratum of \( X_1 - \partial X_1 \) and \( X_2 - \partial X_2 \) (and therefore each regular stratum of \( (|X_i - \partial X_i|)^* \)), contains exactly one \( x_j \), and
3. in \( |\hat{X}_i| \), no \( x_j \) is contained in \( |\hat{N}_1|, |\hat{N}_2|, \) or \( |\hat{N}^*| \).

Let \( M \) be the manifold \( (X_1 - (\partial X_1 \cup \Sigma X_1)) \cap (X_2 - (\partial X_2 \cup \Sigma X_2)) \). Then we have a diagram

\[
\begin{align*}
I^0H_n(X_1, \partial X_1) &\to \oplus_j I^0H_n(X_1, X_1 - \{x_j\}) \\
\Rightarrow &\Rightarrow \\
I^0H_n(\hat{X}_1, \hat{N}_1) &\to \oplus_j I^0H_n(\hat{X}_1, \hat{X}_1 - \{x_j\}) \\
\Rightarrow &\Rightarrow \\
I^0H_n(\hat{X}^*, \hat{N}_1^*) &\to \oplus_j I^0H_n(\hat{X}^*, \hat{X}^* - \{x_j\}) \\
\Rightarrow &\Rightarrow \\
I^0H_n(\hat{X}^*, \hat{N}_2^*) &\to \oplus_j I^0H_n(\hat{X}^*, \hat{X}^* - \{x_j\}) \\
\Rightarrow &\Rightarrow \\
I^0H_n(\hat{X}_2, \hat{N}_2) &\to \oplus_j I^0H_n(\hat{X}_2, \hat{X}_2 - \{x_j\}) \\
\Rightarrow &\Rightarrow \\
I^0H_n(X_2, \partial X_2) &\to \oplus_j I^0H_n(X_2, X_2 - \{x_j\}).
\end{align*}
\]

The diagonal arrows are all excision isomorphisms, and the vertical arrows in the middle column are isomorphisms by excision and topological invariance. By Theorem 8.41, the \( \Gamma_i \) are the unique elements of the \( I^0H_n(X_i, \partial X_i) \) whose images in the \( \oplus I^0H_n(X_1, X_1 - \{x_j\}) \)
are the direct sums of the local orientation classes at the $x_j$. By the compatibility of the orientations of $X_1$ and $X_2$ and the lack of codimension one strata, $M$ carries the orientation information for both $X_i$ (see Section 8.2.2). Thus the composition along the left must take $\Gamma_1$ to $\Gamma_2$, as claimed.

**Remark 8.46.** Notice that the full compactification $\hat{X}$ is used in the proof of Proposition 8.44 only to establish the distinguished neighborhood $\hat{N}^*$ of $\{v\}$. Without an intrinsic stratification of $X$ available, there would not have been a good way to describe $\hat{N}^*$ in terms of collars. However, for the homological parts of the proof, $v$ is always contained within the subset space of a relative homology group, and the arguments would be the same if $\{v\}$ were excised out. This observation will be useful in the proof Theorem 8.49 below.

### 8.4.2 Lefschetz duality

Now we can prove Lefschetz duality theorems for compact $\partial$-stratified pseudomanifolds. We will assume $\partial X = A \amalg B$, where each of $A$ and $B$ is a union of components of $\partial X$ and $A \cap B = \emptyset$. We allow the possibility that $A$ or $B$ might be empty. If $\Gamma_X$ is the fundamental class determined by an orientation on the compact $n$-dimensional $\partial$-stratified pseudomanifold $X$, then we have a duality map

$$D : I_{\bar{p}}H^i(X, B; R) \to I_{\bar{D}\bar{p}}H_{n-i}(X, A; R)$$

given by

$$D(\alpha) = (-1)^{|\alpha|n} \alpha \smile \Gamma_X.$$

Even though $A$ and $B$ are not open subsets of $X$, we will see in the proof of the following theorem that this cap product is well defined by the techniques of Section 7.3.10. We show that, under the appropriate locally torsion free hypotheses, this duality map is an isomorphism. The freedom to work with singularities allows us to provide a somewhat different proof from what is usually done for manifolds, e.g. [53, Theorem 3.43]; for a proof more akin to that one, see [38, Theorem 7.10].

**Theorem 8.47** (Lefschetz duality). Suppose $R$ is a Dedekind domain, and let $X$ be a compact $n$-dimensional $R$-oriented locally ($\bar{p}; R$)-torsion free \footnote{Or, equivalently, $X$ can be locally $(\bar{D}\bar{p}; R)$-torsion free; see Footnote 110 and Corollary 8.36} $\partial$-stratified pseudomanifold. Let $A$ and $B$ be disjoint compact stratified pseudomanifolds with $A \cup B = \partial X$, i.e. each of $A$ and $B$ is a union of components of $\partial X$ and $A \cap B = \emptyset$. Then the duality maps

$$D : I_{\bar{p}}H^i(X, B; R) \to I_{\bar{D}\bar{p}}H_{n-i}(A; R)$$

induced by the cap product with the fundamental class $\Gamma_X$ is an isomorphism.

**Proof.** Using Lemma 8.40, the spaces $A$ and $B$ are compact orientable stratified pseudomanifolds. We can form a new stratified pseudomanifold $X^+$ (without boundary) by coning off $A$ and $B$. Specifically, let $X^+ = \bar{c}(A) \cup_A X \cup_B \bar{c}(B)$. Let $v_A$ and $v_B$ denote the cone vertices of the cones $\bar{c}X$, and let $V = \{v_A, v_B\}$. Using the existence of a stratified collar of $\partial X$ in
X, it is easy to verify that $X^+$ is a stratified pseudomanifold, stratified so that $X^+ - V$ is stratified homeomorphic to the interior of $X$; we can and do assume that this stratified homeomorphism is fixed outside a neighborhood of $\bar{c}(A) \bowtie \bar{c}(B) \subset X^+$. Furthermore, the stratified homeomorphism induces an orientation of $X^+$ that agrees with the orientation on $X$ when restricted to $X$. The set $V$ comprises two 0-dimensional strata.

Let $N_A$ and $N_B$ be disjoint open stratified collars of $A$ and $B$, respectively, in $X$, let $N_A^- = N_A \cup A \bar{c}(A) \cong c(A)$, let $N_B^- = N_B \cup B \bar{c}(B) \cong c(B)$, and let $N^+ = N_A^+ \bowtie N_B^+$. Let $\bar{p}^+$ be a perversity defined on $X^+$ whose value on each stratum of $X^+$ that is not in $V$ agrees with the value of $\bar{p}$ on the corresponding stratum of $X$. Let $\bar{p}^+(\{v_A\}) = -2$ and $\bar{p}^+(\{v_B\}) = n$. Notice that the links of $v_A$ and $v_B$ in $X^+$ are $A$ and $B$, respectively, and we have $\dim(A) - \bar{p}^+(\{v_A\}) - 1 = n$ and $\dim(B) - \bar{p}^+(\{v_B\}) - 1 = -2$. Clearly $I^{\bar{p}^+}H_{-2}(B; R) = 0$, while $I^{\bar{p}^+}H_n(A; R) = 0$ by item $[1]$ of Theorem 8.17. Therefore, $X^+$ is locally $(-\bar{p}, R)$-torsion free; away from $V$, this follows from the hypotheses of our current theorem.

Consider the following diagram:

$$
\begin{array}{cccccc}
I^{\bar{p}^+}H^*(X, B; R) & \longrightarrow & I^{\bar{p}^+}H^*(X^+ - V, N^+_B - \{v_B\}; R) & \longrightarrow & I^{\bar{p}^+}H^*(X^+, N^+_B; R) & \longrightarrow & I^{\bar{p}^+}H^*(X^+; R) \\
\sim \Gamma_X & \longrightarrow & \sim \Gamma_{X^+ - V, N^+_B - V} & \longrightarrow & \sim \Gamma_{X^+, N^+} & \longrightarrow & \sim \Gamma_X \\
I^{D\bar{p}^+}H_{n-i}(X, A; R) & \longrightarrow & I^{D\bar{p}^+}H_{n-i}(X^+ - V, N^+_A - \{v_A\}; R) & \longrightarrow & I^{D\bar{p}^+}H_{n-i}(X^+, N^+_A; R) & \longrightarrow & I^{D\bar{p}^+}H_{n-i}(X^+; R).
\end{array}
$$

We claim that, with appropriate definitions of the various $\Gamma$'s, this diagram commutes and that all the horizontal arrows, which are all induced by inclusions, are isomorphisms. The righthand vertical map is an isomorphism by Poincaré Duality, Theorem 8.34. It would follow that all of the vertical maps are isomorphisms. The lefthand vertical map is, up to sign, our Lefschetz duality map, so this would prove the theorem.

We will work right to left through the diagram. As $D\bar{p}^+(\{v_A\}) = n - 2 = -\bar{p}^+(\{v_A\}) = n$, we have $I^{D\bar{p}^+}H_n(N_A^+; R) = 0$ by the cone formula, Theorem 6.10. Similarly, $\bar{p}^+(\{v_B\}) = n$, so $I^{\bar{p}^+}H^*(N_B^+; R) = 0$ by the cone formula and the Universal Coefficient Theorem, Theorem 7.4. Therefore, the horizontal maps in the righthand square are isomorphisms by the long exact sequences of the pairs. We define $\Gamma_{n+1}^+$ to be the image of $\Gamma_{X^+} \in I^0H_n(X^+; R)$ in $I^0H_{n+1}(X^+, N^+_A; R)$. The right hand square then commutes by naturality of the cap product, Lemma 7.34, with respect to the inclusion map $(X^+; \emptyset, \emptyset) \to (X^+; N^+_A, N_B^+)$. In the middle square, we consider the inclusions $(X^+ - V; N_A^+ - \{v_A\}, N_B^+ - \{v_B\}) \to (X^+; N_A^+, N_B^+)$. By Proposition 8.21 having $D\bar{p}^+(\{v_B\}) = -2 < 0$ implies that the inclusion-induced map $I^{D\bar{p}^+}H_n(X^+ - \{v_B\}; R) \to I^{D\bar{p}^+}H_n(X^+; R)$ is an isomorphism, so $I^{D\bar{p}^+}H_n(X^+ - \{v_B\}, N_A^+; R) \to I^{D\bar{p}^+}H_n(X^+, N_A^+; R)$ is an isomorphism from the long exact sequences and the Five Lemma. We also have that $I^{D\bar{p}^+}H_n(X^+ - V, N_A^+ - \{v_A\}; R) \to I^{D\bar{p}^+}H_n(X^+ - \{v_B\}, N_A^+; R)$ is an isomorphism, by excision of $V$. Together, the composite isomorphism

$$I^{D\bar{p}^+}H_n(X^+ - V, N_A^+ - \{v_A\}; R) \to I^{D\bar{p}^+}H_n(X^+ - \{v_B\}, N_A^+; R) \to I^{D\bar{p}^+}H_n(X^+, N_A^+; R)$$

is the bottom map of the square. The argument that the top map in the square is an isomorphism is essentially the same, using excision in intersection cohomology, and dualizing
Proposition 8.21 using the Universal Coefficient Theorem and the Five Lemma. We also observe that \((N^+_A - \{v_A\}) \cup (N^+_B - \{v_B\}) = N^+ - V\); then the inclusion map \(I^0 H_n(X^+ - V, N^+ - V; R) \to I^0 H_n(X^+; N^+; R)\) is also an excision isomorphism, so we can let \(\Gamma_{X^+ - V, N^+ - V} \in I^0 H_n(X^+ - V, N^+ - V; R)\) be the image of \(\Gamma_{X^+, N^+}\) under the inverse isomorphism. Once again the square commutes by the naturality lemma, Lemma 7.34.

For the leftmost square, we have the inclusion \((x; A, B) \to (X^+ - V; N^+_A - \{v_A\}, N^+_B - \{v_B\})\). All three inclusions are stratified homotopy equivalences (in fact, each image subset is a stratified deformation retract of its codomain), so the horizontal maps are isomorphisms using stratified homotopy invariance and the Five Lemma applied to the long exact sequence to see that the cap product with \(\Gamma\) of the pair. This observation also allows us to invoke Proposition 7.82 and Theorem 7.81 to see that the cap product with \(\Gamma_X\) is well defined. We also claim that the image of \(\Gamma_X \in I^0 H_n(X, \partial X; R)\) in \(I^0 H_n(X^+ - V, N^+ - V; R)\) is \(\Gamma_{X^+ - V, N^+ - V}\). Then we can once again apply Lemma 7.34, which will complete the argument.

To verify the claim, let the set \(\{x_1, \ldots, x_m\} \subset X^+ - N^+\) consist of one point from each regular stratum of \(X^+ - N^+\). Then this set also provides one point in each regular stratum of \(X\). By items (2) and (3) of Theorem 8.17, \(\Gamma_{X^+}\) is the unique element whose images in the \(I^0 H_n(X^+, X^+ - \{x_i\}; R)\) agree with the orientations at \(x_i\). As the \(x_i \notin N^+\), the images of \(\Gamma_{X^+, N^+}\) also give the local orientation at the \(x_i\). But we have an easy commutative diagram

$$
\begin{align*}
I^0 H_n(X, \partial X; R) & \xrightarrow{\cong} I^0 H_n(X^+ - V, N^+ - V; R) \xrightarrow{\cong} I^0 H_n(X^+, N^+; R) \\
\oplus_i I^0 H_n(X, X - \{x_i\}; R) & \xrightarrow{\cong} \oplus_i I^0 H_n(X^+ - V, (X^+ - V) - \{x_i\}; R) \xrightarrow{\cong} \oplus_i I^0 H_n(X^+, X^+ - \{x_i\}; R),
\end{align*}
$$

with the lefthand vertical isomorphism due to item (3) of Theorem 8.41. It follows from the diagram that \(\Gamma_X\) must map across the composition in the top line to \(\Gamma_{X^+ - N^+}\), so the image of \(\Gamma_X\) in the middle term must be \(\Gamma_{X^+ - V, N^+ - V}\) by the definition of \(\Gamma_{X^+ - V, N^+ - V}\).

This concludes the demonstration that \(D : I^0 H^i(X, B; R) \to I^0 \beta^i H_{n-i}(X, A; R)\) is an isomorphism.

With Theorem 8.47 in hand, we can prove an even more general form of Lefschetz duality. The following version of Poincaré duality, as well as the broad strokes of the following proof, can be found for manifolds in [53, Theorem 3.43]. We include the extra details necessary to verify commutativity of the main diagram of the proof for the cap product as we have defined it here; the commutativity is more transparent using the front face/back face construction of cap products.

**Corollary 8.48.** Suppose \(R\) is a Dedekind domain, and let \(X\) be a compact \(n\)-dimensional \(R\)-oriented locally \((\bar{p}; R)\)-torsion free \(\partial\)-stratified pseudomanifold. Let \(A\) and \(B\) be compact stratified pseudomanifolds with \(A \cup B = \partial X\) and such that \(A \cap B = \partial A = \partial B\). Then there

\[\text{[116] Or, equivalently, } X \text{ can be locally } (D\bar{p}; R)\text{-torsion free; see Footnote [110] and Corollary 8.36.}\]
is a duality isomorphism
\[ \mathcal{D} : I_p H^i(X, B; R) \to I_p^D H_{n-i}(X, A; R) \]
induced by the cap product with the fundamental class \( \Gamma_X \).

**Proof.** We will show that there is an up-to-sign commutative diagram

\[
\begin{array}{ccccccc}
I_p H^i(X, \partial X; R) & \longrightarrow & I_p H^i(X, A; R) & \longrightarrow & I_p H^i(\partial X, A; R) & \longrightarrow & I_p H^{i+1}(X, \partial X; R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I^D H_{n-i}(X; R) & \longrightarrow & I^D H_{n-i}(X, B; R) & \longrightarrow & I^D H_{n-i}(X, \partial X; R) & \longrightarrow & I^D H_{n-i}(X, \partial X; R)
\end{array}
\]

in which the rows are long exact. Then the corollary will follow from the Five Lemma, as the leftmost displayed vertical map and the map \( \sim \Gamma_B \) are isomorphisms by Theorem 8.47 (using Lemma 8.40 to ensure that \( \partial X \), and hence A and B, is \( R \)-orientable). The top row here is the long exact sequence of the triple \((X, \partial X, A)\), and the bottom row is the exact sequence of the pair \((X, B)\), so it suffices to check that the rectangles commute.

The square on the left is in fact the composite of two squares:

\[
\begin{array}{ccc}
I_p H^i(X, \partial X; R) & \longrightarrow & I_p H^i(X, A; R) \\
\downarrow & & \downarrow \\
I^D H_{n-i}(X; R) & \longrightarrow & I^D H_{n-i}(X, B; R)
\end{array}
\]

\[
\begin{array}{ccc}
I_p H^i(X, \partial X; R) & \longrightarrow & I_p H^i(X, A; R) \\
\downarrow & & \downarrow \\
I^D H_{n-i}(X; R) & \longrightarrow & I^D H_{n-i}(X, B; R)
\end{array}
\]

These two squares each commute by the naturality of the cap product, Lemma 7.34, applied to the map obvious map \((X; \emptyset, \partial X) \to (X; B, \partial X)\) for the first square and the obvious map \((X; B, A) \to (X; B, \partial X)\) for the second square. Lemma 7.34 applies by Proposition 7.82 and the discussion in Section 7.3.10. We have already seen above how to replace pairs of the form \((X, \partial X)\) with open pairs that stratified deformation retract to them by using open collars and the interior collar around \( \partial X \). For the partial boundaries A and B, we can use the collars of \( \partial A = \partial B \) in A and B to similarly create neighborhoods around A and B in \( \partial X \) that stratified deformation retract to A and B, respectively. These neighborhoods can then be extended to open subsets of \( \partial X \) in an extension of \( X \) to the union of \( X \) with an external collar on \( \partial X \). Such neighborhoods can be used to ensure the existence of necessary cap products. We leave the details to the reader.

The middle diagram commutes up to sign by Lemma 7.61 via Theorem 7.81 and Lemma 7.84; the open neighborhoods for the latter are easily constructed using the stratified collars

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of \( A \cap B = \partial A = \partial B \) in \( A \) and \( B \) and the stratified collar of \( \partial X = A \cup B \) in \( X \). Notice that we are notationally reversed from the statement of Lemma 7.61, i.e., the \( A \) there corresponds to \( B \) here and vice versa. We also note that the map \( i^* \) of the lemma corresponds to the composition of the maps right then down one arrow in our diagram here. We use that, up to signs, \( \partial^* \) of \( \Gamma_X \), treated as an element of \( l^0H_{n-1}(\partial X, A; R) \) maps to \( \Gamma_B \in l^0H_{n-1}(B, \partial B; R) \) under excision; in fact, we know that \( \partial_*\Gamma_X = \Gamma_{\partial X} \) and then the chains representing \( \Gamma_B \) and \( \Gamma_{\partial X} \) represent the same element of \( l^0H_{n-1}(\partial X, A; R) \), as each of these must restrict to the local orientation classes at points in \( B - \partial B \).

For the commutativity up to sign of the final square, we can expand it as

\[
\begin{array}{ccc}
I_{\bar{p}}H^i(\partial X, A; R) & \xrightarrow{\cong} & I_{\bar{p}}H^i(\partial X; R) \\
\downarrow & & \downarrow \\
I_{\bar{p}}H^i(B, \partial B; R) & \xrightarrow{\Gamma_{\partial B}} & I_{\bar{p}}H^i(\partial X; R) \\
\downarrow \cong & & \downarrow \Gamma_{\partial X} \\
I_{\bar{p}}H_{n-i-1}(B; R) & \xrightarrow{\Gamma_X} & I_{\bar{p}}H_{n-i-1}(\partial X; R) \\
\end{array}
\]

The composition along the bottom is clearly equivalent to the intersection homology map induced by the inclusion \( B \hookrightarrow X \), while the composition along the top is equal to the connecting map in the long exact sequence of the triple; this can be seen by looking at the map of cohomology exact sequences of triples induced by \((X, \partial X, \emptyset) \hookrightarrow (X, \partial X, A)\). Then the left triangle commutes by naturality, Lemma 7.34, applied to the map of triples \((B; \emptyset, \partial B) \rightarrow (\partial X; \emptyset, A)\). This uses again that chains representing \( \Gamma_B \) in \( l^0H_{n-1}(B, \partial B; R) \) and \( \Gamma_{\partial X} \) in \( l^0H_{n-1}(\partial X; R) \) both represent the same element in \( l^0H_{n-1}(\partial X, A; R) \cong l^0H_{n-1}(B, \partial B; R) \) as they each represent the local orientation class at each point of \( B - \partial B \). Similarly, the triangle on the right commutes by Lemma 7.34, applied to the map of triples \((\partial X; \emptyset, \emptyset) \rightarrow (\partial X; \emptyset, A)\). Finally, the square on the right commutes up to sign by Lemma 7.60 via Theorem 7.81. Note that the triple \((X; A, B)\) in the statement of the lemma becomes here \((X; \emptyset, \partial X)\).

**Topological invariance.** For Lefschetz duality, we have the following analogue of Theorem 8.37.

**Theorem 8.49.** Suppose \( R \) is a Dedekind domain and \( \bar{p} \) is a GM perversity. Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional compact compatibly \( R \)-oriented \( \partial \)-stratified pseudomanifolds with no codimension one strata and with the same underlying space pairs \((|X_1|, |\partial X_1|) = (|X_2|, |\partial X_2|)\). Suppose \( X_1 \) (or, equivalently, \( X_2 \)) is locally \((\bar{p}; R)\)-torsion free. Then there are canonical diagrams of isomorphisms.

\[
\begin{array}{cccc}
\end{array}
\]

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\[ I^\beta H^i(X_1, \partial X_1; R) \xrightarrow{\mathcal{D}} I^{D^\beta} H_{n-i}(X_1; R) \]

\[ I^\beta H^i(X_2, \partial X_2; R) \xrightarrow{\mathcal{D}} I^{D^\beta} H_{n-i}(X_2; R) \]

and

\[ I^\beta H^i(X_1; R) \xrightarrow{\mathcal{D}} I^{D^\beta} H_{n-i}(X_1, \partial X_1; R) \]

\[ I^\beta H^i(X_2; R) \xrightarrow{\mathcal{D}} I^{D^\beta} H_{n-i}(X_2, \partial X_2; R). \]

Remark 8.50. Analogously to Remark 8.38, it follows from such invariance results that Lefschetz duality is a topological invariant in the following broader sense: Suppose \( X \) and \( Y \) are compact \( n \)-dimensional \( \mathbb{R} \)-oriented \( \partial \)-stratified pseudomanifolds without codimension one strata, and suppose that \( f : (|X|, |\partial X|) \to (|Y|, |\partial Y|) \) is a topological homeomorphism, i.e. that it is a homeomorphism of the underlying spaces without regard to the stratifications. Then \( X \) induces an image stratification, say \( Y' \), on \( Y \), and an image \( \mathbb{R} \)-orientation on \( Y' \). Suppose that \( f \) is orientation preserving in the sense the image \( \mathbb{R} \)-orientation is compatible with the given \( \mathbb{R} \)-orientation on \( Y \) in the sense of Corollary 8.10. Then employing Remark 8.45, Theorem 8.49, and naturality, we arrive at a canonical diagram of isomorphisms of the following form, and analogously for the other duality diagram of Theorem 8.49.

\[ I^\beta H^i(Y, \partial Y; R) \xrightarrow{\mathcal{D}} I^{D^\beta} H_{n-i}(Y; R) \]

Proof of Theorem 8.49. The proof runs through essentially the same sorts of isomorphisms as the proof of Proposition 8.44. Once again, for each \( i = 1, 2 \), let \( \hat{X}_i = X_i \cup_{\partial X_i} \hat{c}(\partial X_i) \), and let \( \hat{X}^* \) denote \( |\hat{X}_1| = |\hat{X}_2| \) with its intrinsic stratification. If \( v \in \hat{X} \) is the cone point,
let \( \hat{N}^* \) be a distinguished neighborhood of \( v \) in \( X^* \). Let \( N_i \) be a stratified collar of \( \partial X_i \) in \( X_i \), let \( \hat{N}_i = N_i \cup_{\partial X_i} \tilde{c}(\partial X_i) \). As noted in the proof of the proof of Proposition 8.44, given that the \( \hat{N}_i \) are homeomorphic to (stratified!) cone neighborhoods of \( v \), we can assume, up to stratified homeomorphisms, that \( |\hat{N}_i| \subset |\hat{N}^*| \) for each \( i \) by shrinking along the cone lines. Finally, let \( \hat{N}_i^* \) denote \( \hat{N}_i \) with its intrinsic stratification, which is inherited from \( \hat{X}^* \). For each of these sets, let the corresponding symbol with a hat replaced by a ring denote the space with \( v \) removed. For example, \( \hat{X}^* = \hat{X}^* - \{v\} \).

Employing the naturality of cap products, Lemma 7.34, and the various manipulations of the fundamental classes from the proof of Proposition 8.44 (see also Remark 8.46), we obtain the following commutative diagram (coefficients tacit):

\[
\begin{array}{c}
I_pH^i(X_1, \partial X_1) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(X_1) \\
\downarrow & & \downarrow \\
I_pH^i(\hat{X}_1, \hat{N}_1) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(\hat{X}_1) \\
\downarrow & & \downarrow \\
I_pH^i(\hat{X}^*, \hat{N}_1^*) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(\hat{X}^*) \\
\downarrow & & \downarrow \\
I_pH^i(\hat{X}^*, \hat{N}_2^*) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(\hat{X}^*) \\
\downarrow & & \downarrow \\
I_pH^i(\hat{X}_2, \hat{N}_2) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(\hat{X}_2) \\
\downarrow & & \downarrow \\
I_pH^i(X_2, \partial X_2) & \xrightarrow{\mathcal{D}} & I^{D_p}H_{n-i}(X_2).
\end{array}
\]

All the vertical arrows marked as isomorphisms are so by stratified homotopy invariance, topological invariance (Theorem 5.52 or Theorem 7.16), and neighborhood invariance (Corol-
All of the horizontal arrows are signed cap products with the fundamental classes or their images as in the proof of Proposition 8.44, as modified by Remark 8.46. The top and bottom horizontal maps are isomorphisms by Lefschetz duality, Theorem 8.47, and so all the arrows in the diagram are isomorphisms. The compositions along the perimeter of the diagram provide our first claimed diagram of isomorphisms.

To see that this diagram is canonical, we must demonstrate independence of choices, in this case of $N_1$, $N_2$, and $\hat{N}^*$. But given any alternative choices, we can interpolate between these and our original choices analogously to the proof that the isomorphisms of Proposition 8.44 are canonical (though with a more elaborate three-dimensional diagram that we leave to the reader’s visualization).

The argument for the second claimed diagram is analogous.

Remark 8.51. We leave the reader to formulate and verify the analogous topological invariance property of Lefschetz duality of the form $\mathcal{D} : I_\varphi H^i(X, B; R) \to I_{\bar{D}\varphi} H_{n-i}(A; R)$.

8.5 Pairings

In this section, we discuss the nonsingular pairings that arise as a consequences of Poincaré and Lefschetz duality. The first, the cup product pairing on the torsion-free quotients of the cohomology groups, is well known from manifold theory and is a standard topic in introductory texts. The torsion product pairing is also classical for manifolds but is not often treated in textbooks; we will provide all the necessary details.

As usual, throughout this section we continue to assume that our base ring $R$ is a Dedekind domain.

8.5.1 Some algebra

We begin by introducing some notation and recalling some algebraic background.

Pairings. First, let us recall what we mean by a nonsingular pairing:

Suppose we have a homomorphism $P : A \otimes B \to C$; such a homomorphism is called a pairing. Recall that there is an adjunction isomorphism $\Lambda : \text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$ [105 Proposition 2.6.3]. To describe the adjunction isomorphism, suppose $P \in \text{Hom}(A \otimes B, C)$ and that $a \in A$ and $b \in B$. Then we typically write $P(a \otimes b) = P(a, b)$. Of course, not every element of $A \otimes B$ has the form $a \otimes b$, but knowing the $P(a, b)$ (in fact, just knowing $P(a, b)$ run over all generators) is enough to determine $P$ completely, as $P$ is a homomorphism. Now, given $P$, we can describe $\Lambda(P)$ as the homomorphism that takes $a \in A$ to what we might write as $P(a, \cdot) : B \to C$, where $P(a, \cdot)$ takes $b \in B$ to $P(a, b)$. Completely symbolically, we could write $((\Lambda(P))(a))(b) = P(a, b)$. Conversely, if $F \in \text{Hom}(A, \text{Hom}(B, C))$, then $(\Lambda^{-1}(F))(a \otimes b) = (F(a))(b)$.

Additionally, as $A \otimes B \cong B \otimes A$, we obtain isomorphisms $\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C) \cong \text{Hom}(B \otimes A, C) \cong \text{Hom}(B, \text{Hom}(A, C))$ [105 Proposition 2.6.3]. Thus any $P \in \text{Hom}(A \otimes B, C)$ corresponds equivalently to a homomorphism $A \to \text{Hom}(B, C)$ and a homomorphism $B \to \text{Hom}(A, C)$.
Definition 8.52. The pairing \( P : A \otimes B \to C \) is called \textit{nonsingular} if the corresponding adjoint maps \( A \to \text{Hom}(B, C) \) and \( B \to \text{Hom}(A, C) \) are both isomorphisms.

A slightly weaker notion that will concern us in Section 8.5.5 is that of a pairing being \textit{non-degenerate}, which means that the corresponding maps \( A \to \text{Hom}(B, C) \) and \( B \to \text{Hom}(A, C) \) are both injective. This is equivalent to saying that \( P(a, b) = 0 \) for all \( b \) if and only if \( a = 0 \) and \( P(a, b) = 0 \) for all \( a \) if and only if \( b = 0 \). One is frequently concerned with pairings of finitely generated vector spaces with image in the ground field. In this case, a pairing is nondegenerate if and only if it is nonsingular, so the two expressions tend to be used interchangeably in that context, sometimes leading to confusion of the terminology when the context changes.

Torsion submodules and torsion free quotients. Next we need some notation:

Definition 8.53. If \( A \) is an \( R \)-module, let \( T(A) \) denote the \( R \)-torsion submodule of \( A \),

\[ T(A) = \{ x \in A | \exists r \in R, r \neq 0, \text{ such that } rx = 0 \}. \]

Let \( F(A) = A/T(A) \) be the torsion-free quotient of \( A \).

The module \( F(A) \) is torsion free. Furthermore, if \( F(A) \) is finitely generated (in particular, if \( A \) is finitely generated), then \( F(A) \) is projective by \([63, \text{Theorem } 4.38]\), using that \( R \) is a Dedekind domain.

Example 8.54. As observed in the proof of Corollary 8.36, \( \text{Hom}(A, R) \) is torsion free for any \( R \)-module \( A \), so \( T(\text{Hom}(A, R)) = 0 \) and \( F(\text{Hom}(A, R)) = \text{Hom}(A, R) \).

Recall next that the cohomological dimension of the Dedekind domain \( R \) is \( \leq 1 \); see \([85, \text{Proposition } 8.1]\) and use that Dedekind domains are hereditary by definition \([85, \text{page } 161]\). In particular, this means that \( \text{Ext}^n(A, B) = 0 \) for \( n > 1 \) with any \( A, B \). This is also our excuse for simply writing \( \text{Ext}(A, B) \) to mean \( \text{Ext}^1(A, B) \) throughout our text. Therefore, the right derived homology exact sequence for the functor \( \text{Hom}(\cdot, R) \) \([85, \text{Corollary } 6.62]\) applied to the short exact sequence

\[
0 \to T(A) \to A \to F(A) \to 0
\]

yields the exact sequence

\[
0 \to \text{Hom}(F(A), R) \to \text{Hom}(A, R) \to \text{Hom}(T(A), R) \to \text{Ext}(F(A), R) \to \text{Ext}(A, R) \to \text{Ext}(T(A), R) \to 0.
\]

We note that \( \text{Hom}(T(A), R) \) must be 0 because if \( x \in T(A) \) with \( rx = 0, r \neq 0 \), then for any \( f \in \text{Hom}(T(A), R) \), \( rf(x) = f(rx) = f(0) = 0 \); this implies \( f(x) = 0 \), as \( R \) is a domain. Additionally, if \( F(A) \) is finitely generated (for example if \( A \) is finitely generated), then \( F(A) \) is projective, so \( \text{Ext}(F(A), R) = 0 \). Thus we have the following lemma, which also incorporates Example 8.54.

Lemma 8.55. If \( R \) is a Dedekind domain and \( A \) is an \( R \)-module, then
1. $\text{Hom}(A, R)$ is torsion free,

2. the canonical map $\text{Hom}(F(A), R) \to \text{Hom}(A, R)$ is an isomorphism,

3. if $F(A)$ is finitely generated (in particular, if $A$ is finitely generated), then the canonical map $\text{Ext}(A, R) \to \text{Ext}(T(A), R)$ is an isomorphism.

Next, let $Q(R)$ denote the field of fractions of $R$ (see [64, Section II.4]); an important special case is $R = \mathbb{Z}$ with $Q(\mathbb{Z}) = \mathbb{Q}$. There is an exact sequence

$$0 \to R \to Q(R) \to Q(R)/R \to 0$$

and, again using that $R$ has cohomological dimension 1, the right derived homology exact sequence of the functor $\text{Hom}(A, \cdot)$ (see [85, Corollary 6.46] or [55, Section IV.8]) yields the six-term exact sequence

$$0 \to \text{Hom}(A, R) \to \text{Hom}(A, Q(R)) \to \text{Hom}(A, Q(R)/R) \to \text{Ext}(A, R) \to \text{Ext}(A, Q(R)) \to \text{Ext}(A, Q(R)/R) \to 0$$

In this sequence, $\text{Ext}(A, Q(R))$ is trivial, as $Q(R)$ is a field. Thus also $\text{Ext}(A, Q(R)/R) = 0$. We will be particularly interested in the case where $A$ is replaced with its torsion submodule $T(A)$. In this case, we have seen just above that $\text{Hom}(T(A), R) = 0$ and $\text{Hom}(T(A), Q(R)) = 0$ by the same argument, as $Q(R)$ is also a domain. This yields the first part of the following lemma.

**Lemma 8.56.** If $R$ is a Dedekind domain and $A$ is an $R$-module, then

1. the connecting map $\text{Hom}(T(A), Q(R)/R) \to \text{Ext}(T(A), R)$ is an isomorphism, and

2. if $A$ is finitely generated, then $\text{Hom}(T(A), Q(R)/R) \cong \text{Ext}(T(A), R) \cong \text{Ext}(A, R)$ is a torsion module.

**Proof.** The first item follows immediately from the discussion just above. For the second, we first observe that if $A$ is finitely generated, then so is $F(A)$, so the isomorphisms come from Lemma 8.55 and the first part of this lemma. Additionally, $T(A)$ will be finitely generated, as $R$ is Noetherian (see [10, Theorem VII.2.2.1] and [64, Section X.1]). So, suppose $f \in \text{Hom}(T(A), Q(R)/R)$, and let $\{x_i\}$ be a finite set of generators of $T(A)$. For each $f(x_i) \in Q(R)/R$, there is some $r_i \in R$, $r_i \neq 0$, such that $r_i f(x_i) = 0$; for example, we can let $r_i$ be the denominator of any fraction in $Q(R)$ representing $f(x_i)$. If we let $r = \prod_i r_i$, then $rf$ takes all generators of $T(A)$ to 0 and so $rf = 0$, $r \neq 0$.

**8.5.2 The cup product pairing**

We now turn to demonstrating that Poincaré and Lefschetz duality implies that the cup product determines a nonsingular pairing. For the sake of generality, we state the result for a $\partial$-stratified pseudomanifold, but of course $\partial X$ may be empty.
Corollary 8.57. Suppose $R$ is a Dedekind domain, and let $X$ be a compact $n$-dimensional $R$-oriented locally $(\bar{p}; R)$-torsion free $\partial$-stratified pseudomanifold. Then the composition

$$F(I_{\bar{p}}H^i(X; R)) \otimes F(I_{\bar{D}p}H^{n-i}(X, \partial X; R)) \xrightarrow{\sim} I_0H^n(X, \partial X; R) \xrightarrow{\mathcal{D}} \partial I^H \sim R_0 \xrightarrow{\partial} R$$

(37)

is a nonsingular pairing.

Definition 8.58. We will refer to the pairing of (37) as the cup product pairing. Evidently, there is a similar cup product pairing

$$F(I_{\bar{p}}H^i(X, \partial X; R)) \otimes F(I_{\bar{D}p}H^{n-i}(X, \partial X; R)) \xrightarrow{\sim} I_0H^n(X, \partial X; R) \xrightarrow{\mathcal{D}} \partial I^H \sim R_0 \xrightarrow{\partial} R.$$  

(38)

Proof of Corollary 8.57. First, we verify that the given composition makes sense. From the definitions, all the perversities have been chosen so that we have well-defined cup products and a well-defined duality map. In case $\partial X \neq \emptyset$, the cup product exists by Corollary 7.83.

If $\alpha \in I_{\bar{p}}H^i(X; R)$ and $\beta \in I_{\bar{D}p}H^{n-i}(X, \partial X; R)$ and either $\alpha$ or $\beta$ is a torsion element, then the composition $a(\mathcal{D}(\alpha \cup \beta))$ must be 0, as $R$ is free. Therefore, the cup product descends to be well defined on $F(I_{\bar{p}}H^i(X; R)) \otimes F(I_{\bar{D}p}H^{n-i}(X, \partial X; R))$. It remains to show that the adjoint maps $F(I_{\bar{p}}H^i(X; R)) \rightarrow \text{Hom}(F(I_{\bar{D}p}H^{n-i}(X, \partial X; R)), R)$ and $F(I_{\bar{D}p}H^{n-i}(X, \partial X; R)) \rightarrow \text{Hom}(F(I_{\bar{p}}H^i(X; R)), R)$ determined by the pairing are isomorphisms.

Let us see how the adjoint to the pairing operates. Given $\alpha \in F(I_{\bar{p}}H^i(X; R))$, its image in $\text{Hom}(F(I_{\bar{D}p}H^{n-i}(X, \partial X; R)), R)$ takes $\beta \in F(I_{\bar{D}p}H^{n-i}(X, \partial X; R))$ to $a(\mathcal{D}(\alpha \cup \beta) = (-1)^n a((\alpha \cup \beta) \cup \Gamma))$. Applying Lemma 7.59, this is equal\(^{117}\) to $(-1)^n a((\alpha \cup (\beta \cup \Gamma))$, and by Lemma 7.49, this is further equivalent to the evaluation $(-1)^n \alpha(\beta \cup \Gamma)$. Note that the dimension indices and perversities are sufficient to invoke these lemmas and that they hold using elements of $F(I_{\bar{D}p}H^{n-i}(X, \partial X; R))$ and $F(I_{\bar{p}}H^i(X; R))$ by noting again that any torsion elements of $I_{\bar{D}p}H^{n-i}(X, \partial X; R)$ or $I_{\bar{p}}H^i(X; R)$ would force these expressions, which live in $R$, to be 0.

Let $\kappa : I_{\bar{p}}H^i(X; R) \rightarrow \text{Hom}(I_{\bar{p}}H_i(X; R), R)$ be the universal coefficient (Kronecker) evaluation map in the universal coefficient sequence of Theorem 7.4.

$$0 \leftarrow \text{Hom}(I^pH_i(X; R), R) \leftarrow I_{\bar{p}}H^i(X; R) \leftarrow \text{Ext}(I^pH_{i-1}(X; R), R) \leftarrow 0.$$ 

Then $(-1)^n \alpha(\beta \cup \Gamma)$ can be written as $(-1)^n(\kappa(\alpha))(\beta \cup \Gamma)$. Furthermore, as $I^pH_i(X; R)$ is finitely generated by Corollary 6.39, the Ext term is a torsion module by Lemma 8.56, while $\text{Hom}(I^pH_i(X; R), R)$ is torsion free. It follows that $\kappa$ induces an isomorphism $F(I_{\bar{p}}H^i(X; R)) \cong \text{Hom}(I^pH_i(X; R), R) \cong \text{Hom}(F(I^pH_i(X; R)), R)$ and that $T(I_{\bar{p}}H^i(X; R)) \cong \text{Ext}(I^pH_{i-1}(X; R), R)$.

Next, we observe that

\(^{117}\)Our conventions for the Poincaré duality map are responsible for the sign $(-1)^n$, which is probably not commonly in use for the cup product pairing. However, the most important uses of the cup product pairing in the literature are, no doubt, those involving the symmetric and anti-symmetric self-pairings on the middle dimensional cohomology of even-dimensional manifolds, in which case the sign vanishes.

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\[-1\]^{\alpha} = (-1)^{n+n(n-i)}\kappa(\alpha)(D(\beta))
= (-1)^{n+n(n-i)+in}D^\ast(\kappa(\alpha))(\beta)
= D^\ast(\kappa(\alpha))(\beta),

where \(D^\ast\) is the Hom(\cdot, R) dual of \(D\) and we have used the Koszul convention for the interchange of the degree \(n\) operator \(D\) with the degree \(i\) operator \(\kappa(\alpha)\).

So, altogether, our pairing adjoint \(F(I_pH(\cdot); R) \rightarrow \text{Hom}(F(I_DpH^{n-i}(\cdot, \partial \cdot; R)), R)\) takes \(\alpha\) to \(D^\ast(\kappa(\alpha))\). But we have just observe that \(\kappa\) is an isomorphism \(\text{Hom}(F(I_pH^i(X; R)) \rightarrow \text{Hom}(F(I_DpH_i(X; R)), R)\), and it follows from Theorem 8.47 that \(D^\ast\) is an isomorphism from \(\text{Hom}(F(I_pH_i(X; R)), R)\) to \(\text{Hom}(F(I_DpH^{n-i}(X, \partial X; R)), R)\). Thus we have shown that the desired map \(F(I_pH^i(X; R)) \rightarrow \text{Hom}(F(I_DpH^{n-i}(X, \partial X; R)), R)\) is an isomorphism.

For the map \(F(I_DpH^{n-i}(X, \partial X; R)) \rightarrow \text{Hom}(F(I_pH^i(X; R)), R)\), we use that, by Lemma 7.40 \(\big(\alpha \sim \beta \sim \Gamma\big) = \big(\beta \sim \alpha \sim \Gamma\big)\), up to sign. From here, we can utilize an equivalent argument to that above, interchanging the roles of \(\alpha\) and \(\beta\) and of \((X, 0)\) and \((X, \partial X)\).

\(\square\)

**Example 8.59.** Let \(X = S\mathbb{R}P^2\) be the suspension of \(\mathbb{R}P^2\), stratified in the natural way with two singular points at the north and south pole. Let \(\bar{0}\) and \(\bar{1}\) be the perversities that take values, respectively, 0 or 1 at both singular points. These are dual Goresky-MacPherson perversities. Using Theorem 6.21 we compute

\[
\begin{align*}
I^0H_3(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I^1H_3(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 \\
I^0H_2(X; \mathbb{Z}_2) &\cong 0 & I^1H_2(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 \\
I^0H_1(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I^1H_1(X; \mathbb{Z}_2) &\cong 0 \\
I^0H_0(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I^1H_0(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2.
\end{align*}
\]

Thus, taking \(\mathbb{Z}_2\) as our ground field, the Universal Coefficient Theorem for cohomology implies

\[
\begin{align*}
I_0H^3(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I_1H^3(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 \\
I_0H^2(X; \mathbb{Z}_2) &\cong 0 & I_1H^2(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 \\
I_0H^1(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I_1H^1(X; \mathbb{Z}_2) &\cong 0 \\
I_0H^0(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2 & I_1H^0(X; \mathbb{Z}_2) &\cong \mathbb{Z}_2.
\end{align*}
\]

Corollary 8.57 now provides nontrivial nonsingular pairings

\[
\begin{align*}
I_0H^3(X; \mathbb{Z}_2) \otimes I_1H^0(X; \mathbb{Z}_2) &\rightarrow \mathbb{Z}_2 \\
I_0H^1(X; \mathbb{Z}_2) \otimes I_1H^2(X; \mathbb{Z}_2) &\rightarrow \mathbb{Z}_2 \\
I_0H^0(X; \mathbb{Z}_2) \otimes I_1H^3(X; \mathbb{Z}_2) &\rightarrow \mathbb{Z}_2.
\end{align*}
\]

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In particular, if $\alpha \in I_0H^1(X;\mathbb{Z}_2)$ and $\beta \in I_1H^2(X;\mathbb{Z}_2)$ are the non-zero elements, then $\alpha \sim \beta \neq 0$.

This is quite different from the situation for cup products in ordinary cohomology on suspensions; in that setting, the cup product on reduced cohomology is always trivial (see, e.g., [101 Corollary 13.66]).

### 8.5.3 The torsion pairing

Suppose again that $X$ is a compact $R$-oriented $\partial$-stratified pseudomanifold. By the Universal Coefficient Theorem,

$$I_\beta^iH^i(X;R) \cong \text{Hom}(I_\beta^iH_i(X;R),R) \oplus \text{Ext}(I_\beta^iH_{i-1}(X;R),R).$$

By Lemmas 8.55 and 8.56, $\text{Ext}(I_\beta^iH_{i-1}(X;R),R)$ is a torsion module and $\text{Hom}(I_\beta^iH_i(X;R),R)$ is torsion free, so $T(I_\beta^iH^i(X;R)) \cong \text{Ext}(I_\beta^iH_{i-1}(X;R),R)$ and $F(I_\beta^iH^i(X;R)) \cong \text{Hom}(I_\beta^iH_i(X;R),R).$

The latter isomorphism is essentially our map $\kappa$ from the proof of Corollary 8.57. In fact, we saw in the proof of Corollary 8.57 that the adjoint of the cup product pairing is (up to signs) the composition of $\kappa$ with the $\text{Hom}(\cdot,R)$ dual of the Poincaré or Lefschetz duality isomorphism.

There is another pairing that can be obtained by a similar procedure, this time on the torsion side of things. Via Lemma 8.56 we have the composite isomorphism

$$\lambda : T(I_\beta^iH^i(X;R)) \cong \text{Ext}(I_\beta^iH_{i-1}(X;R),R) \cong \text{Ext}(T(I_\beta^iH_{i-1}(X;R)),R) \cong \text{Hom}(T(I_\beta^iH_{i-1}(X;R)),Q(R)/R),$$

and then the $\text{Hom}(\cdot,Q(R)/R)$ dual of the Poincaré-Lefschetz duality isomorphism identifies the image of this isomorphism with $\text{Hom}(T(I_{D\beta}H^{n-i+1}(X,\partial X;R)),Q(R)/R)$. So, altogether, we obtain an isomorphism $T(I_\beta^iH^i(X;R)) \rightarrow \text{Hom}(T(I_{D\beta}H^{n-i+1}(X,\partial X;R)),Q(R)/R)$ and the adjoint is a pairing

$$L : T(I_\beta^iH^i(X;R)) \otimes T(I_{D\beta}H^{n-i+1}(X,\partial X;R)) \rightarrow Q(R)/R.$$

It will take a bit more work to unravel this pairing in terms of known operations such as the cup product and to show that the other adjoint $T(I_{D\beta}H^{n-i+1}(X,\partial X;R)) \rightarrow \text{Hom}(T(I_\beta^iH^i(X;R)),Q(R)/R)$ is also an isomorphism. Peeking ahead, the end result of this work will be the following corollary of Poincaré duality:

**Corollary 8.60.** Suppose $R$ is a Dedekind domain and that $X$ is a compact $n$-dimensional $R$-oriented locally $(\bar{p};R)$-torsion free $\partial$-stratified pseudomanifold. Then the composition of isomorphisms

$$T(I_\beta^iH^i(X;R)) \xrightarrow{\lambda} \text{Hom}(T(I_\beta^iH_{i-1}(X;R)),Q(R)/R) \xrightarrow{\text{D}} \text{Hom}(T(I_{D\beta}H^{n-i+1}(X,\partial X;R)),Q(R)/R)$$

determines an adjoint nonsingular pairing

$$L_{\beta,D\beta} : T(I_\beta^iH^i(X;R)) \otimes T(I_{D\beta}H^{n-i+1}(X,\partial X;R)) \rightarrow Q(R)/R.$$
Analogously, there is a nonsingular pairing

\[ L''_{Dp,\beta} : T(I_{Dp}H^{n-i+1}(X,\partial X; R)) \otimes T(I_{p}H^{i}(X; R)) \to Q(R)/R. \]

If \( \alpha, \beta \) are cochains representing elements in \( T(I_{p}H^{i}(X; R)) \) and \( T(I_{Dp}H^{n-i+1}(X,\partial X; R)) \), respectively, and if \( d\beta = \tau \beta \) for \( \tau \in R \), \( t \neq 0 \), and \( d\alpha = r\alpha \) for \( r \in R \), \( r \neq 0 \), then

\[
L''_{Dp,\beta}(\alpha \otimes \beta) = (-1)^{n} \frac{a((\alpha \odot b) \sim \Gamma)}{t}
\]

\[
L''_{Dp,\beta}(\beta \otimes \alpha) = (-1)^{n} \frac{a((\beta \odot a) \sim \Gamma)}{r}.
\]

Furthermore,

\[
L''_{Dp,\beta}(\alpha \otimes \beta) = (-1)^{1+n+i} L''_{Dp,\beta}(\beta \otimes \alpha).
\]

We begin by understanding the map \( \lambda : T(I_{p}H^{i}(X; R)) \to \operatorname{Hom}(T(I_{p}H_{i-1}(X; R)), Q(R)/R) \) in more detail, starting with the identification of the Ext summand guaranteed by the Universal Coefficient Theorem. For this, we first need some notation: Let \( C_{i} = I_{p}S_{i}(X; R) \), let \( Z_{i} = \ker(\partial : I_{p}S_{i}(X; R) \to I_{p}S_{i-1}(X; R)) \), and let \( B_{i-1} = \operatorname{im}(- \partial : I_{p}S_{i}(X; R) \to I_{p}S_{i-1}(X; R)) \).

Further, let \( W_{i} \subset C_{i} \) be the submodule of \textit{weak boundaries},

\[ W_{i} = \{ w \in C_{i} \mid \exists r_{w} \neq 0 \in R \text{ such that } r_{w}w \in B_{i} \}. \]

In other words, elements of \( W_{i} \) need not be boundaries, but they are the elements of \( C_{i} \) some non-zero scalar multiple of which are boundaries. Note that, for a given \( w \in W_{i} \), there is not a unique associated \( r_{w} \), but for the purposes of our current discussion, we can assume that some particular \( r_{w} \) has been fixed for each \( w \). Observe that \( B_{i} \subset W_{i} \subset Z_{i} \), the first inclusion by using \( r_{w} = 1 \) and the second by observing that if \( w \in W_{i} \) then \( 0 = \partial(r_{w}w) = r_{w}(\partial w) \), so \( \partial w = 0 \) as \( C_{i} \) is projective, and hence torsion free.

**Lemma 8.61.** The torsion submodule \( \operatorname{Ext}(I_{p}H_{i-1}(X; R), R) \) of \( I_{p}H^{i}(X; R) \) can be identified with the image of the dual of the boundary map \( \partial^{*} : \operatorname{Hom}(B_{i-1}, R) \to H^{i}(\operatorname{Hom}(C_{i}, R)) \), which is well defined as the image of \( \partial^{*} : \operatorname{Hom}(B_{i-1}, R) \to \operatorname{Hom}(C_{i}, R) \) consists of cocycles. Furthermore, the kernel of \( \partial^{*} : \operatorname{Hom}(B_{i-1}, R) \to H^{i}(\operatorname{Hom}(C_{i}, R)) \) is the image of \( \operatorname{Hom}(W_{i-1}, R) \to \operatorname{Hom}(B_{i-1}, R) \), the dual of the inclusion \( B_{i-1} \hookrightarrow W_{i-1} \). Therefore, we can identify \( T(I_{p}H^{i}(X; R)) \) with \( \frac{\operatorname{Hom}(B_{i-1}, R)}{\operatorname{im}(\operatorname{Hom}(W_{i-1}, R) \to \operatorname{Hom}(B_{i-1}, R))} \) via the dual of the boundary map.

**Proof.** We begin by recalling the proof of the Universal Coefficient Theorem (see [55], [105], or [74]; the discussion is also similar to our look at the algebraic Künneth theorem in Section 6.4.6). Continuing the notation introduced just before the statement of the lemma, let \( Z_{i} \) be the chain complex with modules \( Z_{i} \) and all boundary maps \( 0 \), and let \( B_{i-1} \) be the chain complex with modules \( B_{i-1} \) and all boundary maps \( 0 \). Note that, as \( R \) is a Dedekind domain, each \( Z_{i} \) and \( B_{i-1} \) is projective, being submodules of the projective \( I_{p}S_{i}(X; R) \) and \( I_{p}S_{i-1}(X; R) \), respectively.

We have a short exact sequence of chain complexes

\[ 0 \to Z_{i} \xrightarrow{i} C_{i} \xrightarrow{\partial} B_{i-1} \to 0, \]
where \( j \) is inclusion and \( \partial \) is the boundary map in \( I^p S_*^R(X; R) \). As each module is projective, this dualizes to the short exact sequence of cochain complexes:\(^{118}\)

\[
0 \leftarrow \text{Hom}(Z_i, R) \xrightarrow{j^*} \text{Hom}(C_i, R) \xrightarrow{\partial^*} \text{Hom}(B_{i-1}, R) \leftarrow 0.
\]

All coboundary maps in \( \text{Hom}(B_{i-1}, R) \) are 0, so we see that \( \partial^* \) does indeed take elements of \( \text{Hom}(B_{i-1}, R) \) to cocycles in \( \text{Hom}(C_i, R) \), as claimed in the statement of the lemma.

Taking (co)homology, the resulting long exact sequence includes sections of the form

\[
\leftarrow \text{Hom}(Z_i, R) \xrightarrow{j^*} H^i(\text{Hom}(C_i, R)) = I_p H^i(X; R) \xrightarrow{\partial^*} \text{Hom}(B_{i-1}, R) \leftarrow .
\]

Notice that the cohomology symbols \( H^i \) do not need to appear on the outside terms because the differential of the complexes \( \text{Hom}(Z_i, R) \) and \( \text{Hom}(B_{i-1}, R) \) are trivial.

It turns out (see the cited references) that the image of \( j^* \) can be identified with \( \text{Hom}(H_i(C_i, R) = \text{Hom}(I^p H_i(X; R), R) \) and that the image of \( \partial^* \), which is the cokernel of \( \text{Hom}(Z_{i-1}, R) \to \text{Hom}(B_{i-1}, R) \), is \( \text{Ext}(H_{i-1}(C_i), R) = \text{Ext}(I^p H_{i-1}(X; R), R) \). In fact, this is not difficult to see, as

\[
0 \to B_{i-1} \to Z_{i-1} \to H_{i-1}(C_i) \to 0
\]

is a projective resolution of \( H_{i-1}(C_i) \), so the cokernel of \( \text{Hom}(B_{i-1}, R) \leftarrow \text{Hom}(Z_{i-1}, R) \) is precisely the definition of \( \text{Ext}(H_{i-1}(C_i), R) \). Of course, we should also observe that the boundary map of our long exact cohomology sequence agrees with the dual of the inclusion \( Z_{i-1} \to B_{i-1} \). For this we refer again to the cited references.

If each \( H_i(C_i) \) is finitely generated, as we are assuming in the case at hand, we know from the discussion in the proof of Corollary 8.57 that \( I_p H^i(X; R) \leftarrow \text{Ext}(I^p H_{i-1}(X; R), R) \) takes \( \text{Ext}(I^p H_{i-1}(X; R), R) \) isomorphically onto \( T(I_p H^i(X; R)) \), so every \( \alpha \in T(I_p H^i(X; R)) \) is the image under \( \partial^* \) of some \( f \in \text{Hom}(B_{i-1}, R) \), though \( f \) is only well-defined up to elements in the image of \( \text{Hom}(Z_{i-1}, R) \). This shows that we can identify the Ext summand of \( I_p H^i(X; R) \) with the isomorphic image of \( \text{Hom}(B_{i-1}, R)/\text{im}(\text{Hom}(Z_{i-1}, R) \to \text{Hom}(B_{i-1}, R)) = \text{im}(\text{Hom}(W_{i-1}, R) \to \text{Hom}(B_{i-1}, R)) \).

Since \( W_{i-1} \) consists precisely of those cycles whose scalar multiples are boundaries, \( W_{i-1}/B_{i-1} \cong T(I^p H_{i-1}(X; R)) \). Then

\[
Z_{i-1}/W_{i-1} \cong (Z_{i-1}/B_{i-1})/(W_{i-1}/B_{i-1}) \cong I^p H_{i-1}(X; R)/T(I^p H_{i-1}(X; R)) = F(I^p H_{i-1}(X; R)).
\]

So

\[
0 \to W_{i-1} \to Z_{i-1} \to F(I^p H_{i-1}(X; R)) \to 0
\]

is a projective resolution of \( F(I^p H_{i-1}(X; R)) \), and

\[
\frac{\text{Hom}(W_{i-1}, R)}{\text{im}(\text{Hom}(Z_{i-1}, R) \to \text{Hom}(W_{i-1}, R))} \cong \text{Ext}(F(I^p H_{i-1}(X; R)), R).
\]

\(^{118}\)In the exact sequence of complexes \(^{39}\), the map \( \partial \) is treated as a degree 0 chain map. Therefore, the dual \( \partial^* \) in this sequence acts with no sign as \( (\partial^* f)(x) = f(\partial x) \).
But, as $F(I^pH_{i-1}(X; R))$ is torsion free and finitely generated, it is projective, so
\[ \text{Ext}(F(I^pH_{i-1}(X; R)), R) = 0, \]
and it follows that $\text{Hom}(Z_{i-1}, R) \to \text{Hom}(W_{i-1}, R)$ is surjective. Therefore,
\[ \text{im}(\text{Hom}(Z_{i-1}, R) \to \text{Hom}(B_{i-1}, R)) = \text{im}(\text{Hom}(W_{i-1}, R) \to \text{Hom}(B_{i-1}, R)), \]
as desired.

\[ \square \]

Remark 8.62. Using some work from the proof of Lemma 8.61, we can provide another, perhaps more direct, proof that $\text{Ext}(I^pH_{i-1}(X; R), R) \cong \text{Ext}(T(I^pH_{i-1}(X; R)), R)$: We showed in the proof of the lemma that we have
\[ \frac{\text{Hom}(B_{i-1}, R)}{\text{im}(\text{Hom}(Z_{i-1}, R) \to \text{Hom}(B_{i-1}, R))} \cong \text{Ext}(I^pH_{i-1}(X; R), R), \]
using the definition of Ext and the projective resolution (40). Meanwhile,
\[ 0 \to B_{i-1} \to W_{i-1} \to T(I^pH_{i-1}(X; R)) \to 0 \] (41)
is a projective resolution of $T(I^pH_{i-1}(X; R))$, so, applying $\text{Hom}(\cdot, R)$, we see
\[ \frac{\text{Hom}(B_{i-1}, R)}{\text{im}(\text{Hom}(W_{i-1}, R) \to \text{Hom}(B_{i-1}, R))} \cong \text{Ext}(T(I^pH_{i-1}(X; R)), R), \]
using the definition of Ext and the projective resolution (41). But the proof of the lemma demonstrates that these are the same module, identically.

Next, we look more closely at the isomorphism
\[ \text{Ext}(T(I^pH_{i-1}(X; R)), R) \cong \text{Hom}(T(I^pH_{i-1}(X; R)), Q(R)/R) \]
established in Lemma 8.56. Built upon the exact sequence $0 \to R \to Q(R) \to Q(R)/R \to 0$, we have a commutative diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}(W_{i-1}, R) & \longrightarrow & \text{Hom}(W_{i-1}, Q(R)) & \longrightarrow & \text{Hom}(W_{i-1}, Q(R)/R) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}(B_{i-1}, R) & \longrightarrow & \text{Hom}(B_{i-1}, Q(R)) & \longrightarrow & \text{Hom}(B_{i-1}, Q(R)/R) & \longrightarrow & 0.
\end{array}
\]
The rows are short exact, as $W_{i-1}$ and $B_{i-1}$ are projective. Thinking of the columns as representing the only non-trivial modules in complexes, the resulting long exact homology sequence (or, equivalently, the snake lemma) yields the six-term Hom-Ext sequence, as, for each $R$-module $M$ and using the projective resolution (41), we have
\[ \ker(\text{Hom}(W_{i-1}, M) \to \text{Hom}(B_{i-1}, M)) = \text{Hom}(T(I^pH_{i-1}(X; R)), M) \]
and
\[ \text{cok} \left( \text{Hom}(W_{i-1}, M) \to \text{Hom}(B_{i-1}, M) \right) = \text{Ext}(T(I^p H_{i-1}(X; R)), M). \]

We are interested in the connecting map that takes \( \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R) \) to \( \text{Ext}(T(I^p H_{i-1}(X; R)), R) \). We already know this is an isomorphism by Lemma 8.50 with our standing assumptions, and we would like to construct an explicit inverse.

Given \( \bar{f} \in \text{Ext}(T(I^p H_{i-1}(X; R)), R) \), let \( f \in \text{Hom}(B_{i-1}, R) \) represent \( \bar{f} \). As we already know that the six-term exact sequence degenerates to yield an isomorphism \( \text{Ext}(T(I^p H_{i-1}(X; R)), R) \cong \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R) \), there must be a unique \( g \in \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R) \subset \text{Hom}(W_{i-1}, Q(R)/R) \) that maps to \( \bar{f} \) via the connecting “zig-zag” map. Unwinding the zig-zag map, our \( g \in \text{Hom}(W_{i-1}, Q(R)/R) \) must be the image of some \( h \in \text{Hom}(W_{i-1}, Q(R)) \), and \( h \) must restrict on \( B_{i-1} \) to agree with \( f \). Even though \( \bar{f} \in \text{Ext}(T(I^p H_{i-1}(X; R)), R) \) is only defined up to elements of the image of \( \text{Hom}(W_{i-1}, R) \), if we can find such an \( h \) whose restriction agrees with \( f \), the corresponding \( g \) must map to \( \bar{f} \) in (co)homology by the Zig-Zag Lemma (see [77, Lemma 24.1]).

So let us deduce \( h \) from \( f \). Suppose \( w \in W_{i-1} \). Then for some \( r_w \in R \) with \( r_w \neq 0 \), we have \( r_w w \in B_{i-1} \), so \( f(r_w w) \in R \subset Q(R) \) is defined, and we want \( h(r_w w) = f(r_w w) \). But \( h \) is a \( Q(R) \)-module homomorphism, so we must have \( h(r_w w) = r_w h(w) \), which determines \( h(w) = f(r_w w)/r_w \). Let us show that this yields a well-defined element of \( \text{Hom}(W_{i-1}, Q(R)/R) \), in particular that it does not depend on our choice of \( r_w \). Suppose that \( r'_w \in R \) with \( r'_w \neq 0 \) and \( r'_w w \in B_{i-1} \). Then we want to know that \( f(r_w w)/r_w = f(r'_w w)/r'_w \in Q(R) \). But this is equivalent to asking that \( r'_w f(r_w w) = r_w f(r'_w w) \in Q(R) \), which is certainly true as \( f \) is an \( R \)-module homomorphism so both sides are equal to \( f(r_w r'_w w) \). This latter expression is well defined, as \( r_w r'_w w \in B_{i-1} \).

We can now put this discussion together with Lemma 8.61 to find an explicit description of \( \lambda : T(I^p H^i(X; R)) \xrightarrow{\cong} \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R) \). Suppose \( \alpha \in T(I^p H^i(X; R)) \), and let \( w \in T(I^p H_{i-1}(X; R)) \). We will write down an explicit formula for \( (\lambda(\alpha))(w) \). Let \( z \in C_i \) be such that \( \partial z = rw \) for some \( r \neq 0 \). By Lemma 8.61 there is an \( f \in \text{Hom}(B_{i-1}, R) \) such that \( \partial^*(f) = \alpha \). By the zig-zag argument of the preceding paragraph, we must have
\[
(\lambda(\alpha))(w) = f(rw)/r \\
= f(\partial z)/r \\
= ((\partial^* f)(z))/r \\
= \alpha(z)/r.
\]

So \( (\lambda(\alpha))(w) = \alpha(z)/r \). While this formula should be well defined by our preceding discussion, it is reassuring to observe that this construction is independent of our choice of a cochain representative for \( \alpha \), as, if we replace \( \alpha \) by \( d\gamma \), we have
\[
\frac{(d\gamma)(z)}{r} = \frac{\pm \gamma(\partial z)}{r} = \frac{\pm \gamma(rw)}{r} = \frac{\pm r\gamma(w)}{r} = \pm \gamma(w) = 0 \in Q(R)/R.
\]
Similarly, if \( w \) is a boundary, so that \( w = \partial z \), then \( (\lambda(\alpha))(w) = \alpha(z) = 0 \in Q(R)/R \). So this provides a nice independent verification that our formula is indeed independent of our choices of cochain and chain representatives of our cohomology and homology classes.
Putting together our discussion thus far, we have obtained the following lemma:

**Lemma 8.63.** Let $R$ be a Dedekind domain and $X$ a compact stratified pseudomanifold. Then the universal coefficient theorem and the isomorphism of the six term Hom-Ext sequence result in an isomorphism $\lambda : T(I_p H^i(X; R)) \cong \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R)$. If $\alpha \in T(I_p H^i(X; R))$ and $w \in T(I^p H_{i-1}(X; R))$, we have $\lambda(\alpha)(w) = \alpha(z)/r$, where $z \in I^p S_i(X; R)$ is a chain with $\partial z = rw$, $r \neq 0$.

The map $\lambda$ was the first ingredient in our torsion pairing $L : T(I_p H^i(X; R)) \otimes T(I^p H^{n-i+1}(X, \partial X; R)) \rightarrow Q(R)/R$. The second ingredient is the $\text{Hom}(\cdot, Q(R)/R)$ dual of the Poincaré duality isomorphism. So, altogether, our pairing is determined by the composition

$$T(I_p H^i(X; R)) \overset{\lambda}{\rightarrow} \text{Hom}(T(I^p H_{i-1}(X; R)), Q(R)/R) \overset{\text{dual}}{\rightarrow} \text{Hom}(T(I^p H^{n-i+1}(X, \partial X; R)), Q(R)/R).$$

Now that we have studied $\lambda$ in some detail, we are almost ready to write down a formula for $L(\alpha, \beta)$.

Let us assume that we have chosen a fixed algebraic diagonal on $X$ so that it makes sense to discuss cup and cap products at the chain level. Ultimately, $L(\alpha, \beta)$ is independent of this choice because it is defined at the level of cohomology; we’re using cup and cap products simply to find chain level formulas for the torsion pairing.

Suppose $\alpha \in T(I_p H^i(X; R))$ and $\beta \in T(I^p H^{n-i+1}(X, \partial X; R))$. Then there are $r, t \in R$, $r, t \neq 0$, and some $a \in I_p S^{i-1}(X; R)$ and $b \in I^p S^{n-i}(X, \partial X; R)$ such that $da = r\alpha$ and $db = t\beta$. Then $D(\beta) = (-1)^{(n-i+1)n}b \sim \Gamma$ is a torsion element of $I^p H_{i-1}(X; R)$. In fact, as $t \beta = 0 \in I^p H^{n-i+1}(X, \partial X; R)$, we must have $t D(\beta) = 0 \in I^p H_{i-1}(X; R)$. So $t D(\beta)$ is a boundary. Let us see what it is the boundary of: By Lemma 7.28

$$\partial(\gamma \sim \xi) = (d\gamma) \sim \xi + (-1)^{|\gamma|+|\xi|} \sim \partial \xi,$$

for a cochain $\gamma$ and chain $\xi$ (of appropriate perversities). In our setting, $\xi$ will be $\Gamma$, which is a cycle (in $I^0 S_n(X, \partial X; R)$). So, we have

$$t(\beta \sim \Gamma) = (t\beta) \sim \Gamma$$

$$= (db) \sim \Gamma$$

$$= \partial(b \sim \Gamma).$$

Using this and Lemma 8.63, we now have

$$\text{d}^* \lambda(\alpha)(\beta) = (-1)^{(|\alpha|-1)n} \lambda(\alpha)(D(\beta))$$

$$= (-1)^{(|\alpha|-1)n+|\beta|n} \lambda(\alpha)(\beta \sim \Gamma)$$

$$= (-1)^{|\alpha|+|\beta|-1} \alpha(b \sim \Gamma)$$

$$= (-1)^{|\alpha|\beta} \alpha(b \sim \Gamma)$$

$$= (-1)^{|\alpha|} \alpha(b \sim \Gamma).$$

\[119\]Note that $\lambda(\alpha)$ is an object of degree $|\alpha| - 1$. 415
So, the adjoint pairing to $\mathcal{D}^*\lambda$ is $L : T(I_pH^i(X; R)) \otimes T(I_{Dp}H^{n-i+1}(X, \partial X; R)) \rightarrow Q(R)/R$, defined by

$$L(\alpha, \beta) = (-1)^n \frac{\alpha(b \smile \Gamma)}{t}$$

when $db = t\beta$, $t \neq 0$.

To study the other adjoint to this pairing, and so to verify that $L$ is nonsingular, we would like to move toward a more symmetric expression for $L$, which means rewriting our formula for $L$ in terms of the cup product. This is the purpose of our next proposition.

**Proposition 8.64.** Suppose $R$ is a Dedekind domain and that $X$ is a compact $n$-dimensional $R$-oriented locally $(\bar{p}; R)$-torsion free stratified pseudomanifold. Then the pairing

$$L : T(I_pH^i(X; R)) \otimes T(I_{Dp}H^{n-i+1}(X, \partial X; R)) \rightarrow Q(R)/R$$

adjoint to the composition

$$T(I_pH^i(X; R)) \xrightarrow{\lambda} \text{Hom}(T(I^pH_{i-1}(X; R)), Q(R)/R) \xrightarrow{\mathcal{D}^*} \text{Hom}(T(I_{Dp}H^{n-i+1}(X, \partial X; R)), Q(R)/R)$$

can be computed by

$$L(\alpha, \beta) = (-1)^n \frac{\alpha(b \smile \Gamma)}{t} = (-1)^n \frac{a((\alpha \smile b) \smile \Gamma)}{t},$$

for $\alpha \in T(I_pH^i(X; R))$, $\beta \in T(I_{Dp}H^{n-i+1}(X, \partial X; R))$, and $b \in I_{Dp}S^{n-i}(X, \partial X; R)$ such that $120$ $db = t\beta$ with $t \in R$, $t \neq 0$.

**Proof.** We will employ now the somewhat mysterious formula from Remark 7.50, which states that for $\alpha$ a cochain and $\xi$ a chain, each with the same degree and perversity, we have

$$\alpha(\xi) = a(\alpha \smile \xi) + a(D\partial \xi + \partial D\xi),$$

where $D$ is a chain homotopy from Lemma 7.45, see Remark 7.50 for the details about this chain homotopy, though we will not need any of those details now. Taking $\xi = b \smile \Gamma$,

$$\alpha(b \smile \Gamma) = a(\alpha \smile (b \smile \Gamma)) + a(D\partial(b \smile \Gamma)) + a(\partial D(b \smile \Gamma)).$$

Now, $\alpha(\partial D(b \smile \Gamma)) = 0$ because our $\alpha$ is a cocycle, and, by 121,

$$\alpha(D\partial(b \smile \Gamma)) = \alpha(D((db) \smile \Gamma)) = \alpha(D(t\beta \smile \Gamma)) = ta(D(\beta \smile \Gamma)).$$

So

$$\frac{\alpha(b \smile \Gamma)}{t} = \frac{a(\alpha \smile (b \smile \Gamma))}{t} + \frac{ta(D(\beta \smile \Gamma))}{t} = \frac{a(\alpha \smile (b \smile \Gamma))}{t} + \alpha(D(\beta \smile \Gamma)) = \frac{a(\alpha \smile (b \smile \Gamma))}{t} \in Q(R)/R.$$

120 Warning: the $\beta$ in this last expression is really any cochain representative of $\beta \in T(I_{Dp}H^{n-i+1}(X; R))$.  

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The point at the end here is that, although we have no idea what \( \alpha(D(\beta \sim \Gamma)) \) might be, we do know it’s in \( R \); so it’s 0 in \( Q(R)/R \). Therefore,

\[
L(\alpha, \beta) = (-1)^n \frac{a(\alpha \sim (b \sim \Gamma))}{t}.
\]

Finally, we would like to be able to write \( L(\alpha, \beta) = (-1)^n \frac{a((\alpha \sim b) \sim \Gamma)}{t} \). For this last step, we must have a closer look at the proof of Lemma 7.59. There, we saw that, at the chain level, \( (\alpha \sim \beta) \sim x = \Phi(id \otimes \Theta(\alpha \otimes \beta)) (d \otimes id) \mathbf{d}(x) \), while \( \alpha \sim (\beta \sim x) = \Phi(id \otimes \Theta(\alpha \otimes \beta)) (d \otimes id) \mathbf{d}(x) \). The identification of these in homology follows from \((d \otimes id) \mathbf{d} \) and \((id \otimes \mathbf{d}) \mathbf{d} \) being chain homotopic, by Lemma 7.54. If \( D \) denotes the chain homotopy, we therefore have

\[
(\alpha \sim \beta) \sim \Gamma - \alpha \sim (\beta \sim \Gamma) = \Phi(id \otimes \Theta(\alpha \otimes \beta))((D\partial + \partial D)\Gamma).
\]

As \( \Gamma \) is a cycle, this difference reduces to \( \Phi(id \otimes \Theta(\alpha \otimes \beta)) (\partial D\Gamma) \). But \( \partial D\Gamma \) is a cycle, while \( \Theta(\alpha \otimes \beta) \) is a cocycle, as \( \alpha \) and \( \beta \) are cocycles and \( \Theta \) is a chain map, by Lemma 7.25. Following the computation of Lemma 7.29, where we show that the cap product is well defined, \( \Phi(id \otimes \Theta(\alpha \otimes \beta)) (\partial D\Gamma) \) must therefore be a boundary, so \( a(\Phi(id \otimes \Theta(\alpha \otimes \beta)) (\partial D\Gamma)) = 0 \). Thus \( a((\alpha \sim \beta) \sim \Gamma) = a(\alpha \sim (\beta \sim \Gamma)) \).

Now that we have an expression for the pairing \( L \) in terms of the cap product, we can provide a symmetry formula, which will also allow us to show that \( L \) is nonsingular. More specifically, we have defined \( L \) as the adjoint of \( D^* \lambda \); as \( \lambda \) and \( D \) (and hence \( D^* \)) are isomorphisms, one of the adjoints of \( L \) is an isomorphism directly. We need to show that the other adjoint \( T(I_{\bar{p}} H^{n-i+1}(X, \partial X; R)) \rightarrow \text{Hom}(T(I_{\bar{p}} H^i(X; R)), Q(R)/R) \) is also an isomorphism. For this, we will show that

\[
\frac{a((\alpha \sim b) \sim \Gamma)}{t} = \pm \frac{a((\beta \sim a) \sim \Gamma)}{r},
\]

continuing to assume \( da = r \alpha \) and \( db = t \beta \), \( r, t \neq 0 \), and with the sign depending only on the degrees of \( \alpha \) and \( \beta \). This is then, up to sign, the exact formula we have discovered for \( L(\alpha, \beta) \) but with the roles of \( \alpha \) and \( \beta \) reversed, and so also with the roles of the pairs \( (X, \emptyset) \) and \( (X, \partial X) \) reversed. But each of our preceding computations involving \( I_{\bar{p}} H^* (X; R) \) hold just as well applied to \( I_{\bar{p}} H^* (X, \partial X; R) \). Therefore, this expression must be the adjoint of the composition of isomorphisms

\[
T(I_{\bar{p}} H^{n-i+1}(X, \partial X; R)) \xrightarrow{\lambda} \text{Hom}(T(I_{\bar{p}} H^{-i}_n(X, \partial X; R)), Q(R)/R) \xrightarrow{D^*} \text{Hom}(T(I_{\bar{p}} H^i(X; R)), Q(R)/R),
\]

which defines \( L'_{\bar{p}_{\bar{p}} b} \).

**Lemma 8.65.** Suppose \( R \) is a Dedekind domain and that \( X \) is a compact \( n \)-dimensional \( R \)-oriented locally \( (\bar{p}; R) \)-torsion free \( \partial \)-stratified pseudomanifold. Let \( L_{\bar{p}, p} : T(I_{\bar{p}} H^i(X; R)) \otimes T(I_{\bar{p}} H^{n-i+1}(X, \partial X; R)) \rightarrow Q(R)/R \) and \( L'_{\bar{p}, \bar{p}} : T(I_{\bar{p}} H^{n-i+1}(X, \partial X; R)) \otimes T(I_{\bar{p}} H^i(X; R)) \rightarrow Q(R)/R \) be torsion pairings. Suppose \( \alpha \in T(I_{\bar{p}} H^i(X; R)) \) and \( \beta \in T(I_{\bar{p}} H^{n-i+1}(X, \partial X; R)) \). Then

\[
L_{\bar{p}, p}(\alpha \otimes \beta) = (-1)^{1+n+i} L'_{\bar{p}, \bar{p}}(\beta \otimes \alpha).
\]

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Remark 8.66. The sign \((-1)^{1+n+in}\) in the lemma looks somewhat asymmetric, but recall that we have \(|\alpha| + |\beta| = n + 1\). A quick computation then shows that \((-1)^{1+n+in} = (-1)^{1+(|\alpha|+1)(|\beta|+1)}\), or, equivalently, \((-1)^{1+n+in} = (-1)^{1+|\alpha|}\), if \(d\alpha = r\alpha\) and \(d\beta = t\beta\), \(r,t \neq 0\). A quick comparison with, for example, [22, Exercise 56], demonstrates that this sign yields the correct symmetries for self pairings when \(n = 4m + 1\) or \(4m + 3\).

Proof of Lemma 8.65. We continue to assume our earlier notation, in particular that \(d\alpha = r\alpha\) and \(d\beta = t\beta\) for some \(r,t \in R\), \(r,t \neq 0\), and that we have chosen a fixed algebraic diagonal with which to define cup and cap products.

We begin by observing that

\[
d(a \cup b) = (da) \cup b + (-1)^{|\alpha|}a \cup db = r\alpha \cup b + (-1)^{|\alpha|}a \cup t\beta,
\]

using that the cup product is defined via a composition of chain maps. So we have

\[
L_{\bar{p},D\bar{p}}(\alpha \otimes \beta) = (-1)^{n}a((\alpha \cup b) \cup \Gamma)
\]

\[
= (-1)^{n}r a((\alpha \cup b) \cup \Gamma)
\]

\[
= (-1)^{n}a((r\alpha \cup b) \cup \Gamma)
\]

\[
= (-1)^{n}a((da \cup b) - (-1)^{|\alpha|}a \cup t\beta) \cup \Gamma)
\]

\[
= (-1)^{n}a((da \cup b) \cup \Gamma) - (-1)^{n+|\alpha|}a((a \cup t\beta) \cup \Gamma)
\]

\[
= -(-1)^{n+|\alpha|}a((a \cup \beta) \cup \Gamma).
\]

In the last line, we have used that

\[
d(a \cup b) \cup \Gamma = \pm \partial((a \cup b) \cup \Gamma),
\]

by Lemma [7.28] as \(\Gamma\) is a cycle. But the augmentation of a boundary is 0.

Next, a close look at the proof of Corollary [7.40] shows that, at the chain level,

\[
a \cup \beta - (-1)^{|\beta||\alpha|} \beta \cup a = (Dd + dD)\Theta(a \otimes \beta),
\]

where \(D\) is the chain homotopy between \(d^*\) and \((\tau d)^*\) guaranteed by Lemma [7.37] and using that the dual of a chain homotopy is a chain homotopy.

So

\[
a((a \cup \beta) \cup \Gamma) - (-1)^{|\beta||\alpha|}a((\beta \cup a) \cup \Gamma) = a(((Dd + dD)\Theta(a \otimes \beta)) \cup \Gamma)
\]

\[
= a((Dd\Theta(a \otimes \beta)) \cup \Gamma) + a((dD\Theta(a \otimes \beta)) \cup \Gamma).
\]

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The second expression on the last line is 0, as \( (dD\Theta(a \otimes \beta)) \sim \Gamma ) = \pm \partial((D\Theta(a \otimes \beta)) \sim \Gamma ) \), using again Lemma 7.28 that \( \Gamma \) is a cycle, and that the augmentation of a cycle is 0. For the first expression, we use again that \( \Theta \) is a chain map, by Lemma 7.25. So, as \( \beta \) is a cycle,

\[
\frac{a(((Dd\Theta(a \otimes \beta)) \sim \Gamma )}{r} = \frac{a(((D\Theta((da) \otimes \beta)) \sim \Gamma )}{r} = \frac{a(((D\Theta(r\alpha \otimes \beta)) \sim \Gamma )}{r} = \frac{a(((D\Theta(\alpha \otimes \beta)) \sim \Gamma )}{r},
\]

which must be in \( R \). So this term also vanishes in \( Q(R)/R \).

Altogether, we have now shown that

\[
L_{\bar{p},D\bar{p}}(\alpha \otimes \beta) = (-1)^{t} \frac{a((\alpha \sim b) \sim \Gamma )}{r} = \frac{(-1)^{1+n+|a|+|\beta||a|} a((\beta \sim a) \sim \Gamma )}{r} = (-1)^{1+n+|a|+|\beta||a|} L'_{D\bar{p},\bar{p}}(\beta, \alpha).
\]

To simplify the signs, let us recall that \( |\alpha| = i \) and \( |\beta| = n - i + 1 \), so \( |a| = i - 1 \). Thus, mod 2, we have

\[
1 + |a| + |\beta||a| \equiv 1 + i - 1 + (n - i + 1)(i - 1) \equiv 1 + i + 1 + ni + i + n + i + 1 \equiv 1 + ni + n.
\]

This completes the lemma.

As noted just before the preceding lemma, the symmetry demonstrated by the lemma implies the second condition of nonsingularity for the linking pairing \( L_{\bar{p},D\bar{p}} \). The nonsingularity of \( L'_{D\bar{p},\bar{p}} \) follows analogously. So, summarizing all the work in this section, we arrive at the statement of Corollary 8.60 presented at the beginning of this section.

### 8.5.4 Topological invariance of pairings

**Theorem 8.67.** Suppose \( R \) is a Dedekind domain and \( \bar{p} \) is a GM perversity. Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional compact compatibly \( R \)-oriented \( \partial \)-stratified pseudomanifold stratifications with no codimension one strata of the same underlying space pairs \( (|X_1|, |\partial X_1|) = (|X_2|, |\partial X_2|) \). Suppose \( X_1 \) (or, equivalently, \( X_2 \)) is locally \( (\bar{p}; R) \)-torsion free. Then the cup product and torsion pairings are independent of the stratification in the sense that there are
canonical diagrams

\[
F(I_p H^i(X_1; R)) \otimes F(I_D H^{n-i}(X_1, \partial X_1; R)) \cong \leftarrow \sim \rightarrow \sim \rightarrow R
\]

and

\[
T(I_p H^i(X_1; R)) \otimes T(I_D H^{n-i+1}(X_1, \partial X_1; R)) \cong \leftarrow \sim \rightarrow Q(R)/R
\]

and similarly for the other torsion pairing \(L'\).

Remark 8.68. Yet again, as in Remarks 8.38 and 8.50, this statement can be extended immediately to the observation that the pairings are topological invariants in the sense that, if there is an orientation-preserving topological homeomorphism \(f : (|X|, |\partial X|) \rightarrow (|Y|, |\partial Y|)\) between the compact \(n\)-dimensional \(R\)-oriented \(\partial\)-stratified pseudomanifolds \(X\) and \(Y\) without codimension one strata, then \(f\), together with the isomorphisms of the theorem, induce isomorphisms between the pairings for \(X\) and the pairings for \(Y\).

Proof. For the cup product, we have already established the basic tools of the proof in our prior proofs of the topological invariance of products (Theorem 7.36), fundamental classes (Propositions 8.28 and 8.44), and duality (Theorems 8.37 and 8.49). We did not establish invariance of products for \(\partial\)-pseudo-manifolds, but the methods of Proposition 8.44 and Theorem 8.49 apply: we can replace the pairs \((X_i, \partial X_i)\) with appropriate pairs \((\hat{X}_i, \hat{N}_i)\) or \((\hat{X}_i, \hat{N}_i)\) for comparisons with the intrinsic stratifications. Such arguments are sufficient to demonstrate invariance of the cup product pairing.

For the torsion pairing, the proof is a little trickier as our direct formula for computing the pairing involves a cup product of cochains, not just cohomology classes. At the chain level, maps such as \(I_p S^i(X^*; R) \rightarrow I_p S^i(X_i; R)\) are not, in general, isomorphisms, so we cannot invert them as we have when working at the cohomology level. Rather, let us revert to our original definition of the torsion product by its adjoint. Then we can consider the diagram
Every horizontal map is induced by an inclusion $X_i \rightarrow \tilde{X}^*$ or $(X_i, \partial X_i) \rightarrow (\tilde{X}^*, \tilde{N}^*)$, where the latter spaces are as in the proof of Theorem 8.49. In fact, the top row here is a compressed version of the chain of cohomology isomorphisms corresponding to the homology isomorphisms on the right of diagram (36) in the proof of Theorem 8.49, restricted to the torsion submodules. As in that proof, these maps induce isomorphisms in intersection homology or cohomology. Noting that an isomorphism of modules induces an isomorphism of torsion submodules, the horizontal maps are all isomorphisms by functoriality.

The composition of all vertical maps in each column, except for the bottom map in each column, is the composition we called $\lambda$ in Section 8.5.3. All the squares except those in the bottom row commute by the naturality of the Universal Coefficient Theorem, by functoriality of Hom and Ext, or by the naturality of the six-term exact sequence Section IV.8. The bottom two squares of the diagram can be obtained by taking diagram (36) in the proof of Theorem 8.49 and interchanging $\tilde{p}$ and $\tilde{Dp}$, restricting to torsion submodules, taking the Hom$(\cdot, Q(R)/R)$ dual, and then compressing the diagram.

Now, let $L_{\tilde{p}, \tilde{Dp}}^i$ be the torsion pairing on $X_i$. Let $\phi_{\tilde{p}} : T(I_{\tilde{p}}H^i(X_1; R)) \rightarrow T(I_{\tilde{p}}H^i(X_2; R))$ be the composition isomorphism across the top of the diagram. We can then let $\phi_{\tilde{Dp}} : T(I_{\tilde{Dp}}H^{n-i+1}(X_1, \partial X_1; R)) \rightarrow T(I_{\tilde{Dp}}H^{n-i+1}(X_2, \partial X_2; R))$ be the composition along the left side of diagram (36) (interchanging $\tilde{p}$ and $\tilde{Dp}$ and restricting to torsion), and it follows that $\phi_{\tilde{Dp}}$ is the composition along the bottom of our diagram, right to left\(^{121}\). Let $\lambda_i$ and $D_i^*$ be the appropriate maps on $X_1$ and $X_2$. Then, using the diagram and the definition of the

\(^{121}\) Notice that if we define a map as $g^{-1}f$ for some isomorphism $g$, then we do indeed have $(g^{-1}f)^* = f^*(g^{-1})^*$, so this claim makes sense.
We compute
\[ L^i_{\bar{\phi},D\bar{p}}(\alpha \otimes \beta) = (D^i_1\lambda_1(\alpha))(\beta) = (\phi_{D\bar{p}}D^i_2\lambda_2\phi_p(\alpha))(\beta) = (D^i_2\lambda_2\phi_p(\alpha))(\phi_{D\bar{p}}(\beta)) = (\phi_p(\alpha)) \otimes (\phi_{D\bar{p}}(\beta))). \]

Therefore, letting the map on the left side of diagram (43) be \( \phi_p \otimes \phi_{D\bar{p}} \), the diagram commutes as claimed. This choice is canonical as each of \( \phi_p \) and \( \phi_{D\bar{p}} \) is one the canonical maps involved in Theorem 8.49.

### 8.5.5 Image pairings

When \( M^{2k} \) is a compact \( R \)-oriented even-dimensional manifold, the cup product pairing of Corollary 8.57 gives us a nonsingular pairing
\[ FH^k(M; R) \otimes FH^k(M; R) \to R. \]
Notice that the two input modules to the pairing are identical. In this setting, it is possible to tease out further invariants that have proven important in manifold theory. For example, taking \( R = \mathbb{Q} \) and \( k \) even the symmetric pairing \( H^k(M; \mathbb{Q}) \otimes H^k(M; \mathbb{Q}) \to \mathbb{Q} \) yields the signature invariant by subtracting the number of negative eigenvalues of a matrix representing the pairing from the number of positive eigenvalues of the matrix. Details of the signature will be developed in Section 10.

When \( M \) is a compact \( \tilde{R} \)-oriented \( \partial \)-manifold with non-empty boundary, then the cup product pairing only has the form
\[ FH^k(M; R) \otimes FH^k(M, \partial M; R) \to R. \]

Even though Poincaré duality can be used to show that \( FH^k(M; R) \) and \( FH^k(M, \partial M; R) \) will be abstractly isomorphic, they are not identical, and so we do not obtain the same sort of self-pairing that occurs when \( \partial M = \emptyset \). Nonetheless, there is a way to recover a self-pairing in this setting that allows one to define the signature for manifolds with boundary of dimension 0 mod 4. It turns out that the cup product pairing between \( FH^k(M; R) \) and \( FH^k(M, \partial M; R) \) induces a non-degenerate pairing on \( \text{im}(i^*: FH^k(M, \partial M; R) \to FH^k(M; R)) \). If \( \alpha, \beta \in \text{im}(i^*) \) and \( \bar{\alpha}, \bar{\beta} \) are their preimages in \( FH^k(M, \partial M; R) \), then the pairing takes \( \alpha \otimes \beta \) to
\[ a(D(\bar{\alpha} \sim \bar{\beta})) \in R. \]

We will see below that this is well defined, though it’s worth noting now that, thanks to naturality with respect to the maps \( (M; \partial M, \emptyset) \to (M; \partial M, \partial M) \) and \( (M; \emptyset, \partial M) \to (M; \partial M, \partial M) \), we have \( \bar{\alpha} \sim \bar{\beta} = \bar{\alpha} \sim \beta = \alpha \sim \bar{\beta} \in H^n(M, \partial M; R) \). We will call this pairing the image pairing.

In the world of intersection homology, there are two issues with construction a self-pairing from the cup product pairing
\[ F(I_pH^i(X; R)) \otimes F(I_{D\bar{p}}H^{n-i}(X, \partial X; R)) \to R. \]
We still have the possibility of nontrivial boundaries to contend with, but there is also a lack of symmetry due to the difference between the two perversities. In Section 10, we will discuss conditions that can be imposed on spaces to ensure that \( I_p H^*(X; R) \cong I_{Dp} H^*(X; R) \). In this section, we will look at an intersection homology version of the image pairing that arises if we make the assumption that \( p \leq Dp \).

**Nondegeneracy.** Before moving on to intersection homology, we observe that the best we can hope for image pairings is nondegeneracy, not, in general, nonsingularity.

First, recall from Section 8.5.1 that a pairing \( P : A \otimes B \to C \) is called nonsingular if the adjunct homomorphisms \( A \to \text{Hom}(B, C) \) and \( B \to \text{Hom}(A, C) \) are both isomorphisms, while it is called nondegenerate if these homomorphisms are only assumed injective. Equivalently, \( P \) is nondegenerate if and only if

1. \( P(a, b) = 0 \) for all \( b \in B \) if and only if \( a = 0 \) and
2. \( P(a, b) = 0 \) for all \( a \in A \) if and only if \( b = 0 \).

If \( A \) and \( B \) are finitely-generated vector spaces over a field \( F \) and if \( C = F \), then nonsingularity and nondegeneracy are equivalent as having injections \( A \to \text{Hom}(B, F) \cong B \) and \( B \to \text{Hom}(A, F) \cong A \) implies that \( A \) and \( B \) must have the same dimension and the injections must therefore be isomorphisms. However, when the ground ring is not a field, nondegeneracy and nonsingularity are not equivalent, as the following example shows.

**Example 8.69.** Let \( P : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z} \) be the pairing such that \( P(1, 1) = r \) with \( r \neq 0 \). Then by the bilinearity of \( P \), for any \( a, b \in \mathbb{Z} \) we have \( P(a, b) = abP(1, 1) = rab \). It is easy to observe that this pairing is nondegenerate: clearly \( P(a, b) = 0 \) only if \( a \) or \( b \) is equal to 0. In fact, we can compute that each of the two adjunct maps \( \mathbb{Z} \to \text{Hom}(\mathbb{Z}, \mathbb{Z}) \) takes 1 to \( \phi_1 \in \text{Hom}(\mathbb{Z}, \mathbb{Z}) \) with \( \phi_1(1) = r \). Thus the adjuncts are injective. However, the map \( \mathbb{Z} \to \text{Hom}(\mathbb{Z}, \mathbb{Z}) \) determined by \( 1 \to \phi_1 \) will be an isomorphism only if \( r = \pm 1 \), as otherwise \( \phi_1 \) is not a generator of \( \text{Hom}(\mathbb{Z}, \mathbb{Z}) \).

While we will show below that the image pairing is nondegenerate, the next example shows that it need not be nonsingular, even when working with \( \partial \)-manifolds, which of course constitute a special case of \( \partial \)-stratified pseudomanifolds.

**Example 8.70.** Let \( M \) be the disk bundle associated to the tangent bundle of the sphere \( S^2 \). Then \( \partial M \) is the sphere bundle associated to the tangent bundle of the sphere \( S^2 \), and in fact \( \partial M \cong \mathbb{R} P^3 \). To see this, let \( x \in S^2 \) and let \( v \) be a unit vector in \( T_x S^2 \), the tangent space to \( S^2 \) at \( x \), which we can think of as embedded in \( \mathbb{R}^3 \) in the standard way as the tangent plane to the unit sphere. Then the triple \((x, v, x \times v)\) is an ordered, right-handed, orthonormal triple of vectors in \( \mathbb{R}^3 \), which prescribes the rotation in \( SO(3) \) that takes the standard basis to the triple. In fact, this map describes a homeomorphism from \( \partial M \) to \( SO(3) \) (exercise!), which is well known to be homeomorphic to \( \mathbb{R} P^3 \) (see, e.g., [53 Section 3.D])

The space \( M \) itself is compact 4-dimensional and orientable, as \( M \) is homotopy equivalent to \( S^2 \) and so simply connected. Let us compute some cohomology groups:

- \( M \) is homotopy equivalent to \( S^2 \), so \( H^i(M) \cong \mathbb{Z} \) if \( i = 0, 2 \) and \( H^i(M) = 0 \) otherwise.
By Lefschetz duality, \( H_i(M, \partial M) \cong \mathbb{Z} \) if \( i = 2, 4 \) and \( H_i(M, \partial M) = 0 \) otherwise. So, by the Universal Coefficient Theorem, the only nontrivial \( H^i(M, \partial M) \) are \( H^2(M, \partial M) \cong H^4(M, \partial M) \cong \mathbb{Z} \).

As \( \partial M \cong \mathbb{R}P^3 \), we have \( H_0(\partial M) \cong H_3(\partial M) \cong \mathbb{Z} \), \( H_1(\partial M) \cong \mathbb{Z}_2 \), and \( H_i(\partial M) = 0 \) otherwise, by standard computations (e.g. \cite{53} Example 2.42). By Poincaré duality, we have \( H^3(\partial M) \cong H^0(\partial M) \cong \mathbb{Z} \), \( H^2(\partial M) \cong \mathbb{Z}_2 \), and \( H^1(\partial M) = 0 \).

The middle portion of the exact sequence of the pair thus looks like

\[
\begin{array}{cccccc}
H^1(\partial M) & \hookrightarrow & H^2(M, \partial M) & \overset{i^*}{\longrightarrow} & H^2(\partial M) & \longrightarrow & H^3(M, \partial M) \\
0 & \quad & \mathbb{Z} & \quad & \mathbb{Z} & \quad & \mathbb{Z}_2 & \quad & 0,
\end{array}
\]

and it follows that \( i^* \) must take a generator of \( H^2(M, \partial M) \) to twice a generator of \( H^2(M) \). We can assume we have chosen generators \( \bar{\alpha} \in H^2(M, \partial M) \) and \( \gamma \in H^2(M) \) so that \( i^*(\bar{\alpha}) = 2\gamma \), which is a generator of \( \text{im}(i^*) \).

The nonsingular pairing \( H^2(M) \otimes H^2(M, \partial M) \to \mathbb{Z} \) guarantees that if \( \bar{\alpha} \) is a generator of \( H^2(M, \partial M) \) and \( \gamma \) is a generator of \( H^2(M) \), then \( \bar{\alpha} \smile \gamma = \pm 1 \in H^4(M, \partial M) \cong \mathbb{Z} \). So then, by definition, the image pairing acting on the generators \( 2\gamma \) of \( \text{im}i^* \) takes \( 2\gamma \otimes 2\gamma \) to the augmentation of the Lefschetz dual of \( \bar{\alpha} \smile \bar{\alpha} \). By naturality of the cup product with respect to the map \( (M; \partial M, \emptyset) \to (M; \partial M, \partial M) \), we have \( \bar{\alpha} \smile \bar{\alpha} = \bar{\alpha} \smile i^*(\alpha) = \bar{\alpha} \smile 2\gamma = \pm 2 \in H^4(M, \partial M) \cong \mathbb{Z} \). Thus the image pairing takes the tensor product of generators of \( \text{im}(i^*) \) to \( \pm 2 \in \mathbb{Z} \).

So by Example \ref{8.69} the image pairing is nondegenerate, but it is not nonsingular.

**The intersection homology image pairings.** Suppose \( X \) is a \( \partial \)-stratified pseudomanifold with a perversity \( \bar{p} \) such that \( \bar{p} \leq D\bar{p} \), i.e. \( \bar{p}(S) \leq D\bar{p}(S) \) for all \( S \). This is certainly possible; for example is \( X \) is a classical \( \partial \)-stratified pseudomanifold then \( \bar{0} \leq D\bar{0} = \bar{t} \). We also have \( \bar{m} \leq \bar{n} \), where \( \bar{m} \) and \( \bar{n} \) are the lower- and upper-middle perversities of Definition \ref{3.8}.

Given this assumption, the identity map \( X \to X \) is \((\bar{p}, D\bar{p})\)-stratified. If we let \( i : (X, \emptyset) \to (X, \partial X) \) be the identity/inclusion, we can then consider

\[
\begin{align*}
\text{i}^* : F(I_{D\bar{p}}H^*(X, \partial X; R)) & \to F(I_{\bar{p}}H^*(X; R)).
\end{align*}
\]

This is well defined, as \( \text{i}^* \) takes torsion elements of \( I_{D\bar{p}}H^*(X, \partial X; R) \) to torsion elements of \( I_{\bar{p}}H^*(X; R) \).

In the following proposition, we use the notation \( \text{i}^*_i \) to specify the map \( \text{i}^* \) in degree \( i \).

**Proposition 8.71.** Suppose \( R \) is a Dedekind domain, and let \( X \) be a compact \( n \)-dimensional \( R \)-oriented locally \((\bar{p}; R)\)-torsion free \( \partial \)-stratified pseudomanifold. Suppose that \( \bar{p} \leq D\bar{p} \), and
Let $\alpha \in \text{im}(i^*_i : F(I_D^i H^i(X, \partial X; R)) \to F(I_D^i H^i(X; R)))$ and $\beta \in \text{im}(i^*_{n-i} : F(I_D^{n-i} H^{n-i}(X, \partial X; R)) \to F(I_D^i H^{n-i}(X; R)))$. Let $\bar{\alpha}$ and $\bar{\beta}$ be preimages of $\alpha$ and $\beta$ in $F(I_D^i H^i(X, \partial X; R))$. Then the pairing $\text{im}(i^*_i) \otimes \text{im}(i^*_{n-i}) \to R$ given by
\[
\alpha \otimes \beta \to aD(\alpha \sim \bar{\beta}) = aD(\bar{\alpha} \sim \bar{\beta}) = aD(\bar{\alpha} \sim \beta)
\]
is well defined and nondegenerate.

**Proof.** Notice that $\alpha \in F(I_D^i H^i(X; R))$ and $\beta \in F(I_D^{n-i} H^{n-i}(X, \partial X; R))$, so $aD(\alpha \sim \bar{\beta})$ is the well-defined image of the cup product pairing of Corollary 8.57. Similarly for $aD(\bar{\alpha} \sim \beta)$. So to show that our image pairing is well defined, we must demonstrate independence of the choices of $\bar{\alpha}$ and $\bar{\beta}$ and show that the claimed equalities are valid.

Consider the following diagram:
\[
\begin{array}{ccc}
I_D^i H^i(X, \partial X; R) \otimes I_D^i H^i(X, \partial X; R) & \xrightarrow{i \otimes \text{id}} & I_D^i H^i(X, \partial X; R) \\
& \text{id} \downarrow & \\
I_D^i H^i(X; R) \otimes I_D^i H^i(X, \partial X; R) & \xrightarrow{i} & I_D^i H^i(X, \partial X; R).
\end{array}
\]

Follow Definition 7.23, the cup product on the top line is well-defined as
\[
D\bar{0} = \bar{i} = \bar{p} + D\bar{p} \geq \bar{p} + \bar{p} = DD\bar{p} + DD\bar{p},
\]
the inequality is via our assumption that $\bar{p} \leq D\bar{p}$. Also, we use Corollary 8.36 to know that $X$ is locally $(D\bar{p}; R)$-torsion free. The vertical maps are induced by the inclusion/identity maps $(X; \emptyset, \partial X) \to (X, \partial X, \partial X)$. The diagram commutes by naturality of the cup product, Lemma 7.33. But this says that $\bar{\alpha} \sim \bar{\beta} = i^*(\bar{\alpha}) \sim \bar{\beta} = \alpha \sim \beta$. A similar diagram shows that $\bar{\alpha} \sim \beta = \bar{\alpha} \sim \bar{\beta}$. So $\alpha \sim \beta = \bar{\alpha} \sim \bar{\beta} = \bar{\alpha} \sim \beta$, showing that our pairing formulas are equal. Additionally, as the left side of the equation does not depend on the choice of preimage of $\alpha$ and the right hand side does not depend on the choice of the preimage of $\beta$, our pairing is independent of these choices.

Next, we must show that the image pairing is nondegenerate. For this, first suppose that $\alpha \in F(i^*_i) \subset F(I_D^i H^i(X; R))$ with $\alpha \neq 0$. By the nonsingularity of the cup product pairing demonstrated in Corollary 8.57, there exists some $\bar{\beta} \in F(I_D^{n-i} H^{n-i}(X; R))$ such that $aD(\alpha \sim \bar{\beta}) \neq 0$. But then if $\beta = i^*(\bar{\beta})$, the image pairing is non-zero on $\alpha \otimes \beta$. Thus the image pairing takes $\alpha \otimes \beta$ to 0 for all $\beta$ only if $\alpha = 0$. The equivalent argument holds interchanging the roles of $\alpha$ and $\beta$, so we see that the image pairing is nondegenerate. 

Similarly, we can consider the image torsion pairing:

**Proposition 8.72.** Suppose $R$ is a Dedekind domain, and let $X$ be a compact $n$-dimensional $R$-oriented locally $(\bar{p}; R)$-torsion free $\partial$-stratified pseudomanifold. Suppose that $\bar{p} \leq D\bar{p}$, and let $\alpha \in \text{im}(i^*_i : T(I_D^i H^i(X, \partial X; R)) \to T(I_D^i H^i(X; R)))$ and $\beta \in \text{im}(i^*_{n-i+1} : T(I_D^{n-i+1} H^{n-i+1}(X, \partial X; R)) \to$
\[ T(I_pH^{n-i+1}(X; R)). \] Let \( \bar{\alpha} \) and \( \bar{\beta} \) be the preimages of \( \alpha \) and \( \beta \) in \( T(I_pH^*(X, \partial X; R)) \). Then the pairing on the image modules given by
\[ \alpha \otimes \beta \rightarrow L_{p,DP}(\alpha, \beta) = L'_{p,DP}(\bar{\alpha}, \bar{\beta}) \]
is well defined and nondegenerate.

**Proof.** The proof is analogous to that of Proposition \[8.71\]. In particular, assuming the pairing is well defined, we obtain nondegeneracy as follows: Suppose that \( \alpha \in \text{im}(t^*: T(I_pH^i(X, \partial X; R)) \rightarrow T(I_pH^i(X; R))) \) with \( \alpha \neq 0 \). By the nonsingularity of the torsion pairing demonstrated in Corollary \[8.60\] there exists some \( \bar{\beta} \in T(I_pH^{n-i+1}(X; R)) \) such that \( L_{p,DP}(\alpha, \bar{\beta}) \neq 0 \). But then if \( \beta = \bar{i}^*(\bar{\beta}) \), the image pairing is non-zero on \( \alpha \otimes \beta \) by definition. Thus the image pairing takes \( \alpha \otimes \beta \) to 0 for all \( \beta \) only if \( \alpha = 0 \). The equivalent argument holds interchanging the roles of \( \alpha \) and \( \beta \), so we see that the image pairing is nondegenerate.

To show that the pairing is well defined, we need to show that \( L_{p,DP}(\alpha, \bar{\beta}) = L'_{p,DP}(\bar{\alpha}, \beta) \). This will demonstrate the independence of the choices of \( \bar{\alpha} \) and \( \bar{\beta} \), as the left hand term does not depend on the choice of \( \bar{\alpha} \) and the right hand side does not depend on the choice of \( \bar{\beta} \).

Suppose we let \( \bar{b} \in I_{DP}S^{n-i}(X, \partial X; R) \) such that \( \bar{d}\bar{b} = t\bar{\beta} \). Then \( d\bar{b} = t\bar{\beta} \). By definition, we thus have \( L_{p,DP}(\alpha, \bar{\beta}) = (-1)^n a(\frac{(\alpha - \bar{b})}{t}) \) and \( L'_{p,DP}(\bar{\alpha}, \beta) = (-1)^n a(\frac{(\bar{\alpha} - \beta)}{t}) \). So, it suffices to show that both these expressions are equal to \( (-1)^n \frac{a((\bar{\alpha} - \beta) - \Gamma)}{t} \). We will provide the argument for \( \frac{a((\alpha - \bar{b}) - \Gamma)}{t} \), the other argument being equivalent.

We begin by observing that the cup product in these formulas is the chain level cup product, as \( \bar{b} \) and \( \bar{\beta} \) are not cocycles; therefore the naturality of the cup product stated in Lemma \[7.33\] does not apply. However, we do still have a chain level version of Diagram \[14\], and, using Lemma \[7.30\] and the definitions, this diagram commutes up to chain homotopy. If we let \( D \) denote the chain homotopy, we therefore have that
\[ \frac{a((\alpha - \bar{b}) - \Gamma)}{t} = \frac{a((\bar{\alpha} - \bar{b}) - \Gamma)}{t} \frac{a((Dd + dD)(\bar{\alpha} \otimes \bar{b})) - \Gamma)}{t} \].

To evaluate the right hand side, we use that \( \bar{\alpha} \) is a cocycle, that \( \bar{d}\bar{b} = t\bar{\beta} \), that the augmentation of a boundary must be 0, and the formula \( \partial(\gamma \otimes \xi) = (d\gamma) \otimes \xi + (\xi) \otimes \partial \xi \) from Lemma \[7.28\], noting that \( \Gamma \) is a cycle in \( I^0S_n(X, \partial X; R) \):
\[
\frac{a((Dd + dD)(\bar{\alpha} \otimes \bar{b})) - \Gamma)}{t} = \frac{a((Dd(\bar{\alpha} \otimes \bar{b})) - \Gamma)}{t} + \frac{a((d(D(\bar{\alpha} \otimes \bar{b}))) - \Gamma)}{t} \\
= \frac{a((D((d\bar{\alpha}) \otimes \bar{b} \pm \bar{\alpha} \otimes d\bar{b})) - \Gamma)}{t} + \frac{a(\partial(D(\bar{\alpha} \otimes \bar{b}) - \Gamma) \pm (D(\bar{\alpha} \otimes \bar{b}) - \partial \Gamma)}{t} \\
= \pm \frac{a((D(\bar{\alpha} \otimes t\bar{\beta})) - \Gamma)}{t} \\
= \pm \frac{a((D(\bar{\alpha} \otimes \bar{\beta})) - \Gamma)}{t}.
\]
This remaining expression is in \( R \), and therefore \( \frac{a((\alpha - \bar{b}) - \Gamma)}{t} = \frac{a((\bar{\alpha} - \bar{b}) - \Gamma)}{t} \) in \( Q(R)/R \). \( \square \)
9 CHAPTER ON THE PL INTERSECTION PAIRING?? (TENTATIVE PENDING RESULTS)

10 Witt spaces and IP spaces

If $M$ is a compact oriented $4k$-dimensional manifold, then we have seen in Section 8.5.2 that we have a nonsingular cup product

$$H^{2k}(M; \mathbb{Q}) \otimes H^{2k}(M; \mathbb{Q}) \xrightarrow{\alpha \otimes \beta} H^{4k}(M; \mathbb{Q}) \xrightarrow{D} H_0(M; \mathbb{Q}) \xrightarrow{\nu} \mathbb{Q}.$$ 

By Corollary 7.40, this pairing is symmetric, i.e. $\alpha \otimes \beta$ and $\beta \otimes \alpha$ have the same image in $\mathbb{Q}$. This algebraic situation yields an invariant, the signature of the pairing, which turns out to be an important topological invariant of $M$. According to Gromov [49, Section 7], the signature “is not just ‘an invariant’ but the invariant which can be matched in beauty and power only by the Euler characteristic.”

In this chapter, we will explore when signatures, and related invariants, can be extended to pseudomanifolds. It will turn out that some restrictions are needed on the space, but that on such spaces we obtain invariants with very nice properties.

10.1 Witt spaces

When we turn to expanding the signature invariant to stratified spaces, there is an immediate problem: the signature of a closed oriented manifold $M^{4n}$ is defined using the nonsingular symmetric cup product pairing $H^{2n}(M; \mathbb{Q}) \otimes H^{2n}(M; \mathbb{Q}) \rightarrow H^{4n}(M; \mathbb{Q}) \rightarrow \mathbb{Q}$. But for closed oriented pseudomanifolds, the Poincaré duality studied in Chapter 8 provides nonsingular pairings $I^\bar{p}H^{2n}(X; \mathbb{Q}) \otimes I^{D\bar{p}}H^{2n}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$. There is no reasonable way to interpret this as a symmetric pairing unless $I^\bar{p}H^{2n}(X; \mathbb{Q})$ and $I^{D\bar{p}}H^{2n}(X; \mathbb{Q})$ are naturally isomorphic. So at first glance, we need only find a perversity $\bar{p}$ such that $\bar{p} = D\bar{p}$. However, this would require $\bar{p}(S) = D\bar{p}(S) = \text{codim}(S) - 2 - \bar{p}(S)$, or $\bar{p}(S) = \frac{\text{codim}(S) - 2}{2}$ for all singular strata $S$. Clearly this is not possible if $X$ has strata of odd codimension.

In fact, an early solution to this problem [42] was to work with spaces with only even codimension singularities. For example, every complex algebraic variety can be given such a stratification [44, Section 1.7], so certainly this limitation was not completely unreasonable. However, one can do better.

To start, let us consider perversities $\bar{p}$ so that $\bar{p}(S) = \frac{\text{codim}(S) - 2}{2}$ when the codimension of $S$ is even. For such a $\bar{p}$, we have $\bar{p} = D\bar{p}$ on spaces with only singularities of even codimension. Now, what should we do when the codimension of $S$ is odd? To stay as close to self-dual as possible, we should choose perversities that round $\frac{\text{codim}(S) - 2}{2}$ up or down to the nearest integer. If we round down, we obtain $\bar{m}$, the lower-middle perversity, defined by $\bar{m}(S) = \left\lfloor \frac{\text{codim}(S) - 2}{2} \right\rfloor$. This extends the Goresky-MacPherson lower middle perversity [123]

$$\bar{m} = [0, 0, 1, 1, 2, 2, 3, \ldots].$$

122 Recall Definition 8.52
123 Recall from Definition 3.3 that Goresky-MacPherson perversities are function $\{2, 3, \ldots\} \rightarrow \mathbb{Z}$. In this
When we round up, we get the dual perversity $D\bar{m} = \bar{n}$, the upper-middle perversity with $\bar{n}(S) = \left\lceil \frac{\text{codim}(S) - 2}{2} \right\rceil$ that extends the Goresky-MacPherson upper-middle perversity

$$\bar{n} = [0, 1, 1, 2, 2, 3, \ldots].$$

These two perversities are as close to each other as it is possible for two dual perversities to be, assuming we wish to make a consistent choice of which one is larger than the other on the odd codimension strata. This is a useful assumption because it allows us to construct a “comparison map” via the inclusion $I^mS_*(X; G) \to I^nS_*(X; G)$ for any coefficient system; in fact, if $\bar{p}, \bar{q}$ are any perversities with $\bar{p}(S) \leq \bar{q}(S)$ for all $S$, then $I^mS_*(X; G) \subset I^nS_*(X; G)$.

One can then pose the natural question: for what spaces beyond those with only even codimension strata does the inclusion $I^mS_*(X; G) \hookrightarrow I^nS_*(X; G)$ induce isomorphisms on homology? For $\mathbb{Q}$ coefficients, such spaces are candidates to possess signatures, and for other ring coefficients there are other invariants of self-duality that might be exploited.

A natural class of spaces for which the inclusion $I^mS_*(X; \mathbb{Q}) \to I^nS_*(X; \mathbb{Q})$ induces an isomorphism was discovered by Siegel [95] in his thesis. He named these spaces “Witt spaces” because he was able to prove that the $4n$-dimensional ($n > 0$) oriented bordism groups of PL Witt spaces are isomorphic to the Witt group $W(\mathbb{Q})$ via the map that takes the self-dual cup product pairing $I_mH^{2n}(X; \mathbb{Q}) \otimes I_mH^{2n}(X; \mathbb{Q}) \to \mathbb{Q}$ to its class as an element of $W(\mathbb{Q})$.

The defining condition for Witt spaces arises from the desire to have $I^mH_*(X; \mathbb{Q}) \cong I^nH_*(X; \mathbb{Q})$. As we have seen many times, intersection homology is in many ways controlled by what happens on cones, and if $L$ is $n - 1$ dimensional, for $n$ odd, then we have

$$I^mH_i(cL; \mathbb{Q}) \cong \begin{cases} 0, & i \geq n - \left\lceil \frac{n-2}{2} \right\rceil - 1, \\ I^mH_i(L; \mathbb{Q}), & i < n - \left\lceil \frac{n-2}{2} \right\rceil - 1, \end{cases}$$

$$I^nH_i(cL; \mathbb{Q}) \cong \begin{cases} 0, & i \geq n - \left\lceil \frac{n-2}{2} \right\rceil - 1, \\ I^nH_i(L; \mathbb{Q}), & i < n - \left\lceil \frac{n-2}{2} \right\rceil - 1. \end{cases}$$

Now, if we assume (by an induction on depth) that $I^mH_i(L; \mathbb{Q}) \cong I^nH_i(L; \mathbb{Q})$, then the only difference between these two formulas is in dimension

$$n - \left\lceil \frac{n-2}{2} \right\rceil - 1 = n - \frac{n-1}{2} - 1 = \frac{2n - 2 - (n - 1) - 2}{2} = \frac{n - 1}{2},$$

case, where $\bar{m}$ depends only on the codimension, the only greater generality of our definition is to include the possibility codim$(S) = 1$.

124By definition, for a field $F$, the Witt group $W(F)$ is the group generated by isomorphism classes of symmetric pairings on vector spaces, with the group operation being direct sum and with additional relations such that pairings with matrices of the form $\begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$, with $A$ arbitrary and $I$ an identity matrix, are set to 0. See [72] for more details.

125Technically, Siegel worked with the dual intersection pairing on homology, which is equivalent. REF!!
in which \( I^n H_{n+1}(cL; \mathbb{Q}) \cong I^n H_{n+1}(L; \mathbb{Q}) \), but \( I^n H_{n+1}(cL; \mathbb{Q}) = 0 \). So to ensure that \( I^n H_*(cL; \mathbb{Q}) \cong I^n H_*(cL; \mathbb{Q}) \), we also need to have \( I^n H_{n+1}(cL; \mathbb{Q}) = 0 \). This requirement is called the *Witt condition* for \( \mathbb{Q} \).

**Definition 10.1.** Let \( G \) be an abelian group. Then a \( \partial \)-stratified pseudomanifold \( X \) is a *\( G \)-Witt space* if, for any point \( x \in X \) contained in a stratum of odd codimension, \( I^n H_{\dim(L)}(L; G) = 0 \) for any link \( L \) of \( x \).

**Remark 10.2.** Since \( x \) is assumed to be in a stratum of odd codimension \( S \), \( L \) must have even dimension because \( \dim(X) = \dim(L) + \dim(S) + 1 \) and odd codimension means precisely that \( \dim(X) - \dim(S) \) is odd. Also, notice that the condition on links is really a condition on the stratum itself by Corollary 6.28 which extends easily to \( \partial \)-pseudomanifolds and which showed us that any two links of the same stratum have the same intersection homology.

**Remark 10.3.** Since the boundary of a \( \partial \)-stratified pseudomanifold is assumed to have a collar that is stratified homeomorphic to \([0, 1) \times \partial X\), the stratified spaces \( X \) and \( X - \partial X \) have the same links. Therefore, \( X \) will be \( G \)-Witt if and only if \( X - \partial X \) is \( G \)-Witt.

**Remark 10.4.** It is not uncommon for other underlying condition, such as orientability, compactness, empty boundary, or being PL (as in [95]), to be assumed as part of the definition for being \( G \)-Witt. We will not make such assumptions here in order to allow for maximal flexibility, but the reader should pay careful attention to such assumptions when reading the literature. Furthermore, although we have formulated the definition of a \( G \)-Witt space for pseudomanifolds, the definition clearly extends to CS sets.

**Example 10.5.** Suppose \( X \) is a pseudomanifold whose non-empty strata all have even codimension. Then \( X \) is automatically a \( G \)-Witt space for any \( G \). Although this example appears somewhat trivial, all irreducible complex algebraic varieties can be given such stratifications! See [44, Section 1.7].

**Proposition 10.6.** If \( X \) is a \( G \)-Witt space, then the inclusion \( I^n S_*(X; G) \to I^n S_*(X; G) \) induces a homology isomorphism \( I^n H_*(X; G) \to I^n H_*(X; G) \).

**Proof.** For a stratified pseudomanifold \( X \), we can use Theorem 5.3 with \( F_*(U) = I^m H_*(U; G) \), \( G_* = I^n H_*(U; G) \), and \( \Phi \) induced by the inclusion of chain groups. Using stratified homotopy invariance and our above computation for cones on links (modified to use \( G \) coefficients), the conditions of the theorem are nearly immediate. Note that, since \( X \) is a pseudomanifold, the only open subsets of strata homeomorphic to Euclidean space are Euclidean subsets of the top stratum, on which both functors reduce to ordinary homology. We can also apply Lemma 5.5 together with Lemma 6.23 for the ascending chain condition.

If \( X \) is a \( \partial \)-stratified pseudomanifold with non-empty boundary, then we can use that \( X \) and \( X - \partial X \) are stratified homotopy equivalent so that the inclusion map \( I^p H_*(X - \partial X; G) \to I^p H_*(X; G) \) is an isomorphism. The proposition then follows from the empty-boundary case.
via the commutative diagram

\[
\begin{array}{ccc}
I^\bar{m}H_*(X - \partial X; G) & \xrightarrow{\cong} & I^\bar{m}H_*(X; G) \\
\downarrow & & \downarrow \\
I^\bar{n}H_*(X - \partial X; G) & \xrightarrow{\cong} & I^\bar{n}H_*(X; G).
\end{array}
\]

**Corollary 10.7.** If \( R \) is a Dedekind domain and \( X \) is an \( R \)-Witt space, then the inclusion \( I^\bar{m}S_*(X; G) \to I^\bar{n}S_*(X; G) \) induces a cohomology isomorphism \( I^\bar{n}H^*(X; R) \to I^\bar{m}H^*(X; R) \).

**Proof.** This follows from Proposition 10.6, the Universal Coefficient theorem (Theorem 7.4), and the Five Lemma.

**Remark 10.8.** The proof of Proposition 10.6 given in Siegel is more elaborate, utilizing spectral sequences (though, to be fair, it took us a certain amount of work with machinery to set up Theorem 5.3). Siegel also notes that this proposition is proven via the sheaf-theoretic formulation of intersection homology in [43]; that proof is also very straightforward, once again using only the local cone computation together with the sheaf-theoretic machinery developed in [43].

**Example 10.9.** Let \( M \) be a compact \( 2k + 1 \) dimensional \( \partial \)-manifold, and let \( X = M \cup_{\partial M} \bar{c}(\partial M) \) as in Example 4.44. Then the cone vertex \( v \) is the only singularity, and it is of odd codimension. Since \( \partial M \) is a manifold, \( I^\bar{m}H_k(\partial M; G) \cong H_k(\partial M; G) \), and so \( X \) is a \( G \)-Witt space if and only if \( H_k(\partial M; G) = 0 \).

**Example 10.10.** As another easy example, let \( M \) be a compact \( 2k \)-dimensional manifold with \( H_k(M) \) finite but non-zero. Then \( H_k(M; \mathbb{Q}) = 0 \), but for some prime \( p \), \( H_k(M; \mathbb{Z}_p) \neq 0 \). Thus the suspension \( SM \) will be a \( \mathbb{Q} \)-Witt space, but not a \( \mathbb{Z}_p \)-Witt space. Using the universal coefficient theorem, \( SM \) will also be \( \mathbb{Z}_p \)-Witt for any prime \( p \) that does not divide the order of any element of \( H_k(M) \) or \( H_{k-1}(M) \).

These examples demonstrate that the coefficient choice matters in Definition 10.1. We will explore such issues further in Section 10.1.1. For now, we return to more general properties of Witt spaces.

**Remark 10.11.** Since the upper- and lower-middle perversities, and GM perversities in general, depend only on the codimensions of strata, we may write, for example, \( \bar{m}(\ell) \) rather that \( \bar{m}(S) \) when \( \text{codim}(S) = \ell \).

Here is a lemma that will be useful below:

**Lemma 10.12.** Let \( X, X' \) be \( F \)-Witt spaces for a field \( F \). Then \( X \times X' \) is an \( F \)-Witt space.

**Proof.** The product \( X \times X' \) is a \( \partial \)-stratified pseudomanifold by Lemma 2.124. For the Witt condition, we need to examine the links of odd-codimension strata. Since the product of a
stratum of $X$ of codimension $k$ and a stratum of $Y$ of codimension $\ell$ has codimension $k + \ell$ in $X \times X'$, in order for a stratum of $X \times X'$ to have odd codimension, it must be a product $S \times S'$ of strata of $X$ and $X'$ such that one of $S, S'$ has odd codimension and the other has even codimension.

We first assume $S$ and $S'$ are singular strata. Without loss of generality, we assume $S$ has odd codimension. Let $L$ be the link of a point in $S$, and let $L'$ be the link of a point of $S'$. Then the corresponding link in the product will be the join $L \ast L'$; see Section 2.9.

We have computed the intersection homology of joins in our K"unneth theorem computation. This required using the K"unneth theorem (inductively, at the time). Here we wish to use $\bar{m}$ for all perversities, so we need to check that this is consistent with the hypotheses of the K"unneth theorem. We compute in the following table, using codimensions $k$ and $\ell$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\ell$</th>
<th>$\bar{m}(k)$</th>
<th>$\bar{m}(\ell)$</th>
<th>$\bar{m}(k + \ell)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>$\frac{k}{2} - 1$</td>
<td>$\frac{\ell}{2} - 1$</td>
<td>$\frac{k + \ell}{2} - 1$</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
<td>$\frac{k+1}{2} - 2$</td>
<td>$\frac{\ell}{2} - 1$</td>
<td>$\frac{k + \ell + 1}{2} - 2$</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
<td>$\frac{k}{2} - 1$</td>
<td>$\frac{\ell + 1}{2} - 2$</td>
<td>$\frac{k + \ell + 1}{2} - 2$</td>
</tr>
<tr>
<td>odd</td>
<td>odd</td>
<td>$\frac{k+1}{2} - 2$</td>
<td>$\frac{\ell + 1}{2} - 2$</td>
<td>$\frac{k + \ell + 1}{2} - 1$</td>
</tr>
</tbody>
</table>

Thus $\bar{m}(k) + \bar{m}(\ell) \leq \bar{m}(k + \ell) \leq \bar{m}(k) + \bar{m}(\ell) + 2$, and since we are working with field coefficients, this is sufficient for Theorem 6.56.

Now, supposing $\dim(L) = k - 1$, $\dim(L') = \ell - 1$, we need to compute $I^m H_{\frac{k+\ell-1}{2}}(L \ast L'; F)$. Without loss of generality, we may assume that $k$ is odd and $\ell$ is even. We wish to use formula (14) on page 220 which, with the notation of the current situation, holds in dimensions $< k + \ell - \bar{m}(k) - \bar{m}(\ell) - 2$. But we observe that

$$k + \ell - \bar{m}(k) - \bar{m}(\ell) - 3 = k + \ell - \left(\frac{k + 1}{2} - 2\right) - \left(\frac{\ell}{2} - 1\right) - 3$$

$$= k + \ell - \frac{k + \ell + 1}{2}$$

$$= \frac{2k + 2\ell}{2} - \frac{k + \ell + 1}{2}$$

$$= \frac{k + \ell - 1}{2}.$$

So, to compute $I^m H_{\frac{k+\ell-1}{2}}(L \ast L'; F)$, formula (14) on page 220 applies to give us

$$I^m H_{\frac{k+\ell-1}{2}}(L \ast L'; F) \cong \bigoplus_{j+\ell = \frac{k+\ell-1}{2}, \ell < \bar{m}(\ell)-1} I^m H_j(X; F) \otimes_F I^m H_{\ell}(Y; F)$$

But $k - \bar{m}(k) - 1 = k - \left(\frac{k+1}{2} - 2\right) - 1 = \frac{k-1}{2} + 1$, and $\ell - \bar{m}(\ell) - 1 = \ell - \left(\frac{\ell}{2} - 1\right) - 1 = \frac{\ell}{2}$. So we need $j + \ell = \frac{k+\ell-1}{2}$ with $j < \frac{k-1}{2} + 1$ and $\ell < \frac{\ell}{2}$. But there are no such non-trivial terms.

\footnote{Recall that links are not unique, but their intersection homology is by Lemma 6.27, so we may blur this technical point with definite articles.}
as the largest possible sum satisfying the last two conditions is \( \frac{k-1}{2} + \frac{\ell}{2} - 1 = \frac{k+\ell-1}{2} - 1 \). So the sum is trivial, and \( I^m H_{\frac{k+\ell}{2}}(L * L'; F) = 0 \).

If \( S' \) is a regular stratum and \( S \) has odd codimension \( k \), then the link is \( L \) and we need \( I^m H_{\frac{k-1}{2}}(L; F) = 0 \), which follows from \( X \) being \( F \)-Witt. Similarly, if \( S \) is regular and \( S' \) has odd-codimension \( \ell \), then the link is \( L' \) and \( I^m H_{\frac{\ell-1}{2}}(L'; F) = 0 \) because \( X' \) is Witt.

It is interesting to see that the portions of the products over the regular strata are the only places where we need to utilize the assumption that \( X \) and \( X' \) are \( F \)-Witt; we did not need that for the portion of the argument over the products of singular strata. \( \square \)

To end this section, we demonstrate that, suitably interpreted, the condition of being \( G \)-Witt is a property of a space and not of its stratification. First, notice that a space with codimension one strata can never be \( G \)-Witt for any (non-trivial) \( G \) since the link \( L \) of a codimension one stratum of a pseudomanifold is a disjoint union of points, trivially filtered and with formal dimension 0, and so \( I^m H_{\dim(L)}(L; G) \cong H_0(L; G) \neq 0 \). On the other hand, the next proposition says that if \( X \) is a \( \partial \)-stratified pseudomanifold without codimension one strata, then the property of being \( G \)-Witt turns out to depend only on the underlying space and not on the choice of stratification, assuming we rule out stratifications with codimension one strata. The proof is quite analogous to that of Proposition 5.61.

**Proposition 10.13.** If \( X \) and \( X' \) represent two different \( \partial \)-pseudomanifold stratifications, without codimension one strata, of the same underlying space, then \( X \) is a \( G \)-Witt space if and only if \( X' \) is.

**Proof.** The property of being a \( G \)-Witt space is contingent only on the intersection homology of the links of the strata. Since the links of points in the boundary of a \( \partial \)-pseudomanifold are the same as the links of interior points, it suffices to prove the proposition for pseudomanifolds without boundaries. In that setting, we will show that \( X \) is \( G \)-Witt if and only if \( X^* \) is a CS \( G \)-Witt space, where \( X^* \) is \( |X| \) with its intrinsic stratification (see Section 2.8). As \( X \) and \( X' \) have the same intrinsic stratification, the result will follow. Recall that, as \( \check{m} \) is a GM perversity, it depends only on the codimensions of strata, so if \( \operatorname{codim}(S) = \ell \), we can write \( \check{m}(\ell) \) for \( \check{m}(S) \).

First, assume that \( X \) is \( G \)-Witt. Recall that every stratum \( S \) of \( X^* \) is a union of strata of \( X \) of dimension \( \leq \dim(S) \) (see Section 2.8). So let \( S \) be a stratum of \( X^* \) of odd codimension, so that the dimension of its link is even, and let \( x \) be a point of \( X \) contained in a stratum \( T \) of \( X \) with \( T \subset S \) and \( \dim(S) = \dim(T) \); such a stratum \( T \) must exist by the definition of a CS set, in particular the compatibility of the filtration with the local conical structure. Let \( L \) be a link of \( x \) in \( X \) and let \( \mathcal{L} \) be a link of \( x \) in \( X^* \). As \( \dim(S) = \dim(T) \), we have \( \dim(L) = \dim(\mathcal{L}) \), and, using the computation at the beginning of this section, \( I^m H_{\frac{\dim(L)}{2}}(cL; G) \cong I^m H_{\frac{\dim(L)}{2}}(L; G) \) and \( I^m H_{\frac{\dim(L)}{2}}(c, \mathcal{L}; G) \cong I^m H_{\frac{\dim(\mathcal{L})}{2}}(\mathcal{L}; G) \). It follows that if \( \check{N} \) and \( N^* \) are regular neighborhoods of \( x \) in \( X \) and \( X^* \), respectively, we have

\[
I^m H_{\frac{\dim(L)}{2}}(L; G) \cong I^m H_{\frac{\dim(L)}{2}}(N; G) \cong I^m H_{\frac{\dim(L)}{2}}(N^*; G) \cong I^m H_{\frac{\dim(L)}{2}}(\mathcal{L}; G),
\]

using Corollary 5.56 for the middle isomorphism. As we have assumed that \( X \) is \( G \)-Witt, the link \( L \) of \( x \) in \( X \) satisfies the \( G \)-Witt condition \( I^m H_{\frac{\dim(L)}{2}}(L; G) = 0 \), and so \( \mathcal{L} \) also satisfies
the $G$-Witt condition. Since the $G$-Witt condition is satisfied for a link at one point in $S$, it is satisfied at all points in $S$ by Corollary 6.28.

Conversely, suppose $X^*$ is CS $G$-Witt. Let $x \in X$ be a point with distinguished neighborhood $N \cong \mathbb{R}^k \times cL$. Suppose $\dim(L) = \ell$ is even. As observed in the preceding paragraph, we have $I^mH_{\ell/2}(L;G) \cong I^mH_{\ell/2}(N;G)$. Now, let $N^*$ be a distinguished neighborhood of $x$ in $N^*$. By Corollary 5.56, $I^mH_{\ell/2}(N;G) \cong I^mH_{\ell/2}(N^*;G)$. But $N^* \cong \mathbb{R}^m \times cL$ for some link $\mathcal{L}$ and some $\mathbb{R}^m$ with $m \geq k$, since the stratification of $X^*$ is coarser than that of $X$. If $m = k$, then $\dim(\mathcal{L}) = \ell$ as well, and, by the same argument as above, $I^mH_{\ell/2}(N^*;G) \cong I^mH_{\ell/2}(\mathcal{L};G)$, which is 0 by the assumption that $X^*$ is CS $G$-Witt. So suppose $m > k$, which implies that $d = \dim(\mathcal{L}) < \ell$. By stratified homotopy invariance, $I^mH_{\ell/2}(N^*;G) \cong I^mH_{\ell/2}(\ell;G)$, which, by the cone formula, is 0 if $\ell/2 \geq d - \tilde{m}(d + 1)$. To see that this is indeed the case, we use that $\frac{d}{2} = \ell - \tilde{m}(\ell + 1) - 1$, which is easy to verify. We need to show that $\ell - \tilde{m}(\ell + 1) \geq d - \tilde{m}(d + 1) + 1$. Since $d < \ell$, let’s see what happens to the quantity $i - \tilde{m}(i + 1)$ if we start with $i = \ell$ and step down from $\ell$ to $d$ with step size 1. Obviously, the $i$ summand will decrease by one every time, while the term $\tilde{m}(i + 1)$ alternates between decreasing by one and not changing at all. So as we decrease from $i$ to $i - 1$, the expression $i - \tilde{m}(i + 1)$ either decreases by one or stays the same. Let’s see what happens in the first step from $\ell$ to $\ell - 1$. We already know that $\ell - \tilde{m}(\ell + 1) = \frac{\ell}{2} + 1$. Since $\ell$ is even, we have $\tilde{m}(\ell) = \lfloor \frac{\ell}{2}\rfloor = \frac{\ell}{2} - 1$. So $\ell - 1 - \tilde{m}(\ell) = \ell - 1 - (\frac{\ell}{2} - 1) = \frac{\ell}{2}$. So indeed $\ell - \tilde{m}(\ell + 1) \geq \ell - 1 - \tilde{m}(\ell) + 1$, and thus $\ell - \tilde{m}(\ell + 1) \geq i - \tilde{m}(i + 1) + 1$ for all $i < \ell$, including $i = d$. Therefore, $0 = I^mH_{\ell/2}(N^*;G) \cong I^mH_{\ell/2}(N^*;G) \cong I^mH_{\ell/2}(L;G)$. Altogether, we see that in all circumstances, the $G$-Witt condition is satisfied for all links of points in $X$, so $X$ is $G$-Witt. 

10.1.1 Dependence of Witt spaces on coefficient choices

In Example 10.10, we saw how to construct spaces that are $\mathbb{Q}$-Witt but not $\mathbb{Z}_p$-Witt for some $p$. In the remainder of this section, we will explore related issues. The reader eager to get on to our application of Witt spaces to defining signature invariants can safely skip forward to the next section.

In the next example, we show that there are spaces that are $\mathbb{Z}_p$-Witt for some $p$ but not $\mathbb{Q}$-Witt, though the construction is a bit more elaborate.

Example 10.14. To find spaces that are $\mathbb{Z}_p$-Witt but not $\mathbb{Q}$-Witt, we need to take advantage of the failure of the universal coefficient theorem for intersection homology. It will suffice for us to find a $2k$-dimensional stratified pseudomanifold $X$ such that $I^mH_k(X;\mathbb{Z}_p) = 0$ but $I^mH_k(X;\mathbb{Q}) \neq 0$. Then the suspension $SX$ will be a $\mathbb{Z}_p$-Witt space but not a $\mathbb{Q}$-Witt space. To construct $X$, let $M$ be a compact connected oriented $k$-manifold equipped with a $k$-dimensional vector bundle $V$ with Euler number $p$. For example, since the Euler number of the tangent bundle is the Euler characteristic [74 Corollary 11.12], we could use the complex projective space $\mathbb{C}P^{p-1}$ with its tangent bundle. Let $X$ be the Thom space of $V$, which is the one-point compactification of the bundle. Then $X$ is a stratified pseudomanifold with one singular stratum corresponding to the point at infinity. In fact, $X$ can be identified as the disk bundle of $V$ with a cone joined on the boundary, so computations analogous to those
of Example 4.44 apply. Using the homotopy equivalence of $V$ and $M$ and the computation $\hat{m}(2k) = k - 1$, we get

\[ I^mH_i^{GM}(X; G) \cong \begin{cases} H_i(X; G), & i > k, \\ \text{im}(H_i(M; G) \to H_i(X; G)), & i = k, \\ H_i(M; G), & i < k. \end{cases} \]

So the key term is $\text{im}(H_k(M; G) \to H_k(X; G))$. We claim that $H_k(M; G) \cong H_k(X; G) \cong G$ and that the image $\text{im}(H_k(M; G) \to H_k(X; G))$ is isomorphic to $pG$. This claim is proven in the Lemma [10.15] below. Assuming the lemma, we see that if $G = \mathbb{Z}$ and $k > 0$, then $I^mH_i^{GM}(X; G) = 0$, while if $G = \mathbb{Q}$, $I^mH_i^{GM}(X; G) \cong \mathbb{Q}$. This is our desired result.

**Lemma 10.15.** If $M$ is a compact connected oriented $k$-manifold, $k > 0$, and $X$ is the Thom space of a $k$-dimensional oriented vector bundle over $M$ with Euler number $\chi$, then $H_k(M; G) \cong H_k(X; G) \cong G$, and the inclusion map $i : M \to X$ induces a homology homomorphism corresponding (up to sign) to multiplication by $\chi$.

**Proof.** Since $M$ is closed and oriented, $H_k(M) \cong \mathbb{Z}$ and $H_{k-1}(M)$ is torsion free: By Poincaré duality and the universal coefficient theorem $H_{k-1}(M) \cong H^1(M) \cong \text{Hom}(H_1(M), \mathbb{Z}) \oplus \text{Ext}(H_0(M), \mathbb{Z})$, but $\text{Ext}(H_0(M), \mathbb{Z}) = 0$ due to $H_0(M)$ being free and $\text{Hom}(H_1(M), \mathbb{Z})$ is torsion free. So $H_k(M; G) \cong (H_k(M) \otimes G) \oplus (H_{k-1}(M) \ast G) \cong H_k(M) \otimes G \cong G$, by the homology universal coefficient theorem.

Next, let $\infty$ be the point at infinity in the Thom space $X$, let $V_0$ be the vector bundle $V$ with the zero section deleted, and let $D(V)$ and $S(V)$ be respectively the unit disk and unit sphere bundles associated with $V$. Then

\[
H_k(X; G) \cong H_k(X, \infty; G), \quad \text{since } k > 0
\]

\[
\cong H_k(X, X - M; G), \quad \text{homotopy equivalence}
\]

\[
\cong H_k(D(V), D(V) - M; G), \quad \text{excision}
\]

\[
\cong H_k(D(V), S(V); G), \quad \text{homotopy equivalence}
\]

\[
\cong H^k(D(V); G), \quad \text{Poincaré-Lefschetz duality}
\]

\[
\cong H^k(M; G), \quad \text{homotopy equivalence}
\]

\[
\cong \text{Hom}(H_k(M), G) \oplus \text{Ext}(H_{k-1}(M), G), \quad \text{universal coefficient theorem}
\]

\[
\cong \text{Hom}(\mathbb{Z}, G) \cong G,
\]

where the last line follows from our preceding computations.

For the claim regarding the map, consider the map of universal coefficient sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}(H^{k+1}(M), G) & \longrightarrow & H_k(M; G) & \longrightarrow & \text{Hom}(H^k(M); G) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}(H^{k+1}(X), G) & \longrightarrow & H_k(X; G) & \longrightarrow & \text{Hom}(H^k(X); G) & \longrightarrow & 0;
\end{array}
\]
see [74] Theorems 56.1 and 56.2] and note that all the homology groups of $M$ are finitely generated since $M$ is a compact manifold, and all the homology groups of $X$ are finitely generated since $\tilde{H}_*(X) \cong \tilde{H}_*(D(V), S(V))$ and $D(V)$ and $S(V)$ are compact $\partial$-manifold. $H^{k+1}(M)$ vanishes, as $M$ is $k$-dimensional, and by an argument similar to the above computation, $H^{k+1}(X) \cong H^{k+1}(D(V), S(V)) \cong H_{k-1}(D(V)) \cong H_{k-1}(M)$, which is torsion-free. So the two left hand terms vanish and we see that the map $H_k(M; G) \to H_k(X; G)$ is isomorphic to the dual to the restriction map $i^*: H^k(X) \to H^k(M)$.

Now $H^k(X) \cong H^k(D(V), S(V))$ by homotopy equivalence, excision, and the long exact sequence of the pair, and by homotopy equivalence and the Thom isomorphism theorem (see [74, Theorem 9.1]), there is an isomorphism $\tilde{Z} \cong H^0(M) \cong H^0(D(V)) \xrightarrow{\cup u} H^k(D(V), S(V))$. Since $1 \in H^0(D(V))$ is the generator, the generator of $H^k(D(V), S(V))$ is the Thom class $u$. The restriction of the Thom class to $H^k(M)$ is precisely the Euler class $e \in H^k(M)$ by the definition on page 98 of [74]. Since $H^k(M) \cong \text{Hom}(H_k(M), \tilde{Z}) \cong \tilde{Z}$, we can determine the class $e$ by computing $e([M])$, where $[M]$ is the fundamental class of $M$. But $e([M])$ is precisely the Euler number $\chi$ by definition (if $V$ is the tangent bundle of $M$, this is the Euler characteristic by [74, Corollary 11.2]). So we conclude that the image of $i^*: H^k(X) \to H^k(M)$ is $\chi \tilde{Z}$. The lemma now follows from the universal coefficient diagram.

Generalizing these examples, the following theorem is proven in [32].

**Theorem 10.16.** Let $K$ denote a field, and let $\mathbb{Z}_p$ denote the field of $p$ elements.

1. If $K$ has characteristic $p > 0$, then $X$ is $K$-Witt if and only if $X$ is $\mathbb{Z}_p$-Witt; if $K$ has characteristic 0, then $X$ is $K$-Witt if and only if $X$ is $\mathbb{Q}$-Witt.

2. If $n > 4$ and $P$ is a finite set of primes, then there is a compact orientable $n$-dimensional stratified pseudomanifold that is $\mathbb{Z}_p$-Witt for any $p \in P$ but that is not $\mathbb{Q}$-Witt and not $\mathbb{Z}_p$-Witt for $p \notin P$.

3. If $n > 4$ and $P$ is a finite set of primes, then there are $\mathbb{Q}$-Witt spaces that are not $\mathbb{Z}_p$-Witt for any $p \in P$ and are $\mathbb{Z}_p$-Witt for $p \notin P$.

4. If $X$ is a 3- or 4-dimensional $\mathbb{Z}_p$-Witt space, then $X$ is a $\mathbb{Q}$-Witt space.

5. If $X$ is a 3- or 4-dimensional $\mathbb{Q}$-Witt space, then $X$ is a $\mathbb{Z}_p$-Witt space for any $p \neq 2$. If $X$ is also $\mathbb{Q}$-orientable, then it is also a $\mathbb{Z}_2$-Witt space. However, there are non-orientable 3- and 4-dimensional $\mathbb{Q}$-Witt spaces that are not $\mathbb{Z}_2$-Witt spaces.

6. All 0-, 1-, and 2-dimensional pseudomanifolds are $K$-Witt for all $K$.

We refer the reader to [32] for the construction of these examples, many of which are in the same vein as the Thom space example above. We will, however, provide a proof of the first fact of this theorem:

**Proposition 10.17.** Let $X$ be a $\partial$-stratified pseudomanifold and $K$ a field of characteristic $p$, possibly with $p = 0$. Then $X$ is $K$-Witt if and only if $X$ is $\mathbb{Z}_p$-Witt (taking $\mathbb{Z}_0 = \mathbb{Q}$).
Proof. Let $L$ be an even-dimensional link of $X$. As we work with field coefficients, every space is locally torsion free, so we can apply Theorem 6.30 to compute that $I^mH_i(L;K) \cong H_i(I^mS^*(X;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} K)$. But now the algebraic Universal Coefficient Theorem [105 Theorem 3.6.1] shows that the latter is isomorphic to $I^mH_i(X;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} K$; to verify the hypotheses, we use that each $I^mS^*(X;R)$ is a vector space over $\mathbb{Z}_p$. It follows immediately that if $I^mH_{\dim(L)}(L;\mathbb{Z}_p) = 0$, then so does $I^mH_{\dim(L)}(L;K)$. But also if $I^mH_{\dim(L)}(X;\mathbb{Z}_p)$ is not 0, it is a $\mathbb{Z}_p$-vector space of some dimension $m > 0$, and so $I^mH_{\dim(L)}(X;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} K$ is a $K$-vector space of dimension $m$. Thus if $I^mH_{\dim(L)}(X;K)$ vanishes, so does $I^mH_{\dim(L)}(X;\mathbb{Z}_p)$. \hfill \Box

### 10.2 Self pairings and IP spaces

As we have seen in Corollary [10.7], for an $R$-Witt space $X$ with $R$ a Dedekind domain, the natural map $I^\ell H^*(X;R) \rightarrow I^\ell H^*(X;R)$ is an isomorphism. If, furthermore, we take $R$ to be the field $F$, then $X$ is automatically locally $(\bar{p},F)$-torsion free for any $\bar{p}$. Thus, if the $F$-Witt space $X$ is also a compact $F$-oriented stratified pseudomanifold of dimension $4k$, Poincaré duality also guarantees that there is a nonsingular dual pairing $I^\ell H^{2k}(X;F) \otimes I^\ell H^{2k}(X;F) \rightarrow F$. Putting this together with the isomorphism guaranteed by $X$ being $F$-Witt, we obtain a pairing $I^\ell H^{2k}(X;F) \otimes I^\ell H^{2k}(X;F) \rightarrow F$, which turns out to be symmetric by the following proposition. Thus we have the ingredients to define a signature, which we will provide in detail in the next section.

**Proposition 10.18.** Suppose that the compact stratified pseudomanifold $X$ is $2\ell$-dimensional, $F$-oriented, and $F$-Witt for some field $F$. Let $i : X \rightarrow X$ be the identity map, thought of as an $\bar{m}$-stratified map. Then the composition

$$I^\ell H^\ell(X;F) \otimes I^\ell H^\ell(X;F) \xrightarrow{id \otimes i^*} I^\ell H^{2k}(X;F) \otimes I^\ell H^{2k}(X;F) \xrightarrow{\text{P}} I^\ell H_0(X;F) \xrightarrow{a} F$$

is a nonsingular $(-1)^\ell$-symmetric pairing.

If we let $P$ denote the pairing, then $(-1)^\ell$-symmetric means that $P(\alpha, \beta) = (-1)^\ell P(\beta, \alpha)$.

**Proof.** By Corollary [10.7] the first map is an isomorphism, and the composition of the remaining maps is a nonsingular pairing by Corollary [8.57] noting that all pseudomanifolds are locally torsion free over a field. It follows that the given pairing is nonsingular. For the symmetry, we consider the diagram

$$\begin{array}{ccc}
I^\ell H^\ell(X;F) \otimes I^\ell H^\ell(X;F) & \xrightarrow{id \otimes i^*} & I^\ell H^{2k}(X;F) \otimes I^\ell H^{2k}(X;F) \\
\downarrow & & \downarrow \\
I^\ell H^{2k}(X;F).
\end{array}$$

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According to Definition 7.23, to show that the diagonal cup product is well defined, we need to show that $\tilde{D} \geq D\tilde{n} + D\tilde{m} = \tilde{m} + \tilde{m} = 2\tilde{m}$. Equivalent, we show that $\tilde{0} \leq D(2\tilde{m})$. Let $S \subset X$ be a singular stratum; we compute:

$$(D(2\tilde{m}))(S) = \text{codim}(S) - 2 - (2\tilde{m})(S)$$

$$= \text{codim}(S) - 2 - 2 \left\{ \begin{array}{ll}
\frac{\text{codim}(S) - 2}{2}, & \text{codim}(S) \equiv 0 \mod 2, \\
\frac{\text{codim}(S) - 3}{2}, & \text{codim}(S) \equiv 1 \mod 2,
\end{array} \right.$$  

$$= \begin{cases}
0, & \text{codim}(S) \equiv 0 \mod 2, \\
1, & \text{codim}(S) \equiv 1 \mod 2,
\end{cases} \geq 0.$$  

Therefore, the diagonal cup product is well defined.

The diagram commutes by Lemma 7.33, as the identity map $X \rightarrow X$ is $(\tilde{p}, \tilde{q})$-stratified with respect to any pair of perversities with $\tilde{p} \leq \tilde{q}$. The appropriate symmetry properties now arise from the diagonal cup product by the commutativity of the cup product, Corollary 7.40.

When Siegel first introduced Witt spaces in [95], he was interested exclusively in rational coefficients as these (or, equivalently for these purposes, $\mathbb{R}$) are the coefficients one typically uses to define signatures. At the time, intersection homology Poincaré duality was only known for field coefficients, and so it was natural to consider only $\mathbb{Q}$. However, the work of Goresky and Siegel in [46] made it apparent that duality could be extended to more general rings (at least to the integers) by introducing local torsion free conditions. This led to Pardon’s formulation in [81] of the definition for what he called $IP$ spaces, short for intersection homology Poincaré spaces. In our language, this definition amounts to the following:

**Definition 10.19.** Let $R$ be a Dedekind ring. Then an $R$-oriented $\partial$-stratified pseudomanifold $X$ is an IP space (with respect to $R$) if

1. $X$ is $R$-Witt, and
2. $X$ is locally $(\tilde{m}, R)$-torsion free (or, equivalently by Corollary 8.36, locally $(\tilde{n}, R)$-torsion free).

If we need to specify the ring, we will refer to an $R$-IP space. If we simply say IP space with no ring specified by context, then we assume $R = \mathbb{Z}$.

**Remark 10.20.** As mentioned in Remark 10.4, we have tried to keep the definition of IP spaces as general as possible, though other authors often include other assumptions, such as orientability, compactness, empty boundary, or being PL.
Remark 10.21. For $X$ to be locally $(\bar{m}, R)$-torsion free, we need each $I^\bar{m}H_{\dim(L)-\bar{m}(S)-1}(L; R)$ to be flat for each singular stratum $S$ with link $L$. Let us compute these dimensions more explicitly, recalling that $\bar{m}(S) = \left\lfloor \frac{\text{codim}(S) - 2}{2} \right\rfloor$ for each singular stratum $S$. Using that $\dim(L) + 1 = \text{codim}(S)$, when $\text{codim}(S) = 2k$, we have

$$\dim(L) - \bar{m}(S) - 1 = (2k - 1) - \left\lfloor \frac{2k - 2}{2} \right\rfloor - 1 = (2k - 1) - (k - 1) - 1 = k - 1 = \frac{\dim(L) - 1}{2}.$$ 

When $\text{codim}(S) = 2k + 1$, we have

$$\dim(L) - \bar{m}(S) - 1 = 2k - \left\lfloor \frac{2k - 1}{2} \right\rfloor - 1 = 2k - (k - 1) - 1 = k = \frac{\dim(L)}{2}.$$ 

In the latter case, we know that $I^pH_{\dim(L)}(L; R)$ in fact vanishes if $X$ is $R$-Witt, so there is some redundancy in Definition 10.19. Some authors therefore only state the torsion-free condition for IP spaces in terms of the links of the even codimension strata. Our current formulation does, however, have the advantage of clearly enunciating the two nice things that the definition is doing: guaranteeing duality with the torsion-free condition and guaranteeing $I^mH_\ast = I^nH_\ast$ with the Witt condition.

Remark 10.22. By definition, every $R$-IP space is also $R$-Witt. Conversely, if $F$ is a field, then any $F$-Witt space is automatically an $F$-IP space, as all spaces are automatically locally $(\bar{p}, F)$-torsion free for all $\bar{p}$. Thus the terms “Witt space” and “IP space” as defined here are identical when working over field coefficients. Due to the historical development described above, it remains common in the literature to utilize the expression “Witt space” when working with field coefficients and to reserve the expression “IP spaces” for work over principal ideal domains; our treatment here is likely the first extension to include also Dedekind domains.

The point of IP spaces is that, like $F$-Witt spaces for a field $F$, an IP space (over $R$) satisfies both Poincaré duality and $I_nH^\ast(X; R) \cong I_mH^\ast(X; R)$. Thus, one obtains for compact $R$-orientable IP spaces without boundary both a cup product self-pairing (now over $R$, which might provide more delicate information) and a torsion self-pairing. In particular, we have the following version of Proposition 10.18.

**Proposition 10.23.** Suppose that $R$ is a Dedekind domain and that the compact stratified pseudomanifold $X$ is $2\ell$-dimensional, $R$-oriented, and an $R$-IP space. Let $i : X \to X$ be the identity map, thought of as an $(\bar{m}, \bar{n})$-stratified map. Then the composition

$$F(I_nH^\ell(X; R)) \otimes F(I_nH^\ell(X; R)) \xrightarrow{i \otimes i^*} F(I_nH^\ell(X; R)) \otimes F(I_mH^\ell(X; R)) \xrightarrow{\cong} I_0H^{2\ell}(X; R) \xrightarrow{\partial} I^\ell H_0(X; R) \xrightarrow{\alpha} R$$

is a nonsingular $(-1)^\ell$-symmetric pairing. Similarly, if $\dim(X) = 2\ell - 1$, then we have a $(-1)^\ell$-symmetric pairing $T(I_nH^\ell(X; R)) \otimes T(I_nH^\ell(X; R)) \to Q(R)/R$ that takes $\alpha \otimes \beta$ to $L_{\bar{n}, \bar{m}}(\alpha, i^*(\beta))$. 

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Proof. The proof for the cup product pairing is analogous to that for Proposition 10.18.

For the linking pairing, the composition
\[ T(I_\delta H_*(X; R)) \otimes T(I_\delta H_*(X; R)) \xrightarrow{\text{id} \otimes \iota^*} T(I_\delta H_*(X; R)) \otimes T(I_\delta H_*(X; R)) \xrightarrow{L_{\bar{m}, \bar{n}}} Q(R)/R \]
is the composition of an isomorphism and a nonsingular pairing, so it is nonsingular. For the symmetry, we need to show that \( L_{\bar{m}, \bar{n}}(\bar{\beta}, \iota^*(\alpha)) = (-1)^{\ell} L_{\bar{m}, \bar{n}}(\bar{\alpha}, \iota^*(\beta)) \), but, by Corollary 8.60,
\[ L_{\bar{m}, \bar{n}}(\bar{\beta}, \iota^*(\alpha)) = (-1)^{1+(2^\ell-1)+(2^\ell-1)} L'_{m,n}(\iota^*(\alpha), \beta) = (-1)^{\ell} L'_{m,n}(\iota^*(\alpha), \beta). \]
So it suffices to show \( L_{\bar{m}, \bar{n}}(\alpha, \iota^*(\beta)) = L'_{\bar{m}, \bar{n}}(\iota^*(\alpha), \beta) \).

Suppose \( db = t\beta \in I_\delta S^\ell(X; R) \). Then \( dt^*(b) = ti^*(\beta) \in I_\delta S^\ell(X; R) \), and, by Corollary 8.60, we have
\[ L_{\bar{m}, \bar{n}}(\beta, \iota^*(\alpha)) = (-1)^n \frac{a((\alpha \sim \iota^*(b)) \sim \Gamma)}{t} \]
and
\[ L'_{m,n}(\iota^*(\alpha), \beta) = (-1)^n \frac{b(\iota^*(\alpha) \sim b \sim \Gamma)}{t}. \]

From here, we can proceed as in the proof of Proposition 8.72 by showing that each of these expressions is equal to \( \frac{a((\alpha \sim b) \sim \Gamma)}{t} \) in \( Q(R)/R \). In fact, by the arguments of Proposition 10.18, this cup product with image in \( I_\delta S^{2\ell-1}(X; R) \) is well defined as far as the perversities are concerned, though of course, working at the chain level, there is ambiguity in our choice of IAW map up to chain homotopy. But now the argument that our two expressions both equal \( \frac{a((\alpha \sim b) \sim \Gamma)}{t} \) in \( Q(R)/R \) is completely analogous to the argument at the end of the proof of Proposition 8.72, using chain homotopies that arise when considering naturality of the cup product at the chain level.

As part of the definition, an \( R \)-IP space is also an \( R \)-Witt space. To finish this section, we show that \( R \)-IP spaces are also Witt with respect to the fraction field of \( R \), among other fields, though the converse is not necessarily true.

Lemma 10.24. Let \( R \) be a Dedekind domain and \( K \) a field that is also a flat \( R \)-module. Then if \( X \) is an \( R \)-IP space, it is also a \( K \)-Witt space. In particular, every \( R \)-IP space is a \( Q(R) \)-Witt space, where \( Q(R) \) is the field of fractions of \( R \), and so every \( \mathbb{Z} \)-IP space is a \( \mathbb{Q} \)-Witt space.

Proof. By definition, if \( X \) is an \( R \)-IP space then \( X \) is \( R \)-Witt and locally \( (\bar{m}, R) \)-torsion free. Let \( L \) be an even-dimensional link of \( X \). As every link of \( L \) is also a link of \( X \) by Remark 2.55, \( L \) is itself locally \( (\bar{m}, R) \)-torsion free, and we can apply Theorem 6.30 to compute that \( I^m H_i(L; K) \cong H_i(I^m S_*(X; R) \otimes_R K) \). But now the algebraic Universal Coefficient Theorem 105 Theorem 3.6.1 shows that the latter is isomorphic to \( I^m H_i(X; R) \otimes_R K \); to verify the hypotheses, we use that each \( I^m S_*(X; R) \) is projective over \( R \) (and so each submodule of \( I^m S_*(X; R) \) is projective over \( R \), as \( R \) is Dedekind). We also use that the torsion product \( I^m H_{i-1}(X; R) \otimes_R K = 0 \) as we have assumed that \( K \) is flat over \( R \). It follows now that \( I^m H_{\sim \iota(m)(L; K)} = I^m H_{\sim \iota(m)(L; R) \otimes_R K} = 0 \), as \( I^m H_{\sim \iota(m)(L; R)} \) vanishes by the assumption that \( X \) is \( R \)-Witt. This shows that \( X \) is \( K \)-Witt.

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For the second statement of the theorem, we utilize that any localization of \( R \) is a flat \( R \)-module [59, Theorem 4.80].

The following example provides a \( \mathbb{Q} \)-Witt space that is not \( \mathbb{Z} \)-IP.

Example 10.25. Let \( X = X^4 \) be the suspension of \( \mathbb{R}P^3 \) with the natural stratification as \( X^0 \subset X^4 \) with \( X^0 \) equal to the set of suspension points. As \( \mathbb{R}P^3 \) is \( \mathbb{Z} \)-orientable, so is \( X \). The only singular strata have even codimension, so \( X \) is a Witt space for any coefficient system. However, the links of the singular points, which are homeomorphic to \( \mathbb{R}P^3 \), are 3-dimensional, and so, using Remark 10.21, the groups we need to check for the torsion-free condition are \( I_{n}^1H_{\frac{2}{2}}(\mathbb{R}P^3) \cong H_1(\mathbb{R}P^3) \). But this group is \( \mathbb{Z}_2 \), so \( X \) cannot be a \( \mathbb{Z} \)-IP space.

10.3 Signatures

Since \( I_{n}^mH_*(X; \mathbb{Q}) \cong I_{n}^mH_*(X; \mathbb{Q}) \) for \( \mathbb{Q} \)-Witt spaces (which include \( \mathbb{Z} \)-IP spaces by Lemma 10.24), it is possible to define signatures. We review the algebraic background regarding signature invariants in the appendix to this section. Here we recall just the basic definition:

Definition 10.26. If \( M \) is a symmetric matrix of rational numbers, then the signature \( \sigma(M) \) is defined to be

\[
\sigma(M) = \#\{\text{positive eigenvalues of } M\} - \#\{\text{negative eigenvalues of } M\}.
\]

Now, let \( (V, (.,.)) \) be a rational vector space together with a symmetric bilinear pairing \( (.,.): V \times V \to \mathbb{Q} \), and let \( M \) be the matrix of this pairing with respect to a given basis. Then the signature of the pairing \( \sigma(V, (.,.)) \) is defined to be \( \sigma(M) \). As changing the basis will have the effect of changing the pairing matrix from \( M \) to a congruent matrix \( MQ \), the signature \( \sigma(V, (.,.)) \) is independent of the choice of basis by Sylvester’s law of inertia; see [102, 54, Theorem 6.11.1], or Lemma 10.34 in the appendix to this section.

Definition 10.27. Suppose \( X \) is a closed (compact without boundary) oriented \( \mathbb{Q} \)-Witt space of dimension \( 4k \). Then the Witt signature \( \sigma(X) \) is defined to be the signature of the symmetric pairing

\[
I_{n}^mH_{2k}(X; \mathbb{Q}) \otimes I_{n}^mH_{2k}(X; \mathbb{Q}) \xrightarrow{1 \otimes i^*} I_{n}^mH_{2k}(X; \mathbb{Q}) \otimes I_{n}^mH_{2k}(X; \mathbb{Q}) \xrightarrow{\sim} I_{n}^mH_{4k}(X; \mathbb{Q}) \xrightarrow{D} I^0H_0(X; \mathbb{Q}) \xrightarrow{a} \mathbb{Q},
\]

where \( a \) is the augmentation map and \( i : I_{n}^mH_{2k}(X; \mathbb{Q}) \to I_{n}^mH_{2k}(X; \mathbb{Q}) \) is the isomorphism induced by inclusion of chains.

If \( X \) has dimension \( \neq 0 \mod 4 \), we set \( \sigma(X) = 0 \).

Remark 10.28. If \( M \) is a manifold, then all intersection homology/cohomology groups in the definition reduce to ordinary homology/cohomology, and we recover the classical manifold signature. Hence the notation is unambiguous in this case. It is also standard in manifold theory to define the signature to be 0 when the dimension is not a multiple of 4; this is primarily a convenience for the formulas below.

\[\text{The symmetry assures that all eigenvalues will be real by elementary linear algebra.}\]
The Witt signature possesses the properties one would expect of the signature from manifold theory:

**Theorem 10.29.** If $X, X'$ are closed oriented $\mathbb{Q}$-Witt spaces and $\amalg$ denotes disjoint union, then

1. If $-X$ is the same stratified space as $X$ but with the opposite orientation, $\sigma(-X) = -\sigma(X)$,
2. $\sigma(X \amalg X') = \sigma(X) + \sigma(X')$,
3. $\sigma(X \times X') = \sigma(X)\sigma(X')$,
4. if $X$ is the boundary of a compact oriented $\mathbb{Q}$-Witt space, then $\sigma(X) = 0$.

The proof of this theorem is essentially identical to that for manifolds, and in fact, as manifolds are $\mathbb{Q}$-Witt spaces, our proof will reduce to the classical case. For completeness, we will provide all the details of this beautiful theorem.

**Proof of Theorem 10.29.1.** Let $M$ be the pairing matrix for the pairing on $X$ induced by the cup product followed by the cap product with the fundamental class and augmentation. Clearly $\tilde{I}_n^{\mathbb{Q}}(X) \cong \tilde{I}_n^{\mathbb{Q}}(-X)$, so choosing the same basis for $-X$, the only difference in computing the pairing is reversing the sign of the fundamental class. So the pairing matrix for $-X$ is $-M$. The eigenvalues of $-M$ are the negatives of the eigenvalues of $M$, so $\sigma(-M) = -\sigma(M)$ and the result follows.

**Proof of Theorem 10.29.2.** The proof of (2) is straightforward: since the cup product between an element of $I_n^{\mathbb{Q}}(X; \mathbb{Q})$ and an element of $I_n^{\mathbb{Q}}(X'; \mathbb{Q})$ must be trivial, the pairing matrix must have block form $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and so $\sigma(M) = \sigma(A) + \sigma(B)$, with $A$ and $B$ corresponding to the pairing matrices on $X$ and $X'$ respectively. See Lemma 10.40 in the appendix.

**Proof of Theorem 10.29.3.** We first recall that the signature of a space that is not of dimension $4k$ is $0$ by definition. So if $\dim(X \times X')$ is odd, then so must be the dimension of one of $X$ or $X'$ and the statement is true trivially. If $\dim(X \times X') \equiv 2 \mod 4$, then it can’t be that both $\dim(X) \equiv 0 \mod 4$ and $\dim(X') \equiv 0 \mod 4$, so again the statement holds trivially. So the only nontrivial situation is when $\dim(X \times X') \equiv 0 \mod 4$. Here we must look at the pairing.

We will use the Künneth theorem to obtain $I_\delta H^*(X \times X'; \mathbb{Q}) \cong I_\delta H^*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} I_\delta H^*(X'; \mathbb{Q})$. By Theorem 6.56, this requires verifying that if $S, T$ are respective singular strata of $X, X'$, then $\tilde{n}(S) + \tilde{n}(T) \leq \tilde{n}(S \times T) \leq \tilde{n}(S) + \tilde{n}(T) + 2$ (the condition when one stratum is regular is trivial). But for a codimension $k$, $\tilde{n}(k) = \left\lfloor \frac{k-2}{2} \right\rfloor$. So suppose $\text{codim}_X(S) = k$, 441
codim\(_{X'}(T) = \ell\); we have the following table:

<table>
<thead>
<tr>
<th>(k)</th>
<th>(\ell)</th>
<th>(\bar{n}(k))</th>
<th>(\bar{n}(\ell))</th>
<th>(\bar{n}(k + \ell))</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>(\frac{k}{2} - 1)</td>
<td>(\frac{\ell}{2} - 1)</td>
<td>(\frac{k + \ell}{2} - 1)</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
<td>(\frac{k + 1}{2} - 1)</td>
<td>(\frac{\ell}{2} - 1)</td>
<td>(\frac{k + \ell + 1}{2} - 1)</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
<td>(\frac{k}{2} - 1)</td>
<td>(\frac{\ell + 1}{2} - 1)</td>
<td>(\frac{k + \ell}{2} - 1)</td>
</tr>
<tr>
<td>odd</td>
<td>odd</td>
<td>(\frac{k + 1}{2} - 1)</td>
<td>(\frac{\ell + 1}{2} - 1)</td>
<td>(\frac{k + \ell}{2} - 1)</td>
</tr>
</tbody>
</table>

Notice that in each case \(\bar{n}(k) + \bar{n}(\ell) \leq \bar{n}(k + \ell) \leq \bar{n}(k) + \bar{n}(\ell) + 1\), and so the Künneth theorem applies. Therefore, each group \(I_nH^*(X \times X'; \mathbb{Q})\) is generated by elements of the form \(\alpha \times \beta\) with \(\alpha \in I_nH^*(X; \mathbb{Q})\) and \(\beta \in I_nH^*(X'; \mathbb{Q})\).

More specifically, we assume \(\dim(X \times X') = 4K\), and we are interested in \(I_nH^{2K}(X \times X'; \mathbb{Q}) \cong \bigoplus_{i+j=2K} I_nH^i(X; \mathbb{Q}) \otimes \bigoplus I_nH^j(X'; \mathbb{Q})\). An important fact to observe is that many of the cup products of elements in \(I_nH^{2K}(X \times X'; \mathbb{Q})\) are automatically 0. In fact, suppose \(\alpha_1 \times \beta_1, \alpha_2 \times \beta_2\) are two elements of \(I_nH^{2K}(X \times X'; \mathbb{Q})\) corresponding to \(\alpha_1 \otimes \beta_1 \in I_nH^{i_1}(X; \mathbb{Q}) \otimes \bigoplus I_nH^{j_1}(X'; \mathbb{Q})\) and \(\alpha_2 \otimes \beta_2 \in I_nH^{l_2}(X; \mathbb{Q}) \otimes \bigoplus I_nH^{l_2}(X'; \mathbb{Q})\). Then by the interchange rule for cross products and cup products, \((\alpha_1 \times \beta_1) \circ (\alpha_2 \times \beta_2) = (-1)^{i_1l_2}(\alpha_1 \circ \alpha_2) \times (\beta_1 \circ \beta_2)\). But this will be 0 if either \(i_1 + i_2 > \dim(X) = n\) or \(j_1 + j_2 > \dim(X') = n'\). But since we must have \(i_1 + j_1 + i_2 + j_2 = n + n' = 4K\), we see that we can only have non-zero cup products if in fact \(i_1 + i_2 = n\) and \(j_1 + j_2 = n'\). Thus for each summand \(I_nH^i(X; \mathbb{Q}) \otimes I_nH^j(X'; \mathbb{Q})\), all corresponding cup products are zero except with terms from \(I_nH^{n-i}(X; \mathbb{Q}) \otimes I_nH^{n-j}(X'; \mathbb{Q})\), and, therefore, the pairing matrix on \(X \times X'\) has a block sum decomposition where the blocks corresponding to the subgroups \(I_nH^i(X; \mathbb{Q}) \otimes I_nH^j(X'; \mathbb{Q}) \oplus I_nH^{n-i}(X; \mathbb{Q}) \otimes I_nH^{n-j}(X'; \mathbb{Q})\), except if \(i = n/2\) and \(j = n'/2\), in which case \(I_nH^{n/2}(X; \mathbb{Q}) \otimes I_nH^{n/2}(X'; \mathbb{Q})\) pairs with itself. The signature of \(X \times X'\) will be the sum of the signatures on these blocks.

First, let’s let \(W_{i,j} = I_nH^i(X; \mathbb{Q}) \otimes \bigoplus I_nH^j(X'; \mathbb{Q})\), and let’s consider the pairings on \(W_{i,j} \oplus W_{n-i,n'-j}\), where it is not the case that both \(i = n/2\) and \(j = n'/2\). By the separate dualities on the \(\mathbb{Q}\)-Witt spaces \(X\) and \(X'\), \(\dim(I_nH^i(X; \mathbb{Q})) = \dim(I_nH^{n-i}(X; \mathbb{Q}))\) and \(\dim(I_nH^j(X'; \mathbb{Q})) = \dim(I_nH^{n-j}(X'; \mathbb{Q}))\). So \(\dim(W_{i,j}) = \dim(W_{n-i,n'-j})\). But since we are assuming that we do not have both \(i = n/2\) and \(j = n'/2\), one of the following must be true:

- \(i + i > n\),
- \((n - i) + (n - i) > n\),
- \(j + j > n'\),
- \((n' - j) + (n' - j) > n'\).

Without loss of generality, let’s assume that it’s the first situation that holds. But then cup product \(I_nH^i(X; \mathbb{Q}) \otimes I_nH^j(X; \mathbb{Q}) \rightarrow I_nH^{2i}(X; \mathbb{Q})\) must be trivial, so the cup product on \(X \times X'\) restricted to \(W_{i,j}\) must be trivial. Since \(W_{i,j}\) and \(W_{n-i,n'-j}\) have equal dimensions, it follows from Lemma \[10.37\] that the signature of the pairing on \(W_i \oplus W_{n-i}\) is 0.
So the only possibly non-zero signature in our block decomposition of the cup product on \( I_n H^{2K}(X \times X'; \mathbb{Q}) \) comes from the self pairing on \( W_{n/2,n'/2} \). This can only happen if both \( n \) and \( n' \) are even, so if one of \( n \) or \( n' \) is odd, we must have \( \sigma(X \times X') = 0 \), which then agrees with \( \sigma(X)\sigma(X') \). So now suppose that \( n, n' \) are both even. In fact, since \( n + n' = 4K \), a quick mod 4 computation shows that either \( n \equiv n' \equiv 0 \) mod 4 or \( n \equiv n' \equiv 2 \) mod 4. In the latter case, \( n/2 \) and \( n'/2 \) are both odd. But that implies that the cup product pairing on \( I_n H^{n/2}(X; \mathbb{Q}) \) is antisymmetric. So by Lemma \[10.42\] there is a subspace of \( I_n H^{n/2}(X; \mathbb{Q}) \) of half its dimension on which the cup product pairing is trivial. Let \( \{a_1, \ldots, a_n\} \) be a basis for this subspace. But then if \( \{c_j\} \) is a basis for \( I_n H^{n'/2}(X'; \mathbb{Q}) \), the collection \( \{a_i \otimes c_j\} \) is a basis for a subspace of half the dimension of \( I_n H^{n/2}(X; \mathbb{Q}) \otimes \mathbb{Q} I_n H^{n'/2}(X'; \mathbb{Q}) \). But for the corresponding \( a_i \times c_j \), we then have \( (a_i \times c_j)^{-} - (a_k \times c_k) = \pm (a_i - a_k) \times (c_j - c_k) = 0 \). So the pairing is trivial on this subspace, and the signature of \( X \times X' \) is thus 0, which equals \( \sigma(X)\sigma(X') \), which is a product of 0s.

The last remaining case is that for which \( \dim(X) \equiv \dim(X') \equiv 0 \mod 4 \), and the signature of \( X \times X' \) reduces to that of the cup product pairing on \( I_n H^{n/2}(X; \mathbb{Q}) \otimes \mathbb{Q} I_n H^{n'/2}(X'; \mathbb{Q}) \). In this case, the separate pairings, say \((\cdot, \cdot)_X \) and \((\cdot, \cdot)_{X'} \) on \( I_n H^{n/2}(X; \mathbb{Q}) \) and \( I_n H^{n'/2}(X'; \mathbb{Q}) \) are symmetric and non-degenerate, so by Lemma \[10.35\] we can find respective orthogonal bases \( \{a_1, \ldots, a_r\} \) and \( \{b_1, \ldots, b_{r'}\} \) such that \( (a_i, a_i)_X > 0 \) for \( i \leq r \), \( (b_i, b_i)_{X'} > 0 \) for \( i \leq r' \), \( (a_i, a_i)_X < 0 \) for \( i > r \), and \( (b_i, b_i)_{X'} < 0 \) for \( i > r' \). We then observe that the collection \( \{a_i \otimes b_i\} \) is a basis for \( I_n H^{n/2}(X; \mathbb{Q}) \otimes \mathbb{Q} I_n H^{n'/2}(X'; \mathbb{Q}) \) that is orthogonal with respect to the pairing on this space. Hence the corresponding pairing matrix \( M \) is diagonal, and we compute the signature by counting the positive and negative elements on the diagonal. If we put the basis in the dictionary order, then we can decompose \( M \) into blocks corresponding to the subspaces obtained by fixing and \( a_i \) and considering the span of basis elements \( \{a_i \otimes b_1, \ldots, a_i \otimes b_{r'}\} \). If \( (a_i, a_i)_X = m_i \), then this matrix has the form \( m_i B \), where \( B \) is the pairing matrix for \( X' \) in the basis \( \{b_i\} \). So if \( m_i > 0 \), the signature of this block is just \( \sigma(X') \), and if \( m_i < 0 \), the signature of the block is \(-\sigma(X')\). But then the signature of all of \( M \) is \( \sum_{i=1}^{r+s} \text{sgn}(m_i)\sigma(X') = \left( \sum_{i=1}^{r+s} \text{sgn}(m_i) \right) \sigma(X') = \sigma(X)\sigma(X') \).

This completes the proof.

\[ \Box \]

\textbf{Proof of Theorem \[10.29\].} We need to establish that if \( \dim(X) = 4k \) and \( X = \partial W \), then \( \sigma(X) = 0 \). We will employ Lemma \[10.37\] by finding a self-annihilating subspace \( A \) of \( I_n H^{2k}(X; \mathbb{Q}) \) of half the dimension. In fact, we let \( A \) be the image of the restriction map \( j^* : I_n H^{2k}(W; \mathbb{Q}) \to I_n H^{2k}(X; \mathbb{Q}) \).

Let us first verify that \( A \) is self-annihilating. Let \( \alpha, \beta \in I_n H^{2k}(W; \mathbb{Q}) \). From the definitions, we must compute \( a(((j^* \alpha)^- - (i^* j^* \beta))^{-}) \), where \( \Gamma \) is the fundamental class of \( X \), \( i : I^n S_*(X; \mathbb{Q}) \to I^n S_*(X; \mathbb{Q}) \) is the inclusion. Notice, however, that the chain maps induced by inclusion of spaces \( X \to W \) commute with the inclusion of perversities \( I^n S_* \Rightarrow I^n S_* \), so \( i^* j^* \beta = j^* i^* \beta \), and also \( a_X \) agrees with the composition \( I^i H_0(X; \mathbb{Q}) \to I^i H_0(W; \mathbb{Q}) \to \mathbb{Q} \). Now, via the properties of cup and cap products, we compute
\[ a_x((j^*\alpha \sim (i^*j^*\beta)) \sim \Gamma) = a_w j((j^*\alpha \sim (j^*i^*\beta)) \sim \Gamma) \]
\[ = a_w j(j^*(\alpha \sim i^*\beta) \sim \Gamma) \]
\[ = a_w((\alpha \sim i^*\beta) \sim j\Gamma). \]

But since \( X = \partial W \), it follows that \( \partial[W] = j\Gamma \), so \( j\Gamma \) is homologically trivial and this expression is 0.

Next we must show that \( \dim(A) = \frac{1}{2} \dim(I_n H^{2k}(X; \mathbb{Q})) \). For this, consider the long exact sequence of the pair \((W, X)\). Leaving coefficients tacit, this has the following form:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & I_n H^0(W, X) & \longrightarrow & I_n H^0(W) & \longrightarrow & I_n H^0(X) & \longrightarrow & I_n H^1(W, X) & \longrightarrow & I_n H^1(W) & \longrightarrow & \cdots \\
& & \cdots & \longrightarrow & I_n H^{4k}(W) & \longrightarrow & I_n H^{4k}(X) & \longrightarrow & I_n H^{4k+1}(W, X) & \longrightarrow & I_n H^{4k+1}(W) & \longrightarrow & 0.
\end{array}
\]

Now, by duality, and since all spaces are Witt spaces, \( I_n H^0(W, X; \mathbb{Q}) \) is dual to \( I_n H^{4k+1}(W; \mathbb{Q}) \), \( I_n H^0(W; \mathbb{Q}) \) is dual to \( I_n H^{4k+1}(W, X; \mathbb{Q}) \), \( I_n H^0(X; \mathbb{Q}) \) is dual to \( I_n H^{4k}(X; \mathbb{Q}) \), and so on symmetrically inward until we arrive at \( I_n H^{2k}(X; \mathbb{Q}) \), which is dual to itself. Since each pair of dual spaces has the same dimension, we can complete the argument using a linear algebra argument that is presented as Lemma \[10.30\]. Note that to apply the lemma, we relabel the terms of the sequence so that \( C_0 = I_n H^{2k}(X; \mathbb{Q}) \) and the maps of our long exact sequence correspond to the boundary maps of the lemma, which lower degree.

**Lemma 10.30.** Let \( C_i \) be an exact sequence of finite-dimensional vector spaces such that \( C_i = 0 \) for \( |i| > m \) and \( \dim(C_i) = \dim(C_{i-1}) \) for all \( i \). Let \( d_i : C_i \to C_{i-1} \) denote the maps in this sequence. Then \( \dim(\text{im}(d_i)) = \dim(\text{im}(d_{i+1})) \) for \( i > 0 \), and \( \dim(\text{im}(d_1)) = \frac{1}{2} \dim(C_0) \).

**Proof.** This is clearly true for \( i > m \). We will apply downward induction.

As the base case, since \( C_m \to C_{m-1} \) is injective, \( \dim(\text{im}(d_m)) = \dim(C_m) \). Similarly, since \( C_{m-1} \to C_{m-2} \) is surjective, \( \dim(\text{im}(d_{m-1})) = \dim(C_m) \). But \( \dim(C_m) = \dim(C_{m-1}) \), so this case holds.

Now, assume that the claim has been verified for \( i > n > 0 \), and consider \( d_n : C_n \to C_{n-1} \). By elementary algebra, \( \dim(\text{im}(d_n)) = \dim(\text{cok}(d_{n+1})) = \dim(C_n) - \dim(\text{im}(d_{n+1})) \), while \( d_{n+1} : C_{n+1} \to C_n \). Hence, \( \dim(\text{im}(d_n)) = \dim(C_n) - \dim(\text{im}(d_{n+1})) \). So, as \( \dim(C_n) = \dim(C_{n-1}) \), we obtain \( \dim(\text{im}(d_n)) + \dim(\text{im}(d_{n+1})) = \dim(\text{im}(d_{n+1})) + \dim(\text{im}(d_{n-1})) \). But by induction hypothesis, \( \dim(\text{im}(d_{n+1})) = \dim(\text{im}(d_{n-1})) \), so \( \dim(\text{im}(d_n)) = \dim(\text{im}(d_{n-1})) \). This establishes the first claim of the lemma by induction.

For the last claim, we have by elementary linear algebra that \( \dim(C_0) = \dim(\text{im}(d_1)) + \dim(\text{im}(d_0)) \). But we have established that \( \dim(\text{im}(d_1)) = \dim(\text{im}(d_0)) \), so the dimension of each of these is half the dimension of \( C_0 \).

**Theorem 10.31.** The signature of a \( \mathbb{Q} \)-Witt space \( X \) does not depend on the stratification of \( X \). In fact, it is a topological invariant of \( X \).

**Proof.** This follows immediately from the topological invariance of the cup product pairing Theorem 8.67 (see also Remark 8.68) and from the invariance of the signature on the isomorphism class of the pairing, which is established in Lemma 10.34 below.
10.3.1 Appendix: linear algebra of signatures

In this section, we collect some material from linear algebra regarding the signature of a symmetric bilinear pairing. We will work primarily with the rational numbers as our ground field, but, unless stated otherwise, all results are equally valid for any ground field $F$ with $\mathbb{Q} \leq F \leq \mathbb{R}$. All vector spaces in this section are assumed to be finite dimensional.

**Definition 10.32.** If $M$ is a symmetric matrix of rational numbers, then the signature $\sigma(M)$ is defined to be

$$\sigma(M) = \#\{\text{positive eigenvalues of } M\} - \#\{\text{negative eigenvalues of } M\}.$$  

Notice that this makes sense because the spectral theorem tells us that all eigenvalues of a real symmetric matrix will be real.

We will be interested in signatures that arise from symmetric bilinear pairings on vector spaces. So, let $(V, (\cdot, \cdot))$ be a rational vector space together with a symmetric bilinear pairing $(\cdot, \cdot) : V \times V \to \mathbb{Q}$. In other words, for $u, v, w \in V$ and $c \in \mathbb{Q}$, $(u + v, w) = (u, w) + (v, w)$, $(u, v + w) = (u, v) + (u, w)$, $(cu, v) = (u, cv) = c(u, v)$, and $(u, v) = (v, u)$. If we choose a basis $\{e_i\}$ for $V$, then we obtain the pairing matrix $M$ with $M_{i,j} = (e_i, e_j)$. If $u, v \in V$, then $u = \sum_i a_i e_i$ and $v = \sum_i b_i e_i$ for some $a_i, b_i \in \mathbb{Q}$, so using the bilinearity of the pairing,

$$(u, v) = \left( \sum_i a_i e_i, \sum_j b_j e_j \right)$$

$$= \sum_{i,j} a_i b_j (e_i, e_j)$$

$$= \sum_{i,j} a_i M_{i,j} b_j$$

$$= u^t M v,$$

where $u^t$ is the transpose of $u$.

**Definition 10.33.** We define the signature of the pairing $\sigma(V, (\cdot, \cdot))$ to be the signature $\sigma(M)$ of the pairing matrix $M$.

Of course, we need to know that this is independent of the choice of basis. Suppose $\{f_i\}$ is another basis of $V$, that $N$ is the pairing matrix with respect to this basis, i.e. $N_{i,j} = (f_i, f_j)$
and \(Q\) is the matrix such that \(e_i = \sum_j Q_{i,k}f_k\). Then

\[
M_{i,j} = (e_i, e_j)
= \left( \sum_k Q_{i,k}f_k, \sum_\ell Q_{j,\ell}f_\ell \right)
= \sum_{k,\ell} Q_{i,k}Q_{j,\ell}(f_k, f_\ell)
= \sum_{k,\ell} Q_{i,k}Q_{j,\ell}N_{k,\ell}
= \sum_{k,\ell} Q_{i,k}N_{k,\ell}(Q^T)_{\ell,j},
\]

where \(Q^t\) is the transpose of \(Q\). This computation shows that \(M = QNQ^t\). Thus a change of basis changes the pairing matrix to a congruent matrix, which must have the same signature by Sylvester’s law of inertia (see [102] or [54, Theorem 6.11.1]).

There is a useful alternative characterization of the signature in terms of positive definite and negative definite subspaces, and this provides another proof of the independence of the signature of the choice of basis. Given a symmetric bilinear pairing \((V, (.,.))\), a subspace \(W \subset V\) is called positive definite if for any \(w \in W, w \neq 0\), \((w, w) > 0\). Similarly, a subspace \(W \subset V\) is called negative definite if for any \(w \in W, w \neq 0\), \((w, w) < 0\).

**Lemma 10.34.** Let \((V, (.,.))\) be a rational vector space with symmetric bilinear pairing, and let \(M\) be the pairing matrix with respect to some basis. Then the maximal dimension for a positive definite subspace is equal to the number of positive eigenvalues of \(M\), and the maximal dimension for a negative definite subspace is equal to the number of negative eigenvalues of \(M\). It follows that

\[
\sigma(V, (.,.)) = \max_{W^+ \text{ positive definite}} \dim(W^+) - \max_{W^- \text{ negative definite}} \dim(W^-),
\]

which does not depend on the choice of basis. It also demonstrates that two pairings \((V, (.,.))\) and \((V', (.,.))\) have the same signature if they are isomorphic in the sense that there is an isomorphism \(\phi : V \to V'\) and a diagram of the form:

\[
\begin{array}{ccc}
V \otimes V & \xrightarrow{(.,.)} & Q \\
\phi \otimes \phi & \downarrow & \\
V' \otimes V' & \xleftarrow{(.,.)} & \\
\end{array}
\]

To prove this lemma, it helps to use another elementary lemma:
Lemma 10.35. Let \((V, \langle \cdot, \cdot \rangle)\) be a rational vector space with symmetric bilinear pairing. Then there is a basis of \(V\) with respect to which the pairing matrix \(N\) is a diagonal matrix.

Proof. We will show that there is some basis \(\{b_i\}\) of \(V\) that is orthogonal, in the sense that \((b_i, b_j) = 0\) if \(i \neq j\). Then the pairing matrix \(N\) with respect to this basis will be diagonal.

Let \(v \in V\) be such that \((v, v) \neq 0\) (if there is no such \(v\), it follows from the identity \((v + w, v + w) = (v, v) + 2(v, w) + (w, w)\) that \((v, w) = 0\) for all \(v, w \in V\), and then any pairing matrix is the 0 matrix, so we are done). If we let \(\langle v \rangle\) be the span of \(v\), we will show that

\[ V = \langle v \rangle \oplus \langle v \rangle^\perp, \]

where for any subspace \(W \subseteq V\), \(W^\perp = \{u \in V \mid (u, w) = 0\text{ for all }w \in W\}\). For this, let \(u \in V\), and let \(u_1 = \frac{\langle u, v \rangle}{\langle v, v \rangle} v\) and \(u_2 = u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v\). In Euclidean space with the standard inner product, these would correspond to the projections of \(u\) to \(\langle v \rangle\) and to its perpendicular subspace. Clearly \(u = u_1 + u_2\), and

\[ (u_2, v) = (u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v, v) = (u, v) - \frac{\langle u, v \rangle}{\langle v, v \rangle} (v, v) = 0. \]

Thus any \(u\) is contained in \(\langle v \rangle + \langle v \rangle^\perp\). Next, suppose \(w \in \langle v \rangle \cap \langle v \rangle^\perp\). Then \(w = \lambda v\) for some \(\lambda \in \mathbb{Q}\). But then, since \(w \in \langle v \rangle^\perp\), we have \(0 = (w, w) = (v, \lambda v) = \lambda(v, v)\), which is impossible unless \(\lambda = 0\). Hence \(V = \langle v \rangle \oplus \langle v \rangle^\perp\).

Now we can complete the lemma using induction on \(\dim(V)\). If \(\dim(V) = 1\), there is nothing to prove. So suppose we have proven the result whenever the dimension of the vector space is \(< n\), and let \(\dim(V) = n\). Again, we will also be done trivially if there is no \(v\) with \((v, v) \neq 0\). If there is a \(v \in V\) with \((v, v) \neq 0\), then we have seen that \(V = \langle v \rangle \oplus \langle v \rangle^\perp\). But by induction there will be an orthogonal basis \(\{b_1, \ldots, b_{n-1}\}\) of \(\langle v \rangle^\perp\), and so \(\{b_1, \ldots, b_{n-1}, v\}\) is the desired orthogonal basis for \(V\).

Proof of Lemma 10.34. The preceding lemma provides a basis for \((V, \langle \cdot, \cdot \rangle)\) with respect to which the pairing matrix \(N\) is diagonal. We are free to reorder the basis \(\{b_i\}\) so that \(\{b_1, \ldots, b_r\}\) satisfy \((b_i, b_i) > 0\), \(\{b_{r+1}, \ldots, b_{r+s}\}\) satisfy \((b_i, b_i) < 0\) and \(\{b_{r+s+1}, \ldots, b_n\}\) satisfy \((b_i, b_i) = 0\). In this case, \(N\) is a diagonal matrix with \(r\) positive entries and \(s\) negative entries, and so \(r\) and \(s\) are the respective number of positive and negative eigenvalues.

It is then clear that \(\{b_1, \ldots, b_r\}\) span a positive definite subspace \(W^+\) of dimension \(r\). On the other hand \(\{b_{r+1}, \ldots, b_n\}\) span a subspace \(W^{\leq 0}\) such that no \(w \in W^{\leq 0}\) has the property that \((w, w) > 0\). Since \(W^{\leq 0}\) has dimension \(n - r\), every subspace of \(V\) of dimension \(r + 1\) must intersect \(W^{\leq 0}\) in a subspace of dimension \(\geq 1\) and hence possesses a \(w\) with \((w, w) \leq 0\). So no positive definite subspace can have dimension \(> r\). Thus \(r\) is the maximal dimension for a positive definite subspace. A similar argument shows that \(s\) is the maximal dimension for a negative definite subspace.

To complete our argument, we must now transition to work over the real numbers. Notice that \(V_\mathbb{R} = V \otimes_\mathbb{Q} \mathbb{R}\) is a real vector space. If \(\{e_i\}\) is any basis for \(V\), then \(e_i \otimes_\mathbb{Q} 1\) is a basis for \(V_\mathbb{R}\). Furthermore, using this basis, we can easily extend the pairing to a pairing \((\cdot, \cdot)_\mathbb{R}\) on \(V_\mathbb{R}\) using \((e_i \otimes_\mathbb{Q} x, e_j \otimes_\mathbb{Q} y)_\mathbb{R} = xy(e_i, e_j)\) for \(x, y \in \mathbb{R}\) and extending by bilinearity. Once
defined, the pairing \((.,.)_R\) is independent of the choice of basis, though we notice that if \(M\) is the pairing matrix for \((.,.)\) with respect to \(\{e_i\}\), then \(M\) is also the pairing matrix for \((.,.)_R\) with respect to \(\{e_i \otimes \mathbb{Q}\ 1\}\). Additionally, we can observe that if we decompose \(V\) as \(V = W^+ \oplus W^{\leq 0}\) as we did above, then \(V_\mathbb{R} = (W^+ \otimes \mathbb{R}) \oplus (W^{\leq 0} \otimes \mathbb{R})\), which we can denote \(W_\mathbb{R}^+ = W_R^+ \oplus W_R^{\leq 0}\). By the same arguments, these spaces have the same properties; in particular \(W_\mathbb{R}^+\) is a positive definite subspace with respect to \((.,.)_\mathbb{R}\) of maximal dimension, and this dimension is \(r\), the number of positive eigenvalues of the matrix \(N\), which is also a pairing matrix for \((.,.)_R\). Similarly, there is a \(W_\mathbb{R}^-\), which is a negative definite subspace of maximal dimension, which must be \(s\), the number of negative eigenvalues of \(N\).

Now, let us return to \(M\), which was a pairing matrix of \((V,(.,.))\) with respect to some basis \(\{e_i\}\). We have seen that \(M\) is also a pairing matrix for \((V_\mathbb{R},(.,.)_\mathbb{R})\) with respect to the corresponding basis \(\{e_i \otimes \mathbb{Q}\ R\}\). Since \(M\) is symmetric, it follows from the spectral theorem of elementary linear algebra (see, e.g., [99, Section 5.5]) that there is a real matrix \(Q\) such that \(Q^t = Q^{-1}\) and \(QMQ^{-1} = QMQ^t\) is diagonal with the eigenvalues of \(M\) along the diagonal. Since \(Q^t\) is congruent to \(M\), treating \(Q\) as a change of basis matrix, we see that \(QMQ^t\) is also a pairing matrix for \((V_\mathbb{R},(.,.)_\mathbb{R})\). So now, the number of positive eigenvalues for \(M\) is equal to the number of positive diagonal entries in \(QMQ^t\), which is a diagonal pairing matrix for \((V_\mathbb{R},(.,.)_\mathbb{R})\). But we have seen (using the same argument over the rationals or reals) that the number of positive entries in a diagonal pairing matrix for \((V_\mathbb{R},(.,.)_\mathbb{R})\) is equal to the dimension of a maximal positive definite subspace for \((V_\mathbb{R},(.,.)_\mathbb{R})\), and in turn, using a diagonal pairing matrix for \((V,(.,.))\), we have seen that the dimension of a maximal positive definite subspace for \((V_\mathbb{R},(.,.)_\mathbb{R})\) is the same as the dimension of a maximal positive definite subspace for \((V,(.,.))\). So, taken altogether, we see that the number of positive eigenvalues of \(M\) equals the maximal dimension for a positive definite subspace of \((V,(.,.))\). The same argument yields the analogous result for negative eigenvalues and negative definite subspaces.

The final statement of the lemma follows from observing that \(\phi\) must take positive-definite subspaces to isomorphic positive-definite subspaces and negative-definite subspaces to isomorphic negative-definite subspaces.

\[\square\]

**Definition 10.36.** A symmetric pairing \((V,(.,.))\) on a rational vector space is called\(^{128}\) **nondegenerate** if \((v,w) = 0\) for all \(w \in V\) implies \(v = 0\).

By Lemmas \[10.34\] and \[10.35\] if \((V,(.,.))\) is nondegenerate, the sum of the maximal dimension for a positive definite subspace and the maximal dimension for a negative definite subspace must be \(\dim(V)\).

\(^{128}\)This condition is also sometimes called **nonsingular**. Technically, a pairing being nondegenerate means that the homomorphism \(V \to \text{Hom}(V, \mathbb{Q})\) described by \(v \to (v,\cdot)\) is an injective, while being nonsingular means that it is an isomorphism. Of course these conditions are equivalent with field coefficients, as we are currently using. However, it is possible to define symmetric bilinear pairings on free modules over other rings, such as \(\mathbb{Z}\), in which case these become different conditions. For example, the pairing \(\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}\) with pairing matrix \(M = (2)\) is a nondegenerate pairing, but it is not nonsingular.
Lemma 10.37. Suppose $(V,(.,.))$ is a rational vector space together with a nondegenerate symmetric bilinear pairing $(.,.) : V \times V \to \mathbb{Q}$. If there is a subspace $A \subset V$ of dimension $\dim(A) = \frac{1}{2} \dim(V)$ such that $(x,y) = 0$ for all $x,y \in A$, then $\sigma(V,(.,.)) = 0$.

Proof. Assume that such a subspace $A$ exists. Let $V^+$ and $V^-$ be respectively positive definite and negative definite subspaces of $V$ of maximal dimensions. Let $\dim(V^+) = r$, $\dim(V^-) = s$, and $\dim(V) = n$. Since $(V,(.,.))$ is nondegenerate, $r + s = n$. Now, from the definitions, we must have $\dim(A \cap V^+) = \{0\}$ and $\dim(A \cap V^-) = \{0\}$. From the first equation, we must have that $\dim(A) \leq s$, and from the second, we must have $\dim(A) \leq r$. But since we have also assumed $\dim(A) = \frac{n}{2}$, this forces

$$n = 2 \dim(A) \leq r + s = n.$$ 

So in fact all the inequalities of the discussion must be equalities, and $\dim(A) = r = s = \frac{n}{2}$. Thus $\sigma(V,(.,.)) = r - s = 0$. □

Definition 10.38. Given a nondegenerate pairing $(V,(.,.))$, a subspace $A \subset V$ such that $(x,y) = 0$ for all $x,y \in A$ and $\dim(A) = \frac{1}{2} \dim(V)$ is called a Lagrangian subspace. Such subspaces are not generally unique.

Remark 10.39. The converse of Lemma 10.37 is true if we work with ground field $\mathbb{R}$. To see this, suppose we have used Lemma 10.35 to find an orthogonal basis for $(V,(.,.))$. Let us order and name the basis so that $\{a_1, \cdots, a_r\}$ are the orthogonal basis vectors with $(a_i,a_i) > 0$ and $\{b_1, \cdots, b_s\}$ are the orthogonal basis vectors with $(b_i,b_i) < 0$. We continue to assume that the pairing is nondegenerate so that the $\{a_i\}$ and $\{b_i\}$ together constitute a full basis. We can now normalize the basis by setting $c_i = \frac{1}{\sqrt{(a_i,a_i)}} a_i$ and $d_i = \frac{1}{\sqrt{(b_i,b_i)}} b_i$. With respect to this new basis consisting of the $c_i$ and $d_i$, we have $(c_i,c_i) = 1$, $(d_i,d_i) = -1$, and all other pairings between basis elements yield 0.

Now suppose the signature is 0, so that $r = s = \frac{1}{2} \dim(V)$. Let $f_i = c_i + d_i$, $1 \leq i \leq \frac{1}{2} \dim(V)$. Then

$$(f_i, f_j) = (c_i + d_i, c_j + d_j)$$

$$= (c_i, c_j) + (c_i, d_j) + (d_i, c_j) + (d_i, d_j)$$

$$= \delta_{i,j} + 0 + 0 - \delta_{i,j}$$

$$= 0.$$ 

It follows that the $f_i$ span a Lagrangian subspace of $V$.

The converse of Lemma 10.37 is not true over $\mathbb{Q}$. For example, consider the form with pairing matrix $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to some basis $\{a,b\}$. This pairing is nondegenerate with index 0. In order to have a Lagrangian subspace, there would have to be rational numbers $x,y$ such that $(xa + yb, xa + yb) = 2x^2 - y^2 = 0$. But since 2 is not the square of any rational number, this is impossible. This fact forms part of a rich theory of nondegenerate symmetric forms over $\mathbb{Q}$, as can be found, for example, in [72], particular in Section IV.2.
An important situation that arises in practice is one for which a pairing matrix for \((V, (\cdot, \cdot))\) has a block sum form, meaning that it has the form

\[
M = \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & & \\
& \ddots & \ddots & \\
0 & & \ldots & A_k
\end{pmatrix}.
\]

Here the \(A_i\) are square matrices of any size with their diagonals lying along the diagonal of \(M\), and all other entries of \(M\) not in the \(A_i\) are zero. Such a form corresponds to a decomposition of \(V\) as a direct sum of subspaces \(V = \bigoplus_i W_i\) such that the \(W_i\) are orthogonal to each other, i.e. \((w_i, w_j) = 0\) if \(w_k \in W_k\) and \(i \neq j\). In this case, each \(A_i\) represents the pairing restricted to \(W_i\). Now, we can find an orthogonal basis spanning each \(W_i\) by Lemma 10.35. If we do this for all subspaces simultaneously, we can find an orthogonal basis for \(V\). In other words, in this new basis we having a pairing matrix

\[
M' = \begin{pmatrix}
A'_1 & 0 & \ldots & 0 \\
0 & A'_2 & & \\
& \ddots & \ddots & \\
0 & & \ldots & A'_k
\end{pmatrix}
\]

in which all the \(A'_i\) are diagonal matrices. It now follows easily that \(\sigma(M') = \sum_i \sigma(A'_i)\). But since these invariants are independent of basis, in fact \(\sigma(M) = \sum \sigma(A_i)\). So we have shown:

**Lemma 10.40.** If \((V, (\cdot, \cdot))\) is a direct sum of orthogonal subspaces \(W_i\), then \(\sigma(V, (\cdot, \cdot)) = \sum_i (W_i, (\cdot, \cdot))\).

While the results so far in this section have been about symmetric bilinear pairings, there is one result we will need about antisymmetric pairings. An antisymmetric bilinear pairing consists of a vector space \(V\) and a pairing 

\[(\cdot, \cdot) : V \times V \to \mathbb{Q}\]

such that \((u + v, w) = (u, w) + (v, w)\), \((u, v + w) = (u, v) + (u, w)\), \((cv, w) = c(v, w) = (v, cw)\), and \((v, w) = -(w, v)\). As in the symmetric case, a choice of basis determines a pairing matrix, though now it will be antisymmetric, i.e. \(M^t = -M\).

In analogy with the symmetric case, an antisymmetric pairing is called nondegenerate if the assignment \(v \to (v, \cdot)\) is an injection (and hence an isomorphism) \(V \to \text{Hom}(V, \mathbb{Q})\). This corresponds to each pairing matrix having non-zero determinant.

In this setting, we have the following analogue of Lemma 10.35.

**Lemma 10.41.** Any nondegenerate antisymmetric pairing \((V, (\cdot, \cdot))\) of rational numbers has a basis with respect to which the pairing matrix is a block sum of \(2 \times 2\) matrices of the form \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

It follows as a consequence that \(V\) must be even dimensional to have a nondegenerate antisymmetric pairing.
Proof. We will construct a basis \( \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\} \) such that \( (a_i, b_i) = 1, (a_i, b_j) = 0 \) for \( i \neq j \), and \( (a_i, a_j) = (b_i, b_j) = 0 \) for all \( i, j \). Then the pairing matrix \( N \) with respect to this basis will have the desired form.

Let \( b_1 \) be an arbitrary vector in \( V \). Since nondegeneracy implies \( V \cong \text{Hom}(V, \mathbb{Q}) \), there is another vector in \( V \), which we will label \( a_1 \), such that \( (a_1, b_1) = 1 \). The pairing matrix restricted to the span of \( \{a_1, b_1\} \) is now \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), using \( \{a_1, b_1\} \) as a basis. Note that antisymmetry says that \( (v, v) = -(v, v) \), implying that \( (v, v) = 0 \) for any vector \( v \in V \).

If \( V \) is 2-dimensional, we are done. Otherwise, choose \( f_2 \) not in the span of \( \{a_1, b_1\} \), and let \( b_2 = f_2 + (b_1, f_2)a_1 - (a_1, f_2)b_1 \). Then

\[
(a_1, b_2) = (a_1, f_2 + (b_1, f_2)a_1 - (a_1, f_2)b_1) \\
= (a_1, f_2) + (b_1, f_2)(a_1, a_1) - (a_1, f_2)(a_1, b_1) \\
= (a_1, f_2) + 0 - (a_1, f_2) \\
= 0.
\]

Similarly

\[
(b_1, b_2) = (b_1, f_2 + (b_1, f_2)a_1 - (a_1, f_2)b_1) \\
= (b_1, f_2) + (b_1, f_2)(b_1, a_1) - (a_1, f_2)(b_1, b_1) \\
= (b_1, f_2) - (b_1, f_2) - 0 \\
= 0.
\]

So \( b_2 \) is orthogonal to span\( \{a_1, b_1\} \), but there must be some \( e_2 \) with \( (e_2, b_2) = 1 \) (note that \( e_1 \) is not in the span of \( \{a_1, b_1, b_2\} \)). Let \( a_2 = e_2 + (b_1, e_2)a_1 - (a_1, e_2)b_1 \). Then by the same calculations as for \( b_2 \), we see that \( a_2 \) is orthogonal to span\( \{a_1, b_2\} \), while

\[
(a_2, b_2) = (e_2 + (b_1, e_2)a_1 - (a_1, e_2)b_1, b_2) \\
= (e_2, b_2) + (b_1, e_2)(a_1, b_2) - (a_1, e_2)(b_1, b_2) \\
= 1 + 0 + 0 \\
= 1.
\]

It is now clear that we can continue in this manner: once we have found \( \{a_1, b_1, \ldots, a_k, b_k\} \), we can let \( f_{k+1} \) be any vector not in the span\( \{a_1, b_2, \ldots, a_k, b_k\} \) and then let \( b_{k+1} = f_{k+1} + \sum_{i=1}^{k}(b_i, f_{k+1})a_i - \sum_{i=1}^{k}(a_i, f_{k+1})b_i \). By analogous computations to those above, this will be orthogonal to all the previous \( a_i \) and \( b_i \). Then there must be some \( e_{k+1} \) not in the span of the established \( a_i \) and \( b_i \) such that \( (e_{k+1}, b_{k+1}) = 1 \). Let \( a_{k+1} = e_{k+1} + \sum_{i=1}^{k}(b_i, e_{k+1})a_i - \sum_{i=1}^{k}(a_i, e_{k+1})b_i \). Again this vector will be orthogonal to all \( a_i \) and \( b_i \) for \( i \leq k+1 \), but \( (a_{k+1}, b_{k+1}) = 1 \).

Eventually, the \( a_i \) and \( b_i \) span the space, and we are done. \( \square \)
Corollary 10.42. If \((V, \langle \cdot, \cdot \rangle)\) is an vector space with a nondegenerate antisymmetric pairing, there is a vector space of half the dimension of \(V\) on which the pairing is trivial.

Proof. Continuing to use notation of the proof of the preceding lemma, the subspace spanned by \(\{a_1, \ldots, a_n\}\) is such a subspace. \qed

10.4 L-classes

WARNING: THIS SECTION CONTAINS A SIGN DISCREPANCY (PRIMARILY IN SECTION 10.4.7, THOUGH POSSIBLY DUE TO AN ERROR ELSEWHERE) THAT STILL NEEDS TO BE RECONCILED.

A remarkable fact about signatures is that they provide the means to define other invariants of spaces. To see why this should be so, let us perform a thought experiment with the warning that the details will be largely incorrect but with the enticement that they can still be modified to obtain something useful:

Suppose \(X\) is a space, and let’s imagine that all its homology classes can be represented by manifolds. In other words, suppose that if \(\xi \in H_k(X)\), then there exists a closed oriented \(k\)-dimensional manifold \(M^k\) and map \(M^k \to X\) such that \(\xi\) is the image of the fundamental class of \(M\). Furthermore, let’s assume that all homologies in \(X\) can also be represented by manifold bordisms (so if \(f : M \to X\) and \(f' : M' \to X\) represent the same homology class, there is a \(W^{m+1} \to X\) that restricts to \(f\) and \(f'\) on \(\partial W = M \sqcup -M'\)). Then we could assign to \(\xi\) the signature of \(M\), and it would follow from our assumptions that this assignment \(\xi \to \sigma(M)\) is well-defined, since if we represent \(\xi\) by \(f' : M' \to X\) instead, \(M\) and \(M'\) are bordant and hence have the same signature. It is not difficult to see that this induces a homomorphism \(H_k(X) \to \mathbb{Z}\), since reversing orientation of a manifold changes the sign of its signature and signature is additive over disjoint unions. Tensoring with \(\mathbb{Q}\) makes this a homomorphism \(H_k(X; \mathbb{Q}) \to \mathbb{Q}\), i.e. an element of \(\text{Hom}(H_k(X; \mathbb{Q}), \mathbb{Q})\), which can then be identified as an element of \(H^k(X; \mathbb{Q})\). So we obtain an invariant of \(X\) in \(H^k(X; \mathbb{Q})\).

One major problem with this argument is that our assumptions that we can identify homology classes in terms of bordisms of manifolds is simply not true, even when \(X\) is itself a manifold. This was historically an important question, which was answered in the negative by Thom [104]. However, there turns out to be another interesting way to find manifolds and bordisms within a compact PL manifold \(M\): the rational cohomology classes determine embedded bordisms within \(M\). The idea is to find a correspondence between cohomology classes and homotopy classes of maps \(M^n \to S^m\); this is possible in certain dimension ranges and then the inverse images of generic points will be embedded manifolds in \(M\) and homotopies will generate embedded bordisms between these manifolds. Taking signatures will generate an element of \(\text{Hom}(H^m(M; \mathbb{Q}); \mathbb{Q})\), and since \(\text{Hom}(H^m(M; \mathbb{Q}); \mathbb{Q}) \cong H_m(M; \mathbb{Q})\) if \(M\) is compact, this gives us a class \(L_m(X)\) in homology. Remarkably, if \(M\) is smooth, this class will turn out to be Poincaré dual to the classical \(L\) classes that arise as suitable polynomials in the Pontrjagin classes. This construction is due to Thom [103]; an exposition is contained in Chapter 20 of Milnor-Stasheff [74].

This construction of \(L\)-classes can be extended to closed (compact without boundary) oriented PL \(\mathbb{Q}\)-Witt spaces. In this setting, it is still possible to identify rational cohomology
with maps to spheres (this part works for any space of the homotopy type of a CW complex),
but now the inverse images of generic points will be PL $\mathbb{Q}$-Witt spaces. As these have
signatures, and since these signatures are invariants of $\mathbb{Q}$-Witt bordism, we can enact the
same program to get an element of $H_*(X; \mathbb{Q})$ that deserves to be called a characteristic class
of $X$. This basic idea was enacted already in [42] for stratified PL pseudomanifolds with
only even codimension strata (before the discovery of more general Witt spaces in [95]).

In section 10.4.1 below, we will carefully formulate the details of the construction of
$L$-classes, and in the following sections we will provide the proofs. We will very roughly
follow the exposition of [74, Chapter 20], though we adapt several results as necessary for
our needs, fill in some additional details, and make a more serious departure in invoking
some basic stable homotopy theory in order to simplify some arguments.

To see where the stable homotopy comes in, the $m$-dimensional $L$-class on a PL manifold
$M$ is defined in detail in [74] when $m > \frac{\dim(M)+1}{2}$. This is the dimension requirement
mentioned above in order to ensure (roughly speaking) that rational cohomology classes can
be represented by maps to $S^m$ (see below for details). To define $L$-classes for smaller $m$, one
can utilize instead the space $S^k \times M$ for some large enough $k$ so that $m+k > \frac{\dim(M)+k+1}{2}$. This
yields an $L$-class in $H_{m+k}(S^k \times M; \mathbb{Q})$, and then one can transfer the $L$-class from
$S^k \times M$ to $H_m(M; \mathbb{Q})$ using the Künneth theorem; again, we will describe the precise procedure below.
However, one point not provided in detail in [74] is the independence of this construction
of the precise choice of sufficiently large $k$. Furthermore, while crossing with a sphere is
the natural construction in manifold theory, once one allows singularities, it is more natural
to utilize instead the suspension $S^k M$. Thus we hope that a stable homotopy approach
might be considered a bit more natural while providing easier access to the needed stability
(independence of $k$) results. And once we have gone over to stable homotopy theory, it is
very easy to prove the necessary relation between cohomology and homotopy classes of maps
to spheres by making some light use of spectra and generalized (co)homology theory and by
invoking a deep, but well-known, result of Serre concerning the homotopy groups of spheres.
For the most part, we will need only elementary aspects of stable homotopy theory, and we
develop most of these as we go along. The one more substantial use of stable homotopy
theory will be our brief use of spectra in Section 10.4.4, though the reader will have the
option of accepting the result of that argument as a black box. Section 10.4.6 contains our
results showing that suspensions and products with spheres give the same $L$-classes, and then
independence of $k$ when crossing with spheres follows from the independence of number of
suspension that we develop as part of our initial approach.

10.4.1 Outline of the construction of $L$-classes (without proofs)

In this section, we describe the construction of $L$-classes $Z_m(X) \in H_m(X; \mathbb{Q})$ for a closed
oriented $n$-dimensional PL $\mathbb{Q}$-Witt space $X$. The construction relies on a variety of results
that will be stated here and then proven in subsequent sections.

There is one exceptional case to deal with first, the $L$-class in $H_0(X; \mathbb{Q})$, which will
 correspond in our notation below to the case $m = 0$. At various points the homotopy
machinery we will develop breaks down when $m = 0$. However, as we’ll see in Lemma 10.52
the $L$-classes we define will be Poincaré dual to the classical Hirzebruch $L$-classes when $X$ is a smooth manifold. So the dual to our $\mathcal{L}_0(X)$ in this situation should be the $L^n(X)$ of [74] for an $n$-manifold $X$. Then by the Hirzebruch signature theorem, $L^n(X)[X] = \sigma(X)$. Since $L^n(X)[X] = a(L^n(X) \cap [X])$, where $a$ is the augmentation, we see that, if $X$ is connected, the Poincaré dual to $L^n$ corresponds to the element of $H_0(X; \mathbb{Q}) \cong \mathbb{Q}$ given by the signature. Using additivity of homology with respect to disjoint unions, it is thus reasonable when $X$ is a closed PL $\mathbb{Q}$-Witt space to define $\mathcal{L}_0(X) \in H_0(X; \mathbb{Q})$ to be the class represented by a point in each component carrying the coefficient corresponding to the signature of the component. This will be consistent with the manifold scenario.

We now outline how to proceed for $m > 0$. We begin with the following proposition, which implies that, in suitable situations, we can assign Witt signature invariants to maps $f : X \to S^m$.

**Proposition 10.43.** Let $X$ be a closed oriented $n$-dimensional PL stratified pseudomanifold without codimension one strata and with a PL subspace $A \subset X$ such that $X - A$ is a $\mathbb{Q}$-Witt space. Suppose that $S^m$, $m > 0$, has been given an orientation and that either (1) $f : X \to S^m$ is a PL map with $m > \dim(A) + 1$ or (2) $f : (X, A) \to (S^m, \{s_0\})$ is a PL map of pairs, for $s_0$ some basepoint of $S^m$. Then for almost all $y \in S^m$, the inverse image $f^{-1}(y)$ can be stratified as a compact oriented $n - m$-dimensional PL $\mathbb{Q}$-Witt space embedded. Furthermore, for almost all $y, y' \in S^m$, the Witt spaces $f^{-1}(y)$ and $f^{-1}(y')$ have the same signature; this common signature depends only on the PL homotopy class of $f$ in, respectively $[X, S^m]_P L$ (in case (1)) or $[(X, A), (S^m, \{s_0\})]_P L$ (in case (2)).

Here, $[\cdot, \cdot]_P L$ denotes the set of PL homotopy classes of PL maps, and, as in [74], “almost all $y$” means that the statement applies to all $y$ not belonging to the simplicial $m - 1$ skeleton of some appropriate triangulation of $S^m$, in particular one with respect to which $f : X \to S^m$ is simplicial. Let $\sigma(f)$ denote the common value of $\sigma(f^{-1}(y))$ for almost all $y \in S^m$. We defer for now the proof of Proposition 10.43. The basic idea is that for almost all $y$, the preimages of points of $S^m$ will have stratifications with respect to which they are PL $\mathbb{Q}$-Witt spaces, and changing from $y$ to $y'$ or changing from $f$ to a homotopic map will yield PL $\mathbb{Q}$-Witt space bordisms between these point inverse $\mathbb{Q}$-Witt spaces. Since signature is independent of stratification and $\mathbb{Q}$-Witt bordism class, almost all point inverses will have the same signature.

Let us briefly describe how the orientation of $f^{-1}(y)$ is chosen. We will see in Lemma 10.54 that in fact for almost all $y \in S^m$, there is a Euclidean neighborhood $U$ of $y$ such that $f^{-1}(U) \cong U \times f^{-1}(y)$. If $X$ were a smooth manifold and $f$ a smooth map, this would provide a tubular neighborhood of $f^{-1}(y)$ and we could associate the “fibers” $U \times \{z\}$, $z \in f^{-1}(y)$, with the fibers of the normal bundle of $f^{-1}(y)$. Then the orientation of this normal bundle, induced by the orientation of the normal bundle to $y \in S^m$, along with the orientation of $X$, would provide an orientation of $f^{-1}(y)$. In the PL case, we can still do this locally, which is sufficient when discussing orientation. We give $U$ an orientation inherited from a canonically chosen orientation on $S^m$. Then if $x$ is a point in a regular stratum $T$ of $f^{-1}(y)$ contained in a regular stratum $S$ of $X$ (we will see that regular strata of $f^{-1}(y)$ are always contained in regular strata of $X$), we can define an orientation on $T$ at $x$ so that the orientation of $U$
followed by the orientation of $T$ agrees with the orientation of $S$. Since $S$ is oriented, we can make consistent such choices of orientation at all points of $T$.

So if $X$ is a Q-Witt space, Proposition \[10.43\] shows how to assign a number, $\sigma(f)$, to each element of $[X, S^m]_{PL}$, the PL homotopy set of PL maps from $X$ to $S^m$. If we have a map $g : X \to S^m$ that is not necessarily PL, then by the PL approximation theorem \[97\] Theorem 3.5.8, $g$ is homotopic (by a small homotopy) to a PL map. Furthermore, applying \[97\] Lemma 3.5.8, any homotopy of PL maps is homotopic to a PL homotopy. Therefore, $[X, S^m]_{PL} \cong [X, S^m]$, the full set of topological homotopy classes of maps $X \to S^m$, so we in fact obtain a well-defined function $[X, S^m] \to \mathbb{Z}$.

For obvious reasons, $[X, S^m]$ is also called the cohomotopy set $\pi^m(X)$. Since $S^m$ is simply connected for $m > 1$ and since $S^1$ is a $K(\mathbb{Z}, 1)$, $[X, S^m] \cong [X, S^m]_0$, the basepoint preserving homotopy set, by \[53\] Proposition 4A.2 and Example 4A.3 (using that $K(\mathbb{Z}, 1)$ can be taken to be $S^1$, which is an $H$-space). So for $m > 0$, we do not need to take care with basepoints when discussing $\pi^m(X)$. This is very different from the case of homotopy groups.

Putting together the discussion so far together, we have functions

$$F_m : \pi^m(X) \to \mathbb{Z},$$

for $m > 0$, such that $[f] \in \pi^m(X)$ gets taken to $\sigma(f^{-1}(y))$ for a generic basepoint $y$ in $S^m$. We can only call this a function as $\pi^m(X)$ will only be a group for certain values of $m$, depending on $n = \dim(X)$:

**Lemma 10.44.** If $m > \frac{n+1}{2}$, $F : \pi^m(X) \to \mathbb{Z}$ is a homomorphism of abelian groups.

Our next order of business is to remove the dimension restriction. The key will be to examine the behavior of the cohomotopy sets under suspension.

Suppose we have a map $f : X \to S^m$. By suspending, we obtain a map $Sf : SX \to SS^m = S^{m+1}$. Explicitly, if we think of the suspension of space $Y$ as $[0, 1] \times Y/\sim$, where $\sim$ is the relation $(0, x) \sim (0, y)$ and $(1, x) \sim (1, y)$ for all $x, y \in Y$, then we can describe $Sf$ by $(Sf)(t, x) = (t, f(x))$. It is not difficult to see that if $f, g : X \to S^m$ are homotopic, then so are $Sf$ and $Sg$, so we obtain a function $S : [X, S^m] \to [SX, S^{m+1}]$, or $S : \pi^m(X) \to \pi^{m+1}(SX)$.

**Lemma 10.45.** If $X$ is any $n$-dimensional CW complex and $m > \frac{n+1}{2}$, $X \neq \emptyset$, then the suspension map $S : \pi^m(X) \to \pi^{m+1}(SX)$ is a group isomorphism.

Suppose now that $X$ is any compact non-empty $n$-dimensional CW complex. So will be any iterated suspension $S^iX$, where $S^iX$ denotes that we have repeated the suspension process $i$-times. The cohomotopy set $\pi^m(X) = [X, S^m]$ might not even be a group, but suppose we use repeated suspensions

$$[X, S^m] \to [SX, S^{m+1}] \to [S^2X, S^{m+2}] \to \cdots.$$ 

Once $m + k > \frac{n+k+1}{2}$, i.e. once $k > \dim(X) - 2m + 1$, $[S^kX, S^{m+k}] = \pi^{m+k}(S^kX)$ will be a group, and furthermore, by Lemma \[10.45\] this group will be independent of $k$ for all $k$ in this range. The resulting group is called the $m$th stable cohomotopy groups of $X$ and denote
\[ \pi^m_s(X). \] It is common to write \( \pi^m_s(X) \) as the direct limit under the suspension function \( \lim[S^k X, S^{m+k}] \), but since the direct sequence of sets stabilizes for \( k > \text{dim}(X) - 2m + 1 \) this language isn’t completely necessary.

**Lemma 10.46.** Let \( X \) be a non-empty closed oriented PL \( \mathbb{Q} \)-Witt space. For any \( m > 0 \), there is a group homomorphism \( F : \pi^m_s(X) \to \mathbb{Z} \). It is obtained by assigning to an element \( \alpha \in \pi^m_s(X) \) the signature of the inverse image of a generic point under a PL map \( f : S^k X \to S^{m+k} \) representing \( \alpha \) for a sufficiently large \( k \), assuming the spheres \( S^{m+k} \) are oriented consistently.

The assumption concerning orientation means that we assume that if \( S^i \) is oriented, then \( S^{i+1} \) is oriented consistent with thinking of it as the suspension of \( S^i \) and so as the quotient of the product \([0,1] \times S^i\), where \([0,1]\) is given the standard orientation.

**Remark 10.47.** If we suspend a \( \mathbb{Q} \)-Witt space \( X \), it might not be \( \mathbb{Q} \)-Witt anymore as there is no reason to assume the links of the suspension points (which will each be \( X \), itself) satisfy the necessary condition. So, after iterating, \( S^k X \) has a subset homeomorphic to \( S^{k-1} \), the \((k-1)\)st suspension of the suspension points of \( SX \), along which it might not be \( \mathbb{Q} \)-Witt. This is the reason we could not simply assume in Proposition [10.43] that our space was \( \mathbb{Q} \)-Witt, as the greater generality is needed here to ensure that point inverses of maps \( S^k X \to S^{m+k} \) are \( \mathbb{Q} \)-Witt spaces. In this case, \( A \cong S^{k-1} \) and the requirement of the proposition translates to \( m + k > k \), which is met provided \( m > 0 \). If \( m = 0 \), the lemma does not apply. We will see precisely why this condition on the dimension of \( A \) is necessary in the proof of Proposition [10.43].

So now we have constructed for any non-empty closed oriented PL \( \mathbb{Q} \)-Witt space \( X \) and any \( m > 0 \) a homomorphism \( F : \pi^m_s(X) \to \mathbb{Z} \), which, by tensoring with \( \mathbb{Q} \) over \( \mathbb{Z} \), can be made into a map \( F \otimes \text{id}_\mathbb{Q} : \pi^m_s(X) \otimes \mathbb{Q} \to \mathbb{Q} \). On the other hand, there is a homomorphism \( \alpha : \pi^m_s(X) \to \bar{H}^m(X) \) that becomes an isomorphism when tensored with \( \mathbb{Q} \):

**Theorem 10.48.** If \( X \) is a compact CW complex, there is a homomorphism \( \alpha : \pi^m_s(X) \to \bar{H}^m(X) \) such that \( G = \alpha \otimes \text{id}_\mathbb{Q} : \pi^m_s(X) \otimes \mathbb{Q} \to \bar{H}^m(X) \otimes \mathbb{Q} \) is an isomorphism.

The proof of Theorem [10.48] relies upon deep results of Serre concerning stable homotopy groups of spheres. We will see that the map \( \alpha \) can be described concretely. To every \([f] \in \pi^m_s(X)\), which we can identify with \([S^k X, S^{m+k}]\) for large enough \( k \), we can assign the element \( f^*(u) \in \bar{H}^{m+k}(S^k X) \cong \bar{H}^m(X) \), where \( u \) is a chosen generator of \( \bar{H}^{m+k}(S^{m+k}) \). We can assume these generators are chosen consistently as the images for the various \( k \) of the suspension isomorphisms \( \bar{H}^m(S^m) \to \bar{H}^{m+k}(S^{m+k}) \). Then \( \alpha([f]) \) is the image of \( f^*(u) \) under the cohomology suspension isomorphism \( \bar{H}^{m+k}(S^k X) \cong \bar{H}^m(X) \). The details will be discussed later.

For a closed oriented PL \( \mathbb{Q} \)-Witt space and \( m > 0 \), we thus obtain by composition the homomorphism \((F \otimes \text{id}_\mathbb{Q})G^{-1} : \bar{H}^m(X) \otimes \mathbb{Q} \to \mathbb{Q} \). Since \( X \) is compact, and thus \( H^*(X) \) is finite generated, the universal coefficient theorem [77] Theorem 56.1] says that \( \bar{H}^m(X) \otimes \mathbb{Q} \cong H^m(X; \mathbb{Q}) \). So \((F \otimes \text{id}_\mathbb{Q})G^{-1} \) can be thought of as an element of \( \text{Hom}(\bar{H}^m(X; \mathbb{Q}), \mathbb{Q}) \). But then, again by [77] Theorem 56.1], \( \text{Hom}(\bar{H}^m(X; \mathbb{Q}), \mathbb{Q}) \cong H_m(X; \mathbb{Q}) \). So \((F \otimes \text{id}_\mathbb{Q})G^{-1} \) represents an element of \( H_m(X; \mathbb{Q}) \). This is our \( L \)-class.

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**Definition 10.49.** The element $\mathcal{L}_m(X) \in H_n(X; \mathbb{Q})$ corresponding to $(F \otimes \text{id}_\mathbb{Q})G^{-1}$ under the universal coefficient isomorphism $\text{Hom}(H^m(X; \mathbb{Q}), \mathbb{Q}) \cong H_m(X; \mathbb{Q})$ is called the $m$th homology $L$-class of $X$.

**Remark 10.50.** If $n - m$ is not a multiple of 4, then the signature of any point inverse of any map $X \to S^m$ (or $S^kX \to S^{m+k}$) must be 0, by definition. In this case the $L$-class is trivial. Therefore the $L$-classes are typically only defined in dimensions $m = n - 4k$, $k \geq 0$.

**Remark 10.51.** Taking into account Koszul sign conventions, the evaluation isomorphism $ev : H_m(X; \mathbb{Q}) \to \text{Hom}(H^m(X; \mathbb{Q}), \mathbb{Q})$ of the universal coefficient theorem takes an element $[\xi] \in H_m(X; \mathbb{Q})$, represented by a rational chain $\xi$, to a homomorphism that acts on the class of a cocycle $\alpha$ by $ev([\xi])[\alpha] = (-1)^m \alpha(\xi)$.

**Proposition 10.52.** If $M^n$ is a closed oriented smooth $n$-manifold, then for each $m = n - 4k$, $(-1)^m \mathcal{L}_m(M)$ is the Poincaré dual of the rational Thom-Hirzebruch $L$-class $L^k(M) \in H^{4k}(X; \mathbb{Q})$, i.e. $L^k(X)$ is the degree $k$ term of the multiplicative sequence belonging to the power series of $\frac{\sqrt{1 - \tanh \sqrt{t}}}{\tanh \sqrt{t}}$ and taking as its variables the Pontrjagin classes of the tangent bundle of $M$ (see [74, Chapter 19]).

In our development of the $L$-classes $\mathcal{L}_m(X)$, we eliminated the dimension restriction $m > \frac{n+1}{2}$ early on by working with the stable cohomotopy groups $\pi^m_*(X)$, which are always groups, rather than the cohomotopy sets $\pi^m(X)$, which are only groups when $m > \frac{n+1}{2}$. The minor cost to us was that using the groups $\pi^m_*(X) = \lim_{\to}[S^kX, S^m]$ necessitates working with suspensions. Of course, in manifold theory, one would want to avoid introducing singularities, and so another alternative for defining $L$-classes for $m \leq \frac{n+1}{2}$ is presented in [74]. We now describe this alternative. We will demonstrate below that one arrives at the same $L$-classes either way.

Rather than considering the suspension $S^kX$, one can work with the product $S^k \times X$. Of course if $X$ is an $n$-manifold, $S^k \times X$ will remain a manifold. Next, consider the cohomotopy set $\pi^{m+k}(S^k \times X)$. This will be a group if $m + k > \frac{n+k+1}{2}$, i.e. if $k > n - 2m + 1$. In this range, $\pi^{m+k}(S^k \times X) \otimes \mathbb{Q} \cong H^{m+k}(S^k \times X) \otimes \mathbb{Q}$, and we’ll have

$$Q \xleftarrow{\frac{F_{m+k} \otimes \text{id}_Q}{G_{m+k}}} \pi^{m+k}(S^k \times X) \otimes \mathbb{Q} \xrightarrow{G_{m+k}} H^{m+k}(S^k \times X) \otimes \mathbb{Q},$$

where $G_{m_k}$ is induced by pulling back the generator $u \in H^{m+k}(S^m)$ via the homotopy class $[f] \in \pi^{m+k}(S^k \times X)$. But, from the Künneth theorem, the cross product with the preferred generator $\alpha \in H^k(S^k)$ induces a homomorphism $\alpha \times : H^k(X) \to H^{m+k}(S^k \times X)$. So precomposing with this map, tensored with $\mathbb{Q}$, yields a homomorphism $H^m(X) \otimes \mathbb{Q} \to \mathbb{Q}$, and once again this gives an element of $H_m(X; \mathbb{Q})$, which we will denote $\ell_m(X)$.

**Proposition 10.53.** For $m > 0$, $\ell_m(X) = \mathcal{L}_m(X) \in H_m(X; \mathbb{Q})$.

We now turn to proving the various claims of this section.
10.4.2 Maps to spheres yield integer invariants.

We first turn to proving Proposition [10.43].

We will need the following lemma, which is a filtered version of Lemma 20.5 of [74]. We give a slightly more detailed proof.

**Lemma 10.54.** If \( f : K \to L \) is a simplicial mapping, with \( K \) a simplicial filtered space, and \( y \) is contained in the interior \( U \) of a simplex of \( L \), then \( f^{-1}(U) \) is stratified homeomorphic to \( U \times f^{-1}(y) \), where \( f^{-1}(y) \) inherits its stratification from \( K \) (shifting formal dimensions as necessary). Furthermore, \( f^{-1}(y) \) is stratified homeomorphic to \( f^{-1}(y') \) for \( y, y' \in U \).

**Proof.** Suppose that \( U \) is the interior of a simplex \( \tau \) of \( L \) with \( \tau = [w_0, \ldots, w_M] \). Let \( y = \sum_{i=0}^{M} t_i w_i \) represent \( y \) in barycentric coordinates. Let \( \sigma = [v_0, \ldots, v_N] \) be a simplex of \( K \). If the mapping \( f \) is not surjective from \( \sigma \) onto \( \tau \), then \( f(\sigma) \cap U = \emptyset \) and the lemma holds vacuously. So suppose instead that \( f \) maps \( \sigma \) onto \( \tau \).

Assuming \( f \) maps \( \sigma \) onto \( \tau \), then for each vertex \( w_k \) of \( \tau \), there is some vertex \( v_i \) of \( \sigma \) that maps to \( w_k \). Let us relabel the vertices of \( \sigma \) as \( \{v_{ij}\} \) so that \( f(v_{ij}) = w_i \) for all \( i \). Then every element \( x \in \sigma \) can be written \( x = \sum_{i=0}^{M} \sum_{j=1}^{k(i)} a_{ij} v_{ij} \) in barycentric coordinates, where \( k(i) \) is the number of vertices of \( \sigma \) that map to \( w_i \). Then \( f(x) = \sum_{i=0}^{M} (\sum_{j=1}^{k(i)} a_{ij}) w_i \). So \( f^{-1}(y) \cap \sigma = \{ x \in \sigma \mid \sum_{j=1}^{k(i)} a_{ij} = t_i \text{ for each } i \} \). This is a system of \( M+1 \) linear equations in \( N+1 \geq M+1 \) unknowns. The solution set will be a linear subspace of \( \sigma \) of dimension \( N-M \).

Next, let us rewrite \( x = \sum_{i=0}^{M} \sum_{j=1}^{k(i)} a_{ij} v_{ij} \) as

\[
x = \sum_{i=0}^{M} \left( \sum_{\ell=1}^{k(i)} a_{i\ell} \right) \left( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} v_{ij} \right).
\]

This will be possible so long as \( \sum_{\ell=1}^{k(i)} a_{i\ell} \neq 0 \) for each \( i \), which is equivalent to assuming \( x \) maps to \( U \). The point here is that now \( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} = 1 \), so each \( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} v_{ij} \) is a description in barycentric coordinates of a point on a face of \( \sigma \) spanned by \( \{v_{ij}\}_{j=1}^{k(i)} \); let us denote this point by \( u_i(x) \), noting that \( u_i(x) \) depends continuously on the barycentric coordinates of \( x \). Then

\[
x = \sum_{i=0}^{M} \left( \sum_{\ell=1}^{k(i)} a_{i\ell} \right) \left( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} v_{ij} \right) = \sum_{i=0}^{M} \left( \sum_{\ell=1}^{k(i)} a_{i\ell} \right) u_i(x).
\]

Furthermore,

\[
f(u_i(x)) = f \left( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} v_{ij} \right) = \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} w_i = w_i.
\]
Define \( g : f^{-1}(U) \cap \sigma \to U \times (f^{-1}(y) \cap \sigma) \) by \( g(x) = (f(x), \sum_{i=0}^{M} t_i u_i(x)) \). This is well-defined because \( f(x) \in U \) by assumption and \( f(\sum_{i=0}^{M} t_i u_i(x)) = \sum_{i=0}^{M} t_i w_i = y \). To see that \( g \) is a stratified map, recall that \( x \in \partial \sigma \) if and only if there is some pair \( i, j \) such that \( a_{ij} = 0 \) in the barycentric coordinates for \( x \). In fact, the face containing \( x \) in its interior is completely determined by which \( a_{ij} \) are 0 (\( x \) will be in the interior of the face spanned by the \( v_{ij} \) such that \( a_{ij} \neq 0 \)). Whenever some \( a_{ij} = 0 \), this will result in the coefficient for \( v_{ij} \) being 0 in \( u_i(x) \). But since the second coordinate of \( g(x) \) is written in terms of the \( u_i(x) \), the coefficient of \( v_{ij} \) will continue to be 0 in \( \sum_{i=0}^{M} t_i u_i(x) \in f^{-1}(y) \). Therefore, since we assume \( t_i \neq 0 \) for all \( i \), \( g \) maps a point \( x \) in \( \sigma \) to a point of \( U \times f^{-1}(y) \) whose second coordinate is contained in the same open face of \( \sigma \) that contains \( x \). In other words, every open face of \( \sigma \) goes to the same stratum in \( U \times f^{-1}(y) \), giving \( U \times f^{-1}(y) \) the product filtration. Therefore, \( g \) must be a stratified map as the filtration on \( K \) must be compatible with the triangulation.

The map \( g \) is surjective: Choose any \( x \in f^{-1}(y) \) and a point \( y' = \sum_{i=0}^{M} s_i w_i \) in \( U \). Write \( x \) as \( x = \sum a_i u_i(x) \). Since \( f(x) = \sum_i a_i w_i = y \), by assumption, and since barycentric coordinates are unique, we must have \( a_i = t_i \). Therefore we actually have \( x = \sum t_i u_i(x) \). Now, let \( z = \sum s_i u_i(x) \). Then \( f(z) = \sum s_i f(u_i(x)) = \sum s_i w_i = y' \). On the other hand, we claim that \( u_i(z) = u_i(x) \), which case \( g(z) = (y', \sum t_i u_i(z)) = (y', \sum t_i u_i(x)) = (y', x) \); since the choices of \( x \) and \( y' \) were arbitrary, surjectivity follows. To prove the claim, notice that, by definition, \( u_i(x) = \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{s(i)} a_{i\ell}} v_{ij} \), where the \( a_{ij} \) are the coefficients of \( v_{ij} \) in the barycentric coordinates for \( x \). The corresponding coordinate for \( z \) is \( s_i \frac{a_{ij}}{\sum_{\ell=1}^{s(i)} a_{i\ell}} \). But then the coefficient for \( v_{ij} \) in \( u_i(z) \) is

\[
\frac{s_i \frac{a_{ij}}{\sum_{\ell=1}^{s(i)} a_{i\ell}}}{\sum_{\ell=1}^{s(i)} s_i a_{i\ell}} = \frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}}.
\]

So the coefficient of \( v_{ij} \) in \( u_i(z) \) is identical to the corresponding coefficient in \( u_i(x) \), which proves the claim.

The map \( g \) is injective: Suppose \( x, x' \in f^{-1}(U) \cap \sigma \) with \( x = \sum b_i u_i(x) \) and \( x' = \sum b'_i u_i(x') \) and that \( g(x) = g(x') \). This implies that \( f(x) = \sum b_i w_i = \sum b'_i w_i = f(x') \), so we must have \( b_i = b'_i \) for all \( i \) by uniqueness of barycentric coordinates. It also implies that \( \sum_{i=0}^{M} t_i u_i(x) = \sum_{i=0}^{M} t_i u_i(x') \). Writing out \( u_i \) and \( u'_i \), this becomes

\[
\sum_{i=0}^{M} t_i \left( \sum_{j=1}^{k(i)} \frac{a_{ij}}{\sum_{\ell=1}^{s(i)} a_{i\ell}} v_{ij} \right) = \sum_{i=0}^{M} t_i \left( \sum_{j=1}^{k(i)} \frac{a'_{ij}}{\sum_{\ell=1}^{s(i)} a'_{i\ell}} v_{ij} \right),
\]

where \( a_{ij} \) and \( a'_{ij} \) are the barycentric coordinates of \( x \) and \( x' \) with respect to the \( v_{ij} \). By the uniqueness of barycentric coordinates, we must have

\[
\frac{t_i a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} = \frac{t_i a'_{ij}}{\sum_{\ell=1}^{k(i)} a'_{i\ell}}
\]

for each \( i, j \), so

\[
\frac{a_{ij}}{\sum_{\ell=1}^{k(i)} a_{i\ell}} = \frac{a'_{ij}}{\sum_{\ell=1}^{k(i)} a'_{i\ell}}
\]

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for each $i, j$. But now comparing the definitions of $u_i(x)$ and $u_i(x')$, we see that we must actually have $u_i(x) = u_i(x')$ for all $i$. So altogether,

$$x = \sum b_iu_i(x) = \sum b_i' u_i(x') = x'.$$

So, we have constructed a stratified homeomorphism from $f^{-1}(U) \cap \sigma$ to $U \times (f^{-1}(y)) \cap \sigma$.

Now, returning to the full map $f : K \to L$. We see that we can piece these stratified homeomorphisms together over all the simplices of $K$ that intersect $f^{-1}(y)$ in order to obtain a global stratified homeomorphism $f^{-1}(U) \cong U \times f^{-1}(y)$.

For the last statement of the proposition, that $f^{-1}(y)$ is stratified homeomorphic to $f^{-1}(y')$ for any $y, y' \in U$, let $G : f^{-1}(U) \to U \times f^{-1}(y)$ be the stratified homeomorphism constructed above. Within each simplex $\sigma$ of $K$, $G$ restricts to the stratified homeomorphism $g : f^{-1}(U) \cap \sigma \to U \times (f^{-1}(y) \cap \sigma)$ defined by $g(x) = (f(x), \sum_{i=0}^{M} t_iu_i(x))$, where $y = \sum t_iw_i$. So we see that $f$ maps a point $x \in \sigma$ to a point $z \in U$ if and only if the first component of $g(x)$ is $z$. Since $\sigma$ was chosen arbitrarily, we see that, more generally, $f$ maps a point $x \in K$ to a point $z \in U$ if and only if the first component of $G(x) \in U \times f^{-1}(y)$ is $z$. In particular, then $f^{-1}(y') = G^{-1}((\{y\} \times f^{-1}(y))$, which is a stratified homeomorphic image of $f^{-1}(y)$. □

Now we can return to Proposition [10.43]

**Proof of Proposition [10.43]** Since $X$ is compact, the map $f : X \to S^m$ is proper. Therefore, by [57] Theorem 3.6.C], there are triangulations $K$ of $X$ and $L$ of $S^m$ with respect to which $f$ is simplicial. Let $y \neq s_0$ be a point in the interior of an $m$-simplex $\tau$ of $L$, and let $U$ be the interior of $\tau$. By the preceding Lemma, $f^{-1}(U) \cong U \times f^{-1}(y)$. Since $U$ is $m$-dimensional, each $n - m$ stratum of $f^{-1}(y)$ must be contained in an $n$-dimensional stratum of $X$. Let $T$ be such a stratum of $f^{-1}(y)$. If we choose an orientation of $S^m$, and hence an induced orientation on $U$, then we have that $U \times T$ is an open subset of $X - \Sigma_X$, and so the orientations on $X - \Sigma_X$ and $U$ induce a compatible orientation on $T$, the one such that at each point the orientation of $U$ followed by the local orientation on $T$ is compatible with the local orientation on $X - \Sigma_X$. Since $U$ and $X - \Sigma_X$ are globally orientable, this determines a global orientation on each regular stratum $T$ of $f^{-1}(y)$. So $f^{-1}(y)$ is orientable, with its orientation determined by a choice of orientation on $S^m$.

$f^{-1}(y)$ is a PL $Q$-Witt space. Now, we must show that $f^{-1}(y)$ is an $n - m$ dimensional PL $Q$-Witt space. It is clearly compact, being the inverse image of a closed point in a compact space $X$. Furthermore, $f^{-1}(y)$ can be triangulated. In fact, subdivide $L$ to a triangulation $L'$ so that $y$ is a vertex. Then, by [57] Theorem 3.6.C], we can subdivide $K$ to $K'$ and $L'$ to $L''$ so that $f : K' \to L''$ is simplicial. Then $f^{-1}(y)$ is a simplicial complex.

Next, we will invoke Proposition [2.111] to show that the intrinsic stratification of $f^{-1}(y)$ has the structure of a classical PL stratified pseudomanifold; we do not claim that the proposition makes $f^{-1}(y)$ a stratified PL pseudomanifold with the filtration inherited from $X$ (this won’t be necessary). According to Proposition [2.111] it suffices to show that $f^{-1}(y)$ contains a dense $n - m$ dimensional manifold $M$ such that $f^{-1}(y) - M$ has dimension $\leq n - m - 2$; then $f^{-1}(y)$ will be a classical pseudomanifold with respect to its intrinsic stratification. For this, let $M = f^{-1}(y) \cap (X - \Sigma_X) = (f|_{X-\Sigma_X})^{-1}(y)$. Since $X - \Sigma_X$
is a manifold, so is \((f|_{X-\Sigma_X})^{-1}(y)\); this follows from \([21\text{, Proposition 5.6.1}]\), using \([21\text{, Proposition 5.2}]\) and the definition in \([21\text{]}\) of the dual cell \(D(\alpha, L)\), noting that the dual cell of a top dimensional simplex is its barycenter. We observe that \(M\) does indeed have dimension \(n-m\) if it is not empty: we saw in the proof of Lemma \([10.54]\) that if a simplicial map takes a \(k\)-simplex \(\sigma\) onto an \(m\)-simplex \(\tau\), then the inverse image of an interior point of \(\tau\) will be a \(k-m\) dimensional subspace of \(\sigma\). So in the case at hand, if \(\sigma\) is an \(n\)-simplex of \(X\) that maps onto the face of \(S^m\) containing \(y\), then \(f^{-1}(y) \cap \sigma\) is \(n-m\) dimensional. Thus \(M\) must be \(n-m\) dimensional. Note that if no \(n\)-simplex of \(X\) maps onto the face of \(S^m\) containing \(y\), then \(f^{-1}(y)\) is empty, and so it is trivially a \(\mathbb{Q}\)-Witt space of any dimension.

Next we must show that \(M\) is dense in \(f^{-1}(y)\) and that \(f^{-1}(y) - M\) has dimension \(\leq n-m-2\). If \(x \in \Sigma_X\), then \(x\) is contained in a simplex \(\tau\) of \(K\) with \(\tau \subset \Sigma_X\). Let \(\sigma\) be an \(n\)-simplex of \(K\) with \(\tau\) as a face. Such a simplex must exist by the definition of PL pseudomanifolds. In order for \(f^{-1}(y)\) to be non-empty, \(\sigma\) must map onto the \(m\)-simplex of \(S^m\) containing \(y\), so the intersection \(\sigma \cap f^{-1}(y)\) is an \(n-m\) plane through \(\sigma\). This plane must intersect the interior of \(\sigma\) because \(f\) is continuous and simplicial, so if the interior of \(\sigma\) does not have any points that map to \(y\), which lies in the interior of a face of \(L\), \(f(\hat{\sigma})\) cannot intersect the interior of the face containing \(y\) at all, and therefore no point of \(\sigma\) can map to \(y\), a contradiction. Thus every point of \(\sigma\) in \(f^{-1}(y)\) must be in the closure of a plane that runs through the interior of \(\sigma\), and every point in \(f^{-1}(y) \cap \sigma\) is therefore in the closure of \(f^{-1}(y) \cap \hat{\sigma}\). This suffices to show that \(M\) is dense in \(f^{-1}(y)\). To see that \(M = f^{-1}(y) - M\) has dimension \(\leq n-m-2\), we need only note that \(f^{-1}(y) - M = f^{-1}(y) \cap \Sigma_X = (f|_{\Sigma_X})^{-1}(y)\); since every simplex of \(\Sigma_X\) has dimension \(\leq n-2\), \(f^{-1}(y) \cap \sigma\) has dimension \(\leq n-m-2\) for every simplex \(\sigma \in \Sigma_X\). So \(\dim((f|_{\Sigma_X})^{-1}(y)) \leq n-m-2\).

So, we have now shown that \(f^{-1}(y)\) is a classical pseudomanifold with respect to its intrinsic stratification, and we know that \(f^{-1}(y)\) has a neighborhood \(W\) in \(X\) (not-necessarily stratified) homeomorphic to \(U \times f^{-1}(y)\), where \(U\) is the interior of the \(m\)-simplex of \(S^m\) containing \(y\). Pick a pseudomanifold stratification of \(f^{-1}(y)\), and let \(U \times f^{-1}(y)\) have the product stratification. Let the homeomorphic \(W\) have the stratification inherited from \(X\). Notice that if \(\dim(A) < m-1\), for \(A\) as in the statement of Proposition \([10.43]\) all of \(A\) must map to the \(m-2\) skeleton of \(L\), so \(W \cap A = \emptyset\). Similarly, if \(f : (X, A) \to (S^m, \{s_0\})\), we can assume \(s_0\) is a vertex so that \(W \cap A = \emptyset\). Therefore, \(W\) is a \(\mathbb{Q}\)-Witt space, since the \(\mathbb{Q}\)-Witt condition is local. Since the property of being a \(\mathbb{Q}\)-Witt space is independent of the stratification by Proposition \([10.13]\) we see that \(U \times f^{-1}(y)\) is a \(\mathbb{Q}\)-Witt space. But the links of \(f^{-1}(y)\) are all also links of \(U \times f^{-1}(y)\), so \(f^{-1}(y)\) is a PL \(\mathbb{Q}\)-Witt space.

**Homotopies of \(f\).** Next, we investigate the effect on \(f^{-1}(y)\) of changing \(f\) by a homotopy. Suppose that \(f\) and \(g\) are PL homotopic PL maps \(X \to S^m\) (or relatively homotopic maps \((X, A) \to (S^m, \{s_0\})\)) by a PL homotopy \(H : I \times X \to M\). By adding closed collars to \(I \times X\) if necessary, we can assume that there is an \(\epsilon > 0\) such that \(H(t, x) = f(x)\) for \(t \in [0, \epsilon]\) and \(H(t, x) = g(x)\) for \(t \in [1-\epsilon, 1]\).

Since \(I \times X\) is compact, we can find triangulations \(\bar{K}\) and \(\bar{L}\) of \(I \times X\) and \(S^m\) with

\[129\text{Although this seems to be a fundamental fact of PL manifold theory, I don’t know of any other explicit proofs. Of course, the corresponding result is well-known for smooth manifolds and smooth maps.}\]
respect to which $H$ is simplicial, again using \cite[Theorem 3.6.C]{57}. Without loss of generality, we can assume that $y \in S^m$ is contained in the interior of an $m$-simplex of $\bar{L}$ (if not, replace $y$ with some $y'$ from the interior of the simplex of $L$ containing $y$ such that $y'$ is in the interior of an $m$-simplex of $\bar{L}$, and relabel $y'$ to $y$; by Lemma 10.54, $f^{-1}(y) \cong f^{-1}(y')$ if $y, y'$ are both in the interior of the same $m$-simplex of $L$). Consider $Y = (H|_{(0,1) \times X})^{-1}(y)$. The above arguments demonstrate that the intrinsic stratification of $Y$ gives it the structure of a stratified pseudomanifold, and, in fact, a $Q$-Witt space.\footnote{Here is where we use the full condition $m > \dim(A) + 1$ to ensure that $I \times A$ maps into the $m - 1$ skeleton of $\bar{L}$ (in the case of maps $(X, A) \to (S^m, \{s_0\})$, of course all of $A$ maps into the 0-skeleton of $L$.
} As spaces, $(H|_{(0,\epsilon) \times X})^{-1}(y)$ and $(H|_{(1-\epsilon,1) \times X})^{-1}(y)$ are respectively homeomorphic to $(0, \epsilon) \times f^{-1}(y)$ and $(1 - \epsilon, 1) \times g^{-1}(y)$. By Lemma 2.115, the intrinsic PL stratifications on these spaces (which are compatible by restriction with the intrinsic PL stratification on $Y$, as intrinsic stratifications are defined by local properties) have the forms $(0, \epsilon) \times f^{-1}(y)^*$ and $(1 - \epsilon, 1) \times g^{-1}(y)^*$, where $f^{-1}(y)^*$ and $g^{-1}(y)^*$ are the intrinsic PL stratifications of $f^{-1}(y)$ and $g^{-1}(y)$. This implies that if we filter $H^{-1}(y)$ with the intrinsic PL stratification on $Y$ and then glue on the collars $[0, 1) \times f^{-1}(y)^*$ and $(1 - \epsilon, 1) \times g^{-1}(y)^*$, then the filtrations will all be compatible so that we obtain a stratified PL $\partial$-pseudomanifold with boundary $g^{-1}(y)^* \amalg -f^{-1}(y)^*$.

Furthermore, as we have already seen that $Y, f^{-1}(y)^*$, and $g^{-1}(y)^*$ are PL $Q$-Witt space, $H^{-1}(y)$ is a PL $Q$-Witt space providing a $Q$-Witt bordism between $f^{-1}(y)^*$ and $g^{-1}(y)^*$. Thus $f^{-1}(y)^*$ and $g^{-1}(y)^*$ have the same Witt signature. We observe that by REF!!!, these signatures are also independent of the choices of stratification we have made.

Independence of $y$. To finish the proof of Proposition 10.43 we now need only see what happens if we change our choice of $y \in S^m$. Let $y, y'$ be two points of $S^m$ that are each in the interior of an $m$-simplex of the triangulation $L$ (not necessarily the same simplex); in particular, neither is $s_0$. We can find a PL homeomorphism $h$ that takes $y$ to $y'$, that fixes $s_0$, and that is PL homotopic to the identity by homotopies fixing $s_0$ (this follows from the material in \cite[Chapter VI]{57}). Therefore, $f$ and $hf$ are PL homotopic. So for almost all points $z'$ in $S^m$, $f^{-1}(z')$ is $Q$-Witt bordant to $(hf)^{-1}(z') = f^{-1}(h^{-1}(z'))$. In particular, we can choose $z'$ to lie both in the interior of the simplex of $L$ containing $y'$ and also so that neither $z'$ nor $z = h^{-1}(z')$ lie in the $m - 1$ dimensional simplicial skeleton of $S^m$ in a triangulation chosen to make the homotopy simplicial. If we now choose such a $z'$ close enough to $y'$, then $h^{-1}(z')$ will be close to $y$, i.e. in the same simplex interior in $L$. With this assumption, we will have that $f^{-1}(y) \cong f^{-1}(z)$ and $f^{-1}(z') \cong f^{-1}(y')$, and $f^{-1}(z)$ and $f^{-1}(z')$ will be $Q$-Witt bordant to each other (with appropriate stratifications). Therefore, we conclude that for almost all $y'$, $f^{-1}(y)$ and $f^{-1}(y')$ are $Q$-Witt bordant with respect to their intrinsic stratifications; thus all these spaces have the same signature. Finally, since $y$ itself was chosen to be any point lying in the interior of an $m$-simplex of $L$, we see that the signature is independent of $y$ for almost all $y$.

\[\square\]

10.4.3 Cohomotopy.

We now turn to a discussion of the cohomotopy sets $\pi^m(X)$. We will sketch the results that we need with the idea of providing for the reader enough of an introduction to get the basic
ideas, leaving the more advanced material for other sources. More thorough discussions of cohomotopy in general can be found in \[96\] and \[56\] Chapter VII. As mentioned above, for \(m > 0\), \([X, S^m] \cong [X, S^m]_0\), using \[53\] Proposition 4A.2 and Example 4A.3 (since \(K(\mathbb{Z}, 1)\) can be taken to be \(S^1\), which is an \(H\)-space). In this section, it will be more convenient to think of \(\pi^m(X)\) as \([X, S^m]\), though in later sections basepoints will become more important. We will deal with that issue when the time comes.

If \(f : X \to S^m\) is a map, we let \([f]\) denote the homotopy class of \(f\). The notation \(f \sim g\) will mean that \(f\) and \(g\) are homotopic.

**Introduction to cohomotopy and proof of Lemma 10.44.** We begin by proving the part of Lemma 10.44 that says that the cohomotopy sets become abelian groups when \(m\) is large enough compared to the dimension of \(X\).

The basic idea for turning the set of homotopy classes \(\pi^m(X) = [X, S^m]\) into a group is the following: Let \(S^m\) have a basepoint \(s_0\), and suppose \(f, g : X \to S^m\) are two maps from an \(n\)-dimensional CW complex to \(S^m\). We need to define a product \([f] + [g]\). For this we suppose that the product map \((f, g) : X \to S^m \times S^m\) misses a point \((z, z) \neq (s_0, s_0) \in S^m \times S^m\); this can be ensured up to homotopy when \(n < 2m\) (so, in particular, we need \(m > 0\)), using standard cellular approximation arguments. In fact, using the structure of \(S^m\) as a CW complex with two cells, one in dimension 0 and one in dimension \(m\), \(S^m \times S^m\) can be written as a product CW complex with four cells. The Cellular Approximation Theorem \[53\] Section 4.1] allows us to deform \((f, g)\) to a map to the \(m\)-skeleton, which is homeomorphic to \(S^m \vee_0 S^m\) (the union of two copies of \(S^m\), joined at their copies of \(s_0\)). Let us call the deformed map \(h : X \to S^m \vee S^m\), and let us denote by \(H : I \times X \to S^m \times S^m\) the homotopy from \((f, g)\) to \(h\). Next, we employ the fold map \(\Omega : S^m \vee S^m \to S^m\), which is the identity on each copy of \(S^m\). Let \(f + g\) denote the composition \(X \xrightarrow{h} S^m \vee S^m \xrightarrow{\Omega} S^m\); then \(f + g\) depends on \(h\), but we’d like to show that the homotopy class of \(f + g\) is well-defined so that we can then define the group operation on \(\pi^m(X)\) by \([f] + [g] = [f + g]\). For this we need to consider what happens if we use an alternative homotopy to \(H\), or, for that matter, alternative representatives of the homotopy classes of \(f\) and \(g\). For this, it is necessary to strengthen the assumption on dimension to \(n + 1 < 2m\). But now suppose that \(f \sim f', g \sim g'\) and that \(H'\) is a homotopy from \((f', g')\) to \(h'\) with the image of \(h'\) in \(S^m \vee S^m\). Then \(h \sim (f, g) \sim (f', g') \sim h'\) as maps \(X \to S^m \times S^m\), and so there is a homotopy \(K : I \times X \to S^m \times S^m\) from \(h\) to \(h'\). But now by another application of the Cellular Approximation Theorem, there is a homotopy rel \((\{0\} \times X)\) \(\Pi (\{1\} \times X)\) from \(K\) to a map \(K' : I \times X \to S^m \vee S^m\), which then provides a homotopy \(\Omega \circ K'\) from \(f + g\) to \(f' + g'\).

To see that our proposed group operation is commutative, let \(T : S^m \times S^m\) interchange coordinates; we also let \(T\) denote the restriction of \(T\) to \(S^m \vee S^m\). Then with the notation above, \(TH\) is a homotopy from \((g, f)\) into \(Th\), which has image in \(S^m \vee S^m\). But evidently \(\Omega T = \Omega\). So \(f + g = \Omega h = \Omega Th = g + f\).

Associativity is technically more difficult, though the concepts are not more complicated; one needs to use suitable maps in \(S^m \times S^m \times S^m\). We refer the reader to \[96\] or \[56\] Chapter VII.

The identity of \(\pi^m(X)\) is homotopy class of the map \(e\) that takes \(X\) to \(s_0 \in S^m\). Then

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(f, e) and (e, f) already map into \( S^m \setminus S^m \), without even the need for cellular approximation, and clearly \( f + e = \Omega(f, e) = f = \Omega(e, f) = e + f \).

The inverse \([-f]\) is represented by the composition \( X \xrightarrow{f \sim} S^m \xrightarrow{r} S^m \), where \( \rho \) is a map of degree \(-1\). To see this, let us fix \( \rho \) more precisely as a reflection map across the plane separating two hemispheres \( E_1 \) and \( E_2 \) of \( S^m \). We can assume \( s_0 \in E_1 \cap E_2 \). Let \( R : I \times S^m \to S^m \) be a homotopy that retracts \( E_1 \) into \( s_0 \), and let \( r = R|_{\{1\} \times S^m} \) be the non-identity end of the homotopy. Then \( rf \sim f \), and similarly \( r\rho f \sim \rho f \). Notice that \( rf \) maps \( M_1 = f^{-1}(E_1) \) into \( s_0 \), and \( r\rho f \) maps \( M_2 = f^{-1}(E_2) \) into \( s_0 \). Since every point of \( X \) is either \( M_1 \) or \( M_2 \) (possibly in their intersection), it follows that \((rf, r\rho f)\), which is homotopic to \((f, \rho f)\), has image in \( S^m \setminus S^m \). Then \( \Omega(rf, r\rho f) \) is well-defined. Furthermore, \( \Omega(rf, r\rho f)|_{M_1} = r\rho f|_{M_1} \sim \rho f|_{M_1} \), and \( \Omega(rf, r\rho f)|_{M_2} = rf|_{M_2} \sim f|_{M_2} \). Since in both cases we use the same homotopy \( R \) and since points in \( M_1 \cap M_2 \) map under \( f \) to \( E_1 \cap E_2 \), which is fixed by \( \rho \), these homotopies agree on \( M_1 \cap M_2 \). So \( \Omega(rf, r\rho f) \), which represents \([rf] + [r\rho f] = [f] + [\rho f] \), is homotopic to a map \( G \) that is \( f \) on \( M_2 \) and \( \rho f \) on \( M_1 \). But by definition, \( f \) takes \( M_2 \) into \( E_2 \) and \( M_1 \) into \( E_1 \), and so \( \rho f \) takes \( M_1 \) into \( E_2 \). Together, then, \( G \) takes all of \( X \) to \( E_2 \), which implies that \( G \) is homotopically trivial and hence represents the identity.

We have now established that \( \pi^n(X) \) is an abelian group whenever \( X \) is a CW complex of dimension \( n < 2m - 1 \). It is also worth observing that any map \( \phi : X \to Y \) induces a map \( \phi^* : \pi^n(Y) \to \pi^n(X) \) by taking the homotopy class \([g] \in \pi^n(Y) = [Y, S^m] \) to \([g \phi] \in [X, S^m] = \pi^n(X) \). Of course, if \( \phi \) is a homotopy equivalence then \( \phi^* \) is a bijection. If \( \dim(X) \) and \( \dim(Y) \) are in the dimension ranges such that \( \pi^n(X) \) and \( \pi^n(Y) \) are groups, then \( \phi^* \) is a homomorphism (see [36], Proposition VII.5.4]): if \( f, g : Y \to S^m \), let \( H \) denote the homotopy from \((f, g)\) to a map \( h : Y \to S^m \setminus S^m \). Then \( \Omega h \) represents \([f] + [g] \), and \( \Omega h \phi \) represents \( \phi^*([f] + [g]) \). On the other hand, \( f \phi \) and \( g \phi \) represent \( \phi^*[f] \) and \( \phi^*[g] \), and \( H \circ (id_Y \times \phi) \) is a homotopy from \((f, g) \phi = (f \phi, g \phi) \) to \( h \phi \), so \( \Omega h \phi \) also represents \( \phi^*\phi^*[g] \).

To finish proving Lemma 10.44, we want to show that, when \( X \) is a closed PL oriented \( \mathbb{Q} \)-Witt space and \( m \) is in the dimension range where \( \pi^n(X) \) is a group, the assignment \( F : \pi^n(X) \to \mathbb{Z} \) that takes \([f] \) to the signature of \( f^{-1}(y) \), for a sufficiently generic \( y \), is a homomorphism. We have already shown that \( F \) is well-defined on individual elements of \( \pi^n(X) \). We first observe that if \( H \) is the homotopy from \((f, g) : X \to S^m \times S^m \) to \( h : X \to S^m \setminus S^m \) and \( \pi_1, \pi_2 : S^m \times S^m \to S^m \) are the two projections, then \( \pi_1 H \) is a homotopy from \( f \) to, say, \( \bar{f} \), and \( \pi_2 H \) is a homotopy from \( g \) to \( \bar{g} \). In fact, \( h|_{h^{-1}(S^m \setminus \{s_0\})} = \bar{g}|_{h^{-1}(S^m \setminus \{s_0\})} \) and \( h|_{h^{-1}(S^m \setminus \{s_0\}) \setminus S^m} = \bar{g}|_{h^{-1}(S^m \setminus \{s_0\}) \setminus S^m} \). By further approximation, we can assume that all maps involved are simplicial. So, for a generic \( y \), \( \sigma((f^{-1}(y)) = \sigma(\bar{f}^{-1}(y)) \) and \( \sigma((g^{-1}(y)) = \sigma(\bar{g}^{-1}(y)) \). But now \( \Omega h^{-1}(y) = \bar{f}^{-1}(y) \Pi \bar{g}^{-1}(y) \), as \( \Omega^{-1}(y) \) consists of the two copies of \( y \) in \( S^m \setminus S^m \); the inverse image of this set under \( h \) correspond to the two inverse images under \( \bar{f} \) and \( \bar{g} \). So \( \sigma((\Omega h)^{-1}(y)) = \sigma(\bar{f}^{-1}(y)) + \sigma(\bar{g}^{-1}(y)) = \sigma(f^{-1}(y)) + \sigma(g^{-1}(y)) \), which established that \( F \) is a homomorphism.

More basic cohomotopy and the proof of Lemma 10.45. We turn toward proving Lemma 10.45. To prove the lemma, we will invoke some further results of basic cohomotopy.

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theory without proof. The main reason we do not simply cite a reference to Lemma 10.45 itself is that we were not able to find a clean statement and proof in the expository literature of elementary cohomotopy theory\textsuperscript{131} though this result is certainly well-known. However, the results we invoke are also useful to be aware of and demonstrate that the reason for the lemma is essentially equivalent to the well-known analogous property for homology and cohomology (e.g. [77, Theorem 33.2]).

First, we remark that one can define relative cohomotopy sets \( \pi^m(Y, A) = [(Y, A), (S^m, \{s_0\})] \), where \( s_0 \) is a chosen basepoint of \( S^m \). If \( (Y, A) \) is a CW pair, then \( \pi^m(Y, A) \) is also a group if \( \dim(Y) < 2m - 1 \); the construction of the group operation is essentially the same as for cohomotopy of a single space (see [56, Theorem VII.5.2]). Given such a CW pair, one can define the sequence of maps

\[
\rightarrow \pi^m(Y, A) \xrightarrow{j^*} \pi^m(Y) \xrightarrow{i^*} \pi^m(A) \xrightarrow{\delta} \pi^{m+1}(Y, A) \xrightarrow{j^*} \pi^{m+1}(Y) \rightarrow .
\]

The map \( j^* \) is induced by restriction a map \( (Y, A) \to (S^m, \{s_0\}) \) to a map \( (Y, \emptyset) \to (S^m, \{s_0\}) \) (so, essentially, by ignoring \( A \)), and the map \( i^* \) is induced by restricting a map \( Y \to S^m \) to a map \( A \to S^m \). To describe \( \delta \), let \( f : A \to S^m \) be a map representing an element of \( \pi^m(A) \), and let \( S^{m+1} = E^m_+ \cup_{S^m} E^m_- \), where we identify \( S^m \) as the equator of \( S^{m+1} \) and \( E^m_\pm \) are the two hemispheres, which we can identify with unit \( m+1 \) balls. Let \( s \), for “south pole”, denote the center of \( E^{m+1}_- \), which we make the basepoint of \( S^{m+1} \). Since \( E^{m+1}_+ \) is contractible, the composition of \( f \) with the embedding of \( S^m \) into \( S^{m+1} \) can be extended to a map \( \tilde{f} : (Y, A) \to (E^{m+1}_+, S^m) \). Let \( H \) be a homotopy \( I \times (E^{m+1}_+, S^m) \to (S^{m+1}, E^{m+1}_\pm) \) such that \( H \) is the identity at time 0 and such that \( H(\cdot, 1) \) takes \( S^m \) to the south pole \( s \in S^{m+1} \) and takes \( E^{m+1}_+ - S^m \) homeomorphically onto \( S^{m+1} \setminus \{s\} \). In [56, Section VII.4], the corresponding homotopy \( H \) retracts \( E^{m+1}_- \) instead to the basepoint \( s_0 \in S^m \subset S^{m+1} \); our construction is equivalent up to homotopy and will be more convenient for our purposes. It will be useful below to assume that each \( H(\cdot, t) \) is the identity map on \( E^{m+1}_+ \) outside of some \( \epsilon \) neighborhood of \( S^m \); it is not difficult to arrange this. We let \( \delta[f] = [H(\cdot, 1) \circ F] \). One can show that this process yields a well-defined map \( \delta : \pi^m(A) \rightarrow \pi^{m+1}(Y, A) \) [56, Section VII.4] and that it is a group homomorphism if \( H^i(Y, A) = 0 \) for all \( i \geq 2m + 1 \) and \( H^i(A) = 0 \) for all \( i \geq 2m - 1 \) [56, Proposition VII.5.5]; in particular, this will be true if \( \dim(Y) \leq 2m \) and \( \dim(A) \leq 2m - 2 \). By [56, Corollary VII.9.3], if \( H^i(Y, A) = H^i(A) = 0 \) for all \( i \geq 2m \), the sequence of cohomotopy groups is exact beginning with the map \( \pi^m(Y, A) \xrightarrow{j^*} \pi^m(Y) \xrightarrow{i^*} \pi^m(A) \) and continuing to the right (no claim is made about the kernel of \( j^* : \pi^m(Y, A) \to \pi^m(Y) \)). It will be useful to record this fact in the following form as a lemma so that we can refer to it later.

**Lemma 10.55.** If \( (Y, A) \) is a CW pairs such that \( H^i(Y, A) = H^i(A) = 0 \) for all \( i \geq 2m \), in particular if \( \dim(Y) < 2m \), the long exact cohomotopy sequence is exact, starting with the exact sequence \( \pi^m(Y, A) \xrightarrow{j^*} \pi^m(Y) \xrightarrow{i^*} \pi^m(A) \) and continuing to the right.

Now we can prove Lemma 10.45\textsuperscript{131} which states that if \( m > \frac{\dim(X)+1}{2} \), \( X \neq \emptyset \), then the suspension map \( S : \pi^m(X) \to \pi^{m+1}(SX) \) is a group isomorphism.

\textsuperscript{131}Admittedly, this may be a research failing on the author’s part.
Proof of Lemma 10.45. In the long exact cohomotopy sequence, consider the pair \((\bar{c}X, X)\), where \(X\) is a compact CW complex (to be our stratified PL pseudomanifold in applications) and \(\bar{c}X\) is the closed cone on \(X\). As \(\bar{c}X\) is contractible, \(\pi^m(\bar{c}X) = 0\) for all \(m > 0\), letting 0 represent the set with one element in the range where \(\pi^m(\bar{c}X)\) is not a group. In particular, if \(\text{dim}(X) \leq 2m - 2\), then \(\text{dim}(\bar{c}X) \leq 2m - 1\), so \(\pi^m(X) \xrightarrow{\delta} \pi^{m+1}(\bar{c}X, X)\) is an isomorphism, using Lemma 10.55.

Next, let us write \(SX = \bar{c}_+X \cup_X \bar{c}_-X\), where \(\bar{c}_\pm X\) are the “north and south” cones on \(X\). We claim that, for any \(i\), the restriction \(\pi^i(SX, \bar{c}_-X) \rightarrow \pi^i(\bar{c}_+X, X)\) is a bijection and an isomorphism when these are both groups. Notice that both sets consist of homotopy classes of maps that take everything except the open cone \(c_\pm X\) to a basepoint of \(S^m\). This is not sufficient in itself to provide an isomorphism of the cohomotopy sets, but it is not difficult to fill in the rest of the argument from here. A full proof can be found in [56], where the claim follows from [56] Theorem VII.3.2. Since we will use it again later, we state the relevant theorem here:

Lemma 10.56 (Theorem VII.3.2 of [56]). If \(f : (X, A) \rightarrow (Y, B)\) takes \(X - A\) homeomorphically onto \(Y - B\), \(X\) is compact, \(A\) is non-empty, \(Y\) is regular Hausdorff, and \(B\) is closed, then the induced map \(f^* : \pi^m(Y, B) \rightarrow \pi^m(X, A)\) is a bijection (and hence a group isomorphism in the range where these are groups).

As \(\bar{c}_-X\) contracts in \(SX\) to the south pole \(s\), we must have \(\pi^i(SX, \bar{c}_-X) \cong \pi^i(SX, \{s\})\). Notice that, \(\pi^i(\{s\})\) must be trivial for all \(m\), so again from the exact cohomotopy sequence of Lemma 10.55, we must have \(\pi^{m+1}(SX, \{s\}) \cong \pi^{m+1}(SX)\) if \(\text{dim}(X) \leq 2m - 2\).

For a compact CW complex with \(\text{dim}(X) \leq 2m - 2\), we have now provided a string of group isomorphisms

\[
\pi^m(X) \cong \pi^{m+1}(\bar{c}_+X, X) \cong \pi^{m+1}(SX, \bar{c}_-X) \cong \pi^{m+1}(SX, \{s\}) \cong \pi^{m+1}(SX).
\]

Lemma 10.45 will follow if we can verify that this chain of isomorphisms agrees with the suspension map. Let \(f : X \rightarrow S^m\) represent an element of \(\pi^m(X)\). The first isomorphism is the coboundary map \(\delta\). From the definition of \(\delta\), we can think of representing \(\delta[f]\) as first forming the cone on \(f\) to get \(\bar{c}f : \bar{c}_+X \rightarrow \bar{c}_+S^m = E^m_+\) and then homotoping \(\bar{c}f\) so that \(X\) gets taken to the south pole \(s\) of \(S^{m+1}\). As observed above, we can choose this homotopy so that it is the identity outside an \(\epsilon\) neighborhood of \(X\) in \(\bar{c}_+ X\). The next isomorphism simply extends \(\delta[f]\) to take \(\bar{c}_-X\) to \(s\). The next two isomorphisms are restrictions, and so can also be represented by \(SF\).

This completes the proof of Lemma 10.45.

Proof of Lemma 10.46. Now we prove Lemma 10.46 which says that, for a non-empty closed oriented PL Q-Witt space \(X\) and for any \(m > 0\), assigning signatures to generic point inverses yields a group homomorphism \(F : \pi^m_s(X) \rightarrow \mathbb{Z}\). In other words, suspension does not change our point-inverse signature invariant in the stable range:

To define \(F : \pi^m_s(X) \rightarrow \mathbb{Z}\), we first need to suspend \([X, S^m]\) enough times to get into the stable range with \(k\) sufficiently large to make \([S^k X, S^{m+k}]\) a group that is stable under
further suspensions, i.e. if \( m + k > \frac{\dim(X) + k + 1}{2} \) or \( k \geq \dim(X) - 2m - 2 \). As we suspend, \( S^kX \) might no longer be a \( \mathbb{Q} \)-Witt space, even if \( X \) is. However, \( S^kX \) has a subspace \( Z \) homeomorphic to the sphere \( S^{k-1} \) comprised of the \( k-1 \) times iterated suspension of the suspensions points of \( SX \). The space \( S^kX - Z \) is homeomorphic to \( X \times \mathbb{R}^k \) and so is a \( \mathbb{Q} \)-Witt space by Lemma [10.12]. So long as \( m + k > (k-1) + 1 \), which is assured by our assumption that \( m > 0 \), Proposition [10.43] continues to guarantee that the map \( F_{m+k} : \pi^{m+k}(S^kX) \to Z \) obtained by taking the Witt signature of the inverse image of a generic point is well-defined.

Next we will show that it is also independent of \( k \) for \( k \geq \dim(X) - 2m + 2 \). Observe that if \( f : X \to S^m \) is any map, and if we denote points in \( S^{m+1} = SS^{m+1} \) as \((t, y)\) for \( t \in [0, 1] \) and \( y \in S^m \) and similarly for points of \( SX \), then for \( t \in (0, 1) \), \((Sf)^{-1}((t, y)) = \{t\} \times f^{-1}(y)\). So, up to possible orientation issues, the point inverses of \( f \) for almost all points in \( S^m \) will be the same as the point inverses of \( Sf \) for almost all \((t, y) \in S^{m+1}\). Hence, up to sign, suspension does not change the signature of almost all point inverses.

Finally, let’s consider the orientation issues. Given a point inverse \( f^{-1}(y) \) in \( X \), for a generic point, the orientation of \( f^{-1}(y) \) is assigned at a point \( x \) in a regular stratum so that, identifying a neighborhood of \( f^{-1}(y) \) with \( U \times f^{-1}(y) \) for \( U \) a neighborhood of \( y \) in \( S^m \), the orientation of \( U \) followed by the orientation of \( f^{-1}(y) \) at \( x \) is consistent with the orientation for \( X \). Now, as we suspend, identifying \( S^{m+1} \) as \( SS^m = [0, 1] \times S^m / \sim \) provides an orientation of \( S^{m+1} \) via the product orientation at all points of \((0, 1) \times S^m\), and this extends to an orientation of \( S^{m+1} \). In particular, if \( U \) was our neighborhood of \( y \) in \( S^m \), we now have a neighborhood homeomorphic to \( \mathbb{R} \times U \) of any \((t, y) \) in \( SS^m \), \( t \in (0, 1) \). Correspondingly, we can extend the neighborhood \( U \times f^{-1}(y) \) of \( f^{-1}(y) \) in \( X \) to a neighborhood \( \mathbb{R} \times U \times f^{-1}(y) \) in \( SX \), again by appending an \( \mathbb{R} \) factor in the first coordinate. But this is consistent with also orienting \( SX \) via the product structure on \((0, 1) \times X \subset SX \). In all cases, the new \( \mathbb{R} \) factor is in the first component, and all these new \( \mathbb{R} \) factors are consistently oriented. Therefore, \( f^{-1}(y) \) and \((Sf)^{-1}((t, y)) \) are oriented homeomorphic, so the signature is preserved exactly under the suspension.

\[ 10.4.4 \text{ Rational cohomology and rational stable cohomotopy} \]

We turn to Theorem [10.48] In the treatment of \( L \)-classes in Milnor-Stasheff [73], the isomorphism over \( \mathbb{Q} \) between cohomotopy and cohomology is established in a certain dimension range by invoking results of Serre from [91]. However, it seems more reasonable for us to utilize the corresponding stable result in order to more directly define \( L \)-classes in the dimension ranges where the ordinary cohomotopy sets are not necessarily groups. Unfortunately, the fastest (and perhaps also most reasonable) route to the stable result requires some use of results about spectra and generalized cohomology theory, on top of some results of Serre concerning the homotopy theory of spheres. Despite the level of sophistication, we consider this a good expositional option. Later, when we need to compare the suspension approach to the product-with-spheres approach to \( L \)-classes, we will need some results at the space (as opposed to spectrum) level; we will develop these a consequence of the stable approach. We take our results concerning spectra from [87], which we recommend as an accessible modern text, and the results of Serre that we will need are extremely well-known. The reader who is
limits of pointed homotopy classes to the homotopy classes of maps of spectra.

From the spectral point of view, the $\pi_{m+1}SS$ above, where we wanted to use unreduced suspensions of PL spaces so that they would from the unpointed unreduced suspension maps we utilized to define stable cohomotopy $\pi_S S$ on $SS$ orientation and $0$ is given a negative orientation. Then for each $SS$ spectrum compatibly as follows: let $SS$ by the homeomorphism $SS$, which says that $SS$ do this now to get it out of the way.

Recall that, for CW complex $X$, the smash product $S^k \land X$ is defined to be the quotient $S^k \times X/(\{s_0\} \times X) \cup (S^k \times \{x_0\}) = (S^k \times X)/(S^k \lor X)$ for chosen basepoints $x_0 \in X$, $s_0 \in S^k$. The basepoint of $S^k \land X$ is the image of $S^k \lor X$. Also, recall that $S^k \land X$ is homeomorphic to the reduced $k$th suspension of $X$ [101, Corollary 2.28], which is homotopy equivalent to the $k$th suspension of $X$ via a quotient map $S^kX \to S^k \land X$ that collapses $S^k\{x_0\}$ to a point [53, Propositions 0.16 and 0.17]. In fact, $SX \to S^1 \land X$ is a homotopy equivalence by [53 Propositions 0.16 and 0.17], and since the suspension functor takes homotopy equivalences to homotopy equivalences, we have a sequence of homotopy equivalences

$$S^kX \to S^{k-1}(S^1 \land X) \to \cdots \to S^1 \land \cdots S^1 \land X \xrightarrow{\sim} S^k \land X,$$

using for the last isomorphism the associativity of smash products and [101, Lemma 2.27], which says that $S^1 \land S^n \cong S^{n+1}$ for all $n \geq 0$. We can also form the iterated suspension $S^kX$ all at once as the quotient of $D^k \times X$ such that $(t, x) \sim (t, y)$ for each $t \in \partial D^k$ and for all $x, y \in X$. When necessary to use coordinates, we can write $(s, x) \in S^kX$ to denote the image of $(s, x) \in D^k \times X$; similarly, we will write $[s, x] \in S^k \land X$.

In this section, we will assume, unless noted otherwise, that spaces and maps are pointe; this is necessary for our spectral machinery. Recall that, up to isomorphism, this will not alter our cohomotopy groups, as discussed in Section 10.4.3, though we will have to be a little careful to verify that the maps induced by reduced suspensions in stable homotopy theory are compatible with the unreduced cohomotopy suspensions maps already considered. Let’s do this now to get it out of the way.

We let $S$ denote the sphere spectrum with $S_i = S^i$ and spectrum structure map given by the homeomorphism $S^1 \land S^i \to S^{i+1}$. To be specific, we orient the spheres in the sphere spectrum compatibly as follows: let $S^0 = \{0, 1\}$ be oriented so that $1$ is given a positive orientation and $0$ is given a negative orientation. Then for each $S_i$, $i > 0$, orient $S^i = SS^{i-1} = ([0, 1] \times S^{i-1})/ \sim$ using the product orientation. This induces also the orientation on $S^1 \land S^{i-1}$, as the quotient $S S^{i-1} \to S^1 \land S^{i-1}$ is a homeomorphism from a dense set of $SS^{i-1}$ onto the complement of a point in $S^1 \land S^{i-1} \cong S^i$.

\(\pi^m_s(X)\) via spectra. Let $X$ denote the suspension spectrum of the finite CW complex $X$. From the spectral point of view, the $m$th stable cohomotopy groups of a space $X$ correspond to the homotopy classes of maps of spectra $[\mathbb{X}, S^m \land S]$, which in turn are the direct limits of pointed homotopy classes $[S^k \land X, S^{m+k}]_0$ under the reduced suspension functor $[S^k \land X, S^{m+k}]_0 \to [S^1 \land S^k \land X, S^1 \land S^{m+k}]_0 \cong [S^{k+1} \land X, S^{m+k+1}]_0$. This is a bit different from the unpointed unreduced suspension maps we utilized to define stable cohomotopy above, where we wanted to use unreduced suspensions of PL spaces so that they would

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132We will use the notation $S^m \land E$ for the $m$th suspension of a spectrum $E$. This is consistent with the idea that the suspension of a spectrum is achieved by taking reduced suspensions of the spaces that make up a spectrum.
remain PL spaces. So to utilize spectral results, we need to show compatibility between these two different notions of stable cohomotopy.

**Lemma 10.57.** If $X$ is a finite CW complex, the direct limit under suspension $\lim_{k \to \infty} [S^kX, S^{m+k}]$ is isomorphic to the spectrum cohomology groups $[X, S^m \wedge \mathbb{S}]$. In fact, this isomorphism is the direct limit of isomorphisms for sufficiently large $k$.

**Proof.** Since the maps in spectra are pointed, we will want to use the pointed version of the cohomotopy sets, which are isomorphic to the unpointed version as discussed at the beginning of Section [10.4.3](#). If we have a space $Z$ with a basepoint $z_0$, we can suppose that $SZ$ has basepoint corresponding to $(1/2, z_0) \in SZ = [0,1] \times Z/\sim$, so that the suspension functor provides a function $[X, S^m]_0 \to [SX, S^{m+1}]_0$. Now, consider the diagram

$$
\begin{array}{ccc}
[S^{k+1}X, S^{m+k+1}]_0 & \xrightarrow{q_{k+1}} & [S^{k+1}X, S^{k+1} \wedge S^m]_0 & \xrightarrow{p_{k+1}^*} & [S^{k+1} \wedge X, S^{k+1} \wedge S^m]_0 \\
| & | & | & | & | \\
[S^kX, S^{m+k}]_0 & \xrightarrow{q_k} & [S^kX, S^k \wedge S^m]_0 & \xrightarrow{p_k^*} & [S^k \wedge X, S^k \wedge S^m]_0 \\
\end{array}
$$

Here the maps $q_r$ are quotients from the $r$th suspension $S^r S^m$ to the reduced suspension $S^r \wedge S^m$, for $r = k, k + 1$; $q_r$ is a homotopy equivalence, and both $S^r S^m$ and $S^r \wedge S^m$ are homeomorphic to $S^{r+m}$. The map $q'$ is the quotient from the suspension of $S^k \wedge S^m$ to the reduced suspension, which is homeomorphic to $S^1 \wedge S^k \wedge S^m \cong S^{k+1} \wedge S^m$. The maps $p_r^*$ are the pullbacks induced by the quotients $p_r : S^r X \to S^r \wedge X$; they are isomorphisms because $p_r$ is a homotopy equivalence. Let us verify the diagram commutes.

Let us write elements of $S^{m+k}$ as $(t_k, \ldots, t_1, z)$, where $z \in S^m$ and the $t_i$ are the suspension coordinates. This utilizes our above notation for a suspension, though we remove interior parentheses from the iterated notation for clarity. Suppose $f$ represents an element of $[S^kX, S^{m+k}]$ and that, for some $x \in S^kX$, $f(x) = (t_k, \ldots, t_1, z)$. The map $Sf$ takes $(s, x) \in S(S^kX) = S^{k+1}X$ to $(s, f(x)) = (s, t_k, \ldots, t_1, z) \in SS^{m+k} = S^{m+k+1}$, and $\pi_{k+1} Sf$ takes this to $[s, t_k, \ldots, t_1, z]$ in the image of the iterated suspension of $S^m$ in the reduced iterated suspension. On the other hand, $q_k f$ takes $x$ to $[t_k, \ldots, t_1, z]$, $S q_k f$ takes $(s, x)$ to $(s, [t_k, \ldots, t_1, z])$, and $q'$ then takes $(s, [t_k, \ldots, t_1, z])$ to $[s, t_k, \ldots, t_1, z]$, since the $k + 1$ times iterated reduced suspension is the same as the smash product with $S^{k+1}$. This shows that the left side of the diagram commutes.

On the right side, suppose we start with a map $f : S^k \wedge X \to S^k \wedge S^m$ representing a homotopy class. If $[s_k, \ldots, s_1, x]$ represents an element in $S^k \wedge X$ with image $[t_k, \ldots, t_1, z]$
under $f$, $S^1 \land f$ takes $[s, s_k, \ldots, s_1, x]$ to $[s, t_k, \ldots, t_1, z]$, and $p_{k+1}^*(S^1 \land f)$ is the map that takes $(s, s_k, \ldots, s_1, x)$ to $[s, t_k, \ldots, t_1, z]$. On the other hand, $p_k^*f$ takes $(s, s_k, \ldots, s_1, x)$ to $[t_k, \ldots, t_1, z]$. So the right-hand diagram also commutes.

Define now maps $(p_k^*)^{-1} \pi_k : [S^k, S^{m+k}_0] \to [S^k \land X, S^k \land S^m_0]$. Commutativity of the diagram shows that these form a map of direct systems. The maps are isomorphisms for sufficiently large $k$, and so they induce isomorphisms on the direct limits. Therefore the two possible ways of defining $\pi_m^*(X)$ are isomorphic. □

**Remark 10.58.** The commutativity of the right side of diagram (46) also demonstrates that $S^1 \land : [S^k \land X, S^k \land S^m_0] \to [S^{k+1} \land X, S^{k+1} \land S^m_0]$ is a group homomorphism in the appropriate range, since we already know this of the other maps in the diagram by Lemma 10.45 and [56, Proposition VII.5.4], which says that maps of spaces induce homomorphisms of cohomotopy groups.

**Stable cohomotopy and cohomology.** Let $HZ$ denote the integer Eilenberg-MacLane prespectrum. The prespectrum $HZ$ consists of Eilenberg-MacLane spaces $K(\mathbb{Z}, i)$. It will be useful to have specific CW models of the $K(\mathbb{Z}, i)$ in mind for $i > 0$. To this end we may assume that $K(\mathbb{Z}, i)$ is a CW complex whose $i + 1$ skeleton is $S^i$, using the standard CW construction of $K(\mathbb{Z}, i)$ by iteratively attaching cells of dimension $\geq i + 2$ to $S^i$ to kill higher homotopy. Then the $i + 2$ skeleton of $S^1 \land K(\mathbb{Z}, i)$ is the sphere $S^{i+1}$. The identity map $S^{i+1} \to S^{i+1}$ extends uniquely (up to homotopy) to a map $s_i : S^1 \land K(\mathbb{Z}, i) \to K(\mathbb{Z}, i + 1)$ by elementary obstruction theory (e.g. apply [53, Lemma 4.7] to $(S^k \land K(\mathbb{Z}, m), S^{m+k})$ to show existence of an extension and to $(I \times (S^k \land K(\mathbb{Z}, m))), ((\partial I) \times (S^k \land K(\mathbb{Z}, m))) \cup (I \times S^{m+k}))$ to get uniqueness up to homotopy). We claim that $s_i$ is adjoint to a homotopy equivalence, and so this construction corresponds to the standard construction of $H(\mathbb{Z})$ as an $\Omega$-prespectrum, e.g. [87, Example II.3.24].

**Lemma 10.59.** For $i \geq 1$, $s_i : S^1 \land K(\mathbb{Z}, i) \to K(\mathbb{Z}, i + 1)$ is adjoint to a (basepoint preserving) homotopy equivalence $h_i : K(\mathbb{Z}, i) \to \Omega K(\mathbb{Z}, i + 1)$.

**Proof.** Since we have already observed that maps $S^1 \land K(\mathbb{Z}, i) \to K(\mathbb{Z}, i + 1)$ are determined uniquely up to homotopy by their restrictions $S^{i+1} \to S^{i+1}$ to the $i + 2$ skeletons, this

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133In older terminology (e.g. [11, Part III]), a spectrum is a sequence of (pointed) spaces $X_i$ together with structure maps $s_i : S^1 \land X_i \to X_{i+1}$. In newer terminology (including the treatment in [87]), such objects are called “prespectra”, with the term spectrum being reserved for the case where the $X_i$ are all CW complexes and $s_i$ is the inclusion of a subcomplex. Every prespectrum is homotopy equivalent (space-by-space) to a spectrum [87, Lemma-Definition II.1.19], so we will not be too careful about this detail. In the few places where we need to make use of spectrum arguments, we will tacitly assume prespectra to be replaced by homotopy equivalent spectra.

134Seasoned homotopy theorists will no doubt claim that the following proof puts many carts before many horses, using various theorems from homotopy theory to prove results on which those theorems were based. We take as our excuse that our goal is to provide for the reader an argument that proves the theorem but that relies solely on major results with which the reader is likely to be acquainted or can look up easily. We agree that this is no substitute for becoming more deeply acquainted with elementary homotopy theory, but we hope that the reader with other interests will find the argument both accessible and reasonably swift, and so somewhat satisfying within context.
means that \([S^{i} \wedge K(Z, i), K(Z, i+1)]_0 \cong [S^{i+1}, S^{i+1}]_0 \cong \mathbb{Z}\). Since \(s_i\) corresponds to a generator of \([S^{i+1}, S^{i+1}]_0 \cong \mathbb{Z}\), its adjoint \(h_i \in [K(Z, i), \Omega K(Z, i+1)]_0 \cong [S^{i} \wedge K(Z, i), K(Z, i+1)]_0 \cong \mathbb{Z}\) is a generator.

We know abstractly that \(\Omega K(Z, i+1)\) is homotopy equivalent to \(K(Z, i)\) since \(\pi_k(\Omega K(Z, i+1)) = [S^{k}, \Omega K(Z, i+1)]_0 \cong [S^{k+1}, K(Z, i+1)]_0 = \pi_{k+1}(K(Z, i+1))\), which is 0 unless \(i+1 = k+1\) in which case it is \(\mathbb{Z}\). But the homotopy type of an Eilenberg-MacLane space is determined completely by this property \[\text{Proposition 4.30]}. Thus \(\mathbb{Z} \cong [K(Z, i), \Omega K(Z, i+1)]_0 \cong H^i(K(Z, i))\), since any space homotopy equivalent to an Eilenberg-MacLane space represents reduced cohomology. Furthermore, there is a universal object \(u \in H^i(\Omega K(Z, i+1))\) such that the isomorphism \([K(Z, i), \Omega K(Z, i+1)] \cong H^i(K(Z, i))\) takes a homotopy class \([f]\) to \(f^*(u)\); see \[\text{Theorem 4.57]}. The element \(u\) must be a generator of \(H^i(\Omega K(Z, i+1))\) (else it could not be a universal element), and since \(h_i\) generates \([K(Z, i), \Omega K(Z, i+1)]\), we must have that \(h_i^*(u)\) is a generator of \(H^i(K(Z, i))\). Thus \(h_i^*\) induces an isomorphism on cohomology in dimension \(i\). As \(K(Z, i)\) and \(\Omega K(Z, i+1)\) are \(i-1\) connected, by the universal coefficient theorem, \(h_i\) also induces an isomorphism on homology in dimension \(i\). So if \(i > 1\), the Hurewicz theorem \[\text{Theorem 4.37] implies that \(h_i\) induces an isomorphism on \(\pi_i\). Since all other homotopy groups of \(K(Z, i)\) and the homotopy equivalent \(\Omega K(Z, i+1)\) vanish, this is enough to conclude by the Whitehead theorem \[\text{Theorem 4.5] that \(h_i\) is a homotopy equivalence. In general, we cannot apply the Hurewicz theorem for \(i = 1\), but in the present case we know that \(\pi_1(K(Z, 1)) = \mathbb{Z}\) (in fact \(K(Z, 1) = S^1\)), so \(\pi_1(K(Z, 1)) \cong H_1(K(Z, 1))\) and similarly for the homotopy equivalent \(\Omega K(Z, 2)\), so here again the that \(h_i\) induces isomorphisms on homology is enough to conclude that it also induces isomorphisms on homotopy. Then again we can invoke the Whitehead theorem.

Let \(\alpha : S \to H\mathbb{Z}\) be the map of prespectra such that, for \(i > 0\), \(\alpha_i : S^i \to K(Z, i)\) is precisely the inclusion of the \(i+1\) skeleton of \(K(Z, i)\). Each \(\alpha_i \in [S^i, K(Z, i)]_0 = \pi_i(S^i, K(Z, i)) \cong \mathbb{Z}\) corresponds to a preferred generator. Furthermore, the \(\alpha_i\) are consistent with suspension and the structure maps of the prespectra, so \(\alpha \in [S, H\mathbb{Z}]\) generates \(\pi_0(H\mathbb{Z}) = \lim_{\to} \pi_i(S^i, K(Z, i))_0\), the limit taken with respect to the maps \([S^i, K(Z, i)]_0 \to [S^{i+1}, K(Z, i+1)]_0\), \(f \mapsto s_i[S^i \wedge f]\). Since \(H\mathbb{Z}\) represents cohomology, \(\alpha\) also corresponds to a generator of \(H^0(\mathbb{S}) \cong H^0(S^0) \cong \mathbb{Z}\).

We next want to show that \(\alpha\) induces an isomorphism between rational stable cohomotopy and rational cohomology. For this, recall that, if \(E\) is any spectrum, we can form the spectrum \(E_Q = E \wedge M(\mathbb{Q})\), the smash product of \(E\) with the Moore spectrum \[\text{of } \mathbb{Q}\). Also, recall that for a spectrum \(E\), \(E_*\) and \(E^*\) denote the respective (reduced) homology and cohomology theories associated to \(E\).

**Lemma 10.60.** \(\alpha\) induces an equivalence of spectra between \(S\mathbb{Q}\) and \((H\mathbb{Z})\mathbb{Q}\). Both of these spectra are equivalent to \(H\mathbb{Q}\), the rational Eilenberg-MacLane spectrum.

---

135Here we need to invoke Milnor’s result that a loop space of a CW complex has the homotopy type of a CW complex \[\text{Corollary 3].}

136For an abelian group \(G\), the Moore spectrum \(M(G)\) is characterized by the properties \(\pi_i(M(G)) = G = H_0(M(G))\), and \(H_i(M(G)) = 0\) for \(i \neq 0\). See \[Theorem-Definition II.4.32].
Proof. To prove the lemma, we will need the following fact from the theory of spectra, which we present here without proof. It is a special case of [571, Theorem II.5.4 and Corollary II.5.5].

Lemma 10.61. If \( E, F \) are spectra, then

1. there are natural isomorphisms

\[
\pi_*(E_Q) \cong \pi_*(E) \otimes \mathbb{Q}
\]

so that the composition \( \pi_*(E) \xrightarrow{i} \pi_*(E_Q) \cong \pi_*(E) \otimes \mathbb{Q} \) has the form \( x \rightarrow x \otimes 1 \),

2. there are isomorphisms

\[
t : (E_Q)^*(F) \cong E^*(F) \otimes \mathbb{Q}.
\]

that are natural in \( E \) and \( F \). Furthermore, there is a natural commutative triangle

\[
\begin{array}{ccc}
E^*(F) & \xrightarrow{j} & (E_Q)^*(F) \\
\downarrow{\ell} & & \downarrow{\chi} \\
E^*(F) \otimes \mathbb{Q} & & \\
\end{array}
\]

In the statement of the Lemma 10.61, the map \( j \) is induced by the map \( E = E \wedge \mathbb{S} \xrightarrow{id \wedge i} E \wedge M(\mathbb{Q}) = E_Q \) and \( i : \mathbb{S} \rightarrow M(\mathbb{Q}) \) represents \( 1 \in \pi_0(M(\mathbb{Q})) = \mathbb{Q} \). The map \( \ell \) takes \( x \in E^*(F) \) to \( x \otimes 1 \in E^*(F) \otimes \mathbb{Q} \). If \( X \) is a CW complex, Lemma 10.61 applies to \( E^*(X) \) by replacing \( X \) with its suspension spectrum \( X \); in fact, we recall that this is the definition of \( E^*(X) \) for \( X \) a space.

By the naturality of (47), there is a diagram

\[
\begin{array}{ccc}
\pi_*(\mathbb{S}_Q) & \xrightarrow{\cong} & \pi_*(\mathbb{S}) \otimes \mathbb{Q} \\
(\alpha \wedge id_{M(\mathbb{Q})})_* & & \alpha_* \otimes id_{\mathbb{Q}} \\
\pi_*(HZ)_Q & \xrightarrow{\cong} & \pi_*(HZ) \otimes \mathbb{Q}.
\end{array}
\]

Now, we quote Serre’s well-known result that \( \pi_0(\mathbb{S}_Q) = \pi_0^*(S^0) \otimes \mathbb{Q} \cong \mathbb{Q} \), while \( \pi_i(\mathbb{S}_Q) = \pi_i^*(S^0) \otimes \mathbb{Q} = 0 \) for \( i \neq 0 \). Of course for \( i < 0 \), all \( [S^{i+k}, S^k] \) are trivial by elementary algebraic topology, e.g. [53, Corollary 4.9], while \( [S^{i+k}, S^{i+k}] \cong \mathbb{Z} \) for \( i + k > 0 \), determined by the degree of the map of spheres [53, Corollary 4.25]; the result for \( i > 0 \) is less elementary and can be found, for example, in [52] or as a consequence of [97, Theorems 9.7.7 or 9.7.9] after stabilizing by a sufficient number of suspensions. On the other hand, directly from the definition of \( HZ \), the groups \( \pi_i(HZ) \) are trivial except for \( i = 0 \), in which case \( \pi_0(HZ) \cong \mathbb{Z} \). Since we have already chose \( \alpha : \mathbb{S} \rightarrow HZ \) to represent the generator of \( \pi_0(HZ) \), and since \( \pi_0(\mathbb{S}) \) is generated by the identity map \( \mathbb{S} \rightarrow \mathbb{S} \), \( \alpha_* : \pi_0(\mathbb{S}) \rightarrow \pi_0(HZ) \) takes the generator

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id$_S$ to the generator $\alpha$ of $\pi_0(H\mathbb{Z})$. So $\alpha_* \otimes \text{id}_S$ is an isomorphism in dimension 0, and so trivially an isomorphism in all dimensions. Therefore, $(\alpha \land \text{id}_{M(Q)})_* : \pi_*(S_Q) \to \pi_*((H\mathbb{Z})_Q)$ is an isomorphism in all dimensions. Since $S$ and $H\mathbb{Z}$ are bounded below (see [87 Definition II.4.4a]), $\alpha \land \text{id}_{M(Q)} : S_Q \to (H\mathbb{Z})_Q$ is an equivalence of spectra by [87 Corollary II.4.7iii].

Even more simply, we observe that since $\pi_0((H\mathbb{Z})_Q) \cong \mathbb{Q}$ and $\pi_i(H\mathbb{Z}) = 0$ for $i \neq 0$, the spectrum $(H\mathbb{Z})_Q$, and hence also $S_Q$, is equivalent to the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$ by the arguments in [87 Example II.3.24] □

Now we look at how elements of $\pi_*^m(X)$ yield elements of $H^m(X)$. Given a finite CW complex $X$ and an element $[f] \in \pi_*^m(X) = [X, S^m \land S]$, the composition of maps of spectra $X \xrightarrow{[f]} S^m \land S \xrightarrow{S^m\alpha} S^m \land H\mathbb{Z}$ yields an element of $[X, S^m \land H\mathbb{Z}] \cong \tilde{H}^m(X)$.

Next, consider the diagram

$$
\begin{array}{ccc}
\pi_*^m(X) \otimes \mathbb{Q} = S^m(X) \otimes \mathbb{Q} & \xrightarrow{\alpha_* \otimes \text{id}_Q} & H^m(X) \otimes \mathbb{Q} = (H\mathbb{Z})^m(X) \otimes \mathbb{Q} \\
&(\alpha \land \text{id}_{M(Q)})_* & \\
(S_Q)^m(X) & \xrightarrow{(\alpha \land \text{id}_{M(Q)})_*} & ((H\mathbb{Z})_Q)^m(X).
\end{array}
$$

The square commutes by the naturality of the isomorphism $\alpha_* \otimes \text{id}_Q$ with respect to the map $\alpha : S \to H\mathbb{Z}$. We have seen above that the map $\alpha \land \text{id}_{M(Q)} : S_Q \to (H\mathbb{Z})_Q$ is an equivalence, and hence $(\alpha \land \text{id}_{M(Q)})_* : (S_Q)^m(X) \to ((H\mathbb{Z})_Q)^m(X)$ is an isomorphism. It now follows from the diagram that we have an isomorphism $\alpha_* \otimes \text{id}_Q : \pi_*^m(X) \otimes \mathbb{Q} \to H^m(X) \otimes \mathbb{Q}$. This completes the proof of Theorem 10.48.

10.4.5 Unstable homotopy and cohomology

Theorem 10.48 provides an isomorphism $G = \alpha_* \otimes \text{id}_Q : \pi_*^m(X) \otimes \mathbb{Q} \to H^m(X) \otimes \mathbb{Q}$. It will be useful for us to extract from this isomorphisms $\pi^m+k(S^k \land X) \otimes \mathbb{Q} \to H^m(X) \otimes \mathbb{Q}$ for sufficiently large $k$. We can get this by looking “under the hood” of the spectrum-level result a bit.

Suppose $[f] \in \pi_*^m(X) = [X, S^m S]$. At the level of spaces, $[f]$ is represented by homotopy classes $[f_k] \in [S^k \land X, S^{m+k}]_0$, for large enough $k$, such that $[f_k] = [S^1 \land f_{k-1}]$. We have seen in Lemma 10.57 that the $[S^k \land X, S^{m+k}]_0$ are isomorphic to our $\pi^m+k(S^k X)$ for large enough $k$. Furthermore, by Lemma 10.45 and Lemma 10.57 suspension is an isomorphism for large enough $k$, so the sequence of groups $[S^k \land X, S^{m+k}]_0$ stabilizes. Similarly, we have defined $\alpha : S \to H\mathbb{Z}$ so that it is represented by maps $S^{m+k} \to K(\mathbb{Z}, m+k)$ that represent the generator of $\tilde{H}^{m+k}(S^{m+k})$. We will see that this sequence of groups is also stable with respect to suspension (this corresponds to the standard suspension isomorphism in cohomology). So, for each large enough $k$, we obtain a map $\pi^m+k(S^k \land X) \to \tilde{H}^{m+k}(S^k \land X)$ that assigns to $[f_k] \in \pi^m+k(S^k \land X)$ the pullback under $f_k$ of the generator of $\tilde{H}^{m+k}(S^{m+k})$. In particular, if $\dim(X)$ is in the stable range $\dim(X) < 2m - 1$, then we can take $k = 0$ so that an element
$[f_0] \in \pi^m(X)$ yields an element of $H^m(X)$ by pulling back the generator of $H^m(S^m)$. We claim that these maps are compatible as $k$ increases and induce in the limit the map $\alpha$ of Theorem 10.48. Since the groups all stabilize, this implies that we have compatible rational isomorphisms $\pi^{m+k}(S^k \land X) \otimes \mathbb{Q} \to H^{m+k}(S^k \land X) \otimes \mathbb{Q} \simeq H^m(X) \otimes \mathbb{Q}$.

To verify our claims, for each $k > 0$ we have a commutative diagram

$$
\begin{array}{ccc}
[S^{k+1} \land X, S^{k+1} \land S^m]_0 &=& [S^{k+1} \land X, S^m \land S^{m+k} + 1]_0 \\
S^1 \land &\cong& S^1 \land \\
[S^k \land X, S^k \land S^m]_0 &=& [S^k \land X, S^k \land K(\mathbb{Z}, m + k + 1)]_0 \\
\end{array}
$$

in which the composition along the top corresponds to the map induced by $\alpha_{m+k+1}$.

We will refer to the diagonal map of the diagram as the cohomology suspension map and denote it by $s_k$. As we saw in Lemma 10.57, the lefthand vertical map is equivalent, up to isomorphisms, to the suspension map in cohomotopy that we discussed in Section 10.4.3. This diagram yields a commutative square

$$
\begin{array}{ccc}
\pi^{m+k+1}(S^{k+1} \land X) &\cong& H^{m+k+1}(S^{k+1} \land X) \\
S^1 \land &\cong& s_k \\
\pi^{m+k}(S^k \land X) &\cong& H^{m+k}(S^k \land X). \\
\end{array}
$$

It is the direct limit over such diagrams that yields our map $\alpha : \pi^m_\ast(X) \to H^m(X)$. But we have already observed in Lemmas 10.44 and 10.45 that the sets $\pi^{m+k}(S^k \land X)$ become groups and stabilize once $k + \dim(X) \leq 2(m + k) - 2$, i.e. $k \geq \dim(X) - 2m + 2$. Furthermore, the righthand maps in these diagrams $H^{m+k}(S^k \land X) \to H^{m+k+1}(S^{k+1} \land X)$ are isomorphisms for all $k$. In this setting, it is perhaps easiest to see this using the following lemma:

**Lemma 10.62.** Let $\mu : Y \to \Omega Z$ be a map, and let $\nu : S^1 \land Y \to Z$ be its adjoint. Then the following diagram commutes:

$$
\begin{array}{ccc}
[B, Y]_0 &\cong& [S^1 \land B, S^1 \land Y]_0 \\
\mu_* &\cong& \nu_* \\
[B, \Omega Z]_0 &\cong& [S^1 \land B, Z]_0, \\
\end{array}
$$

where $A$ is the adjointness isomorphism.
Proof. Recall that if \( \phi : B \to \Omega Z \) is basepoint-preserving then for \( [t, x] \in S^1 \wedge B \), \( (A\phi)[t, x] = \phi(x)(t) \). So for \( f : B \to Y \), \( ((A\mu_+)(f))[t, x] = (A(\mu f))[t, x] = (\mu f(x))(t) \). On the other hand, \( ((\nu_+(S^1))(f))[t, x] = (\nu(\text{id}_\mathbb{S} \wedge f))[t, x] = \nu[t, f(x)] \). But now \( (\mu f(x))(t) = \nu[t, f(x)] \) by the definition of the adjointness relation.

In the case at hand, we plug into the lemma \( B = S^k \wedge X \), \( Y = K(\mathbb{Z}, m + k) \), \( Z = K(\mathbb{Z}, m + k + 1) \), \( \nu = s_{m+k} \), \( \mu = h_{m+k} \) to get

\[
[S^k \wedge X, K(\mathbb{Z}, m + k)]_0 \xrightarrow{h_{m+k}} [S^{k+1} \wedge X, S^1 \wedge K(\mathbb{Z}, m + k)]_0 \xrightarrow{s_{m+k}} [S^k \wedge X, \Omega K(\mathbb{Z}, m + k + 1)]_0 \xrightarrow{A} [S^{k+1} \wedge X, K(\mathbb{Z}, m + k + 1)]_0.
\]

The lefthand vertical map is an isomorphism because \( h_{m+k} \) is a homotopy equivalence by Lemma 10.59. So tracing right then down in the diagram gives an isomorphism that corresponds to the cohomology suspension map \( s_k : H^{m+k}(S^k \wedge X) \to H^{m+k+1}(S^{k+1} \wedge X) \).

Putting together the stability of the cohomotopy and cohomology groups under suspension with diagram (51), we see that the maps \( \alpha_{m+k} : \pi^{m+k}(S^k \wedge X) \to H^{m+k}(S^k \wedge X) \) stabilize once \( k \geq \dim(X) - 2m + 2 \). We have seen in Theorem 10.48 that the maps \( \pi^{m+k}(S^k \wedge X) \to H^{m+k}(S^k X) \) become isomorphisms in the direct limit after tensoring with \( \mathbb{Q} \). But since direct limits commute with tensor products, the stabilization of the diagram now implies the following corollary:

**Corollary 10.63.** If \( X \) is a finite CW complex and \( k \geq \dim(X) - 2m + 2 \), \( k \geq 0 \), the maps \( \alpha_{m+k} \otimes \text{id}_\mathbb{Q} : \pi^{m}(S^k \wedge X) \otimes \mathbb{Q} \to H^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \) are isomorphisms, so we have isomorphisms

\[
\pi^{m}(S^k \wedge X) \otimes \mathbb{Q} \xrightarrow{\alpha_{m+k} \otimes \text{id}_\mathbb{Q}} H^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \to H^m(X) \otimes \mathbb{Q},
\]

using the cohomology suspension isomorphism. Furthermore, the isomorphism \( \alpha_{m+k} \otimes \text{id}_\mathbb{Q} \) “commute with suspension” in the sense that, for \( k \) in this range, diagram (51) becomes a commutative square of isomorphisms after tensoring with \( \mathbb{Q} \). Additionally, these isomorphisms are all natural with respect to maps \( X \to Y \) provided also \( k \geq \dim(Y) - 2m + 2 \).

The last statement about naturality follows from the functoriality of all the constructions involved.

Putting together Corollary 10.63 with Lemma 10.46 and our other work in this section, we see that the \( L \)-class \( \mathcal{L}_m(X) \) of a closed oriented PL \( \mathbb{Q} \)-Witt space can be described via the map

\[
\mathbb{Q} \xleftarrow{F_{m+k} \otimes \text{id}_\mathbb{Q}} \pi^{m+k}(S^k X) \otimes \mathbb{Q} \cong \pi^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \xrightarrow{\alpha_{m+k} \otimes \text{id}_\mathbb{Q}} H^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \cong H^m(X) \otimes \mathbb{Q},
\]

(52)
where the arrow on the right is the iterated cohomology suspension isomorphism. $\mathcal{L}_n(X)$ corresponds under the universal coefficient theorem to the element of $\text{Hom}(H^m(X) \otimes \mathbb{Q}, \mathbb{Q})$ represented by the homomorphism, right-to-left, of (52).

Before moving on to the next section, we stop to observe that it is possible to achieve iterates of the cohomology suspension maps “all at once”.

Recall that $s_i : S^1 \wedge K(\mathbb{Z}, i) \to K(\mathbb{Z}, i + 1)$ is defined to be the unique-up-to-homotopy extension of the identity map $S^{i+1} \to S^{i+1}$ on the $i + 2$ skeleta of these spaces. Similarly, we can define a map $\sigma_{i,a} : S^a \wedge K(\mathbb{Z}, i) \to K(\mathbb{Z}, i + a)$ for any $a > 0$ as the unique-up-to-homotopy extension of the identity map $S^{i+a} \to S^{i+a}$ on the $i + a + 1$ skeleta of these spaces. We then have a composition, for any $k \geq 0$,

$$[Z, K(\mathbb{Z}, i)]_0 \overset{S^a}{\longrightarrow} [S^a \wedge Z, S^a \wedge K(\mathbb{Z}, i)]_0 \overset{\sigma_{m+k,a}}{\longrightarrow} [S^a \wedge Z, S^a \wedge K(\mathbb{Z}, i + a)]_0,$$

which we can think of as a cohomology suspension operator that raises the degree by $a$. We claim this is the same isomorphism as that obtained by iterating $a$ times the cohomology suspension $s$.

**Lemma 10.64.** The homomorphism $\tilde{H}^i(Z) \to \tilde{H}^{i+a}(S^a \wedge Z)$ corresponding to the composition (53) is the same as that obtained by iterating the cohomology suspension isomorphism $a$ times.

**Proof.** We can form the commutative diagram

The composition along the diagonals is the iteration of the cohomology suspension map.

On the other hand, the composition along the lefthand vertical maps takes the class $[f]$ to the class of $[\text{id}_{S^a} \wedge f]$, which is the class of the $a$th reduced suspension of the map $f$. The composition along the top of the diagram is induced by the suspensions of the various maps $s_{i+j}$, $0 \leq j \leq a - 1$. But on the $i + a + 1$ skeleta of the spaces involved, each of these is homeomorphic to the identity map $S^{i+a} \to S^{i+a}$. So this is also true of their composition, and we see that the composition is precisely $\sigma_{i,a}$, up to homotopy. So following clockwise around the diagram corresponds to the sequence of maps (53), which we then see is equivalent to the iterated cohomology suspension.

\[ \square \]
10.4.6 Suspensions versus products.

We turn to proving Proposition [10.53]. In the last section, we saw that the $L$-class $L_m(X)$ we have defined is the dual of the right-to-left composite homomorphism

$$Q \xleftarrow{F_m \otimes \mathrm{id}_Q} \pi^{m+k}(S^k X) \otimes \mathbb{Q} \cong \pi^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \xrightarrow{\alpha_{m+k} \otimes \mathrm{id}_Q \cong} H^{m+k}(S^k \wedge X) \otimes \mathbb{Q} \leftarrow H^m(X) \otimes \mathbb{Q}$$

for large enough $k$ (and that the construction is independent of $k$ CHECK!!).

The claim of Proposition [10.53] is that instead we can define $L_m(X)$ as the dual of the composition

$$Q \xleftarrow{F_m \otimes \mathrm{id}_Q} \pi^{m+k}(S^k \times X) \otimes \mathbb{Q} \xrightarrow{\alpha_{m+k} \otimes \mathrm{id}_Q \cong} H^{m+k}(S^k \times X) \otimes \mathbb{Q} \leftarrow H^m(S^k X) \otimes \mathbb{Q},$$

(55)

where here $\alpha_k \in H^k(S^k) = [S^k, K(Z, k)]_0$ is our standard generator and $\alpha_k \times$ is the cohomology cross product operator.

Throughout this section, all maps and homotopy classes are basepoint-preserving. As we have seen at the beginning of Section [10.4.3], this does not affect our cohomotopy group computations. In the case of ordinary (non-reduced) suspensions, we can assume that the basepoint of $SZ$ is $(1/2, z_0)$, where $z_0$ is a basepoint of $Z$, and this is consistent with the construction of the maps $F: \pi^m(X) \to Z$, $X$ a $\mathbb{Q}$-Witt space, of Lemma [10.46]. The proof will proceed via a collection of diagrams that we will put together at the end.

Let $q: S^k \times X \to S^k \wedge X = \frac{S^k X}{S^k X}$ and $p: S^k X \to S^k \wedge X$ denote the quotient maps.

These maps induce the following diagram:

$$\begin{array}{ccc}
\pi^{m+k}(S^k \times X) & \xrightarrow{\alpha_{m+k}} & H^{m+k}(S^k \times X) \\
q^* & & q^* \\
\pi^{m+k}(S^k \wedge X) & \xrightarrow{\alpha_{m+k}} & H^{m+k}(S^k \wedge X) \\
p^* & \cong & p^* \cong \\
\pi^{m+k}(S^k X) & \xrightarrow{\alpha_{m+k}} & H^{m+k}(S^k X).
\end{array}$$

(56)

That each $p^*$ is an isomorphism is due to $p$ being a homotopy equivalence. To see that the diagram commutes, recall that the map $\alpha_{m+k}$ corresponds to a generator of $H^{m+k}(S^{m+k}) \cong [S^{m+k}, K(Z, m+k)]_0$, where $K(Z, m+k)$ is the Eilenberg-MacLane space. So for any $f: X \to Y$, there is a commutative diagram

$$\begin{array}{ccc}
[X, S^{m+k}]_0 & \xrightarrow{\alpha} & [X, K(Z, m+k)]_0 \\
f^* & & f^* \\
[Y, S^{m+k}]_0 & \xrightarrow{\alpha} & [Y, K(Z, m+k)]_0.
\end{array}$$
as both directions around the square take an element \([g] \in [Y, S^{m+k}]_0\) to the element \([\alpha g f] \in [X, K(\mathbb{Z}, m + k)]_0\). Diagram \([56]\) is just two copies of such a commutative square, corresponding to the maps \(p\) and \(q\).

Our next goal is to show that all the groups on the left side of Diagram \([56]\) yield compatible maps to \(\mathbb{Q}\) by taking signatures of inverse images of points. One technical problem is that in Proposition \([10.43]\) we have defined such a map beginning with a PL stratified pseudomanifold that is \(\mathbb{Q}\)-Witt off a PL subspace of sufficiently small dimension. However, \(S^k \land X\) would not seem to have such a structure, at least not in any obvious way. To get around this difficulty, we utilize a little easy cohomotopy theory along with the geometric properties of suspensions, products, and smash products.

We begin by considering the space \(D^k \times X\), where \(D^k\) is the closed \(k\)-dimensional unit disk. We will see that all three spaces \(S^k \times X\), \(S^k \land X\), and \(S^k X\) are quotients of \(D^k \times X\). We will also want to keep our eyes on a certain subspace. Let \(x_0\) be a basepoint of \(X\) contained in the interior of the top stratum, and let \(U\) be a neighborhood of \(x_0\) that is PL homeomorphic to an open \(n\)-ball (where \(n = \dim(X)\)) so that the closure \(\bar{U}\) is PL homeomorphic to a closed \(n\)-ball and so that \((X - U, \bar{U} - U)\) is a PL \(\partial\)-stratified pseudomanifold with boundary. In particular, we can choose some Euclidean neighborhood of \(x_0\) and then let \(U\) be the image of the open unit disk. Let \(D^k\) be the subspace of \(D^k\) consisting of the closed disk with radius \(r\), \(0 < r < 1\). We notice that \(Y = D^k \times (X - U)\) is a closed PL subspace of \(D^k \times X\). Furthermore, denote the complement of the interior of \(D^k \times (X - U)\) by \(B = (D^k \times \bar{U}) \cup ((D^k - \bar{D}^k) \times X)\); the space \(B\) deformation retracts to \((D^k \times \{x_0\}) \cup (\partial(D^k \times X))\).

Now, we first notice that \(S^k \times X\) is the quotient of \(D^k \times X\) by the quotient map \(Q : D^k \times X \to S^k \times X\) that identifies \((\partial D^k) \times \{x\}\) to a point for each \(x \in X\). The quotient \(Q\) restricts to a homeomorphism off of \((\partial D^k) \times X\), in particular on \(Y\), and since \(B\) deformation retracts to \((D^k \times \{x_0\}) \cup (\partial(D^k \times X))\) in \(D^k \times X\), \(Q(B)\) deformation retracts to \((S^k \times \{x_0\}) \cup \{s_0\} \times X\) = \(S^k \land X\), where we let \(s_0\) be the quotient point in \(S^k = D^k / \sim\). If \(m + k > \max(k, n)\) (i.e. if \(k > n - m\) and \(m > 0\)) then any map \(S^k \land X \to S^{m+k}\) will be null-homotopic, so if also \(n + k < 2(m + k - 1)\) (i.e. if \(k > n - 2m + 2\)), then \(\pi^{m+k}(S^k \land X, Q(B)) \xrightarrow{j} \pi^{m+k}(S^k \times X)\) is an isomorphism by Lemma \([10.55]\). We also notice that since \(Q\) is a homeomorphism on \(Y\), there is an inclusion \((Y, \partial Y) \hookrightarrow (S^k \times X, Q(B))\) that is a homeomorphism onto \((S^k \times X - \text{int}(Q(B)), Q(B) - \text{int}(Q(B)))\). By Lemma \([10.56]\), this induces an isomorphism \(\pi^{m+k}(S^k \times X, Q(B)) \to \pi^{m+k}(Y, \partial Y)\). So, altogether, \(\pi^{m+k}(S^k \times X) \cong \pi^{m+k}(Y, \partial Y)\) if \(k > n - m\).

Next, \(S^k X\) is the quotient of \(D^k \times X\) by the quotient map \(P : D^k \times X \to S^k X\) that identifies \(\{y\} \times X\) to a point for each \(y \in \partial D^k\). The quotient \(P\) restricts to a homeomorphism off of \((\partial D^k) \times X\), in particular on \(Y\), and since \(B\) deformation retracts to \((D^k \times \{x_0\}) \cup (\partial(D^k \times X))\) in \(D^k \times X\), \(P(B)\) deformation retracts to \(S^k \{x_0\}\). Since \(S^k \{x_0\}\) is contractible, any map \(S^k \{x_0\} \to S^{m+k}\) will be null-homotopic, so if also \(n + k < 2(m + k - 1)\) (i.e. if \(k > n - 2m + 2\)), then \(\pi^{m+k}(S^k X, P(B)) \xrightarrow{i} \pi^{m+k}(S^k X)\) is an isomorphism by Lemma \([10.55]\). We also notice that since \(P\) is a homeomorphism on \(Y\), there is an inclusion \((Y, \partial Y) \hookrightarrow (S^k X, P(B))\) that is a homeomorphism onto \((S^k X - \text{int}(P(B)), P(B) - \text{int}(P(B)))\). By Lemma \([10.56]\) this induces an isomorphism \(\pi^{m+k}(S^k X, P(B)) \to \pi^{m+k}(Y, \partial Y)\). So, alto-
together, $\pi^{m+k}(S_kX) \cong \pi^{m+k}(Y, \partial Y)$ if $k > n - 2m + 2$.

Finally, the smash product $S^k \times X$ is a quotient of both $S^k \times X$ and $S^k X$. It is also a direct quotient of $D^k \times X$ by the quotient map $R : D^k \times X \to S^k X$ that identifies all of $(D^k \setminus \{x_0\}) \cup (\partial(D^k \times X))$ to a single point. The quotient $R$ restricts to a homeomorphism off of $(D^k \setminus \{x_0\}) \cup (\partial(D^k \times X))$, in particular on $Y$, and since $B$ deformation retracts to $(D^k \setminus \{x_0\}) \cup (\partial(D^k \times X))$ in $D^k \times X$, $P(B)$ deformation retracts to the base point, say $z_0$ of $S^k \times X$. Since $z_0$ is a single point, any map $\{z_0\} \to S^{m+k}$ will be null-homotopic, so if also $n + k < 2(m + k - 1)$ (i.e. if $k > n - 2m + 2$), then $\pi^{m+k}(S^k \times X, R(B))$ is an isomorphism by Lemma 10.55. We also notice that since $R$ is a homeomorphism on $Y$, there is an inclusion $(Y, \partial Y) \hookrightarrow (S^k \times X, R(B))$ that is a homeomorphism onto $(S^k \times X - \text{int}(R(B)), R(B) - \text{int}(R(B)))$. By Lemma 10.56, this induces an isomorphism $\pi^{m+k}(S^k \times X, R(B)) \to \pi^{m+k}(Y, \partial Y)$. So, altogether, $\pi^{m+k}(S^k \times X) \cong \pi^{m+k}(Y, \partial Y)$ if $k > n - 2m + 2$.

So now consider the diagram

$$
\begin{array}{ccc}
\pi^{m+k}(Y, \partial Y) & \overset{i^*}{\cong} & \pi^{m+k}(S^k \times X, Q(B)) \\
\downarrow & & \downarrow q^* \\
\pi^{m+k}(S^k \times X, R(B)) & \overset{j^*}{\cong} & \pi^{m+k}(S^k \times X) \\
\end{array}
$$

Here $F_Y$ is the map guaranteed by Proposition 10.43, noting that $(Y, \partial Y)$ is a PL pair with the interior of $Y$ a $\mathbb{Q}$-Witt space (using Lemma 10.12) and that $\pi^{m+k}(Y, \partial Y)$ consists of homotopy classes of maps of pairs $(Y, \partial Y) \to (S^{m+k}, \{s_0\})$, so the hypotheses of the proposition are satisfied. We claim that the composition along the top of the diagram from $\pi^{m+k}(S^k \times X)$ to $\mathbb{Z}$ agrees with $F_{S^k \times X}$, the map guaranteed by Proposition 10.43 on $S^k \times X$, and similarly that the composition along the bottom of the diagram agrees with $F_{S^k \times X}$, the map guaranteed by Proposition 10.43 on $S^k X$. Indeed, on the top, the isomorphism $j^*$ says that any map $f \in S^k \times X \to S^{m+k}$ is homotopic to a map $f_1$ taking $Q(B)$ to $s_0$. Since $Q(B)$ is a PL subspace of $S^k \times X$, we can also assume that $f_1$ is a PL map by the relative PL approximation theorem. The map $i^*$ is just the restriction of $f_1$ to $Y$; call this restriction $f_2$. Now, by Proposition 10.43 and the definition of the map $F$, $F_{S^k \times X}$ assigns to $[f] = [f_2] \in \pi^{m+k}(S^k \times X)$ the signature of the $\mathbb{Q}$-Witt space $f_2^{-1}(y)$, for a sufficiently generic $y \in S^m$. But, similarly, $F_Y$ assigns to $[f_2] \in \pi^{m+k}(Y, \partial Y)$ the signature of the $\mathbb{Q}$-Witt space $f_2^{-1}(y)$ for a sufficiently generic $y \in S^m$. But since $f_2$ is the restriction of $f_1$ to $Y$, and since $\partial Y$ maps to $s_0 \in S^{m+k}$, $f_2^{-1}(y) = f_2^{-1}(y) \subset Y$ for almost all $y \in S^m$. This argument shows that $F_{S^k \times X} = F_Y i^*(j^*)^{-1}$. A completely analogous argument shows that $F_{S^k X} = F_Y i^*(j^*)^{-1}$. 

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We obtain a commutative diagram

\[
\begin{array}{ccc}
\pi^{m+k}(S^k \times X) & \xrightarrow{p^*} & \pi^{m+k}(S^k) \\
\pi^{m+k}(S^k \wedge X) & \xrightarrow{\mu} & \pi^{m+k}(S^k) \\
\end{array}
\]

Lastly, we need a commutative diagram of the form

\[
\begin{array}{ccc}
H^{m+k}(S^k \times X) & \xrightarrow{q^*} & H^{m+k}(S^k \wedge X) \\
& \xrightarrow{\alpha \times} & H^m(X) \\
\end{array}
\]

where \(\sigma\) is the “all-at-once” cohomology suspension map. By Lemma 10.64 this is the same as iterating the one-degree suspension map \(k\) times.

For this, we use the fact that there are canonically (up to homotopy) defined maps \(\mu : K(Z, m) \wedge K(Z, n) \to K(Z, m+n)\) (see [53, Section 4.3]). These can be defined by assuming that the \(r+1\) skeleton of \(K(Z, r)\) is the sphere \(S^r\) (using the standard CW construction of \(K(Z, r)\) by iteratively attaching cells to \(S^r\) to kill higher homotopy) and then extending to \(K(Z, m) \wedge K(Z, n)\) the map \(S^m \wedge S^n \to K(Z, m+n)\). As \(S^m \wedge S^n\) is the \(m+n+1\) skeleton of \(K(Z, m) \wedge K(Z, n)\), such an extension exists and is unique up to homotopy by elementary obstruction theory, e.g. apply [53, Lemma 4.7] to \((K(Z, m) \wedge K(Z, n), S^m \wedge S^n)\) to show existence and to \((I \times (K(Z, m) \wedge K(Z, n))), ((\partial I) \times (K(Z, m) \wedge K(Z, n))) \cup (I \times (S^m \wedge S^n)))\) to get uniqueness up to homotopy.

Let \(\alpha_k \in [S^k, K(Z, k)]_0 = H^k(S^k)\) correspond to the chosen generator, and consider the diagram

\[
\begin{array}{ccc}
H^{m+k}(S^k \times X) & \xrightarrow{q^*} & H^{m+k}(S^k \wedge X) \\
& \xrightarrow{\alpha \times} & H^m(X) \\
\end{array}
\]

If \([\beta] \in [X, K(Z, m)]_0 = H^m(X)\), the first two horizontal maps take \([\beta]\) to \([\alpha \wedge \beta]\). Then \(\mu_*\) and \(q^*\) takes this to the class of the composite

\[
S^k \times X \xrightarrow{q} S^k \wedge X \xrightarrow{\alpha \wedge \beta} K(Z, k) \wedge K(Z, m) \xrightarrow{\mu} K(Z, m+n).
\]
But using that \((\alpha \wedge \beta)q\) is the same as the composite \(S^k \times X \xrightarrow{\alpha \times \beta} S^k \times X \xrightarrow{q} K(\mathbb{Z}, k) \wedge K(\mathbb{Z}, m)\) by the functoriality of the smash product, the composite (61) is precisely the homotopy theoretic definition of the cross product. See [53, Section 4.3], [2, Definition 7.2.9], or [137, 69, Section 22.3]. So the diagram commutes.

Putting together Diagrams (56), (57), and (58), we have a commutative diagram

If tensor this diagram with \(\mathbb{Q}\), then the maps \(\alpha_{m+k} \otimes \text{id}_{\mathbb{Q}}\) all become isomorphisms for large enough \(k\). Then tracing the diagram right to left around the top gives the homomorphism that is dual to the version of \(L^m(X)\) constructed via taking a product with a sphere, while tracing the diagram from the right, two steps left, down, then diagonally up gives our the homomorphism dual to our initial construction of \(L^m(X)\). Therefore, we have established that the two approaches are equivalent. This completes the proof of Proposition 10.53.

10.4.7 Relation with smooth L-classes

WARNING: THIS SECTION CONTAINS A SIGN DISCREPANCY THAT STILL NEEDS TO BE RECONCILED.

In this section, we will prove Lemma 10.52 which says that if \(M^n\) is a closed oriented smooth \(n\)-manifold, then for each \(m = n - 4k\), then \((-1)^m L_m(M)\) is the Poincaré dual of the rational Thom-Hirzebruch L-class \(L^k(X) \in H^{4k}(X; \mathbb{Q})\), as constructed, for example, in [74].

The following lemma will be needed for the computation.

Lemma 10.65. Let \(N^{n-m}\) be an oriented smooth closed submanifold of the closed oriented smooth manifold \(M^n\), \(n > m\), with trivial normal bundle \(\nu\), which we identify with a tubular neighborhood of \(N\) in \(M\). Identify \(S^m\) with the one point compactification of \(\mathbb{R}^m\) with \(z_0\) being the point at infinity, and let \(u \in H^m(S^m)\) be the generator such that \(u([S^m]) = 1\), where \([S^m]\) is the fundamental class. Suppose that \(f : M \to S^m\) is the map that takes each fiber \(\mathbb{R}^m\) of the normal bundle identically to \(\mathbb{R}^m = S^m - \{z_0\}\) and maps \(M - \nu\) to \(z_0\). Then \(f^*(u) \cap [M]\) equals the image of the fundamental class of \(N\) in \(H_{n-m}(M)\).

\[\begin{align*}
\pi^{m+k}(S^k \times X) & \xrightarrow{\alpha_{m+k}} H^{m+k}(S^k \times X) \\
\pi^{m+k}(S^k \wedge X) & \xrightarrow{\alpha_{m+k}} H^{m+k}(S^k \wedge X) \\
\pi^{m+k}(S^k X) & \xrightarrow{\alpha_{m+k}} H^{m+k}(S^k X).
\end{align*}\]

137 The latter reference, [69, Section 22.3], also provides an explanation for why this procedure agrees with the standard chain-theoretic cross product.
Proof of Lemma 10.65. We will use a nice diagram. Let $y \in S^{n-k}$ be the image of $N$ under $f$, so $N = f^{-1}(y)$. Also, let $\nu_1$ be the unit disk bundle in $\nu$. Let $i : (\nu_1, \nu_1 - N) \hookrightarrow (M, M - N)$ be the inclusion map. We have a diagram

\[
\begin{array}{ccc}
H^m(D^m, D^m - \{0\}) & \cong & H^m(S^m, S^m - \{y\}) \\
\downarrow f^* & & \downarrow f^* \\
H^m(M, M - N) & \xrightarrow{\cap [M]} & H^m(M) \\
\downarrow i^* & & \downarrow \ \\
H^m(\nu_1, \nu_1 - N) & \cong & H_{n-m}(M) \\
\downarrow \cap [\nu_1, \nu_1 - N] & & \\
H_{n-m}(\nu) & & \end{array}
\]

We will explain the notation and commutativity of the diagram and show that all the maps down the left hand side are isomorphisms. The path counterclockwise around the outside of the diagram from the top right to the bottom right will take $u$ to the fundamental class of $H_{n-m}(N)$. It then follows from the commutativity of the diagram that the image of the fundamental class $[N]$ in $H_{n-m}(M)$ is equal to the composite $f^*(u) \cap [M]$, which proves the claim.

The map $j^*$ is induced by the inclusion $(M, \emptyset) \hookrightarrow (M, M - N)$. The top square evidently commutes by naturality of cohomology, induced by the map $f : (M, M - N) \rightarrow (S^m, S^m - \{y\})$.

For the uppermost triangle, $[M]$ is the fundamental class of $M$. The class $[M, M - N] \in H_n(M, M - N)$ is the image of $[M]$ in the standard map from absolute to relative homology. Commutativity then follows from the naturality properties of the cap product. In particular, recall [23, Section VII.12] that for an excisive triad $(X; A_1, A_2)$, the general form of the cap product is

\[H^n(X, A_2) \otimes H_m(M, M - N) \cong H_{n-m}(M, M - N) \xrightarrow{\alpha'} H_{n-m}(M, A_1).\]

Furthermore, given a map of excisive triads $g : (X; A_1, A_2) \rightarrow (X'; A_1', A_2')$, the formula $g^*(\alpha' \cap \xi) = \alpha' \cap g^*(\xi)$ holds [23, Section VII.12.6]. But this is exactly the situation of the triangle if we let $g$ be the inclusion/identity $g : (M, \emptyset, \emptyset) \rightarrow (M, \emptyset, M - N)$, $\alpha' \in H^m(M, M - N)$, and $\xi = [M] \in H_n(M)$.

Similarly, for the middle triangle, we will use the inclusion $g : (\nu_1, \emptyset, \nu_1 - N) \hookrightarrow (M, \emptyset, M - N)$. Notice that inclusion induces an isomorphism by excision $H_n(\nu_1, \nu_1 - N) \rightarrow H_n(M, M - N)$.\]
N). We let \([\nu_1, \nu_1 - N]\) be the image of \([M, M - N]\) under this isomorphism. Then again this square commutes by naturality of the cap product \cite{23} Section VII.12.6] with \(\alpha' \in H^m(M, M - N)\) and \(\xi = [\nu_1, \nu_1 - N]\).

Finally, the bottom triangle is just a triangle of inclusions; the bottom inclusion is a homotopy equivalence.

It remains only to show that the counterclockwise composition takes \(u\) to the fundamental class of \(N\). Since \(\nu\) is the trivial bundle, \((\nu_1, \nu_1 - N) \cong (D^m \times N, D^m \times N - N) = (D^m, D^m - \{0\}) \times N\), and we can identify \(fi : (\nu_1, \nu_1 - N) \to (S^m, S^m - \{y\})\) with projection to the first factor followed by inclusion \((D^m, D^m - \{0\}) \hookrightarrow (S^m, S^m - \{y\})\). Let \(\mu\) be the image of \(u\) in \(H^m(\nu_1, \nu_1 - N)\), and let \(a\) be the isomorphic image of \(u\) in \(H^m(D^m, D^m - \{0\})\) under restriction. Then \(\mu = (fi)^*a\), which, since \(fi\) is a projection, corresponds to \(\mu = a \times 1 \in H^m((D^m, D^m - \{0\}) \times N)\); see \cite{23} Section VII.7.10.

If \([\nu_1, \nu_1 - N]\) is oriented consistent with the orientation of \(M\), which is oriented consistently with the product orientation on \(D^m \times N\), so \([\nu_1, \nu_1 - N] = [D^m, D^m - \{0\}] \times [N]\), we can compute explicitly

\[
\mu \cap [\nu_1, \nu_1 - N] = (a \times 1) \cap [D^m, D^m - \{0\}] \times [N] \\
= (a \cap [D^m, D^m - \{0\}]) \times (1 \cap [N]) \quad \text{see \cite{23} Section VII.12.17}
\]

where \(\xi_0\) is the canonical generator of \(H_0(D^m)\) since we assumed that \(u\) is the canonical generator of \(H^m(S^m)\). This class lives in \(H_0(D^m) \otimes H_{n-m}(N) \cong H_{n-m}(N)\), and it is the image of \([N]\) under the inclusion.

\[\text{Remark 10.66.} \text{Although we have chosen to provide this elementary direct argument, this lemma is really just an application of a special case of the Thom isomorphism theorem, up to being careful with signs: Recall \cite{74} Theorem 10.4 that if \(E\) is an oriented \(m\)-dimensional disk bundle over the space \(B\), then there is a unique class \(\tau \in H^m(E, E - B)\) that restricts on each fiber to the generator of \(H^m(D^m, D^m - \{0\})\) compatible with the orientation of \(E\). If \(B = M^n\) is a closed oriented \(n\)-manifold, then \(E\) is a closed oriented \(\partial\)-manifold, and if \(M\) is connected (which we can assume without loss of generality by restricting our argument to connected components), then homotopy equivalence and Poincaré duality yield an isomorphism \(H^m(E, E - B) \cong H^m(E, \partial E) \cong H_n(E) \cong H_n(M) \cong \mathbb{Z}\). Therefore, \(H^m(E, E - B) \cong \mathbb{Z}\), and \(\tau\) must be a generator, or else its restriction to \(H^m(D^m, D^m - \{0\})\) could not be a generator.}

In the lemma, the class \(\mu\) is the Thom class of the normal bundle to \(N\), so \(\mu \cap [\nu_1, \nu_1 - N]\) is a generator of \(H_n(\nu_1) \cong H_n(N)\), which is represented (up to sign) by the image of the fundamental class of \(N\). But then we can argue

\[
i[N] = i(\mu \cap [\nu_1, \nu_1 - N]) \\
= i(i^* f^*(a) \cap [\nu_1, \nu_1 - N]) \\
= f^*(a) \cap i[\nu_1, \nu_1 - N] \\
= f^*(a) \cap [M, M - N] \\
= f^*(u) \cap [M],
\]

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where the last equality follows from the commutativity of the top of the diagram in the proof of the lemma.

Now we prove Lemma 10.52 under the assumption that $m > \frac{n+1}{2}$ and $n - m \equiv 0 \mod 4$, i.e. that $L^{\frac{n-m}{4}} \cap [M] = \mathcal{L}_m$, where $L^{\frac{n-m}{4}}$ is the rational Thom-Hirzebruch $L$-class in $H^{n-m}(M; \mathbb{Q})$. Notice that since the dimension of $L^{\frac{n-m}{4}}$ is even, the Poincaré map is simply the cap product with the fundamental class, with no sign. To show that $L^{\frac{n-m}{4}} \cap [M] = \mathcal{L}_m$, we will show that for any $\beta \in H^m(M; \mathbb{Q}) = \text{Hom}(H_m(M; \mathbb{Q}), \mathbb{Q})$ we have $\beta(L^{\frac{n-m}{4}} \cap [M]) = \beta(\mathcal{L}_m)$.

On the one hand, by our work in Section 10.4.5, $\beta = rf^*(u)$ for some rational number $r$, where $f : M \to S^m$ and, by abuse of notation, $u \in H^m(S^m; \mathbb{Q})$ is the image of the generator of Lemma 10.65 under the change of coefficients. Then

$$\beta(\mathcal{L}_m) = (-1)^m ev(\mathcal{L}_m)(\beta) = (-1)^m r \sigma([f^{-1}(y)])$$

see Remark 10.51

On the other hand, let $a : H_0(M; \mathbb{Q}) \to \mathbb{Q}$ be the augmentation map, we compute

$$\beta(L^{\frac{n-m}{4}} \cap [M]) = a(\beta \cap (L^{\frac{n-m}{4}} \cap [M])) \quad \text{see [77, Theorem 66.3]}$$

$$= a((\beta \cup L^{\frac{n-m}{4}}) \cap [M])$$

$$= a((L^{\frac{n-m}{4}} \cup \beta) \cap [M]) \quad \text{since } L^{\frac{n-m}{4}} \text{ has even degree}$$

$$= a((L^{\frac{n-m}{4}} \cup rf^*(u)) \cap [M])$$

$$= ra(L^{\frac{n-m}{4}} \cap (f^*(u) \cap [M]))$$

$$= ra(L^{\frac{n-m}{4}}(f^*(u) \cap [M]))$$

$$= ra(L^{\frac{n-m}{4}}([f^{-1}(y)]))$$

$$= r \sigma([f^{-1}(y)])$$

by Lemma 10.65

This completes the proof of Proposition 10.52 for $m > \frac{n+1}{2}$.

Now we turn to the cases where $m \leq \frac{n+1}{2}$. For this it will help to rework our definition a little bit for these dimension ranges. Recall that when $m > \frac{n+1}{2}$, we define $\mathcal{L}_m(X)$ as the inverse under the evaluation isomorphism $ev : H_m(X; \mathbb{Q}) \to \text{Hom}(H_m(X; \mathbb{Q}), \mathbb{Q})$ of a certain homomorphism, the one obtained by identifying $H^m(X; \mathbb{Q})$ with $\pi^m(X) \otimes \mathbb{Q}$ and utilizing the map $\pi^m(X) \to \mathbb{Z}$ (tensored with $\mathbb{Q}$) that sends a map $X \to \mathbb{Q}$ to the signature of a point inverse. Let us denote this homomorphism $\Psi : H^m(X; \mathbb{Q}) \to \mathbb{Q}$. When $m \leq \frac{n+1}{2}$, we have shown that we can define $\mathcal{L}_m(X)$ as the inverse under $ev$ of the homomorphism $H^m(X; \mathbb{Q}) \to \mathbb{Q}$ that first takes the cross product $\alpha_k \times : H^m(X; \mathbb{Q}) \to H^m(S^k \times X; \mathbb{Q})$, for large enough $k$ and $\alpha_k$ a canonically chosen generator of $H^k(S^k)$, and then applies $\Psi : H^{m+k}(S^k \times X; \mathbb{Q}) \to \mathbb{Q}$. So consider then the following diagram:

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By definition, \( \mathcal{L}_m(X) \) is the image of \( \Psi \in \text{Hom}(H^{k+m}(S^k \times X; \mathbb{Q}), \mathbb{Q}) \) under the composition \( (ev)^{-1}(\alpha \times *) \). Let us check the commutativity of the diagram. Suppose \( \xi \in H_m(X; \mathbb{Q}) \) and \( \beta \in H^m(X; \mathbb{Q}) \). Then

\[
(\alpha \times *)ev([S^k] \times (\xi)(\beta)) = ev([S^k] \times \xi)(\alpha \times \beta)
\]

\[
= (-1)^{m+k}(\alpha \times \beta) \cdot ([S^k] \times \xi)
\]

\[
= (-1)^{m+k}(-1)^{nk} \alpha([S^k]) \beta(\xi)
\]

\[
= (-1)^{m+k} \beta(\xi)
\]

\[
= (-1)^{k+mk} ev(\xi)(\beta).
\]

Hence the diagram commutes up to the sign \((-1)^{k+mk}\), and we can conclude from this that \([S^k] \times \mathcal{L}_m(X) \cong (-1)^{k+mk} \mathcal{L}_{m+k}(S^k \times X)\).

On the other hand, we can obtain information about how the classical \(L\)-classes behave under products using the multiplicativity of the Pontryagin classes. Let \( \pi_M : M \times S^k \to M \) and \( \pi_S : M \times S^k \to S^k \) be the projections; then if \( \tau_M, \tau_S, \) and \( \tau_{M \times S^k} \) are the respective tangent bundles, then \( \tau_{S^k \times M} = \pi_S^* \tau_{S^k} \oplus \pi_M^* \tau_M \). Then it follows from [74, Theorem 15.3] that \( p(\tau_{S^k \times M}) = p(\pi_S^* \tau_{S^k} \circ p(\pi_M^* \tau_M)) \), where \( p \) is the total Pontrjagin class. By naturality, \( p(\pi_S^* \tau_{S^k}) = \pi_S^* (p(\tau_S^k)) \). But clearly \( p^i(\tau_S^k) = 0 \) if \( i \neq 0, k/4 \). Furthermore, even if \( k/4 \) is an integer, \( \langle p^k(\tau_S), [S^k] \rangle \) is a Pontrjagin number and so must be 0 as \( S^k \) is the boundary of the \( k + 1 \) ball [74, pages 185-186]. But then we must have \( p^i(\tau_S^k) = 0 \). It follows that \( p(\tau_S^k) = 1 \in H^0(S^k; \mathbb{Q}) \), and so \( \pi_S^*(p(\tau_S^k)) = 0 \). Therefore, \( p(\tau_{S^k \times M}) = p(\pi_S^* \tau_{S^k}) = \pi_M^* p(\tau_M) \). Since the \( L \) classes \( L^i \) are polynomials in the Pontrjagin classes, it follows that \( L^i(S^k \times M) = \pi_M^* L^i(M) \).

Now, we want to show that \( L^{n-m} \cap [M] = \mathcal{L}_m(M) \). Consider the following diagram

\[
H_{k+m}(S^k \times X; \mathbb{Q}) \xrightarrow{ev} \text{Hom}(H^{k+m}(S^k \times X; \mathbb{Q}), \mathbb{Q})
\]

\[
H_m(M; \mathbb{Q}) \xrightarrow{[S^k] \times} H_{k+m}(S^k \times M).
\]

If \( \beta \in H^{n-m}(M; \mathbb{Q}) \), then \( \pi^*_M \beta = 1 \cup (\pi^*_M \beta) = \pi_S^*(1) \cup (\pi^*_M \beta) = 1 \times \beta \), using standard properties of the cup and cross products, e.g. [97, Corollary 5.6.14]. So

\[
\pi^*_M \beta \cap [S^k \times M] = (1 \times \beta) \cap ([S^k] \times [M])
\]

\[
= (-1)^{(n-m)k}[S^k] \times (\beta \cap [M])
\]

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by [23 Section VII.12.17]. In the case at hand, we wish to consider $L$-classes, so we can assume $n - m \equiv 0 \mod 4$ so that the diagram commutes in this case.

So, now let $\beta = L^{\frac{n-m}{4}}(M) \in H^{n-m}(M; \mathbb{Q})$. From the commutativity of the diagram and our work above, we have

$$[S^k] \times (L^{\frac{n-m}{4}}(M) \cap [M]) = (\pi_M^*(L^{\frac{n-m}{4}}(M))) \cap [S^k \times M]$$

$$= L^{\frac{n-m}{4}}(S^k \times M) \cap [S^k \times M]$$

$$= (-1)^{m+k} \mathcal{L}_{m+k}(S^k \times M)$$

But the map $[S^k] \times : H_m(M; \mathbb{Q}) \to H_{m+k}(S^k \times M; \mathbb{Q})$ is injective, by the Künneth theorem, and we have seen $[S^k] \times \mathcal{L}_m(M) = (-1)^{k+m} \mathcal{L}_{m+k}(S^k \times M)$. Therefore,

$$L^{\frac{n-m}{4}}(M) \cap [M] = (-1)^{m+k}(-1)^{k+m} \mathcal{L}_m(M) = (-1)^{m+k} \mathcal{L}_m(M).$$

### 10.5 Bordism computation? K-homology? (TENTATIVE)

Yet to be written

### 10.6 Perverse signatures (PLANNED)

Yet to be written

### 11 Differential forms approach (TENTATIVE)

Yet to be written; see [11 88 89].

#### 11.1 Unfoldable spaces (TENTATIVE)

Yet to be written; see [11 88 89].

#### 11.2 Perverse differential forms (TENTATIVE)

Yet to be written; see [11 88 89].

#### 11.3 de Rham theorem (TENTATIVE)

Yet to be written; see [11 88 89].
12 Appendix

We here briefly compile some some useful facts from homological algebra that are often useful, though many other bits of homological algebra are developed elsewhere in the main body of the text closer to where they are needed. This section can be thought to comprise “background results”, including facts that are well known but not always easy to pinpoint in the literature or facts that are easy to pinpoint with certain restrictions, e.g. $R$ being a PID, but for which we need slight generalizations, e.g. $R$ being a Dedekind domain. We also recall some standard definitions for the reader’s convenience. Some of this material overlaps with material treated in more detail elsewhere in the text.

We always assume that our rings $R$ are commutative with unity.

12.1 Koszul sign conventions

Following the Koszul sign conventions, as elaborated in [23, Section VI.10], if $C_\ast$ is a chain complex of $R$-modules with boundary operator $\partial$, then the dual complex $\mathrm{Hom}^\ast(C_\ast, R) = \mathrm{Hom}(C_\ast, R)$ consists of modules $\mathrm{Hom}^i(C_\ast, R) = \mathrm{Hom}(C_i, R)$ with coboundary operator $d$ such that, for $\alpha \in \mathrm{Hom}(C_i, R)$ and $x \in C_{i+1}$,

$$(d\alpha)(x) = (-1)^{|\alpha|+1}\alpha(\partial x).$$

Notice that this is one of the rare times when the Koszul convention does not yield a sign corresponding to the product of the degrees of the elements being interchanged.

We also remind the reader that the Koszul conventions decree that

$$\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{|x|}x \otimes \partial y.$$ 

In fact, if $f$ and $g$ are any collections of homomorphisms of chain complexes, which includes the possibility that $f$ and $g$ shift degrees, then

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|}f(x) \otimes g(y).$$

Here, the degree $|g|$ of $g$ is $n$ if $g : C_\ast \to D_\ast$ with $g(C_i) \subset D_{i+n}$. In particular, $|\partial| = -1$. There will be no conflict between the notation $|g|$ and the the absolute value symbols, as these degree symbols only appear of consequence when working with exponents of $-1$, in which case the sign of $|g|$ is irrelevant.

12.2 Shifts and mapping cones

Here, we briefly review some facts about shifting of complexes and algebraic mapping cones. The shift notation is useful for describing the mapping cones, which are used below in the proof of Lemma 12.7. More about these objects is contained in the text proper in Section 7.3.5.
Shifts. It is useful to be able to reindex chain complexes. Most often, one sees this done for cohomologically indexed complexes, in which case, if $D^*$ is such a chain complex (of $R$-modules) with (co)boundary map $d_{D^*}$, the shifted complex $D[k]^*$ is defined such that $(D[k])^i = D^{k+i}$ and $d_{D[k]^*} = (-1)^k d_{D^*}$; see [103] Section III.3. For homological indexing, using the standard bijection between cohomologically indexed complexes and homologically indexed complexes such that $C_i = C^{-i}$, we see that if $C_*$ is a homologically indexed complex, then we should have

$$(C[k]_*)_i = (C[k]^*)_i = C^{k-i} = C_{i-k}.$$ 

In other words, given $C_*$, we should let $C[k]_*$ be the chain complex with $(C[k]*)_i = C_{i-k}$ and $d_{C[k]_*} = (-1)^k d_{C_*}$. 

Taking $k = 1$ and $C_*$ a chain complex (of $R$-modules), we obtain $C[1]_*$ with $(C[1]*)_i = C_{i-1}$ and $d_{C[1]_*} = -d_{C_*}$. Let us define $s : C[1]_* 	o C_*$ so that it takes $C[1]_*$ identically to the corresponding module $C_{i-1}$. Then from the definition of the boundary map on $C[1]_*$, we see that $sd_{C[1]_*} = -d_{C_*} s$, which is consistent with $s$ being a (homological) degree $-1$ chain map. Unfortunately, it is easy to get confused when attempting to consider $C_{i-1}$ and $(C[1]*)_i$ as two separate entities, especially when working with individual elements. Indeed, it is very tempting to write things like $s(x) = x$, which is right and wrong; right because $C_{i-1}$ and $(C[1]*)_i$ are identical modules, but wrong because they live in different chain complexes. In an attempt to mitigate the confusion, if $x$ is an element of $C_{i-1}$, we will write $\bar{x}$ for the corresponding element of $(C[1]*)_i$, i.e. $s(\bar{x}) = x$. Of course we could also write $s^{-1}(x)$ instead of $\bar{x}$, but it is convenient to have both notations available.

Algebraic mapping cones. Suppose $f : C_* \hookrightarrow D_*$ is a chain map of chain complexes. We let $E^f_*$ (or simply $E_*$ if there’s no ambiguity) denote the algebraic mapping cone of $f : C_* \to D_*$. Recall [103] Section III.3 that that this means that $E_i = D_i \oplus C_{i-1} = D_i \oplus C[1]_i$ and $\partial(x, y) = (f(y) + \partial x, -\partial y)$. This is a chain complex, as

$$\partial(\partial(x, y)) = \partial(f(y) + \partial x, -\partial y) = (-f(\partial y) + \partial f(y) + \partial(\partial x), \partial(\partial y)) = 0.$$ 

This construction mimics algebraically the chain complex one obtains from a topological mapping cone; the shift can be thought of as being due to taking the cone on the domain space, and so increasing the dimension by one. Lemma 7.62 in Section 7.3.5 should provide a more technically convincing version of this claim. We should also note that there are alternative conventions for the algebraic mapping cone construction; see, for example, [105] Section 1.5.

There is a short exact sequence of chain complexes

$$0 \to D_* \to E_* \to C[1]_* \to 0 \quad (63)$$ 

with $e(x) = (x, 0)$ and $b(x, y) = \bar{y}$, where $\bar{y}$ uses our notation for shifted elements from just above. It is immediate to verify that $e$ and $b$ are both chain maps. Notice, however, that it is not true that $E_* = D_* \oplus C[1]_*$ as chain complexes, since the boundary map of $E_*$ is not a direct sum of the boundary maps of the summands.
12.3 Some basic results of homological algebra

This section contains a variety of results, mostly concerning chain homotopies and chain homotopy equivalences.

Lemma 12.1. Suppose $f, g : C_* \to D_*$ are chain homotopic (degree zero) chain maps. Then $f^*, g^* : \text{Hom}(D_*, R) \to \text{Hom}(C_*, R)$ are chain homotopic.

Proof. Suppose $D : C_* \to D_{*+1}$ is the chain homotopy so that $\partial D + D \partial = f - g$. To be more precise, we can let $D_i$ be the map $C_i \to D_{i+1}$ induced by $D$. So if $x \in C_i$, we have $\partial D_i(x) + D_{i-1} \partial(x) = f(x) - g(x)$.

As $D$ is not a chain map (of any degree), let us avoid thinking too much about conventions and simply choose to denote by $D_i^*$ the homomorphism $\text{Hom}(D_i, R) \to \text{Hom}(C_{i-1}, R)$ such that for $\alpha \in \text{Hom}(D_i, R)$ and $x \in C_{i-1}$, we have $(D_{i-1}^*(\alpha))(x) = \alpha(D_{i-1}(x))$. Define $D_i = (-1)^{i+1}D_i^*$ for all $i$.

Now, suppose $x \in C_i$ and $\alpha \in \text{Hom}(D_i, R)$. Then we compute

\[(D_i d + dD_i^{-1})(\alpha)(x) = (D_i d \alpha + dD_i^{-1} \alpha)(x)\]
\[= (-1)^{i+1}(D_i^*(\partial \alpha))(x) + (-1)^i d((D_i^* \alpha)(x))\]
\[= (-1)^{i+1}(\partial \alpha)(D_i(x)) + (-1)^i(-1)^{i-1}(D_i^*(\alpha))(\partial x)\]
\[= (-1)^{i+1}(-1)^{i-1}\alpha(\partial D_i(x)) + \alpha(D_{i-1}(\partial(x))\]
\[= \alpha(\partial D_i + D_{i-1} \partial)x\]
\[= \alpha((f - g)(x))\]
\[= \alpha(f(x)) - \alpha(g(x))\]
\[= f^*(\alpha)(x) - g^*(\alpha)(x)\]
\[= (f^*(\alpha) - g^*(\alpha))(x)\]
\[= ((f^* - g^*)(\alpha))(x).\]

Varying over all $x \in C_i$, we see that $(D_i d + dD_i^{-1})(\alpha) = (f^* - g^*)(\alpha)$, and so varying over $\alpha \in \text{Hom}(D_i, R)$, we see that $D_i d + dD_i^{-1} = f^* - g^*$. This provides the desired chain homotopy.

Corollary 12.2. If $f : C_* \to D_*$ is a chain homotopy equivalence, then so is $f^* : \text{Hom}(D_*, R) \to \text{Hom}(C_*, R)$.

Proof. By definition, there is a $g : D_* \to C_*$ such that $gf$ and $fg$ are chain homotopic to the respective identity maps $\text{id}_C$ and $\text{id}_D$. But then, by the lemma, $(gf)^* = f^* g^*$ and $(fg)^* = g^* f^*$ are chain homotopic to the respective maps $\text{id}_C^*$ and $\text{id}_D^*$. But the dual of an identity map is an identity map, by functoriality. So $f^*$ and $g^*$ are chain homotopy inverses.

Lemma 12.3. Suppose $f, g : C_* \to D_*$ are chain homotopic (degree zero) chain maps. Then $f \otimes \text{id}, g \otimes \text{id} : C_* \otimes E_* \to D_* \otimes E_*$ are chain homotopic and $\text{id} \otimes f, \text{id} \otimes g : E_* \otimes C_* \to E_* \otimes D_*$ are chain homotopic.

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Proof. Suppose \( D : C_* \to D_{*+1} \) is the chain homotopy so that \( \partial D + D \partial = f - g \). Let \( x \in C_i \) and \( y \in D_j \). Recall that \( C_i \otimes D_j \) is generated by elements of the form \( x \otimes y \) (though not every element has this form). Then

\[
((f \otimes \mathrm{id}) - (g \otimes \mathrm{id}))(x \otimes y) = (f \otimes \mathrm{id})(x \otimes y) - (g \otimes \mathrm{id})(x \otimes y) \\
= f(x) \otimes y - g(x) \otimes y \\
= ((f(x) - g(x))) \otimes y \\
= ((f - g)(x)) \otimes y \\
= (\partial D + D \partial)(x) \otimes y \\
= (\partial D(x)) \otimes y + (D \partial(x)) \otimes y \\
= \partial D(x) \otimes y - (-1)^{|i+1|}(D(x)) \otimes \partial y + (D \otimes \mathrm{id})((\partial x) \otimes y) \\
= \partial(\partial D \otimes \mathrm{id})(x \otimes y) - (-1)^{|i+1|}(D \otimes \mathrm{id})(x \otimes \partial y) \\
+ (D \otimes \mathrm{id})(\partial(x \otimes y) - (-1)^i x \otimes \partial y) \\
= \partial(\partial D \otimes \mathrm{id})(x \otimes y) + (D \otimes \mathrm{id})\partial(x \otimes y) \\
= (\partial D \otimes \mathrm{id} + (D \otimes \mathrm{id})\partial)(x \otimes y).
\]

So \( D \otimes \mathrm{id} \) provides a chain homotopy from \( f \otimes \mathrm{id} \) to \( g \otimes \mathrm{id} \).

Similarly,

\[
((\mathrm{id} \otimes f) - (\mathrm{id} \otimes g))(x \otimes y) = (\mathrm{id} \otimes f)(x \otimes y) - (\mathrm{id} \otimes g)(x \otimes y) \\
= x \otimes f(y) - x \otimes g(y) \\
= x \otimes (f(y) - g(y)) \\
= x \otimes ((f - g)(y)) \\
= x \otimes (\partial D + D \partial)(y) \\
= x \otimes (\partial D(y)) + x \otimes (D \partial(y)) \\
= (-1)^i \partial(x \otimes D(y)) - (-1)^j(\partial x) \otimes D(y) + (-1)^i(\mathrm{id} \otimes D)(x \otimes \partial y) \\
= (-1)^{i+j} \partial(\mathrm{id} \otimes D)(x \otimes y) - (-1)^{i+j-1}(\mathrm{id} \otimes D)((\partial x) \otimes y) \\
+ (-1)^i(\mathrm{id} \otimes D)((-1)^i \partial(x \otimes y) - (-1)^i(\partial x) \otimes y) \\
= \partial(\mathrm{id} \otimes D)(x \otimes y) + (\mathrm{id} \otimes D)((\partial x) \otimes y) \\
+ (\mathrm{id} \otimes D)(\partial(x \otimes y) - (\mathrm{id} \otimes D)((\partial x) \otimes y) \\
= \partial(\mathrm{id} \otimes D)(x \otimes y) + (\mathrm{id} \otimes D)(\partial(x \otimes y) \\
= (\partial(\mathrm{id} \otimes D) + (\mathrm{id} \otimes D)\partial)(x \otimes y).
\]

Thus \( (\mathrm{id} \otimes f) - (\mathrm{id} \otimes g) = \partial(\mathrm{id} \otimes D) + (\mathrm{id} \otimes D)\partial \), so \( \mathrm{id} \otimes D \) provides a chain homotopy from \( \mathrm{id} \otimes f \) to \( \mathrm{id} \otimes g \).

Corollary 12.4. If \( f, g : C_* \to D_* \) and \( h, k : E_* \to F_* \) are chain homotopic chain maps, then \( f \otimes h, g \otimes k : C_* \otimes E_* \to D_* \otimes F_* \) are chain homotopic chain maps.

Proof. We can write \( f \otimes h \) as the composition \( f \otimes h = (f \otimes \mathrm{id})(\mathrm{id} \otimes h) \). Recall that if \( a, b \) are chain homotopic chain maps, then precomposing or postcomposing \( a \) and \( b \) with the same
chain map yields chain homotopic compositions, i.e. \( a \sim b \) implies \( cad \sim cbd \) for chain maps \( c,d \) (by an easy argument). Applying this fact and the lemma, \((f \otimes \text{id})(\text{id} \otimes k)\) is chain homotopic to \((f \otimes \text{id})(\text{id} \otimes h)\), which is chain homotopic to \((g \otimes \text{id})(\text{id} \otimes h) = g \otimes h\).

**Corollary 12.5.** If \( f : C_* \to D_* \) and \( h : E_* \to F_* \) are chain homotopy equivalences, then so is \( f \otimes h : C_* \otimes E_* \to D_* \otimes F_* \).

**Proof.** Let \( g : D_* \to C_* \) and \( k : F_* \to E_* \) be chain homotopy inverses to \( f \) and \( h \). Then, applying the preceding corollary, \((f \otimes h)(g \otimes k) = fg \otimes hk \sim \text{id}_D \otimes \text{id}_F = \text{id}_{D \otimes F}\) and \((g \otimes k)(f \otimes h) = gf \otimes kh \sim \text{id}_C \otimes \text{id}_E = \text{id}_{C \otimes E}\).

**Lemma 12.6.** Suppose \( 0 \to A \to B \to C \to 0 \) is a short exact sequence of \( R \)-modules and that \( C \) is projective. Then the sequence splits and, in particular, \( B \cong A \oplus C \).

**Proof.** We have a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{s} & C \\
\downarrow & & \downarrow \\
B & \to & C,
\end{array}
\]

and, by the definition of projective, the map \( s \) exists, making the diagram commute. The map \( s \) provides a splitting of the exact sequence by standard homological algebra. See, e.g., [77, Theorem 23.1] or [53, Section 2.2].

**Lemma 12.7.** Let \( f : C_* \to D_* \) be a chain map of complexes of projective \( R \)-modules such that \( C_i = D_i = 0 \) if \( i < 0 \). If \( f \) induces isomorphisms in homology of all dimensions, then \( f \) is a chain homotopy equivalence.

**Remark 12.8.** This lemma is a basic fact of algebraic topology, but it is surprisingly hard to pin down a clean textbook reference. Munkres proves this in [77, Theorem 46.2] under the additional assumption that \( C \) and \( D \) are chain complexes of free modules. Hilton and Stammbach leave this fact as [55, Exercise IV.4.2]. This fact also follows immediately from more elaborate theorems, such as the fact that if a category \( \mathcal{A} \) has enough projectives, then the derived category of cochain complexes \( D^{-} (\mathcal{A}) \) is equivalent to the homotopy category \( K^{-}(\mathcal{P}) \), whose objects are bounded above cochain complexes of projectives; see [105, Theorem 10.4.8] and note that the bounded below condition of the lemma becomes a bounded above condition when thinking of complexes as cochain complexes. This last fact is a somewhat big hammer that is not really necessary for this lemma. Really, all of the major pieces of the proof are provided between [77] and [55], but we will provide the details here for the convenience of the reader, beginning by assuming [55, Theorem IV.4.1], which is proven in detail.
Proof of Lemma \ref{lemma12.7}. First, recall \cite[Theorem IV.4.1]{Munkres}, which states that if $A_*$ is a complex of projectives and $B_*$ is acyclic (assuming $A_i = B_i = 0$ if $i < 0$), then for every homomorphism $\phi_0 : H_0(A_*) \to H_0(B_*)$, there is a chain map $\phi : A_* \to B_*$ inducing $\phi_0$ and, furthermore, any two such chain maps are chain homotopic.

Now, consider the algebraic mapping cone of $E_*$ of $f : C_* \to D_*$ and, in particular, the long exact sequence associated to \eqref{longexactsequence63}; also see Section \ref{section12.2} for the notation used here. The zig-zag construction, taking $(0, y)$ as a preimage of a cycle $\bar{y} \in C[1]_*$, shows that $\partial_*(\bar{y})$ is represented by $f(y)$ so that $\partial_* = \phi q$. Thus, given our hypotheses, $\partial_*$ is an isomorphism and, from the long exact sequence, $E_*$ is acyclic. It therefore follows from \cite[Theorem IV.4.1]{Munkres} that the maps $\text{id} : E_* \to E_*$ and the zero map $0 : E_* \to E_*$ are chain homotopic, as they both induce the 0 map $H_0(E) \to H_0(E)$. By definition, this means that there is a map $D : E_* \to E_{*+1}$ such that $\partial D + D \partial = \text{id}$. From here, we follow the proof from Munkres \cite[Theorem 46.2]{Munkres} and define $\theta, \psi, \lambda, \mu$ such that if $x \in D_i$ and $y \in (C_*[1])_i = C_{i-1}$, then

$$
D(x, 0) = (\theta(x), \psi(x)) \in E_i = D_{i+1} \oplus C_i \\
D(0, y) = (\lambda(y), \mu(y)) \in E_i = D_{i+1} \oplus C_i.
$$

Now, in the words of Munkres, “we compute like mad!”

$$
\partial D(x, 0) = D(\partial x, 0) = (\theta(\partial x), \psi(\partial x)) \\
\partial D(0, y) = D(0, -\partial y) + D(f(y), 0) = (-\lambda(\partial y), -\mu(\partial y)) + (\theta(f(y)), \psi(f(y))) \\
\partial D(0, y) = \partial(\lambda(y), \mu(y)) = (f(\mu(y)) + \partial(\lambda(y)), -\partial(\mu(y)).
$$

Since $\partial D + D \partial = \text{id}$, adding the first two equations implies that

$$
(\theta(\partial x), \psi(\partial x)) + (f(\psi(x)) + \partial \theta(x), -\partial(\psi(x))) = (\theta(\partial x) + f(\psi(x)) + \partial \theta(x), \psi(\partial x) - \partial(\psi(x))) = (x, 0).
$$

Therefore, $\psi(\partial x) = \partial \psi(x)$, so $\psi$ is a chain map, and $\theta(\partial x) + f(\psi(x)) + \partial \theta(x) = x$, which implies that $\theta$ is a chain homotopy between $f \psi$ and the identity. Adding the last two equations gives

$$
(-\lambda(\partial y), -\mu(\partial y)) + (\theta(f(y)), \psi(f(y))) + (f(\mu(y)) + \partial(\lambda(y)), -\partial(\mu(y)) = \\
(-\lambda(\partial y) + \theta(f(y)) + f(\mu(y)) + \partial(\lambda(y)), -\mu(\partial y) + \psi(f(y)) - \partial(\mu(y)) = (0, y).
$$

In the second coordinate, we obtain $y = -\mu(\partial y) + \psi(f(y)) - \partial \mu(y)$, which shows that $\mu$ provides a chain homotopy between the identity and $\psi f$. The first coordinate tells us that $-\lambda(\partial y) + \theta(f(y)) + f(\mu(y)) + \partial(\lambda(y)) = 0$. I have no idea what this represents, but anyway, we have seen that $\psi f$ and $f \psi$ are each homotopic to the identity, and so $f$ is a chain homotopy equivalence.  

\footnote{The terminology in \cite{Munkres} is actually “projective complex”, by which is meant that each $C_i$ is projective; see \cite[page 126]{Munkres}. However, there is some danger of confusing “projective complex” with the requirement that $C_*$ be projective as an object in the category of chain complexes, which is not the same thing. Thus we use the more precise terminology.} 

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12.4 Some facts about Dedekind domains

In several sections later in the book, it is necessary to work with Dedekind domains. A Dedekind domain is an integral domain with the property that every submodule of a projective $R$-module is projective. This is essentially taken as the definition of a Dedekind domain in Cartan-Eilenberg [17] Section VII.5 and Theorem I.5.4]. Exercise 20 to Section 4 of Chapter VII of Bourbaki’s Commutative Algebra [10] shows that this property can be derived from other, alternative, defining properties of Dedekind domains. A short literature search reveals that there are a very large number of equivalent definitions for Dedekind domains! Principal ideal domains and fields are Dedekind domains. It is also true that any torsion-free module over a Dedekind domain is flat. In fact, this is true more generally of Prüfer domains, which satisfy the weaker property that that submodules of finitely-generated projective modules are projective; a module over a Prüfer domain is torsion free if and only if it is flat [63, Proposition 4.20].

Lemma 12.9. Let $R$ be a Dedekind domain. Suppose $E_\ast$ is a complex of $R$-modules with $E_i = 0$ for $i < 0$. Then there is a complex $C_\ast$ of projective $R$-modules that is chain homotopy equivalent to $E_\ast$. Furthermore, if $H_i(E_\ast)$ is finitely generated for all $i$, then we can choose $C_i$ finitely generated for all $i$.

Proof. The construction of the chain complex $C_\ast$ and a homotopy equivalence $f : C_\ast \to E_\ast$ that induces homology isomorphisms proceeds exactly as in the proof of [55, Proposition V.2.4], replacing the free modules in that discussion with projective ones. As in [55], the proof is the consequence of two slightly more general lemmas we will prove below:

Lemma 12.10. Let $R$ be a Dedekind domain. Suppose $E_\ast$ is a complex of $R$-modules. Then there is a complex $C_\ast$ of projective $R$-modules such that $H_i(C_\ast) \cong H_i(E_\ast)$ for all $i$. Furthermore, if $H_i(E_\ast)$ is finitely generated for all $i$, then we can choose $C_i$ finitely generated for all $i$, and if $H_i(E_\ast) = 0$ for all $i < 0$, then we can choose $C_i = 0$ for all $i < 0$.

Lemma 12.11. Let $R$ be a Dedekind domain. Suppose $E_\ast$ is a complex of $R$-modules and that $C_\ast$ is a complex of projective free modules. Suppose $g_i : H_i(C_\ast) \to H_i(D_\ast)$ is any collection of homomorphisms. Then there is a chain map $f : C_\ast \to D_\ast$ that induces the $g_i$.

It follows from the Lemma 12.10 that, given $E_\ast$ as in the statement of Lemma 12.9, there is a chain complex $C_\ast$ with the desired characteristics ($C_i$ projective, $C_i = 0$ for $i < 0$, $C_i$ finitely generated if $H_i(E_\ast)$ is, and $H_i(C_\ast) \cong H_i(E_\ast)$), and it follows from Lemma 12.11 that there is a chain map $f : C_\ast \to E_\ast$ inducing the isomorphisms $H_i(C_\ast) \to H_i(E_\ast)$. The map $f$ is then a chain homotopy equivalence by Lemma 12.7.

Proof of Lemma 12.10. Let $F_\ast$ to be a free $R$-module that surjects onto $H_p(E_\ast)$ by $q_p : F_\ast \to H_p(E_\ast)$. Then, as $R$ is Dedekind, $R_p = \ker(q_p)$ is projective, so $0 \to R_p \to F_\ast \to H_p(E_\ast)$ is a projective resolution. If $H_p(E_\ast)$ is finitely generated, we can choose $F_\ast$ to be finitely generated. It follows that $R_p$ will also be finally generated. This uses that Dedekind domains are Noetherian [10, Theorem VII.2.2.1], so submodules of finitely-generated modules are finitely generated (see [64, Section X.1]).

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Now, let $C_p = F_p \oplus R_{p-1}$, with $\partial(x, y) = (y, 0)$. If $H_p(E) = 0$ for $p < 0$, we can have $F_p = 0$ for $p < 0$ and so $C_p = 0$ for $p < 0$. This provides the desired chain complex $C_*$ with $H_p(C_*) \cong H_p(E_*)$, as the cycle modules of $C_*$ are the $F_p \oplus 0 \cong F_p$ and the boundary map corresponds to the natural embedding of $R_p$ into $F_p$.

Proof of Lemma 12.11. Let $Z_p, B_p$ be the cycle and boundary submodules of $C_p$, and let $\bar{Z}_p, \bar{B}_p$ be the corresponding submodules for $E_*$. As the $C_p$ are projective and $B_p, Z_p \subset C_p$, the modules $B_p$ and $Z_p$ are projective, and so, using Lemma 12.6, we have splittings $C_p \cong Z_p \oplus Y_p$, where $Y_p$ maps isomorphically onto $B_p$ by $\partial$. In fact, if $(z, y) \in Z_p \oplus Y_p$ is the general element, then $\partial(z, y) = \partial y \in B_{p-1} \subset Z_{p-1} \subset C_{p-1}$, corresponding to $(\partial y, 0)$ in the decomposition $Z_{p-1} \oplus Y_{p-1}$. Consider now the diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & B_p & \rightarrow & Z_p & \rightarrow & H_p(E_*) & \rightarrow & 0 \\
\downarrow \phi & & \downarrow \theta & & \downarrow g_p & \\
0 & \rightarrow & \bar{B}_p & \rightarrow & \bar{Z}_p & \rightarrow & H_p(C_*) & \rightarrow & 0.
\end{array}
$$

(64)

As $Z_p$ is projective, the definition of projectivity yields the desired dashed map $\theta : Z_p \rightarrow \bar{Z}_p$, and this, in turn, induces the map of kernels $\phi : B_p \rightarrow \bar{B}_p$. We also have a diagram

$$
\begin{array}{ccc}
Y_p & \xrightarrow{\partial} & B_{p-1} \\
\downarrow \psi & & \downarrow \phi \\
E_p & \xrightarrow{\partial} & B_{p-1},
\end{array}
$$

and, once again, projectiveness of $Y_p \cong B_{p-1}$ yields the map $\psi$.

We now define $f : C_p \cong Z_p \oplus Y_p \rightarrow E_p$ by $f(z, y) = \theta(z) + \psi(y)$. To see that $f$ is a chain map, we check

$$
f\partial(z, y) = f(\partial y, 0) = \theta(\partial y) + \psi(0) = \theta(\partial y) = \phi(\partial y) = \partial \psi(y) = \partial \theta(z) + \partial \psi(y) = \partial (\theta(z) + \psi(y)) = \partial f(z, y).
$$

We have used here that $\theta(z)$ is a cycle in $E_*$. It follows from Diagram (64) that $f$ induces the desired isomorphism on homology. □
References


[41] Stegan Geschke, *Convex open subsets of \( \mathbb{R}^n \) are homeomorphic to \( n \)-dimensional open balls*, http://relaunch.hcm.uni-bonn.de/fileadmin/geschke/papers/ConvexOpen.pdf.


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Several diagrams in this book were typeset using the TeX commutative diagrams package by Paul Taylor.