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The Eighth annual Workshop in Geometric Topology was held at the University of Wisconsin-Milwaukee on June 13-15, 1991. The participants were:

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These proceedings contain a summary of the three one-hour talks given by principal speaker Andrew Casson as well as writeups of many of the 25 minute presentations given by other participants. Also included is a list of questions posed at the problem session held at the conclusion of the conference. We wish to acknowledge financial support received from the National Science Foundation (DMS-9101515) and the University of Wisconsin-Milwaukee which contributed greatly to the success of this conference.

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Convergence Groups and Seifert Fibered 3-manifolds

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February 24, 1993

1 Introduction

In this paper, we sketch a proof of the following result, often referred to as the Seifert Fiber Space Conjecture.

Theorem 1.1 *If M is a compact, orientable and irreducible 3-manifold, and $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is Seifert fibered.*

David Gabai has given a different proof; however, both proofs make use of the same reduction (due to Geoffrey Mess and Peter Scott) to a problem about discrete convergence groups acting on the circle. Our proof also uses the solution of the Seifert Fiber Space Conjecture for Haken manifolds, due to Waldhausen and Gordon-Heil.

The notion of a *discrete convergence group* acting on the circle is due to Gehring and Martin [GM]; here we give an equivalent definition due to Tukia [T].

Let T denote the set of ordered triples (x, y, z) of distinct points ocurring in positive order on S^1 (with respect to a fixed orientation on S^1). Observe that T is homeomorphic to $S^1 \times \mathbb{R}^2$. The group $Homeo_+(S^1)$ of all orientation-preserving homeomorphisms of the circle acts naturally on T. A subgroup $\Gamma \subset Homeo_+(S^1)$ is a *discrete convergence group* if the projection

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$T \to T/\Gamma$ is a covering map. An equivalent condition is that the action of Γ on T be fixed-point free and properly discontinuous. Mess [M] and Scott [S] showed that Theorem 1.1 follows from:

Theorem 1.2 *If $\Gamma \subset Homeo_{+}(S^1)$ is a discrete convergence group, then T/Γ is Seifert fibered.*

Tukia [T] proved Theorem 1.2 in the case that T/Γ is non-compact and also in the case that Γ contains no torsion element of order greater than 3. We will therefore assume that $M = T/\Gamma$ is compact, and that Γ contains an element of finite order $m \geq 3$.

The basic strategy is to construct a simple closed curve $C \subset M = T/\Gamma$ such that $\pi_1(M \setminus C)$ has an infinite cyclic normal subgroup. The complement N of a product neighborhood of C in M has torus boundary (which is easily seen to be incompressible), and it follows from the work of Waldhausen, Gordon and Heil referred to above that N is Seifert fibered. This implies easily that M is Seifert fibered. The key to the proof that $\pi_1(M \setminus C)$ has an infinite cyclic normal subgroup is to show that $\widetilde{C}$, the pre-image of C, is a product link in the solid torus T. This depends on a somewhat delicate characterization of product links in $S^1 \times \mathbb{R}^2$. By a (pure) possibly-infinite closed braid, we mean a disjoint, locally finite union L of circles in $S^1 \times \mathbb{R}^2$ such that each component of L is the graph of a function from S^1 to $\mathbb{R}^2$.

Theorem 1.3 *Let L be a possibly-infinite closed braid in $S^1 \times \mathbb{R}^2$. Suppose that with respect to the projection $\pi : S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$, each pair of components of L has exactly tw crossings, both with positive sign. Then L is is a product link in $S^1 \times \mathbb{R}^2$.*

2 Constructing a Fiber

In this section we show how Theorem 1.2 follows from Theorem 1.3. By Tukia's result [T], it suffices to prove Theorem 1.2 under the assumption that Γ contains an element e of finite order $m \geq 3$. Some power of e has rotation number $1/m$, so we may choose e to be this element.

The following technical result contains all the information about convergence groups that we shall need.

Lemma 2.1 *Let Γ and e be as above, and suppose that for some $f, g \in \Gamma$, $f = geg^{-1} \neq e$. Then $e^{-1}f$ has exactly two fixed points a and b, and the points a, $e(a)$, b, $e(b)$ are distinct and positively ordered on S^1.*

Sketch of proof: Since $e^{-1}f$ is a non-identity element of Γ, it has no fixed points in T, so it has at most 2 fixed points in S^1.

Observe that a is a fixed point of $e^{-1}f$ if and only if $e(a) = f(a)$. If $e^{-1}f$ has no fixed points on S^1, then for all $x \in S^1$ the triple $(x, e(x), f(x))$ has the same orientation, which may be assumed to be positive (by interchanging e and f if necessary). By applying this inequality repeatedly, we see that e has smaller rotation number than $f = gfg^{-1}$, a contradiction.

A similar argument applies if the fixed points of $e^{-1}f$ are all contained in a fundamental interval for the action of e on S^1. The conclusion of the lemma follows quickly.

For every element $f \in \Gamma$ that is conjugate to e, there is a well-defined embedding $C_f : S^1 \to T$ given by $C_f(x) = (x, f(x), f^{-1}(x))$. Let C be the image of $C_e(S^1)$ in M. Let

$$\tilde{C} = \bigcup_{g \in \Gamma} C_{geg^{-1}}(S^1) \subset T$$

be the full pre-image of C in T. The set S of pairs of distinct points on S^1 is homeomorphic to $S^1 \times \mathbb{R}$. Let $\pi : T \to S$ be the projection given by $(x, y, z) \mapsto (x, y)$.

The following two results are easy corollaries of Lemma 2.1

Corollary 2.2 *C is a simple closed curve C in $M = T/\Gamma$.*

Corollary 2.3 *If f is conjugate to e is Γ, and $f \neq e$, then $\pi(C_e(S^1))$ intersects $\pi(C_f(S^1))$ in exactly two points, both positive crossings of $C_e(S^1)$ and $C_f(S^1)$ with respect to the planar projection π.*

Proof of Theorem 1.2: Observe that $\pi_1(M \setminus C)$ contains $\pi_1(T \setminus \tilde{C})$ as a normal subgroup (with quotient group isomorphic to Γ). Using Corollary 2.2, one sees that $\tilde{C}$ is a possibly-infinite closed braid in $S^1 \times \mathbb{R}^2$. By Theorem 1.3 and Corollary 2.3, $\pi_1(T \setminus \tilde{C})$ is the direct product of a free group of infinite rank and an infinite cyclic group Z. In particular, Z is the center of $\pi_1(T \setminus \widetilde{C})$, and is therefore a normal subgroup of $\pi_1(M \setminus C)$.

Even though the action of Γ on S^1 is not given to be smooth, it is not hard to show that C has a product neighborhood in M. The complement N of such a neighborhood is Haken, so by Waldhausen's theorem [W] it is Seifert fibered. Now M can be reconstructed as the union of M and a solid torus, and the generic fiber is homotopically non-trivial in $\pi_1(M)$ (in fact it generates the given infinite cyclic normal subgroup). It follows that M is Seifert fibered.

3 Possibly-Infinite Closed Braids

Let L be a pure closed braid in $S^1 \times \mathbb{R}^2$ such that, after projection by π, each pair of components of L has exactly two crossings, both positive. First we consider the case when L has only finitely many components; this will serve as a model for the general case.

Theorem 3.1 *If L has only finitely many components, then it is a full twist; in particular, it is homeomorphic to a product.*

Choose coordinates (x, y, z) on $S^1 \times \mathbb{R}^2$ such that the projection

$$\pi : S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$$

is given by $(x, y, z) \mapsto (x, y)$. The terms "left" and "right" will refer to the x-direction; terms like "up" and "above" will refer to the y-coordinate, while terms such as "in front of" or "forward" will refer to the z-coordinate. (This proof was developed on a vertical blackboard.)

Lemma 3.2 *Suppose that some component S of L has a point V that is above every other component. (See Figure 1.) Then, while fixing all other components of L, S can be isotoped so that its projection $\pi(S)$ assumes any desired position provided V stays fixed, the x-coordinates of points in S stay fixed, and the y-coordinates of points in S are not increased.*

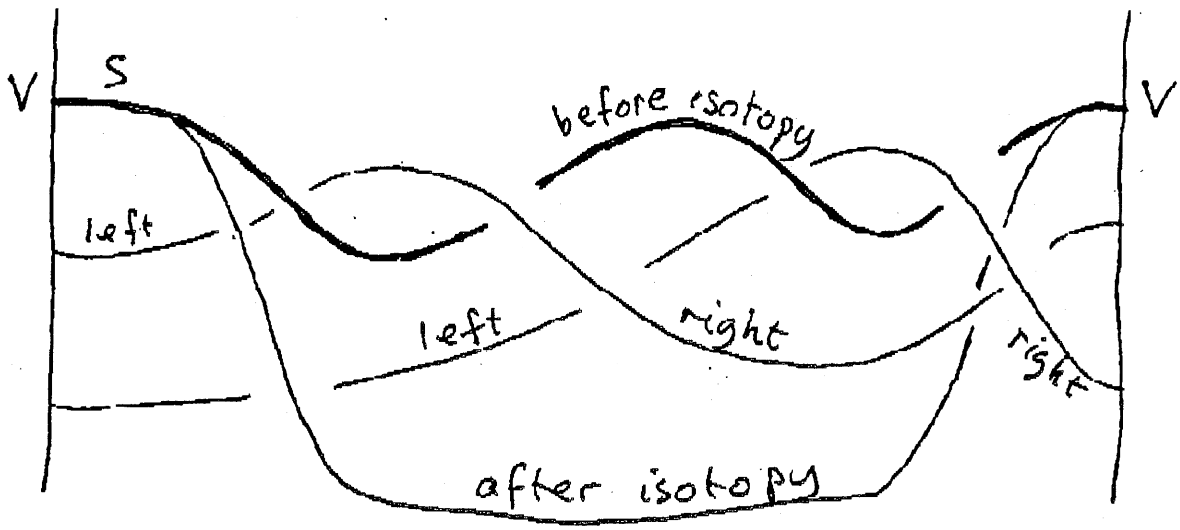

Figure 1:

Proof: Cut $S^1 \times \mathbb{R}^2$ along the yz-plane through V. For each component of $L \setminus S$, let the "left segment" (respectively right segment) denote the part of the component to the left (right) of any crossing with S. (See Figure 1.) Note that left and right segments are precisely the parts of components that are below S. No left segment can cross in front of a right segment, otherwise (since each segment must cross S), the two segments would have to cross at least three times. (See Figure 2.) Thus there is an isotopy I that pulls forward all left segments and pushes backwards all right segments. After applying I, S can be isotoped to the desired position (provided V stays fixed). Transforming this isotopy by I^{-1} yields an isotopy lowering S as required and fixing $L \setminus S$.

It should be noted that the property that each pair of components crosses exactly twice is not necessarily preserved at all stages of the isotopy, even if the final configuration has this property.

Proof of Theorem 3.1: Use Lemma 3.2 to isotop some component S of L so that it winds around all of the other components (in the sense described below; see Figure 3). The remaining components form a closed braid with

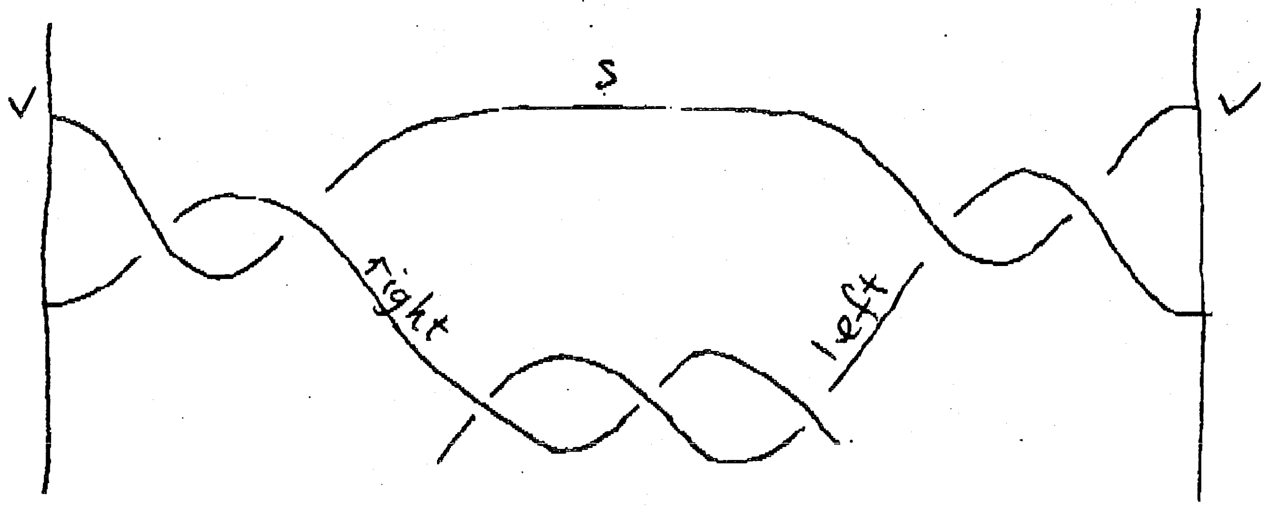

Figure 2:

one fewer component; by induction they form a full twist. It follows that L is a full twist.

Now let L be any possibly-infinite closed braid satisfying the hypotheses of Theorem 1.3. Let K be a union of finitely many components of L. We will say that another component S *winds around* K if S crosses in front of every component of K exactly once, and then crosses behind every component of K exactly once. (See Figure 3.) The closed arc of S between the first and last crossing in front of K is called the *descending arc* of K, and the *ascending arc* is defined similarly. Observe that if all components of $L \setminus K$ wound around K, then these components could simply be isotoped away from K, leaving an isolated finite closed braid. The following lemma follows from the assumption of local finiteness.

Lemma 3.3 *Let $Z \subset Y$ be the union of finitely many components of a closed braid L satisfying the hypotheses of Theorem 1.3. Then all but finitely many components of L wind around K.*

We will say a component S *tangles* K *from above* if some point of S is above all components of K, but no point of S is below all components of K.

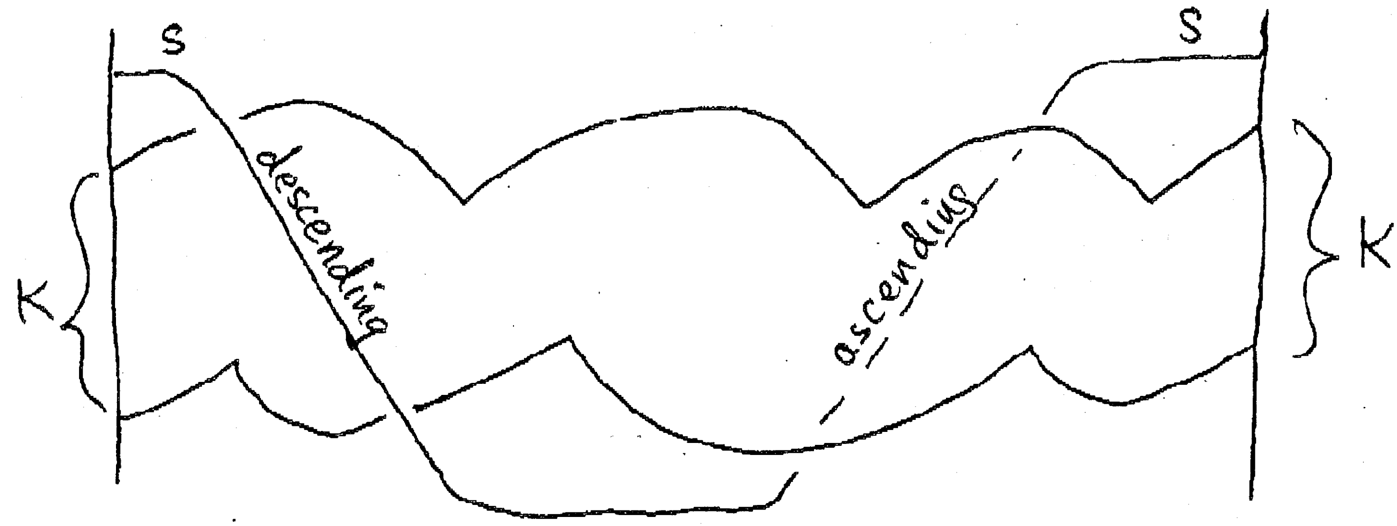

Figure 3:

If the part of S that is above all components of K is a single interval, then S *neatly tangles* K *from above* (See Figure 4.) We define the term *tangling from below* similarly.

Lemma 3.4 *Any finite union K_0 of components of L is contained in a finite union K of components of L such that every component of $L \setminus K$ either winds around K or neatly tangles K from above.*

Sketch of proof: Let K_1 be the union of K_0 and the finitely many components of L that tangle K_0 from above. Let K_2 be the union of K_1 and the finitely many components of L that tangle K_1 from below. It is not hard to verify that K, the union of K_2 and the finitely many components of L that neither wind around nor tangle K_2, has the required properties.

Let S be a component of L that neatly tangles K from above. let A be the first crossing of S with K, let D be the last crossing of S with K; then the closed arc $[A, D]$ will be called the *tangled arc* of S. The definitions of ascending and descending arcs given above for winding components generalize naturally to neatly tangling components as follows. Let B be the first point

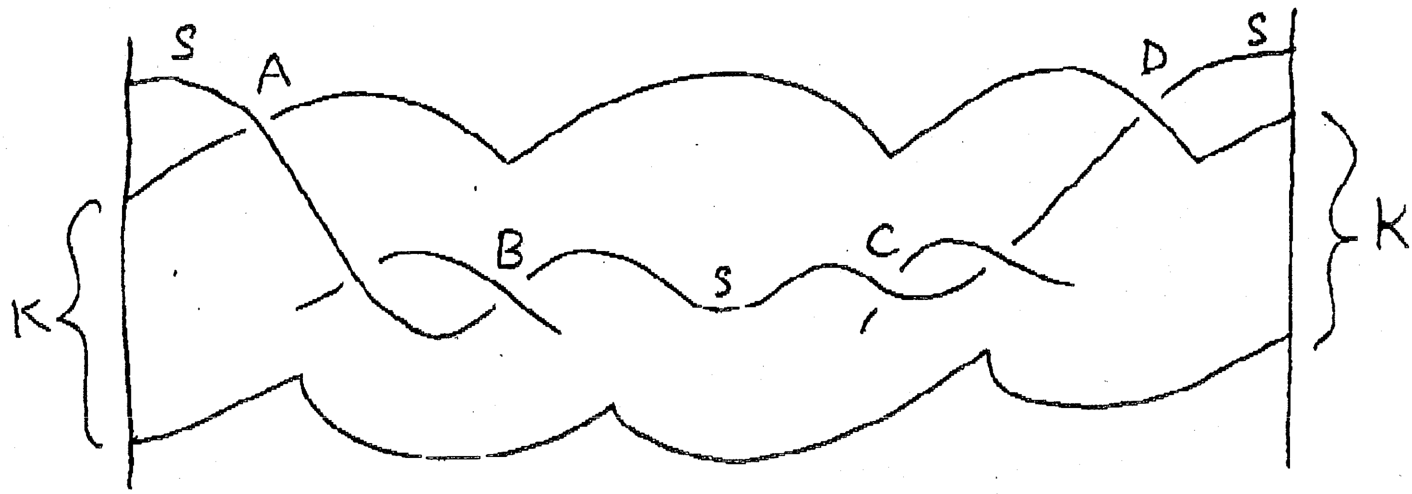

Figure 4:

where S crosses a component of K for the second (and last) time, and let C be the last point where S crosses a component of K for the first time. The half-open arcs $[A, B)$ and $(C, D]$ will be called the *descending* and *ascending* arcs of S respectively. (See Figure 4.) The following property is easily verified.

Lemma 3.5 *Suppose that each of S and T either winds around k or tangles Z from above. If the descending arc of S crosses behind a point Q of T, then Q is on the descending arc of T. Similarly, if the ascending arc of S crosses in front of a point Q of T, then Q is on the ascending arc of T.*

An embedded curve γ in $S^1 \times \mathbb{R}^2$ is a *connecting path* if γ is contained in some yz-plane, the endpoints of γ lie on components of L, and the interior of γ is disjoint from L. A *connecting annulus* is an annulus in $S^1 \times \mathbb{R}^2$ whose intersection with any yz-plane is a connecting path. We will say that a connecting path can be *spun* if it is contained in some connecting annulus. Clearly, any connecting path in a product braid can be spun.

Lemma 3.6 *Let L be a closed braid satisfying the hypotheses of Theorem 1.3, and let $P \subset S^1 \times \mathbb{R}^2$ be a yz-plane. Let α, β, and γ be connecting paths in P*

with common endpoints but disjoint interiors, and let D be the disc bounded by $\alpha \cup \beta$. If α and β can be spun and D contains γ, then γ can be spun.

Sketch of proof: The union of the connecting annuli containing α, β can be modified to form an embedded torus T that is disjoint from L and bounds a solid torus V containing γ. By Theorem 3.1, $L \cap V$ is a product braid, so any connecting path in V, in particular γ, can be spun.

Theorem 3.7 *Let L be a closed braid satisfying the hypotheses of Theorem 1.3. Then every connecting path γ that is parallel to the z-axis can be spun.*

Sketch of proof: Let S_1 and S_2 be the components containing the endpoints of γ. Set $K_0 = S_1 \cup S_2$ and construct K as in Lemma 3.4. We can pull the descending arcs of all winding components forward from K, and push the corresponding ascending arcs backwards, while only moving the ascending and descending portions of the tangled arcs.

Once the winding components and descending and ascending portions of the tangling components are pulled away from K, we can apply Lemma 3.2 to "untangle" the tangled arcs. If there were only one tangling component we could use Lemma 3.2 to pull it down to the position illustrated in Figure 5, thus converting it into a winding component.

If there are several tangling components, we simultaneously pull down all of the tangled arcs to the position illustrated in Figure 5. We move the tangled arcs in such a way that crossings remain crossings at all stages of the motion. Furthermore, if two points P, Q on tangled arcs initially have different projections (that is, they are not a crossing), then at no time may their trajectories have the same projection. It remains to be proved that the tangled arcs do not interfere with each other, either at a crossing or at the endpoint of a descending or ascending arc. An argument similar to that used in Lemma 3.2 shows that no such interference occurs.

Thus we can pull down all tangled arcs, so that tangling components become winding components. We can then pull all of these new winding components away from Z leaving Z as an isolated finite closed braid. Let I be the isotopy that pulls away the winding components, pulls down the tangled arcs, and then pulls away the new winding components. Observe that this isotopy, I, never increases the y-coordinate of any point.

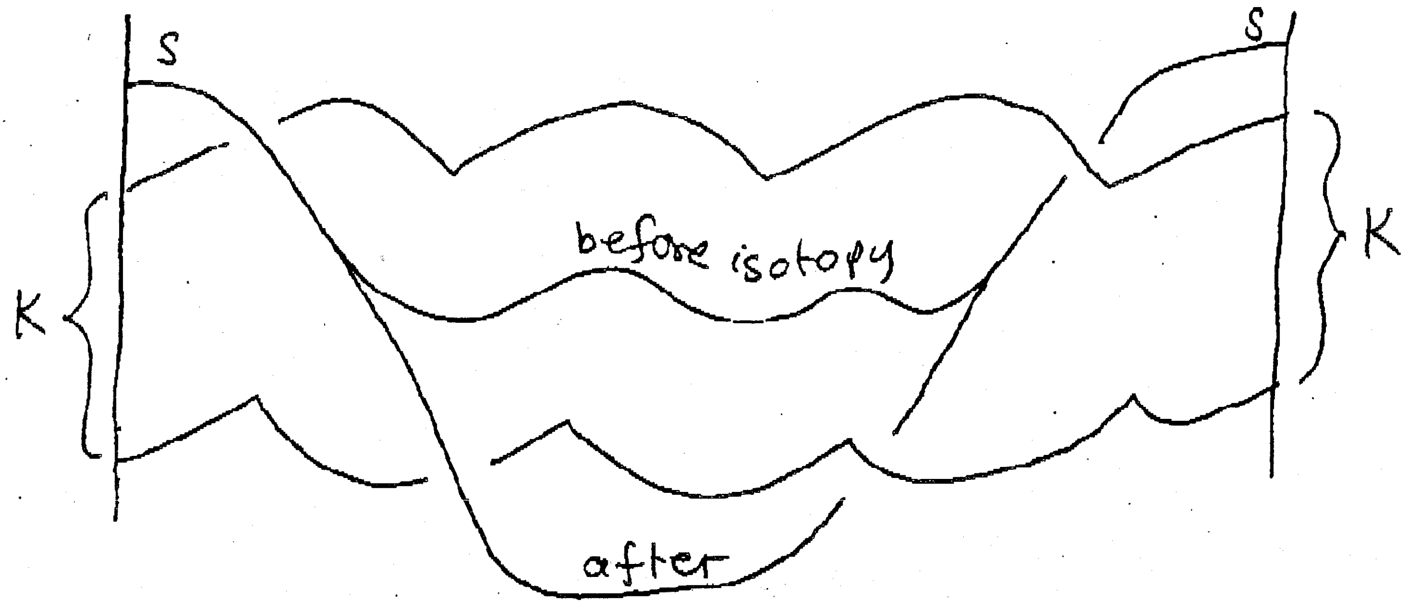

Figure 5:

Apply I to the closed braid (X, Y), leaving Z isolated. Recall that γ is the horizontal connecting path at a simple crossing of the components S_1 and S_2 of Z. By Theorem 3.1, Z is a trivial closed braid, so γ can be spun to give a connecting annulus in the braid $I(Y)$. It follows that $\alpha = I^{-1}(\gamma)$ can be spun in the given braid Y. Observe that α has the same endpoints as γ, since I leaves Z fixed, and α is above γ, since I never increases y-coordinates.

By symmetry, we can use the same construction to find a connecting path β that has the same endpoints as γ, that is entirely below γ, and that can be spun. In the plane containing α and β, let D be the disc bounded by $\alpha \cup \beta$. γ is contained in D, so by Lemma 3.6, γ can be spun.

4 Completion of Proof (Sketch)

Lemma 4.1 *Suppose that α and β are connecting paths that intersect in exactly one point, either a common end-point or a transverse interior intersection. Let P be the yz-plane containing α and β, and let $N \subset P$ be a regular neighborhood of $\alpha \cup \beta$. If α and β can be spun, then any connecting*

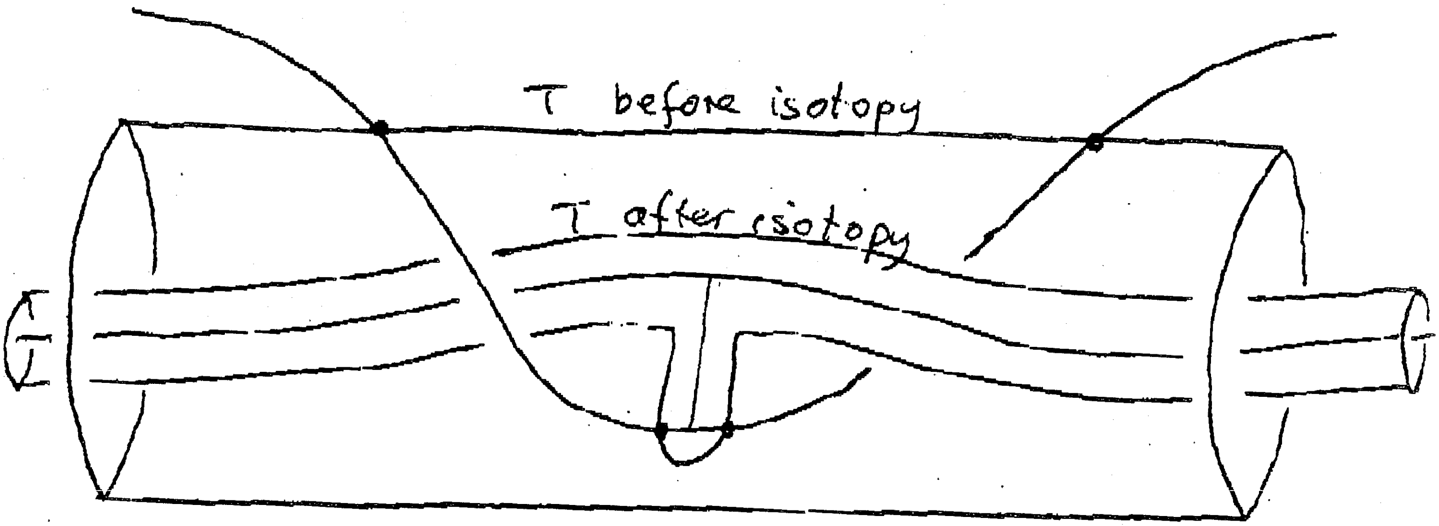

Figure 6:

path contained in N can be spun.

This lemma is proved by taking the existing connecting annuli to have minimum istersection, and then fattening the union of these connecting annuli. This gives a solid torus whose boundary is disjoint from L, and therefore encloses a finite braid. Every connecting path in this braid can thus be spun.

We have already shown that every horizontal connecting path can be spun. Clearly, as we spin such a path, every connecting path along its trajectory can be spun. We can use these connecting paths and Lemma 4.1 to construct a large family of connecting paths that can be spun. It can be shown that this family includes all connecting paths, so we have:

Theorem 4.2 *Let (X, Y) be a closed braid satisfying the hypotheses of Theorem 1.3. Then every connecting path for (X, Y) can be spun.*

With this fact about connecting paths, we are now ready to generalize Theorem 3.1 (the finite case) to the infinite case.

Sketch of proof: Using the fact that every connecting path can be spun, it is not difficult to show that the union of finitely many components of L and finitely many connecting paths joining these components can be engulfed in a compact solid torus with boundary disjoint from L. It follows that any compact solid torus $T \in S^1 \times \mathbb{R}^2$ can be engulfed in a compact solid torus with boundary disjoint from L. (Homotop T to any component in its interior, as in Figure 6). We can then express $S^1 \times \mathbb{R}^2$ as a nested union of compact solid tori with boundaries disjoint from L. Applying the finite version of the theorem to each torus in the sequence shows that the original braid was a full twist.

References

[G] D. Gabai, *Convergence groups are Fuchsian groups*, preprint.

[GH] C. Gordon and W Heil, *Cyclic normal subgroups of fundamental groups of 3-manifolds*, Topology **14** (1975), 305-309.

[GM] F. W. Gehring and G. Martin, *Discrete quasiconformal groups I*, Proc. London Math. Soc. (3) **55** (1987), 331-358.

[M] G. Mess, *Centers of 3-manifold groups and groups which are coarse quasiisometric to planes*, preprint.

[S] P. Scott, *There are no fake Seifert fibre spaces with infinite π_1*, Ann. of Math. **117** (1983), 35-70.

[T] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 1-54.

[W] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. **87** (1968), 56-88.

THE DYNAMICS OF CIRCLE MAPS AND THE ONE-WAY PROPERTY

by Lauren Weinstock Ancel

Abstract. A *circle map* f is a continuous transformation from a circle C into itself. An interval in C is *one-way* with respect to f if under repeated applications of f all points of the interval move in the same direction. Examples are given of circle maps which admit intervals that are locally one-way (i.e., covered by one-way subintervals) but are not one-way. Also, it is proved that for *onto* circle maps, all locally one-way intervals are one-way.

1. Introduction.

A continuous transformation from a circle into itself is called a *circle map.* The *dynamics* of a circle map concerns the behavior of points of the circle under *iteration* (repeated application of the map). This paper studies an aspect of the dynamics of circle maps called the *one-way property.* Let f be a circle map. If x is a point of the circle, then the points obtained by applying f repeatedly to x are called the *iterates* of x. Let J be an oriented open interval in the circle. J is *free* if no iterate of a point of J returns to J. J is *positive* if it is not free and if for every point x in J, any iterate of x which returns to J lies to the positive side of x. J is *negative* if it is not free and if for every point x in J, any iterate of x which returns to J lies to the negative side of x. J is *one-way* if it is either free, positive, or negative. J is *locally one-way* if every point of J lies in a one-way open subinterval of J. Since the dynamical behavior of a map on a one-way interval is relatively uncomplicated and well understood, it is desirable to find conditions under which one-way intervals exist. The main question addressed here is the following. For which maps are all locally one-way intervals one-way?

The analogous question for maps from the real line to itself has a simple answer. For any such map, all locally one-way intervals are one-way. The proof of this fact is elementary. [1, Chapter 4, Lemma 6] The definition of one-way for subintervals of the circle first appears in [2] along with a result (Lemma 3.2) which, under a strong hypothesis on the dynamics of the map, implies that all locally one-way intervals are one-way. (Lemma 3.2 of [2] is repeated here as Lemma 4.) The 1990 Ph.D. thesis of M. Hero [3] explores this question carefully and extensively, but does not completely resolve it. Hero's thesis provided motivation and background for the work in this paper.

My research into this question began with the discovery of an example of a circle map with respect to which there is a subinterval of the circle which is locally one-way but not one-way. This map is not onto and my efforts to construct examples of onto maps failed. Since circle maps that are not onto are degree zero and circle maps of non-zero degree are onto, I originally hypothesized that the answer to this question would depend on the degree of the map. I have proved a theorem which has the corollary that for all circle maps of non-zero degree, every locally one-way interval is one-way. However, the theorem does not directly involve the notion of the degree of the map. Instead, it is based on the more fundamental concept of whether or not the map is onto. The theorem simply states that for every *onto* circle map, every locally one-way interval is one-way. One corollary of this theorem is that an onto circle map which has periodic points has the property that every subinterval of the circle which contains no periodic points is one-way. This corollary provides a simple characterization of the one-way intervals of an onto circle map with periodic points. The proof of the theorem reveals four situations in which a non-onto circle map might admit an interval which is locally one-way but not one-way. The example mentioned at the beginning of this paragraph illustrates only three of these four situations. I searched for and found a second example illustrating the fourth situation. The theorem together with the two examples are the principal results of this paper.

The remainder of this paper is divided into six sections. Section 2 contains definitions, statements of the theorem and four corollaries, and brief descriptions of the two examples. Section 3 contains several observations and lemmas used in the proof of the theorem. Section 4 presents proofs of the theorem and the corollaries. Section 5 presents the two examples in detail. Section 6 contains a concluding summary and states several open questions.

2. Definitions and Statements of Results

Definition. Let $f : X \to X$ be a function from a set X to itself. For each integer $n \geq 0$, define the function $f^n : X \to X$ as follows. Let f^0 be the identity function on X: $f^0(x) = x$. Let $f^1 = f$. For each integer $n \geq 2$, define f^n by $f^n(x) = f(f^{n-1}(x))$. For each integer $n \geq 1$, f^n is called the *n^{th} iterate* of f. If $x \in X$, then for each integer $n \geq 1$, $f^n(x)$ is called the *n^{th} iterate* of x with respect to f. If A is a subset of X, then for each integer $n \geq 1$, $f^n(A)$ is called the *n^{th} iterate* of A with respect to f.

Definition. Let S^1 be the circle in $\mathbb{R}^2$ centered at the origin with radius 1, i.e. $S^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$. Orient S^1 in a counterclockwise direction. So from now on, the "positive direction" on S^1 means the counterclockwise direction.

Definition. Let $n \geq 3$ and let $x_1, x_2, x_3, \cdots, x_n$ be points of S^1. Write $x_1 < x_2 < x_3 < \cdots < x_n$ if $x_1, x_2, x_3, \cdots, x_n$ are distinct points and if moving away from x_1 in S^1 in the positive direction, one encounters $x_1, x_2, x_3, \cdots, x_n$ in that order before one encounters x_1 again. If in the expression $x_1 < x_2 < x_3 < \cdots < x_n$ one or more of the <'s is replaced by $\leq$, then let this expression have the obvious meaning.

Definition. Let a and b be distinct points of S^1. Define $(a,b) = \{ x \in S^1 : a < x < b \}$, and call (a,b) an *open interval.* Define $[a,b] = \{ x \in S^1 : a \leq x \leq b \}$, and call [a,b] a *closed interval.* Define $[a,b) = \{ x \in S^1 : a \leq x < b \}$ and $(a,b] = \{ x \in S^1 : a < x \leq b \}$ and call [a,b) and (a,b] *half-open intervals.*

Definition. Let $f : S^1 \to S^1$ be a map. Let (a,b) be an open interval in S^1. (a,b) is *free* (with respect to f) if $f^n(x) \notin (a,b)$ for every $x \in (a,b)$ and every positive integer n. (a,b) is *positive* (with respect to f) if (a,b) is not free and if $a < x < f^n(x) < b$ whenever x, $f^n(x) \in (a,b)$ for some positive integer n. (a,b) is *negative* (with respect to f) if (a,b) is not free and if $a < f^n(x) < x < b$ whenever $x, f^n(x) \in (a,b)$ for some positive integer n. (a,b) is *one-way* if it is either free, positive or negative. (a,b) is *locally one-way* if every point of (a,b) is contained in a one-way open interval.

Definition. Define the map $e : \mathbb{R} \to S^1$ by $e(x) = (\cos(2\pi x), \sin(2\pi x))$. If $x \in S^1$, $y \in \mathbb{R}$ and $e(y) = x$, then it is said that y *covers* x. If J is an interval in S^1, K is an interval in $\mathbb{R}$ and $e(K) = J$, then it is said that K *covers* J. If $f : S^1 \to S^1$ and $g : \mathbb{R} \to \mathbb{R}$ are maps such that $f(e(x)) = e(g(x))$ for all $x \in S^1$, then it is said that g *covers* f. If J is an interval in S^1, $f : J \to S^1$ is a map, K is an interval in $\mathbb{R}$ which covers J, and $g : K \to \mathbb{R}$ is a map such that $f(e(x)) = e(g(x))$ for all $x \in K$, then it is said that g *covers* f.

Definition. Let $f : S^1 \to S^1$ be a map. The *degree* of f is an integer d with the following property. If $g : \mathbb{R} \to \mathbb{R}$ is any map which covers f, then $g(x+1) = g(x)+d$ for every $x \in \mathbb{R}$.

Definition. Let X be a set and let $f : X \to X$ be a function. A point x of X is a *fixed point* of f if $f(x) = x$. A point x of X is a *periodic point* of f if $f^n(x) = x$ for some integer $n \geq 1$. Thus a periodic point of f is a fixed point of f^n for some integer $n \geq 1$.

The main results of this paper are now stated.

Theorem. If $f : S^1 \to S^1$ is an onto map, then with respect to f, every locally one-way open interval in S^1 is one-way.

The Theorem has two immediate corollaries.

Corollary 1. If $f : S^1 \to S^1$ is a map of non-zero degree, then with respect to f, every locally one-way open interval in S^1 is one-way.

Corollary 2. If $f : S^1 \to S^1$ is an onto map such that the set P of periodic points of f is non-empty, and every open interval in S^1 which is disjoint from P is one-way.

The proof of the Theorem concerns the situation in which $f : S^1 \to S^1$ is a map and $a < b < c < d$ are points of S^1 such that (a,c) and (b,d) are one-way with respect to f. The proof analyzes the question of whether (a,d) is one-way with respect to f. Since (a,c) and (b,d) can be either positive, negative or free, there are nine cases to consider. The outcome of this analysis is interesting in its own right and is described in the following two corollaries to the proof of the Theorem.

Corollary 3. Let $f : S^1 \to S^1$ be a map, and let $a < b < c < d$ be points of S^1.
a) If (a,c) and (b,d) are positive, then (a,d) is positive.
b) If (a,c) and (b,d) are negative, then (a,d) is negative.
c) It is impossible for (a,c) to be positive and (b,d) to be negative.
d) If (a,c) is positive and (b,d) is free, then (a,d) is positive.
e) If (a,c) is free and (b,d) is negative, then (a,d) is negative.

Corollary 4. Let $f : S^1 \to S^1$ be an onto map, and let $a < b < c < d$ be points of S^1.

a) It is impossible for (a,c) to be negative and (b,d) to be positive.

b) If (a,c) is free and (b,d) is positive, then (a,d) is positive.

c) If (a,c) is negative and (b,d) is free, then (a,d) is negative.

d) If (a,c) and (b,d) are free, then (a,d) is one-way.

Corollary 3 was already known to M. Hero. It appears in [3] as Lemma 2.4 with a different proof.

Suppose $f : S^1 \to S^1$ is a map and $a < b < c < d$ are points of S^1 such that with respect to f, (a,c) is either negative or free, and (b,d) is either positive or free. In Corollary 4, these four cases considered under the hypothesis that f is onto, and it is concluded that (a,d) is one-way with respect to f. However, if f is not onto, then it is possible that (a,d) is not one-way with respect to f. Examples 1 and 2 show that these possibilities can be realized. In both examples, $e(0) = a_0 < a_1 < a_2 < \cdots < a_8$ are points in S^1.

Example 1. There is a map $f : S^1 \to S^1$ with respect to which (a_1,a_5) is negative, (a_4,a_8) is positive, and (a_1,a_8) is not one-way. Also (a_1,a_7) is negative and (a_6,a_8) is free, and (a_1,a_3) is free and (a_2,a_8) is positive.

Example 2. There is a map $f : S^1 \to S^1$ with respect to which (a_1,a_4) and (a_3,a_6) are free and (a_1,a_6) is not one-way. Also (a_1,a_5) is negative and (a_2,a_6) is positive. So (a_1,a_5) is negative and (a_3,a_6) is free, and (a_1,a_4) is free and (a_2,a_6) is positive.

Example 1 illustrates that locally one-way does not imply one-way in the three cases: negative-positive, negative-free and free-positive. A similar example was found independently by M. Hero. Example 1 does not provide an illustration of the fourth possible case: free-free. Example 2 illustrates all four cases.

3. Observations and Lemmas

First, some general observations about the covering map $e : \mathbb{R} \to S^1$ are stated. For each $x \in \mathbb{R}$, e maps the half-open interval $[x,x+1)$ one-to-one onto S^1; and if x moves in the positive direction in $\mathbb{R}$, then e(x) moves in the positive direction in S^1. Hence, if $x_1, x_2, \cdots, x_n$ are points in $\mathbb{R}$ such that $x_1 < x_i < x_{i+1}$ for $2 \le i \le n$, then if $x_1 < x_2 < \cdots < x_n$ if and only if $e(x_1) < e(x_2) < \cdots < e(x_n)$. Also, if $x < y$ in $\mathbb{R}$, then $e([x,y]) \supset [e(x),e(y)]$; if $x < y < x+1$, then $e([x,y]) = [e(x),e(y)]$; and if $x+1 \le y$, then $e([x,y]) = S^1$.

The following comments are observations about the existence and uniqueness of covers of points, intervals and maps. Each point of S^1 is covered by a point of $\mathbb{R}$; and two points x and y of $\mathbb{R}$ cover the same point of S^1 if and only if $y = x+n$ for some integer n. Each interval in S^1 is covered by an interval in $\mathbb{R}$, and two intervals [a,b] and [c,d] in $\mathbb{R}$ cover the same interval in S^1 if and only if there is an integer n such that c =

$a+n$ and $d = b+n$. Each map from S^1 to itself is covered by a map from $\mathbb{R}$ to itself, and two maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ cover the same map from S^1 to itself if and only if there is an integer n such that $h(x) = g(x)+n$ for all $x \in \mathbb{R}$. If $f : S^1 \to S^1$ is a map, then there is a unique integer d with the property that if $g : \mathbb{R} \to \mathbb{R}$ is any cover of f, then $g(x+1) = g(x)+d$ for every $x \in \mathbb{R}$; recall that d is the *degree* of f. Let J be an interval in S^1 and let K be an interval in $\mathbb{R}$ which covers J. Then every map from J to S^1 is covered by a map from K to $\mathbb{R}$, and two maps $f : K \to \mathbb{R}$ and $g : K \to \mathbb{R}$ cover the same map from J to S^1 if and only if there is an integer n such that $h(x) = g(x)+n$ for all $x \in K$.

Next four useful observations with very simple proofs are stated.

Observation 1. a) For each integer n, define the map $h_n : \mathbb{R} \to \mathbb{R}$ by $h_n(x) = x+n$. Then the h_n's are the maps from $\mathbb{R}$ to itself which cover the identity map on S^1.
b) Let J be an interval in S^1, let $f : J \to S^1$ be a map, let K be an interval in $\mathbb{R}$ which covers J, and let $g : K \to \mathbb{R}$ be a map which covers f. For a point $x \in K$, $e(x)$ is a fixed point of f if and only if $g(x) = h_n(x)$ for some integer n.
c) Let J_1 be another interval in S^1, and let K_1 be an interval in $\mathbb{R}$ which covers J_1. If $g(K) \supset K_1$, then $f(J) \supset J_1$.

Proof. a) Each h_n covers the identity map of S^1 because $e(h_n(x)) = e(x+n) = e(x)$. Also if $h : \mathbb{R} \to \mathbb{R}$ is a map which covers the identity map on S^1, then there is an integer n such that $h(x) = h_0(x)+n = h_n(x)$ for all $x \in \mathbb{R}$.

b) If $f(e(x)) = e(x)$, then $e(g(x)) = e(x)$. So there is an integer n such that $g(x) = x+n = h_n(x)$. Also, if $g(x) = h_n(x)$ for some integer n, then $f(e(x)) = e(g(x)) = e(h_n(x)) = e(x)$.

c) $f(J) = f(e(K)) = e(g(K)) \supset e(K_1) = J_1$. □

Observation 2. Let $f : S^1 \to S^1$ be a map. If J is an open interval in S^1 which is locally one-way, then no point of J is a periodic point.

Proof. If $x \in J$, then x lies in a one-way open interval. Then clearly $f^n(x) \neq x$ for all $n \geq 1$. □

Observation 3. Let $f : S^1 \to S^1$ be a map. If J and K are open intervals in S^1 such that $J \subset K$ and J is not one-way, then K is not one-way. If (a,b) is an open interval in S^1 which is not one-way, then there are $c, d \in (a,b)$ such that $a < c < d < b$ and (c,d) is not one-way.

Proof. The first sentence is obvious. If (a,b) is an open interval in S^1 which is not one-way, then there are points $x, y \in (a,b)$ and positive integers m and n such that $a < x < f^m(x) < b$ and $a < f^n(y) < y < b$. Choose points $c, d \in (a,b)$ such that $a < c < d < b$ and (c,d) contains x, y, $f^m(x)$ and $f^n(y)$. Then (c,d) is not one-way. □

Observation 4. Let $f : S^1 \to S^1$ be a map. If (a,b) is a positive open interval in S^1, then $f^n(b) \notin (a,b)$ for every integer $n \geq 0$.

Proof. Assume $f^n(b) \in (a,b)$ for some positive integer n. Let $x \in (f^n(b),b)$. Then $f^n(b) \in (a,x)$. Since f^n is continuous, there is a $y \in (x,b)$ such that $f^n([y,b)) \subset (a,x)$. This is a contradiction because $a < f^n(y) < x < y < b$ and (a,b) is positive. □

Lemma 1. Let a and b be distinct points of S^1 and let $g : [a,b] \to S^1$ be a map with no fixed points.

a) If $a < g(a) \leq g(b) < b$, then $g([a,b]) = S^1$.
b) If $a \leq g(b) < g(a) \leq b$, then $g([a,b]) \supset [g(a),g(b)]$.
c) If $a < b \leq g(a) < g(b)$, then $g([a,b]) \supset [g(a),g(b)]$.
d) If $a < b < g(b) < g(a)$, then $g([a,b]) \supset [g(b),g(a)]$.
e) If $a < g(a) \leq b < g(b)$, then $g([a,b]) \supset [g(a),g(b)]$.
f) If $a \leq g(b) < b \leq g(a)$, then $g([a,b]) \supset [g(a),g(b)]$.

Proof. There are points a' and $b' \in \mathbb{R}$ such that $a' < b' < a'+1$ and $[a',b']$ covers $[a,b]$. There is a map $g' : [a',b'] \to \mathbb{R}$ which covers g. For each integer n, let $h_n : [a',b'] \to \mathbb{R}$ be the map defined in Observation 1a) by the formula $h_n(x) = x+n$; then h_n covers the identity map on S^1. Since g has no fixed points, then by Observation 1b), $g'(x) \neq h_n(x)$ for all integers n and all $x \in [a',b']$. So the graph of g' is disjoint from the graph of h_n for each integer n. Hence, there is an integer m such that the graph of g' lies above the graph of h_m and below the graph of h_{m+1}. So $a'+m < g'(a') < a'+m+1$ and $b'+m < g'(b') < b'+m+1$.

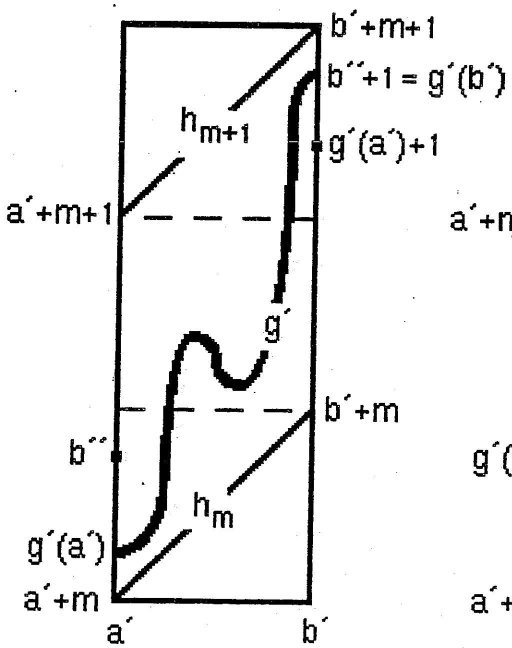

Figure 1a

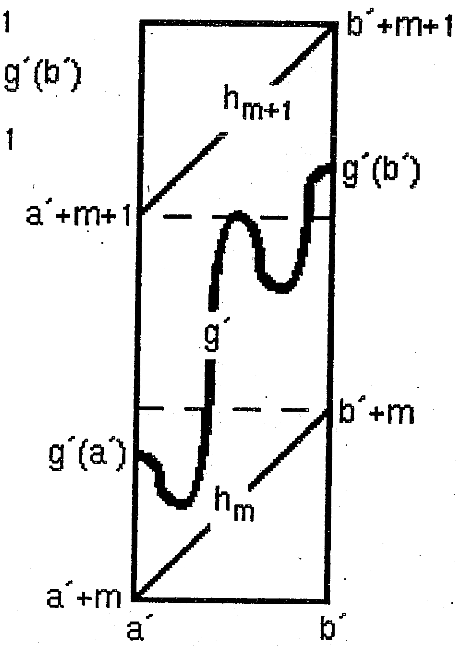

Figure 1b

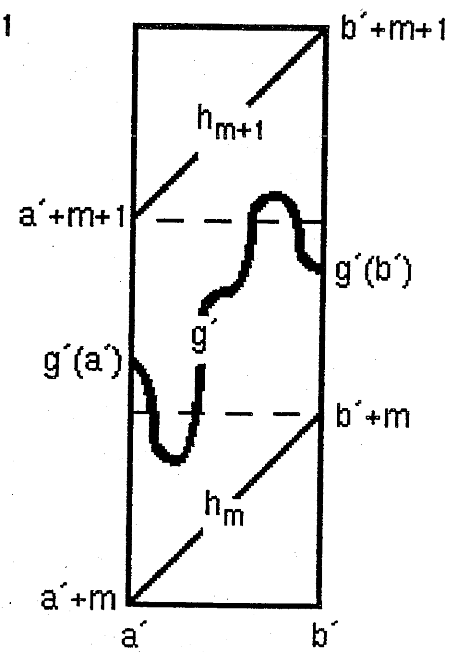

Figure 1c

a) Since $a < g(a) \le g(b) < b$, then there is a $b'' \in \mathbb{R}$ such that $a'+m < g'(a') < b'' < b'+m$ and b'' covers $g(b)$. Since $b'' < b'+m < a'+m+1 < g'(b')$, and both b'' and $g'(b')$ cover $g(b)$, then $g'(b') = b''+1$. So $g'(b') \ge g'(a')+1$. See Figure 1a. Hence, the graph of g' extends vertically from $g'(a')$ to $g'(b')$. Therefore, $g'([a',b']) \supset [g'(a'),g'(b')] \supset [g'(a'),g'(a')+1]$. (The last step uses the following Intermediate Value Theorem. Let ϕ map a closed interval $[p,q]$ into $\mathbb{R}$. If $\phi(p) \le \phi(q)$, then $\phi([p,q]) \supset [\phi(p),\phi(q)]$; and if $\phi(q) \le \phi(p)$, then $\phi([p,q]) \supset [\phi(q),\phi(p)]$.) Thus, $g([a,b]) = g(e([a',b']) = e(g'([a',b'])) \supset e([g'(a'),g'(a')+1]) = S^1$.

b) Since $a < g(a) \le b$, then $a'+m < g'(a') \le b'+m$. So $g'(a') \le b'+m < g'(b')$. See Figure 1b. Hence, the graph of g' extends vertically from $g'(a')$ to $g'(b')$. Therefore, $g'([a',b']) \supset [g'(a'),g'(b')]$. So, by Observation 1c), $g([a,b]) \supset [g(a),g(b)]$.

c) Since $a < b \le g(a) < g(b)$, then there is a $b'' \in \mathbb{R}$ such that $a'+m < b'+m \le g'(a') < b'' < a'+m+1$ and b'' covers $g(b)$. So both b'' and $g'(b')$ lie in the interval $(b'+m,b'+m+1)$ and both cover $g(b)$. Therefore, $b'' = g'(b')$. So $g'(a') < g'(b')$. See Figure 1c. Hence, the graph of g' extends vertically from $g'(a')$ to $g'(b')$. Therefore, $g'([a',b']) \supset [g'(a'),g'(b')]$. So, by Observation 1c), $g([a,b]) \supset [g(a),g(b)]$.

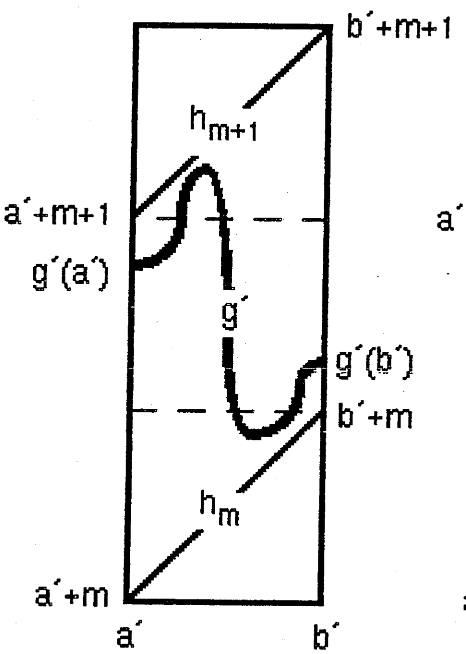

Figure 1d

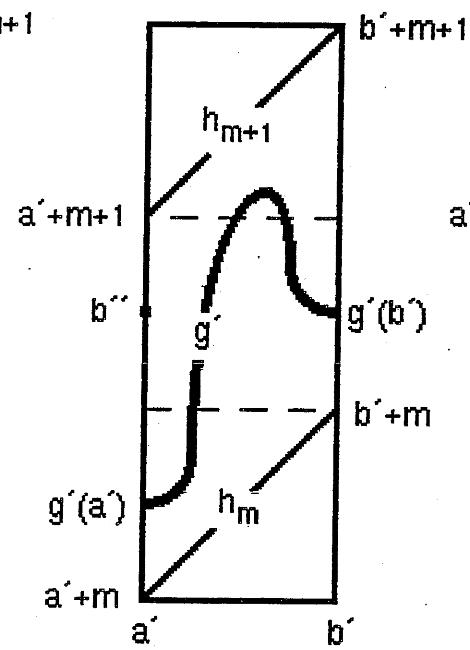

Figure 1e

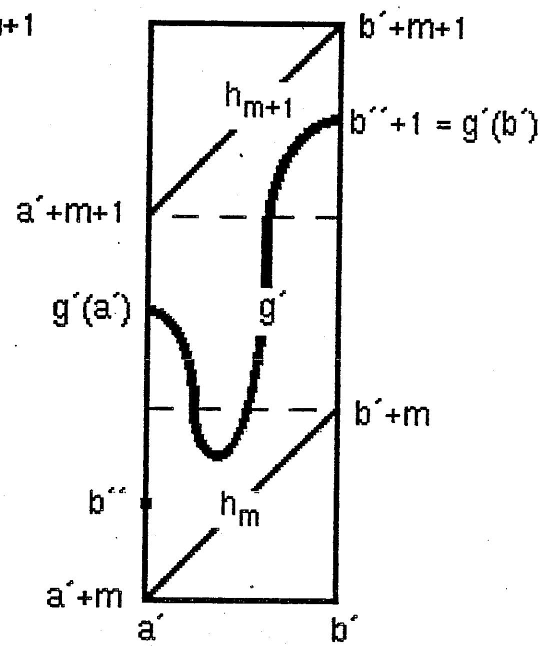

Figure 1f

d) Since $a < b < g(b) < g(a)$, then there is a $b'' \in \mathbb{R}$ such that $a'+m < b'+m < b'' < g'(a') < a'+m+1$ and b'' covers $g(b)$. So both b'' and $g'(b')$ lie in the interval $(b'+m,b'+m+1)$ and both cover $g(b)$. Therefore, $b'' = g'(b')$. So $g'(b') < g'(a')$. See

Figure 1d. Hence, the graph of g' extends vertically from $g'(b')$ to $g'(a')$. Therefore, $g'([a',b']) \supset [g'(b'),g'(a')]$. So, by Observation 1c), $g([a,b]) \supset [g(b),g(a)]$.

e) Since $a < g(a) \leq b$, the argument given in case b applies here. See Figure 1e.

f) Since $a \leq g(b) < b \leq g(a)$, then there is a $b'' \in \mathbb{R}$ such that $a'+m \leq b'' < b'+m \leq g'(a') < a'+m+1$ and b'' covers $g(b)$. Since $b'' < a'+m+1 < g'(b')$, and both b'' and $g'(b')$ cover $g(b)$, then $g'(b') = b''+1$. So $g'(b') \,\tilde{}\, a'+m+1 > g'(a')$. See Figure 1f. Hence, the graph of g' extends vertically from $g'(a')$ to $g'(b')$. Therefore, $g'([a',b']) \supset [g'(a'),g'(b')]$. So, by Observation 1c), $g([a,b]) \supset [g(a),g(b)]$. □

Lemma 2. Let $f : S^1 \to S^1$ be a map. If (a,b) is a positive open interval in S^1, then for every $x \in (a,b)$, there is a $y \in (a,b)$ and a positive integer n such that $f^n(y) \in (x,b)$.

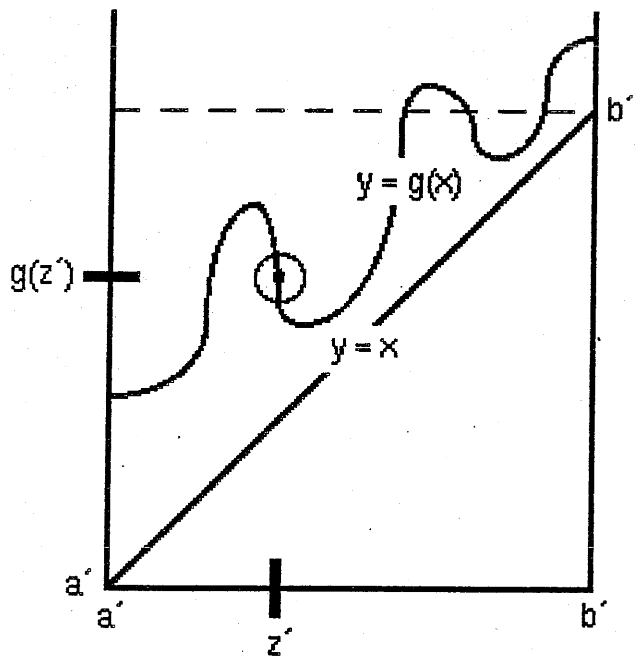

Figure 2

Proof. Since (a,b) is positive with respect to f, then $a < z < f^n(z) < b$ for some integer n and some $z \in (a,b)$.

Let $[a',b']$ be an interval in $\mathbb{R}$ which covers $[a,b]$. Let $z' \in (a',b')$ cover z. There is a map $g : [a',b'] \to \mathbb{R}$ which covers f^n such that $a' \leq g(z') < a'+1$. Since $a < z < f^n(z) < b$, then $a' < z' < g(z') < b'$. See Figure 2.

Since f^n has no fixed points in (a,b), then, by Observation 1b), g has no fixed points in (a',b'). Since, in addition $g(z') > z'$, then the graph of g restricted to (a',b') lies above the line $y = x$. Therefore $g(b') \geq b'$. See Figure 2. Hence, the graph of g extends vertically from $g(z')$ to $g(b')$. Therefore $g(z',b') \supset (g(z'),g(b')) \supset (g(z'),b')$. Hence, by Observation 1c), $f^n(z,b) \supset (f^n(z),b)$.

For any $x \in (a,b)$, $(x,b) \cap (f^n(z),b)$ is a non-empty subset of $f^n(z,b)$. So there is a $y \in (z,b)$ such that $f^n(y) \in (x,b)$. □

Lemma 3. Let $f : S^1 \to S^1$ be a map, and let (a,b) be a positive open interval in S^1. If $x, y \in (a,b)$ such that $a < x < y < b$ and n is a positive integer, then there is a positive integer i such that $f^{jn}(x) \in (a,y]$ for $0 \leq j < i$ and $f^{in}(x) \notin (a,y]$.

Proof. Suppose otherwise. Since $f^{0n}(x) = x \in (a,y]$, then $f^{in}(x) \in (a,y]$ for all integers $i \geq 0$. Since (a,b) is positive, then $a < x < f^n(x) < f^{2n}(x) < \cdots < y$. Since bounded increasing sequences converge, then $\{ f^{in}(x) \}_{i \geq 0}$ converges to a point $z \in (a,y]$. Therefore, $\{ f^n(f^{in}(x)) \}_{i \geq 0}$ converges to $f^n(z)$. However, since $\{ f^n(f^{in}(x)) \}_{i \geq 0}$ is a subsequence of $\{ f^{in}(x) \}_{i \geq 0}$, then $\{ f^n(f^{in}(x)) \}_{i \geq 0}$ converges to z. So $f^n(z) = z$. This is a contradiction, because (a,b) is positive and $z \in (a,b)$. □

Lemma 4. Let $f : S^1 \to S^1$ be a map. If J is an open interval in S^1 which contains no periodic points and J is not one-way, then $\bigcup_{n \geq 0} f^n(J) = S^1$.

Proof. This is Lemma 3.2 of [2]. □

4. Proofs

We restate and prove the Theorem and the four corollaries.

Theorem. If $f : S^1 \to S^1$ is an onto map, then with respect to f, every locally one-way open interval in S^1 is one-way.

Proof. Assume there is a locally one-way subinterval (a_0,b_0) of S^1 which is not one-way with respect to f. By Observation 2, no point of (a_0,b_0) is periodic.

Claim. There are points a, b, c, d in (a_0,b_0) such that $a_0 < a < b < c < d < b_0$, (a,c) and (b,d) are one-way, and (a,d) is not one-way.

Proof of claim. Observation 3 implies there are points $a_1, b_1 \in (a_0,b_0)$ such that $a_0 < a_1 < b_1 < b_0$ and (a_1,b_1) is not one-way. Hence, $[a_1,b_1] \subset (a_0,b_0)$ and $[a_1,b_1]$ is not one-way. Since (a_0,b_0) is locally one-way, then each $x \in [a_1,b_1]$ is contained in a one-way open interval $J(x) \subset (a_0,b_0)$. $\{ J(x) : x \in [a_1,b_1] \}$ is an open cover of $[a_1,b_1]$. $[a_1,b_1]$ is compact because it is a closed subset of S^1 which is compact. Therefore some finite subset $\{ J_1, J_2, \cdots, J_n \}$ of $\{ J(x) : x \in [a_1,b_1] \}$ covers $[a_1,b_1]$. Assume that n is the smallest positive integer for which such a finite cover exists. Then for $i \neq j$, J_i does not contain J_j. This implies that $J_1, J_2, \cdots, J_n$ can be reindexed so that $J_i \cap J_{i+1} \neq \varnothing$ for $1 \leq i < n$. J_1 is one-way; however, $\bigcup_{1 \leq i \leq n} J_i$ is not one-way because it contains $[a_1,b_1]$. So there is an integer k such that $1 \leq k < n$, $\bigcup_{1 \leq i \leq k} J_i$ is one-way, and $\bigcup_{1 \leq i \leq k+1} J_i$ ßis not one-way. There are points a, b, c, d $\in (a_0,b_0)$ such that $a_0 < a < b < c < d < b_0$, $(a,c) = \bigcup_{1 \leq i \leq k} J_i$ and $(b,d) = J_{k+1}$.. So $(a,d) = \bigcup_{1 \leq i \leq k+1} J_i$. Therefore, (a,c) and (b,d) are one-way, but (a,d) is not one-way.

There are nine cases to consider because (a,c) can be positive, negative or free, and (b,d) can be positive, negative or free. In each case, a contradiction is derived from the assumption that (a,d) is not one-way.

Case 1: (a,c) and (b,d) are positive.

Since (a,c) is positive, then Lemma 2 implies there is an $x \in (a,c)$ and a positive integer m such that $f^m(x)$ [(b,c). Since (a,c) is positive, then $a < x < f^m(x) < c$.

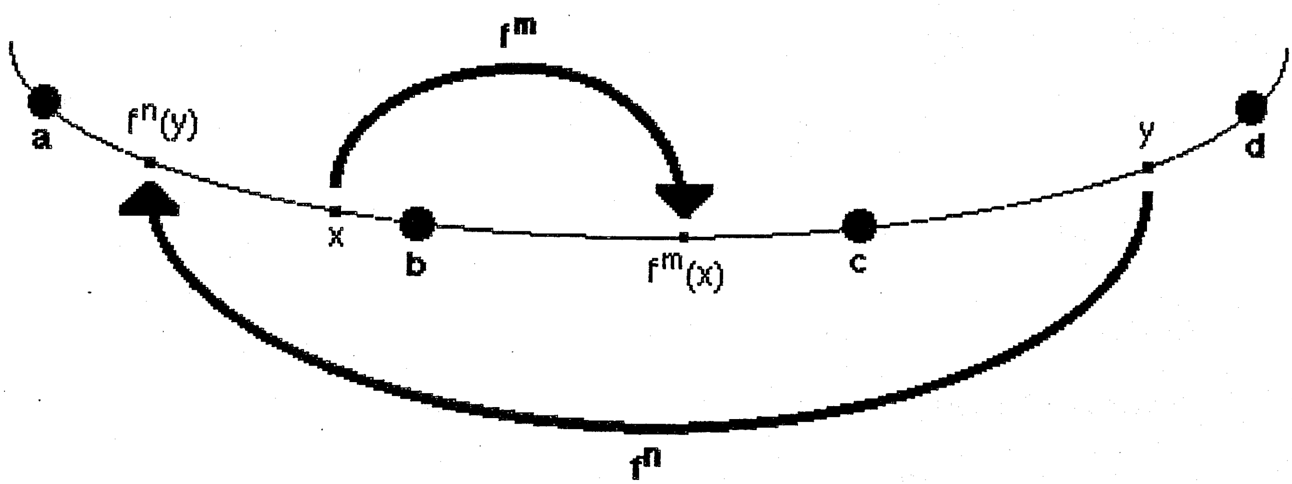

Figure 3a

Since (a,d) is not one-way, then Observation 3 implies there is a $y' \in (a,d)$ such that $a < y' < x$ and (y',d) is not one-way. Since $(y',d) \subset (a,d)$ and (a,d) is locally one-way, then Observation 2 implies that (y',d) contains no periodic points. Hence, Lemma 4 implies that (y',d) and its iterates cover S^1. Therefore, there is a $y \in (y',d)$ and a positive integer n such that $f^n(y) = y'$. So $a < f^n(y) < x$ and $a < f^n(y) < y < d$. Since (a,c) is positive, then $y \in [c,d)$. So $a < f^n(y) < x < f^m(x) < y < d$. See Figure 3a.

Since (b,d) is positive and $b < f^m(x) < y < d$, then Lemma 3 implies there is a positive integer k such that $f^{(k-1)m}(f^m(x)) \in (b,y]$ and $f^{km}(f^m(x)) \notin (b,y]$. Thus, $f^{km}(x) \in (b,y]$ and $y \in [f^{km}(x), f^{km}(f^m(x)))$. $b < f^m(x) \leq f^{km}(x) < d$ because (b,d) is positive. Therefore, $x < f^m(x) \leq f^{km}(x)$. $f^{km}(f^m(x)) \notin [x,f^m(x)]$ because $[x,f^m(x)] \subset (a,c)$ and (a,c) is positive. $f^{km}(f^m(x)) \notin [f^m(x),f^{km}(x)]$ because $[f^m(x),f^{km}(x)] \subset (b,y]$ and $f^{km}(f^m(x)) \notin (b,y]$. So $f^{km}(f^m(x)) \notin [x,f^{km}(x)]$. Hence, $x < f^m(x) \leq f^{km}(x) \leq y < f^{km}(f^m(x))$.

$[x,f^m(x)]$ contains no fixed points of f^{km} because it contains no periodic points. Hence, Lemma 1 c) implies that $f^{km}([x,f^m(x)]) \supset [f^{km}(x),f^{km}(f^m(x))]$. Since $y \in [f^{km}(x),f^{km}(f^m(x))]$, then $y \in f^{km}([x,f^m(x)])$. See Figure 3b. Thus, there is a $z \in [x,f^m(x)]$ such that $f^{km}(z) = y$. Therefore, $f^{km+n}(z) = f^n(y)$. So $a < f^{km+n}(z) < x \leq z \leq f^m(x) < c$. This is a contradiction because (a,c) is positive.

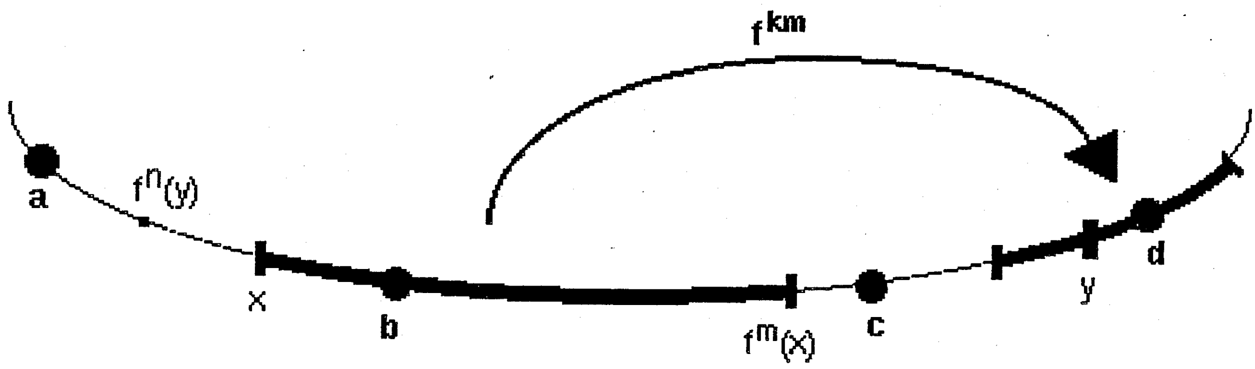

Figure 3b

Case 2: (a,c) and (b,d) are negative.

Reverse the orientation on S^1. Then $d < c < b < a$, (d,b) and (c,a) are positive, and (d,a) is not one-way. So case 1 applies here.

Case 3: (a,c) is positive and (b,d) is negative.

Since (a,c) is positive and (a,d) is not one-way, then as in case 1 there are $x, y \in (a,d)$ and positive integers m and n such that $f^m(x) \in (b,c)$ and $a < f^n(y) < x < f^m(x) < y < d$. Now $f^m(f^m(x)) \notin (a,f^m(x)]$ because (a,c) is positive, and $f^m(f^m(x)) \notin [f^m(x),d)$ because (b,d) is negative. So $f^m(f^m(x)) \notin (a,d)$. Therefore $x < f^m(x) < y < f^m(f^m(x))$. Set $k = 1$. Then $x < f^m(x) = f^{km}(x) < y < f^{km}(f^m(x))$. From this point the proof is the same as the proof of case 1.

Case 4: (a,c) is positive and (b,d) is free.

In case 3, the hypothesis that (b,d) is negative can be replaced by the hypothesis that (b,d) is free. Then the argument for case 3 applies to this case.

Case 5: (a,c) is free and (b,d) is negative.

By reversing orientation as in case 2, the argument for case 4 applies to this case.

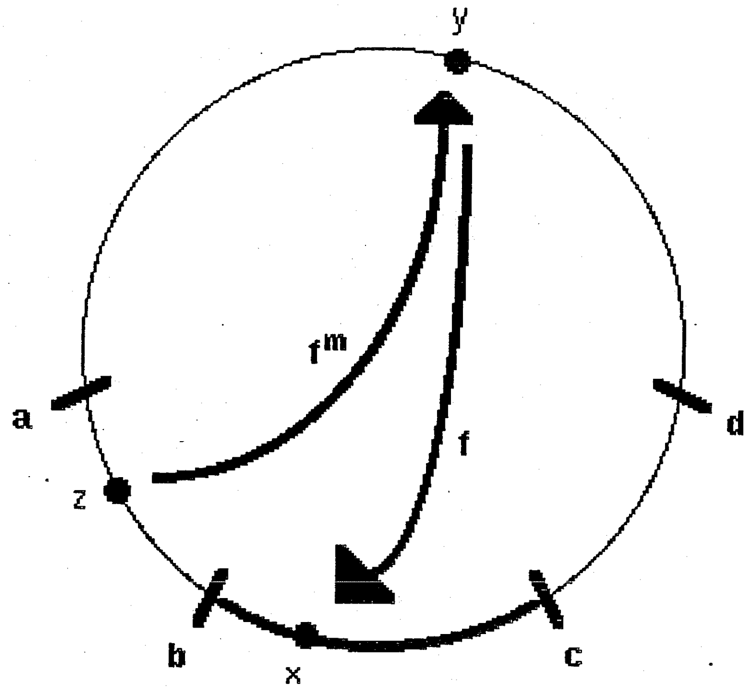

Figure 3c

Case 6: (a,c) is negative and (b,d) is positive.

Let $x \in (b,c)$. Since (a,c) is negative, then $x \notin \bigcup_{n \geq 1} f^n(a,x]$; and since (b,d) is positive, then $x \notin \bigcup_{n \geq 1} f^n[x,d)$. So $x \notin \bigcup_{n \geq 1} f^n(a,d)$.

Since f is onto then $f(y) = x$ for some $y \in S^1$.

Since (a,d) is locally one-way, then Observation 2 implies that (a,d) contains no periodic points. Since (a,d) is not one-way, then by Lemma 4, (a,d) and its iterates cover S^1. In particular, there is a $z \in (a,d)$ and an integer $m \geq 0$ such that $f^m(z) = y$. So $f^{m+1}(z) = f(y) = x$. Hence, $x \in f^{m+1}(a,d)$. This is a contradiction. See Figure 3c.

Case 7: (a,c) is free and (b,d) is positive.

In case 6, the hypothesis that (a,c) is negative can be replaced by the hypothesis that (a,c) is free. Then the argument for case 6 applies to this case.

Case 8: (a,c) is negative and (b,d) is free.

By reversing orientation as in case 2, the argument for case 7 applies to this case.

Case 9: (a,c) and (b,d) are free.

In case 6, the hypothesis that (a,c) is negative and (b,d) is positive can be replaced by the hypothesis that (a,c) and (b,d) are free. Then the argument for case 6 applies to this case. □

Corollary 1. If $f : S^1 \to S^1$ is a map of non-zero degree, then with respect to f, every locally one-way open interval in S^1 is one-way.

Proof. Every map of non-zero degree is onto. □

Corollary 2. If $f : S^1 \to S^1$ is an onto map such that the set P of periodic points of f is non-empty, and every open interval in S^1 which is disjoint from P is one-way.

Proof. Let J be an open interval in S^1 which is disjoint from P. Then Corollary 2.17 of [3] implies that J is locally one-way. Since f is onto, the Theorem implies that J is one-way. □

In the proof of the Theorem, $f : S^1 \to S^1$ is a map and $a < b < c < d$ are points of S^1 such that (a,c) and (b,d) are one-way. Contradictions arise from assuming (a,d) is not one-way. So (a,d) must be one-way. Therefore, if one of (a,c) or (b,d) is positive, then (a,d) must be positive; and if one of (a,c) or (b,d) is negative, then (a,d) is negative. Moreover, it is impossible for one of (a,c) and (b,d) to be positive and the other negative. Also, the hypothesis that f is onto is used only in cases 6, 7, 8 and 9. These comments yield the following two corollaries to the proof of the Theorem.

Corollary 3. Let $f : S^1 \to S^1$ be a map, and let $a < b < c < d$ be points of S^1.

a) If (a,c) and (b,d) are positive, then (a,d) is positive.
b) If (a,c) and (b,d) are negative, then (a,d) is negative.
c) It is impossible for (a,c) to be positive and (b,d) to be negative.
d) If (a,c) is positive and (b,d) is free, then (a,d) is positive.
e) If (a,c) is free and (b,d) is negative, then (a,d) is negative.

Corollary 4. Let $f : S^1 \to S^1$ be an onto map, and let $a < b < c < d$ be points of S^1.

a) It is impossible for (a,c) to be negative and (b,d) to be positive.
b) If (a,c) is free and (b,d) is positive, then (a,d) is positive.
c) If (a,c) is negative and (b,d) is free, then (a,d) is negative.
d) If (a,c) and (b,d) are free, then (a,d) is one-way.

Corollary 3 was already known to M. Hero. It appears in [3. Lemma 2.4] with a different proof.

5. Two Examples

Suppose $f : S^1 \to S^1$ is a map and $a < b < c < d$ are points of S^1 such that one of the following four cases holds: **1)** (a,c) is negative and (b,d) is positive, **2)** (a,c) is negative and (b,d) is free, **3)** (a,c) is free and (b,d) is positive, or **4)** (a,c) and (b,d) are both free. Corollary 4 implies that if f is onto, then (a,d) is one-way. If f is not onto, then (a,d) may not be one-way. Examples 1 and 2 illustrate this possibility. Example 1 is the original example of an interval which is locally one-way but not one-way. A similar example was found independently by M. Hero. Example 1 shows that (a,d) can fail to be one-way in cases 1), 2) and 3). Example 2 illustrates this failure in all four cases.

In both examples, let $e(0) = a_0 < a_1 < a_2 < \cdots < a_8$ be points in S^1, and let $0 = b_0 < b_1 < b_2 < \cdots < b_8 < 1$ be points in $[0,1]$ such that b_i covers a_i for $0 \le i \le 8$.

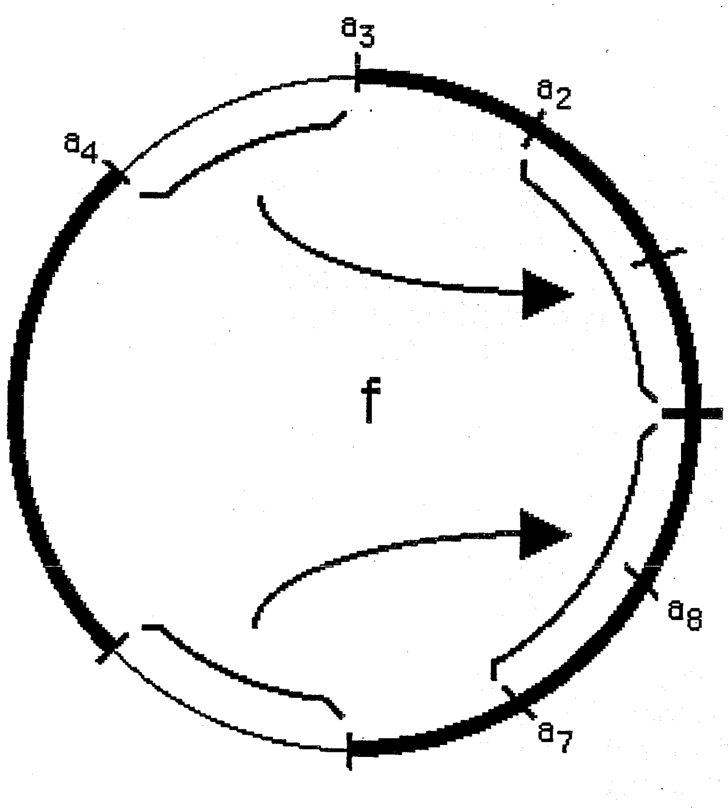

Figure 4a

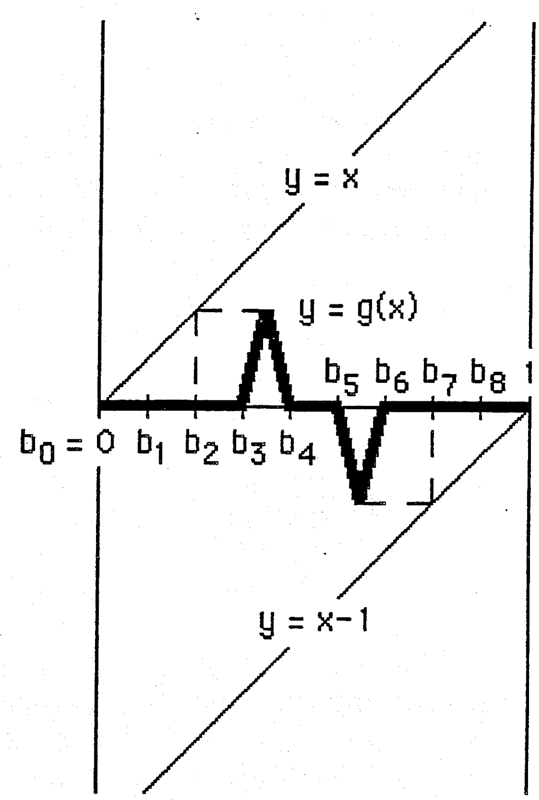

Figure 4b

Example 1. There is a map $f : S^1 \to S^1$ with with respect to which (a_1,a_5) is negative, (a_4,a_8) is positive, and (a_1,a_8) is not one-way. Also (a_1,a_7) is negative and (a_6,a_8) is free; and (a_1,a_3) is free and (a_2,a_8) is positive.

To construct f, let $g : [0,1] \to \mathbb{R}$ be a map such that $g([0,b_3]\cup[b_4,b_5]\cup[b_6,1]) = \{0\}$, $g((b_3,b_4)) = (0,b_2]$ and $g((b_5,b_6)) = [b_7-1,0)$. See Figure 4b. Since $g(0) = g(1)$, there is a unique map from S^1 to itself which is covered by g. Let $f : S^1 \to S^1$ be the unique map which is covered by g; i.e., $f(e(x)) = e(g(x))$ for $x \in [0,1]$. Then

$$f([a_6,a_3]\cup[a_4,a_5]) = f(e([0,b_3]\cup[b_4,b_5]\cup[b_6,1])) =$$

$$e(g([0,b_3]\cup[b_4,b_5]\cup[b_6,1])) = \{e(0)\} = \{a_0\}$$

$$f((a_3,a_4)) = f(e((b_3,b_4))) = e(g((b_3,b_4))) = e((0,b_2]) = (a_0,a_2], \text{ and}$$

$$f((a_5,a_6)) = f(e((b_5,b_6))) = e(g(b_5,b_6))) = e([b_7-1,0)) = [a_7,a_0).$$

See Figure 4a.

Next it is proved that $f^n(S^1) = \{a_0\}$ for $n \geq 2$. Since $f([a_6,a_3]\cup[a_4,a_5]) = \{a_0\}$ and $f(a_0) = a_0$, then $f^n([a_6,a_3]\cup[a_4,a_5]) = \{a_0\}$ for $n \geq 1$. Since $f((a_3,a_4)\cup(a_5,a_6)) = (a_0,a_2]\cup[a_7,a_0) \subset [a_6,a_3]$, then $f^{n+1}((a_3,a_4)\cup(a_5,a_6)) \subset f^n([a_6,a_3]) = \{a_0\}$ for $n \geq 1$. So $f^n((a_3,a_4)\cup(a_5,a_6)) = \{a_0\}$ for $n \geq 2$. Therefore, $f^n(S^1) = \{a_0\}$ for $n \geq 2$.

The fact that (a_1,a_5) and (a_1,a_7) are negative with respect to f is a consequence of the next three statements. $f((a_1,a_3]\cup[a_4,a_7)) = [a_7,a_0]$ and $[a_7,a_0]\cap(a_1,a_7) = \varnothing$. $f((a_3,a_4))\cap(a_1,a_7) = (a_0,a_2]\cap(a_1,a_7) = (a_1,a_2]$. For $n \geq 2$, $f^n((a_1,a_7)) = \{a_0\}$ and $a_0 \notin (a_1,a_7)$. The fact that (a_2,a_8) and (a_4,a_8) are positive with respect to f is proved similarly. (a_1,a_3) and (a_6,a_8) are free with respect to f because for $n \geq 1$, $f^n((a_1,a_3)\cup(a_6,a_8)) \subset f^n([a_6,a_3]) = \{a_0\}$ and $a_0 \notin (a_1,a_2)$. □

Example 2. There is a map $f : S^1 \to S^1$ with respect to which (a_1,a_4) and (a_3,a_6) are free and (a_1,a_6) is not one-way. Also (a_1,a_5) is negative and (a_2,a_6) is positive. So (a_1,a_5) is negative and (a_3,a_6) is free, and (a_1,a_4) is free and (a_2,a_6) is positive.

The construction here is similar to Example 1. First a map $g : [0,1] \to \mathbb{R}$ is defined such that $g([0,b_2]\cup[b_3,b_4]\cup[b_5,1]) = \{0\}$, $g((b_2,b_3)) = [b_5-1,0)$, and $g((b_4,b_5)) = (0,b_2]$. See Figure 5b. Then $f : S^1 \to S^1$ is defined to be the unique map which is covered by g. Therefore, $f([a_5,a_2]\cup[a_3,a_4]) = \{a_0\}$, $f((a_2,a_3)) = [a_5,a_0)$, and $f((a_4,a_5)) = (a_0,a_2]$. See Figure 5a.

Arguments similar to those given in Example 1 prove that $f^n(S^1) = \{a_0\}$ for $n \geq 2$, and that (a_1,a_4) and (a_3,a_6) are free, and that (a_1,a_5) is negative and (a_2,a_6) is positive. Thus, (a_1,a_6) is not one-way. □

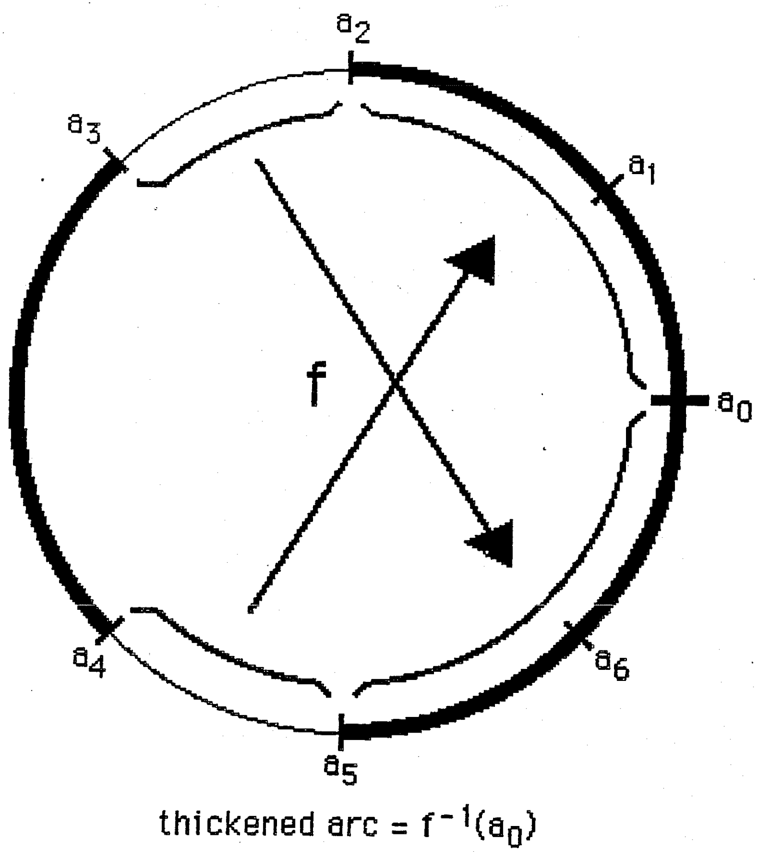

Figure 5a

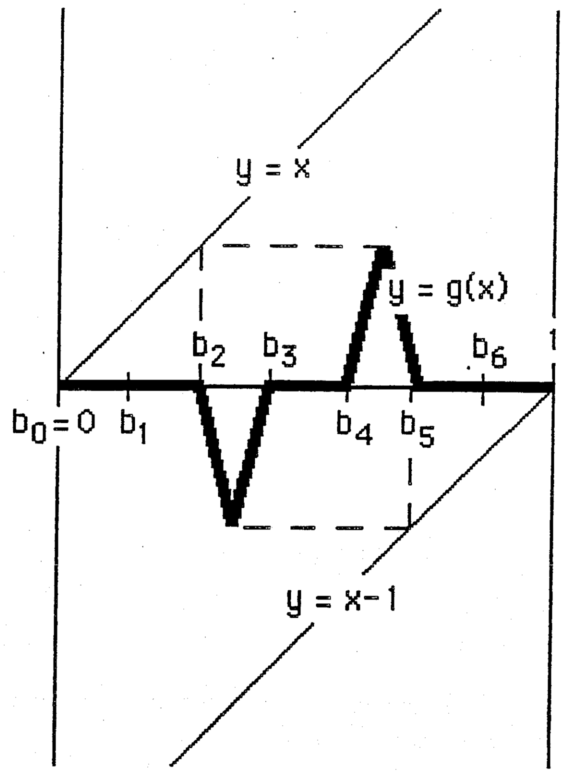

Figure 5b

6. Conclusions

In this paper it has been proved that for any onto circle map, every locally one-way interval is one-way. One corollary of this theorem is that for a circle map of non-zero degree, every locally one-way interval is one-way. Another corollary is that for every onto circle map which has a periodic point, every interval which contains no periodic points is one-way. Also two examples of non-onto circle maps are given which admit subintervals of the circle that are locally one-way but not one-way. The proof of the theorem reveals four different ways in which such examples can arise. The two examples illustrate these four possibilities.

Some questions for future research are now posed.

Questions. Suppose $f : S^1 \to S^1$ is a non-onto map such that there is an open interval J in S^1 which is locally one-way but not one-way with respect to f.
1) Must the dynamical behavior of f on J resemble the dynamical behavior of the two examples given in this paper?
2) If not, can J be broken into smaller intervals on which f resembles one or the other of the two examples?

3) If not, is there some simple characterization of the non-onto maps which have intervals that are locally one-way but not one-way?

References

[1] W. A. Coppel, *Continuous maps of an interval,* photocopied notes.

[2] E. M. Coven and I. Mulvey, Transitivity and the centre for maps of the circle, *Ergodic theory and dynamical systems* 6 (1986), 1-8.

[3] M. W. Hero, *A characterization of the attracting center for dynamical systems on the interval and circle,* Ph.D. Thesis, University of Wisconsin - Milwaukee, 1990.

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THE BING CONJECTURE

Matthew Brahm

Abstract. R.H. Bing conjectured that if a (wild) simple closed curve in a 3-manifold shrinks in its own complement, then it bounds a nonsingular disk in the 3-manifold. The conjecture is related to some important problems in the topology and geometry of 3-manifolds, like the recognition problem, the free surface problem, and the Plateau problem. In this talk we will present progress towards proving the conjecture. Namely, we can show that the simple closed curve bounds a singular disk, where the singularities can be pushed arbitrarily close to the boundary. We will indicate that a new general position property of 3-manifolds can be derived from this result. Finally, we will describe ideas for approaching the general case.

Conjecture 1 (The Bing Conjecture) *Let $f : D^2 \to M^3$ be a map of a 2-disk into a 3-manifold where $f_{|\partial D^2}$ is one to one and $f(\partial D^2) \cap f(IntD^2) = \emptyset$. Then there exists an embedding $g : D^2 \to M^3$, where $g_{|\partial D^2} = f_{|\partial D^2}$.*

We can think of the conjecture as a generalization of Dehn's lemma, where we allow the singularites to get arbitrarily close to the boundary. When Bing published this conjecture in 1961 (page 10 [Bi]), he was interested in properties of 3-manifolds which could be used to detect when certain cell-like upper semicontinuous decompositions of 3-manifolds were not actual 3-manifolds. If the Bing conjecture were true, it could be used as follows. Suppose there is a real 2-disk in the (3-manifold) domain of a cell-like map, where the boundary of the disk misses the nondegenerate elements of the induced decomposition. Then the map is one-to-one over the boundary of the disk, just as in the conjecture. If the image of the map, the decomposition space, were a real 3-manifold, then the Bing conjecture would tell us that the image of the boundary must bound a real disk. So, in order to prove that the decomposition space is not a real 3-manifold, it would be sufficient to prove (say, using the geometry of the nondegenerate elements), that the embedded boundary does not bound a nonsingular disk in the decomposition space.

Since we imagine an application in which the image of the disk is arbitrary, in a space which is not necessarily a manifold, we make no assumptions on the tameness of the image of the boundary of the disk. Therefore, it is best to think of the boundary as wildly embedded. The conjecture is, in fact, true if the boundary is tamely embedded. This fact was known by Bing but never published. One of Bing's students, D. Henderson, later proved a slight strengthening of this, that the conjecture is true if the image of the boundary is "nicely" wild (Theorem V.4 [Hen]). A nicely wild simple closed curve has either a finite number of wild points with finite penetration index, or a tame 0-dimensional set of wild points with penetration index 2.

The Bing conjecture has some characteristics of a famous unsolved problem about topologically embedded surfaces in a 3-manifold. In proving the Bing conjecture, one imagines having to control the growth of long feelers produced by standard cut and paste techniques. This is a difficulty often faced in approaching the free surface problem, stated below.

Conjecture 2 (The Free Surface Problem) *Let $f : S^2 \to \mathbf{R}^3$ be an embedding of a 2-sphere in 3-space, such that for each $\epsilon > 0$ there exists a PL map $g_\epsilon : S^2 \to IntS^2$ such that for any $x \in S^2, d(f(x), g_\epsilon(x)) < \epsilon$ (where $IntS^2$ denotes the bounded component of $\mathbf{R}^3 - f(S^2)$). Then $f(S^2)$ bounds a 3-ball (i.e. $f(S^2)$ is tame on the inside).*

The Bing conjecture is also related to a recent result in the theory of noncompact 3-manifolds. E. M. Brown and C. D. Feustel's plane theorem (Theorem 2.2 [BnFe]), stated below, shows that given a map of a disk, as in the hypothesis of the Bing conjecture, we can replace the image of the interior of the disk with a nonsingular open disk which satisfies an essentiallity condition. In all likelihood this nonsingular map of the open disk can not be extended to a map of the whole disk. The proof of the plane theorem relies on an infinite number of applications of the loop theorem. It is the lack of control in the loop theorem which prevents the map on the interior of the disk from converging at the boundary.

Theorem 1 (The Plane Theorem) *Let $f : IntD^2 \to M^3$ be a map of an open (unit) 2-disk into a noncompact 3-manifold, and let $C \subset M^3$ be a compact set where $f(\partial B(0;r))$ is not null homotopic in $M^3 - C$ for r very close to 1. Then there exists an embedding $g : IntD^2 \to M^3$, where $g(\partial B(0;r))$ is not null homotopic in $M^3 - C$ for r very close to 1.*

Next we present progress towards proving the Bing conjecture. The following theorem, which we call the modified Bing conjecture (Lemma 3

[Br]), shows that we can replace the map in the hypothesis of the Bing conjecture with a new map which hits a compact sub-3-manifold in a pairwise disjoint collection of embedded disks.

Theorem 2 (The Modified Bing Conjecure) *Let $f : D^2 \longrightarrow M^3$ be a map of a 2-disk into a 3-manifold where $f_{|\partial D^2}$ is one to one and $f(\partial D^2) \cap f(IntD^2) = \emptyset$, and let $L \subset M^3 - f(\partial D^2)$ be a compact 3-manifold with boundary. Then there exists a map $g : D^2 \longrightarrow M^3$, where $g_{|\partial D^2} = f_{|\partial D^2}$, and a compact 3-manifold with boundary, L', where $L \subset L' \subset M^3 - f(\partial D^2)$, such that $g^{-1}(L')$ is a collection of pairwise disjoint disks, F, where $g_{|F}$ is an embedding.*

Instead of giving a proof of the modified Bing conjecture we'll give the following application of the theorem. The recognition problem for n-manifolds asks for a general position property of an n-manifold which forces a decomposition space of an n-manifold to be a real n-manifold. In higher dimensions (greater than 4) the appropriate property turns out to be the following; any map of a 2-manifold can be approximated arbitrarily closely by an embedding. Clearly this is a trivial property of manifolds of dimension greater than 4. R. D. Edwards showed that an equivalent property, called DDP, is enough to insure that a finite dimensional decompostion space of a 5 or higher dimensional manifold is an actual manifold [Ed].

In dimension 3, a generic map of a 2-manifold is obviously not approximable by an embedding. A map of a 2-manifold into a 3-manifold which is *almost* approximable by an embedding is a map with a 0-dimensional singular set. By *almost* approximable by an embedding we mean that we can approximate by a map where the diameter of the preimage of any point in the 3-manifold can be made arbitrarily small. It is conjectured that this particular general position property of 3-manifolds is strong enough to insure that decompositions of 3-manifolds are real 3-manifolds (Conjecture 5.4 [DaRe]). We'll now show that the modified Bing conjecture can be used to establish this nontrivial property of 3-manifolds (Theorem 4 [Br]).

Theorem 3 (The Almost Embedding Theorem) *Let $f : M^2 \longrightarrow M^3$ be a map of a 2-manifold into a 3-manifold where $\{x : f^{-1}(f(x)) \neq \{x\}\}$ is 0-dimensional, and $\epsilon, \delta > 0$. Then there exists a map $g : M^2 \longrightarrow M^3$ such that for any $x \in M^2$, $d(f(x), g(x)) < \epsilon$, and for any $y \in M^3$, diam $f^{-1}(y) < \delta$.*

OUTLINE OF PROOF. We take a triangulation of the 2-manifold which is of fine mesh with respect to the given ϵ and δ. We assume that the 1-skeleton of the triangulation has been pushed off the 0-dimensional singular set. Thus, the map restricted to each 2-simplex in the triangulation is set up for the modified Bing conjecture. Applying the modified Bing conjecture, we replace the map on each of the 2-simplexes by a map with "large" nonsingular cutting off disks in the interior of each 2-simplex. Finally, we apply a standard cut and paste argument to remove intersections between the image of any two disjoint disks in the 2-skeleton of the triangulation. □

We end by drawing the Bing conjecture into context with a famous problem in geometry, the Plateau problem. Given a simple closed curve in 3-space, the Plateau problem asks for a minimal area surface whose boundary is the curve. In 1931 J. Douglas solved the problem by proving that with no additional assumptions on the simple closed curve (i.e. it could be wild) it bounds a singular disk of minimal area [Do]. It is of course true, that an arbitrary simple closed curve can be knotted, so the interior of the surface guaranteed by Douglas might hit the boundary. W. Meeks and S. Yau proved much later that if the simple closed curve is contractible and on the boundary of a convex 3-manifold, then the curve bounds an embedded minimal area disk [MeYa].

The connection between these results and the Bing conjecture may be tenuous at best, but it seems that the additional structure that a minimal surface affords may be a valuable tool in approaching the Bing conjecture. First, it seems difficult to find a solution to the Bing conjecture by standard cut and paste techniques. The machinery of the theory of minimal surfaces give us access to many more surfaces than could be generated by just rearranging the simplices in the image of the map. Also, it seems inherent in the nature of a minimal surface that excessively long feelers can't occur. More concretely, there is a PL mininmal surface version of the loop theorem by W. Jaco and J. H. Rubinstein [JaRu] which, when adapted in the right way, might provide that elusive extra control necessary to make progress on the Bing conjecture.

So we see that the Bing conjecture is a very rich problem. It is interesting in its own right as a generalization of Dehn's lemma. It could have applications in the study of decomposition spaces of 3-manifolds, as well as, play a role in the characterization of 3-manifolds by general position properties. It is similar to a famous unsolved problem in topology, the free surface problem. Finally, we might hope to bring to bear on the problem tools from various areas of topology and geometry. The standard techniques of PL geometric topology like cutting and pasting, Dehn's lemma and the the loop theorem are obviously applicable. In addition, some tricks from the theory of noncompact 3-manifolds could prove useful. Finally, the additional structure of minimal surfaces could provide the necessary control missing in previous approaches to the problem.

References

[Bi] R. H. Bing, "Decompositions of E^3", Topology of 3-Manifolds and Related Topics, ed. by M. K. Fort Jr., Prentice-Hall, Englewood Cliffs, N. J., (1962) 5-21.

[Br] M. Brahm, Approximating Maps of 2-Manifolds with 0-Dimensional Nondegeneracy Sets, to appear in Topology and Its Applications.

[BnFe] E.M. Brown and C.D. Feustel, On Properly Embedding Planes in Arbitrary 3-Manifolds, Proc. Amer. Math. Soc. 94, (1985) 173-178.

[DaRe] R.J. Daverman and D. Repovš, A New 3-Dimensional Shrinking Criterion, Trans. Amer. Math. Soc. 315 (1989) 219-230., preprint.

[Do] J. Douglas, Solution of the Problem of Plateau, Trans. Amer. Math. Soc. 33 (1931) 263-321.

[Ed] R.D. Edwards, Approximating certain cell-like maps by homeomorphisms, (1977), manuscript.

[Hen] D.W. Henderson, Self-Unlinked Simple Closed Curves, Trans. Amer. Math. Soc. 120 (1965), 470-480.

[JaRu] W. Jaco and J. H. Rubinstein, PL Minimal Surfaces in 3-Manifolds, J. Diff. Geom. 27 (1988) 493-524.

[MeYa] W. Meeks and S. Yau, The Classical Plateau and the Topology of Three Dimensional Manifolds, Topology, 21 (1982) 409-440.

Geometrically tame hyperbolic 3-manifolds

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In this note we will discuss three notions of tameness for infinite volume hyperbolic 3-manifolds. These three types of tameness (topological, geometrical and analytical) are all conjectured to be equivalent, but each definition emphasizes different features of the manifolds.

A complete (orientable) Riemannian 3-manifold N is said to be hyperbolic if it has constant sectional curvature -1. In this case, N is the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a discrete, torsion-free subgroup Γ of the group $Isom_+(\mathbf{H}^3)$ of orientation-preserving isometries of $\mathbf{H}^3$. The limit set L_Γ is defined to be the smallest closed Γ-invariant subset of the sphere at infinity S^2_∞ for hyperbolic 3-space.

We will assume throughout this note that N has finitely generated fundamental group and that every homotopically non-trivial curve is homotopic to a closed geodesic. The first assumption is essential, but the second is only made for ease of exposition.

We will say that N is *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. It is conjectured that all hyperbolic 3-manifolds (with finitely generated fundamental group) are topologically tame. This conjecture was first posed as a question by Al Marden [6].

The basic geometric object associated to N is its convex core $C(N)$. $C(N)$ is defined to be the smallest convex submanifold of N whose inclusion into N is a homotopy equivalence. Explicitly, $C(N)$

is the quotient of the convex hull of the limit set by the action of Γ. Ahlfors finiteness' theorem [1] asserts that the boundary $\partial C(N)$ is a finite collection of closed hyperbolic surfaces. (In general if N has finitely generated fundamental group, then $\partial C(N)$ is a finite collection of finite area hyperbolic surfaces.) There exists a retraction $R : N \to C(N)$ which simply takes a point in N to the nearest point in $C(N)$. R induces a product structure on the complement of $C(N)$, in particular $N - C(N)$ is homeomorphic to $\partial C(N) \times (0, \infty)$ and the metric is K-quasiisometric to $cosh^2 t ds^2_{\partial C(N)} + dt^2$.

If $C(N)$ is compact, N is said to be convex cocompact and the manifold is clearly topologically tame. (In general, N is said to be geometrically finite if $C(N)$ has finite volume.) However, not all hyperbolic 3-manifolds are convex cocompact. In fact, the convex core may be the entire manifold (see Jorgensen [5] for example.) But there always exists a compact submanifold M such that the inclusion map is a homotopy equivalence and the ends of N are in one-to-one correspondence with the components of $N - M$ (see Scott [8]). An end E is said to be geometrically finite if some neighborhood of E misses $C(N)$.

An end is said to be *simply degenerate* if it is homeomorphic to $S \times [0, \infty)$ and there exists a sequence of surfaces $\{f_n : S \to E\}$ such that $\{f_n(S)\}$ leaves every compact set, $f_n(S)$ is homotopic to $S \times \{0\}$ (within E) and the induced geometry on $f_n(S)$ has curvature ≤ -1. N is said to be *geometrically tame* if all its ends are either simply degenerate or geometrically finite.

Our first existence theorem has the most desirable form: group theoretic conditions imply a strong geometric consequence:

Theorem 1: (Bonahon [2]) *If $\pi_1(N)$ is freely indecomposable, then N is geometrically tame.*

One may use this to prove:

Theorem 2: (Canary [3]) *A hyperbolic 3-manifold is geometrically tame if and only if it is geometrically tame.*

Theorem 2 reduces many questions in the theory of Kleinian groups to purely topological issues, for example Ahlfors' measure conjecture.

Also whenever one can uses topological means to guarantee topological tameness, then one can use theorem 2 to derive geometric consequences. For example, if N is a closed hyperbolic 3-manifold and G is a finitely generated subgroup of $\pi_1(N)$ whose abelianization has infinite index in $H_1(N)$, then $\mathbf{H}^3/G$ is topologically and hence geometrically tame.

We now turn to a third notion of tameness which carries the analytic information in the definition of geometric tameness. A hyperbolic 3-manifold is said to be analytically tame if $C(N)$ can be exhausted by a sequence $\{C_i\}$ of compact submanifolds such that $C_i \subset C_j$ if $i < j$ and there exists K and L such that the area of ∂C_i is less than K for all i and the volume of the neighborhood of radius one of ∂C_i is less than L for all i. We may easily observe that

Theorem 3: (Canary [3]) *If N is geometrically tame then it is also analytically tame.*

One of the first consequences of analytical tameness is that the Ahlfors' measure conjecture holds for such manifolds.

Theorem 4: (Thurston [11], Canary [3]) *If $N = \mathbf{H}^3/\Gamma$ is an analytically tame hyperbolic 3-manifold then its limit set L_Γ either has measure zero or is all of S^2_∞. If $L_\Gamma = S^2_\infty$, then Γ acts ergodically on L_Γ.*

One may make use of work of Sullivan [9] to observe:

Theorem 5: (Thurston [11], Canary [3]) *Let N be an analytically tame hyperbolic 3-manifold. Then N's geodesic flow is ergodic if and only if $L_\Gamma = S^2_\infty$.*

Analytically tame hyperbolic 3-manifolds are also a good setting in which to do spectral theory. Let $\lambda_0(N)$ denote the bottom of the spectrum of the Laplacian acting on N. One nice statement is:

Theorem 6: (Canary [4]) *Let N be an analytically tame hyperbolic 3-manifold. Then N is geometrically finite if and only if $\lambda_0(N) = 0$.*

For a further discussion of spectral theory on analytically tame hyperbolic 3-manifolds and its relationship to the Hausdorff dimension of the limit set (among other things), see Sullivan [10], Patterson [7] or Canary [4].

References

[1] L.V. Ahlfors, "Finitely generated Kleinian groups," *Amer. J. of Math.* **86**(1964), pp. 413-29.

[2] F. Bonahon, "Bouts des variétés hyperboliques de dimension 3," *Ann. of Math.* **124**(1986), pp. 71-158.

[3] R.D. Canary, "Ends of hyperbolic 3-manifolds," preprint.

[4] R.D. Canary, "The Laplacian and the geometry of hyperbolic 3-manifolds," in preparation.

[5] T. Jorgensen, "Compact 3-manifolds of constant negative curvature fibering over the circle," *Ann. of Math.* **98**(1977) pp. 61-72.

[6] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99**(1974), pp. 383-462.

[7] S.J. Patterson, "Lectures on limit sets of Kleinian groups," in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, pp. 281-323.

[8] P. Scott, "Compact submanifolds of 3-manifolds," *J. London Math. Soc.* **7**(1973), pp. 246-250.

[9] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions," *Publ. I.H.E.S.* **50**(1979), pp. 419-450.

[10] D. Sullivan, "Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups," *Acta. Math.* **153**(1984), pp. 259-277.

[11] W. Thurston, *The Geometry and Topology of 3-manifolds*, lecture notes.

MANIFOLDS WITH NON-ZERO EULER CHARACTERISTIC ARE CODIMENSION 2 FIBRATORS

by R. J. Daverman

Manifolds are taken to be connected, metrizable, boundaryless and orientable. A closed n-manifold N is a codimension 2 fibrator if, whenever $p:M \to B$ is a proper mapping from an (n+2)-manifold M to a metric space B such that each $p^{-1}b$ is shape equivalent to N, p is an approximate fibration. The somewhat surprising point is: manifolds with this desireable feature are widely prevalent. Background appears in [D1], which this supercedes.

A closed orientable n-manifold N is hopfian if all degree one maps $N \to N$ inducing fundamental group isomorphisms are homotopy equivalences. Whether all such degree one maps $N \to N$ are homotopy equivalences is an old problem, due of course to Hopf (cf. [Ha]). Recall that a group Γ is hopfian if every epimorphism $\Gamma \to \Gamma$ is an automorphism. The first paragraph of the proof hints at the relevance of these concepts.

The topic is the title statement, limited a bit by hopfian data. The main ideas appear ahead in the breakdown into cases depending on subgroup index size, with reference to [D2] for technical matters (the detailed proof also has been incorporated into an earlier version of [D2]).

Theorem. Every closed, hopfian n-manifold N with $\pi_1(N)$ hopfian and $\chi(N) \neq 0$ is a codimension 2 fibrator.

Proof. Simplifying a little, let $p:M \to B$ be a proper map defined on an (n+2)-manifold M such that each $p^{-1}b$ is a copy of N. Just as in [DW, Prop. 4.8] or [D1, Prop. 2.8], B is a 2-manifold, B contains a dense subset C (the continuity set of p), points of $B\backslash C$ are isolated in B, so one can immediately localize to the situation in which $B \approx E^2$ and p is an approximate fibration over the complement of the origin, 0. (Remark: over $E^2\backslash 0 = C$ each point preimage of p has a neighborhood retraction transporting nearby preimages in degree one fashion, and the hypothesis about N being a hopfian manifold with hopfian group ensures that the restricted retractions are the homotopy equivalences needed to produce approximate fibrations.)

Set $g_0 = p^{-1}0$. Properties of ANRs and relative lifting properties of the approximate fibration $p|M\backslash g_0$ give rise to a strong deformation retraction $R:M \to g_0$.

Treating the contrapositive, we show $\chi(N) = 0$ when p fails to be an approximate fibration. In view of an idea of Im (or see [D2, Lemma 5.2]), we can suppose $R|g$ does not induce an surjection $H_1(g) \to H_1(g_0)$ for other $g = p^{-1}b$ $(b\in E^2\backslash 0)$. The exact homotopy sequence for approximate fibrations [CD, Cor. 3.5] gives

$$1 = \pi_2(E^2\backslash 0) \to \pi_1(g) \to \pi_1(M\backslash g_0) \to \pi_1(E^2\backslash 0) \to 1 ,$$

Hence, there exists an epimorphism of $\pi_1(M\backslash g_0)$ to Z whose kernel equals the image of $\pi_1(g)$. Because g_0 has codimension 2, inclusion $M\backslash g_0 \to M$ induces an H_1-epimorphism, which makes $H_1(M)/j_*(H_1(g)) \cong H_1(g_0)/R_*(H_1(g))$ a nontrivial cyclic group T ($j:g \to M$ denotes inclusion). Form the cyclic covering $\theta:M' \to M$ corresponding to the kernel K of the natural composite epimorphism.

$$\pi_1(g) \to \pi_1(M) \to H_1(M) \to T .$$

Since R lifts to another deformation retraction $R':M' \to \gamma_0 = \theta^{-1}(g_0)$, θ restricts to a cyclic covering $\theta_0:\gamma_0 \to N$.

Case 1: $[\pi_1(M):K] = \infty$. Upon verification that γ_0 has finitely generated homology, work of Milnor [Mi, Assertion 6] will provide the conclusion $\chi(N) = 0$.

Here γ_0 has the homotopy type of M' , and $M'\backslash\gamma_0$ is partitioned into copies of N (i.e., the components of the various sets $(p\theta)^{-1}(z)$, $z\neq 0$). The associated decomposition map φ (which again is proper) makes the following diagram commutative:

$$\begin{array}{ccc} M'\backslash\gamma_0 & \xrightarrow{\varphi} & B' \\ \downarrow{\scriptstyle \theta|} & & \downarrow{\scriptstyle \mu} \\ M\backslash g_0 & \xrightarrow{p} & E^2\backslash 0 \end{array}$$

The induced map $\mu : B' \to E^2\backslash 0$ is an infinite cyclic covering and $M'\backslash\gamma_0 \to B' \approx E^2$ is an approximate fibration, as before; thus, $M'\backslash\gamma_0$ has the homotopy type of N.

An inductive argument indicates $H_k(\gamma_0)$ is finitely generated. Obvious for $k=0,1$, suppose it holds for all $j<k\geq 2$. Then $H_{k-2}(\gamma_0)$ finitely generated implies the same about each of the following, in turn: $H_c^{n+2-k}(\gamma_0)$, by Poincaré duality in γ_0; $H_k(M', M'\backslash\gamma_0)$, by Poincaré-Lefshetz duality in M'; and $H_k(M') \cong H_k(\gamma_0)$, by inspection of the long exact sequence for the pair $(M', M'\backslash\gamma_0)$.

Case 2: $[\pi_1(M):K] = m > 1$. This time $R|g : g \to g_0$ has positive degree [D2, Lemma 5.2'] (the argument is virtually identical to the older [D2, Lemma 5.2], using rational coefficients for homology). Therefore, $R|g$ lifts to a positive degree map $r_g : g \to \gamma_0$ with $R|g = \theta r_g$. From naturality of cap products, any positive degree map $N_1 \to N_2$ between closed, orientable n-manifolds yields $\beta_i(N_1) \geq \beta_i(N_2)$ for arbitrary Betti numbers β_i. Application with both $\theta|\gamma_0$ and r_g establishes

$$\beta_i(N) = \beta_i(g) \geq \beta_i(\gamma_0) \geq \beta_i(g_0) = \beta_i(N)$$

for all i, which implies $\chi(N) = \chi(\gamma_0) = m\cdot\chi(N)$ and $\chi(N) = 0$.

Corollary 1. Every closed 4-manifold N with $\pi_1(N)$ hopfian and $\chi(N) \neq 0$ is a codimension 2 fibrator.

Proof. Hausmann [Ha] has shown that the hopfian fundamental group condition implies the hopfian manifold condition for $n<5$.

As a result, fundamental group properties alone most emphatically do not classify codimension 2 fibrators.

Corollary 2. Every closed 4-manifold N with $\pi_1(N)$ hopfian and $H_1(N)$ finite is a codimension 2 fibrator.

REFERENCES

[CD] D. S. Coram and P. F. Duvall, "Approximate fibrations," Rocky Mtn. J. Math. **7** (1977), 275-288.

[D1] R. J. Daverman, "Submanifold decompositions that induce approximate fibrations," Topology Appl. **33** (1989), 173-184.

[D2] _______, "Hyperhopfian groups and approximate fibrations," preprint.

[DW] R. J. Daverman and J. J. Walsh, "Decompositions into codimension 2 submanifolds," Trans. Amer. Math. Soc. **288** (1985), 273-291.

[Ha] J. C. Hausmann, "Geometric Hopfian and non-Hopfian situations," in Geometry and Topology (C. McCrory and T. Shiflin, eds.) Marcel Dekker, Inc., New York, 1987, 157-166.

[Im] Y. H. Im, "Decompositions into codimension two submanifolds that induce approximate fibrations," preprint.

[Mi] J. Milnor, "Infinite cyclic coverings," in Conference on the Topology of Manifolds (J. G. Hocking, ed.), Prindle Weber & Schmidt, Inc., Boston, 1968, 115-133.

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The Topology of Minimal Surfaces in R^3

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This talk is on joint work with Bill Meeks.

Let A and B be subsets of space X. We say that A and B are topologically equivalent if there is a homeomorpism H : X --> X so that H(A) = B. We say that A and B are homeomorphic if there is a homeomorphism h: A --> B .By minimal surface we mean a connected properly embedded surface in R^3 with mean curvature zero. We are interested in giving useful necessary and sufficient conditions for two minimal surfaces to be topologically equivalent. A minimal surface has nonpositive gauss curvature. The first theorem along these lines is due to Meeks.

Theorem. Two properly embedded surfaces of finite genus in R^3 with one end are topologically equivalent if and only if they are homeomorphic.

The theorem is proved by noting that distance from a point in R^3 generically induces a morse function on the surface that has only index 0 and index 1 critical points. This is used to show that there is a large ball so that the intersection of the surface with the ball is a Heegaard surface for the ball with one boundary component and that the part of the surface exterior to the ball is a standardly embedded once punctured disk. The topological unicity of the surface inside the ball is deduced from Waldhausen's classification of Heegaard splittings of the three-sphere.

The next result along these lines is a theorem of mine. A surface in R^3 is said to be triply periodic if it is invariant under the action of some cocompact lattice in R^3.

Theorem. Any two triply periodic minimal surfaces in R^3 are topologically equivalent.

This theorem led to the following theorem of Meeks and Frohman.

Theorem. Any two minimal surfaces in R^3 having infinite genus and one end are topologically equivalent.

This theorem is proved by first proving that a minimal surface with one end in R^3 is a Heegaard surface. It is still an open question whether a surface of nonpositive gauss curvature with one end in R^3 is a Heegaard surface, the unsettled case being exactly when the surface has infinite genus.

A Heegaard surface in a three-manifold is said to be infinitely reducible if there exists a disjoint proper family of balls each intersecting the Heegaard surface in a surface of genus one with one boundary component so that each end representative of the manifold contains infinitely many of these balls. It is an easy to prove consequence of the Reidemeister -Singer theorem that any two infinitely reducible Heegaard surfaces of a manifold a topologically equivalent. Furthermore it is easy to see that any Heegaard surface of R^3 of infinite genus is infinitely reducible.

Next we come to a theorem of Meeks and Yau.

Theorem. Any two minimal surfaces of finite type are topologically equivalent if and only if they are homeomorphic.

The easiest way to see this result is to apply a theorem of Frohman. If F is a surface that is properly embedded in the ball B then the graph of F is defined to have as vertices the connected components of ∂B - F where two vertices are connected by an edge if and only if their closures have nonempty intersection.

Theorem. Two Heegaard surface of the ball are topologically equivalent if and only if they have the same genus and isomorphic graphs. Furthermore for each finite graph and each genus there is a Heegaard splitting having that graph and genus.

To prove Meeks and Yau's result first prove that there is a large ball so that the part of the minimal surface inside the ball is a Heegaard surface and the part of the surface outside the ball is a family of standardly embedded once punctured disks. Then prove that the graph of the surface inside the ball must be a line segment . Then apply my result.

Recently Meeks and I have proven the following.

Theorem. If F is a minimal surface in R^3 then there is a geometric ordering on the ends of the surface. The betweenness relation induced by this ordering is a topological invariant of the surface.

Let me describe the associated betweenness relation. Let A, B and C be ends of F. Then B is between A and C if and only if there exist disjoint properly embedded planes P and Q in R^3 having compact intersection with F and end representatives a, b and c so that any arc joining b to a or c has Z_2 intersection number one with $P \cup Q$.

An end is called a limit end if it is a limit point of the ordering, this will be true if and only if the end is a limit point in the topology induced on the space of ends of F by construction. In fact the order topology is equivalent to the end topology. We say that E is an interior limit end of F if it is a limit end and it is not the minimum or maximum in the ordering. We conjecture that a minimal surface can have no interior limit ends. We say an end is even if it bounds the end of one of the components of its complement in R^3 . Otherwise we say that the end is odd.

Theorem. Let F and F' be two minimal surfaces in R^3 and assume that F has no limit ends. Then F is topologically equivalent to F' if and only if there is a homeomorphism h: F --> F' so that the induced map on ends preserves the parity of the ends and the induced betweenness relation.

This theorem allows us to classify all known examples of minimal surfaces up to topological equivalence. It would be interesting to see how much of a parallel theory exists for surfaces of nonpositive surfaces. One of the major properties of minimal surfaces, that they carry the fundamental groups of the components of their complement definitely fails for nonpositively curved surfaces with more than one end. This means that there should be a much richer classification theory.

Three Level Forms in S^4

Eerik Harms

August 23, 1991

1 Introduction

William Eaton and I have been studying PL embeddings of 3-manifolds in S4. William Eaton is especially interested in the 4-Dimensional PL Schoenflies Conjecture. The purpose of this talk is to introduce the notion of a 3-level form and to describe the setting for the following theorem:

Theorem 1 Any Morse ordered critical level embedding $e : M^3 \hookrightarrow S^4$ such that $e(M^3) \cap S^3 \times 0$ is a Heegaard splitting of the equatorial 3-sphere $S^3 \times 0$ is ambiently isotopic to a 3-level form.

A *3-level form* is a special type of piecewise linear embedding of a connected 3-manifold which consists of three flat spots connected by vertical collars. A more precise definition will be given below.

William Eaton has proven the following theorem:

Theorem 2 If a piecewise linear embedding $e : M^3 \hookrightarrow S^4$ of a connected 3-manifold has a simply connected complement then e is ambiently isotopic to a Morse ordered critical level embedding such that $e(M^3) \cap S^3 \times 0$ is a Heegard splitting of $S^3 \times 0$.

It follows from these theorems that any embedding $f : \Sigma^3 \hookrightarrow S^4$ where Σ^3 is either S^3 or an arbitrary homotopy 3-sphere is isotopic to a 3-level form. As a consequence the study of 3-level forms is a possible starting point for investigations of the following conjectures:

The 4-Dimensional Piecewise Linear Schoenflies Conjecture:
If $e : S^3 \hookrightarrow S^4$ is a piecewise linear embedding then a complementary component of e is a piecewise linear 4-ball.

The Codimension One Poincaré Conjecture:
If $e : \Sigma^3 \hookrightarrow S^4$ is a piecewise linear embedding of a homotopy 3-sphere Σ^3 then Σ^3 is piecewise linearly homeomorphic to S^3.

2 The Woodwork

We think of S^4 as the double suspension of S^3. So S^4 is the two point compactification of $S^3 \times (-2, 2)$. The two points at infinity are refered to as the north and south poles respectively and are denoted by $\mathcal{N}$ and $\mathcal{S}$. For an embedded 3-manifold M^3 we define $EXT(M^3)$ to be the closure of the complementary component which contains $\mathcal{S}$ and $INT(M^3)$ to be the closure of the other component.

It follows from a result of Kearton and Lickorish [1] that any 3-manifold embedded in S^4 is isotopic to a critical level embedding in $S^3 \times [-1, 1]$. Some references dealing with the theory of PL critical level embeddings are listed in the bibliography.

A critical level embedding $e : M^3 \hookrightarrow S^3 \times [-1, 1] \subsetneq S^4$ is an embedded manifold which for some handle-collar decomposition embeds each handle H_i horizontally in a critical level $S^3 \times t_i$ and each collar vertically in such a way that the attaching tube of each handle attaches to the collar below and the belt tube of each handle attaches to the collar above. It is a standard result that any critical level embedding of a 3-manifold in $S^3 \times [-1, 1]$ can be arranged so that the handles appear in order of increasing index. We can also arrange that the 0 and 1-handles have critical values less than zero and that the 2 and 3-handles have critical values more than zero. A critical level embedding which has been arranged in this way is called a Morse ordered form. The image of a Morse ordered form will intersect $S^3 \times 0$ in a surface F whose preimage $e^{-1}(F)$ is a Heegaard splitting of M^3. The surface F might not in general be a Heegaard splitting of $S^3 \times 0$.

There is a slight generalization of the notion of a critical level embedding refered to as a collar connected embedding in which the critical levels may contain several handles and some of the collars. A 3-level form is a collar connected embedding $e : M^3 \hookrightarrow S^4$ whose image has the form

$$e(M^3) = (K^3_- \times -1) \cup (\partial K^3_- \times [-1,0]) \cup (R^3 \times 0) \cup (\partial K^3_+ \times [0,1]) \cup (K^3_+ \times 1)$$

where

1. K^3_- and K^3_+ are standard cubes with handles in S^3
2. $cl(S^3 - K^3_+) \subseteq int(K^3_-)$.

Several pictures of 3-level forms are given in figures 3,4 and 5. Notice that a 3-level form will also satisfy $cl(S^3 - K^3_-) \subseteq int(K^3_+)$. The cobordism R^3 refered to as the residue cobordism of the 3-level form. Notice that R^3 is a 3-manifold in S^3 with two boundary components each of which is a Heegaard splitting of S^3. Notice also that the submanifold M^3 is completely determined by the residue cobordism R^3.

A schematic picture of a 3-level form is given in figure 1. It is important that the space immmediately preceding $R^3 \times 0$ lies in $INT(M^3)$ and the space immediately preceeding $K^3_+ \times 1$ lies in $EXT(M^3)$ as indicated in figure 1. If this were not the case the the embedding would be trivially isotopic to the standard embedding of S^3 in S^4.

It is a fact that if M^3 is a homology sphere then $INT(M^3)$ and $EXT(M^3)$ are homology 4-balls and that R^3 is a homology product[1]. If R^3 is an actual product then $M^3 \cong S^3$ and the embedding is isotopic to the standard embedding of S^3 in S^4. The first picture in figure 3 shows a 3-level form whose residue cobordism is a product and figure 5 shows a 3-level form of S^3 whose residue cobordism is not a product.

References

[1] C. Kearton, W.B.R. Lickorish: *Piecewise Linear Critical Levels and Collapsing,*
Trans. Am. Math. Soc., Vol. 170, 1972, pp. 415-424

[1]See Spellman [4].

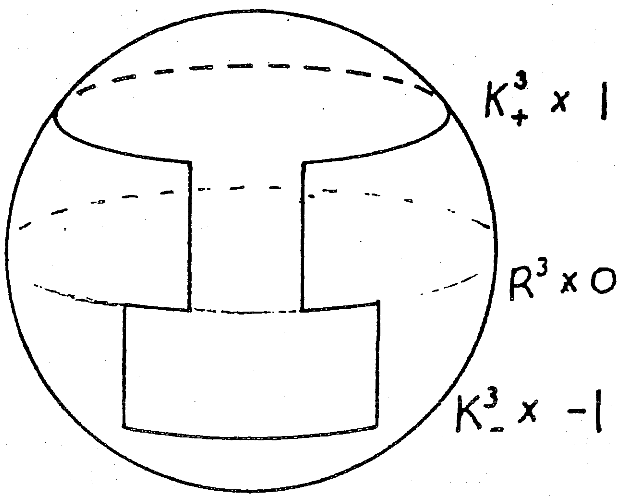

Figure 1: A schematic picture of a 3-level form

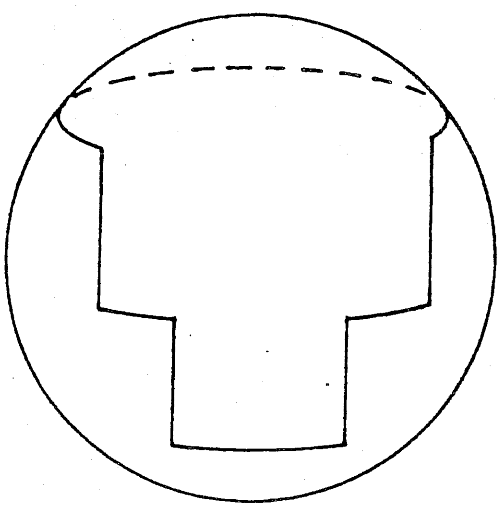

Figure 2: A schematic picture of a trivial embedding

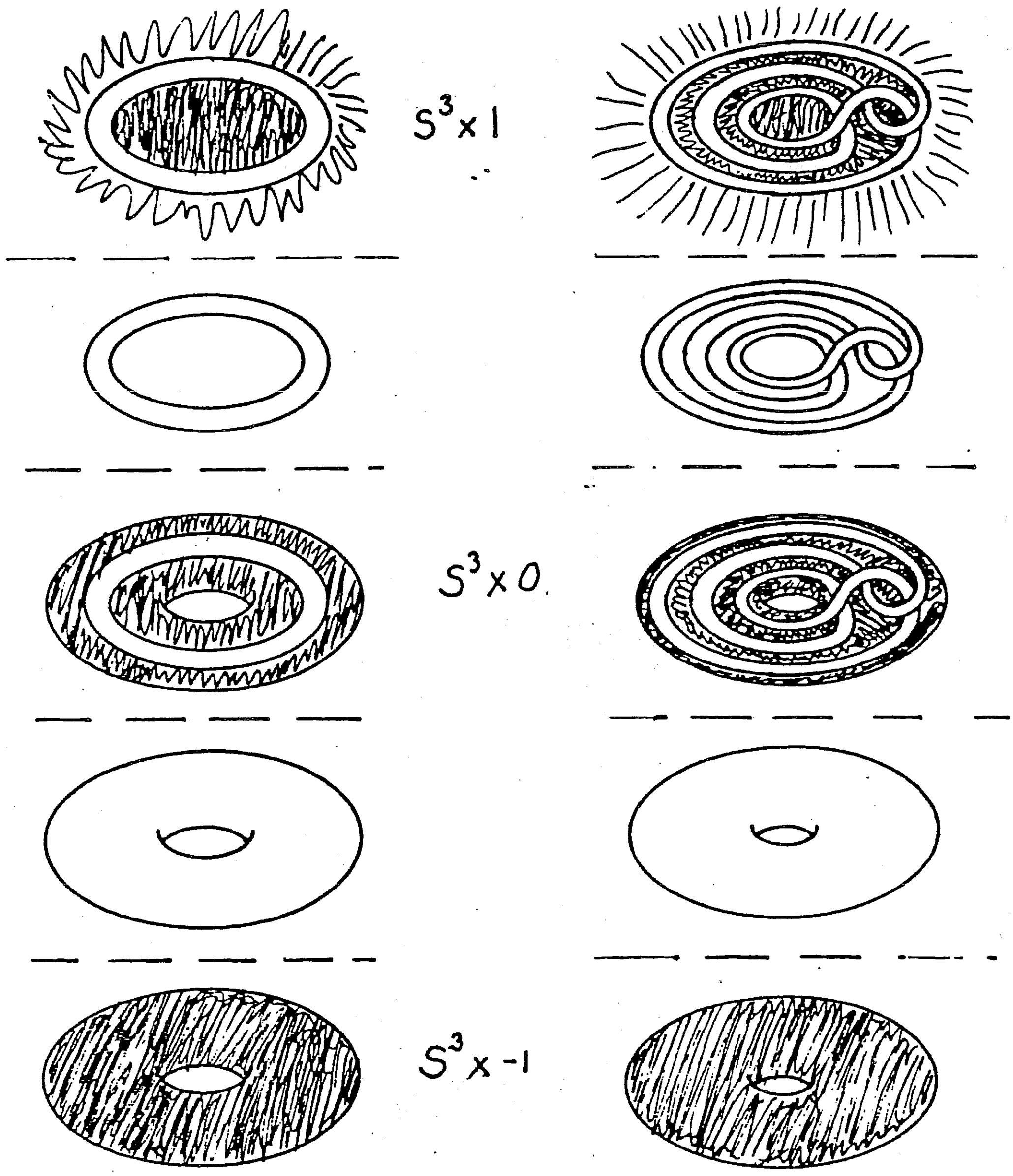

Figure 3: Product Residue

Mazur's 3-Manifold

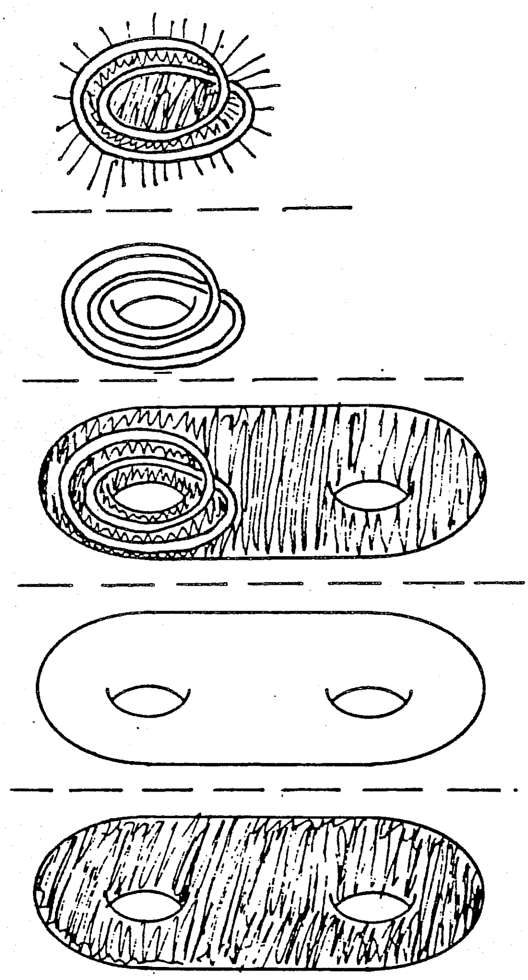

Figure 4: $\partial N(RP^2)\#S^1 \times S^2$

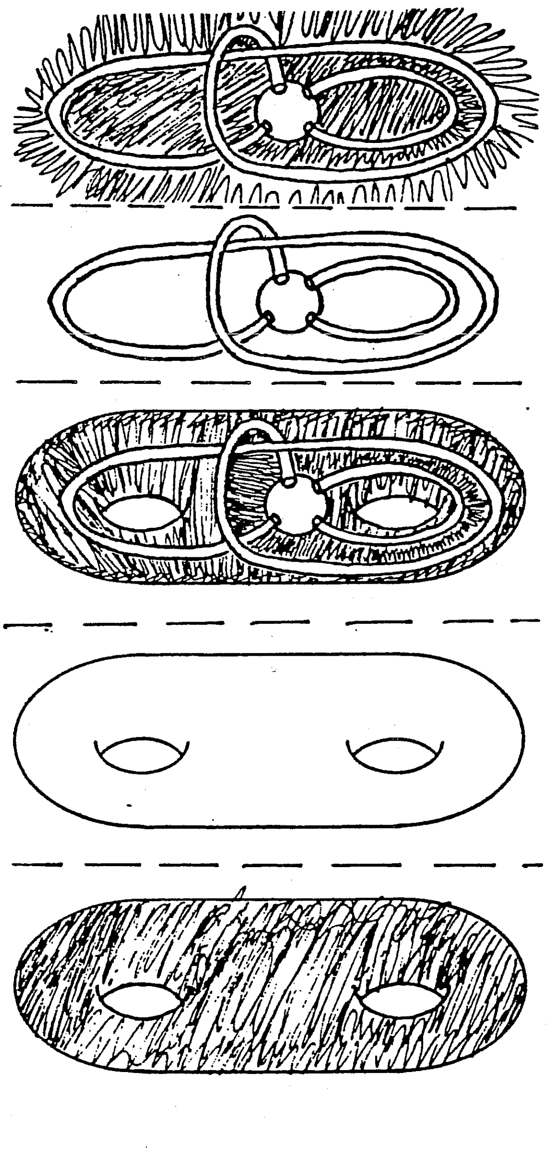

Figure 5: A 3-level form of S^3

[2] C.P. Rourke, B.J. Sanderson: *Introduction to Piecewise-Linear Topology*,
Springer-Verlag, Heidelberg, 1972,1982

[3] M. Scharlemann: *The Four Dimensional Schoenflies Conjecture is True for Genus Two Embeddings.*
Topology Vol. 4, No. 2 (1984) pp. 211-217

[4] D.L. Spellman: *Critical Level Embeddings of S^3 in S^4.*
Ph.D. Thesis, University of Texas, 1986

Irreducible Representations of Strongly Countable Dimensional Spaces

Richard P. Millspaugh, Leonard R. Rubin,

and Philip J. Schapiro

Irreducible polyhedral representations for metric compacta were introduced in a 1937 paper [F] of H. Freudenthal. The idea is to represent a metric compactum X as the inverse limit of a sequence $\{P_i, f_{ij}\}$ of compact polyhedra each having covering dimension [Na1] no more than that of X and such that the bonding maps are PL and irreducible. In 1960, E.G. Sklyarenko [Sk] extended these results to strongly countable dimensional metric compacta.

Recall that a space X is said to be strongly countable dimensional (scd) if it can be written as $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is a closed finite dimensional subspace of X.

For a simplicial complex N, a space X, and a map $f: X \to |N|$, we say that a map $g: X \to |N|$ is an N-modification of f if whenever $f(x)$ is contained in a simplex σ of N, so is $g(x)$. A map $f: X \to |N|$ is said to be N-irreducible if every N-modification of f is surjective. A map $f: X \to P$ from a space X to a polyhedron P is said to be irreducible if there is a triangulation N of P such that f is N-irreducible.

Let X be a metric space and $\mathrm{P} = \{P_i, f_{ij}\}$ an inverse sequence of metric spaces. A sequence of maps $f_i : X \to P_i$ is said to be an ω-representation of X in P if the map $f : X \to \prod_{i=1}^{\infty} P_i$ given by $f(x) = (f_i(x))$ embeds X on a dense subspace of lim P and is called a faithful ω-representation if f is a homeomorphism onto lim P. The ω-representation is strongly countable dimensional (scd) if there is a fixed triangulation K_i for each P_i such that for each $x \in X$ $\sup\{\dim \sigma_i\} < \infty$, where σ_i denotes the carrier of $f_i(x)$ in K_i.

Translated into these terms, Sklyarenko's result is the following,

Theorem. A metric compactum X is scd if and only if it has a faithful scd ω-representation in a sequence $\mathrm{P} = \{|K_i|, f_{ij}\}$ of compact polyhedra.

We have extended this result in the following ways,

Theorem 1. Let X be a metric space. Then X is scd if and only if X has an scd ω-representation.

Theorem 2. Let X be a topologically complete metric space. Then X is scd if and only if X has a faithful scd ω-representation.

To prove sufficiency in the case of theorem 1, let $f_i : X \to |K_i|$ be an scd ω-representation of X in P. Fix $n \geq 0$. Define a sequence of polyhedra $Y_{i,n}$ as follows:

Let $Y_{1,n}$ be the n-skeleton $|K_1^{(n)}|$ of K_1.

For $i \geq 1$ let $Y_{i+1,n} = (f_{i\,i+1})^{-1}(Y_{i,n}) \cap |K_{i+1}^{(n)}|$.

Then $\{Y_{i,n}, f_{ij}\}$ forms an inverse sequence of polyhedra of dimension $\leq n$. By 27.9 of [Na], $Y_n = \lim\{Y_{i,n}, f_{ij}\}$ has dimension at most n. Since each Y_n is closed in lim P, the desired result is obtained by using the definition of an scd ω-representation to show that $f(X) \subset \bigcup_{n=1}^{\infty} Y_n$.

The chief tool used in constructing an scd ω-representation for an scd metric space is the following lemma, which is proved using an extension of ideas extracted from the proof of Theorem 5.3 of [Na2].

Lemma. Let X be an scd metric space. Then there exists a function $n:X \to \mathbb{Z}$ so that for every open cover $\mathcal{U}$ of X there is a locally finite open cover $\mathcal{V}$ of X refining $\mathcal{U}$ satisfying

1) $\mathrm{ord}_x \mathcal{V} \le n(x)+1$ for all $x \in X$,
2) the nerve N of $\mathcal{V}$ is locally finite dimensional, and
3) there is a normal N-irreducible map $b:X \to |N|$.

Note: the map b is normal means that for every $V \in \mathcal{V}$, $b^{-1}(\mathrm{star}(V,N)) = V$, and b is essential on each simplex of N.

The lemma is used to define inductively a sequence of fine open covers of X whose nerves will be the required polyhedra. At the same time, the normality of the maps b_i so obtained will guarantee that we can find irreducible bonding maps between these polyhedra (provided the covers of X are chosen at each stage to refine covers which consist of inverse images of stars of vertices in fine subdivisions of the previously constructed nerves). Uniform limits of compositions of these bonding maps with b_j's will provide the desired scd ω-representation for X. The details can be found in [MRS].

The second theorem requires only noticing that the construction given in the first will yield a faithful scd ω-representation if the space X is topologically complete.

It should be noted that similar results can be obtained in the class of normal spaces using representations in approximate inverse systems (see [MR]) rather than inverse sequences.

References

[F] H. Freudenthal, Entwicklungen von Räumen und ihren Gruppen, Compositio Math. 4 (1937), 145-234

[MR] S. Mardešić and L.R. Rubin, Approximate Inverse Systems of Compacta and Covering Dimension, Pacific J. of Math. (to appear)

[MRS] R.P. Millspaugh, L.R. Rubin, and P.J. Schapiro, Irreducible Representations of Strongly Countable Dimensional Spaces, preprint

[Na] K. Nagami, Dimension Theory, Academic Press, New York, 1970

[Na1] J. Nagata, Modern Dimension Theory (revised and extended edition), Heldermann Verlag, Berlin, 1983

[Na2] J. Nagata, On the Countable Sum of Zero-dimensional Metric Spaces, Fundamenta Mathematicae 48 (1960), 1-14

[Sk] E.G. Sklyarenko, Representation of Infinite-dimensional Compacta as an Inverse Limit of Polyhedra, Dokl. Akad. Nauk SSSR 155 (1960), 1147-1149

A GEOMETRIC APPROACH TO THE DIMENSION THEORY OF INFINITE-DIMENSIONAL SPACES

JAN J. DIJKSTRA AND JERZY MOGILSKI[1]

Abstract. In this note we discuss the problem of preserving some "dimensionality properties" of infinite-dimensional spaces under hereditary shape equivalences.

1. Introduction. The spaces in this note are assumed to be compact metric. Let us recall that a (proper) surjection $f : X \to Y$ is a *cell-like* map if, for every $y \in Y$, $f^{-1}(y)$ is of trivial shape (i.e. $f^{-1}(y)$ is a cell-like set in X). We say that f is a *hereditary shape equivalence* if for every closed subset A in Y $f|f^{-1}(A) : f^{-1}(A) \to A$ is a shape equivalence (i.e., for each ANR Z and each closed $A \subset Y$ the map f produces a one-to-one correspondence between the homotopy classes of $C(A, Z)$ and $C(f^{-1}(A), Z)$). The map f is a *fine homotopy equivalence* if for every open cover $\mathcal{U}$ of Y there exists a map $g : Y \to X$ such that $f \circ g$ is $\mathcal{U}$-homotopic to the identity on Y and $g \circ f$ is $f^{-1}(\mathcal{U})$-homotopic to the identity on X.

According to Haver [**12**] and Kozlowski [**15**] a cell-like map $f : X \to Y$ between absolute neighbourhood retracts is a fine homotopy equivalence and a hereditary shape equivalence. Hereditary shape equivalence is the natural extension to arbitrary compacta of fine homotopy equivalence between ANRs.

The first example of a cell-like map which was not a hereditary shape equivalence was constructed by Taylor [**23**]. It was shown recently by Dranišnikov [**8**] that a cell-like image of a finite-dimensional compactum is not necessarily finite-dimensional. It is well known, however, that finite dimension cannot be raised by hereditary shape equivalences. In this note we will discuss the behaviour of certain "dimensionality properties" of infinite-dimensional spaces under hereditary shape equivalences.

2. Dimension theory of infinite-dimensional spaces. If a space X is finite dimensional we write $X \in \mathbf{FD}$. A space X is infinite-dimensional ($X \in \mathbf{ID}$) if it is not finite-dimensional.

We say that X is *strongly infinite-dimensional* ($X \in \mathbf{SID}$) if there exists a sequence $(A_i, B_i)_{i=1}^{\infty}$ of pairs of closed disjoint subset of X such

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that for every sequence of separators $(S_i)_{i=1}^{\infty}$ of X between A_i and B_i we have $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$.

The space X is *weakly infinite-dimensional* ($X \in$ **WID**) if X is not strongly infinite-dimensional.

A space X is *countable-dimensional* ($X \in$ **CD**) if $X = \bigcup_{i=1}^{\infty} X_i$, where $X_i \in$ **FD** for $i = 1, 2, \ldots$.

We say that X is *strongly countable-dimensional* ($X \in$ **SCD**) if $X = \bigcup_{i=1}^{\infty} X_i$, where $X_i \in$ **FD** and X_i is closed in X for $i = 1, 2, \ldots$.

A space X has *the property $\mathcal{C}$* ($X \in \mathcal{C}$) if for every sequence of positive numbers $(\varepsilon_n)_{n=1}^{\infty}$ there exists an open cover $\mathcal{U}$ of X such that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, where $\mathcal{U}_n$ is a pairwise disjoint family consisting of open sets with diameters $< \varepsilon_n$ (see [**1**] and [**11**]).

We have

$$\mathbf{FD} \subset \mathbf{SCD} \subset \mathbf{CD} \subset \mathcal{C} \subset \mathbf{WID} \quad \text{and} \quad \mathbf{FD} \neq \mathbf{SCD} \neq \mathbf{CD} \neq \mathcal{C}.$$

PROBLEM 2.1. *Is it true that $\mathcal{C} \neq$ **WID**?*

There is one known example of a space with the property $\mathcal{C}$ which is not countable dimensional (Pol [**19**]) but that space has a strongly infinite dimensional subspace. It suggests the following question (cf. [**10**, Problem 8.6] and [**25**, D 12]).

PROBLEM 2.2. *Is there a metrizable compact space every subset of which has the property $\mathcal{C}$ but that is not countable dimensional?*

There are natural extensions of small and large inductive dimensions over countable dimensional spaces (see [**14, 22, 24**]). If α is a countable ordinal ($\alpha < \omega_1$), then $\operatorname{ind} X \leq \alpha$ if for every $x \in X$ and for every open set U, with $x \in U$, there exists an open set V such that $x \in V \subset U$ and $\operatorname{ind}(\partial V) < \alpha$; $\operatorname{Ind} X \leq \alpha$ if for every closed subset A of X and for every open set U, with $A \subset U$, there exists an open set V such that $A \subset V \subset U$ and $\operatorname{Ind}(\partial V) < \alpha$.

We have $\operatorname{ind} X = \operatorname{Ind} X$ for $X \in$ **FD** and $\operatorname{ind} X \leq \operatorname{Ind} X$ in general. The following result was proved by Hurewicz [**14**] and Smirnov [**22**].

THEOREM 2.3. *If X is a compactum then $X \in$ **CD** if and only if $\operatorname{ind} X < \omega_1$, and if and only if $\operatorname{Ind} X < \omega_1$.*

3. Hereditary shape equivalence and transfinite dimension. The following question was posed by Henderson, Kozlowski and Walsh at the problem session of the AMS meeting in Norman, 1983.

PROBLEM 3.1. *Do hereditary shape equivalences preserve countable dimensionality?*

In view of results of Kozlowski [**15**] this question is equivalent to

PROBLEM 3.2. *Let $f : X \to Y$ be a cell-like map of a countable dimensional ANR X onto an ANR Y. Is Y countable dimensional?*

For infinite-dimensional spaces the following is known: hereditary shape equivalences preserve weak infinite-dimensionality ([**17**, **21**]) and the property $\mathcal{C}$ ([**16**, **4**]) but they do not preserve strong countable dimensionality ([**7**]).

The behaviour of transfinite dimension under hereditary shape equivalences is not clearly understood. However, it turns out that it is closely related to Problem 3.1. Using the fact that a complete space X is countable dimensional if and only if $\operatorname{ind} X$ exists and that a cell-like map with a countable dimensional range is a hereditary shape equivalence ([**3**])we define

$$\eta(X) = \sup\{\operatorname{ind} Y : Y \text{ is a countable dimensional cell-like image of } X\}.$$

We have the following equivalence the sufficiency of which was proved in [**7**] and the necessity was recently proved by the authors.

THEOREM 3.3. *For hereditary shape equivalences to preserve countable dimensionality it is necessary and sufficient that every countable dimensional compactum X has countable $\eta(X)$.*

Using absorbers Dijkstra recently showed the following:

THEOREM 3.4. *There exists a cell-like map from an ω-dimensional compact AR onto an $(\omega+1)$-dimensional compact AR.*

This is an improvement of the main result in [**6**] and it shows that the transfinite dimension functions ind and Ind are not preserved under hereditary shape equivalences. Consider now Alexandroff's Essential Mapping Theorem ([**2**]):

THEOREM 3.5. *A space has dimension greater than a finite n if and only if it admits an essential map onto the $(n+1)$-cell.*

Generalising an idea of Henderson [**13**] we define: if M is an AR with a closed subset S then a map f from a space X onto M is called essential if every map $g : X \to M$ that coincides with f on $f^{-1}(S)$ is also surjective. It is easily verified that the composite map of hereditary shape equivalence and an essential map is again essential. Theorem 3.4 implies that essential maps cannot distinguish between certain certain ω-dimensional and $(\omega+1)$-dimensional spaces. This solves a problem of Henderson [**13**] and Pol [**18**]. These results show that the transfinite ind and Ind are "geometrically incorrect" dimension functions. In contrast, Pol's index [**18**] and Borst's transfinite covering dimension [**5**] are examples of dimension functions that can be characterised by essential maps onto Henderson's "cubes" [**13**] and that are preserved by hereditary shape equivalences. Theorem 3.3 and Theorem 3.4 suggest a strategy of attacking Problem 2.2 and Problem 3.1. We summarize this strategy in the following conjectures.

CONJECTURE 3.6. *For every countable ordinal ordinal α there is a hereditary shape equivalence from an ω-dimensional compactum onto a compactum with* ind $> \alpha$.

Conjecture 3.6 combines with Theorem 3.3 and a result of Pol [**20**] to:

CONJECTURE 3.7. *Countable dimensionality is not preserved under hereditary shape equivalences.*

In fact, Theorem 3.3 implies that Conjecture 3.7 is equivalent to:

CONJECTURE 3.8. *There is a countable ordinal α such that* $\sup\{\eta(X) : \operatorname{ind}(X) < \alpha\}$ *is uncountable.*

Conjecture 3.7 implies:

CONJECTURE 3.9. *There is a compact space every subset of which has property C but that is not countable dimensional.*

A proof for this conjecture would also solve the open problems [**10**, 8.6] and [**25**, D 12].

REFERENCES

1. D. F. Addis and J. H. Gresham, *A class of infinite dimensional spaces. Part I: Dimension theory and Alexandroff's problem*, Fund. Math. **101** (1978), 195–205.
2. P. S. Alexandroff, *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), 161–238.
3. F. D. Ancel, *The role of countable dimensionality in the theory of cell-like relations*, Trans. Amer. Math. Soc. **287** (1985), 1–40.
4. F. D. Ancel, *Proper hereditary shape equivalences preserve Property C*, Topology Appl. **19** (1985), 71–74.
5. P. Borst, *Classification of weakly infinite-dimensional spaces. Part I: A transfinite extension of the covering dimension*, Fund. Math **130** (1988), 1–25.
6. P. Borst and J. J. Dijkstra, *Essential mappings and transfinite dimension*, Fund. Math. **125** (1985), 41–45.
7. J. J. Dijkstra, J. van Mill and J. Mogilski, *An AR-map whose range is more infinite-dimensional than its domain*, Proc. Amer. Math. Soc. (to appear).
8. A. N. Dranišnikov, *On a problem of P. S. Alexandrov*, Matem. Sbornik **135** (1988), 551–557.

9. R. Enkelking, *Transfinite dimension*, in "Surveys in General Topology," G. M. Reed ed., New York, 1980, pp. 131–161.
10. R. Enkelking and E. Pol, *Countable dimensional spaces. A survey*, Dissertationes Math. **216** (1983), 1–45.
11. W. E. Haver, *A covering property for metric spaces*, Lecture Notes in Math. **375** (1974), 108–113.
12. W. E. Haver, *Mappings between ANR's that are fine homotopy equivalences*, Pacific J. Math. **58** (1975), 457–461.
13. D. W. Henderson, *A lower bound for transfinite dimension*, Fund. Math. **64** (1968), 167–173.
14. W. Hurewicz, *Ueber unendlich-dimensionale Punktmengen*, Proc. Akad. Amsterdam **31** (1928), 916–922.
15. G. Kozlowski, *Images of ANR's*, unpublished manuscript.
16. J. van Mill and J. Mogilski, *Property C and fine homotopy equivalences*, Proc. Amer. Math. Soc. **90** (1984), 118–120.
17. R. Millspaugh, *Proper hereditary shape equivalences preserve small weak infinite dimensionality*, Proc. Amer. Math. Soc. **110** (1990), 1055–1061.
18. R. Pol, *On a classification of weakly infinite-dimensional compacta*, Topology Proc. **5** (1980), 231–242.
19. R. Pol, *A weakly infinite-dimensional compactum which is not countable dimensional*, Proc. Amer. Math. Soc. **82** (1981), 634–636.
20. R. Pol, *Countable dimensional universal sets*, Trans. Amer. Math. Soc. **297** (1986), 255–268.
21. D. M. Rohm, *Hereditary shape equivalences preserve weak infinite dimensionality*, preprint.
22. Ju. M. Smirnov, *On universal spaces for certain classes of infinite dimensional spaces*, Amer. Math. Soc. Transl. Ser 2 **21** (1962), 21–33.
23. J. L. Taylor, *A counterexample in shape theory*, Bull. Amer. Math. Soc. **81** (1975), 629–632.
24. P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fund. Math. **7** (1925), 30–137.
25. J. E. West, *Open problems in infinite dimensional topology*, in "Open problems in topology," J. van Mill and G. M. Reed eds., North-Holland, Amsterdam, 1990, pp. 523–597.

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Deforming Reducible $SL_2(\mathbf{C})$–Representations of Knot Groups

Douglas Shors

0. Introduction

In this paper we ask the following questions: First, given a reducible representation of a knot group π in $SL_2(\mathbf{C})$ (these representations are easy to characterize), when are there nearby irreducible representations of π? And second, if there are such representations, what can be said about the component(s) of the character variety which they comprise?

Along similar lines, Thurston [T, theorem 5.6] obtains a lower bound for the dimension of the space of small deformations of a hyperbolic structure (with irreducible holonomy) on a 3-manifold. In particular, if X is a component of the $SL_2(\mathbf{C})$–character variety of the fundamental group of a hyperbolic link complement (with r components) containing a lift of a hyperbolic structure with irreducible holonomy, then $\dim_{\mathbf{C}} X \geq r$.

In [FK], Frohman and Klassen show how, in certain cases, an abelian $SU(2)$–representation of a knot group deforms into an arc of irreducible $SU(2)$–representations and an arc of irreducible $SL_2(\mathbf{R})$–representations.

We obtain pictures of the character variety of a knot groups near reducible representations (when the representation corresponds to a simple root of the Alexander polynomial, and more generally, when the second Alexander polynomial is nonzero—see below). We also obtain results about deforming reducible $SL_2(\mathbf{R})$–representations, and reobtain the aforementioned theorem of [FK].

Some of the results stated for knot groups generalize to link groups. For simplicity, only the knot case is discussed here.

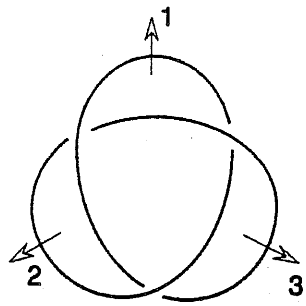
We also determine the topological types of some $SL_2(\mathbf{C})$–character varieties: the nontrivial part of the character variety for the k-twist knot is a smooth genus $k-1$ surface with k punctures. Lastly, we compute the boundary classes corresponding to the ideal points of these character varieties.

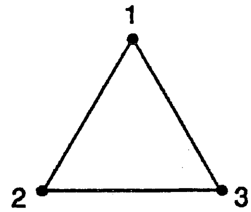
1. Some affine representations of knot groups

A representation $\Gamma \to SL_2(\mathbf{C})$ is reducible if and only if it is conjugate into the planar affine group $\mathrm{Aff}(\mathbf{C}) \simeq \{\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mid a \in \mathbf{C} - \{0\},\, b \in \mathbf{C}\} \subset SL_2(\mathbf{C})$.

We begin with some simple examples of such planar affine representations of knot groups. The figures at the right show the fixed points of the respective Wirtinger generators of the group. These examples can be kept in mind in the next section, when questions of deforming reducible representations are addressed.

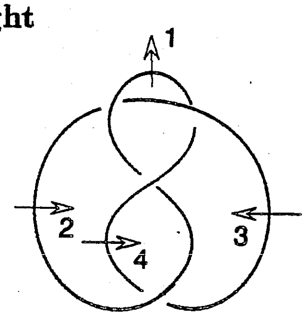
trefoil

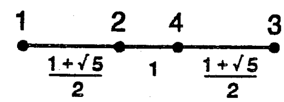




Alexander polynomial $\Delta(t) = t^2 - t + 1$

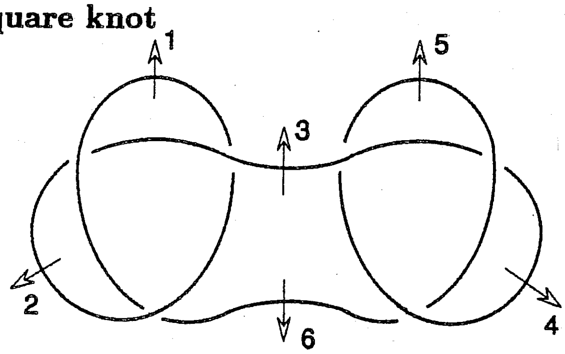
figure eight

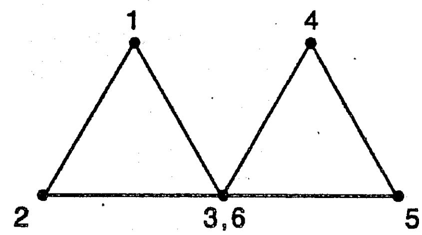




$\Delta(t) = t^2 - 3t + 1$

square knot





$\Delta(t) = (t^2 - t + 1)^2$

2. Results

The following notation will be used:

π	$\pi_1(S^3 - K) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$, a Wirtinger presentation; $K \subset S^3$ a PL knot.
$R(\pi)$	$\mathrm{Hom}(\pi, SL_2(\mathbf{C}))$, with the compact-open topology.
$X(\pi)$	maximal Hausdorff quotient space of $R(\pi)/SL_2(\mathbf{C})$.
$[\rho]$	equivalence class of $\rho \in R(\pi)$ in $X(\pi)$.
$A(t)$	Alexander matrix of some knot $K \subset S^3$.
$\Delta(t)$	Alexander polynomial of $K \subset S^3$.
$\Delta_k(t)$	k^{th} Alexander polynomial of K.

Below we list some elementary facts about $SL_2(\mathbf{C})$–representations of knot groups:

1. $X(\pi)$ is an algebraic variety. (See [CS].)

2. $X(\pi)$ has an algebraic component $X_{\mathbf{ab}} \simeq \mathbf{C}$ consisting of characters of abelian representations $\pi \to \pi/[\pi,\pi] \simeq \mathbf{Z} \to SL_2(\mathbf{C})$.

That $X_{\mathbf{ab}}$ is an algebraic component of $X(\pi)$ follows from a computation of its Weil tangent space at the trivial character (i. e. the character of $\rho(x_i) = I$), which shows that its dimension is equal to the dimension of $X_{\mathbf{ab}}$. (See [W] or [H] for a discussion of the Weil tangent space.)

3. $\rho(x_i)(z) = \alpha(z - \beta_i)$ is a (reducible) representation of π in $SL_2(\mathbf{C}) \iff (\beta_1, \beta_2, \ldots, \beta_n)^T \in \ker A(\alpha)$. Such a representation can be nonabelian (i. e. the β_i's are not all equal) only if $\Delta(\alpha) = 0$.
We will call α the *multiplier* of the representation ρ.

This is seen by solving for $\beta_1, \beta_2, \ldots, \beta_n$ in the linear equations imposed by the Wirtinger relations. The coefficient matrix of the system turns out to be $A(\alpha)$, which has a nontrivial solution (the β_i's not all equal) $\iff$ the nullity $n(A(\alpha))$ of $A(\alpha) \geq 2 \iff \Delta(\alpha) = 0$.

Letting $Z^1(\pi; Ad \circ \rho)$ denote the Weil tangent space to $R(\pi)$ at ρ,

4. If ρ is abelian with multiplier α, then $\dim_{\mathbf{C}} Z^1(\pi; Ad \circ \rho) = 2n(A(\alpha)) + 1$.

For this one solves the system of linear equations which follow from the cocycle relation in Z^1. (See [B].) With appropriate basis the coefficient matrix of the system is $A(0) \oplus A(\alpha) \oplus A(1/\alpha)$. The result follows.

5. $\Delta(\alpha) \neq 0 \Longrightarrow \rho$ doesn't deform into irreducible representations.

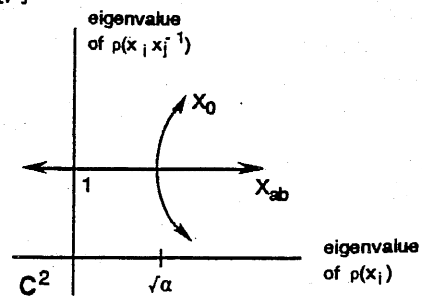
This follows from (4): $\Delta(\alpha) \neq 0 \Longrightarrow \dim Z^1(\pi; Ad \circ \rho) = 3 = \dim_\rho R_{\mathbf{ab}}(\pi)$. A theorem from algebraic geometry says that in such a situation $R_{\mathbf{ab}}(\pi)$ is the only component of $R(\pi)$ through ρ.

The next theorem is a partial converse to (5).

Theorem 2.1. *Suppose ρ is abelian (or reducible nonabelian) with multiplier α.*

(1) *If $\Delta(\alpha) = 0$ and $\Delta_2(\alpha) \neq 0$ then ρ deforms into irreducible representations.*

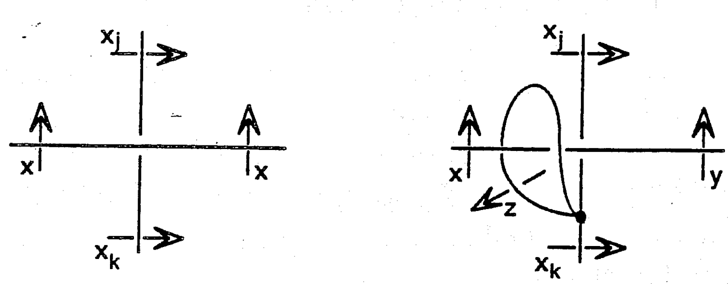
(2) *If α is a simple root of Δ then we have the following picture of $X(\pi)$ near $[\rho]$:*



Remark. The result is sharp in the sense that there are examples of reducible nonabelian representations with $\Delta_2(\alpha) = 0$ which don't deform into irreducible representations.

Idea of proof. (1) Suppose ρ is abelian. The fact that $\Delta(\alpha) = 0$ means that there are reducible nonabelian representations near ρ; let ρ_0 be one such. The argument will show that ρ_0 deforms into irreducible representations, and since ρ is in the closure of the orbit of ρ_0, so does ρ. Without loss ρ_0 fixes $\infty \in \hat{\mathbb{C}} = S^2_\infty$, so upon choosing a Wirtinger presentation for π, $\rho_0(x_i)(z) = \alpha(z - \beta_i)$ for $z \in \hat{\mathbb{C}}$. We will begin by boring out a hole from the knot complement, à la Thurston [T, chapter 5].

$(\beta_1, \beta_2, \ldots, \beta_n) \in \ker A$, and the β_i's are not all equal since ρ_0 is nonabelian. Let $(\beta^1, \beta^2, \ldots, \beta^n) \in \ker A(1/\alpha)$, with the β^i's not all equal (so that $\rho_0'(x_i)(z) = \frac{1}{\alpha}(z - \beta^i)$ is a reducible nonabelian representation of π). By passing to a new knot projection, if necessary, it is possible to choose a crossing with Wirtinger relation $x_i x_j x_i^{-1} = x_k$ so that $\beta_j \neq \beta_k$ and $\beta^j \neq \beta^k$ (so that ρ_0 and ρ_0' each represent x_j and x_k differently). To this crossing add a loop, as shown, and call the fundamental group of the new space π':



An argument along the lines of Thurston's can be used to establish (1): For elementary reasons each algebraic component of the representation space of π'

has dimension at least 6. (Note that we continue to make a distinction between the representation variety and the character variety—this is an issue because ρ_0 is not a stable point of $R(\pi)$ and so a neighborhood of $[\rho_0] \in X(\pi)$ may not be nice.) One can write down collection of polynomial equations which vanish on a nontrivial arc of representations of $\pi \subset R(\pi')$. One difficulty is that there are lots of reducible representations of π' (and π) near ρ_0; one must be careful that the arc obtained above does not consist solely of these representations. (The role of the assumption $\Delta_2(\alpha) = 0$ is to give some control over the dimension of the space of reducible representations of π'.) It should be noted that the sorts of equations Thurston uses don't work because they vanish on the reducible representations of π'.

(2) If α is a simple root of Δ then $\Delta_2(\alpha) \neq 0$, so by (1) ρ deforms into irreducible representations.

The strategy of the proof is to obtain an upper bound on the dimension of $R(\pi)$ at ρ_0, where ρ_0 is a reducible nonabelian representation near ρ as above; this can then be translated into a statement about $X(\pi)$ near $[\rho]$.

In fact, $X(\pi)$ has a single nontrivial branch X_0 through $[\rho]$; it can be shown that $[\rho] \mapsto \mathrm{tr}\rho(x_i x_j^{-1})$ is a smooth parameter on a neighborhood of $[\rho] \subset X_0$, for i and j as above. □

In the following corollaries, ρ is an abelian representation with multiplier α.

Corollary 2.2. *If α is a simple root of Δ and $\alpha \in \mathbf{R}$, then there exists a unique arc of $SL_2(\mathbf{R})$–characters through $[\rho]$.*

Corollary 2.3. *(Frohman–Klassen) If α is a simple root of Δ and $|\alpha| = 1$, then there exists a unique half-arc of $SL_2(\mathbf{R})$–characters and a unique half-arc of $SU(2)$–characters meeting at $[\rho]$.*

Idea of proofs. One shows that complex conjugation $[\rho] \mapsto [\bar{\rho}]$ is a smooth involution on X_0 at $[\rho_0]$. Its fixed point set is a smooth arc consisting of real characters; these correspond to $SU(2)$– and $SL_2(\mathbf{R})$–characters of π. □

Let $A(L, M)$ be the Cooper-Long polynomial and let $\tilde{\Delta}$ be the product of the multiplicity 1 Z–irreducible factors of the Alexander polynomial. (Briefly, $A(L, M)$ is defined as follows: if X_1 is a nontrivial algebraic component of $X(\pi)$, project X_1 to $\mathbf{C}^2$ by $p : [\rho] \to (L_\rho, M_\rho)$, where L_ρ and M_ρ are corresponding eigenvalues of ρ(longitude) and ρ(meridian), respectively. (Of course this map is not well-defined, but this is a minor difficulty.) Assuming X_1 projects to curve in $\mathbf{C}^2$, the closure of $p(X_1)$ is the zero set of a single polynomial $f_{X_1}(L, M)$ in $\mathbf{C}^2$. $A(L, M)$ is the product of these polynomials over all nontrivial algebraic components of $X(\pi)$ which project to curves in $\mathbf{C}^2$. See [CL].) Then

Corollary 2.4. *For any $K \in S^3$, $\tilde{\Delta}(t^2)|A(1,t)$.*

Idea of Proof. Theorem 2.1 shows that if $\tilde{\Delta}(t^2) = 0$ then there is a nontrivial 1-dimensional component of $X(\pi)$ passing through the abelian character with $L = 1$ and $M = t$. This component projects down to a curve in (L, M)–space. So $\tilde{\Delta}(t^2) = 0 \Longrightarrow A(1,t) = 0$. Since $\tilde{\Delta}(t^2)$ is separable, $\tilde{\Delta}(t^2)|A(1,t)$. □

3. Character varieties of twist knots

Let π_k be the fundamental group of the complement of the k-twist knot, and let $X_0(\pi_k)$ be the union of the nontrivial components of $X(\pi_k)$. F_g denotes the closed surface of genus g.

Proposition 3.1. *$X_0(\pi_k)$ is diffeomorphic to $F_{k-1} - \{k \text{ points}\}$.*

Idea of proof. π_k has the following Wirtinger presentation:

$$\pi_k = \langle x_1, x_2, \ldots, x_{k+2} \,|\, x_{i+1} = x_i^{-\epsilon_i} x_{i-1} x_i^{\epsilon_i}\ (2 \le i \le k+1), x_{k+2} = x_1 x_{k+1} x_1^{-1} \rangle,$$

where $\epsilon_i = (-1)^i$.

Suppose ρ is a representation of π_k with $\rho(x_i)$ hyperbolic (this is the case for all but finitely many ρ). One shows that there is an isometry ζ of $\mathbf{H}^3$ which carries the axis of $\rho(x_i)$ to the axis of $\rho(x_{i+1})$ (reversing orientation) for $i = 1, 2, \ldots, k+1$. $PSL_2(\mathbf{C})$–characters of π_k can be classified by $\operatorname{tr} \zeta$, and the $SL_2(\mathbf{C})$–character variety of π_k can be realized as a 2-sheeted branched cover of the $PSL_2(\mathbf{C})$–character variety, branched over dihedral characters and ideal points. □

Lastly, since the above gives a fairly complete understanding of $X(\pi_k)$, it is possible to compute the boundary slope s associated with each of the k ideal points of $X_0(\pi_k)$. (For background, see [CS].) We only give the result of the computation: If k is odd, it turns out that $\frac{k-1}{2}$ of them have $s = 0$, $\frac{k-1}{2}$ have $s = 4$, and one has $s = -2k - 4$. For k even, $\frac{k}{2} - 1$ of them have $s = 0$, $\frac{k}{2}$ have $s = 4$, and one has boundary slope $s = -2k$. (These are the only *strict* boundary slopes, by [HT].)

We end with some questions (some of these may have been posed elsewhere):

1. When do reducible representations with $\Delta(\alpha) = \Delta_2(\alpha) = 0$ deform into irreducible representations? Is each such representation equivalent in $X(\pi)$ to a representation which does deform into irreducible representations?
2. Does $X(\pi)$ have a nontrivial 1-dimensional component for every knot group π?
3. Does $\Delta(t^2)|A(1,t)$ for $K \subset S^3$?
4. When α is a simple root of Δ, what is the slope of $p(X_0) \subset \mathbf{C}^2_{L,M}$ at $p([\rho])$?

REFERENCES

[B] K. S. Brown, *Cohomology of Groups*, Springer-Verlag, New York–Heidelberg– Berlin, 1982.

[CL] D. Cooper and D. D. Long, *Algebraic geometry and knot polynomials*, preprint.

[CS] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109–146.

[FK] C. Frohman and E. Klassen, *Deforming representations of knot groups in $SU(2)$*, preprint.

[H] C. Hodgson, *Geometric structures on 3-dimensional orbifolds: Notes on Thurston's proof*, preprint.

[HT] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Inv. Math. **79** (1985), 225–246.

[T] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University Lecture Notes.

[W] A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.

THE GEOMETRY OF THE OUTER SPACE

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Culler and Vogtmann [C-V] initiated a study of the outer automorphism group $Out(F_n)$ of the free group F_n on n letters by constructing a space X_n upon which $Out(F_n)$ acts properly discontinuously. This space, which has since come to be known as the "outer space", consists of free minimal actions of F_n on simplicial metric trees, where two such trees are identified if they are equivariantly isometric. The quotient of each such tree by such an action is a finite marked metric graph, in which each vertex has valence ≥ 3. Here, "metric" means that each edge has a length, and a "marking" is a preferred homotopy equivalence of the wedge R_n of n circles to this graph. One thus has two possible views of X_n; these are interchangeable. The group $Out(F_n)$ acts in the obvious way: one represents an automorphism of F_n as a self-map of R_n, and precomposes the marking with this map.

Culler and Vogtmann showed that X_n is a finite-dimensional contractible space, and that the $Out(F_n)$-action has finite stabilizers and finite quotient. Furthermore, X_n has a natural compactification $\overline{X}_n$ in which the boundary consists of actions of F_n on **R**-trees with cyclic edge stabilizers. This is analogous to the situation which occurs in Thurston's classification of surface automorphisms; in particular, it suggests a means to study $Out(F_n)$. It is natural to look for parallels between the two theories. In this note we announce two new results in this vein.

1. WEIGHTED LENGTH FUNCTIONS AND FINITE SUBGROUPS OF $Out(F_n)$

It is known that $\overline{X}_n$ is contractible; this was demonstrated, for example, by Skora [Sk], who developed a method to construct a path between two **R**-trees, given a morphism between them. (A *morphism* is a map such that each segment in the domain **R**-tree contains an initial arc which is mapped isometrically.) We study the problem of constructing equivariant maps between arbitrary pairs of **R**-trees, and apply Skora's methods to prove the following theorem:

Theorem. *There is a continuous function $F : X_n \times \overline{X}_n \times I \to \overline{X}_n$ such that*

(1) $F(T_0, T_1, 0) = T_0$;
(2) $F(T_0, T_1, 1) = T_1$;
(3) $F(T_0, T_0, t) = T_0$ *for all t; and*
(4) *F is equivariant under the diagonal action of $Out(F_n)$ on $X_n \times \overline{X}_n$.*

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It is known that any finite subgroup of $Out(F_n)$ possesses a fixed point in X_n; see, for example, [Cu]. It then follows from the above theorem that the subset of $\overline{X}_n$ fixed by a finite subgroup of $Out(F_n)$ is contractible.

The first step in the proof of the theorem is to give a canonical construction for maps between **R**-trees. The following suffices for the present purpose:

Let T and T' be **R**-trees, equipped with F_n-actions, and suppose that T is simplicial and that its action is free. For each $x \in T$, we first define a function $\lambda : F_n \to \mathbf{R}$ via $\lambda(g) = 1/d_T(x, gx)$ if $g \neq \mathrm{id}$, and $\lambda(\mathrm{id}) = 0$. We now set $f(x)$ to be the center of the finite subtree of T' which minimizes

$$\sup_{g \in F_n} \lambda(g) d_{T'}(y, gy).$$

(One must first ascertain that this supremum is finite for all $y \in T'$, and that it is minimized on a compact subtree.) This defines an F_n-equivariant map from T to T'.

One shows that f varies continuously with T and T'. Although f is not a morphism, one can use f to obtain a morphism as follows. Use f on the vertices of T; this map has a unique extension to an edgewise-linear map from T to T'. This map can be converted into a morphism simply by rescaling the edges of T. There is a natural path in the outer space from T to the domain T_0 of this morphism, and Skora's construction gives a path from T_0 to T'. These constructions being continuous, we arrive at the theorem. Details can be found in [W1].

2. Irreducible Automorphisms and Train-Tracks

An outer automorphism Φ of F_n is *reducible* if there exists a graph G together with a self-map f which induces Φ, and which leaves invariant (up to homotopy) a proper, homotopically non-trivial subgraph. In case Φ is irreducible, Bestvina and Handel [B-H] demonstrate the existence of a "train-track" representative for Φ; that is, a self-map f of a graph G such that f^n is locally injective on the interior of each edge for each $n \geq 1$.

There is a natural (asymmetric) distance function on X_n, in which two marked **R**-graphs G_1 and G_2 satisfy $d(G_1, G_2) \leq \ln K$ if there exists a K-Lipschitz map from G_1 to G_2 which is compatible with the markings. (This distance was originally considered by Thurston in connection with earthquakes on hyperbolic surfaces [Th].) A map which realizes the least possible Lipschitz constant in its (free) homotopy class is called a "minimal-stretch" map. $Out(F_n)$ acts on X_n by isometries with respect to this distance; each marked graph is displaced a certain amount under the action of Φ on X_n. It turns out that the Bestvina-Handel train-track representatives are precisely those which minimize this displacement, and the train-track maps are the corresponding minimal-stretch maps.

It is then natural to study the behavior of this displacement function. The analogous object in Teichmüller theory is the displacement of a conformal structure by a pseudo-Anosov mapping class, with respect to the Teichmüller metric. C. J. Earle [E] proves that this displacement function has no critical points, except for its absolute minima. We give an independent proof of the existence of train-track representatives, and obtain the outer space analogue of Earle's result:

Theorem. *For any irreducible outer automorphism Φ of F_n, the associated displacement function on outer space has no critical points, except for absolute minima at the train-track maps.*

In our case, since outer space is not a manifold, we take this statement to mean that given any point G_0 of outer space, either G_0 admits a train-track map inducing Φ, or else we can define a continuous deformation of outer space which strictly decreases the displacement $d(G, \Phi(G))$ for G in a neighborhood of G_0.

We also give a new proof of the existence of train-track representatives for an irreducible automorphism, modelled on Bers' proof of the existence of absolutely extremal mappings between Riemann surfaces [B]. For details on the proofs, we refer the reader to [W2].

References

[B] L. Bers, *An Extremal Problem for Quasicomformal Mappings and a Theorem by Thurston*, Acta Math. **141** (1978), 73-98.

[B-H] M. Bestvina and M. Handel, *"Train Tracks and Outer Automorphisms of Free Groups"*, Ann. Math. (to appear).

[Cu] M. Culler, *Finite Groups of Automorphisms of a Free Group*, Cont. Math. **33** (1984), 197-207, *Contributions to Group Theory*, AMS, Providence, R. I..

[C-V] M. Culler and K. Vogtmann, *Moduli of Graphs and Automorphisms of Free Groups*, Invent. Math. **84** (1986), 91-119.

[E] C. J. Earle, *The Teichmüller distance is differentiable*, Duke Math. J. **44** (1977), 389-97.

[Sk] R. Skora, *"Deformations of Length Functions in Groups"*, (preprint).

[Th] W. Thurston, *"Minimal stretch maps of surfaces"*, (preprint).

[W1] T. White, *"Finite Groups of Automorphisms of Free Groups"*, (preprint).

[W2] T. White, *"A Minimal-Stretch Metric on Outer Space"*, (preprint).

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A Special Ratchet Lemma

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DEFINITION 1. A topological space X is said to be *eventually π_1-injective at infinity* if there is a fixed compact set K (a *core*) of X so that for every compact set A there is a compact set B so that loops in $X - B$ which are inessential in $X - K$ are also inessential in $X - A$.

Informally, we think of this property as stating that loops close to infinity which contract missing the core contract close to infinity. This condition is really a very mild condition which is satisfied by all the classical contractible manifolds. In 3-manifold theory, a manifold that is eventually π_1-injective at infinity is called *eventually end irreducible* [1], [2].

In a previous paper [6], we proved the following *Ratchet Lemma* that was extremely useful for showing that certain contractible manifolds could not be non-trivial covering spaces of another manifold.

LEMMA 2 (*Ratchet Lemma*). *Let $h\colon W \to W$ be a homeomorphism of a space that is eventually π_1-injective at infinity. If K is a core of W, then there is a compact set L in W so that a loop γ in $W - \bigcup_{i=-\infty}^{\infty} h^i(L)$ is inessential in $W - K$ if and only if γ is inessential in $W - h^i(K)$ for each i.*

In this paper we prove another version of this lemma which we call the *Special Ratchet Lemma*. We hope that this lemma will be helplful in solving the long standing conjecture [5, p. 96], [1] that the universal covering space of a closed, P^2-irreducible 3-manifold with infinite fundamental group must be R^3.

LEMMA 3 (*Special Ratchet Lemma*). *Let M be a manifold and W be an open subset of M so that W is eventually π_1-injective at infinity. Furthermore, suppose $\overline{W}$, the closure of W, is a manifold with simply connected boundary. Let $h: M \rightarrow M$ be a homeomorphism with the property that for a core K of W, $h(K)$ and $h^{-1}(K)$ can be isotoped into W. Then there is a compact set L in M so that a loop γ in $M - \bigcup_{i=-\infty}^{\infty} h^i(L)$ is inessential in $M - K$ if and only if γ is inessential in $M - h^i(K)$ for each i.*

Proof. By hypothesis there are isotopies of M that take $h(K)$ and $h^{-1}(K)$ to sets K^+ and K^-, respectively,which lie in W. Let T^+ and T^-, be the respective tracks of $h(K)$ and $h^{-1}(K)$ under these isotopies. Consider the compact set $A = K^+ \cup K \cup K^-$ which lies in W. Since W is eventually π_1-injective at infinity, there is a a compact set B in W so that loops in $W - B$ which are inessential in $W - K$ are also inessential in $W - A$. Note that since $\overline{W}$ has simply connected boundary, this also implies that loops in $M - B$ which are inessential in $M - K$ are also inessential in $M - A$. Let L be the compact set $T^+ \cup T^- \cup B$.

Now let γ be a loop in $M - \bigcup_{i=-\infty}^{\infty} h^i(L)$. If γ is inessential in $M - K$, then γ is inessential in $M - A$. Since $K^+ \subset A$, γ is inessential in $M - K^+$. Now $h(K)$ is isotopic to K^+ by an isotopy so that the track of $h(K)$ misses misses γ. Hence, the Covering Isotopy Lemma [3], [4] implies that γ is inessential in $M - h(K)$. So far we have shown that if γ is inessential in $M - K$, then γ is inessential in $M - h(K)$.

If γ is inessential in $M - h(K)$, then γ is inessential in $M - h(A)$. Since $K^- \subset A$, γ is inessential in $M - h(K^-)$. Now K is isotopic to $h(K^-)$ by an isotopy so that the track of K misses γ. As before, this implies that γ is inessential in $M - K$.

We have thus shown that γ is inessential in $M - K$ if and only if γ is inessential in $M - h(K)$. The rest of the proof now follows by induction.

References

[1] M. G. Brin and T. L. Thickstun, *3-manifolds which are end 1-movable*, Memoirs Am. Math. Soc. **81** , American Mathematical Society (1989).

[2] E. M. Brown, *Contractible 3-manifolds of finite genus at infinity*, Trans. Amer. Math. Soc. **245** (1978), 503-514.

[3] A. V. Cernavskii, *Local contractibility of the group of homeomorphisms of a manifold*, Math. USSR-Sb., 8 (1969), 287-333.

[4] R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. of Math., (2), 93 (1971), 63-88.

[5] J. P. Hempel, *3-manifolds*, Ann. of Math. Studies **86,** Princeton University Press (1976).

[6] D. G. Wright, *Contractible open manifolds which are not covering spaces,* Topology, (to appear).

Problem Session

1. (P. Scott, problem attributed to Jorgensen)) Does there exist a surface M with complete hyperbolic metric, and a closed geodesic γ on M with a transverse triple point, such that the triple point persists for all complete hyperbolic structures on M? Can a triple point persist on an open set? Does there exist a closed geodesic on the 3-cusped sphere with a triple point? If so, does the triple point persist as one deforms the cusps to non-cusp ends?

2. (K. Johanson) Does every minimal marking of a hyperbolic surface determine a unique base point?

3. (J. Hempel) Let Γ be your favorite universal link in S^3. (a) How does one get from one representation of a 3-manifold M as a branched cover $p_1: M \to S^3$ branched over Γ to another such representation $p_2: M \to S^3$. In particular, are there "moves" which always do this? (b) Can one get from a Heegard splitting for M to a representation $p: M \to S^3$ as a branched cover over Γ by some systematic procedure?

4. (R. Daverman) If G is the fundamental group of a closed hyperbolic 3-manifold and H is a finite index subgroup of G ($H \neq G$), can there exist an epimorphism ψ: $G \to H$?

5. (C. Guilbault and F. Ancel) A compactum X contained in the interior of an n-manifold M^n is a "spine" of M if there exists a map f: $\partial M \to X$ for which (Map(f),X) is homeomorphic to (M^n,X). (a) Do any (or all) of the Mazur 4-manifolds contain a pair of disjoint spines? (b) Does there exist a compact contractible n-manifold M^n ($\neq B^n$) which contains a pair of disjoint spines? (c) Does every compact contractible n-manifold M^n contain a pair of disjoint spines?

6. (R. Daverman - the "Chogoshvili problem") If X is a compact subset of $\mathbb{R}^n$ such that for any (n-k-1)-dimensional hyperplane $H \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exists a map f: $X \to \mathbb{R}^n - H$ with $d(x,f(x)) < \varepsilon$ for all $x \in X$, is $\dim X \leq k$?

7. (J. West, problem attributed to Torunczyk and Spiez) Given metric compacta X and Y, when is it true that for any maps $f: X \to \mathbb{R}^n$ and $g: Y \to \mathbb{R}^n$ and any $\varepsilon > 0$, there exist ε-approximations to f and g having disjoint images? Is the assumption that $\dim(X \times Y) < n$ sufficient?

8. (E. Harms) If $M^3 \subset S^4$ is a genus 1 3-level form where M^3 is homeomorphic to S^3, is the middle level cobordism a product?

9. (W. Menasco, with A. Reid) Let M^3 denote a 3-manifold and $K \subset M^3$ a knot for which $M^3 - K$ is hyperbolic. For what M^3 does there exist a K such that there exists a totally geodesic embedding of a surface $S \subset M^3 - K$? (<u>Suspicion:</u> No such K exists in S^3. It is known, for example, that in this case K cannot be alternating or a 3-braid. There do, however, exist link complements which work.)

10. (F. Ancel) Let U be a contractible open n-manifold which covers no compact manifold. Patch together $\mathbb{Z}$ copies of U to form a contractible n-manifold $\mathbb{Z}U$. (A careful description of $\mathbb{Z}U$ is given below.) $\mathbb{Z}U$ admits a properly discontinuous free $\mathbb{Z}$-action. <u>Conjecture</u>: $\mathbb{Z}U$ covers no compact manifold. (<u>Definition of $\mathbb{Z}U$</u>: For i=1,2, let $e_i: \mathbb{R}^{n-1} \times [0,1) \to U$ be proper tame embeddings with disjoint images. Let $V_i = e_i(\mathbb{R}^{n-1} \times [0,1))$ (i=1,2). Let $\tau: \mathbb{R}^{n-1} \times (0,1) \to \mathbb{R}^{n-1} \times (0,1)$ be the orientation reversing homeomorphism $\tau(x,t) = (x, 1-t)$. Define the homeomorphism $g: \mathrm{int}V_2 \to \mathrm{int}V_1$ by $g = e_1 \circ \tau \circ (e_2 | \mathrm{int}V_2)^{-1}$. $\mathbb{Z}U$ is the quotient space obtained from $\mathbb{Z} \times U$ by identifying $\{n\} \times (\mathrm{int}V_2)$ with $\{n+1\} \times (\mathrm{int}V_1)$ via the map $(n,x) \to (n+1, g(x))$, for each $n \in \mathbb{Z}$.)