PROCEEDINGS

THE SEVENTH ANNUAL WESTERN WORKSHOP
IN
GEOMETRIC TOPOLOGY

MAY 31 – JUNE 2, 1990

Department of Mathematics
Oregon State University
Corvallis, Oregon

The Seventh annual Western Workshop in Geometric Topology was held at Oregon State University in Corvallis, Oregon from May 31st to June 2nd, 1990. The participants were:

F. Ancel	University of Wisconsin-Milwaukee
R. Andersen *	Oregon State University
S. Bleiler	Portland State University
E. Boleyn	Portland State University
M. Burdon	Portland State University
R. Daverman *	University of Tennessee
E. Fredon	Portland State Unmiversity
D. Garity *	Oregon State University
C. Guilbault *	University of Wisconsin-Milwaukee
J. Hempel *	Rice University
J. Henderson	Colorado College
Y. Im *	University of Tennessee
I. Jubran	Oregon State University
M. Kelly *	University of Idaho
P. Latiolais	Portland State University
S. Peterson	Oregon State University
D. Rohm	University of Wisconsin-Stevens Point
R. Schori *	Oregon State University
D. Silver *	University of South Alabama
E. Swanson	Brigham Young University
F. Tinsley *	Colorado College
J. Van Buskirk	University of Oregon
J. Walsh *	U. C. Riverside
D. Wright *	Brigham Young University
T. Lay	Idaho State Univ.

A "*" next to a name in the list above indicates a speaker at the conference.

These proceedings contain the notes of two one-hour talks given by the principal speaker, Robert Daverman, and summaries of the talks given by other participants. The proceedings were edited by Dennis Garity. The success of the conference was due in large part to funding provided by the National Science Foundation (DMS-8802424) and by the Mathematics Department at Oregon State University. I would like to thank the National Science Foundation and the Mathematics Department at O.S.U for their support.

Dennis Garity

Additional copies of these proceedings can be obtained by contacting:

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Corvallis, Oregon 97330

To: Participants in Topology Workshop in Corvallis May 21, 1990

From: Dennis Garity

Lodging is being provided for out of state participants at the Towne House Motor Inn, 350 SW 4th Street, Corvallis. Reservations have already been made for the nights you indicated on your registration form. The Airport limousine from both Portland and Eugene stops at this motel. Call Airport Express (503-757-7731) or Away Travel Agency (503-757-9792) to make reservations. If you are driving, take Interstate 5 to U.S. 20 (or to U.S. 34) to Corvallis.

The conference will begin at 8:30 Thursday morning. Maps are included indicating the route from the motel to the conference center (Stewart Center on campus at 875 SW 26th). It is about a 20 minute walk from the motel to the conference center. A university van will pick up participants who need a ride at the motel at 7:55 A. M. and transport them to the conference center.

I am including a list of restaurants in Corvallis. The motel is in downtown Corvallis and there are a number of places to eat nearby. The following is a tentative schedule. Let me know if any corrections need to be made. An overhead projector and dry erase boards will be available for the talks.

Thursday:

8:00 - 8:30 Refreshments
8:30 - 9:30 R. Daverman
9:45 - 10:15 F. Tinsley
10:20 - 10:50 R. Schori
10:55 - 11:25 Y. Im
11:30 - 12:00 D. Silver
12:00 -2:00 lunch
2:00 - 2:30 J. Walsh

Friday:

8:00 - 8:30 Refreshments
8:30 - 9:00 D. Garity
9:15 - 9:45 R. Andersen
9:50 - 10:20 C. Guilbault
10:25 - 10:55 D. Wright
11:00 - 11:30 M. Kelly
11:30 - 2:00 lunch
2:00 - 2:30 J. Hempel
evening- after dinner gathering at R. Schori's house.

Saturday:

8:30 - 9:00 Refreshments
9:00 - 10:00 R. Daverman
10:15 - 11:15 Problem session
afternoon- Outing to Mary's Peak.

If there are any problems, please contact me at 612-255-4734, 612-255-3001, or by e-mail at Garity@math.orst.edu.

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A General Position Property for Recognizing 3-Manifolds

Robert Daverman

(from notes transcribed by Dennis Garity)

1. History

In 1977, R. Edwards proved the following characterization theorem:

Theorem: A resolvable generalized n-manifold $(n > 4)$ is a real manifold if and only if X has the Disjoint Discs Property.

A generalized n-manifold X is resolvable if there exists a cell-like map $P:M^n \to\to X$ where M^n is a genuine n-manifold. A space X has the disjoint discs property, DDP, if any two singular 2-cells in X can be approximated arbitrarily closely by disjoint singular 2-cells. So the DDP is a minimal sort of general position property that distinguishes resolvable generalized n-manifolds from the real thing when $n > 4$. The following theorem, due to F. Quinn in 1987, deals with the question of when a generalized n-manifold is resolvable.

Theorem: A generalized n-manifold X, $(n > 3)$, is resolvable if and only if a certain obstruction $i(X) = 1$.

J. Cannon's conjecture from 1977 is that a generalized n-manifold X, $n > 4$, is a real n-manifold if and only if X has the DDP.

2. Generalized 3-manifolds

The results presented here represent joint work with Dusan Repovš.

Main Theorem: A resolvable generalized 3-manifold M is a real 3-manifold if and only if it has the Spherical Simplicial Approximation Property.

A space X has the Spherical Simplicial Approximation Property, SSAP, if and only if each map $f:S^2 \to X$ can be arbitrarily closely approximated by maps g such that X contains a 2-complex K with $g(S^2)$ in K and with each 2-simplex of K 1-LCC embedded in X. (This formulation works when K has no local cut points. If v is a local cut point of K, we need K to be 1-FLG in X at v – see the final paragrphs of this writeup for an explanation.) A subset C of X is 1-LCC in X if each neighborhood U in X of an arbitrary point p in C contains another neighborhood V in X of p such that all maps from S^1 into $V \setminus C$ extend to maps of I^2 into $U \setminus C$.

R. H. Bing, S. Armentrout, M. Starbird and others have theorems about when cell-like decompositions G of 3-manifolds M yield 3-manifold decomposition spaces M/G. The conditions in their theorems involve general position properties in the source manifold M. Cannon has a result with conditions depending on the target M/G only. This is the first general position condition giving results about decompositions of 3-manifolds with no hypothesis on the singular set of M/G. In general, it is difficult to find an appropriate general position property in the 3-manifold setting.

We point out some additional results involving the SSAP:

Proposition: A generalized n-manifold, $n \geq 4$, has the DDP if and only if it has the SSAP.

Proposition: If a generalized 3-manifold has the SSAP, then X has a cellular resolution. That is, each $p^{-1}(x)$ is cellular in the domain.

Proposition: A cellular resolution $p:M \rightarrow X$ is a near homeomorphism if and only if X has the SSAP.

Corollary: Suppose X is a resolvable generalized 3-manifold with the dimension of the singular set of X less than three. Suppose for each map $f:B^2 \rightarrow X$ and for each $\epsilon > 0$, there is a 2-complex K with no local cut points and an ϵ-approximation f' to f with $f'(B^2)$ in K and with all 2-simplices of K 1-LCC in X. Then X is a 3-manifold.

3. Ingredients and Outlines of Proof:

The following Theorem of Cannon uses Bing 3-space techniques in its proof.

Theorem: If X is a resolvable generalized 3-manifold, and if the singular set of X, S(X), is in a 2-cell D that is 1-LCC embedded in X, then X is a 3-manifold.

Definition: An upper semi-continuous decomposition G of a metric space X is locally semi-controlled shrinkable if for each g in G and for each neighborhood U of g, there is a neighborhood W of g such that for all $\epsilon > 0$ and for all homeomorphisms $h: X \rightarrow X$ there is another homeomorphism $h':S \rightarrow S$ satisfying:

1) h and h′ agree on $X \setminus U$,
2) the diameter of $h'(g)$ is less than ϵ for all g in W, and
3) the diameter of $h'(g)$ is less than $\epsilon + \text{diameter}(h(g))$ for all g.

The following two theorems are also useful:

Theorem: (Cannon/Woodruff) If G is a locally semi-controlled shrinkable decomposition of a locally compact space metric space S with S/G finite dimensional, then p:S$\rightarrow$S/G is a near homeomorphism.

Theorem: (Edwards Countable Shrinking property) Suppose f is a cellular map from an n-manifold M to X and $\mathcal{S} = \{A_j : 1 \leq j < \infty\}$ is a collection of closed sets in X such that f can be arbitrarily closely approximated by cellular maps f_j that are one-to-one over A_j. Then f can be arbitrarily closely approximated by a cellular map F that is one-to-one over $\cup \mathcal{S}$.

Outline of Proof of Cannon-Woodruff Theorem in compact case:

Let m be the dimension of S/G. Fix $\epsilon > 0$. Cover S/G by open sets of diameter less than $\frac{\epsilon}{m+1}$. Find a refinement $\mathcal{U}$ of this cover which splits into $m+1$ pairwise disjoint collections $\mathcal{U}_1, \ldots, \mathcal{U}_{m+1}$. For each g in G find a $p^{-1}(U)$ for some U in $\mathcal{U}$ containing g and find a W as in the definition of locally semi-controlled shrinkable decomposition. Order the sets W thus obtained based on the subcollections $\mathcal{U}_i$ above.

Apply the definition to obtain homeomorphisms h_j:S$\rightarrow$S such that :

1) h_j and h_{j+1} agree on $S \backslash p^{-1}(U_j)$;
2) the diameter of $h_j(g)$ is less than $\frac{\epsilon}{2^j}$ for all g in W_j;
3) the diameter of $h_j(g) <$ diameter$(h_{j-1}(g)) + \frac{\epsilon}{2^j}$ for all g in G.

The final homeomorphism h_k shrinks all g in G to small size and $p \circ h_k$ is ϵ close to p. □

Outline of Proof that a cellular resolution p:M$\rightarrow$X is a near homeomorphism if and only if X has the SSAP.

As an initial simplification, apply the SSAP to find a collection of 2-complexes $\{K_j\}$ in X that are 1-LCC embedded such that all singular 2-spheres in X can be approximated by a singular 2-sphere in some K_j. Apply the Countable Shrinking Principle to approximate p by a cellular map p′:M$\rightarrow$X that is one-to-one over the union of the K_j.

Let $N_{p'}$ be the union of $\{y \in M \mid (p')^{-1}(p'(y) \neq y\}$. Note that $\dim(p'(N_{p'})) = 0$ and that singular discs in X can be approximated by singular disks missing $(p'(N_{p'}))$. this by itself does not imply that p′ is a near homeomorphism since every countable cellular decomposition has this property.

The plan is to now show that the decomposition of M induced by p′ is locally semi-controlled shrinkable. Fix a g_0 in this decomposition and let x_0 be $p'(g_0)$. Fix an open 3-cell neighborhood U_0 of g_0. Choose a neighborhood V in X of x_0 with $(p')^{-1}(V)$ contained in U_0. Find a map ψ from S^2 into $V\backslash x_0$ that is null homotopic in V, but not in $V\backslash x_0$ such that the image S separates x_0 from $X\backslash V$. Let W denote the component of $X\backslash S$ containing x_0 and let W_0 be $(p')^{-1}(W)$.

Note that p′ is one-to-one over K_0. Let K′ be the preimage of K_0 under p′. The 2-simplices of K′ are 1-LCC embedded in M since the 1-LCC property lifts to M. If K′ has no local cut points, a result of Nicholson from 1972 now implies that K′ is tame in M. Construct a regular neighborhood N of K′ and a cell-like map $q:M \rightarrow M$ collapsing N to K′ while fixing g_0. The map q should also be a homeomorphism of $M\backslash N$ onto $M\backslash K'$.

Now apply the Sphere theorem to find a P.L. 2-sphere S in $q^{-1}(K')$ separating g_0 from $M\backslash W_0$. The 2-sphere S bounds a P.L. 3-cell C in W_0 containing g_0. Find a homeomorphism h from M to itself fixed outside of C so that the diameter of $(p' \circ q)^{-1}(x)$ is less than ϵ for all x in $(p' \circ q)(C)\backslash K_0$.

Next find a map f from M to itself that shrinks out nontrivial point preimages of $h \circ q^{-1}$ without allowing the sizes of $h((p' \circ q)^{-1}(x))$ to grow too much. This can be done by a sequence of small moves. Anytime some $(p' \circ q)^{-1}(x)$ begins to grow dangerously large, don't adjust it in subsequent moves. The nullity of these $(p' \circ q)^{-1}(x)$ makes this possible.

We need to consider what happens when complexes in X have local cut points. We need such a complex K to be 1-FLG (to have free local fundamental groups). This means that for each neighborhood U of a local cut point v, there is a neighborhood V of v such that the homomorphism from $\pi_1(V\backslash K)$ to $\pi_1(U\backslash K)$ has free image, and that this image also equals the image of $\pi_1(W\backslash K)$ for any smaller neighborhood W.

Nicholson uses this as a characterization of tameness for 2-complexes in 3-manifolds. We then need to make sure that near v the 2-sphere S meets N is a disc. We can then excise the unnecessary part of K′ and reform N so that the previous argument applies. This completes the outline of the proof. □

Some questions related to this material are listed in the problems at the end of these proceedings.

DECOMPOSITIONS INTO SUBMANIFOLDS OF FIXED CODIMENSION: AN UPDATE

R. J. Daverman

Given an arbitrary generalized n-manifold X, $n \geq 4$, one would like to know whether X is the cell-like image of an n-manifold. Quinn [1987] reduced this resolvability matter to the the triviality of a certain obstruction $i(X) \in 1+8Z$. Whatever the obstruction, Quinn has also shown [1979] that there is an approximate fibration $f:M \to X$ defined on some 2n-manifold where each point preimage has the shape of S^n. The latter fact provides exemplary motivation for studying the larger class of closed maps with manifold domain such that all point preimages (up to shape) are manifolds of a fixed codimension. It is relevant to observe that this class also includes all orbit maps of free, compact (connected) Lie group actions on manifolds and all locally trivial fiber bundle projections where both the total space and the fiber are manifolds. An earlier survey [1986] listed two other, more personal reasons for my interest in this topic.

Standard Notation: M is an $(n+k)$-manifold; G is a usc decomposition of M into closed connected n-manifolds (up to shape); B is the decomposition space M/G; and $p:M \to B$ is the decomposition map. For simplicity, G will be called a <u>codimension</u> k (<u>manifold</u>) <u>decomposition</u>, and unless other hypotheses are given, you are supposed to assume this is always the sort of decomposition under discussion. Beware: some results also require the extra hypothesis that both M and the elements of G are orientable, while others presume the finite-dimensionality of B.

Broadly viewed, a basic issue here is, given full information about two of the following items -- (i) M, (ii) B, and (iii) the elements of G -- to describe the third. Other key problems are to characterize the manifolds M admitting codimension k decompositions and to determine the spaces B arising as images.

We end this introduction by outlining the contents of this survey. The first section presents a philosophy for addressing these issues. The next 3 set forth known results for small values of k (there the reader should note that the overriding codimension hypothesis is provided only by the section title and is not explicitly repeated in statements of theorems). §5 moves to the case of general codimension, where typically to derive interesting conclusions about B the results include hypotheses on triviality of some homology groups (for all the decomposition elements). §6 and §7 treat special topics, one about approximate fibrators, and the other about PL maps p to polyhedra. §8 provides conditions leading to information about the structure of M. §9 briefly deals with generalizations to decompositions involving closed manifolds of variable dimensions. Finally, §10 offers a list of questions (presumeably unsolved). The bibliography is intentionally extensive.

1. Approaches

A fulcrum giving initial leverage in this area involves a concept loosely referred to as the <u>continuity set</u> of $p:M \to B$. It is the maxiaml open subset C of B over which the nth cohomology sheaf of the map p is locally constant. Alternatively, C consists of all $b_0 \in B$ such that for a (shape) retraction $r:U \to p^{-1}b_0$ defined on some neighborhood U of $p^{-1}b_0$ and for all $b \in B$ with $p^{-1}b \subset U$, $r|b:p^{-1}b \to p^{-1}b_0$ is a degree one map. Coram and Duvall [1979] have shown C to be dense (and open) in B. A spectral sequence analysis leads to the following fundamental result [Daverman 1991]:

Theorem 1.1. The continuity set C of $p:M \to B$ $(\dim C < \infty)$ is a generalized k-manifold.

Under additional hypotheses one can hope for stronger conclusions, such as, for example, that B itself is a manifold. Branching out from 1.1, one has at least two directions to pursue: further analysis of C; and investigation of the <u>discontinuity set</u> $D = B \setminus C$. The latter has a secondary subdivision, organized about the degeneracy set K consisting of those points $b_1 \in B$ such that for a retraction $r:U \to p^{-1}b_1$ as above, arbitrarily close to b_1 is some $b \in B$ such that $r|b:b \to b_1$ induces the trivial homomorphism on nth cohomology. A troublesome spot, the degeneracy set is a breeding ground for pathology.

Another device that comes up, not as often but obviously beneficial when it does, is the concept of approximate fibration, also introduced by Coram and Duvall [1977]. A map $f:E \to A$ is called an <u>approximate fibration</u> if given any homotopy $h_t:X \to A$, initial lift $\psi:X \to A$ (where $f\psi=h_0$), and $\varepsilon>0$, there is an approximate lifting $\psi_t:X \to E$ with $\psi_0=\psi$ and with $f\psi_t, h_t$ ε-close for all t. Coram and Duvall proved that all point preimages in approximate fibrations have the same homotopy type (shape) and, just as with Hurewicz fibrations, the homotopy groups are related by the exact sequence:

$$\cdots \to \pi_{i+1}(A) \to \pi_i(f^{-1}a) \to \pi_i(E) \to \pi_i(A) \to \cdots$$

When $p:M \to B$ is an approximate fibration, this homotopy exact sequence provides desireable interrelations among M, B, and the point preimages under p.

Theorem 1.2 [Daverman-Husch 1984]. If $\dim B < \infty$, then B contains a dense, open subset A over which p is an approximate fibration.

Clearly $A \subset C$.

2. The Codimension One Case

Theorem 2.1 [Daverman 1985]. $B=M/G$ is a 1-manifold, possibly with boundary, and $\partial B=\emptyset$ provided both M and the elements of G are orientable.

Theorem 2.2. If $B\approx R^1$, then each inclusion $g \to M$ induces homology isomorphisms $H_*(g) \to H_*(M)$.

The two preceding results combine to give that the continuity set $C \supset \mathrm{Int}B$ (in fact, $C = \mathrm{Int}B$).

Theorem 2.3. If each $g\in G$ is 2-sided and locally flatly embedded in M , then $p:M \to B$ is an approximate fibration; moreover, if M is noncompact, then for each $g\in G$ M is homeomorphic to $g\times R^1$ (via a homeomorphism sending $g\subset M$ to $g\times 0$).

Without the local flatness hypothesis of 2.3, the elements of G can be homotopically inequivalent for $n>3$ (not so for $n=2$). Illustrative examples basically stem from the following laminated plus construction:

Theorem 2.4 [Daverman-Tinsley 1986]. Suppose N is a closed n-manifold $(n>4)$ and P is a finitely generated perfect subgoup of $\pi_1(N)$. Then there exists a compact (n+1)-manifold M having two boundary components N,N^+ such that (1) $\pi_1(N^+) \cong \pi_1(N)/P$, (2) the inclusion $N^+ \to M$ is a homotopy equivalence,

(3) the inclusion $N \to M$ is an homology equivalence, and (4) $M = A \cup (M \setminus A)$ where A denotes a collar on N^+ and $M \setminus A$ is homeomorphic to $N \times [0,1)$.

Tinsley and I have a new example showing that this structure can occur even when P is just the normal closure of a finite set, rather than being finitely generated, but the extent to which comparable structure can be identified in this more general setting remains undecided.

3. The Codimension Two Case

Theorem 3.1 [Daverman-Walsh 1985b; Daverman 1988]. $B=M/G$ is a 2-manifold, possibly with boundary if M is nonorientable.

Theorem 3.2. The complement of the continuity set C of $p:M \to B$ is locally finite (in B).

Theorem 3.3 [Daverman-Walsh 1985a]. If all elements of G have the shape of S^n , then $p:M \to B$ is an approximate fibration for $n>1$ and it is an approximate fibration over the complement of a locally finite set for $n=1$.

See also 8.8 for information about the structure of M .

4. The Codimension Three Case

Theorem 4.1 [Daverman 1986]. $\dim(B)=3$.

Example 4.2. B need not be a generalized 3-manifold; it could be, say, the open cone over a torus. This results from the decomposition of $S^1 \times S^1 \times R^3$ into the torus $S^1 \times S^1 \times 0$ and 2-spheres of the form $s_1 \times s_2 \times rS^2$ $(r>0)$. A similar codimension $n+1$ decomposition of $N^n \times R^{n+1}$ yields the open cone over any preassigned manifold N^n .

5. The Case of Higher Codimension

Example 5.1. Dranishnikov [1988] has an example of a cell-like decomposition K of S^7 with infinite-dimensional decomposition space. Composition of the obvious maps $M=S^7 \times N^n \to S^7 \to S^7/K$ induces a codimension 7 manifold (up to shape) decomposition of M for which B ($\approx S^n/K$) is infinite dimensional. Recently Dydak and Walsh announced this could be done with S^5 in place of S^7, which in turn implies dimension-raising can occur with codimension 5 decompositions.

Theorem 5.2 [Dydak 1977]. B is locally 1-connected.

Say that a decomposition G is m-<u>acyclic</u> if the reduced Steenrod homology (integer coefficients) of each $g \in G$ is trivial in dimensions less than $m+1$.

Theorem 5.3 [Daverman-Walsh 1987]. Suppose G is a $(k-1)$-acyclic codimension k manifold decomposition of a manifold M with $k \geq 2$ and $\dim B < \infty$. Then B is a generalized k-manifold.

Corollary 5.4. Under the hypotheses of 5.3, if additionally each $g \in G$ has the shape of a simply connected manifold, then $p: M \to B$ is an approximate fibration.

Theorem 5.5 [Snyder 1988]. Any nondegenerate $(k-2)$-acyclic codimension k decomposition G of an orientable $(n+k)$-manifold

M $(3 \leq k \leq n+1)$ yields a generalized k-manifold B as decomposition space, provided $\dim B < \infty$ and $B \setminus C$ does not locally separate B. (Addendum: for $n=k+1$ the hypothesis that G is nondegenerate, which amounts to requiring Hausdorffness in the nth-cohomology sheaf of $p: M \to B$, is not necessary.)

Corollary 5.6. If G is a usc decomposition of a $(2n+1)$-manifold M into compacta with the shape of homology n-spheres $(n>2)$ such that $\dim B < \infty$ and $B \setminus C$ nowhere locally separates B, then B is a generalized $(n+1)$-manifold.

6. Approximate fibrators

An n-manifold N is called a <u>codimension</u> k <u>fibrator</u> if whenever G is a decomposition of an $(n+k)$-manifold M such that each $g \in G$ has the shape of N and $\dim(M/G) < \infty$, then $p: M \to M/G$ is an approximate fibration. If their appeal is not transparent, recall that associated with any approximate fibration is an exact sequence relating homotopy groups of M, B, and the typical fiber.

Examples. By 5.4, S^n is a codimension k fibrator for $n \geq k \geq 2$; furthermore, all simply connected manifolds are codimension 2 fibrators. On the other hand, no $N \approx S^1 \times X$ is a codimension 2 fibrator; neither is any manifold that is a regular cyclic cover of itself. Since the join of any homology n-sphere Σ with itself is S^{2n+1}, Lacher's construction [Lacher 1975] shows that Σ is not a codimension $n+1$ fibrator.

Aside from the consequences of 5.4, information on this score pertains largely to $k \leq 2$. Manifolds with finite first homology frequently possess the codimension 2 property, and it is possible for those with infinite first homology to have it as well.

Theorem 6.1 [Daverman 1989]. Real projective n-space is a codimension 2 fibrator.

Theorem 6.2 [Daverman 1990]. Suppose N is an n-manifold such that for any usc decomposition of an (n+2)-manifold M into copies of N, the map $p:M \to B$ is an approximate fibration over its continuity set. Suppose $H_1(N)$ is finite and for each integer $d>1$ equal to the order of some $x \in H_1(N)$, N admits no map $N \to N$ of degree d. Then N is a codimension 2 fibrator.

In the setting of 6.2 p will be an approximate fibration over C if every degree 1 map $N \to N$ is a homotopy equivalence. For fairly trivial reasons, this is the case provided either:

1) $\pi_1(N)$ is finite, or

2) N is aspherical and $\pi_1(N)$ is Hopfian.

Theorem 6.3 [Im]. Any finite product of orientable surfaces of genus at least 2 is a codimension 2 fibrator.

7. PL MAPS

This section treats work in progress. Throughout it we suppose $p:M \to B$ is a PL map from an orientable PL $(n+k)$-manifold M onto a polyhedron B such that each $p^{-1}b$ has the homotopy type of a closed (connected, orientable) n-manifold. This category has particularly useful advantages because it allows induction arguments (based on k).

Let $B^{(t)}$ denote the t-skeleton of B.

Theorem 7.1. $B\backslash B^{(k-2)}$ is a k-manifold and $p| : p^{-1}(B^{(k-2)}) \to B^{(k-2)}$ is an approximate fibration.

Theorem 7.2. $B\backslash C$ separates no open susbet of B.

Theorem 7.3. For $n+k=4$ and $n=1$ B is a 3-manifold.

The argument relies on analysis of possible (generalized) Seifert fiberings of the 3-manifold boundaries of regular neighborhoods of $p^{-1}b$, inspired by [Seifert 1932].

Theorem 7.4. If $H_i(p^{-1}b)\cong 0$ for $1 \le i \le (k-1)/2$, then B is a generalized k-manifold.

Corollary 7.5. If $p:M \to B$ has homology n-spheres as point inverses and $k \le 2n$, then B is a generalized k-manifold.

Theorem 7.6. If $k=3$ and each $H_1(p^{-1}b;Z_2)\cong 0$, then B is a 3-manifold.

Example 7.7. There is a PL map $p:S^1\times R^4 \to B$ such that each $p^{-1}b\approx S^1$ but B is not a manifold. Here B can be the open cone over any Lens space.

Theorem 7.8. If $p:M \to B$ is a PL map such that $\chi(p^{-1}b)=0$ for all $b\in B$ and M is compact, then $\chi(M) = 0$.

Corollary 7.9. $\chi(M) = 0$ when n is odd.

Theorem 7.10. If M is a closed 4-manifold with $H_*(M) \cong H_*(S^4)$ and $p:M \to B$ is PL, then $n=0$.

Theorem 7.11. Suppose $p:M^{2+k} \to B$ is a PL map with each $p^{-1}b$ homotopy equivalent to an orientable surface of fixed genus at least 2. Then p is an approximate fibration.

8. STRUCTURE THEOREMS

Theorem 8.1 [Liem 1985]. If G is a usc decomposition of M^{n+1} into n-spheres ($n\geq 5$), then $p:M^{n+1} \to B$ can be approximated by a locally trivial n-sphere bundle map.

A technical structure theorem, very general and partially unsatisfying, about arbitrary codimension one decompositions is provided in [Daverman 1985]. It implies:

Theorem 8.2. If a closed manifold M admits a codimension 1 manifold decomposition, then $\chi(M)=0$.

Theorem 8.3 [Husch 1977]. An approximate fibration $p:M^{n+1} \to S^1$ ($n\geq 5$) can be approximated by locally trivial bundle maps if and only if p is homotopic to one.

Example 8.4 [Husch 1977]. There exists a closed (n+1)-manifold M and an approximate fibration $p:M \to S^1$ such that each $p^{-1}s$ is homeomorphic to a certain closed n-manifold F but p is not a locally trivial F-bundle over S^1.

Example 8.5 [Chapman-Ferry 1983]. For $n\geq 4$ there exist a closed (n+2)-manifold M and approximate fibration $p:M \to S^2$ with n-manifold fiber such that p is homotopic to a bundle projection but cannot be approximated by one.

Theorem 8.6 [Hughes 1985]. An approximate fibration $p: M \to B$ between closed manifolds $(\dim M \geq 5)$ can be approximated arbitrarily closely by locally trivial bundle projections if and only if p is homotopic through approximate fibrations to a bundle projection.

Theorem 8.7 [Quinn 1979]. If $p: M \to B$ is an approximate fibration, $n \geq 5$, B is a polyhedron, and the fiber F satisfies $Wh(\pi_1(F) \times Z^{\ell}) = 0$ for all ℓ, then M has the structure of a topological block bundle over B; moreover, for every $\varepsilon > 0$ there is an ε-homotopy $h_t: M \to B$ starting at p, ending with a block bundle projection, and for which each h_t is an approximate fibration.

A <u>topological block bundle with fiber</u> F is a map $f: E \to K$ (K triangulated) such that for each simplex $\sigma \in K$ there is a homeomorphism $f^{-1}(\sigma) \approx F \times \sigma$ carrying $f^{-1}(\partial\sigma)$ to $F \times \partial\sigma$.

Corollary 8.8. For codimension 2 manifold decompositions in which all point preimages are 1-connected, M is a topological block bundle over B and $p: M \to B$ can be approximated by block bundle projections.

Using results from [Farrell-Jones 1989], Im has proved:

Theorem 8.9 [Im]. If $p: M \to B$ is an approximate fibration, $n \geq 5$, B is a polyhedron, and the fiber F is a compact Riemannian manifold of nonnegative curvature, then p can be approximated by locally trivial bundle projections with F as fiber.

Corollary 8.10. If B is a polyhedron and $p: M \to B$ is a PL map from a PL manifold M such that $\dim M \geq 5$ and each point preimage is a fixed closed hyperbolic surface F, then M is the total space of a locally trivial F-bundle and p can be approximated by locally trivial bundle projections with F as fiber.

Proof. See also 7.11.

Borel and Serre [1950] have shown euclidean space never has the structure of a locally trivial fiber bundle with compact fiber; Conner [1957] obtained the same sort of result for the complement of a point in any simply connected manifold. One suspects similar nonexistence statements hold for arbitrary codimension k decompositions, but currently this has been verified only for small k values.

Theorem 8.11. E^n admits no codimension 1 or 2 manifold decomposition.

Theorem 8.12. No noncompact manifold having an isolated end ∞ such that $H_1(\infty) \cong 0$ admits a codimension 2 manifold decomposition.

9. OTHER STUFF

Theorem 9.1 [Daverman-Montejano]. An orientable 3-manifold M admits a usc decomposition G into 1-manifolds and 2-manifolds (at least 1 of each dimension) if and only if M is a graph manifold in the sense of Waldhausen (i.e., M contains a locally finite collection of pairwise disjoint, locally flat tori T_i such that $M \setminus \cup T_i$ is an S^1-bundle over some 2-manifold base).

Theorem 9.2. If G is a usc decomposition of any homology 3-sphere Σ^3 into 1-manifolds and 2-manifolds, then Σ^3/G is a cactoid.

Theorem 9.3 [Daverman]. Suppose G is a usc decomposition of an n-manifold M and

$$1 \leq d_0 < d_1 < \cdot \cdot \cdot < d_s \leq n \quad (s>0)$$

are integers such that each $g \in G$ has the shape of a closed, connected, orientable $(n-d_i)$-manifold, for some $i=i(g)$. Then

$$c\text{-}\dim_Z(M/G) \leq (\Sigma d_i) - s \ ;$$

moreover, if $1 < d_0$,

$$c\text{-}\dim_Z(M/G) \leq (\Sigma d_i) - (s+1) \ .$$

The symbolism $c\text{-}\dim_Z$___ in 6.3 stands for "Integral cohomological dimension", which of course coincides with, say, covering dimension provided M/G is finite dimensional.

10. QUESTIONS

1. Is B an ANR? What if the elements of G are pairwise homeomorphic? Describe the various possible image spaces B.

2. What can be said about the structure of M ? Why has no one found any codimension k decomposition of E^{n+k} ?

3. Suppose $n>1$, M is an $(n+2)$-manifold, and the elements of G are all n-spheres. Is M an n-sphere bundle over B ?

4. For which integers n and k is there a decomposition G of the $(n+k)$-sphere into n-spheres? into n-tori? into fixed products of spheres? into closed connected n-manifolds?

5. If G is a usc decomposition of M into n-spheres, where $2<n+1<k<2n+2$, is B a generalized k-manifold? What about into homology n-spheres?

6. If elements of G just have the shape of manifolds, does M admit a related decomposition G^* into genuine n-manifolds?

7. Is B finite-dimensional when elements of G are genuine manifolds? if the elements of G are simple closed curves?

8. Is the set C of continuity points necessarily connected?

9. Assume M is closed and the degeneracy set K is empty. Is there an upper bound to the degrees of maps $g \to g_0$ induced by the restrictions of neighborhood retractions $U \to g_0$?

10. In case $k=3$, is the set of points at which B fails to be a generalized 3-manifold locally finite?

11. In case M is noncompact and $k=1$, must M have the homotopy type of a closed n-manifold?

12. If the elements of G in #11 are pairwise homeomorphic, is M topologically equivalent to $g \times E^1$ $(g \in G)$?

13. If W is a compact $(n+1)$-manifold with $\partial W \neq \emptyset$ and the inclusion $N \to W$ of some component N of ∂W is a homotopy equivalence, does W admit a lamination (i.e., a decomposition into closed n-manifolds)? What if the kernel of the induced π_1-homomorphism is simple (but contains no f.g. perfect groups)?

14. Suppose (W,M,M') is a laminated cobordism. Is there some manifold N admitting acyclic maps $N \to M$ & $N \to M'$?

15. Let $\Theta(M) = \{M'$: there exists a laminated cobordism $(W,M,M')\}$.

Compare $\Theta(M)$ with the group of homology cobordism classes.

16. Is there a 3-manifold $\hat{M}$ whose fundamental group has nontrivial wild group K but M is laminated cobordant to no M' where $\pi_1(M') \cong \pi_1(M)/K$. What if M is the exterior of a knot with trivial Alexander polynomial?

17. Find M^4 with $\pi_1(M^4) \cong A(5)$, the alternating group on 5 symbols, and laminated cobordism to a 1-connected 4-manifold.

18. If a 4-manifold (with ∂) is an h-cobordism, is it laminable? Is its interior a product?

19. Which 2-manifolds result from a decomposition of a fixed $(n+2)$-manifold into n-manifolds? Is there an M yielding all 2-manifolds of genus no larger than half the rank of $H_1(M)$?

20. What is the role of local knottedness in codimension 2 manifold decompositions?

21. Which k-manifolds result from a decomposition of a fixed $(1+k)$-manifold M into 1-spheres?

22. Is there a decomposition of the n-ball into circles? of a compact contractible space? of a cell-like set?

23. Does there exist a usc decomposition of the 5-sphere into simple closed curves and two circles worth of points?

24. If $k=3$, $n=1$, and the degeneracy set K is empty, is B a generalized 3-manifold?

25. Does local constancy of the ith cohomology sheaf associated with p, for $i \leq k-1$, imply B is a generalized k-manifold?

26. Given an arbitrary closed manifold N, does there exist a decomposition G of some $(n+k)$-manifold M into copies of N such that $p:M \to B$ is NOT an approximate fibration. Are there any examples of such N besides those with homology sphere factors and those that regularly, cyclically cover themselves?

27. If B is an ANR, $n=2$, and $p:M \to B$ is an approximate fibration, can p be approximated by a locally trivial bundle map with 2-manifold fibers? Is B resolvable (the 2-sphere fibers case is particularly interesting)?

28. Suppose $p:M \to B$ is an approximate fibration between manifolds, with B aspherical, that is homotopic to a (locally trivial) bundle map. Can p be approximated by such maps?

29. For which n-manifolds N and integers k does the hypothesis that all elements of G' are copies of N imply $p:M \to B$ is an approximate fibration? What if $\pi_1(N)$ is finite and $k=2$? What if N is covered by the n-sphere? What if N is hyperbolic? What if all $g \in G$ are required to be locally flat in M?

30. If a closed 3-manifold fails to be a codimension 2 fibrator, is it a Seifert fiber space?

31. For $n=2m$, $k=2m+1$ does there exist a PL map $p:M \to B$ from a PL $(n+k)$-manifold M to a simplicial complex B which is not a generalized manifold, such that $H_j(g) \cong 0$ when $0 < j < n$?

32. What are the PL locally trivial n-sphere bundles on $S^n \times S^n$?

33. Suppose for $k=n+1>2$ both M and all $g \in G$ are spheres. Is there an example where the degeneracy set $K(B)$ contains more than one point?

34. For odd k is there a codimension k decomposition of some closed $(n+k)$-manifold M having nonzero Euler characteristic?

35. Do all PL M^4 with $\chi(M^4)=0$ admit circle decompositions? What about those that fiber over S^1 ?

36. Is there a closed n-manifold which regularly covers itself via a noncyclic group Γ of deck transformations that also acts freely on some sphere? on some homology sphere?

37. What can be said about decompositions into (n-1)- and (n-2)-manifolds? What if they are all PL manifolds?

38. Can every 4-manifold (n-mfld) be decomposed into 1- and 2-manifolds?

39. If the dimension of (closed manifold) point-preimages of $p:M \to B$ varies, can $\dim B$ exceed $\dim M$?

40. Does there exist a continuous decomposition of Euclidean n-space into solenoids?

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A NEW PROOF THAT $\mathcal{N}_3 = 0$

Fredric D. Ancel and Craig R. Guilbault

In [5], C. Rourke gives a brief clever proof of the classical result of V.A. Rokhlin [4] that every closed orientable 3-manifold bounds a compact orientable 4-manifold (i.e., $\Omega_3 = 0$). The non-orientable version of Rohklin's theorem, originally proven by R. Thom [6], guarantees that every closed 3-manifold (possibly non-orientable) bounds a compact 4-manifold ($\mathcal{N}_3 = 0$). In this presentation, we indicate how Rourke's approach can be extended to give a short proof of this latter theorem.

In [5], $\Omega_3 = 0$ is deduced as a corollary of a stronger theorem (proven earlier in [7] and [2]) that every closed orientable 3-manifold can be reduced to S^3 by a finite number of elementary Dehn surgeries. Here "*elementary*" means that a meridian of the attached solid torus is identified with a curve in the boundary of the removed solid torus that is homotopic to the core of the removed solid torus. Then $\Omega_3 = 0$ follows from the observation that any two closed orientable 3-manifolds which differ by an elementary Dehn surgery cobound a compact orientable 4-manifold.

Similarly we can deduce that $\mathcal{N}_3 = 0$ from a stronger theorem (first proven in [3]) about the reducibility by surgery of every non-orientable 3-manifold to a simple model. In the non-orientable situation the simple model which replaces S^3 is the non-orientable 2-sphere bundle over S^1, which we denote $\mathbb{T}$. Our basic theorem is:

THEOREM *Every closed non-orientable 3-manifold can be reduced to* $\mathbb{T}$ *by a finite number of elementary Dehn surgeries.*

Since $\mathbb{T}$ bounds the non-orientable B^3 bundle over S^1, and since any two closed 3-manifolds (orientable or not) which differ by an elementary Dehn surgery cobound a compact 4-manifold, we have:

COROLLARY. ($\mathcal{N}_3 = 0$) *Every closed 3-manifold bounds a compact 4-manifold.*

We describe how to extend Rourke's techniques to give an elementary proof of the above Theorem . As in [5], we will use an induction argument based on a complexity assigned to Heegaard diagrams.

Suppose $M = H_1 \cup H_2$ is a Heegaard splitting of a non-orientable 3-manifold M. Then H_1 and H_2 are non-orientable handlebodies meeting along a non-orientable surface S. If the H_i's are of genus *n*, then *S* has Euler characteristic 2-2n, and we will call *S* a *non-orientable surface of genus* n. A set of *n* disjoint 2-sided (i.e., having an annular regular neighborhood) simple closed curves on S whose complement is a punctured disk is called a *complete system of curves on S*. (Every non-orientable surface of genus n has a complete system of curves.) It is easy to see that if *X* and *Y* are complete systems of curves on *S* with the property that each element of *X* bounds a disk in H_1 and each element of *Y* bounds a disk in H_2, then *M* is completely determined by *S*, *X* and *Y*. We then call *S(X,Y)* a *Heegard diagram for M*. Moreover, any Heegaard diagram, *S(X,Y)*, uniquely determines a 3-manifold which we will denote *M(X,Y)*.

A 2-sided curve *x* on a non-orientale surface *S* is called *exceptional* if *S - x* is orientable, otherwise it is called *ordinary*. A complete system of curves on *S* is called *uniform* if it contains only ordinary curves, or if genus(*S*) = *1*. It is easy to see that every non-orientable surface of genus n has a uniform complete system of curves. Note that a genus *1* complete system necessarily contains a single exceptional curve. A Heegaard diagram *S(X,Y)* will be called a *uniform* if both *X* and *Y* are uniform systems.

REMARK. The assumption of "2-sidedness" for all curves used in a Heegaard diagram is of utmost importance. While this property is automatic for a curve on an orientable surface, the situation is much different for non-orientable surfaces. On the other hand, our preference for ordinary curves evolved during our work on this problem. Use of uniform Heegard diagrams substantially simplified our original proof. Much of the work done in proving the our theorem is aimed at securing these properties when choosing new curves (see for example the lemma below).

To a uniform Heegaard diagram $S(X,Y)$, where S is non-orientable, assign a complexity $\mathcal{C}(X,Y) = (n,k)$ where $n = genus(S)$ and $k = min\{|x \cap y| : x \in X, y \in Y\}$. Note that since S is non-orientable, then $n \geq 1$. Our proof is by induction on the complexity of these uniform Heegaard diagrams under the lexicographic ordering.

While many facts about surfaces and 3-manifolds must be verified to give a complete proof of the theorem, the key is the following:

LEMMA *Suppose* x *and* y *are two non-separating 2-sided curves on a non-orientable genus* n *surface* S *and that* x *meets* y *transversally. Let* $|x \cap y|$ *denote the number of intersection points.*

(a) *If* $|x \cap y| = 0$ *and both* x *and* y *are ordinary, then there is a (necessarily ordinary) non-separating 2-sided curve* z *on* S *which meets each of* x *and* y *transversally in a single point.*

(b) *If* $|x \cap y| > 1$, *then there is a non-separating 2-sided curve* z *on* S *with* $|x \cap z| < |x \cap y|$ *and* $|y \cap z| < |x \cap y|$. *Moreover, if* x *and* y *are ordinary then* z *can be chosen to be ordinary.*

Proof of this lemma requires careful examination of approximately ten different cases. Its complete proof as well as the remaining ingredients in the proof of the main theorem can be found in [1]

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HIGHER DIMENSIONAL DUNCE HATS

by

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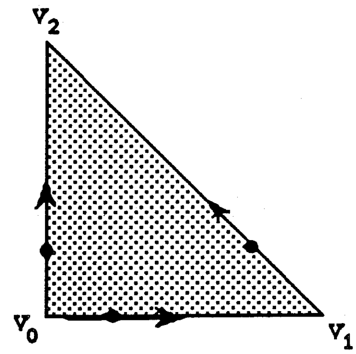
This paper represents joint work with Richard M. Schori, and was presented at the 1990 Workshop in Geometric Topology in Corvallis Oregon in June 1990.

1. Introduction.

Higher dimensional dunce hats, D^{2n}, $n \geq 1$, are defined in [M-S] using the hyperspace of closed subsets version of symmetric products where D^2 is the standard (two-dimensional) topological dunce hat. The standard topological dunce hat is known to be the simplest example of a contractible space, in the sense of homotopy, that is not collapsible in the sense of Whitehead. See Zeeman [Z] for a comprehensive article on the dunce hat and its relation to problems in manifold theory. In [M-S] they give a homology proof that the spaces D^{2n} are contractible. In this paper we give a (non-symmetric product) simple, inductive construction for spaces D^n, $n \geq 0$, and prove, using a homotopy argument, that D^{2n}, $n \geq 0$, is contractible; that D^{2n+1}, $n \geq 0$, has the homotopy type of the $(2n+1)$-sphere, S^{2n+1}; and that D^n, $n \geq 1$, is not collapsible.

2. The contractibility of the Dunce Hat.

The topological dunce hat D is obtained by taking a 2-simplex, Δ^2, and identifying the edges as indicated in Figure 1. The contractibility of D is well known but it is not at all geometrically evident.



Dunce Hat

Fig. 1

We will now give a more formal definition of the dunce hat which will facilitate a proof of the contractibility of D and it will motivate our definition of higher dimensional dunce hats. First we need a few definitions. .

Let Δ^n, $n \geq 0$, be a standard n-simplex $< v_0, \ldots, v_n >$ and if $0 \leq m \leq n$, let $(\Delta^n)^{(m)}$ denote the m-skeleton of Δ^n. Thus the boundary of Δ^n, denoted $Bd\, \Delta^n$, is the space $(\Delta^n)^{(n-1)}$. For $i = 0, \ldots, n+1$, let $d_i^n : \Delta^n \longrightarrow Bd\ \Delta^{n+1}$ be the *face map* which linearly injects Δ^n onto the n-face of Δ^{n+1} opposite the vertex v_i such that the order of the vertices is preserved. Thus, $d_i^n(< v_0, \ldots, v_n >) =< v_0, \ldots, \hat{v}_i, \ldots, v_n >$, where $\hat{v}_i$ means v_i has been deleted.

If X and Y are disjoint topological spaces, let $X \uplus Y$ denote their topological sum, and if A is a closed subset of X and $f : A \longrightarrow Y$ is a map, then let the adjunction space $X \bigcup_f Y$ denote the quotient space $X \uplus Y/\sim$ where $a \sim f(a)$ for each $a \in A$. The following is a basic theorem from homotopy theory and can be found in [V, p. 117].

2.1. Theorem. If $f, g : Bd\, \Delta^n \longrightarrow Y$ are homotopic maps into a space Y, then the adjunction spaces $\Delta^n \bigcup_f Y$ and $\Delta^n \bigcup_g Y$ are of the same homotopy type.

Let D^1 be a copy of S^1 and let $*$ be a specified point of D^1. Let $q_1 : \Delta^1 \longrightarrow D^1$ be a map taking $Bd\ \Delta^1$ to $*$ that is one-to-one on $Int\ \Delta^1$, wrapping Δ^1 around

D^1, in a counterclockwise direction. If we orient Δ^1 with an arrow and take images of Δ^1 under the d_i maps, $i = 0, 1, 2$, then we obtain an identification pattern on the boundary of Δ^2 that yields the dunce hat, D. To realize this with maps, let $\bar{q}_2 : Bd\,\Delta^2 \longrightarrow D^1$ be defined by $\bar{q}_2 | d_i(\Delta^1) = q_1 \circ d_i^{-1}$, for $0 \leq i \leq 2$. The map $\bar{q}_2$ is well-defined since the intersection of any two $d_i(\Delta^1) = \Delta_i^2$, say $\Delta_0^2 \cap \Delta_2^2 = \{v_1\}$, is a vertex of Δ^2 and hence its image under each $q_1 \circ d_i^{-1}$ is the point $*$.

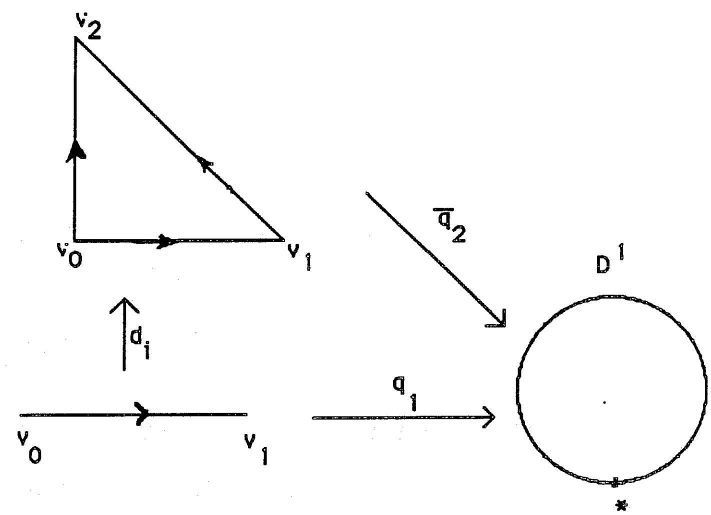


Fig. 2

2.2. Definition. The dunce hat $D^2 = \Delta^2 \bigcup_{\bar{q}_2} D^1$ is the adjunction space where Δ^2 is attached to D^1 by $\bar{q}_2$.

2.3. Theorem. *The Dunce Hat D^2 is contractible.*

Proof. We will argue that $\bar{q}_2$ is homotopic to a homeomorphism. If we start at the vertex v_0 and move counterclockwise around $Bd\,\Delta^2$, $\bar{q}_2$ maps each of $[v_0, v_1]$ and $[v_1, v_2]$ once around D^1 in the same direction and maps $[v_2, v_0]$ around D^2 in the opposite direction. The last two wraps around D^1 are in the opposite direction and hence homotopically cancel each other and consequently $\bar{q}_2$ is homotopic to a homeomorphism $h : Bd\,\Delta^2 \longrightarrow D^1$. Consequently, by Theorem 2.1, $D^2 = \Delta^2 \bigcup_{\bar{q}_2} D^1$ is of the same homotopy type as $\Delta^2 \bigcup_h D^1$, and this latter space is homeomorhic to Δ^2 since h is a homeomorphism and consequently D^2 is contractible since Δ^2 is. ■

3. Defining Dunce Hats Inductively.

In this section we will inductively define spaces D^n, for $n = 0, 1, 2, \ldots$, where D^2 is the dunce hat. For even values of n greater than 0, D^n will be contractible but not collapsible, the important properties of the dunce hat.

We need one additional definition related to an adjunction space, $X \bigcup_f Y$, where $A \subset X$, and $f : A \longrightarrow Y$. If $p : X \uplus Y \longrightarrow X \bigcup_f Y$ is the projection map, define $\hat{f} : X \longrightarrow X \bigcup_f Y$ by $\hat{f} = p|_X$. I.e.

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in X - A. \end{cases}$$

We now begin our inductive definition of spaces D^n.

Let $D^0 = \Delta^0$, which is a single point v_0. Let $q_0 : \Delta^0 \longrightarrow D^0$ be the map from Δ^0 to D^0. Now define a map $\bar{q}_1 : Bd\ \Delta^1 \longrightarrow D^0$ by, $\bar{q}_1(v_0) = \bar{q}_1(v_1) = v_0$.

We now let $D^1 = \Delta^1 \bigcup_{\bar{q}_1} D^0$ and define a map $q_1 : \Delta^1 \longrightarrow D^1$ by $q_1 \equiv \hat{\bar{q}}_1 : \Delta^1 \longrightarrow D^1$.

Assume that $n \geq 1$ and that $q_{n-1} : \Delta^{n-1} \longrightarrow D^{n-1}$ has been defined such that $\bar{q}_{n-1} \circ d_i^{n-2} = q_{n-2}$, for $i = 0, \ldots, n-1$. Define $\bar{q}_n : Bd\ \Delta^n \longrightarrow D^{n-1}$ by letting $\bar{q}_n|_{d_i^{n-1}(\Delta^{n-1})} = q_{n-1} \circ (d_i^{n-1})^{-1}$ for $i = 0, \ldots, n$. See Lemma 3.2 to verify that $\bar{q}_n$ is well-defined. Let $D^n = \Delta^n \bigcup_{\bar{q}_n} D^{n-1}$ and $q_n = \hat{\bar{q}}_n : \Delta^n \longrightarrow D^n$.

We illustrate this inductive definition with the following diagram:

(3.1)
$$\begin{tikzcd}
Bd\ \Delta^{n+1} \arrow[dr, "\bar{q}_{n+1}"] & \\
\Delta^n \arrow[u, "d_i^n"'] \arrow[r, "q_n"] & D^n \\
Bd\ \Delta^n \arrow[u, "in"'] \arrow[dr, "\bar{q}_n"] & \\
\Delta^{n-1} \arrow[u, "d_i^{n-1}"'] \arrow[r, "q_{n-1}"] & D^{n-1} \arrow[uu, "in"']
\end{tikzcd}$$

It is not obvious that the maps $\bar{q}_n$ are well defined. We thus show that these maps are well defined with the following Lemma.

3.1. Lemma. *The maps $\bar{q}_n = q_{n-1} \circ (d_i^{n-1})^{-1}$ are well defined.*

Proof. For $n = 1$, the map $\bar{q}_n$ is clearly well defined. For $n > 1$ we need to check that $\bar{q}_n$ is defined the same way on the intersection, $\sigma =< v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n >$, of $d_i^{n-1}(\Delta^{n-1})$ and $d_j^{n-1}(\Delta^{n-1})$.

We have
$(d_i^{n-1})^{-1}(\sigma) =< u_0, \ldots, \hat{u}_j, \ldots, u_{n-1} >$ and
$(d_j^{n-1})^{-1}(\sigma) =< u_0, \ldots, \hat{u}_i, \ldots, u_{n-1} >$.

By the inductive hypothesis
$q_{n-1}(< u_0, \ldots, \hat{u}_i, \ldots, u_{n-1} >) = \bar{q}_{n-1}(< u_0, \ldots, \hat{u}_i, \ldots, u_{n-1} >) =$
$q_{n-2} \circ (d_i^{n-2})^{-1}(< u_0, \ldots, \hat{u}_i, \ldots, u_{n-1} >) = q_{n-2}(< w_0, \ldots, \ldots, w_{n-2} >$ and
$q_{n-1}(< u_0, \ldots, \hat{u}_j, \ldots, u_{n-1} >) = \bar{q}_{n-1}(< u_0, \ldots, \hat{u}_j, \ldots, u_{n-1} >) =$
$q_{n-2} \circ (d_j^{n-2})^{-1}(< u_0, \ldots, \hat{u}_j, \ldots, u_{n-1} >) = q_{n-2}(< w_0, \ldots, \ldots, w_{n-2} >$. Hence $\bar{q}_n$ is well defined. ■

4. The Main Result.

In this section it is convenient to use some additional tools from homotopy theory.

4.1. Theorem. *Let A, B, X be CW-Complexes. If $A \subset X$, and $h : A \longrightarrow B$ is a homotopy equivalence, then $\hat{h} : X \longrightarrow X \bigcup_h B$ is also a homotopy equivalence.*

The proof of Theorem 4.1 is found in Whitehead [Wh] as Corollary 5.12 in Chapter 1.

4.2. Corollary. *For A, X CW-Complexes, if A is a contractible closed subset of X, then the identification map $p : X \longrightarrow X/A$ defined by*

$$p(x) = \begin{cases} * & \text{if } x \in A \\ x & \text{if } x \in X - A \end{cases}$$

is a homotopy equivalence.

Proof. The map $p : A \longrightarrow *$ is a homotopy equivalence since A is contractible, and $X/A = X \bigcup_p *$. ■

We will use the Homotopy Addition Theorem as found in Hu [Hu].

4.3. Homotopy Addition Theorem. *For any map* $f : (Bd\ \Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \longrightarrow (X, x_0)$, *the homotopy class of* $f, [f] \in \pi_n(X, x_0)$, *and for* $n \geq 2$ *we always have* $[f] = \sum_{i=0}^{n+1} (-1)^i [f \circ d_i^n]$, *where the* $d_i^n : \Delta^n \longrightarrow Bd\ \Delta^{n+1}$ *are the face maps, and for* $n = 1$ *we have* $[f] = [f \circ d_2^1] \cdot [f \circ d_0^1] \cdot [f \circ d_1^1]^{-1}$.

4.4. Theorem. *The space D^{2n} is contractible and $D^{2n+1} \simeq S^{2n+1}$, $n \geq 0$.*

Proof. We have already proved that D^0 and D^2 are contractible and that D^1 is homeomorphic to S^1. Let $n > 2$ be odd and assume that D^{n-1} is contractible. Now $D^n = \Delta^n \bigcup_{\bar{q}_n} D^{n-1}$ and since D^{n-1} is contractible, by Corollary 4.2, the projection $p : D^n \longrightarrow D^n/D^{n-1}$ is a homotopy equivalence. Furthermore, D^n/D^{n-1} is homeomorphic is S^n since it is equivalent to $\Delta^n/Bd\ \Delta^n$. This verifies that $D^n \simeq S^n$, for n odd.

We now apply the Homotopy Addition Theorem to the map of pairs $p \circ \bar{q}_{n+1} : (Bd\ \Delta^{n+1}, (\Delta^{n+1})^{(n-1)}) \longrightarrow (S^n, *)$.

We have

$$[p \circ \bar{q}_{n+1}] = \sum_{i=0}^{n+1} (-1)^i [p \circ \bar{q}_{n+1} \circ d_i^n] = \sum_{i=0}^{n+1} (-1)^i [p \circ q_n].$$

Since n is odd, we have an odd number of maps each of which is the same except for sign, and thus $[p \circ \bar{q}_{n+1}] = [p \circ q_n]$. Thus, $[p \circ \bar{q}_{n+1}]$ as an element of $\pi_n(S^n, *)$ is represented by $p \circ q_n : (\Delta^n, \partial\Delta^n) \longrightarrow (S^n, *)$, which represents the identity element of $\pi_n(S^n)$ since the restriction of $p \circ q_n$ to the interior of Δ^n is a homeomorphism. Consequently, $p \circ \bar{q}_{n+1} : \partial\Delta^{n+1} \longrightarrow S^n$ is homotopic to a homeomorphism and is therefore a homotopy equivalence.

By hypothesis, D^{n-1} is contractible and consequently $p : D^n \longrightarrow D^n/D^{n-1}$ is a homotopy equivalence by Corollary 4.2. Therefore if p' is a homotopy inverse of p, then $\bar{q}_{n+1} \simeq p' \circ p \circ \bar{q}_{n+1} : \partial\Delta^{n+1} \longrightarrow D^n$ is a homotopy equivalence. It follows directly from Theorem 4.1 that $D^{n+1} \equiv \Delta^{n+1} \cup_{\bar{q}_{n+1}} D^n \simeq \Delta^{n+1}$, which is contractible. ■

To see that the n–dimensional dunce hat is not collapsible we follow the polyhedral definition of collapsing in Zeeman [Z].

4.5. Definition. Let X be a polyhedron and Y a subpolyhedron. There is an *elementary collapse* from X to Y if for some n there is an n–ball B^n with face B^{n-1} such that

$$X = Y \cup B^n$$

$$B^{n-1} = Y \cap B^n.$$

We describe the elementary collapse from X to Y by saying *collapse across* B^n *onto* B^{n-1}, or *collapse across* B^n *from* B^{n-1}_*, where B^{n-1}_* is the complementary face of B^n. We say X *collapses to* Y, written $X \searrow Y$ if there is a sequence of elementary collapses

$$X = X_0 \searrow X_1 \searrow \ldots \searrow X_n = Y.$$

If Y is a point we call X *collapsible* and write $X \searrow 0$.

4.6. Theorem. *The n-dimensional dunce hat D^n is collapsible if and only if $n = 0$.*

Proof. For any polyhedron X to be collapsible there must exist a cell $B \subset X$ with a free face on which to begin the collapse. For D^n, $n \geq 1$, we observe that the map $q_n : \Delta^n \longrightarrow D^n$ is a quotient map that identifies all faces of the same dimension to a single face of that dimension. Hence for any triangulation of D^n there will be no free faces on which to begin the collapse. For $n = 0$, D^0 is a single point and thus is trivially collapsible. ■

R. M. Schori has a more detailed proof of this theorem in a paper that was authored by Schori and Marjanovic.

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THIN CODIMENSION ONE DECOMPOSITIONS

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1. Introduction: In [G], for each n greater than 4 and for each positive k less than n, examples of generalized n-manifolds X and cellular maps π from $\mathbb{R}^n$ onto X are constructed having the following properties. The nonmanifold part of X is homeomorphic to a k-cell, and if A is any closed subspace of X of dimension less than k, then the decomposition of $\mathbb{R}^n$ induced over A is shrinkable. In particular, the nonmanifold nature of X is not detectable by examining closed subsets of X of dimension less than k. These examples are produced by combining mixing techniques for producing generalized n-manifolds whose nonmanifold part is a Cantor set, with decompositions arising from special functions from the Cantor set onto a k-cell.

Such spaces are called generalized manifolds arising from *thin decompositions* of $\mathbb{R}^n$. This terminology was suggested by R. J. Daverman and indicates the fact that the nonmanifold nature of X can only be detected by examining large dimensional subspaces of X. This contrasts with other examples of decompositions of $\mathbb{R}^n$ that yield nonmanifolds. In these other examples, the nonmanifold part of the decomposition space is detectable by examining certain closed 0-dimensional subspaces of the decomposition space. The examples produced in [G] generalize a construction of McCauley and Woodruff [MW] in $\mathbb{R}^3$ to higher dimensions.

In this talk, the techniques for producing the examples in [G] are outlined, and ideas for generalizing these techniques to produce analogous thin decompositions of the Hilbert Cube are discussed.

First, note that for a thin decomposition as described above, the decomposition map π from $\mathbb{R}^n$ onto X is cellular since $\mathbb{R}^n/\pi^{-1}(p) \cong \mathbb{R}^n$ for each point p in $\mathbb{R}^n/G$.

Also, note that the thin decomposition G is intrinsically 0-dimensional. For the quotient map $\pi:\mathbb{R}^n \to X$ is approximable by a cell-like map f that is one to one over the manifold part of X and over any (k−1)-dimensional F_σ subset of the nonmanifold part of X by using standard techniques from decomposition theory. This (k−1)-dimensional subset can be chosen so that its complement (with respect to the nonmanifold part of X) is 0-dimensional by an argument from dimension theory. Decompositions that are intrinsically 0-dimensional and have nonmanifold part of dimension k can also be constructed by having 0-dimensional decompositions "limit down" to a k-cell. However, such decompositions would not have the thinness property described above. The nonmanifold nature of such decompositions could be detected by examining certain closed zero-dimensional subsets.

As mentioned above, the two key ingredients in the construction of the examples are mixing techniques for producing generalized n-manifolds whose nonmanifold part is a Cantor set, and decompositions arising from special functions from the Cantor set onto a k-cell. The special functions from the Cantor set onto a k-cell will be described first.

2. Cantor Functions: Let $\mathcal{C}$ be the standard Cantor Set in $I \equiv [0,1]$, and let $f:I \to I$ be the standard Cantor map which is constant on the closure of each component of $I \setminus \mathcal{C}$. Note that $f|_{\mathcal{C}}$ is two to one over the dyadic rationals in I, $f|_{\mathcal{C}}$ is one to one over the complement of the dyadic rationals, and f itself is one to one over the complement of the dyadic rationals.

Next, let $\mathcal{C}^k \subset I^k$ be the product of k copies of $\mathcal{C}$, and let $f^k\colon I^k \to I^k$ be defined by $f^k(x) = (f(x_1), \ldots, f(x_k))$. It follows that $(f^k)^{-1}(p)$ is a cell for each point p in I^k. The dimension of this cell corresponds to the number of dyadic rational coordinates that p has. One can easily check that if G is the decomposition of I^k induced by f^k, then G is cellular and upper semicontinuous, and that if p is a point of $\mathcal{C}^k$ with no triadic rational coordinates, then $(f^k)^{-1} \circ f^k(p) = p$.

A few key properties of the Cantor function and the decomposition G are listed in the following lemmas. Details can be found in [G].

<u>Lemma</u> If A is a nowhere dense subset of I^k then there exists a dense subset D of $\mathcal{C}^k$ so that $f^k(D) \cap A = \emptyset$ and $(f^k)^{-1}(f^k(d)) = d$ for each d in D.

<u>Lemma</u> Let G be the decomposition of $\mathbb{R}^n$ induced by the map f^k. Then π_G from $\mathbb{R}^n$ to $\mathbb{R}^n/G$ is approximable by homeomorphisms and $\pi_G(I^k)$ is a tame k-cell.

In order to have the decompositions from the next section match up in the correct way with the decompositions induced by the Cantor map described above, we need to view $\mathcal{C}^k$ as arising in a special way. The next lemma gives the necessary description.

<u>Lemma</u> The Cantor set $\mathcal{C}^k \subset I^k \times \{0\} \subset I^k \times I^{n-k} \subset \mathbb{R}^n$ can be obtained as $\cap_{i=0}^{\infty} \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}_{i-1}$, $(i > 1)$, and where each $\mathcal{A}_i$ consists of 4^i pairwise disjoint n-cells of the form ((n-1)-cell) $\times$ $[-\frac{1}{i}, \frac{1}{i}]$, and where each n-cell in $\mathcal{A}_i$ has exactly 4 n-cells of $\mathcal{A}_{i+1}$ in its interior.

The n-cells of $\mathcal{A}_k$ are denoted by $A(\epsilon_1, \epsilon_2, \ldots \epsilon_{k-1}, \epsilon_k)$ where ϵ_k is of the form (i_k, j_k) for i_k and j_k in $\{1, 2\}$. We require in addition that the n-cells be chosen so that the following lemma is satisfied:

<u>Lemma</u> If $p = \cap_{j=1}^{\infty} A(\epsilon_1, \epsilon_2, \ldots \epsilon_j)$ and if there exists an N so that past stage N, either the first coordinates of ϵ_i alternate, or the second coordinates of ϵ_i alternate, then p has no triadic rational coordinates, and $f^k(p)$ has no dyadic rational coordinates. Thus $(f^k)^{-1} \circ f^k(p) = p$.

3. Zero-dimensional Decompositions of $\mathbb{R}^n$ obtained by mixing:

The mixing technique referred to was developed independently by Daverman and Eaton. The specific technique used here is more similar to Eaton's construction [Ea]. Fix n greater than or equal to five, and fix k less than n for the rest of this section.

The goal is to produce a cell-like usc decomposition H of $\mathbb{R}^n$ with the following properties:

P1. For each nondegenerate element h of H, $h \cap I^k$ is a point in $\mathcal{C}^k$;

P2. Let f_1 and f_2 be maps from B^2 into $\mathbb{R}^n/H$ and let A be any dense subset of $\mathcal{C}^k$. Then f_1 and f_2 are approximable by maps g_1 and g_2 satisfying: (i) $g_1(B^2) \cap g_2(B^2) \subset \pi_H(A)$, and (ii) if p is a point of $\mathcal{C}^k$ with $\pi_H(p) \in \left(g_1(B^2) \setminus g_2(B^2)\right) \cup \left(g_2(B^2) \setminus g_1(B^2)\right)$, then p has no triadic rational coordinates.

P3. $\mathbb{R}^n/H$ has nonmanifold part equal to $\pi(\mathcal{C}^k)$.

Conditions P1 and P3 are satisfied for the decomposition of $\mathbb{R}^n$ described in [Wr]. Additional care in the construction allows one to construct such a decomposition so that it also satisfies P2. The special description of the Cantor Set from the previous section allows one to carry out the additional details of the construction. See [G] for details.

4. The Examples in $\mathbb{R}^n$:

Let H be the cellular decomposition of $\mathbb{R}^n$ described in the previous section and π_H be the quotient map from $\mathbb{R}^n$ to $\mathbb{R}^n/H$. Let G be the cellular decomposition of $\mathbb{R}^n$ induced by the map $f^k\colon I^k \to I^k$ from section 3, and let π_G be the quotient map from $\mathbb{R}^n$ to $\mathbb{R}^n/G$. Let K be the decomposition of $\mathbb{R}^n$ given by $x \underset{K}{\sim} y$ if and only if $\pi_G \circ \pi_H^{-1} \circ \pi_H(\ y\ ) \cap \pi_G \circ \pi_H^{-1} \circ \pi_H(\ x\ ) \neq \emptyset$. Let $X = \mathbb{R}^n/K$. Let π be the induced decomposition from from $\mathbb{R}^n$ to X.

One can then check that K is the desired decomposition.

Theorem The decomposition K satisfies the following three properties: 1) K is cellular; 2) the nonmanifold part of X is homeomorphic to a k-cell; and 3) If A is any closed subspace of X of dimension $\leq$ k$-$1, then the decomposition of $\mathbb{R}^n$ induced over A is shrinkable.

5. Generalizing the examples to the Hilbert Cube:

In generalizing these examples to the Hilbert Cube, one needs to generalize the constructions of sections two and three. The generalization of the decomposition induced by the Cantor function seems to go through without any problem. One uses the infinite product of the Cantor function to get a function from the Cantor Set onto the Hilbert Cube. Additional notational care is needed to obtain a description of the Cantor Set in the Hilbert Cube analogous to the description from section two.

Generalizing mixing decompositions to the Hilbert Cube has been done in [L]. Modifications of the techniques in this reference so as to mesh with the Cantor decomposition should be possible. The comments here describe work in progress.

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HOMOLOGY OF COVERINGS OF 3–MANIFOLDS BRANCHED OVER LINKS

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It is a theorem of Plans [P] that the first homology of an odd degree cyclic branched cover $\tilde{M}$ of S^3, branched over a knot, is always a direct double: $H_1(\tilde{M}) = A \oplus A$. This extends [VW] to cyclic covers of homology 3-spheres branched over links — modulo p-torsion for primes p dividing the degree of the cover. Some independent information about the first betti number and torsion numbers prime to the degree for such covers (cyclic covers branched over links in homology 3-spheres) is given in [CM].

We show that the arguments of [VW] and [CM] can easily be adapted to treat cyclic covers $\tilde{M} \to M$ branched over a link in an arbitrary 3-manifold, M, and show that the change from $H_1(M)$ to $H_1(\tilde{M})$ follows the same pattern. This allows us to give qualitative information about the homology of branched coverings which can be factored as a composition of cyclic coverings.

We also describe a procedure for calculating the first homology of a general branched covering from a relative Jacobian matrix of free derivatives of a presentation of a certain group pair in much the same way (c.f. [H_2]) as one calculates the first homology of an unbranched covering from a Jacobian of a presentation of the fundamental group of the base.

Statements of the results follow. Proofs and related material will appear in [He_4]

Theorem A. *Let $\rho : \tilde{M} \to M$ be a cyclic branched covering of closed, orientable 3-manifolds branched over a link $L \subset M$ and of prime degree d.*

If d is odd then:

(1) $\beta_1(\tilde{M}) = \beta_1(M) + (d-1)r$ for some $r \geq 0$.

(2) For each prime $q \neq d$

$$q\text{-}torsion(H_1(\tilde{M})) = q\text{-}torsion(H_1(M)) \oplus A^{lcm(2,g_d(q))}$$

for some group A; where $g_d(q)$ is the order of q in the multiplicative group ${\mathbb{Z}_d}^*$.

If $d = 2$ and the nontrivial covering transformation of ρ has an orientation preserving square root then:

(1′) $\beta_1(\tilde{M}) = \beta_1(M) + 2r$ for some $r \geq 0$.

(2′) For each prime $q \neq 2$

$$q\text{-}torsion(H_1(\tilde{M})) = q\text{-}torsion(H_1(M)) \oplus A^2$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

for some group A.

In all cases we have:

(3) Suppose $\beta_1(\tilde{M}) = 0$. If nontrivial branching occurs, then d-torsion($H_1(M)$) is a quotient group of d-torsion($H_1(\tilde{M})$). In any event the image of d-torsion($H_1(\tilde{M})$) in d-torsion($H_1(M)$) has index $\leq d$.

(4) If L is connected, $\beta_1(M) = 0$ and $\beta_1(\tilde{M}) > 0$ then d-torsion($H_1(M)$) $\neq 0$

We regard each finitely generated abelian group A as being decomposed as a direct sum of a free abelian group and cyclic groups of prime power order. Then for a prime q, q-torsion(A) is just the sum of those summands whose orders are powers of q. The result of dividing A by all the q-torsion summands for which q divides a fixed integer n is called the *n-reduction* of A and denoted n-red(A). A branched covering will be called *subsolvable* if it can be factored as a composition of cyclic branched covers — and so can further be factored as a composition of cyclic branched covers of prime degree. Any regular branched cover with solvable covering group is subsolvable; but there are lots of irregular subsolvable branched covers. Theorem A clearly yields

Theorem B. *Let $\rho : \tilde{M} \to M$ be a subsolvable branched cover of odd degree n of closed, orientable 3-manifolds branched over a link $L \subset M$. Then:*

(1) *$\beta_1(\tilde{M}) = \beta_1(M) + 2r$, for some $r \geq 0$;*

(2) *n-red($H_1(\tilde{M})$) $=$ n-red($H_1(M)$) $\oplus A \oplus A$, for some group A;*

(3) *If q^α is the highest power of the prime q which divides n, then $o(q\text{-torsion}(H_1(\tilde{M})) \geq o(q\text{-torsion}(H_1(M))/q^\alpha$.*

Suppose (M, B) is a finite CW-pair with M connected. The associated *joined pair* (M^*, B^*) is obtained from (M, B) by fixing a component of B and joining the remaining components of B to a basepoint in the fixed component by arcs whose interiors are disjoint from M. This gives a well defined *joined fundamental group system* $(\pi_1(M^*), i_\sharp\pi_1(B^*))$. Clearly $\pi_1(M^*) = \pi_1(M) * F$ where F is free of rank $\beta_0(B) - 1$ and $\pi_1(B^*)$ is the free product of the fundamental groups of the components of B.

Given a covering space $\rho : \tilde{M} \to M$ with monodromy $\varphi : \pi_1(M) \to S_d$ and a retraction $f : M^* \to M$ the covering $\tilde{M}^* \to M^*$ whose monodromy is $\varphi \circ f_\sharp$ is an extension of $\rho : \tilde{M} \to M$. In the following it will not matter which retraction f is chosen , but in doing calculations a consistent choice must be made.

We always regard the symmetric group S_d on d symbols as being a subgroup of $GL(d, \mathbb{Z})$ by identifying a permutation with the linear transformation which so permutes the standard basis vectors:

$$\varphi \mapsto (\delta_{i^\varphi, j})$$

Theorem C. *Let*

$$\rho : \tilde{M} \to M$$

be a d-sheeted covering space of a connected CW-complex M, B be a non-empty subcomplex of M, $\tilde{B} = \rho^{-1}(B)$, and

$$\varphi : \pi_1(M^*) \to S_d \subset GL(d, \mathbb{Z})$$

be the monodromy of an extension of ρ to a cover of M^*. If J is the relative Jacobian matrix of a presentation of the joined fundamental group system $(\pi_1(M^*), i_\sharp\pi_1(B^*))$ (c.f. [He$_4$]), then the $m\cdot d\times n\cdot d$ matrix of integers $\varphi(J)$ presents $H_1(\tilde{M}, \tilde{B})$ as an abelian group.

Now consider an orientable 3-manifold M with a k-component link $L \subset Int(M)$. Remove an open regular neighborhood of L from M to obtain a 3-manifold N. Let $\mu = \{\mu_1, \ldots, \mu_k\}$ be a system of meridians for L in ∂N. Let

$$\rho : \tilde{M} \to M$$

be a degree d cover branched over L and with monodromy

$$\varphi : \pi_1(N) \to S_d.$$

Let

$$\begin{aligned} c(\varphi(\mu)) &= \#\text{ of components of } (\rho^{-1}(\mu)) \\ &= \sum c(\varphi(\mu_i)) \end{aligned}$$

where $c(\varphi(\mu_i))$ is the number of cycles of the permutation $\varphi(\mu_i)$. From Theorem C we get:

Theorem D. *Let J be the relative Jacobian of any presentation of the joined fundamental group system $(\pi_1(N^*), i_\sharp\pi_1(\mu^*))$. Then $\varphi(J)$ is a presentation matrix over $\mathbb{Z}$ of*

$$H_1(\tilde{M}) \oplus \mathbb{Z}^{c(\varphi(\mu))-1}.$$

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CODIMENSION 2 FIBRATOR

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A proper map $p\colon M \to B$ between locally compact ANRs is called an approximate fibration if it has the following homotopy property: Given an open cover ϵ of B, an arbitrary space X and two maps $g\colon X \to M$ and $F\colon X\times I \to B$ such that $p\circ g = F_0$, there exists a map $G\colon X\times I \to M$ such that $G_0 + g$ and $p\circ G$ is ϵ-close to F.

To determine whether a proper map $p\colon M \to B$ is an approximate fibration or not, we define the following term.

DEFINITION A closed manifold N^n is a codimension 2 fibrator if whenever G is a usc decomposition of an arbitrary M^{n+2} such that each $g\in G$ is shape equivalent to N^n, then $p\colon M \to B$ $(=M/G)$ is an approximate fibration.

In [D], R.J. Daverman showed that any closed surface N for which $\chi(N)\neq 0$ is a codimension 2 fibrator. We will show the following result extending Daverman's result.

THEOREM 1 Any finite product of closed orientable surfaces of genus at least 2 is a codimension 2 fibrator.

First, we will state the key fact to investigate codimension 2 fibrator, which can be found in [D-W].

THEOREM 2 If G is a usc decomposition of an orientable $(n+2)$-

manifold M into closed orientable n-manifolds, then the decomposition space B is a 2-manifold and $D = B \backslash C$ is locally finite in B, where C represents the continuity set of $p: M \to B$; if either M or some elements of G are non-orientable, B is a manifold with boundary (possibly empty) and $D' = \text{Int } B \backslash C'$ is locally finite in B, where C' represents the mod 2 continuity set.

As a consequence, we can localize the problem to that of an open disk B, where $p: M \to B$ is an approximate fibration over $B \backslash b$ for some $b \in B$.

To prove our result, we need the following several lemmas.

LEMMA 3 Suppose N^n is a closed orientable aspherical manifold with Hopfian fundamental group, G is a usc decomposition of M^{n+k} into copies of N^n, and $\dim M/G < \infty$. Then $p: M \to M/G$ is an approximate fibration over its continuity set C.

LEMMA 4 Let F^2 be a closed orientable surface with g handles, and α_k and β_k be the standard oriented simple closed curves around the k-th handle of F^2 for $1 \le k \le g$. Suppose $(a_1, b_1, \ldots, a_g, b_g)$ is an element of $H_1(F)$. Then by iterations of Dehn twists and the Euclidean algorithm, there is a homeomorphism of F^2 onto itself which induces an automorphism on $H_1(F)$ carrying $(a_1, b_1, \ldots, a_g, b_g)$ to $(d, 0, \ldots, 0, 0)$, where

$$d = \text{g.c.d. } \{a_1, b_1, \ldots, a_g, b_g\}.$$

LEMMA 5 [Z-V-C] Let $f: F_1 \to F_2$ be a continuous map such that the degree of f is zero, where F_i is a closed orientable surface of genus g_i for i = 1, 2. Then we have

$$\text{rank } f_* \leq g_1,$$

where $f_*: H_1(F_1) \to H_1(F_2)$ is the induced homomorphism.

LEMMA 6 Let $\phi: N \to N$ be a continuous map which is not degree 1. Then

$$\text{rank } \phi_* \leq \text{rank } H_1(N) - \min_{1 \leq i \leq n} g_i,$$

where $\phi_*: H_1(N) \to H_1(N)$ and N is a finite product of closed orientable surfaces of genus at least 2.

Sketch of the proof If n = 1, it is simply a consequence of Lemma 5. For simplicity, we assume n = 2. For n ≥ 3, the conclusion follows from the inductive step. Now ϕ_* induces an $m \times m$ matrix of the following form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

By using Dehn twists and cohomology ring of N, we can reduce so that ϕ_* induces either

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$

In the first case, ϕ_* can be written as

$\phi_* = (\phi_1\times\phi_2)_*: H_1(N) \to H_1(N)$. In fact ϕ is the map of composition with homeomorphisms, but we abuse notation. Here ϕ_i: $F_i \to F_i$ is defined by $\phi_i = p_i\circ\phi\circ\iota_i$ for i = 1, 2, and p_i and ι_i are the projection and inclusion, respectively. Also we can easily show that the degree of ϕ is the same as the degree of $\phi_1\times\phi_2$. Since the degree of ϕ is not 1, the degree of ϕ_1 is zero or the degree of ϕ_2 is zero. By Lemma 5, the result follows directly.

In case 2, if $F_1 = F_2$, by a homeomorphism h: N $\to$ N defined by h(x,y) = (y,x) for (x,y) $\in F_1\times F_2$ (=N), we can reduce to the previous case. If $F_1 \neq F_2$ ($g_1<g_2$) where g_i is the genus of F_i for i = 1,2, then the conclusion follows by using deg $p_2\circ\phi\circ\iota_1 = 0$.

Sketch of the proof of Theorem 1 We consider 2 cases.

<u>case 1</u> M is orientable

Theorem 2 implies that the decomposition space B is a 2-manifold and the discontinuity set D=B\C is locally finite. Now we can assume p: M $\to$ B is an approximate fibration over $B\setminus b_0$. Name $g_0 \in G$ such that $p(g_0) = b_0$, specify a retraction r: M $\to G_0$, and fix $g \neq g_0$ in G. Consider the following homotopy exact sequence

$$1 \to \pi_1(g) \to \pi_1(M\setminus G_0) \to \pi_1(B\setminus b_0) \to 1.$$

After abelinization, we have

$$H_1(M\setminus g_0)/\mathrm{im}[i_* H_1(g)] \cong Z,$$

where $i\colon g \to M\backslash g_0$ is the inclusion. Notice that $H_1(M\backslash g_0) \to H_1(M)$ is surjective. Then we can show that

$$\text{rank } (r\circ i)_* \geq \text{rank } H_1(N) - 1.$$

By Lemma 6, $r\circ i$ is a degree 1 map, and hence a π_1-epimorphism. Hopfian property implies that $r\circ i$ induces an isomorphism. Asphericity of N then implies $r\circ i$ is a homotopy equivalence, and the conclusion follows from [C-D_2].

<u>case 2</u> M is non-orientable

We use the mod 2 continuity set C' instead of continuity set C.As above, p is an approximate fibration over the mod 2 continuity set C' of B. By the same way as case 1, we can show $C' \cap \text{Int } B = \text{Int } B$ and B is a 2-manifold without boundary. This completes the proof of Theorem 1.

We have the following corollaries.

COROLLARY 7 Let M^{n+2} be a simply connected manifold,N^n be a finite product of closed orientable surfaces of genus at least 2. Then there is no usc decomposition of M^{n+2} into copies of N.

COROLLARY 8 If G is a usc decomposition of M^{n+2} into copies of N^n, a finite product of closed orientable surfaces of genus at least 2, and $\chi(B) \leq 0$, then M^{n+2} is aspherical and $\pi_1(M)$ is an extension of $\pi_1(N)$ by $\pi_1(B)$.

Also we can extend Theorem 1 so that any finite product of closed orientable surfaces with non-zero Euler characteristic is a codimension 2 fibrator.

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Nielsen Theory and Homeomorphisms of Smooth Manifolds

Michael R. Kelly

0. Introduction

In an attempt to gain an understanding of the dynamics of surface self-maps, J. Nielsen developed a method for estimating the number of fixed points for a given map. This method, now referred to as Nielsen Theory, applies to a large class of spaces which includes all compact polyhedra. The general idea is to use the induced map on the fundamental group to obtain a homotopy invariant called the Nielsen number. It turns out that the Nielsen number is a natural generalization of the Lefschetz number and improves on the classical Lefschetz Theorem by giving a lower bound for the number of fixed points for a given map.

This talk first reviews the definition and properties of the Nielsen number and then mentions some important results concerning this number, focusing on the question of the realizability of the Nielsen number by a given map. Finally, we discuss the following special case: suppose that the space X is a manifold and $h : X \to X$ is a homeomorphism. Then isotopy is a more natural framework than homotopy to work in. Since the Nielsen number itself does not distinguish between homotopy and isotopy, it is still the natural candidate for an optimal lower bound. We present here some results in this direction.

1. Nielsen number

What follows is a brief outline of the definition of the Nielsen number and some of its properties. The reader can find more details in [Br] or [J1]. Let X be a compact polyhedra and $f : X \to X$ a continuous self-map. For convenience of presentation we assume that $Fix(f) = \{x \in X | f(x) = x\}$ is a finite set of points.

The simplicial approximation theorem guarantees the existence of such a map in each homotopy class.

Define an equivalence relation on $Fix(f)$ by saying $x \sim y$ iff there exists a path α in X from x to y such that $f(\alpha)$ is homotopic to α rel $\{x, y\}$. An equivalence class is often referred to as a fixed point class (or sometimes a Nielsen class). Secondly, there exists an integer valued, additive index defined on $Fix(f)$. A special case can be described when X is a manifold. Then the boundary of a neighborhood of a fixed point is a sphere of dimension k. After normalizing, f induces a map $\phi : S^k \to S^k$ and then index$(f, p) \equiv \deg(\phi)$. By additivity, for a fixed point class ψ, index$(f, \psi) = \sum_{p \in \psi}$ index(f, p).

Now define the *Nielsen number*, $N(f)$, to be the number of fixed point classes having a nonzero index. This number satisfies the following:

(1) the sum of indices of the fixed point classes is equal to the Lefschetz number, $L(f)$. In particular, if $N(f) = 0$ then $L(f) = 0$.

(2) if f is homotopic to g then $N(f) = N(g)$.

As a consequence of (2) and the definition we have

Theorem 1. *For any $f : X \to X$, $\#Fix(f) \geq N(f)$.*

It is then natural to ask if this lower bound for the cardinality of the fixed point set can actually be achieved by some map in a given homotopy class. With this in mind we define

$$MF[f] = \min_g \{\#Fix(g) | g \text{ is homotopic to } f\}.$$

Work by a number of authors (see [Bo], [N], [W], [We], [J2]) has lead to the following very concise result:

Theorem 2. *Suppose that X is a compact polyhedron which (1) does not contain any local separating points and (2) is not a surface with negative Euler characteristic, then for any self-map $f : X \to X$, $MF[f] = N(f)$.*

Counterexamples under hypothesis (1) have been known for a long time and are quite straightforward to produce (see [Br] for example). On the other hand, (2) is more subtle as Weier [We], in 1956, presented without proof a potential counterexample but not until Jiang [J4] in 1984 was one verified. Since then more examples have been produced (see [J5], [K1], [K2], [Z], [T]), including verification of Weier's original claim, but the problem is still not well understood.

3. Homeomorphisms of manifolds

One special case of the previous discussion is noteworthy. Suppose that $h : F \to F$ is a homeomorphism of a compact surface (possibly with nonempty boundary). As a result of the Nielsen/Thurston classification of surface automorphisms it can be shown that $MF[h] = N(h)$, (see [J3], [I]). Putting this together with Theorem 2

Corollary 3: If M is a topological manifold and $h : M \to M$ a homeomorphism then $MF[h] = N(h)$.

It is this result that suggests considering the following variation of the minimization problem in Nielsen theory. Under the hypothesis of Corollary 3 it is often more natural to consider isotopy as opposed to homotopy of homeomorphisms which leads us to consider the following quantity

$$MI[h] = \min_g\{\#Fix(g) | g \text{ is isotopic to } h\}.$$

Certainly $N(h)$ is an isotopy invariant and so $MI[h] \geq N(h)$ but what about equality? For example, if $\dim(M) = 2$ then it is well known that homotopy implies isotopy and so from Corollary 3 any surface homeomorphism is isotopic to one having $N(\cdot)$ fixed points. In higher dimensions homotopy does not imply isotopy and so naturally other methods are needed. In fact, it is the purpose of this talk to announce the following partial result.

Theorem 4. *Suppose that M is a smooth manifold with* $\dim(M) \geq 5$ *and let* $h : M \to M$ *be a diffeomorphism. Then* $MI[h] = N(h)$.

The following gives a brief description of the proof of this theorem. By analogy we first consider a proof of corollary 3 when $\dim(M) \geq 3$.

A ball B in M is said to be *homotopy-standard* if $i_{\#} : \pi_1[B \cup h(B)] \to \pi_1(M)$ is the trivial map. By techniques of Wecken [W] and Weier [We] it can be shown that if B is *homotopy-standard* then h is homotopic to h' with support on B so that $Fix(h') \cap B$ is at most one point. Moreover, if $\text{index}(h, B) = 0$ then h' has no fixed points in B. Finally, the definition of Nielsen class allows us to enclose every fixed point class in a *homotopy-standard* ball. Hence, we have exactly $N(h)$ fixed points.

Our approach is to define a special class of balls, called *isotopy-standard.* It is much more restrictive than *homotopy-standard* but we are able to show analogous results. Namely:

(1) if B is *isotopy-standard* and $\text{index}(h, B) = 0$ then h is isotopic to h' (rel $M - B$) so that $Fix(h') \cap B = \phi$ and,

(2) if, in addition, $\dim(M) \geq 5$ then each fixed point class can be enclosed in an *isotopy-standard* ball.

The proof of (1) will appear in [K3] while (2) will appear elsewhere.

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HIGHER DIMENSIONAL DUNCE HATS AS SYMMETRIC PRODUCTS

by

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1. Introduction.

In this talk we will summarize some joint work done jointly with M.M. Marjanovic. We will construct a sequence D^{2n}, $n \geq 1$, of even dimensional CW complexes where D^2 is the traditional topological Dunce Hat and prove that each of the spaces is contractible but not collapsible, the two salient properties of the Dunce Hat. Consequently, we call the spaces D^{2n}, $n \geq 2$, *higher dimensional Dunce Hats.* For the construction we will use the type of symmetric product that corresponds to the hyperspace of subsets of a continuum containing n or fewer points with the Hausdorff metric. Our contractibility proof is based on a homology argument.

The topological Dunce Hat D is discussed in detail by E.C. Zeeman in [Z]. The space D is remarkable because it is the simplest example of a polyhedron that is contractible, in the sense of homotopy, but not collapsible, in the sense of Whitehead. The point of [Z] was to analyse the Dunce Hat, and the manifolds of which it is a spine because of some intimate relations to the Poincare Conjecture. The point of this connection was that the phenomenon of being contractible yet not collapsible had been identified as a primary source of difficulty in the study of manifolds of dimension ≥ 3. In this paper, we will omit most of the proofs in Sections 4 and 5 as they will appear elsewhere. The author thanks Dennis Garity for helpful suggestions.

2. Background and Preliminaries.

We start with giving the necessary background for symmetric products. The following definitions are valid in a more general setting, see [S], but for convenience we will make our definitions for more restrictive spaces.

Definitions 2.1.

a). The n-fold symmetric product.

For a compact metric space X, if X^n is the n-fold cartesian product of X, define the *n-fold symmetric product of X*,

$$X(n) = X^n/\sim,$$

where $\sim$ is the equivalence relation on X^n defined by

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \text{ if and only if } \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}.$$

Thus, two points in X^n are equivalent if the sets consisting of their coordinates are equal. If $n=2$, this means that $(x,y)\sim(y,x)$, but if $n\geq 3$, then not only do we have identification under permutations of coordinates, but we also have the extra identifications as illustrated by the case for $n=3$ where $(a,a,b)\sim(a,b,b)$.

The n-fold symmetric product of X, $X(n)$, is well known, see [S], to be homeomorphic to the space of (closed) subsets of X consisting of n or fewer points topologized with the Hausdorff metric

$$D(A,B)= \inf\{\epsilon>0: A \subset U(B,\epsilon) \text{ and } B \subset U(A,\epsilon)\},$$

where A and B are closed subsets of X and $U(C,\epsilon)$ is the open ϵ-ball about $C \subset X$. In fact, for the rest of this paper we will think of $X(n)$ as the "hyperspace" of n or fewer points of X.

For the closed unit interval $I=[0,1]$ and for $n\geq 2$, we let

$$I_0^1(n) = \{A \in I(n) : 0, 1 \in A\}.$$

That is, $I_0^1(n)$ consists of those subset of I that contain n or fewer points and that contain the points 0 and 1. We remark that the space $I_0^1(3)$ is homeomorphic to the 1-sphere S^1 as can be seen by noting that the generic point of $I_0^1(3)$ is $x=\{0,b,1\}$, for $b \in I$, and as b moves from 0 to 1, the point x moves from the base point $*=\{0,1\}$ around a circle and back to $*$. In the next section we will see that $I_0^1(4)$ is the classical topological Dunce Hat.

b). Contractible and Collapsible.

A topological space X is *contractible* if the identity map from X onto itself is homotopic to a constant map. In order to define collapsible we must first define the notion of an elementary collapse. Let K be a finite simplicial complex with an n-simplex σ with an $(n-1)$-face τ. We say that τ is a *free* face of K if it is a proper face of only one simplex of K. In this case, we say that there is an *elementary collapse* of $|K|$ onto $|K|-(\mathring{\sigma}\cup\mathring{\tau})$. A (compact) polyhedron $P=|K|$ is *collapsible* if P can be reduced to a point after some finite sequence of elementary collapses.

c). Adjunction spaces.

Let X and Y be disjoint topological spaces, let A be a closed subset of X, and let $f : A \to Y$ be a continuous map. Topologize $X \cup Y$ as the topological sum and let $\sim$ be the least equivalence relation on $X \cup Y$ such that $a \sim f(a)$ for all $a \in A$. Then the *adjunction space determined by f* is the quotient space $X \cup Y/\sim$ and is denoted by $X \cup_f Y$. We also say that X is *attached to* Y by f.

We will be constructing spaces by attaching n-cells B^n to a space Y by a map f: $\mathrm{Bd}\ B^n \to Y$. Spaces inductively built this way are CW-complexes.

d). Finite CW-Complex.

Let X^0 be a finite discrete set of points. For $k>0$, X^k is obtained from X^{k-1} by attaching a finite set of n-cells by maps from their boundaries into X^{k-1}. If for some n ,

$$X = \bigcup_{k=0}^{n} X^k$$

then X is called a (finite) *CW-complex.* For each $k = 0, 1, \dots, n$, the space X^k is called the *k-skeleton* of X.

Remark 2.3. Finite CW-complexes are compact metric spaces.

Remark 2.4. If A is a closed subset of the compact metric space Y and

$$f: (B^n, \mathrm{Bd}\ B^n) \to (Y, A)$$

is a *relative homeomorphism* (that is, f is a continuous surjection and $f \mid B^n - \mathrm{Bd}\ B^n$ is one-to-one), then $Y \cong B^n \cup_{\bar{f}} A$ where $\bar{f} = f \mid \mathrm{Bd}\ B^n$.

3. The Traditional Topological Dunce Hat.

The Dunce Hat D is obtained from a solid triangle, say ABC, by identifying the sides AB = AC = BC. By identifying just the first two sides AB = AC we get a cone which has been known as the traditional dunce hat. This one of course is both contractible and collapsible and is topologically a 2-cell.

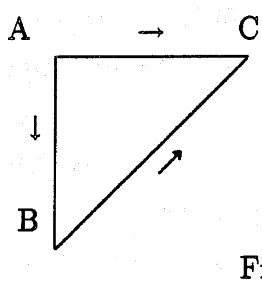

Fig. 1

We state and prove the following theorem to motivate the next section where these ideas are generalized.

Theorem 3.1. *The space* $I_0^1(4)$ *is the Dunce Hat.*

Proof. Let $\sigma^2 = \{(a,b) \in I^2\colon 0 \leq a \leq b \leq 1\}$ and define $q\colon \sigma^2 \to I_0^1(4)$ by $q(a,b) = \{0,a,b,1\}$. The edge of σ^2 labeled BA in Figure 1 is the set $\{(0,b)\colon b \in I\}$ and is mapped onto $I_0^1(3) \doteq S^1$ by q with positive orientation. This is easily seen since $q(0,b) = \{0,0,b,1\} = \{0,b,1\}$. Likewise, the edges $AC = \{(b,1)\colon b \in I\}$ and $BC = \{(b,b)\colon b \in I\}$ are mapped onto $I_0^1(3)$ by q with positive orientations. Consequently, under the map q the edges BA, AC, and BC are identified as indicated in Figure 2. Furthermore, the map q restricted to the interior of σ^2 is one-to-one and hence the diagram at the right in Figure 2 faithfully represents the space $I_0^1(4)$.

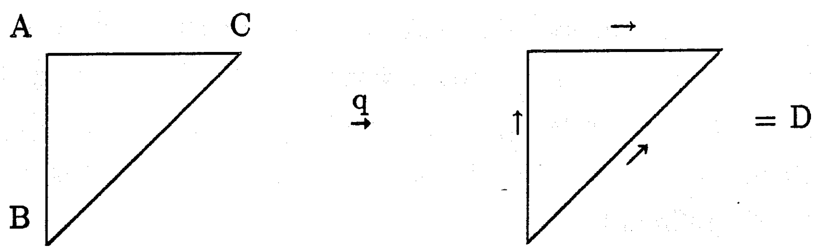

Fig. 2

The orientations of the edges of this figure differ somewhat from the orientations in Figure 1, but the spaces are obviously homeomorphic and hence $I_0^1(4)$ is topologically a Dunce Hat. □

4. Higher Dimensional Dunce Hats.

The symmetric product representation of a Dunce Hat as presented above gives a natural way for defining higher dimensional analogues.

Definition 4.1. For each integer $n \geq 0$, let

$$D^n = I_0^1(n+2) = \{C \in I(n+2) : 0 \in C \text{ and } 1 \in C\}.$$

(We remark the D^0 is a point, and D^1 is homeomorphic to a circle S^1.)

Furthermore, let

$$\sigma^n = \{(a_1,a_2, \ldots ,a_n) : 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1\}$$

and define $q_n\colon \sigma^n \to D^n$ by

$$q_n(a_1,a_2, \ldots ,a_n) = \{0,a_1,a_2, \ldots ,a_n,1\}$$

Lemma 4.2. *The map* $q_n\colon \left(\sigma^n,\ Bd\sigma^n\right) \to \left(D^n,\ D^{n-1}\right)$, $n \geq 1$, *as a map of pairs, is a relative homeomorphism and hence* $D^n \cong \sigma^n \cup_{\overline{q}_n} D^{n-1}$, *the CW-complex obtained by attaching* σ^n *to* D^{n-1} *with the map* $\overline{q}_n = q_n / Bd\sigma^n$.

Thus we have a sequence of CW-complexes

$$D^0 \subset D^1 \subset \cdots \subset D^{n-1} \subset D^n \subset \cdots$$

where D^n is formed by attaching the n-cell σ^n to D^{n-1} with the map $\overline{q}_n : Bd\ \sigma^n \to D^{n-1}$.

The use of rectangular coordinates in the description of

$$\sigma^n = \{(a_1,a_2, \ldots ,a_n) : 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1\}$$

has been very convenient for the construction of D^n. We now switch to barycentric coordinates for an n-simplex in the rest of the paper.

Let $v_0,v_1, \ldots ,v_n$ be independent points in E^n and let $\Delta^n = [v_0,v_1, \ldots ,v_n]$ be the convex hull of $\{v_0,v_1, \ldots ,v_n\}$. Then a point $x \in \Delta^n$ is uniquely determined as $x = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n$ where $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$ and each $\lambda_i \geq 0$. Then $\alpha_n : \Delta^n \to \sigma^n$ defined by $\alpha_n(x) = (\lambda_0,\ \lambda_0+\lambda_1, \ldots ,\ \lambda_0 + \cdots + \lambda_{n-1})$ is a homeomorphism and $p_n : \Delta^n \to D^n$ defined by

$$p_n(x) = \{0,\ \lambda_0,\ \lambda_0+\lambda_1, \ldots ,\ \lambda_0 + \cdots + \lambda_{n-1},\ 1\}$$

is *topologically equivalent* to $q_n\colon \sigma^n \to D^n$. That is, the following diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "p_n"] \arrow[d, "\alpha_n"] & D^n \arrow[d, "id"] \\
\sigma^n \arrow[r, "q_n"] & D^n
\end{tikzcd}$$

commutes ($p_n = q_n \circ \alpha_n$) where the vertical arrows are homeomorphisms.

We remark that in [T], Thomas gives a different but equivalent definition of the spaces D^n but neither states nor proves any properties of the D^n.

We will now set up some notation to deal with a homology proof of the contractibility of D^{2n}. Let $d_i\colon \Delta^n \to \Delta^{n+1}$, $i=0, \ldots ,n+1$, be the *face map* which takes Δ^n simplicially onto the face of $\Delta^{n+1} = [v_0,v_1, \ldots ,v_{n+1}]$ opposite the vertex v_i. The map d_i is defined by

$d_i(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_0 v_0 + \cdots + \lambda_{i-1} v_{i-1} + 0 v_i + \lambda_i v_{i+1} + \cdots + \lambda_n v_{n+1}$. and $d_i[v_0, \ldots, v_n] = [v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]$. The following lemma provides a very convenient relationship between the face maps d_i and the maps p_n.

Lemma 4.3. *The following diagram commutes, where i is injection.*

$$\begin{array}{ccc} \Delta^n & \xrightarrow{p_n} & D^n \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle i} \\ \Delta^{n+1} & \xrightarrow{p_{n+1}} & D^{n+1} \end{array}$$

5. Homology Considerations.

Our proof that D^{2n} is contractible is based on using the following Whitehead theorem [Ma, Cor. 3.3.11, p. 329].

Theorem 5.1. *If X is a connected and simply connected CW-complex and, for all n, $\tilde{H}_n(X) = 0$, then X is contractible.*

Following Munkres [Mu, Section 39] we create the cellular chain complex of D^n as a means of computing the homology of D^n. Here we will use the skeleton structure of D^n, $D^0 \subset D^1 \subset \cdots \subset D^k \subset \cdots \subset D^n$, as constructed in Section 4. Let $C_k(D^n) = H_k(D^k, D^{k-1})$ and define a boundary operator $\partial_k\colon C_k(D^n) \to C_{k-1}(D^n)$ as the composite

$$H_k(D^k, D^{k-1}) \xrightarrow{\partial_*} H_{k-1}(D^{k-1}) \xrightarrow{j_*} H_{k-1}(D^{k-1}, D^{k-2})$$

where ∂_* is the boundary homomorphism in the long exact sequence of the pair (D^k, D^{k-1}), and j is inclusion. Then the chain complex $\mathcal{C}(D^n) = \{C_k(D^n), \partial_k\}$ is called the *cellular chain complex* of D^n. The homology of $\mathcal{C}(D^n)$ yields the homology of D^n, that is, $H_k(D^n) = H_k(\mathcal{C}(D^n))$ for each k.

Lemma 5.2. *The chain group* $C_k(D^n) = H_k(D^k, D^{k-1}) = \begin{cases} Z, & \text{if } 0 \le k \le n \\ 0, & \textit{otherwise.} \end{cases}$

Consequently, $H_k(D^n)$ is the corresponding homology group of the chain complex

$$0 \xrightarrow{\partial_{n+1}} Z \to \cdots \to Z \xrightarrow{\partial_k} Z \to \cdots \to Z \xrightarrow{\partial_0} 0.$$

We will now analyse the boundary operators of this chain complex to compute the homology.

Lemma 5.3. *If $p\colon Bd\Delta^{k+1} \to D^k$ is the map $p_{k+1}/\ Bd\Delta^{k+1}$ and $j\colon (D^k,\emptyset) \to (D^k,D^{k-1})$ is inclusion, then the composite homomorphism*

$$H_k(Bd\Delta^{k+1}) \overset{p_*}{\to} H_k(D^k) \overset{j_*}{\to} H_k(D^k,D^{k-1})$$

is an isomorphism if k is odd and 0 if k is even.

Theorem 5.4. *The homomorphism*

$$\partial_{k+1}\colon H_{k+1}(D^{k+1},D^k) \to H_k(D^k,D^{k-1})$$

is an isomorphism if k is odd and is 0 is k is even.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H_{k+1}(\Delta^{k+1}, Bd\Delta^{k+1}) & \overset{\partial_*}{\to} & H_k(Bd\Delta^{k+1}) & & \\
\Big\downarrow (p_{k+1})_* & & \Big\downarrow p_* & & \\
H_{k+1}(D^{k+1}, D^k) & \overset{\partial_*}{\to} & H_k(D^k) & \overset{j_*}{\to} & H_k(D^k, D^{k-1}).
\end{array}$$

The map $p_{k+1}\colon (\Delta^{k+1}, Bd\Delta^{k+1}) \to (D^{k+1}, D^k)$ is a relative homeomorphism and hence the left vertical arrow in the diagram is an isomorphism. The map ∂_* at the top of the diagram is an isomorphism as seen from looking at the long exact sequence of the pair $(\Delta^{k+1}, Bd\Delta^{k+1})$, and by Lemma 5.3 the homomorphism $j_* \circ p_*$ is an isomorphism if k is odd and 0 if k is even. The theorem follows. □

We are now ready to make our final homology calculations for $H_k(D^n)$.

Theorem 5.5. *The reduced homology of D^n is as follows:*

$$\tilde{H}_k(D^n) = \begin{cases} 0, & \text{if } n \text{ even} \\ Z, & \text{if } n \text{ odd and } k = n \\ 0, & \text{if } n \text{ odd and } k \neq n. \end{cases}$$

Theorem 5.6. *The spaces* D^{2n}, $n \geq 0$, *are contractible, and for* $n \geq 1$, D^n *is not collapsible.*

Remark 5.7. See the paper by Robert N. Andersen in these proceeding representing joint work with this author for additional results concerning these higher dimensional Dunce Hats.

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An Application of Group Endomorphism Growth Rates to Knot Theory

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This paper is a summary and expansion of a talk given at the Seventh Annual Western Workshop on Geometric Topology held at Oregon State University, Corvallis, Oregon, May 31 - June 2, 1990. Details will appear in [S₁] and [S₂].

Introduction. Let K be any oriented spherical or disk n-knot, and denote its exterior by $X(K)$. If $t \epsilon \pi_1(X(K))$ is represented by a meridian of K (with preferred orientation from K), then conjugation by t induces an automorphism μ_t of the commutator subgroup $\pi_1'(X(K))$. We borrow the notion of *exponential growth rate of group endomorphism* from R. Bowen's study [Bn] of topological entropy in order to define an invariant γ_K whenever $\pi_1'(X(K))$ is finitely generated: If the elements $g_1, \ldots, g_r$ generate $\pi_1'(X(K))$, then γ_K is the exponential growth rate $\gamma(\mu_t)$ of μ_t defined by

$$\gamma(\mu_t) = \max_i \ \overline{\lim_{k\to\infty}} \frac{1}{k} \log |\mu_t^k(g_i)|.$$

Here $|x|$ denotes the length of a shortest word in $g_1^{\pm 1}, \ldots, g_r^{\pm 1}$ representing x. Elementary properties of exponential growth rate in [FLP] ensure that γ_K is finite and independent of the choices involved.

An algorithm of M. Bestvina and M. Handel [BH] based on combinatorial techniques of J. Stallings makes precise calculation of γ_K possible whenever $\pi_1'(X(K))$ is free and μ_t is irreducible. Such is the case if K is a hyperbolic fibered 1-knot, and in that case, γ_K is the log of the stretching factor of the pseudo-Anosov monodromy.

Some properties and applications of γ_K. Like pseudo-Anosov stretching factors, γ_K is very sensitive – capable of distinguishing n-knots that have the same Alexander modules. For example, we can prove that if $f(t) = a_0 + a_1 t + \cdots + a_d t^d$ is any polynomial with integer coefficients such that $f(1) = \pm 1$ and $a_0 a_d = \pm 1$, then there exists a sequence of doubly slice fibered ribbon 1-knots $K_n \subset S^3$ with identical Seifert forms, Alexander polynomials equal to $f(t)f(t^{-1})$ and $\lim_{n\to\infty} \gamma_{K_n} = \infty$. (The existence of at least one such knot had previously been established in [AS].)

There are several very useful identities involving the invariant γ_K and well-known knot constructions. For example, $\gamma_{K_1 \sharp K_2} = \max\{\gamma_{K_1}, \gamma_{K_2}\}$, whenever $\gamma_{K_1 \sharp K_2}$ is defined. This is a special case of a general result relating the invariant of any satellite n-knot to those of its pattern and companion. Using it we can prove the following.

Proposition. Let $K \subset S^3$ be any 1-knot. Then $\gamma_K = 0$ if and only if K is a graph knot (i.e., $X(K)$ is a graph manifold.)

The invariant γ_K also yields a useful invertibility obstruction for n-knots. Recall that an oriented n-knot K is *invertible* if K is equivalent to rK, where r is an orientation-reversing diffeomorphism of K. If K is invertible and $\pi_1'(X(K))$ is finitely generated, then it is easy to see that $\gamma(\mu_t) = \gamma(\mu_t^{-1})$. Although $\gamma(\mu_t)$ and $\gamma(\mu_t^{-1})$ are always equal when $n = 1$ [T], they can differ when $n > 1$. We have exploited this in [S_1] to produce an example of a fibered (ribbon) 2-knot that is noninvertible and yet satisfies all of the necessary conditions of J. Hillman [H] and D. Ruberman [Ru] for any fibered even-dimensional knot to be invertible.

It follows from work of W. Thurston [T] that $\log(\gamma_K)$ is an algebraic integer whenever K is a fibered 1-knot. This is also true for n-knots K, $n > 1$, provided that $\pi_1'(X(K))$ is a finitely generated free group and μ_t is irreducible [BH]. Beyond this, nothing is known about the possible values of γ_K.

Question. What are the possible values of γ_K? Is $\log(\gamma_K)$ always an algebraic integer?
In [Fr] D. Fried gave an example of a finitely generated group automorphism α such that $\log(\gamma(\alpha))$ is not an algebraic integer. Such an example in which α is an automorphism μ_t as above would provide a very interesting n-knot.

To date, perhaps the most striking application of the invariant is to the study of ribbon concordance. Recall that a concordance $C \subset S^3 \times I$ between 1-knots $K_i \subset S^3 \times \{i\}$, $i = 0, 1$, is called a *ribbon concordance (from K_1 to K_0)* if the restriction to C of the projection $S^3 \times I \to I$ is a Morse function with no local maxima. In this case we write $K_1 \geq K_0$. The notion of ribbon concordance was introduced by C. Gordon in [G] where he showed that $\geq$ is a partial ordering on the set of all transfinitely nilpotent 1-knots, a collection that includes all fibered 1-knots. In [S_1] we proved that $K_1 \geq K_0$ implies $\gamma_{K_1} \geq \gamma_{K_0}$ for any fibered K_0, K_1. (An essential component of the proof was supplied by Katura Miyazaki.) As a direct consequence we obtain the following.

Proposition. Let K_0 and K_1 be fibered 1-knots. If K_1 is a graph knot and $K_1 \geq K_0$, then K_0 is also a graph knot.

Further results and questions. After completing [S_1] I learned about some new results of O. Kakimizu that provide a natural way to extend the definition of γ_K for all 1-knots. In [K] Kakimizu proved that the projection $K \times S^1 (= \partial X(K)) \to S^1$ always extends to a fibration of a compact codimension-0 submanifold of $X(K)$ that is maximal and unique up to isotopy. The fiber is a subsurface $m(S)$ of any minimal Seifert surface S for K, and $\pi_1(m(S))$ is isomorphic to the intersection $\cap_{k \in \mathbf{Z}} t^k \pi_1(S) t^{-k}$. If we now define μ_t to be the automorphism of $\pi_1(m(S))$ induced by the monodromy of the fibration, then we can define γ_K to be the exponential growth rate $\gamma(\mu_t)$ just as before. This extended invariant is often nonzero: for example, $\gamma_K \neq 0$ whenever the splice decomposition of K (in the sense of [EN]) contains a hyperbolic fibered component.

Question. Does $K_1 \geq K_0$ imply $\gamma_{K_1} \geq \gamma_{K_0}$ for *all* 1-knots K_0, K_1?

A natural question concerning higher dimensions now arises: Can γ_K be usefully defined for all n-knots, $n > 1$? The group $\pi_1(m(S))$ can be described as the union of all finitely generated subgroups H of $\pi_1(S)$ such that $tHt^{-1} = H$. We can ask the following.

Question. Let $\pi_1(X(K))$ be the group of any n-knot K and let H_K be the union of all finitely generated subgroups of $\pi_1(X(K))$ such that $tHt^{-1} = H$. Must H_K be finitely generated?

The interest here is that whenever H_K is finitely generated, we can define γ_K to be the exponential growth rate of the automorphism of H_K induced by conjugation by t. For each n-knot K that we have studied so far, H_K has turned out to be finitely generated. Indeed, the collection $\mathcal{F}$ of all n-knots K for which H_K is finitely generated is large: If K is a satellite n-knot with both pattern and companion belonging to $\mathcal{F}$, then K belongs to $\mathcal{F}$ as well [S_2]. Even if the answer to the preceeding question is no, γ_K still can be defined for any n-knot in the collection $\mathcal{F}$.

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'Almost acyclic degree one maps of manifolds' by F. C. Tinsley

For this summary let G be a finitely presesented group and M an n-manifold with $n > 4$ and $\pi_1(M) \cong G$.

If G contains a finitely generated perfect subgroup, then Daverman and Tinsley have exhibited an acyclic 2-complex K and a locally flat a nice embedding $f : K \mapsto M$ with $\pi_1(M/f(K)) \cong G/ < P >$. Let $\pi : M \mapsto M/f(K)$ be the decomposition map. Now, $M/f(K)$ is a generalized manifold which fails to be a manifold at a single point. $M/f(K)$ has a resolution $d : N \mapsto M/f(K)$ where N is an n-manifold and d has a single non-trivial, cell-like point-preimage. For an appropriate approximate inverse h to d, the composition $h \circ \pi : M \mapsto N$ is of degree one.

We use this construction as a model for building special degree one maps of manifolds where, in fact, G may contain no non-trivial finitely generated perfect subsgroups. In particular, $ker(i_\#)$ is not the normal closure of a finitely generated perfect group. However, we do use maps which in some sense are *almost acyclic.*

Definition 1.0: A finite 2-complex K is *almost acyclic* if $H_2(K;\mathbb{Z}) = 0$ and $H_1(K;\mathbb{Z})$ is free.

Definition 1.1: A finitely presented group, H, is *almost acyclic* if there is a finite, almost acyclic 2-complex, K, with $\pi_1(K) \cong H$.

Definition 1.2: Let G be a group. Denote by $Wild(G)$ (the wild group of G) the unique maximal perfect subgroup of G.

Observe that $Wild(G)$ is also equal to the transfinite intersection of the derived series of G.

Suppose only that K is an almost acyclic 2-complex, $f : K \mapsto M$ is a locally flat embedding, $f_\# : \pi_1(K) \mapsto \pi_1(M)$ is the induced homomorphism on fundamental groups, and that

$$(1.1) \qquad 1 \neq f_\#(\pi_1(K)) \prec ncl(f_\#(Wild(\pi_1(K))), \pi_1(M))$$

Since $f(K)$ is a polyhedron, $M/f(K)$ is an ANR of finite type. Denote the decomposition map by $\pi : M \mapsto M/f(K)$. Now $M/f(K)$ fails to be a manifold only at the point $x = |f(K)|$. However, it also fails to be a generalized manifold at this point. In particular,

$$(1.2) \qquad H_2(M/f(K), M/f(K) - x; \mathbb{Z}) \cong H_1(K) \neq 0$$

Despite this fact, $M/f(K)$ still has a resolution. To see this first observe that

$$(1.3) \qquad \pi_1(M/f(K)) \cong \pi_1(M)/ncl(f_\#(\pi_1(K)), \pi_1(M))$$

By (1.1) necessarily

$$(1.4) \qquad ncl(f_\#(\pi_1(K)), \pi_1(M)) = ncl(f_\#(Wild(\pi_1(K))), \pi_1(M))$$

Since $\pi_1(K)$ is finitely generated, there is a finite collection of elements $y_i \in Wild(\pi_1(K))$ with

$$(1.5) \qquad ncl\,(f_\#(\pi_1(K)), \pi_1(M)) = ncl\,(\{y_1, ..., y_t\}, \pi_1(M))$$

We now obtain the resolution using the technique developed by Cannon. For each y_i there is an open grope, D_i, and a pointed (with basepoint in $bdy(D_i)$) map, $p_i : D_i \mapsto K$ such that $p_i(bdy(D_i))$ is a loop representing $y_i \in Wild(\pi_1(K))$. Since $f : K \mapsto M$ is locally flat , p_i induces a map $\bar{p}_i : D_i^+ \mapsto M/f(K)$ where D_i^+ is the closed grope, $\bar{p}_i(D_i) \subset M/f(K) - x$, and $\bar{p}_i(D_i^+ - D_i) = x$. Since $n > 5$, we may assume $\bar{p}_i|D_i$ is a locally flat embedding and that the images $\bar{p}_i(D_i)$ are mutually disjoint. For each i, identify a pinched topological regular neighborhood, P_i, of each $\bar{p}_i(D_i) \subset M/f(K) - x$ so that these neighborhoods are mutually disjoint.

Identify loops $\{l_1, l_2, ..., l_s\}$ in K representing the generators of $H_1(K;\mathbf{Z})$. Homotopically move each $f(l_j)$ off K in M to an embedded loop, $\hat{l}_j$, so that the homotopy is an isotopy off K and misses $\pi^{-1}\left(\bigcup_{i=1}^{t} P_i\right)$. By (1.5), $\hat{l}_j$ bounds a disk with holes, B_j, so that $\left(B_j - \hat{l}_j\right) \subset \left(M - K - \pi^{-1}\left(\bigcup_{i=1}^{t} P_i\right)\right)$ and each other boundary component of B_j is isotopic in M to $p_i(bdy(D_i))$ for some i.

Replace the interior of each neighborhood by an open cell ([C,]), we obtain a cell-like map $d : Y^{ANR} \mapsto M/f(K)$ which has a single non-trivial point-preimage, $d^{-1}(x)$, homeomorphic to a wedge of arcs, $\bigvee_{\iota \in \mathbf{C}} A_\iota$, where $\mathbf{C}$ is a Cantor set and A_ι is an arc. Let y be the wedge point. Then the map $d : (Y, y) \mapsto (M/f(K), x)$ induces isomorphisms on local homology. In particular,

$$H_2(Y, Y - y; \mathbf{Z}) \cong H_2(M/f(K), M/f(K) - x; \mathbf{Z}) \cong H_1(K; \mathbf{Z}) \tag{1.6}$$

and each is finitely generated. We discuss this fact also in Appendix I.

Now,

$$B'_j = \left(d^{-1} \circ \pi\left(B_j - \hat{l}_j\right) \bigcup \{y\}\right) \subset \left(\left(Y - d^{-1}(x)\right) \bigcup \{y\}\right) \tag{1.7}$$

is a disk with holes with one fewer boundary components than B_j (in effect l_j is mapped to y). The previous discussion allows that each boundary component of B'_j is homotopically trivial in in the manifold $(Y - \{y\})$. General position in $(Y - \{y\})$ yields a wedge of 2-spheres, $S = \bigwedge_{j=1}^{s} S_j^2$ with wedge point y where each S_j^2 arises naturally as B'_j with disks attached to boundary components. By construction, the inclusion of S into Y induces isomorphisms on homology. In particular,

$$H_2(S; \mathbf{Z}) \xrightarrow{\cong} H_2(S, S - y; \mathbf{Z}) \xrightarrow{\cong} H_2(Y, Y - y; \mathbf{Z}) \cong \oplus_{j=1}^{s} \mathbf{Z} \tag{1.8}$$

Let $N = Y/S$, $p : Y \mapsto N$ be the decomposition map, and $z = p(S)$.

Claim: N is a generalized n-manifold.

Proof: A detailed proof would be quite tedious. However, the idea is as follows. Since the 2-complex K is almost acyclic, then the complex, $\bar{K}$, obtained by abstractly attaching disks to loops representing the generators of $H_1(K;\mathbf{Z})$, is acyclic. So if $\bar{K}$ were embedded in an n-manifold, M, then $M/\bar{K}$ would be a generalized manifold. Our construction is homologically equivalent to this. We first collapse K obtaining M/K. Next comes the inverse of a cell-like map which does not affect local homology. Finally, we identify the disks attached to representatives of the generators of $H_1(K;\mathbf{Z})$ (which at this stage are topologically a wedge of 2-spheres) and, in effect, complete the collapse of $\bar{K}$ in two steps.

We now investigate whether N satisfies the DDP (disjoint disks property).

Proof: Since N is a manifold except possibly at z, it is sufficient to show that given any neighborhood U of z in N there exists a neighborhood V of z in N so that any map $g : bdy(B) \mapsto V - z$ extends to a map $\bar{g} : B \mapsto U - z$ where B is a 2-disk.

Case 1: $[\pi_1(K), \pi_1(K)] = Wild(K)$

Claim: N satisfies the DDP .

Let $z \in U^{nbhd} \subset N$. Since N is an ANR, there is a neighborhood V of z in U and a strong deformation retraction of V to z in U. Let $g : bdy(B) \mapsto V$ be a mapping of the boundary of a disk. Since S is simply connected, there is a map $g_1 : B \mapsto p^{-1}(U) \subset Y$ so that and $p \circ g_1 | bdy(B) = g | bdy(B)$. But $(S - y)$ is locally flat in $(Y - y)$ so we may adjust g_1 slightly so that $g_1(B) \bigcap (S - y) = \emptyset$.

Since $Wild(\pi_1(K) = [\pi_1(K), \pi_1(K)]$, we see that

$$\{Wild(\pi_1(K)), |l_1|, ..., |l_s|\} \tag{1.9}$$

generates $\pi_1(K)$. Since $\pi_1(K)$ is finitely generated, we may assume that the elements

$$\{y_1, ..., y_t, |l_1|, ..., |l_s|\} \tag{1.10}$$

generate $\pi_1(K)$. The end of $(p^{-1}(U) - y)$ at y is homeomorphic to $Q \times (0, \infty)$ where Q is a closed (n-1)-manifold and $\pi_1(Q) \cong H_1(K; \mathbb{Z})$. The previous discussion and surface topology allow us to identify a disk with holes, $D \subset B$, so that $bdy(B) \subset bdy(D)$, and for each component, ρD_i, of D not equal to $bdy(B)$, $g_1(\rho D_i) \subset Q \times (R, \infty) \subset p^{-1}(U)$. Thus, each ρD_i itself bounds in $Q \times (R, \infty)$ a disk with holes with each boundary component freely homotopic to $|l_j|$. Finally, we cap off each such component in $(p^{-1}(U) - y)$ with a disk close to $(S_j^2 - y)$ obtaining a map $g_2 : (B \mapsto p^{-1}(U) - y)$ with $p \circ g_2 | bdy(B) = g | bdy(B)$. We take as our map, $\bar{g} = p \circ g_2$.

The other cases require some additional technical details. /end

THE HOMOTOPY THEORY BEHIND THE EXAMPLES OF INFINITE DIMENSIONAL COMPACTA THAT HAVE COHOMOLOGICAL DIMENSION TWO

talk by John Walsh on joint work with Jerzy Dydak

The original examples of infinite dimensional compacta having finite cohomological dimension produced by A. Dranishnikov ("On a problem of P. S. Alexandroff", Sbornik 135, 1989) had their infinite dimensionality detected by complex K-theory with finite coefficients. The central feature used by Dranishnikov is that the complex K-theory of the Eilenberg-MacLane spaces $K(\mathbf{Z}, n)$ had been computed by Anderson and Hodgkin ("The K-theory if Eilenberg-MacLane complexes", Topology 7, 1968) and, independently, by Buchstaber and Mischenko ("K-theory on the category of infinite cell complexes", Mathematics of the USSR Izvestija 32, 1968). These computations include $k^*(K(\mathbf{Z}, n), \mathbf{Z}/\mathbf{p}) \simeq 0$ for $n \geq 3$ and $\mathbf{p}$ a prime. The examples of Dranishnikov include compact metric spaces X with $\dim X = \infty$ and $\dim_{\mathbf{Z}} X = 3$. Since $K(\mathbf{Z}, 2)$ is represented by infinite complex projective space CP^∞ whose K-theory with finite coefficients does not vanish, an alternate homotopy theoretic component is needed to produce examples of compact metric spaces X with $\dim X = \infty$ and $\dim_{\mathbf{Z}} X = 2$. Of course, the representation $K(\mathbf{Z}, 1) \simeq S^1$ is reflected in the equivalence $\dim X \leq 1 \iff \dim_{\mathbf{Z}} X \leq 1$.

The nature of K-theory itself has has no direct impact as any non-trivial generalized cohomology theory which vanishes on $K(\mathbf{Z}, n)$ and whose values on a finite complex are a finite group in each dimension can be used to produce a compactum X with $\dim X = \infty$ and $\dim_{\mathbf{Z}} X = n$. Apparently, the "standard" generalized cohomology theories fail to vanish on CP^∞.

The approach used by Dydak and Walsh evolved from the realization that a full blown generalized theory was not essential. A "truncated" theory can be associated to any CW-complex L by setting $h^k_L(X) = [X, \Omega^{-k} L]$ for any $k = 0, -1, -2, \cdots$. (*All spaces are to have base points and all maps and homotopies are to preserve the base points.*) The notation ΩL refers to the space of loops on L starting and ending at the base point of L with the constant loop serving as the base point. The iterated loops spaces are defined by $\Omega^k L = \Omega(\Omega^{k-1} L)$. For $k \leq -1$, $h^k_L(X)$ is a group and, for $k \leq -2$, an abelian group. These "truncated" theories satisfy a Mayer-Vietoris Theorem and, in turn, a Combinatorial Vietoris-Begle Theorem.

Guided by work of A. Zabrodsky, the Sullivan Conjecture (which was verified by H. Miller) can be used to establish that theories base on the choice $L = S^q$ for $q \geq 2$ are sufficient. Specifically,

$$h^k_{S^2}(K(\mathbf{Z}, 2)) = 0 \text{ for } k \leq -3.$$

This provides the final ingredient needed to produce examples of compact metric spaces X with $\dim X = \infty$ and $\dim_{\mathbf{Z}} X = 2$. Details appear in Dydak-Walsh, "Infinite dimensional compacta having cohomological dimension two: an application of the Sullivan conjecture" (preprint) and, in an expanded form, in "Dimension theory, cohomological dimension theory, and the Sullivan conjecture" (preprint).

Spaces with free fundamental group at infinity

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I am indebted to Mladen Bestvina for suggesting the following definition.

DEFINITION. . The *fundamental group at infinity* of a space X is *free* if for every compact set A in X there is a larger compact set B so than the inclusion induced homomorphism on the level of fundamental groups with any base point from X - B into X - A factors through a free group.

The following lemma will be useful in recognizing when a space has free fundamental group at infinity. It is a generalization of a theorem by R. Brown [1] and is useful in recognizing when a simply connected, locally compact, metric absolute neighborhood retract can be a covering space of a space whose fundamental group has an element of infinite order [3, Theorem 9.1].

LEMMA. *Let U and V be open sets of a metric space W so that loops which lie in U or V are inessential in W. Furthermore assume that U and V are locally path connected. Then the inclusion induced homomorphism on fundamental groups from $U \cup V$ into W factors through a free group.*

Proof. Choose open sets R, S, T so that:

a. $R \subset U$,

b. $S \subset U \cap V$,

c. $T \subset V$,

d. $R \cap T = \varnothing$

e. The components of R, S, and T form an irreducible cover of $U \cup V$.

Let $\mathcal{G}$ be the cover of $U \cup V$ by the components of R, S, and T. The nerve $N(\mathcal{G})$ of $\mathcal{G}$ is a one-dimensional polyhedron whose vertices are the elements of $\mathcal{G}$. For each $g \in \mathcal{G}$, let ϕ_g: $U \cup V \to [0,1]$ be a continuous function so that $\phi_g(x) > 0$ if and only if $x \in g$ and $\sum \phi_g(x) = 1$ for each $x \in U \cup V$. We set $\Phi : U \cup V \to N(\mathcal{G})$ to be the barycentric map induced by the given collection of functions [2, p.70].

For each $g \in \mathcal{G}$, let P_g be a point in g so that $\phi_g(P_g) = 1$. We define a map $f\colon N(\mathcal{G}) \to W$ by sending a vertex g to the point P_g and a one-simplex between vertices g and h to a path between P_g and P_h which lies in $g \cup h$.

We will now show that the composition $f \circ \Phi\colon U \cup V \to W$ induces the same homomorphism on fundamental group as the inclusion from $U \cup V$ into W. Let γ be a loop in $U \cup V$. Without loss of generality, γ is a product of paths γ_i so that a typical γ_i runs between points P_g and P_h and lies in $g \cup h$. But $\Phi \circ \gamma_i$ is a path that runs from g to h and lies in the open star of the one-simplex with vertices g and h; therefore, $\Phi \circ \gamma_i$ is equivalent to a path that lies in the one-simplex with vertices g and h. Hence, $f \circ \Phi \circ \gamma_i$ is equivalent in $U \cup V$ to a path that lies entirely in U or entirely in V. But γ_i also lies in the same set (either U or V). Hence γ_i and $f \circ \Phi \circ \gamma_i$ are equivalent paths in W, and we see that the composition $f \circ \Phi\colon U \cup V \to W$ induces the same homomorphism on fundamental group as the inclusion from $U \cup V$ into W.

Since the fundamental group of a one-dimensional polyhedron is free, and the inclusion induces the homomorphism $f_* \circ \Phi_*$, our theorem is proved.

References

[1] R. Brown, *On a method of P. Olum*, J. London Math. Soc. **40** (1965), 303-304.

[2] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press (1948).

[3] D. G. Wright, *Contractible open manifolds which are not covering spaces,* Topology, (to appear).

PROBLEM SESSION

1. (R. Daverman) If the Poincare Conjecture is true, are all generalized 3-manifolds resolvable? Is it also necessary to hypothesize that Quinn's obstruction is trivial?

2. (R. Daverman) Is any cellular resolution $p:M \to X$ for M a 3-manifold a near homeomorphism if for any two disjoint discs B and B′ in M, p(B) and p(B′) can be approximated by disjoint singular disks? What if $p \mid B$ can be approximated by an embedding?

3. (R. Daverman) Is a resolvable generalized 3-manifold a real 3-manifold if it has the Simplicial Approximation Property (which is the SSAP for 2-cells instead of 2-spheres)?

4. (R. Daverman) What general position properties characterize 4-manifolds among resolvable 4-manifolds?

5. (F. Tinsley) Does the blowing up and blowing down procedure described in my talk work in high dimensions?

6. (F. Ancel) Does there exist a Weakly Infinite Dimensional Space with finite cohomological dimension? Does there exist a weakly infinite dimensional space without Property C? It is known, by R. Pol's example, that there exist strongly infinite dimensional spaces without Property C.

7. (F. Ancel) Can you push the spine of the Mazur manifold off itself by a homeomorphism? Are there disjoint spines for the Mazur manifold? If there exist disjoint spines, is there a free $\mathbb{Z}$ action?

8. (D. Wright) If M is a compact, aspherical irreducible 3-manifold, is the universal covering space of M homeomorphic to Euclidean 3-space?

9. (D. Wright) Give an example of a contractible 3-manifold which covers a non-compact manifold, but does not cover a compact 3-manifold.

10. (D. Wright) Give conditions on contractible open Q-manifolds which imply that they cannot be non-trivial covering spaces.

11. (S. Bleiler) All known examples of finite surgery occur on symmetric knots. Is this true in general?

12. (S. Bleiler) Are finite surgeries on hyperbolic knots integral?

13. (S. Bleiler) What is the maximal geometric intersection number of curves using finite surgeries? The conjecture for hyperbolic knots is 2. The best known example is 22.

14. (S. Bleiler) Lens Space Conjecture. If L(p,q) can be obtained by surgery on a nontrivial knot, then $p \geq 5$. The case $p = 0$ is Property r and is due to Gabai. The case $p = 1$ is that knots are determined by their complement and is due to Gordon and Luecke. The case p=2 is known for symmetric knots and is due to Bleiler and Litherland. The case $p = 4$ is known for strongly invertible knots and is due to Thompson and Scharleman.

15. (D. Garity) Is their a codimension 0 thin decomposition of $\mathbb{R}^n$ or of the Hilbert Cube?

Note: There is an additional list of 40 questions listed in the writeup of R. Daverman's talk on decompositions into submanifolds. There are also questions listed in the writeups of a number of the other talks in these proceedings.