PROCEEDINGS

SIXTH ANNUAL
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Geometric Topology
July 27-29 1989

Department of Mathematics
Brigham Young University
Provo, Utah

The Sixth Annual Western Workshop in Geometric Topology was held at Brigham Young University, Provo, Utah on July 27-29, 1989. The participants were:

Ric Ancel	University of Wisconsin, Milwaukee
Mladen Bestvina	University of California, Los Angeles
Phil Bowers	Florida State University
James W. Cannon	Brigham Young University
Yuanan Diao	Florida State University
Claus Ernst	University of Western Kentucky
Lawrence Fearnley	Brigham Young University
Blake Fordham	Brigham Young University
Dennis Garity	Oregon State University
Craig Guilbault	University of Wisconsin, Milwaukee
John Hempel	Rice University
Jim Henderson	Colorado College
Steve Humphries	Brigham Young University
Jack Lamoreaux	Brigham Young University
John Luecke	University of Texas, Austin
David Snyder	Southwest Texas State University
Michael Starbird	University of Texas, Austin
Eric Swenson	Brigham Young University
Fred Tinsley	Colorado College
Gerard Venema	Calvin College
John Walsh	University of California, Riverside
David Wright	Brigham Young University

The principal speaker at the workshop was John Luecke who spoke on recent joint work with Cameron Gordon on the knot complement problem. These proceedings contain notes by Luecke on his talks. Also included are summaries of talks given by other participants A problem list compiled by R. J. Daverman about finite dimensional manifolds is included here in these informal proceedings. They will appear formally elsewhere. Finally, there is a short list of additional problems in topology.

We express gratitude to funding provided by the National Science Foundation (DMS-8802424) and to Brigham Young University.

David Wright

Previous Workshops

1984 Brigham Young University

1985 Colorado College

1986 Colorado College

1987 Oregon State University

1988 Colorado College

1989 Brigham Young University

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Notes on the Knot Complement Problem

John Luecke

§1. An outline

Let $K \subset S^3$ be a knot, that is, K is a circle smoothly embedded in S^3. Two knots $K, K' \subset S^3$ are called equivalent if there is an orientation preserving homeomorphism of S^3 to itself taking K to K'. The exterior of K, denoted X_K, is the complement in S^3 of an open tubular neighborhood of K. Clearly, equivalent knots have homeomorphic exteriors. In these notes we will be concerned with proving the converse.

Knot Complement Problem. *Two knots are equivalent iff their exteriors are homeomorphic.*

The question of whether or not a knot is determined by its complement was asked as early as 1908 by Tietze and is a natural one in light of the fact that the classical knot invariants were topological invariants of the knot exterior.

Let π be the isotopy class of an essential, simple, closed curve on ∂X_K. $K(\pi)$ is the closed 3-manifold obtained by attaching a solid torus, J_π, to X_K via a homeomorphism of ∂J_π to ∂X_K that sends the boundary of a meridional disk of J_π to π. For example, if π is the meridian of K then $K(\pi)$ is S^3. These notes will be devoted to the proof of the following theorem:

Theorem 1. *If $K(\pi)$ is S^3, then π is the meridian of K.*

Theorem 1 implies the Knot Complement Problem. Let $K, K' \subset S^3$ be two different knots with a common exterior, X. One obtains $K \subset S^3$ from X by attaching a solid torus to X in a particular way (the core of the solid torus becomes K). One obtains $K' \subset S^3$ by attaching a solid torus to X in a different way. But

Theorem 1 says there is only one way to attach a solid torus to X to get S^3. So K and K' must be equivalent.

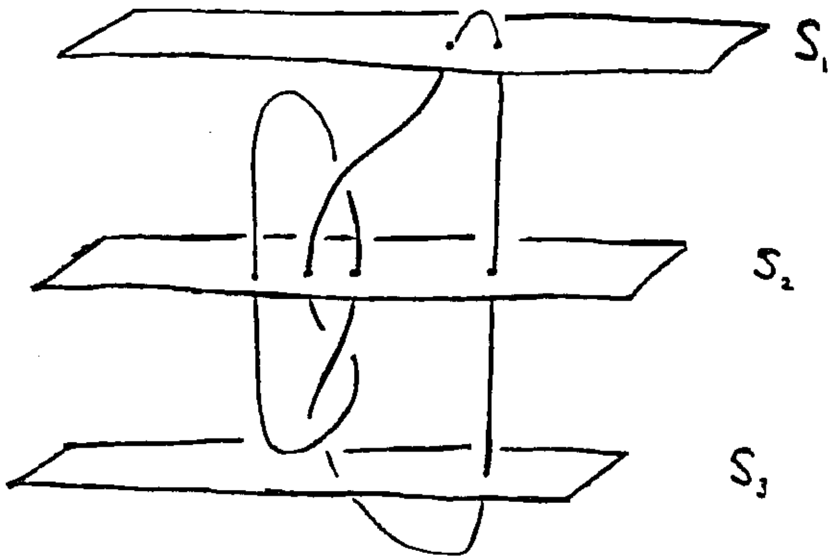
Theorem 1 along with the work of Waldhausen on Haken 3-manifolds gives the following [W, p.26].

Corollary. *Prime knots K and K' are equivalent iff their exteriors have isomorphic fundamental groups.*

Note: The square knot and granny knot are different composite knots whose exteriors have isomorphic fundamental groups.

Theorem 1 is proven in [GL]. These notes are meant to outline this result, to give examples of some of the techniques used, and to record simplifications to the arguments made by Hatcher and Parry. For more details or more precise definitions refer to [GL]. The rest of this section will be devoted to outlining the proof of Theorem 1.

The 3-sphere minus its north and south poles is the product of the 2-sphere with an open interval. This gives a height function $h : S^3 \to \mathbb{R}$ whose level sets (off the north and south pole) are 2-spheres. Let $K \subset S^3$ be a knot. A *Morse presentation* of K is an isotopy of K so that $h|K$ is a Morse function (*i.e.* K is transverse to the level 2-spheres everywhere except at relative maxima and minima which occur at distinct levels). Given a Morse presentation of K, let $S_1, \ldots, S_n$ be level 2-spheres between the consecutive pairs of critical levels of K. Define the complexity of this Morse presentation of K as $\sum_{i=1,n} |S_i \cap K|$ (see figure 1.1). A *thin presentation of* K is a Morse presentation of minimal complexity.



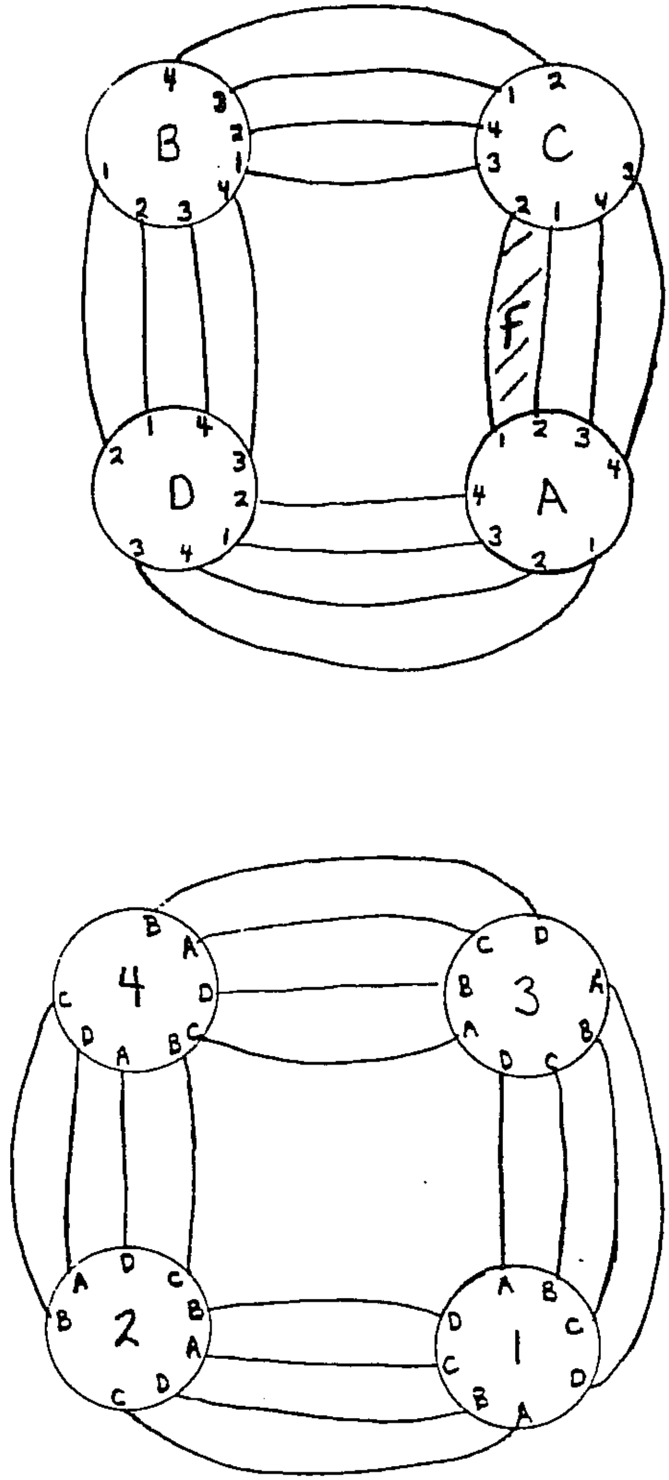
complexity $= 8$

Figure 1.1

Let X be the exterior of K and γ be the meridian of K in X. Assume for contradiction that there is a slope $\pi \neq \gamma$ such that $K(\pi) = S^3 = K(\gamma)$. The cores of the attached solid tori J_γ, J_π become knots K, K_π (resp.) in $K(\gamma), K(\pi)$ (resp.).

Put K in a thin presentation in $K(\gamma)$ under a height function $h : K(\gamma) \to \mathbb{R}$ as described above. A level 2-sphere, $\widehat{Q}$, of h that intersects K transversely gives a punctured 2-sphere, $Q = \widehat{Q} \cap X$, properly embedded in X. Q is called a punctured level sphere for the thin presentation of K. Note that ∂Q is a collection of disjoint, simple, closed curves on ∂X each in the isotopy class γ.

Similarly, put K_π in a thin presentation under a height function $h_\pi : K(\pi) \to \mathbb{R}$. A level 2-sphere, $\widehat{P}$, that intersects K_π transversely gives rise to a punctured sphere, $P = \widehat{P} \cap X$, whose boundary is a collection of disjoint, simple, closed curves in ∂X that lie in the isotopy class π. P is called a punctured level sphere for the thin presentation of K_π.



G_P

G_Q

Figure 1.2

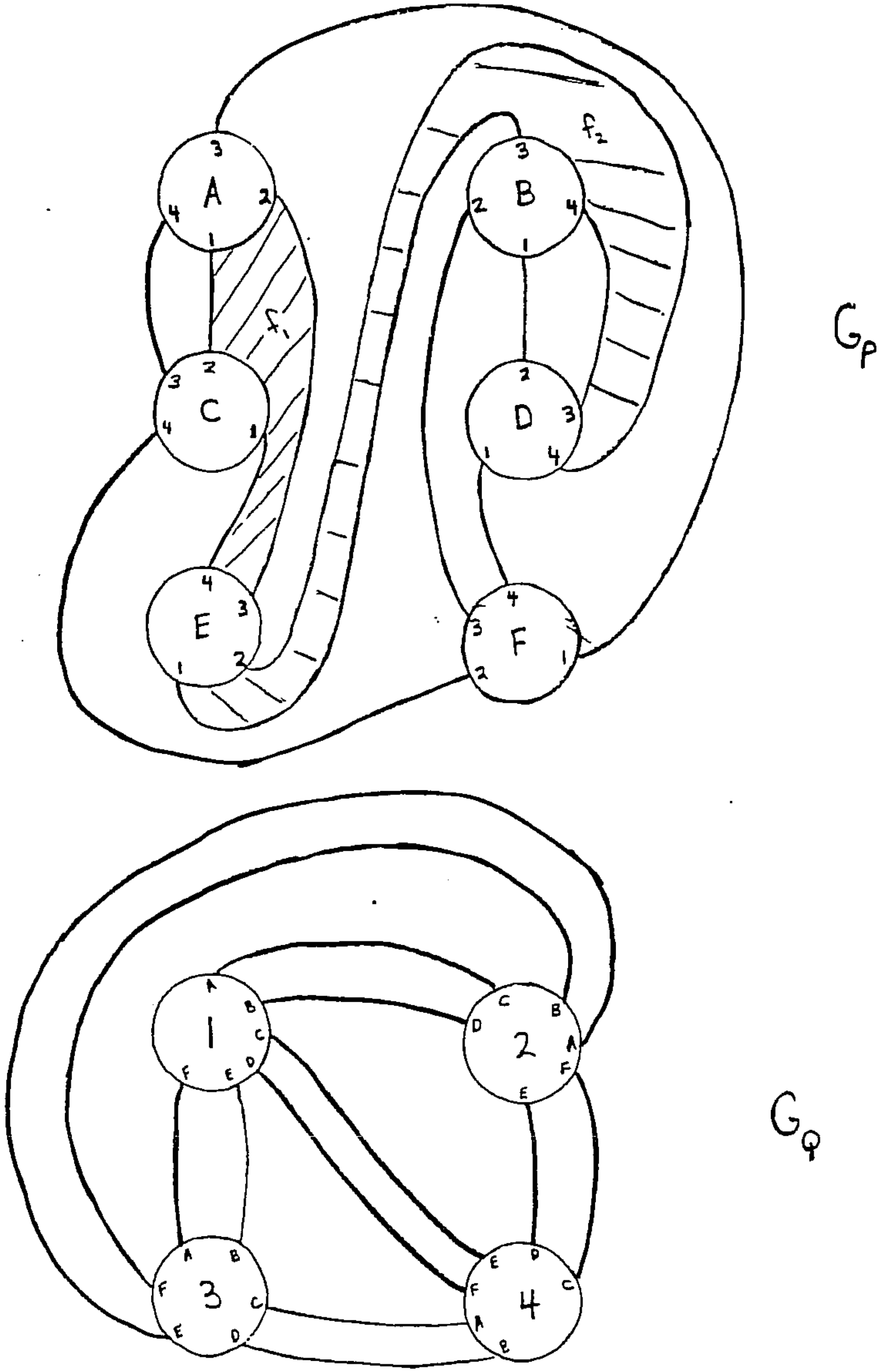


Figure 1.3

We will find a punctured level sphere, P, from the thin presentation of K_π and a punctured level sphere Q from the thin presentation of K that *intersect essentially*. We will define "intersect essentially" shortly but one property is that P and Q intersect transversely. Thus we get a graph G_P in the level 2-sphere $\widehat{P}$ defined by

$$\text{(fat) vertices of } G_P = \text{ components of } \partial P$$

$$\text{edges of } G_P = \text{arc components of } P \cap Q \text{ in } P .$$

Similarly we get the graph G_Q in the level 2-sphere $\widehat{Q}$. Note that the edges of G_P and G_Q are in 1–1 correspondence. Two examples of punctured spheres P, Q and the associated graphs G_P, G_Q are given in figures 1.2 and 1.3.

We number the components of ∂P and ∂Q in the order of their appearance on ∂X. This allows us to label the endpoints of edges in G_P (G_Q) by components of ∂Q (∂P, resp.). To say that P and Q intersect essentially also means that ∂P and ∂Q intersect minimally. Thus around each vertex of G_P (G_Q) we see the vertices of G_Q (G_P, resp.) appearing consecutively as labels, each vertex of G_Q (G_P, resp.) appearing as a label exactly as many times as the algebraic intersection number between γ and π on ∂X (in figure 1.2, $i_{\partial X}(\gamma, \pi) = 2$; in figure 1.3 $i_{\partial X}(\gamma, \pi) = 1$).

We say that a vertex, v, of G_P (G_Q) has a positive sign if the labels around v appear in an anti-clockwise order and has a negative sign if the order is clockwise. Two vertices of G_P (G_Q) are called *parallel* iff they have the same sign. They are called *anti-parallel* otherwise. The orientability of X gives us then the

Parity rule. An edge connects parallel vertices in G_P (G_Q) iff it connects anti-parallel vertices in G_Q (G_P, resp.).

A 1-sided face in G_P (G_Q) is a face in G_P (G_Q) with exactly one edge in its boundary (*i.e.* this edge is an arc of $P \cap Q$ which is parallel into ∂P (∂Q, resp.)).

Definition. P and Q *intersect essentially* iff

1) P and Q intersect transversely and each component of ∂P intersects each component of ∂Q minimally on ∂X.

2) Neither G_P nor G_Q contains a 1-sided face.

Proposition 2. *There are punctured level spheres P and Q where P comes from the thin presentation of K_π and Q comes from the thin presentation of K such that P and Q intersect essentially.*

We will sketch a proof of this in section two.

Let's see that the examples in figures 1.2 and 1.3 cannot arise from level spheres.

Figure 1.2. Here we see a face, f, on G_P called a Scharlemann cycle. A Scharlemann cycle is a face, f, of the graph whose boundary can be oriented so that the tail of each edge (in ∂f) has the same label, p, and the head of each edge has the same label q. In figure 1.2, $p = 1$ and $q = 2$. Now Q separates X and f lies on one side of Q. Let $\widehat{Q} \subset K(\gamma) = S^3$ be the level sphere on which Q lies and let B be the ball bounded by $\widehat{Q}$ that does not contain f. Let A be the annulus in ∂X that runs between components 1 and 2 of Q. Recall that J_γ is the solid torus attached to X to give $K(\gamma)$. Let H be the 3-ball component of $J_\gamma - B$ containing A. Then a regular neighborhood of the union of B (0-handle), H (1-handle), and f (2-handle) is a punctured $\mathbb{R}P^3$. See figure 1.4. But $K(\gamma) = S^3$. $\square$

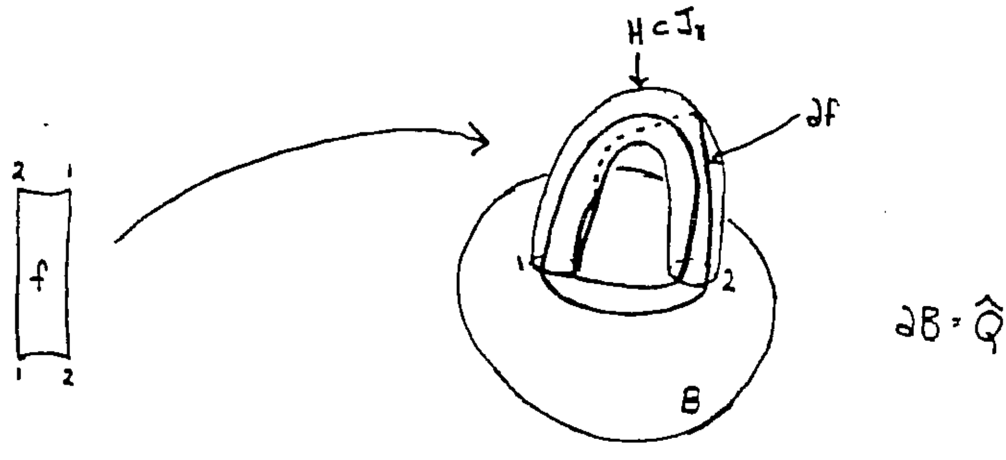


Figure 1.4

The same argument more generally shows that if G_P (G_Q) contains a Scharlemann cycle then $K(\gamma)$ ($K(\pi)$, resp.) contains a punctured lens space, giving a contradiction.

Figure 1.3. Let f_1 and f_2 be the faces of G_P pictured in figure 1.3. Let $\widehat{Q}$ be the level 2-sphere in $K(\gamma)$ on which Q lies. Then f_1 and f_2 are on the same side of $\widehat{Q}$. Let B be the 3-ball bounded by $\widehat{Q}$ that does not contain f_1 and f_2. Let H_{12}, H_{34} be the 3-ball component of $J_\gamma - \text{int}(B)$ that runs between components 1 and 2, 3 and 4 (resp.) of Q. Let N be the submanifold of $K(\gamma)$ that is a regular neighborhood of the union of f_1, f_2 (2-handles) and H_{12}, H_{34} (1-handles) and B (0-handle). See figure 1.5. Then

$$H_1(N) = \frac{\mathbf{Z} \oplus \mathbf{Z}}{\langle (2,1),(1,-2) \rangle}$$

and since

$$\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5$$

we see that $H_1(N)$ has 5-torsion. Since $K(\gamma) = S^3$ cannot contain a codimension 0 submanifold with torsion in first homology, this shows that P and Q cannot arise from level 2-spheres.

□

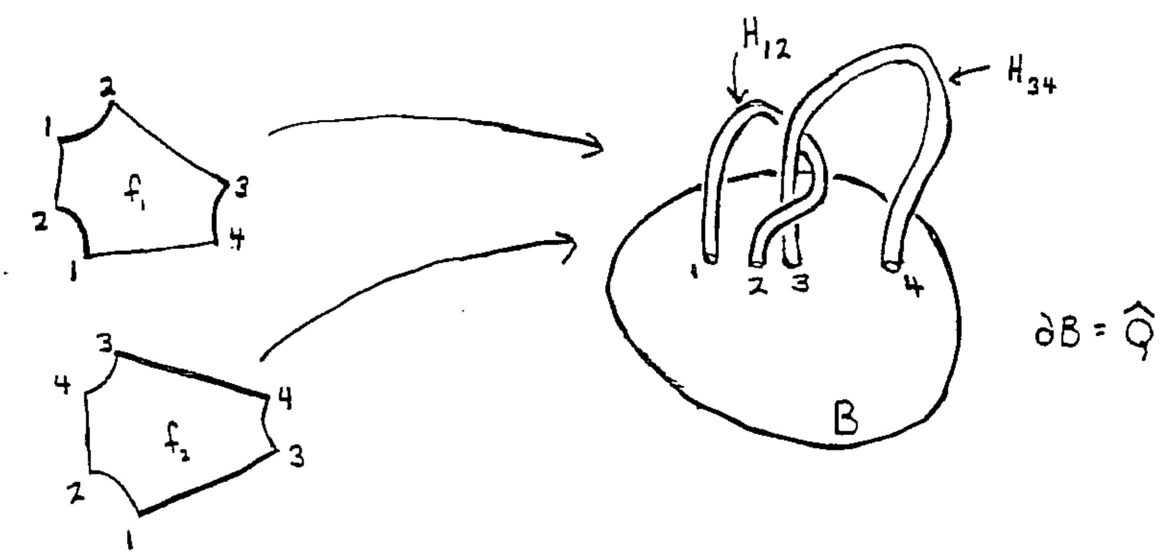


Figure 1.5

In general the plan to show that P and Q cannot exist is to find a collection of faces on G_P or G_Q that will give rise to a submanifold in $K(\gamma)$ or $K(\pi)$ with non-trivial torsion in first homology. Let p be the number of components of ∂P and q be the number of components of ∂Q. Let f be a face of G_P. ∂f is the union of arcs $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ where $a_i \subset P \cap Q$ and $b_i \subset P \cap \partial X$ (figure 1.6). Each b_i runs between some pair of components $j, j+1$ of ∂Q. Orienting ∂f, we say that b_i represents $(j, j+1)$ or $-(j, j+1)$ according to whether b_i runs from j to $j+1$ or vice versa. Given f, we assign an ordered q-tuple $\alpha(f) = (\alpha_1(f), \alpha_2(f), \ldots, \alpha_q(f))$ where $\alpha_j(f)$ is the algebraic number of times ∂f runs over $(j, j+1)$. Note that $\alpha(f)$ is defined only to a multiple of ± 1. Similarly, to each face, f, of G_Q we assign the p-tuple $\alpha(f)$. See figure 1.7.

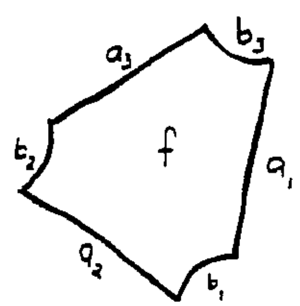
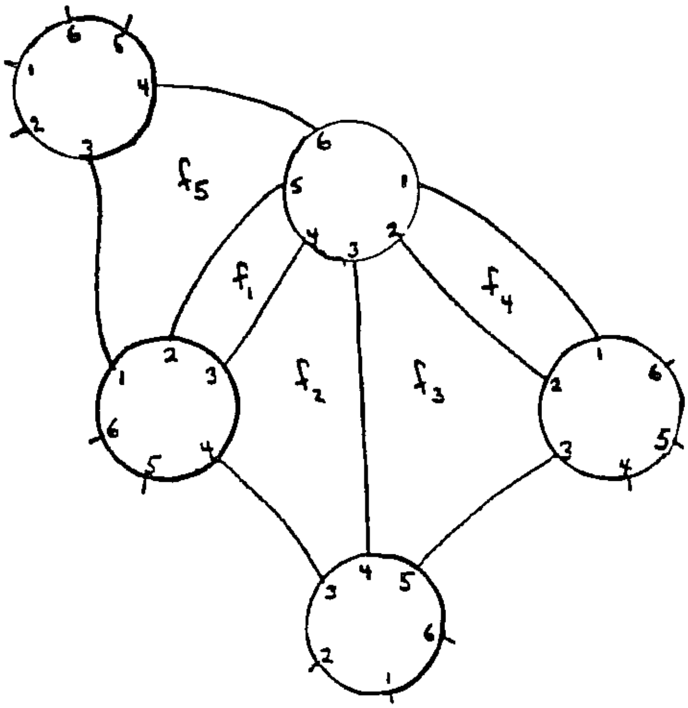


Figure 1.6



$$\begin{array}{rcccccc} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha(f_1) = \pm(& 0, & 1, & 0, & 1, & 0, & 0) \\ \alpha(f_2) = \pm(& 0, & 0, & 3, & 0, & 0, & 0) \\ \alpha(f_3) = \pm(& 0, & 0, & 0, & 1, & 0, & 0) \\ \alpha(f_4) = \pm(& 0, & 0, & 0, & 0, & 0, & 0) \\ \alpha(f_5) = \pm(& 1, & 0, & -1, & 0, & 1, & 0) \end{array}$$

Figure 1.7

We say that G_P *represents all types* if there is a collection, F, of disk faces of G_P such that

1) for each face $f \in F$ and any given orientation of ∂f, all occurrences of $(j, j+1)$ have the same sign (for each j)

2) for each ordered q-type $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q)$ in $\{\pm 1\}^q$ there is a face $f_\varepsilon \in F$ and an $\eta \in \{\pm 1\}$ such that for each $i = 1, q$, $\varepsilon_i = \eta \cdot \mathrm{sign}(\alpha_i(f_\varepsilon))$. [If $\alpha_i(f) = 0$ then we say automatically that $\varepsilon_i = \eta \cdot \mathrm{sign}(\alpha_i(f_\varepsilon))$.]

Similarly we define the concept of G_Q representing all types.

Example 1. If G_P contains a Scharlemann cycle then G_P represents all types.

Example 2. With G_P as in figure 1.3, setting $F = \{f_1, f_2\}$ we see that G_P represents all types:

$$\begin{array}{rl} & \qquad\ \ \underline{12}\quad\ \underline{23}\quad\ \underline{34}\quad\ \underline{41} \\ f_1 \text{ represents types } \pm 1 \cdot & (+\ ,\ +\ ,\ +\ ,\ +) \\ \pm 1 \cdot & (+\ ,\ +\ ,\ +\ ,\ -) \\ \pm 1 \cdot & (+\ ,\ -\ ,\ +\ ,\ +) \\ \pm 1 \cdot & (+\ ,\ -\ ,\ +\ ,\ -) \end{array}$$

$$\begin{array}{rl} & \qquad\ \ \underline{12}\quad\ \underline{23}\quad\ \underline{34}\quad\ \underline{41} \\ f_2 \text{ represents types } \pm 1 \cdot & (+\ ,\ +\ ,\ -\ ,\ +) \\ \pm 1 \cdot & (+\ ,\ +\ ,\ -\ ,\ -) \\ \pm 1 \cdot & (+\ ,\ -\ ,\ -\ ,\ +) \\ \pm 1 \cdot & (+\ ,\ -\ ,\ -\ ,\ -) \end{array}$$

Proposition 3. *Let P and Q be the punctured level spheres given by Proposition 2. Either G_P represents all types or G_Q contains a Scharlemann cycle.*

We will outline the combinatorics involved in the proof of Proposition 3 in section four, after going through its proof in section three in the special case where K is a 2-bridge knot.

Propositions 2 and 3 combine to give a proof of Theorem 1. Suppose that $K(\gamma) = S^3 = K(\pi)$. Let P and Q be the punctured level spheres given by Proposition 2. Apply Proposition 3. Assume that G_P represents all types. In [GL], chapter 3 is devoted to showing that this gives a contradiction to the fact that Q comes from a thin presentation of K in $K(\gamma) = S^3$. However, we can avoid this chapter by appealing to a recent (algebraic) result of Walter Parry [P].

First note that Q separates X into, say, a White side and a Black side and that a face of G_P is consequently either white or black. If F is the collection of faces of G_P representing all types then it is not hard to see that we may assume that F consists entirely of white faces or entirely of black faces. We assume F consists entirely of white faces. Let $\widehat{Q}$ be the level sphere in $K(\gamma)$ on which Q lies. Let B be the 3-ball in $K(\gamma)$ bounded by $\widehat{Q}$ and containing the black side of Q. Recall that J_γ is the solid torus attached to X to give $K(\gamma)$. Let H_j, $j = 1, q$, be the

3-ball component of $J_\gamma - \widehat{Q}$ that runs between components j and $j+1$ of ∂Q. Let $H = \{H_j \mid \alpha_j(f) \neq 0$ for some $f \in F\}$ (note that H is a subset of the components of $J_\gamma - B$, since F consists entirely of white faces). Think of B as a 0-handle, H as a collection of 1-handles, and F as a collection of 2-handles that constitute a submanifold of $K(\gamma)$. The theorem of Parry [P] says that if one has a set of generators, H, of a free abelian group and a set of relations F that represents all types in those generators (in the same way that the faces in F represent all types in the intervals $(j, j+1)$) and if no element of F has length one in the generators H, then one can find a subset, H', of H and a subset, F', of F such that the abelian group with generators H' and relations F' has non-trivial torsion. So let N be a regular neighborhood in $K(\gamma)$ of the union of B with the 1-handles corresponding to H' along with 2-handles corresponding to F' (note that no element of F has length one in H because P contains no 1-sided face). Then N is a codimension 0 submanifold of $K(\gamma)$ with non-trivial torsion in first homology. But this contradicts the fact that $K(\gamma)$ is S^3. Therefore G_P cannot represent all types.

If G_Q, on the other hand, contains a Scharlemann cycle then we can either note that this means that G_Q represents all types and use the previous argument or we can directly construct a lens space summand in $K(\pi)$ as was done in showing that figure 1.2 could not represent the intersection of two level spheres.

In either case, Proposition 3 leads to a contradiction with the fact that $K(\gamma) = S^3 = K(\pi)$.

§2. Finding P and Q

This section will outline a proof of Proposition 2 of section one. Let $K(\gamma) = S^3 = K(\pi)$ and K, K_π be as in section one. Recall that we put K, K_π in thin presentation under the height functions h, h_π (resp.) of $K(\gamma), K(\pi)$ (resp.). Recall

Proposition 2. *There is a punctured level sphere, Q, from the thin presentation of K in $K(\gamma)$ and a punctured level sphere, P, from the thin presentation of K_π in $K(\pi)$ such that*

(1) P and Q intersect transversely and ∂P intersects ∂Q minimally

(2) $P \cap Q$ contains no arc components which are boundary parallel in either P or Q.

The idea behind the proof of Proposition 2 comes from the following beautiful lemma that is taken from Gabai's proof of Property R. In fact, Gabai independently proved Proposition 2 and knew of its application to the knot complement problem.

Lemma 2.1. [Ga, §4A]. *Let Q be a punctured level sphere in the thin presentation of K. There is a punctured level sphere, P, in the thin presentation of K_π such that*

(1) P and Q intersect transversely and ∂P intersects ∂Q minimally

(2) $P \cap Q$ contains no arc component which is boundary parallel on Q.

Proof of lemma. Look at Q in $K(\pi)$. Define a *middle level* of the thin presentation of K_π to be the interval of level 2-spheres between two consecutive critical levels of K_π such that the critical level just above this interval is a relative maximum and the critical level just below the interval is a relative minimum (figure 2.1). Pick one such middle level. Isotop ∂Q on ∂X so that ∂Q intersects minimally the boundary of each level sphere in this middle level. Furthermore, isotop Q so that its projection under h_π in this middle level has only non-degenerate critical points occurring at distinct levels.

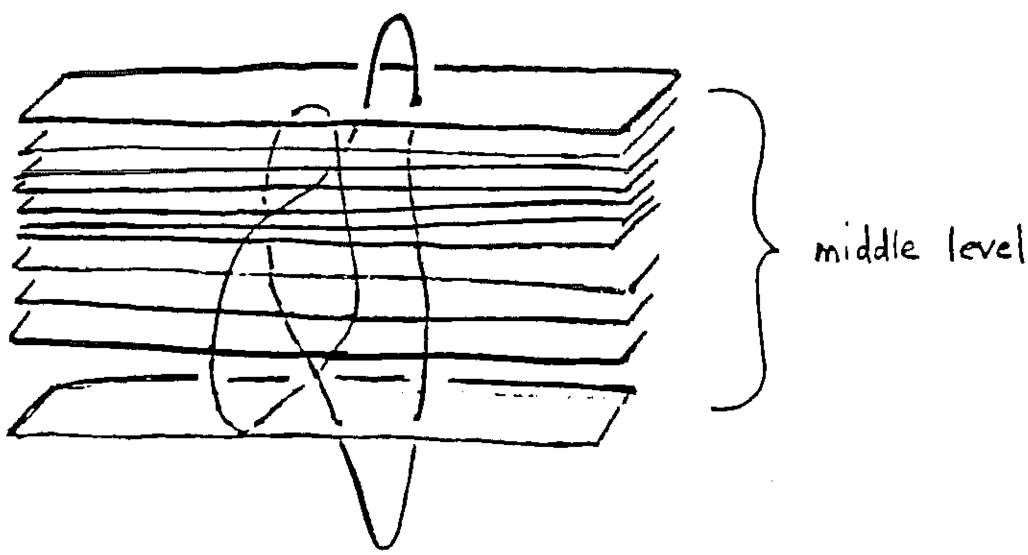


Figure 2.1

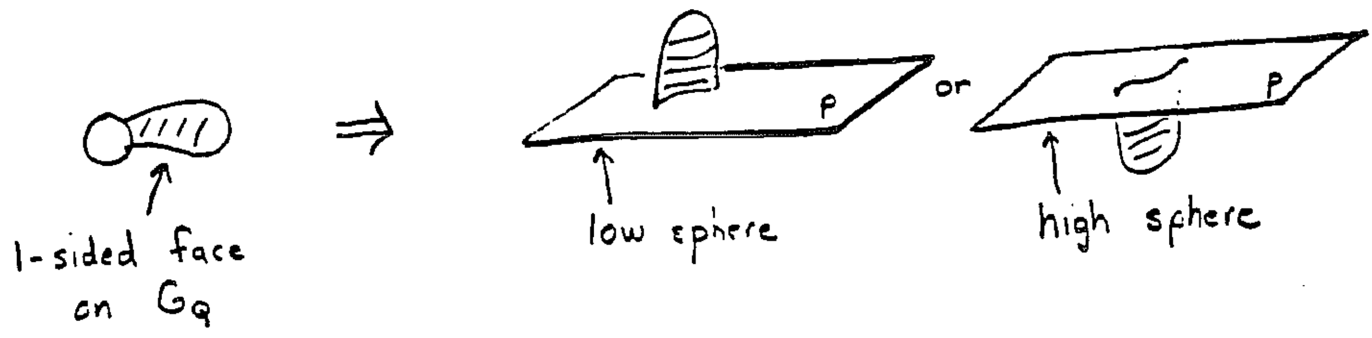


Figure 2.2

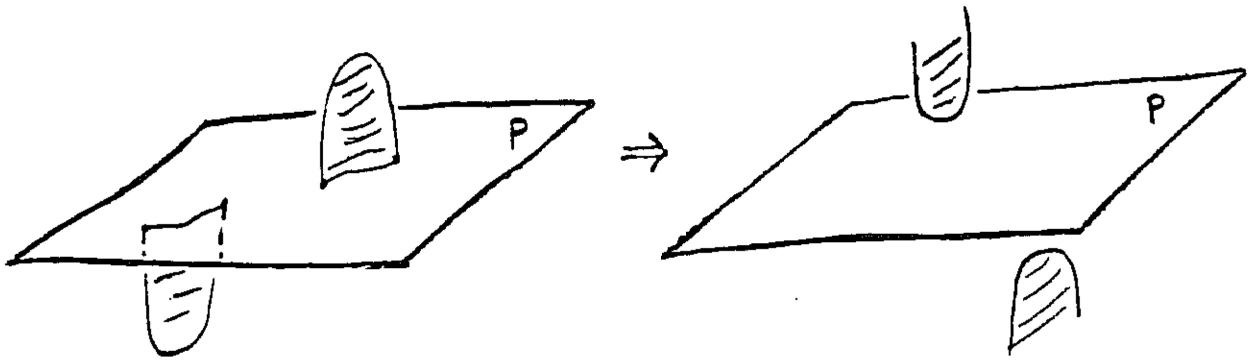


Figure 2.3

Let P be a level sphere in this middle level which is not a critical level of Q. To say that some arc of $P \cap Q$ is boundary parallel in Q gives the picture in $K(\pi)$ of figure 2.2. We call P a *high sphere* or *low sphere* according to figure 2.2. We assume for contradiction that each level sphere in this middle level that is not a critical level of Q is either a high sphere or a low sphere. Note that such a level sphere cannot be both high and low because one could reduce the complexity of K_π. See figure 2.3. Also note that any two level spheres in this middle level of K_π that have no critical levels of Q between them will either both be high spheres or both be low spheres. Thus the critical levels $i_1, i_2, \ldots, i_n$ of Q in this middle level break up the middle level into subintervals of level spheres such that all level spheres in a given subinterval are either high or all are low. See figure 2.4. Furthermore, the subinterval of level spheres above i_1 obviously consists of low spheres and the subinterval below i_n obviously consists of high spheres. Thus there is a critical level, i_k, of Q such that the level spheres just above i_k are low and the level spheres just below i_k are high. But then the picture in $K(\pi)$ at this critical level is either as in figure 2.3 or figure 2.5. In either case we can reduce the complexity of K_π as illustrated in figure 2.3 and figure 2.6 (2.6 corresponds to 2.5). This contradicts the thinness of the presentation of K_π, thereby proving Lemma 2.1. $\square$

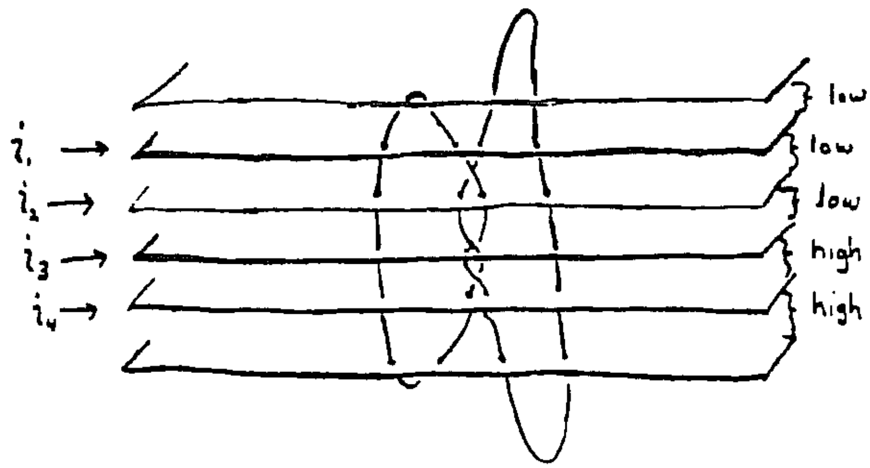


Figure 2.4

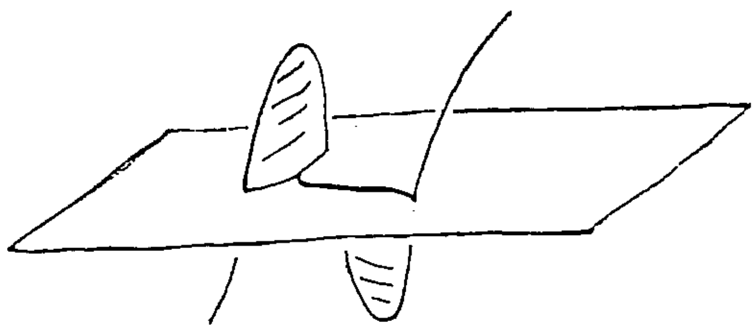


Figure 2.5

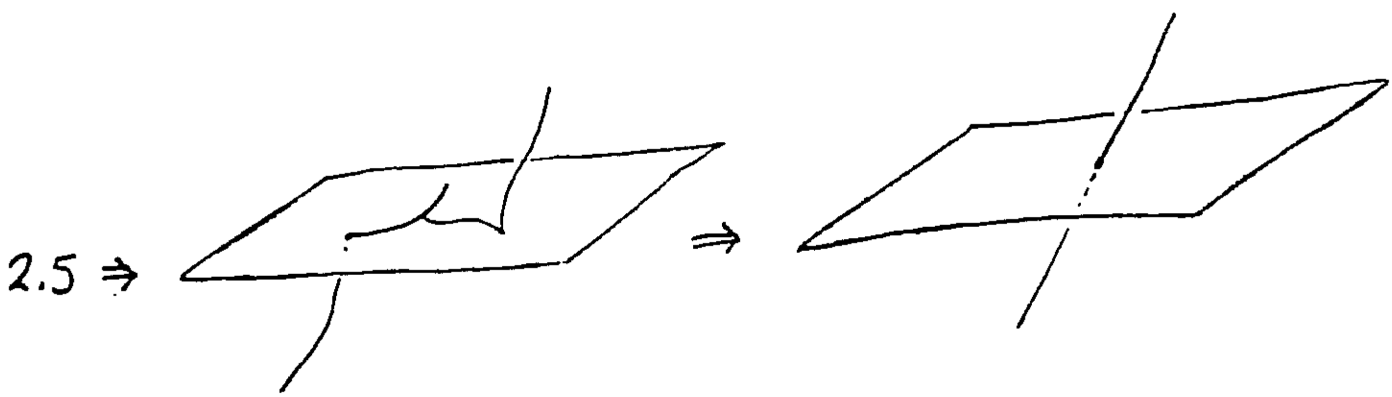


Figure 2.6

Thus, given a level sphere Q of K we can find a level sphere P of K_π that satisfies (1) of Proposition 2 and such that $P \cap Q$ has no arc component which is boundary parallel on Q. Similarly, given a level sphere P of K_π we can find a level sphere Q of K such that (1) of Proposition 2 is satisfied and such that no component of $P \cap Q$ is a boundary parallel arc on P. The additional content of Proposition 2 is that we can find P and Q so that these conditions hold simultaneously.

This is done by taking the argument of Lemma 2.1 and crossing it with $\mathbb{R}$. We pick a one parameter family of level 2-spheres in the thin presentation of K that are between an adjacent local maximum and local minimum of K (*i.e.* we pick a middle level for the thin presentation of K). This becomes a 1-parameter family, $\{Q(\lambda)\}$, of punctured level spheres properly embedded in X. We first isotop the family $\{Q(\lambda)\}$ in X so that $\{\partial Q(\lambda)\}$ intersects the boundaries of the level spheres of h_π minimally. We then perturb the *family* $\{Q(\lambda)\}$ so that it is in general position with respect to h_π. This means that for all but finitely many λ, $h_\pi \mid Q(\lambda)$ is a Morse function and that each $Q(\lambda)$ with $h_\pi \mid Q(\lambda)$ not Morse has non-degenerate singularities (*i.e.* is Morse) except for a single critical value where we see a singularity corresponding to a birth, death, or exchange of tangencies (a "Cerf" singularity).

Assume for contradiction that Proposition 2 is false. The argument of Lemma 2.1 allows one to associate to each $Q(\lambda)$ such that $h_\pi \mid Q(\lambda)$ is Morse a punctured level sphere P_λ of K_π which intersects $Q(\lambda)$ transversely and is such that $Q(\lambda) \cap P_\lambda$ contains an arc component which is boundary-parallel on P_λ (because there are no boundary-parallel arcs of $Q(\lambda) \cap P_\lambda$ on $Q(\lambda)$). If the corresponding arc lies above (below) $Q(\lambda)$ in $K(\gamma)$, then $Q(\lambda)$ is called low (high, resp.) as in Lemma 2.1 (where there it is the P which is either high or low). Again $Q(\lambda)$ cannot be both high or low else we could reduce the complexity of the thin presentation of K. One observes that as λ increases, $Q(\lambda)$ starts off high and ends up low. By the thinness of the presentation of K, a change from high to low in $\{Q(\lambda)\}$ can only occur at a λ_0 such that $h_\pi \mid Q(\lambda_0)$ is not Morse. One analyses what happens at $Q(\lambda_0)$ (*i.e.* at the level of $K(\pi)$ where the Cerf singularity occurs), using the special way in which P_λ is constructed for $Q(\lambda)$, and eventually arrives at a contradiction to the thinness of

K under h.

This completes an outline of the argument for Proposition 2. For more details see Chapter 1 of [GL]. Q.E.D.

For those going through the details of the proof given in Chapter 1 of [GL] I would like to include the following simplification due to Allen Hatcher. The following replaces Lemmas 1.3, 1.4, and 1.5 and is taken almost verbatim from a letter I received from Hatcher. All references are to [GL].

First, starting in the middle of page 378, Hatcher suggests that $p : I^2 - \Gamma \to \mathbb{R}$ be defined so that

$$p(\lambda,\mu) \quad \begin{cases} > 0 & \text{if } P(\lambda) \text{ is high w.r.t. } Q(\mu) \\ < 0 & \text{if } P(\lambda) \text{ is low w.r.t. } Q(\mu) \\ = 0 & \text{otherwise.} \end{cases}$$

Define $q : I^2 - \Gamma \to \mathbb{R}$ similarly. That is, he suggests that > 0, < 0, $= 0$ be used rather than H, L, N. As stated in (P.1) on p.378 [or by Lemma 1.1] p and q are well-defined, p is single-signed on verticals in I^2, and q is single-signed on horizontals in I^2. On the edges of I^2 we have the inequalities pictured in figure 2.7 (this is (P.4), (P.5) on p.378).

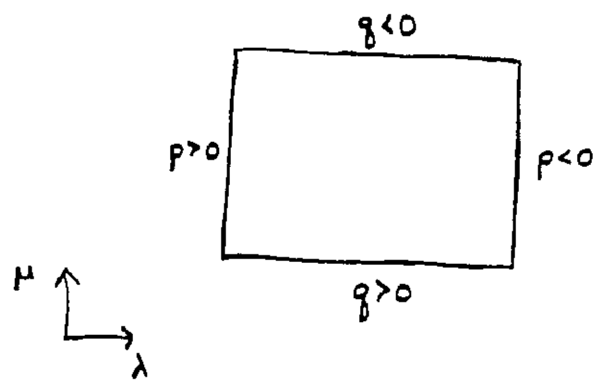


Figure 2.7

To prove Proposition 2 we need to find a component of $I^2 - \Gamma$ with

$p = q = 0$. Choose (s, t) so that

$$p \geq 0 \quad \text{on verticals just to the left of} \quad \lambda = s$$

$$p \leq 0 \quad \text{on verticals just to the right of} \quad \lambda = s$$

$$q \geq 0 \quad \text{on horizontals just below} \quad \mu = t$$

$$q \leq 0 \quad \text{on horizontals just above} \quad \mu = t$$

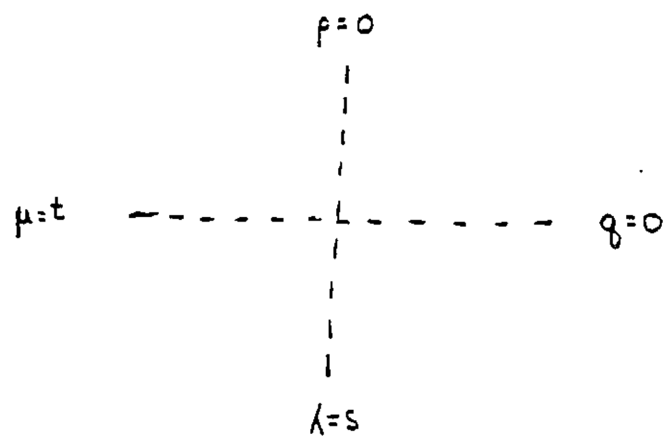


Figure 2.8

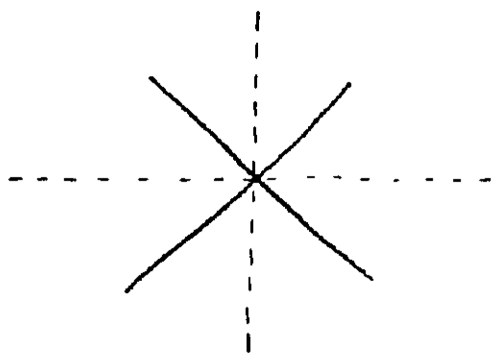


Figure 2.9

Then $p = 0$ on $\lambda = s$, $q = 0$ on $\mu = t$ (see figure 2.8). If there is no region with $p = q = 0$, then the four dotted lines of figure 2.8 must be separated by curves of Γ. That is we must have figure 2.9. Then p, q must take the local values pictured in figure 2.10. This argument takes us through Lemma 1.5 and now continue as in [GL] beginning on page 381 with line 5.

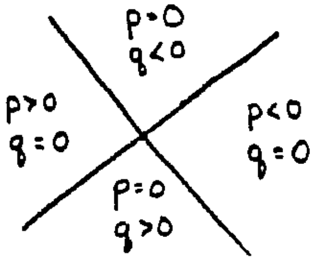


Figure 2.10

I would like to thank Hatcher for this argument.

§3. 2-bridge knots are determined by their complements

Given a Morse presentation of K, the bridge index of this presentation is the number of relative maxima in the presentation. The bridge number of K is the minimum bridge index of all Morse presentations of K.

In this section we prove

Theorem 3.1. *Let K be a knot with bridge number 2. If $K(\pi)$ is the 3-sphere then π is the meridian of K.*

For the rest of this section we assume for contradiction that 3.1 is false; that is, there is a K such that $K(\pi) = S^3 = K(\gamma)$ where $\pi \neq \gamma =$ meridian of K. By applying Proposition 2 of section one we get two punctured level spheres P, Q coming from thin presentations of K_π, K (resp.) such that P and Q intersect essentially. It is not hard to see for 2-bridge knots that a Morse presentation of minimal bridge index is in fact a thin presentation. Thus we may assume that the number of components of $|\partial Q| \leq 4$. In fact, from the proof of Proposition 2 in section two we see that $|\partial Q| = 4$. As in section one we will be done if we can prove:

Theorem 3.2. *Either G_P represents all types or G_Q contains a Scharlemann cycle.*

An example of a possible P and Q is given by figure 1.3. We will assume that at a vertex of P each label appears exactly once, that is, that the algebraic intersection number between π and γ is one. Otherwise we may apply the short argument of [CGLS, Proposition 2.5.6] to conclude that either G_P or G_Q contains a Scharlemann cycle, thereby establishing Theorem 3.2.

The rest of this section is devoted to the proof of Theorem 3.2. We need to find a disk face in G_P representing each of the types listed in figure 3.1. We will assume for simplicity that G_P is a connected graph. In particular, all faces of G_P are disk faces.

Trivial type	Class A	Class B
12 23 34 41	12 23 34 41	12 23 34 41
$\pm(+,+,+,+)$	$\pm(+,+,+,-)$	$\pm(+,-,+,-)$
	$\pm(+,+,-,+)$	
	$\pm(+,-,+,+)$	
	$\pm(-,+,+,+)$	

Non-trivial types (Class A and Class B)

Figure 3.1

Lemma 3.3. *Either there is a face in G_P representing the trivial type or there is a Scharlemann cycle in G_Q.*

Proof. We have two cases:

<u>Case 1</u>: There is a vertex, x, of G_P such that at most one label at x, $y(x)$, say, is the endpoint of an edge of G_P that connects x to a parallel vertex. (Note that there can be no loops based at x.)

Proof in Case 1. Recall the Parity Rule for edges: an edge connects parallel vertices in G_P iff it connects anti-parallel vertices in G_Q.

Let G'_Q be the subgraph of G_Q consisting of all vertices of G_Q plus all edges of G_Q that connect parallel vertices of G_Q. Let Λ be an innermost component of G'_Q that does not contain the vertex $y(x)$. The hypothesis of Case 1 along with the Parity Rule implies that every vertex, z, in Λ has the following property: the edge incident to z with label x is in Λ. Thus, starting at any vertex, z_1, of Λ we can leave that vertex at label x and go to another vertex, z_2, in Λ. We can then leave z_2 on label x and go to another vertex, z_3, in Λ (no edge in Λ can have both endpoints labelled x because of the Parity Rule). Eventually we will get a cycle $z_1, z_2, \ldots, z_n$ whose interior contains only vertices of Λ; that is, its interior contains only parallel vertices. This is what is called a *great x-cycle* in [CGLS], and by inducting on the size of a great v-cycle, v a vertex of G_P, we find that there must be a Scharlemann cycle in the interior of this great x-cycle (see [GL, Lemma 2.0.2]). □

<u>Case 2</u>: Every vertex, x, of G_P has at least two labels $y_1(x), y_2(x)$ such that the edges incident to x at $y_1(x)$ and $y_2(x)$ connect x to parallel vertices.

Proof in Case 2. Let G'_P be the subgraph of G_P consisting of the vertices of G_P along with all edges of G_P connecting parallel vertices. Let Λ be an innermost component of G'_P. The hypothesis of Case 2 guarantees a circuit of edges in Λ. An interior face of Λ will be a face, f, of G_P that touches only parallel vertices of G_P. f then represents the trivial type (+,+,+,+). □

This finishes the proof of Lemma 3.3. Q.E.D.

Lemma 3.4. *If τ is a type of class A (figure 3.1) then τ is represented by a face of G_P.*

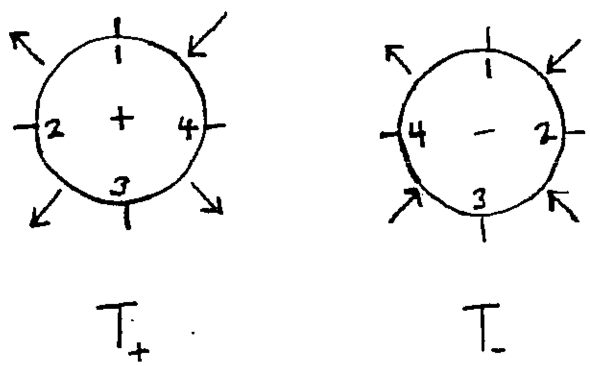


Figure 3.2

Proof. WLOG we assume $\tau = (+,+,+,-)$. Associated to τ we define the "stars" T_+, T_- pictured in figure 3.2. A *star* is an abstract vertex, v, of G_P where to an interval on v between consecutive labels $j, j+1$ on v we assign an arrow based on the sign of τ_j, where $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) = (+,+,+,-)$, and on the sign of v. In figure 3.2 we have stars for a positive and negative vertex of G_P. Note that one gets the star T_- from T_+ by reversing all arrows. The clockwise switch of T_- and that of T_+ correspond to the same label, 1. Similarly, the anti-clockwise switch of T_- and T_+ have the same label, 4. We write the fact that T_+, T_- come from τ by $[T_+] = \tau = [T_-]$. We now construct the *oriented dual graph associated to* τ, which we denote Γ_τ, as follows:

$$\text{Vertices of } \Gamma_\tau = \{\text{"fat" vertices}\} \cup \{\text{"dual" vertices}\}$$

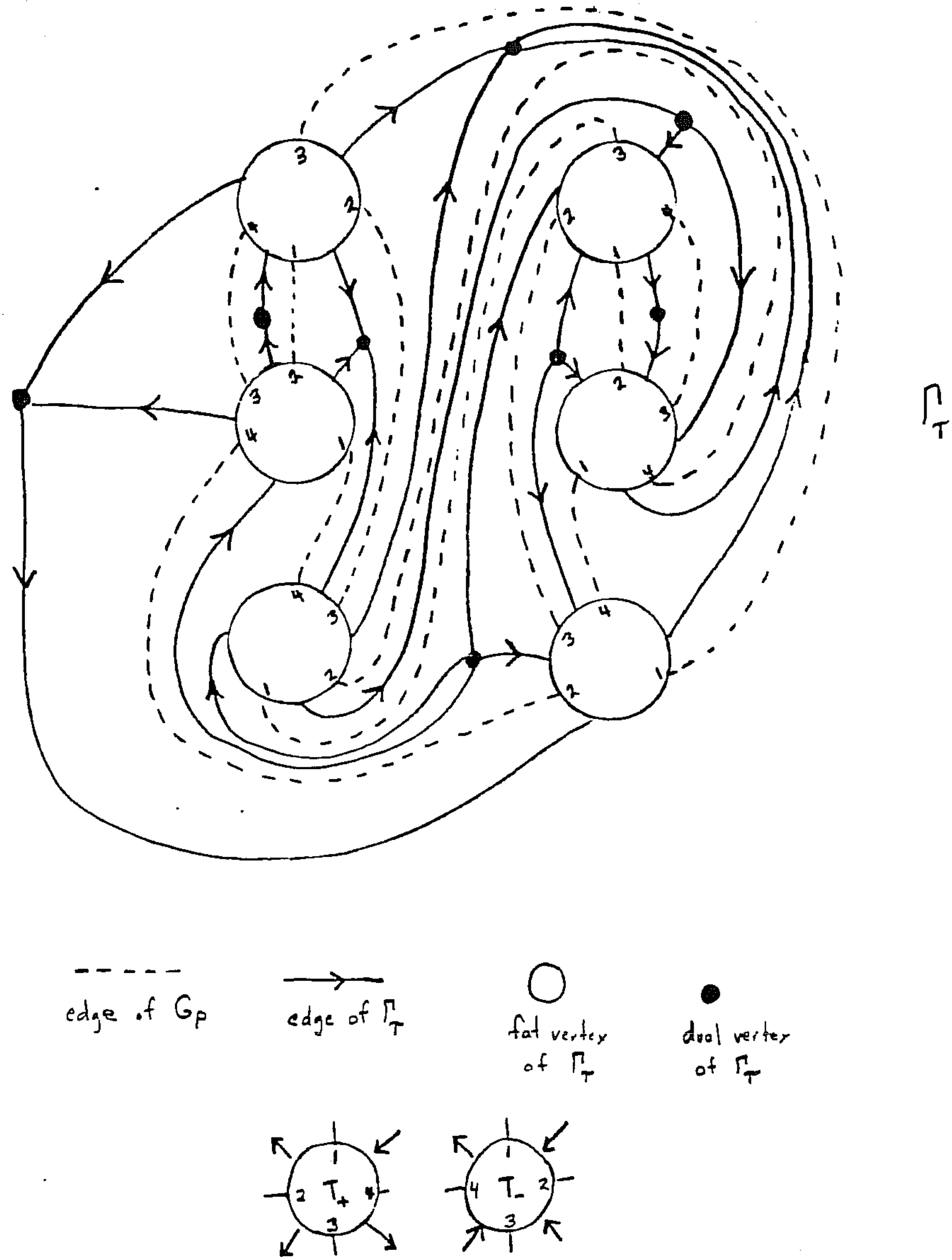
$$\{\text{fat vertices}\} = \{\text{vertices of } G_P\}$$

$$\{\text{dual vertices}\} = \{\text{faces of } G_P\}$$

> Each edge of Γ_τ connects a fat vertex with a dual vertex and is oriented according to T_+ or T_- depending on the sign of the incident fat vertex.

In figure 3.3, Γ_τ is constructed for the example in figure 1.3.

Remark. The notation Γ_τ differs from that of [GL] in that here we don't distinguish between Γ and Γ^*. In that notation our Γ_τ would be Γ^*_τ.



Γ_T
edge of G_P
edge of Γ_T
fat vertex of Γ_T
dual vertex of Γ_T
T_+
T_-

Figure 3.3

Γ_τ is constructed so that the sinks and sources of Γ_τ (which necessarily occur at dual vertices of Γ_τ since τ is a non-trivial type) correspond to faces of G_P that represent τ.

To prove Lemma 3.4 we need to show that Γ_τ contains a sink or source. To do this we do an index calculation.

A switch at a vertex, v, of Γ_τ is a pair of adjacent edges incident to v whose orientations are opposite at v (figure 3.4).

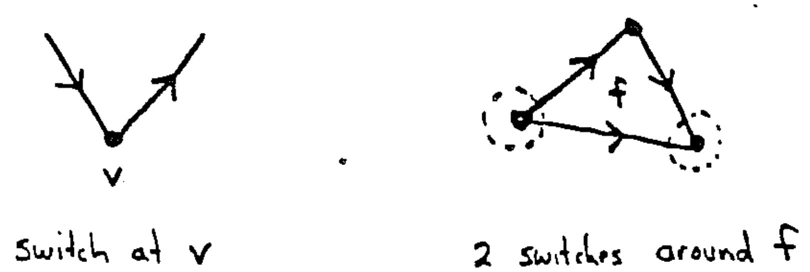


Figure 3.4

A switch of a face, f, of Γ_τ is a pair of adjacent edges of ∂f incident to a vertex v, say, on ∂f whose orientations agree at v (figure 3.4).

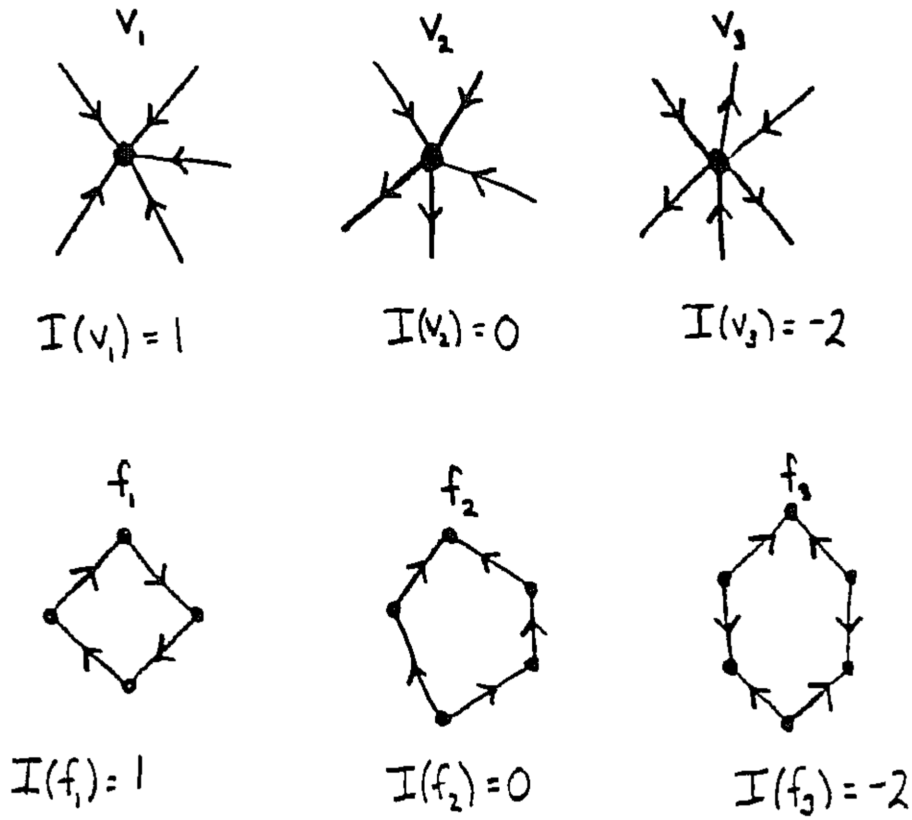


Figure 3.5

The index, $I(v)$, of a vertex v of Γ_τ is defined to be $1 - \frac{s(v)}{2}$ where $s(v)$ is the number of switches at v. The index, $I(f)$, of a face f of Γ_τ is defined to be $1 - \frac{s(f)}{2}$ where $s(f)$ is the number switches of f (see figure 3.5). Note that a vertex of Γ_τ is a sink or source iff its index is 1, and a face of Γ_τ is a cycle iff its index is 1. One now has the following lemma from [Glass], whose proof is an Euler characteristic count.

Index Lemma. $\sum_{\text{vertices}} I(v) + \sum_{\text{faces}} I(f) = 2$

Now assume that there is no sink or source at a dual vertex of Γ_τ. That is, if v is a dual vertex of Γ_τ then $I(v) \leq 0$. Note also that if v is a fat vertex of Γ_τ then $I(v) = 0$. Thus by the Index Lemma there is a face f_1 of Γ_τ such that $I(f_1) = 1$. Since we assume that G_P is connected, f_1 corresponds to an edge, e, of G_P and, in fact, one easily sees that e must be one of the two edges pictured in figure 3.6. Note that e has both endpoints with the same label. This means that e represents a loop in G_Q. Because G_Q has only 4 vertices, the only way G_Q can have a loop is if G_Q contains a 1-sided face. But this contradicts the fact that P and Q intersect essentially. Thus Γ_τ must have a sink or sourse at a dual vertex, which proves Lemma 3.4. Q.E.D.

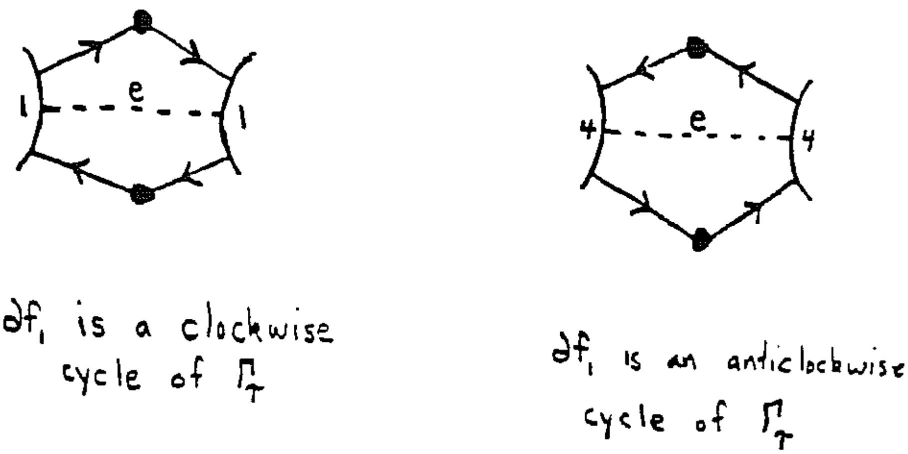


Figure 3.6

Lemma 3.5. *If τ is a type of class B (figure 3.1) then either τ is represented by a face of G_P or G_Q contains a Scharlemann cycle.*

Proof. $\tau = (+,-,+,-)$. Associated to τ we define the stars T_+, T_- pictured in figure 3.7 (*i.e.* $[T_+] = \tau = [T_-]$). Again, note that T_- is obtained from T_+ by reversing the arrows. The clockwise switches of T_- and T_+ have the same labels, and the anti-clockwise switches of T_+ and T_- have the same labels. As in the proof of Lemma 3.4 we construct the oriented dual graph corresponding to τ using the stars of figure 3.7. Figure 3.8 shows Γ_τ for the example of figure 1.3. Again, (because τ is non-trivial) a sink or source in Γ_τ corresponds to a face of G_P representing τ. So we assume that Γ_τ contains no sinks or sources. If v is a fat vertex of Γ_τ then $I(v) = -1$. By assumption if v is a dual vertex of Γ_τ then $I(v) \leq 0$. Let p be the number of vertices of G_P. By the Index Lemma we have $\sum_{\text{faces}} I(f) \geq 2 + p$. Thus Γ_τ must contain more than p faces of index 1. Again, the faces of Γ_τ correspond to edges of G_P and a face, f, of Γ_τ of index one will be one of the edges pictured in figure 3.9.

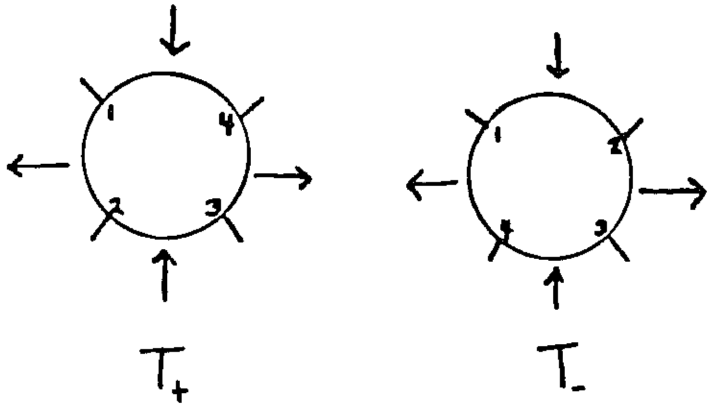


Figure 3.7

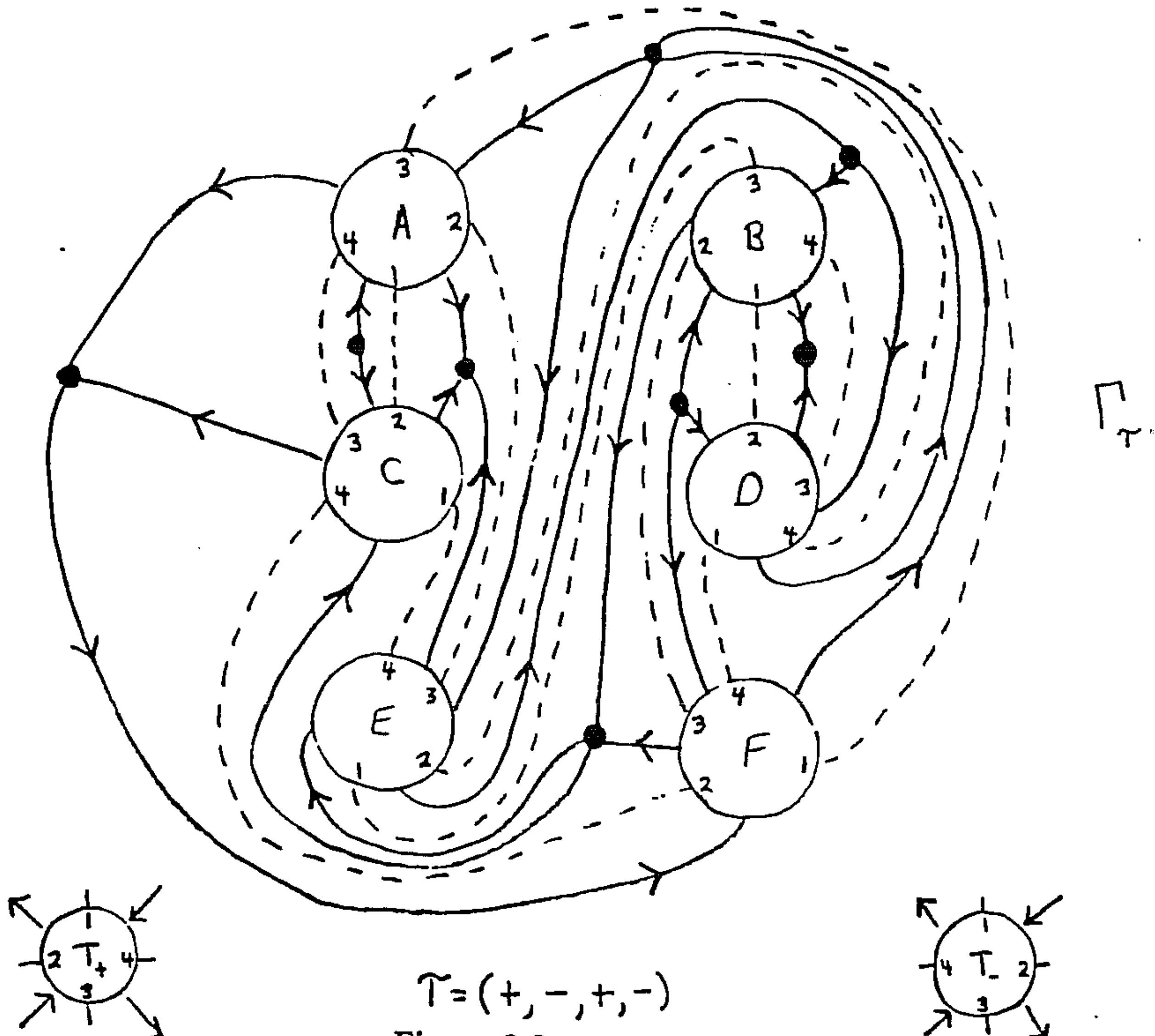


Figure 3.8

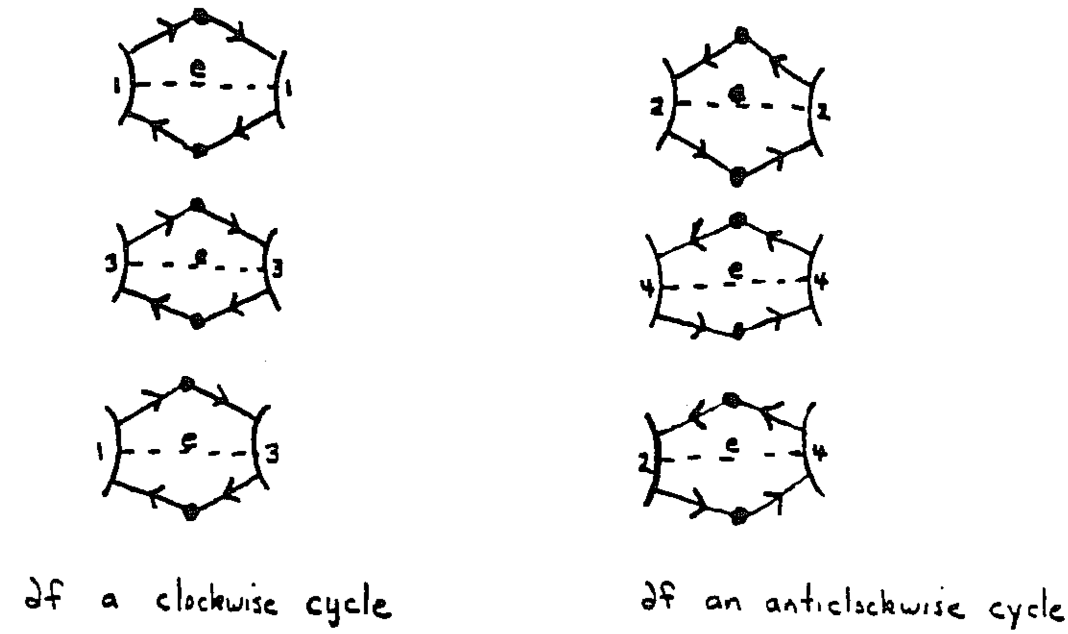


Figure 3.9

Let $\mathcal{F} = \{\text{faces}, f, \text{ in } \Gamma_\tau \mid I(f) = 1 \text{ and } \partial f \text{ is a clockwise cycle}\}$. WLOG we may assume $|\mathcal{F}| > p/2$. Define $\mathcal{E} = \{e \mid e \text{ is an edge of } G_P \text{ corresponding to } f \in \mathcal{F}\}$. Then $|\mathcal{E}| > p/2$. If $e \in \mathcal{E}$ then e is one of the edges of figure 3.9. If e has both endpoints labelled by the same vertex of G_Q then e corresponds to a loop in $\dot{G}_Q$ and this leads to the contradiction that G_Q contains a 1-sided face. Thus we may assume that e has one endpoint labelled 1 and one endpoint labelled 3. Since 1 and 3 are parallel vertices on G_Q, the Parity Rule guarantees that if $e \in \mathcal{E}$ then e cannot be a loop in G_P. Since $|\mathcal{E}| > p/2$ there must be some vertex, x, in G_P with two edges $e_1, e_2 \in \mathcal{E}$ incident to it. See figure 3.10. In G_Q the edges e_1 and e_2 form an x-cycle (figure 3.10). An *x-cycle* in G_Q is a cycle of edges in G_Q that connect only parallel vertices of G_Q and has the property that there is an orientation of the cycle such that the tail of each edge has the label x. Furthermore, an x-cycle, σ, is called a *great x-cycle* if one side contains only vertices that are parallel to the vertices in σ. Because G_Q has only 4 vertices and contains no 1-sided faces, it is easy to see that one side of the x-cycle formed by e_1 and e_2 will have no vertices of G_Q. Thus this x-cycle is a great x-cycle. Now, by induction on the size of a great

v-cycle, where v is a vertex of G_P, one can see that the interior of this great x-cycle contains a Scharlemann cycle [GL, Lemma 2.0.2].

This proves Lemma 3.5. Q.E.D.

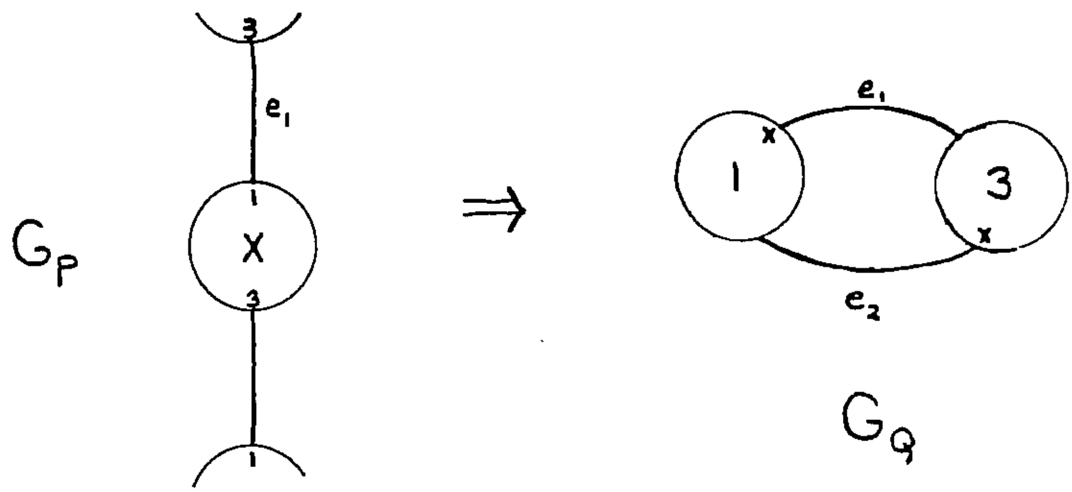


Figure 3.10

Lemma 3.3, 3.4, 3.5 prove Theorem 3.2 and consequently Theorem 3.1.

§4. Proof of Proposition 3

We now consider the proof of Proposition 3 in the general case. We want to show that we can find a collection of faces on G_P representing all types or that we can find a Scharlemann cycle on G_Q.

The argument, in the last section, of Lemma 3.3 (that either G_P represents the trivial type or G_Q contains a Scharlemann cycle) works in general. So we need to show that we can find a face of G_P representing a non-trivial type, τ (or show that G_Q contains a Scharlemann cycle). This is done exactly as in section three. Namely, we construct stars T_+, T_-, and an oriented dual graph, Γ_τ, corresponding to the desired type, then argue that there must be a sink or source in Γ_τ (again, in the notation of [GL] our Γ_τ would be Γ_τ^*). In the 2-bridge case we showed that the lack of a face representing τ (*i.e.* the lack of a sink or source in Γ_τ) gave rise, via an index count, to an x-cycle (note that a loop is automatically an x-cycle) on G_Q. We were then able to see that either this x-cycle gave rise to a 1-sided face, contradicting that P and Q intersected essentially, or was a "great" x-cycle. But then a great x-cycle always contains a Scharlemann cycle in its interior. The fact that K was a two-bridge knot came in two ways:

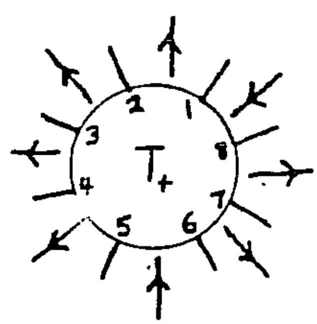
1. When K is 2-bridge, Γ_τ always had the following property:

(⋆1) All of the clockwise switches at a fat vertex of Γ_τ are labelled by parallel vertices of G_Q. Similarly, all anti-clockwise switches were labelled with parallel vertices of G_Q

   The property (⋆1), the absence of a sink or source in Γ_τ, and an index count combined to give rise to an x-cycle on G_Q (see Lemmas 3.4 and 3.5).

2. Because G_Q had only four vertices, this x-cycle had to be a "great" x-cycle (*i.e.* all the vertices on one side were parallel to the vertices in the x-cycle). A great x-cycle always contains a Scharlemann cycle in its interior.

In general one has problems with both 1) and 2) and we outline the techniques used to handle them.



$[T_+] = (+,+,+,+,-,+,+,-)$

Figure 4.1

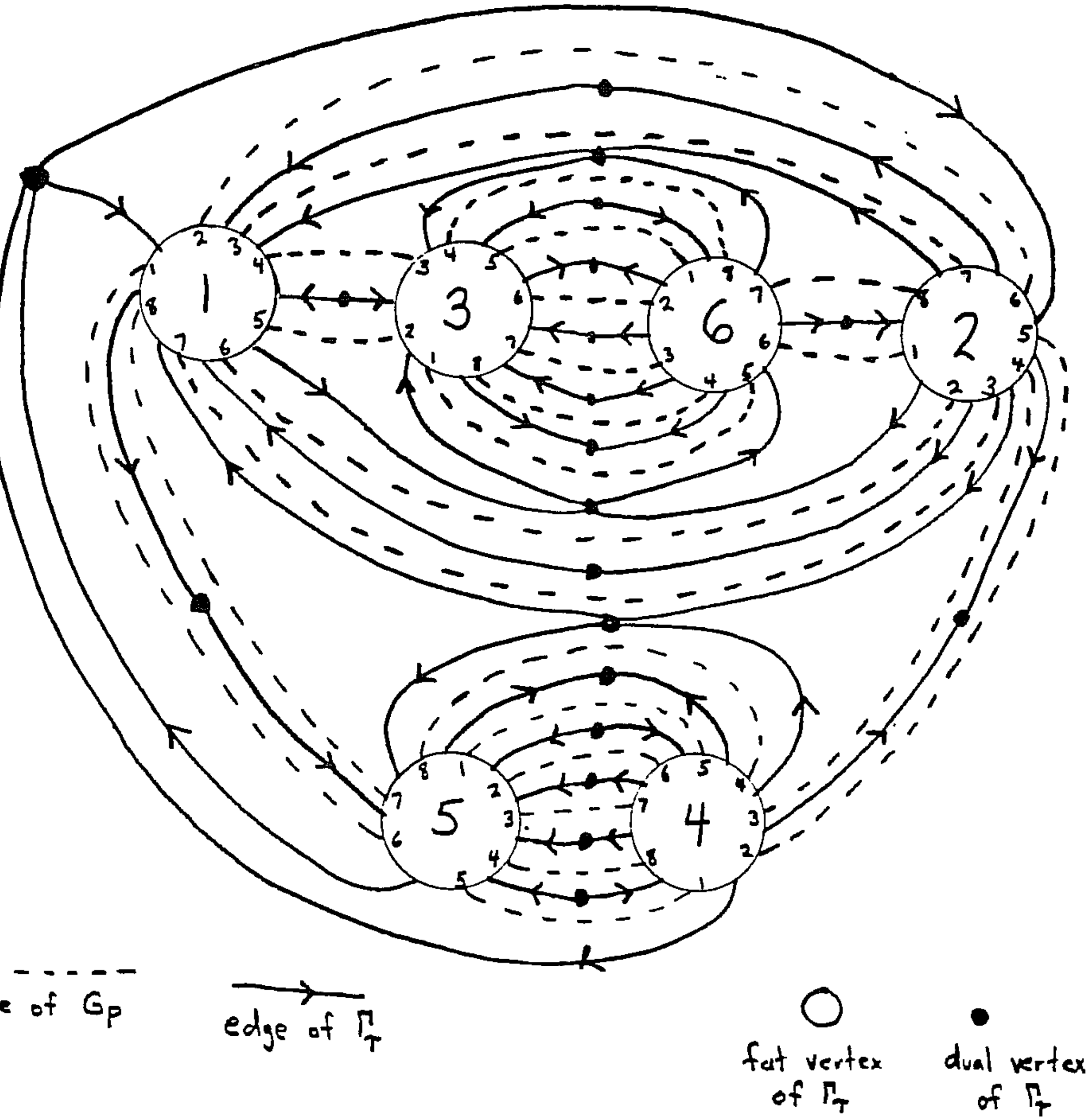


Figure 4.2

1. The oriented dual graph, Γ_τ, corresponding to the non-trivial type, τ, will not in general have property ($\star$1). For example, let τ be the type $\tau = (+,+,+,+,-,+,+,-)$ for the graph G_P pictured in figure 4.2. The star, T_+, with $[T_+] = \tau$ is given in figure 4.1. The oriented dual graph Γ_τ is given in figure 4.2.

We introduce the operation of *taking the derivative of a star* (or type) which when applied inductively reduces the problem to that of trying to find a face of a subgraph of G_P that represents a new type, derived from τ, whose corresponding dual graph (of the subgraph along with the derived type) does satisfy ($\star 1$) (p.389 and p.393 of [GL]). Say as for τ above, one has that the anti-clockwise switches fail ($\star 1$). Let T be such that $[T] = \tau$. Let $C(T)$ be the set of vertices in G_Q that label the clockwise switches of T (in the example $C(T) = \{1, 6\}$). Consider the subgraph, $G_P(C(T))$, of G_P consisting of all vertices of G_P along with all edges of G_P that have at least one endpoint with label in $C(T)$. See figure 4.3. We construct a star, the *derivative of* T, which we denote dT, corresponding to an abstract vertex of $G_P(C(T))$ as in figure 4.4. (The construction of dT from T is given on p.389 of [GL]). In the graph $G_P(C(T))$ we look for a face that represents $[dT]$. We then argue (p.399 of [GL], Corollary 2.4.2) that such a face in $G_P(C(T))$ will contain within it a face in G_P representing type τ. Figure 4.5 depicts a face of $G_P(C(T))$ representing $[dT]$, from figure 4.3, and one sees within it a face representing $[T] = \tau$, from figure 4.2. By repeatedly taking derivatives we eventually arrive at a subgraph $G_P(C(d^{n-1}(T)))$ of G_P and a type d^nT for that subgraph with the property that, $\Gamma(d^nT)$, the oriented dual graph (built from $G_P(C(d^{n-1}(T))))$ corresponding to d^nT, satisfies ($\star 1$). If we find a face in $G_P(C(d^{n-1}(T)))$ that represents $d^n(T)$ then we know, by working backwards inductively, that there is a face in G_P representing τ. Otherwise we may apply the Index Lemma from section three to $\Gamma(d^nT)$ as we did in section three to conclude that there is an x-cycle, σ, in G_Q. Note that the vertices of G_Q in σ label switches of d^nT, hence label clockwise switches of τ.

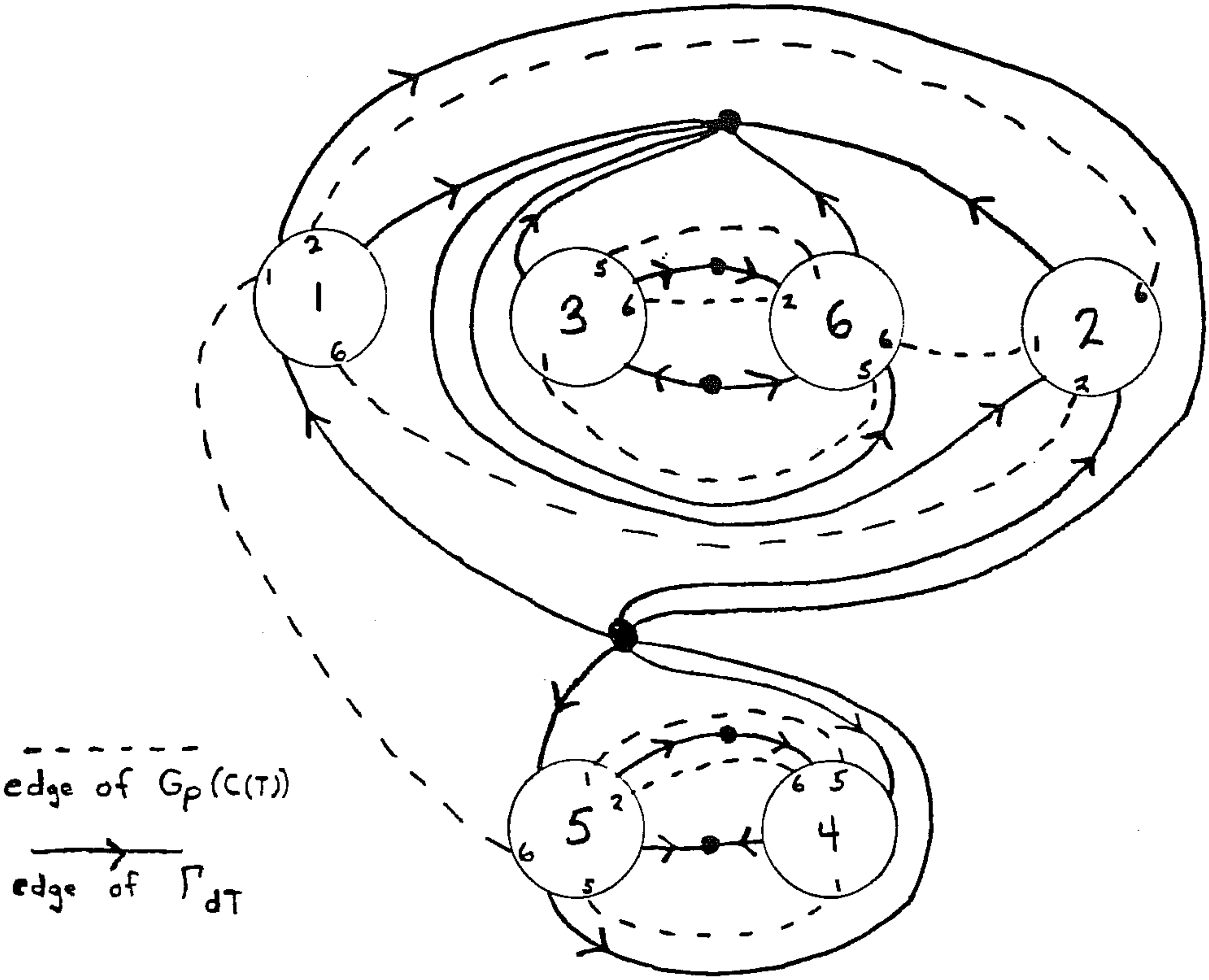


Figure 4.3

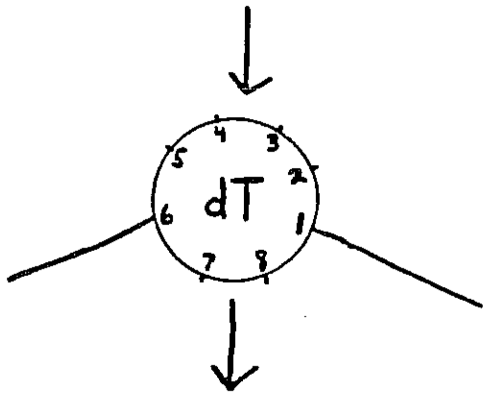


Figure 4.4

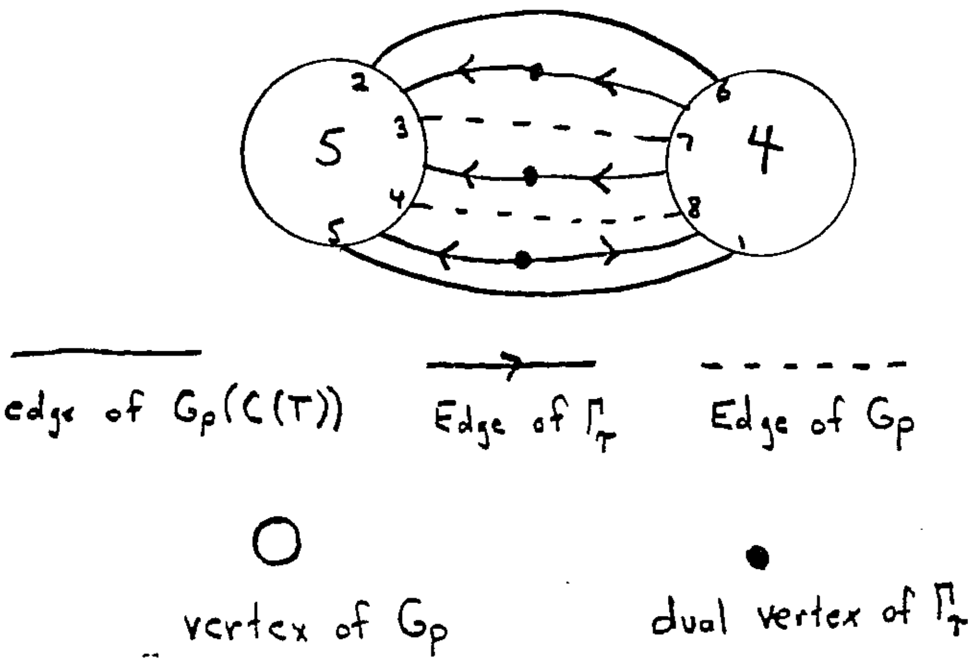


Figure 4.5

If the clockwise switches of T had failed $(\star 1)$ instead of the anti-clockwise switches, then reverse all arrows in T and make this T instead. Now we are in the case above.

2. In general, once we find the x-cycle, σ, in G_Q it will probably not be true that σ is a great x-cycle. So what we would like to do is to "induct" on the vertices inside of σ.

Let L_0 be the set of vertices of G_Q in the interior of σ. Let $G_P(L_0)$ be the subgraph of G_P obtained by taking the vertices of G_P along with only those edges of G_P that have an endpoint labelled with a vertex in L_0. We assume, by induction, that $G_P(L_0)$ represents all types (otherwise we use procedure (1) to find a new x-cycle on G_Q among the vertices of L_0). Note that for $G_P(L_0)$ a type is an ordered $|L_0|$-tuple, called an L_0-type, see p.387 of [GL].

Recall from (1) above that we arrived at the x-cycle, σ, in pursuit of a non-trivial type, τ, by taking a sequence of derivatives of the star, T, corresponding to τ:

$$T_i = dT_{i-1} = d^iT \qquad i = 1, m$$

$$T_0 = T$$

The problem of finding a face in G_P representing τ is reduced (in the general inductive step τ will be an L-type, where L is some subset of the vertices of G_Q, and we will be looking for a face in $G_P(L)$ that represents τ) to that of finding a face in $G_P(C(T_{n-1}))$ [$G_P(C(T_{n-1}))$ is the subgraph of G_P consisting of the vertices of G_P along with all edges in G_P that have an endpoint labelled by a clockwise switch of T_{n-1}] representing $[T_n]$. We want to use the inductive hypothesis, that $G_P(L_0)$ represents all L_0-types, so we change this sequence of derivatives to a sequence which is relative to the subgraph $G(L_0)$. This relative derivative d_{L_0} is defined on p.389 and p.393 of [GL]. Essentially, by taking the relative derivative rather than the absolute derivative we ensure that at each step $G(L_0)$ remains a subgraph of the new derived graph. Thus we now look at the new sequence of stars:

$$R_i = d_{L_0}R_{i-1} \qquad i = 1, m$$

$$R_0 = T_0 = T$$

By repeatedly applying Lemma 2.4.1, p.397, we still have that if we can find a face in $G_P(C(R_{n-1}) \cup L_0)$ [*i.e.* the subgraph of G_P consisting of all edges with at least one endpoint labelled by a vertex of G_Q that is either in L_0 or is a clockwise switch of R_{n-1}] representing $[R_n]$ then we can find a face of G_P representing τ. (To apply 2.4.1 we have to first note the rather subtle point that $\delta_0(\Gamma(R_i)) = \Gamma(d_{L_0}R_i) = \Gamma(R_{i+1})$, which uses the fact that the vertices of σ, the "exceptional labels" of $G_P(L_0)$, are clockwise switches of R_i.) We now use the assumption that $G_P(L_0)$ represents all L_0-types to find a face representing $[R_n]$. Because $G_P(L_0)$ is a subgraph of $G_P(C(R_{n-1}) \cup L_0)$ we must do the following.

Given the star R_n and an interval on the star between two consecutive labels in L_0 (an "L_0-interval"), section 2.6 of [GL] defines that interval to be good or bad according to the sign of the abstract vertex R_n and the switches of R_n in this L_0-interval. That is, each L_0-interval on R_n is either good or bad. This gives us an L_0-type by assigning a $+$ to the L_0-interval if it is good and a $-$ if it is bad. By assumption $G_P(L_0)$ contains a face, F, representing this L_0-type. We now argue that within the face, F, of $G_P(L_0)$ there is a face, f, of $G_P(C(R_{n-1}) \cup L_0)$ that represents R_n. This is done by an index count on the oriented dual graph associated to R_n, restricted to F, and is given in Lemmas 2.7.1 and 2.3.3 of [GL].

Thus the assumption that $G_P(L_0)$ represents all L_0-types implies that there is a face of $G_P(C(R_{n-1}) \cup L_0)$ representing $[R_n]$. This in turn implies that there is a face in $G_P(C(R_{n-2}) \cup L_0)$ representing $[R_{n-1}], \ldots$, which in turn means there is a face of $G_P(C(R_0) \cup L_0)$ representing $[R_1]$, which means there is a face in G_P representing $[R_0] = [T_0] = \tau$. (Again, this last series of inductions comes from Lemma 2.4.1 and uses the fact that $\delta_0 \Gamma(R_{i-1}) = \dot{\Gamma}(d_{L_0} R_{i-1})$. See the top of page 409.) This is what we were looking for.

If $G_P(L_0)$ does not represent all L_0-types, then, as mentioned above, we would apply the procedure of (1) to find a new x-cycle σ' within the vertices of L_0 on G_Q. Replace σ with σ' and let L_0 be the vertices inside σ'. Note that we have reduced the size of L_0, so eventually we will either get that $|L_0| \leq 1$, which will give us a great x-cycle on either G_P or G_Q, or that $G_P(L_0)$ represents all types.

This outlines the proof of Proposition 3.

Remark. The motivating example for the argument in 2) is when G_P is disconnected, *i.e.* when there are no edges of G_Q connecting L_0 with the x-cycle σ. Then $G_P(L_0)$ has no exceptional labels, that is, each endpoint of each edge of $G_P(L_0)$ is labelled by a vertex of G_Q in L_0. It is probably helpful to understand the argument of 2) in this case first. Furthermore, in this case one has that if $G_P(L_0)$ represents all $|L_0|$-types then G_P represents all q-types. The proof of this fact is really the same as the proof of Lemma 3.3 of [GL], though there the context is a little different. We sketch the argument here. Assume $G_P(L_0)$ has no exceptional labels and let $G_P(L_0)$ represent all $|L_0|$-types. Let τ be a q-type.

Claim. *There is a sequence of stars $X_1, X_2, \ldots, X_n$ such that*

(1) $[X_1] = \tau$

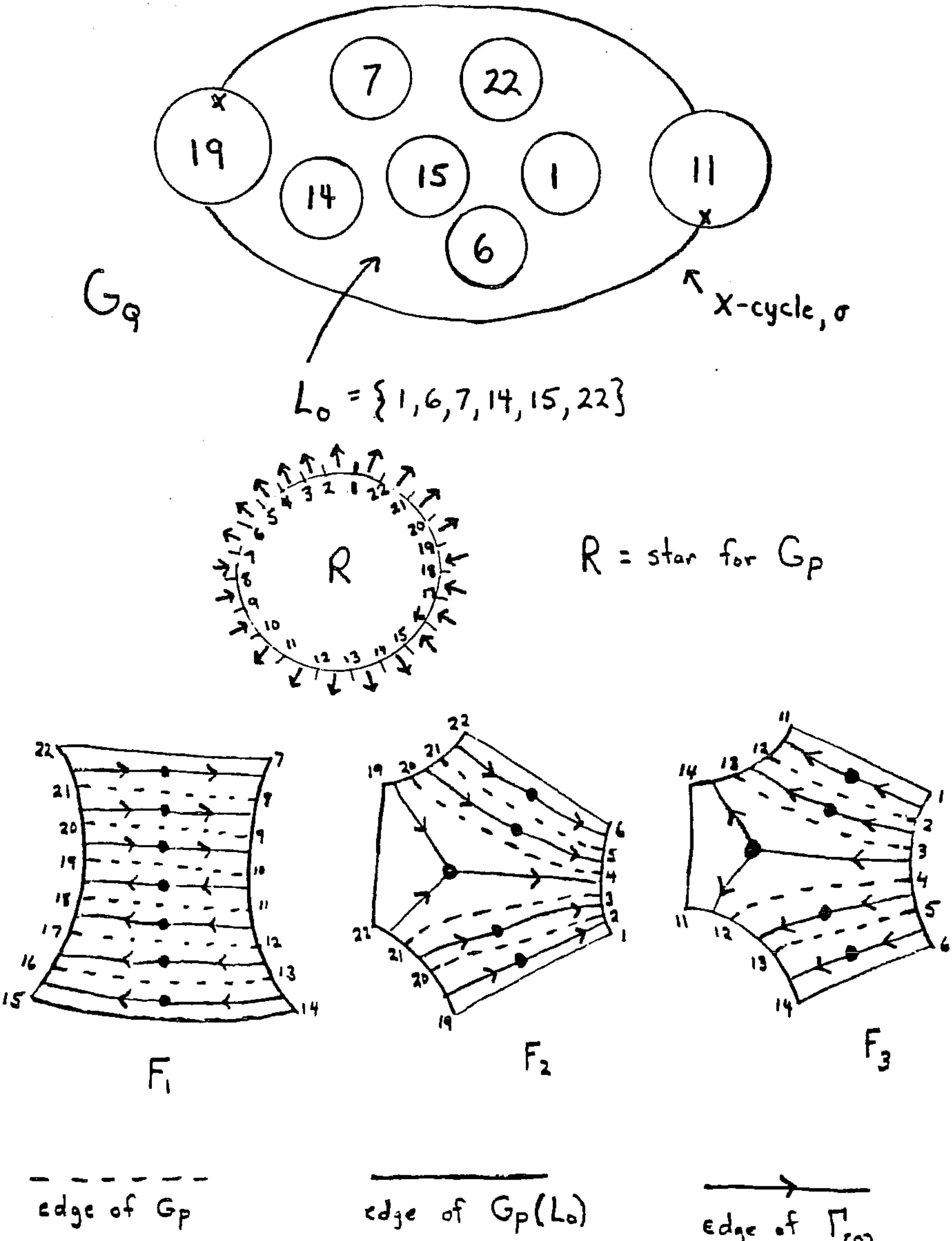
(2) $[X_i] = d_{L_0}X_{i-1}$ or $\overline{d_{L_0}X_{i-1}}$, $\quad 2 \leq i \leq n$,

(3) $[X_n]$ *is an L_0-type*

Proof of claim. Same as the proof of the claim in Lemma 3.3 of [GL]. □

Because $G_P(L_0)$ represents all $|L_0|$-types and $[X_n]$ is an $|L_0|$-type we have that there is a face F of $G_P(L_0)$ representing $[X_n]$. Since $G_P(L_0)$ has no exceptional labels, we have by Lemma 2.2.2 of [GL] that $\delta_{L_0}(\Gamma(X_{i-1})) = \Gamma(d_{L_0}X_{i-1}) = \Gamma(X_i)$. By successively applying Lemma 2.4.1 of [GL] we get a face of G_P representing $[X_1] = \tau$ (it might also be helpful to go through the proof of Lemma 2.4.1 in this setting). This is what we were looking for.

In general when $G_P(L_0)$ has exceptional labels it is not true that the existence of a collection of faces of $G_P(L_0)$ representing all L_0-types implies that G_P represents all types. This is illustrated by figure 4.6. Here the collection $\{F_1, F_2, F_3\}$ of faces of $G_P(L_0)$ represents all L_0-types; however, there is no face of G_P (at least in G_P restricted to the pictured faces of $G(L_0)$) representing the original type $[R]$ (*i.e.* there is no sink or source for $\Gamma_{[R]}$). This example illustrates the necessity of some relationship between the original type $[R]$ and the x-cycle, σ, which defines L_0, *e.g.*, the fact that all the vertices of G_Q in σ (the exceptional labels of $G_P(L_0)$) are clockwise switches of R.



7
22
19
14
15
1
11
6
G_q
x-cycle, σ
$L_0 = \{1, 6, 7, 14, 15, 22\}$
R
R = star for G_p
F_1
F_2
F_3
edge of G_p
edge of $G_p(L_0)$
edge of $\Gamma_{[R]}$

Figure 4.6

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AROUND THE HILBERT-SMITH CONJECTURE

by Mladen Bestvina and Robert D. Edwards

A special case of the Hilbert-Smith Conjecture (for an introduction to the Conjecture and a discussion of our approach see [1]) asserts that the group

$$A_p = \varprojlim (\mathrm{Z/p} \leftarrow \mathrm{Z/p}^2 \leftarrow \cdots)$$

of the p-adic integers does not act freely on the n-dimensional torus T^n. Denote by X the orbit space T^n/A_p of such a presumed action and assume in addition that X is finite-dimensional (in fact it is known that in that case $dim X = n + 2$). Now "suspend" this action in the following way. Pick a large integer m, and a free abelian subgroup $F \subset A_p$ of rank m. Let F act on the Euclidean space R^m in the standard way, and form the orbit space $M = \mathrm{R}^m \times \mathrm{T}^n/\mathrm{F}$ where F acts diagonally on the product. Projection to the second factor induces a map $f: M \to X$ whose point preimages are generalized solenoids $\mathrm{R}^m \times \mathrm{A_p/F}$ which have trivial (integral Čech) homology in positive dimensions. Note that M is a manifold which can be taken to be homeomorphic to the (m+n)-torus. Thus we are led to the following question.

Question. Does for every q there exist $n = n(q)$ such that every map f from the n-torus to a q-dimensional space (e.g. R^q) has a point preimage $f^{-1}(pt)$ such that the inclusion induced homomorphism $\check{H}_1(f^{-1}(pt)) \to H_1(T^n)$ is non-trivial (integer coefficients)?

This question should be contrasted with the following fact (which is a version of the Lusternik-Schnirelman theorem): *If $f: T^n \to X$ is a map with $dim X < n$, then there exists $x \in X$ such that the inclusion induced homomorphism $H^1(T^n) \to \check{H}^1(f^{-1}(x))$ is non-trivial.* (Proof: Otherwise every point preimage is contractible in the torus, and therefore X admits an open cover U such that for every $U \in$ U the set $f^{-1}(U)$ is contractible in T^n. Since $dim X < n$, we can take U to have n elements $U_1, U_2, ..., U_n$ each of which has the property that the closure of its preimage is contractible in the torus. It follows that the i^{th} coordinate projection $p_i: T^n \to S^1$ is homotopic to a map $\bar{p}_i$ such that $\bar{p}_i(f^{-1}(U_i))$ is a proper subset of S^1. But then $\bar{p}_1 \times ... \times \bar{p}_n: T^n \to T^n$ is a non-surjective map homotopic to the identity, a contradiction).

It follows that the answer to the above question is affirmative if we restrict ourselves to "nice" maps, e.g. those that have ANR point preimages. In that case we can take $n = q + 1$ (vanishing homology would imply vanishing cohomology). The following two examples illustrate the subtlety of this question.

Example 1. Recall [1] that A_p acts freely on a 2-dimensional cell-like set C with 2-dimensional orbit space Q. Performing the above construction to C yields $\widehat{T} = \mathrm{R}^m \times \mathrm{C/F}$, which as before maps to Q with generalized solenoids for point preimages. On the other hand, projection to the first coordinate induces a cell-like map $\rho : \widehat{T} \to T^m$. Hence the above question has a negative answer (for $q = 2$) if we pose it in the larger class of "shape tori".

Example 2. Let k be the largest integer with $2(k+2)+2 < m$, and denote by K the k-skeleton of T^m. Let $\widehat{K} = \rho^{-1}(K)$ be the "shape k-skeleton" in the shape torus $\widehat{T}$. Note that $\dim \widehat{K} = k+2$ and hence $\rho|\widehat{K}$ can be approximated by an embedding. We write $\widehat{K} \subset T^m$ and observe that $T^m - \widehat{T}$ is homeomorphic to $T^m - K$ and therefore there is a natural map $g : T^m \to cL$ to the cone over the dual $(m-k-1)$-skeleton L in T^m, sending $\widehat{K}$ to the cone point. Also the map $\widehat{K} \to Q$ (restriction of the map $\widehat{T} \to Q$ from example 1) extends to a map $h : T^m \to I^5$ for an embedding of Q into the 5-cell. The map $g \times h : T^m \to (cL) \times I^5$ has homologically trivial point preimages, and the dimension of the target space is $m - k + 5$, just slightly above $m/2$. This example shows that in general $n = q + 1$ doesn't suffice. An elaboration of this argument yields that n must be bounded below by an exponential function of q (if it exists).

There is another related question. It can be posed so that it more closely resembles the classical Lusternik-Schnirelman theorem, which states that every open cover $U_1, U_2, \cdots, U_n$ of the n-torus has an element that contains a loop essential in the torus.

Question. Does for every q there exist $n = n(q)$ such that the following holds? Suppose that for every $i = 1, 2, \cdots$ we are given an open cover $\mathrm{U}^i = \{\mathrm{U}^i_1, \mathrm{U}^i_2, \cdots, \mathrm{U}^i_q\}$ of the n-torus such that for every $j = 1, 2, \cdots, q$ the sets $U^1_j \supset U^2_j \supset \cdots$ form a shrinking sequence. Then there is an essential loop in the torus which is homologous into some element of each open cover U^i.

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On Knotting of Randomly Embedded n-gons in R^3

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Consider a circle consisting of n line segments which is randomly embedded in $\mathbf{R}^3$, i.e., a randomly embedded n-gon in $\mathbf{R}^3$. This circle can be either knotted or unknotted, so one may ask the question: what is the probability for a randomly embedded n-gon to be knotted? This question is of interest to both mathematicians and scientists. It was first raised by some physists and chemists in early 1960's in the context of estimating topological self-entanglement of molecules ([1], [2]). The importance of studying this problem can be seen from the study of topological constraints in the statistical mechanics of long polymer molecules, and the effects of knots on the long time memory in melts of linear polymers ([1], [2], [3]). Another example is the recent application of knot theory to DNA research, where knotting of circular DNA molecules is used to detect enzyme action ([5]).

Of course, the first thing for us to do is to make clear the meaning of "randomly embedded n-gon in $\mathbf{R}^3$", and it turns out that one may have different models, depending on point of view. We give two models here, one mathematically easier to deal with, and the other one preferred by scientists.

In this field, there is a conjecture raised by Frisch, Wasserman and Delbruck which says that the knot probability for a randomly embedded n-gon goes to 1 as n goes to infinity. The conjecture is usually called FWD conjecture. One would like to express the probability of knotting for a randomly embedded n-gon as a function of n. For example, this probability is always 0 when n is less than or equal to 5. But this becomes very difficult when n is large. In this paper, we discuss a proof of the FWD conjecture for two random polygon models. These models are versions of the continuum case; there is also a discrete model called " self-avoiding walks on the lattice" (SAW), for which the conjectrue has been proved by Sumners and Wittington ([6]).

Definition 1. A *Gaussian random vector* (or *point*) X in R^3 is an ordered triple (x, y, z) such that x, y, z are independent Gaussian random variables, each with $a = 0$ and $\sigma = 1$. (We say that x is a

Gaussian random variable if x is a random variable and its density function is given by $\exp(-(x-a)^2/2\sigma^2)/(2\pi\sigma^2)^{1/2}$, where a is its expectation and σ^2 is its variance.)

Definition 2. A *Gaussian random walk* of n steps is a sequence of n random points $X_1, X_2, ..., X_n$ ($X_i = (x_i,y_i,z_i)$) such that $X_{k+1} - X_k$, k = 0, 1, ..., n-1 (here we take X_0 to be the origin O) is a sequence of independent Gaussian points. We denote it by GW_n.

Definition 3. A *Gaussian random loop* of n steps is a Gaussian random walk with both end points fixed at the origin O. We denote it by GL_n.

Theorem 1. *There exists a constant $\varepsilon > 0$ such that P(GL_n is knotted) $\geq 1 - \exp(-n^\varepsilon)$, provided that n is large enough.*

Definitions 1 to 3 defined the first model. It is nicer to deal with since one can explicitly write down the density function of GL_n and the probability integrations will always be over $\mathbf{R}^{3n}$.

Definition 4. An *equilateral random walk* of n steps is a linear chain consisting of n unit line segments and is denoted by EW_n. If we number the end points of these line segments from one end of the chain by $X_0, X_1, ..., X_n$, then we have $|X_{i+1} - X_i| = 1$ for i =0, 1, ..., n-1. Usually we take X_0 to be the origin. Once X_i is given, the distribution of X_{i+1} will be independent of those end points of EW_n before X_i, and is evenly distributed on $S(X_i,1)$, the unit sphere centered at X_i, in other words, $X_{i+1} - X_i$, i =0, 1, ..., n-1 is a sequence of n independent random points, all are evenly distributed on the unit sphere S(O,1).

Definition 5. An *equilateral random loop* of n steps is an EW_n that with last end point X_n also to be the origin. We denote it by EL_n.

Theorem 2. *There exists a constant $\varepsilon > 0$ such that P(EL_n is knotted) $\geq 1 - \exp(-n^\varepsilon)$, provided that n is large enough.*

When modelling long chain polymers, the equilateral random walk model is preferred to the Gaussian random walk model. The knotting probability results in the two models are similar, but, as one may not expect, we have a better estimation for the number ε in the E-case than in the G-case. We estimate that $\varepsilon > 0.05$ in the E-case but only get $\varepsilon > 0.01$ in the G-case.

The following figure shows an equilateral random walk of 4 steps. The probability for the first vertex X_1 to be in the region A is simply the area of A divided by the total area of the unit sphere, i.e., area(A)/4π.

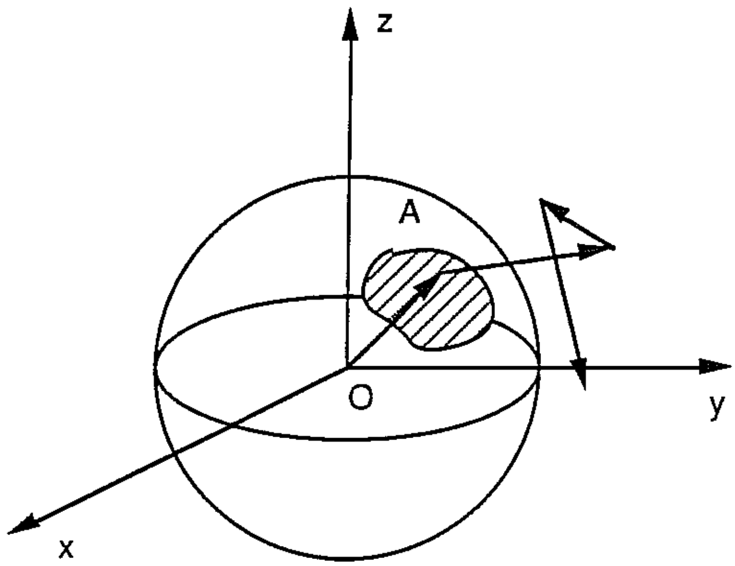

Figure 1

In this paper, we give only the sketch of the proof for the second model. The details for the Gaussian model will appear in [14] and the details for the equilateral model will appear in [15].

The Sketch of The Proof. The effort is to show that with a high probability, a random loop will always contain at least one trefoil (actually any given knot pattern) as a direct sum component when n is large. To see such a summand, one has to look for a (connected) part of the loop that has the given knot pattern and is bounded by a topological 3-ball such that this ball is disjoint from the rest of the loop. It is not difficult for one to show that a given part of the loop (say, the first 10 steps) has a positive probability to form the given knot component (say, a trefoil component) of the loop (provided that this part is long enough to form such a knot pattern). The main difficulty arises when one tries to show that there is at least one such component in the loop with a high probability. The reason is as follows. Take the trefoil pattern as example. Divide the loop into several parts so that each part can form a trefoil pattern. Number them by 1, 2, ..., k. Let T_i be the event that the i-th part forms a trefoil component of the loop in some topological 3-ball so that the rest of the loop will not intersect that ball. If the events T_i's were all independent, we would have no problem. The trouble is that once one of the T_i's has happened, say T_1, then the rest of the loop can not enter the ball that bounds the first part of the loop, hence the rest T_i's are effected, that is, the T_i's are dependent. The following sketch is for the case of equilateral random loops; a similar argument holds for Gaussian random loops. For the sake of simplicity, we suppose that n = 10m and let $Y_i = X_{10i}$, i = 0, 1, ..., m-1. Also, let $EW_n|X_n$ denote an equilateral random walk of n steps under the condition that the last point X_n is fixed.

Lemma 1. *There exists a constant θ such that $0 < \theta < 1/2$ and for any X_n in the ball $B(O,\theta)$, we have*

P(All the vertices $Y_1,\dots, Y_{m-1}$ of $EW_n/X_n \notin B(O,\theta)) \geq 1/2$

where $B(O,\theta)$ is the ball of radius θ that centered at O.

Corollary 1. *No ball of radius θ can contain more than m^{α} of those vertices $Y_1, Y_2,\dots, Y_{m-1}$ of EL_n except with a probability at most $m\exp(-a_1 m^{\alpha})$ where a_1 is some positive constant and $\alpha = 1/20$.*

Corollary 2. *No ball of radius 15 in R^3 can contain more than $a_2 m^{\alpha}$ vertices $Y_0, Y_1, \dots, Y_{m-1}$ of EL_n except with a probability $< m\exp(-a_1 m^{\alpha})$, where a_2 is some positive constant.*

Definition 6. For the vertices $X_0=Y_0(=O)$, $Y_1, \dots, Y_{m-1}$ of EL_n we say that two adjacent ones (Y_k, Y_{k+1}) form a *closing pair* if it happens that the distance between them is less or equal to θ.

Lemma 2. *EL_n has at least bm closing pairs except with a probability at most $\exp(-a_3 m)$, where b and a_3 are some positive constants.*

Corollary 3. *When n is large enough, EL_n will have at least $a_4 m^{1-\alpha}$ special closing pairs such that each of them has a distance at least 14 from any of the rest, except with a probability $< \exp(-a_3 m) + m\exp(-a_1 m^{\alpha})$, where a_4 is also a positive constant. We call these special pairs "far away closing pairs".*

Now we construct EL_n in three steps. First, we determine the vertices $Y_1, ..., Y_{m-1}$. After that we get m parts of EL_n with end points from the Y_i's, we call each such part a *stretch*. Remember that each stretch is an EW_{10} with both end points fixed. By the above lemmas and corollaries, we now have at least $a_4m^{1-\alpha}$ far away closing pairs except with a small probability. Let's take any $a_4m^{1-\alpha}$ far away closing pairs and denote the stretches bounded by them by $S_1, S_2, ..., S_t$ where $t = a_4m^{1-\alpha}$. We then fill in the stretches other than the chosen ones. Once having done so, we will have each remaining stretch bounded in a ball of radius 6 since the end points of all of them are "closing" pairs. On the other hand, these closing pairs are far away from each other, hence the remaining strecthes will not interfere with each other. In other word, what happens to one such stretch is independent of the rest. Without loss of generality, we can suppose that S_1 is bounded by (Y_0, Y_1). Let's try to estimate the probability for it to form a trefoil component. Since each S_i is bounded in a ball of radius 6

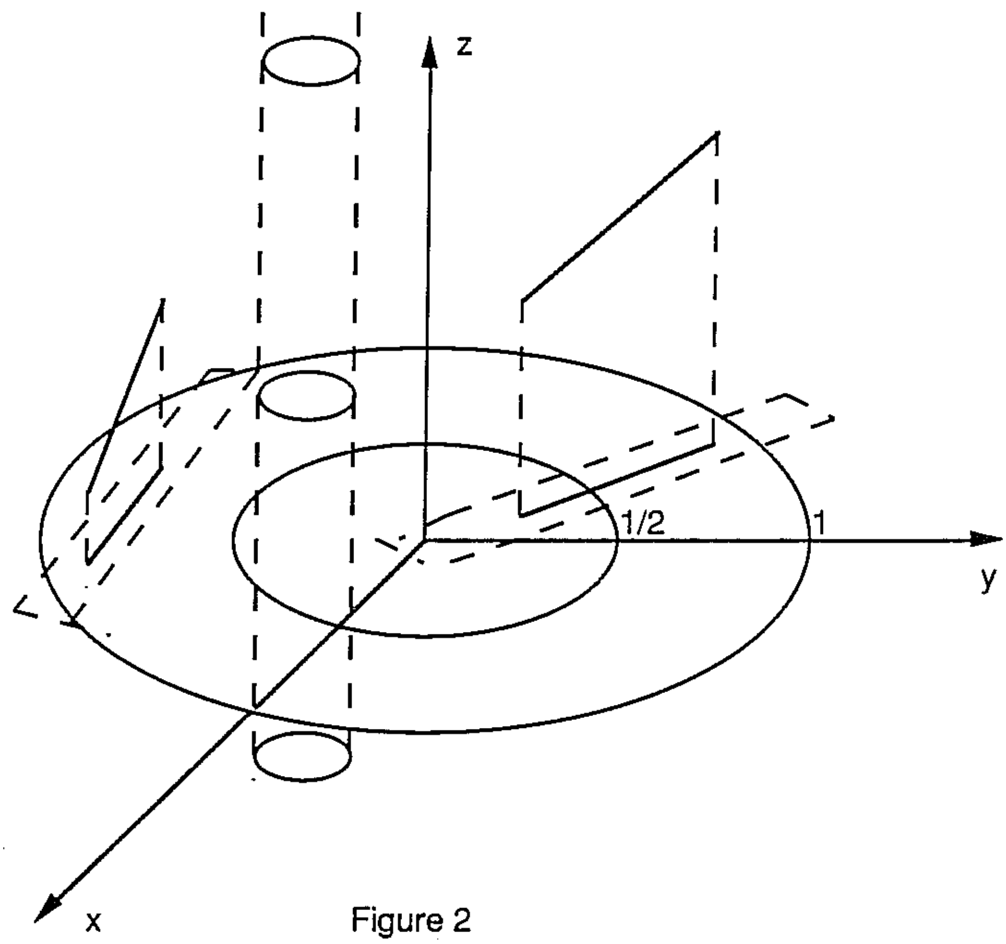

Figure 2

and there are at most $10a_2m^{\alpha}$ steps of EL_n other than S_i intersecting the ball (Cor.2), we can find a cylinder of length 2 and radius $a_5m^{-\alpha}$, where $a_5 = (20a_2)^{-1}$ is a constant, with its center line parallel to the z-axis and the distance of it to the z-axis being between 1/2 and 1, such that it does not intersect any previously fixed step of GL_n.

To see the claim, project all the fixed steps in B(O, 5) onto the annulus $1/4 \leq x^2 + y^2 \leq 1$, $z = 0$. The result is at most $10a_2m^{\alpha}$ line segments, each with length at most 1. Enhance each such line segment to a rectangle with length $1 + 2a_5m^{-\alpha}$ and width $2a_5m^{-\alpha}$(< 1/2 when n is large), such that the line segment lies in the middle of the rectangle. Now the total area of these rectangles is at most $40a_2m^{\alpha}a_5m^{-\alpha} = 2$, which is less than the area of the anulus $(3\pi/4)$, thus we can always find a point on the anulus that is at least $a_5m^{-\alpha}$ away from these line segments. Obviously, the cylinder that has this point as its center, parellel to the z-axis and of radius $a_5m^{-\alpha}$, length 2 is what we want. The situation is shown in fig.2.

We can then prove that the probability for S_1 to form a trefoil pattern in the cylinder we just found is at least $cm^{-15\alpha}$ for some constant $c > 0$. If we let G_i be the event that the i-th stretch forms a trefoil component, then we have seen that they are independent and $P(G_i) > cm^{-15\alpha}$. So the probability for at laest one of them to appear is $1 - (1- cm^{-15\alpha})^t$, where $t = a_4m^{1-\alpha}$. Substituting α by 1/20, we can see that this is greater than $1 - \exp(-n^{3/20})$, provided that n is large enough. Finally, combining all the results together, we have

$$P(EL_n \text{ is knotted}) > 1 - \exp(-a_3m) - m\exp(-a_1m^{\alpha}) - \exp(-n^{3/20})$$

Which is clearly larger than $1 - \exp(-n^{\varepsilon})$ for some $\varepsilon > 0$ when n is large enough. We can take ε to be 0.05 by choosing α a little larger than 1/20 at the begining.

We can state a stronger result as follows:

Theorem 3. *Let K be any knot pattern, then the probability for EL_n to contain K as a dirrect sum component exceeds $1 - \exp(-n^{\varepsilon'})$ provided that n is large enough, where $\varepsilon' > 0$ is a constant (related to K). Similar results hold for GL_n.*

Of course, all the results here hold for GW_n and EW_n, the only difference being that these are open chains, hence it does not make sence to talk about the knotting problem. But one can still discuss this up to local knotting and local knot patterns.

The auther thanks N. Pippenger and D.W. Sumners for helpful conversations.

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Dessert Example
by
Fred Tinsley

During the dessert party at Carolyn and David Wright's residence Bestvina, Walsh, and I constructed the following example of an inclusion map of manifolds which induces isomorphisms on homology groups and fundamental groups but fails to be a homotopy equivalence. Such examples are, of course, well-known. We claim neither originality nor insight. However, since there exists no collection, Counterexamples in Geometric Topology, it seems desireable to have such examples readily available.

The proof is essentially self-contained and depends only on a minimal understanding of the relative homology long exact sequence and duality.

The construction employs the Quillen Plus Construction (see Venema's characterization of knot complements, these proceedings) which very naturally yields the homology equivalence. A modicum of group theory also is needed.

<u>Example</u>: There is an inclusion map of manifolds $i:N \to W$ which induces isomorphisms on homology and fundamental groups but is not a homotopy equivalence.

<u>Fact 1</u>: Suppose (W,M,N) is a cobordism with the inclusion $i:N \to W$ a homotopy equivalence. Let $j:M \to W$ be inclusion. Then the kernel of the induced map $j_{\#}: \pi_1(M) \to \pi_1(W)$ is perfect.

Proof of Fact 1: Let $\tilde{W}$ denote the universal cover of W. This induces a triple $(\tilde{W},\tilde{M},\tilde{N})$ where $\tilde{N}$ is the universal cover of N and $\tilde{M}$ is the cover of M corresponding to $\ker(j_{\#})$. Now $\tilde{i}: \tilde{N} \to \tilde{W}$ is a proper homotopy equivalence so the relative cohomology (with compact supports) is trivial, ie, $H_C^*(\tilde{W},\tilde{N}) = 0$. By duality, we have $H_*(\tilde{W},\tilde{M}) = 0$. In particular, $H_1(\tilde{M}) = H_1(\tilde{W}) = 0$ so $\ker(j_\#)$ is perfect.∎

Let G be the group presented by $\langle x,y \mid y=y^{-1}x^{-1}yx \rangle$.

Fact 2: G has no non-trivial perfect subgroups.

Proof of Fact 2: Let K be the obvious 2-complex with one 0-cell, 2 1-cells, and 1 2-cell with $\pi_1(K) = G$. Let $\tilde{K}$ be the infinite cyclic covering corresponding to the homomorphism $G \xrightarrow{\varphi} \mathbb{Z}$ with $y\varphi = 0$ and $x\varphi = 1$. Then $\pi_1(\tilde{K}) = \ker(\varphi)$ has a presentation given naturally by the cell structure of $\tilde{K}$:

$$\langle \ldots, y_{-1}, y_0, y_1, y_2, \ldots \mid \ldots, y_{-1}^2 = y_0, y_0^2 = y_1, y_1^2 = y_2, \ldots \rangle$$

where the image of y_k in $\pi_1(K)$ is the conjugate $x^{-k}yx^k$. Also, $\ker(\varphi)$ is the commutator subgroup of G and, and therefore, must contain any perfect subgroup of G. But $\ker(\varphi)$ is abelian (the direct limit of embeddings $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$) and so contains no non-trivial perfect subgroups.∎

Let M be your favorite n-manifold ($n>4$) with $\pi_1(M) = G$. Let y^* be a loop in M representing y. Then y^* bounds a disk with one handle, T, in M with spanning curves homotopic to representatives of y and x respectively (use the relation in the fundamental group of M and general position). Attach a 2-handle to $y^*\times\{1\}$ in $M\times[0,1]$ to obtain a cobordism (W',M,M'). It follows that $\pi_1(M') = \mathbb{Z}$ and the inclusion $M' \to W'$ induces an isomorphism on fundamental groups. In particular, the spanning curve of T homotopic to y* bounds in M' a disk, D, which intersects the belt sphere of the 2-handle exactly once.

We identify the singular image of a map of S^2 into M': Take $T\times\{1\}$ together with the core of the 2-handle together with the disk, D. Then D is the image of two mutually disjoint disks in S^2. The algebraic intersection of these two with the belt sphere of the 2-handle is zero. As a result the algebraic intersection of this map of S^2 into M' is ± 1 (depending on orientation). General position gives an embedded 2-sphere which algebraically cancels the 2-handle. Attach a 3-handle to M' along this sphere to obtain the desired cobordism (W,M,N).

By construction, $H_*(W,M) = 0$. So by duality, $H^*(W,N) = 0$. Also, the universal coefficient theorem for cohomology yields that $H_*(W,N) = 0$. Finally, also by construction, $i:N \to W$ induces an isomorphism on fundamental groups. But $i: N \to W$ cannot be a homotopy equivalence by Facts 1 and 2. In particular, $\ker(j_\#)$ is not perfect.

<u>Note</u>: Venema used the Plus Construction on a knot complement whose fundamental group abelianizes to $\mathbb{Z}$ with perfect kernel. In fact, the kernel is equal to the intersection of the derived series of the fundamental group of the knot complement (the wild group, per Cannon). . However, in the example above $\ker(j_\#)$ is equal to the intersection of the lower central series of G (ie, the "omegatators" of G, per McMillan).

TANGLE EQUATIONS AND DEHN SURGERY

by Claus Ernst

1. Introduction

This paper deals with mathematical problems which arise in a topological model for enzyme mechanisms in DNA recombination experiments, [SE]. The mathematics which can be used to model this 2-strand interaction is that of 2-string tangles. When bound to a circular DNA substrate, the enzyme naturally seperates the DNA molecule into two complementary tangles. Enzyme action on circular DNA can be viewed as tangle surgery, the action of the enzyme is to delete one of these tangles, replacing it by another. This leads to equations, where on one side we have a sum of tangles, while on the other we have a known knot or link. The goal is to solve these equations for the unknown tangles.

In general solving tangle equations is a difficult task, and only special cares are mentioned here.

2. Background

A 2-string tangle (or just tangle) is a pair (B,t), where B is a 3-ball and t is a pair of (unoriented) arcs properly embedded in B [C,L1]. A tangle is rational if there exist a homeomorphism of pairs from (B,t) to the trivial tangle $(D^2xI, \{x,y\}xI)$, where D^2 is the unit 2-ball in R^2 and {x,y} are points interior to D^2. A tangle is locally knotted if there exist a local knot in one of its strands, that is, there exists a 2-sphere in B meeting t tranversly in 2 pts., and such that the 3 ball it bounds in B meets t in a knotted spanning arc. A tangle is prime if it is neither rational nor locally knotted.

In order to compare tangles, we need to think of them as having "the same" boundary. As in [BoS], we define a model 2-sphere S^2 in R^3 to be the boundary at the unit 3-ball D^3 in R^3, equipped with 4 distinguished equatorial points P = {NE, SE, SW, NW}. We require that every tangle comes equipped with a boundary parametersation, that is, a homeomorphism $\phi : (\partial B, \partial t) \rightarrow (S^2, P)$. So a tangle is a triple $B = (B, t, \phi)$. Two tangles $B = (B, t, \phi)$ and $B' = (B', t', \phi')$ are isomorphic if there is a homeomorphism $H : (B, t) \rightarrow (B', t')$ such that $\phi = \phi' H$ on ∂B. We write $B = B'$.

Given two tangles {A,B}, we define tangle addition as shown in Figure 1, and denote the result by A + B. Note that A + B may contain a simple closed curve, in which case A + B is not a 2-string tangle. The numerator construction applied to a tangle A is shown in Figure 2. Note that the knot (link) N(A+B) is topologically equivalent to that obtained by glueing A to B along their "common" S^2-boundary.

Rational tangles admit very nice classification schemas [C, ES]. There exists a 1-1 correspondence between isomorphism classes of rational tangles and the extended rational numbers $\beta/\alpha \in Q \cup \{1/0 = \infty\}$, where $\alpha \in N \cup \{O\}, \beta \in Z$ and $\gcd(\alpha, \beta) = 1$. If A and B are rational tangles, then N(A+B) yields an unoriented 4-plat (2 bridge knot or link) [BZ]. The 2-fold branched cover of a rational tangle is a solid torus , see Figure 3.

So N(A+B) has as 2-fold branched cover the Lens space $L(\alpha, \beta)$ obtained as the union of 2 solid tori. The 4 plat covered by the Lens space $L(\alpha, \beta)$ is denoted as $b(\alpha, B)$.

Two 4 plats $b(\alpha, \beta)$ and $b(\alpha', \beta')$ are equivalent iff $\alpha = \alpha'$ and $\beta^{\pm'} \equiv \beta' \pmod{\alpha}$, $[BZ]$. The numbers α and β of $b(\alpha, \beta)$ are standard if $0 < \beta < \alpha$. There are two exceptions. The unknot b(1,0) is covered by S^3 and the unlink of two components b(0,1) is covered by $S' \times S^2$.

4-plats and rational tangles are closely related via the numerator construction. If β/α is a rational tangle with $\beta/\alpha \geq 1$ then $N(\beta/\alpha) = b(\beta, -\alpha)$.

Tangle equations involving only rational tangles are very well understood. In [ES], we prove the following.

Ω THEOREM 1 Let $A_1 \neq A_2$ be rational tangles, and K_1 and K_2 be 4-plats. There are at most 2 distinct rational tangle solutions to the equations

$N(X + A_1) = K_1$

$N(X + A_2) = K_2$

This theorem is sharp as can be seen by the following example

$$A_1 = 1/3, A_2 = 51/7, K_1 = b(5,3) \text{ and } K_2 = b(29,17).$$

The two solutions for X are X = -70/239 and X = -75/254.

It may happen that two equations of the above form have no solutions of any kind (prime, rational, or locally knotted) as we will see.

3. Tangle equations and Dehn Surgery

Let X be a prime tangle with two fold branced cover X'. Then X' is a compact connected, irreducible, orientable 3 manifold with $\partial X'$ a torus. Let A be a rational tangle. Then in equation $N(X+A) = b(\alpha_1 \beta_1)$

gives rise to a decomposition of $L(\alpha,\beta) = X' \cup_F A'$, where A' the two fold branched cover of A is a solid torus and F is a glueing man from $\partial A' \to \partial X'$. In other words $L(\alpha,\beta)$ is obtained by surgery on X'. In the following we will use this to derive necessary algebraic conditions for two tangle equations to have solutions. The next lemma is a generalization of a result of Lickorish [L2].

LEMMA

Let X be any tangle, T and β/α be rational tangles, and b(p,q) be a 4-plat, such that N(X + T) = [1] and $N(X+\beta/\alpha) = b(p,q)$

(1) If T = ∞ then L(p,q) can be obtained by $(\beta+s\alpha)/\alpha$ surgery along a knot in S^3, where s is an integer and p = $\pm$ $(\beta+s\alpha)$.

(2) If T = (0) then L(p,q) can be obtained by $(\alpha+s\beta)/\beta$ surgery along a knot in S^3, where s is an integer and p = $\pm(\alpha + s\beta)$.

Proof: The 2-fold branched cyclic cover X' of X is a knot complement, and the 2-fold branched cover T' of T is a solid torus. For the moment let us assume T = ∞. Then the arcs NW to SW and SW to SE on ∂T lift respectively to a meridian μ' and a longitude λ' on $\partial T'$. The first equation implies $X \cup_g (\infty) = S^3$, where $g : \partial(\infty) \to \partial X$ is a glueing map. Lifting to 2-fold branched covers we have $X' \cup_{g'} T' = S^3$, where $g' : \partial T' \to \partial X'$. Choose a meridian $\mu = g'(\mu')$ and a longitude λ on $\partial X'$. Then λ is isotopic to the curve $g'(\mu') + sg'(\lambda')$ for some integer s. There exist [Mo] orientation preserving homeomorphisms ψ and F, where

(i) F maps the ∞ tangle to the β/α tangle.

(ii) $\psi : \partial T' \to \partial T'$ sends the meridian μ' to a curve isotopic to $\beta\mu' + \alpha\lambda'$,

(iii) The maps $F|_\partial$ and ψ commute with the covering map $p|_\partial : \partial T' \to \partial T$, that is

$(p|_\partial)(F|_\partial) = \psi(p|_\partial)$.

Using the second equation $N(X + \beta/\alpha) = b(p,q)$, the 2-fold branched cover L(p,q) of b(p,q) can be constructed as $X' \cup_{g'\psi} T'$. The glueing map $g'\psi : \partial T' \to \partial X'$ maps μ' to a curve isotopic to $(\beta + s\alpha)\mu + \lambda$. Hence L(p,q) is obtained by $(\beta+s\alpha)/\alpha$ surgery on the knot complement X'. $H_1(L(p,q)) = Z_p$ is generated by the meridan μ, so p = $\pm(\beta + s\alpha)$. The result for the case T = (0) is proved in the same way. The only difference is that the map ψ sends the meridan $\alpha\mu'$ (the lift of the SW SE arc) to a curve isotopic to $\mu' + \beta\lambda'$.

THEOREM 2 Let X be any tangle, T and β/α are rational tangles, and let b(p,q) be a 4-plat, where $N(X+T) = [1]$ and $N(X+\beta/\alpha) = b(p,q)$. If T = ∞ then $q \equiv \pm\alpha t^2$ (mod p) for some integer t. If T = (0) then $q \equiv \pm\beta t^2$ (mod p) for some integer t.

Proof: Let us recall the following facts:

(i) L(p,q) is obtained by p/q surgery on the unknot in S^3

(ii) Suppose M is a 3-manifold and $H_1(M) = Z_p$. If M is obtained by p/q surgery on a knot k in S^3, then the linking form

$L : H_1(M) \times H_1(M) \to Q/Z$ is such that L(g,g) = q/p were g is a generator of $H_1(M)$ representing a meridian of the knot k [L2].

By (i) and (ii) there exist a generator ξ of $H_1(L(p,q))$ such that $L(\xi,\xi) = q/p$. By Lemma 3.6 and (ii) there exists a generator ς of $H_1(L(p,q))$ such that $L(\varsigma,\varsigma) = \alpha/(\beta+s\alpha) = \pm\alpha$ p if $T = \infty$ and $L(\varsigma,\varsigma) = \beta/(\alpha+s\beta) = \pm\beta$/p if $T = (0)$. $H_1(L,(p,q))$ is cyclic, so $\xi = t\varsigma$ for some integer t, and q/p = $t^2 L(\varsigma,\varsigma)$ in Q/Z.

The following corollary (and proof) are due to M. Boileau.

Corollary (Boileau) If T is either ∞ or (0), then there is no tangle X which satisfies the equations

(i) $N(X+T) = [1]$ *and*

(ii) $N(X + (\pm 1)) = b(8,5)$.

Proof: This follows from Theorem 3.7 using $\alpha = \beta = \pm 1, p = 8$ and $q = 5$, since $5 \equiv \pm t^2$ (mod 8) has no solution for t.

Theorem 3 Let X be any tangle, β_1/α_1 and β_2/α_2 are rational tangles, and let $b(p_1,q_1)$ and $b(p_2,q_2)$ be 4-plates, where $N(X+\beta_1/\alpha_1) = b(p_1,q_1)$ and N(X + $\beta_2/\alpha_2) = b(p_2,q_2)$ If $|\alpha_1\beta_2 - \beta_1\alpha_2| > 1$ then X is a Seifert Fiber space.

Proof: Let X', T_1 and T_2 be the 2-fold branched cover of X, $\beta_1/_1$ and β_2/α_2, respectively. Using the parametersation of the tangles we can assume $\partial(\beta_1/\alpha_1) = \partial(\beta_2/\alpha_2) = S^2$ (the unit sphere in R^3) and $\partial T_1 = \partial T_2 = \partial T$. Then the arcs NW to SW and SW to SE on S^2 lift respectively to a meridian μ' and a longitude λ' on ∂T.

There exists [Mo] orientation preserving homeomorphisms ψ_i and F_i i = 1, 2 where

(i) F_i mans the ∞ tangle to the β_i/α_i tangle.

(ii) $\psi_i : \partial T \to \partial T$ sends a meridian μ' and a longitude λ' to curves isotopic to the elements given by matrix multiplication

$$(\mu'\lambda')\begin{pmatrix} \beta_i & \alpha_i' \\ \alpha_i & \beta_i' \end{pmatrix} = (\beta_i\mu' + \alpha_i\lambda',\ \alpha_i'\mu' + \beta_i'\lambda') \text{ where } \beta_i\beta_i' - \alpha_i\alpha_i' = 1.$$

(iii) The mans $F_i|\partial$ and ψ_i commute with the covering man

$p|_\partial \partial T \to \partial(\beta_i/\alpha_i)$.

Therefore $F = F_2 F_1^{-1}$ sends the β_1/α_1 tangle to the B_2/α_2 tangle. The lift $\lambda = \psi_2\,\psi_1^{-1} : \partial T \to \partial T$ sends μ' to a curve isotopic to $(-\beta_1'\beta_2 + \alpha_1'\,\alpha_2)\,\mu' + (\alpha_1\beta_2 - \beta_1\alpha_2)\lambda'$.

The first equation implies $X \cup_g (\beta_1/\alpha_1) = S^3$, where $g : \partial(\infty) \to \partial X$ is a glueing man. Lifting to 2-fold branched covers we have

$$X' \cup_{g'} T_{\prime} = L(p_1, q_{\prime}) \text{ where } g' : \partial T \to \partial X'$$

Choose a meridian $\mu = g'(\mu')$ and a longitude $\lambda = g'(\lambda')$ *on* ∂X.

Thus $L(p_{\prime}q_{\prime})$ is obtained from X' by surgery sending μ' to μ.

Using the second equation $N(X + \beta_2/\alpha_2) = b(p_2, q_2)$, the two fold cover $L(p_2, q_2)$ can be constructed as $X' \cup_{g'\lambda} T_2$. The glueing man $g'\lambda : \partial T \to \partial X'$ maps μ' to a curve isotopic to $C = (-\beta_1'\ \alpha_2\ + \alpha_2'\ \alpha_2)\ \mu + (\alpha_1\ \beta_2\ - \beta_1\ \alpha_2)\lambda$.

If $|\alpha_1\ \beta_2\ - \beta_1\ \alpha_2| > 1$ then the minimal intersection number between C and μ is greater one. The Cyclic Surgery Theorem [CG] implies that X' is a Seifert Fiber Space.

In order to solve the equations of Theorem 3 under the conditions $|\alpha_{\prime}\ \beta_2\ - \alpha_2\beta_1| > 1$ it would be useful to know that X is a Montesinos tangle, that is a tangle made out of national tangles as shown, in Figure 4.

If X' is a Seifert Fiber space then adding a torus to it can only increase the number of exceptional fibers. Since $X' \cup T_i$ is a Lens space, X' can have at most two exceptional fibers and its surface it a disk. This leads to the following question. Suppose X' is a Seifert Fiber Space with two exceptional fibers and orbit surface a disk. If X' is a 2-fold branched cover of a two string locally unknotted tangle X, is X the partial sum of two national tangles? The answer is not know to the author, but a yes is conjectured.

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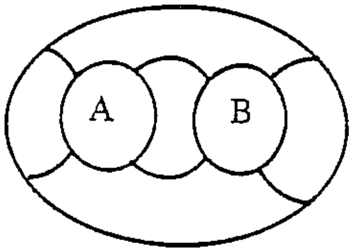

Figure 1 Tangle addition A + B

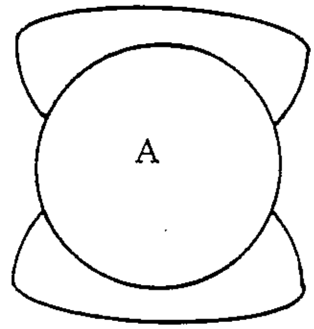

Figure 2 Numerator N(A)

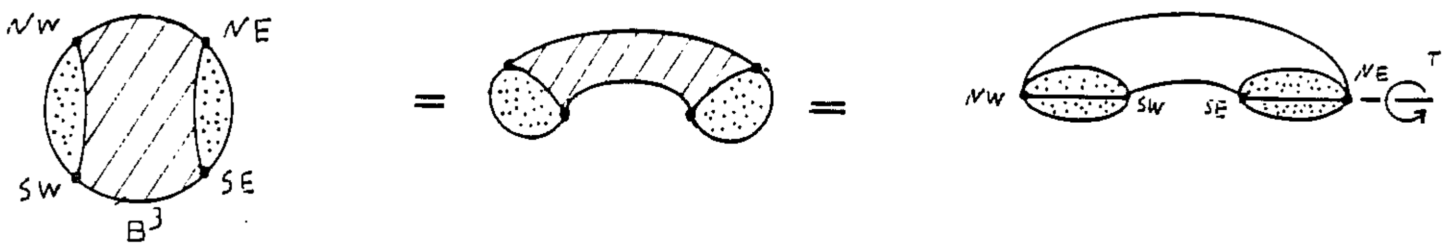

Figure 3 The 2-fold branched cover of a rational tangle

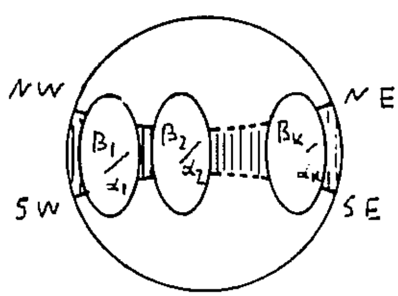

Figure 4 Montesinos Tangle

BRAIDS, GRAPHS AND REPRESENTATIONS

by

STEPHEN P. HUMPHRIES

§1 INTRODUCTION For $n > 1$ let B_n be the group of braids on n strings. Then B_n has a presentation as a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } 1 \leq i,j \leq n-1 \text{ and } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } 1 \leq i < n-1 .$$

It is also well-known that B_n has a faithful representation in Aut(F(n)), the group of automorphisms of the free group F(n) of rank n. If $x_1, \ldots, x_n$ are fixed free generators for F(n), then the action of B_n on F(n) is given by the following actions of the generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n on the generators $x_1, \ldots, x_n$ of F(n) :

$$\sigma_i(x_j) = \begin{cases} x_j & \text{if } j \neq i,\ i+1, \\ x_{i+1} & \text{if } j = i, \\ x_{i+1}{}^{-1}x_ix_{i+1} & \text{if } j = i+1. \end{cases}$$

It is easy to check that with this action the word $x_1x_2 \ldots x_n$ is fixed and that if $\alpha \in B_n$, then $\alpha(x_i)$ is a conjugate of some x_j (we call such a word $\alpha(x_i)$ a simple word). For proofs of these results and more information on braid groups see ([Bi], [Ma]).

In this paper we associate to each braid σ a certain kind of graph which completely determines σ and study the combinatorics of these graphs. We use these graphs to show (i) that certain words in F(n) are never subwords of freely reduced simple words; (ii) that certain representations of B_n are faithful; (iii) that there is a way of associating to every n-braid an (n-1)-braid; and (iv) to give a new normal form for braids.

$2. GRAPHS In this section we describe the graphs which we can associate to braids. All words referred to will be words in F(n) having $x_1, x_2, \ldots, x_n$ and their inverses as letters. Note that if we can define a pairing π of the letters of such a word $w = w_1 w_2 \ldots w_m$ such that $\pi(\pi(w_i)) = w_i$ for all $1 \leq i \leq n$, then this defines a graph having $w_1, \ldots, w_m$ as vertices and an edge between two letters w_i and w_j if and only if $\pi(w_i) = w_j$. Our graphs will all be constructed in this way.

1. The conjugacy graph Let $\alpha \in B_n$ and suppose that $\alpha(x_i) = y_i z_i y_i^{-1}$, and that $w = \alpha(x_1 x_2 \ldots x_n) = y_1 z_1 y_1^{-1} y_2 z_2 y_2^{-1} \ldots y_n z_n y_n^{-1}$, where each $z_i \in \{x_1, x_2, \ldots, x_n\}$, then there is a pairing of the letters of the subwords $y_1, y_2, \ldots, y_n$ in w with the letters of the subwords $y_1^{-1}, y_2^{-1}, \ldots, y_n^{-1}$ in which a letter is paired with its inverse letter in the obvious way. Further, each z_i is paired with itself. This pairing determines the graph $C(\alpha)$ for such a word w, where each z_i is an isolated vertex.

2. The free reduction graph For $\alpha \in B_n$ let $w = w(\alpha)$ be the word that we obtained above and let $w_0 = w, w_1, w_2, \ldots, w_q = x_1, \ldots, x_n$ be words such that

w_i is obtained from w_{i-1} by deleting a pair of adjacent letters of the form $x_ix_i^{-1}$ or $x_i^{-1}x_i$. Keeping track of how the letters of w cancel gives a pairing of the letters of w which allows us to construct the graph FR(α).

Assume that $w = y_1z_1y_1^{-1}y_2z_2y_2^{-1} \ldots y_nz_ny_n^{-1}$. Then CFR($\alpha$) will denote the union of the graphs C(α) and FR(α). To be specific we will think of CFR(α) as lying in the plane $R^2 \subset S^2$ with its vertices on the x-axis X, the graph C(α) being drawn above X while the graph FR(α) is drawn below. Notice that this can be done in such a way that the graph is planar. We will always think of CFR(α) in this way. The basic property of CFR(α) is indicated in the next result:

<u>Proposition 2.1</u> Let $\alpha \in B_n$. Then CFR(α) has exactly n components.

<u>§3 THE (n-1)-BRAIDS ASSOCIATED TO n-BRAIDS</u> Let $\sigma \in B_n$ and assume that $\sigma(x_i)= y_iz_iy_i^{-1}$ in freely reduced form as in §2. If $z_i \neq x_n$, then we let w_i be the largest subword of $y_iz_iy_i^{-1}$ which does not contain x_n or x_n^{-1} and which is symmetric with respect to z_i. Thus for example, if n=4 and $y_iz_iy_i^{-1}=x_3x_4^{-1}x_3^{-1}x_2x_3x_4x_3^{-1}$, then $w_i=x_3^{-1}x_2x_3$. This gives n-1 words each of which is a conjugate of some x_i, $i \neq n$.

<u>Theorem 3.1</u> In some order the product of $w_1, \ldots, w_{n-1}$ is equal to $x_1x_2 \ldots x_{n-1}$ and they determine an (n-1)-braid.

Call this particular (n-1)-braid $(\sigma)_n$. We can do the same thing with respect to x_1 and similarly obtain another (n-1)-braid which we call $(\sigma)_1$. We first note that σ is equal to the identity braid if and only if both $(\sigma)_1$ and $(\sigma)_n$ are equal to the identity braids. We use this idea to give another 'normal form' for elements of B_n. Specifically this is

$$\sigma = (\sigma)_n{}^{-1}((\sigma)_n)_1{}^{-1}(((\sigma)_n)_1)_n{}^{-1}((((\sigma)_n)_1)_n)_1{}^{-1} \dots .$$

Since each $(\sigma)_k$, k=1,n, is an (n-1)-briad, we proceed by induction to give our normal form (the case n=2 is simple).

<u>$4 REPRESENTATIONS</u> Using the graph CFR(α) we can prove the following result of Birman and Hilden [Bi-Hi]:

<u>Theorem 4.1</u> Let N_k be the normal closure in F(n) of the elements $x_1{}^k, \dots ,x_n{}^k$. Then N_k is B_n-invariant and the composite homomorphism $B_n \dashrightarrow Aut(F(n)) \dashrightarrow Aut(F(n)/N_k)$ is injective.

Let M_k be the kernel of the canonical projection $F(3) \dashrightarrow Z_k = \langle x \mid x_k \rangle$, $x_1,x_2,x_3 \dashrightarrow x$. Note that M_k is B_3-invariant. Let M_k' be the commutator subgroup of M_k. Using properties of CFR(α) we can also prove:

<u>Theorem 4.2</u> The representation $B_3 \dashrightarrow Aut(F(3)/M_k')$ is faithful if and only if k is even.

Let G_n be the normal subgroup generated by the elements $x_1^2, x_2^2, \ldots, x_n^2, (x_1 x_2 \ldots x_n)^2$. Note that G_n is B_n-invariant and that the representation $B_n \dashrightarrow Aut(F(n)/G_n)$ is not faithful since the square of the generator of the centre of B_n belongs to the kernel. However we prove

Theorem 4.3 The representation $B_n \dashrightarrow Aut(F(n)/G_n')$ is faithful.

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Characterization of knot complements in the 4-sphere: a special case[1]

Vo Thanh Liem and Gerard A. Venema[2]

In [4], knot complements in S^4 are characterized as follows:

THEOREM 1. *Let W be a connected open subset of S^4. Then W is homeomorphic with $S^4 - K$ for some locally flat 2-sphere $K \subset S^4$ if and only if*

(1.1) $H_1(W) \cong \mathbb{Z}$, *and*

(1.2) *W has one end ϵ with $\pi_1(\epsilon)$ stable and $\pi_1(\epsilon) \cong \mathbb{Z}$.*

The purpose of this note is to sketch the proof of a special case of Theorem 1. The proof of the general case is indirect and relies on a great deal of algebraic machinery, so it seems worthwhile to give a direct, geometric argument which works in at least some nontrivial cases. The proof given here is similar to the "plus construction" (cf. [6] and [2, §11.1]). Before stating the special case, we establish some notation and make a definition.

NOTATION: $W \subset S^4$ is always an open subset of the 4-sphere which satisfies (1.1) and (1.2). Define $\Sigma = S^4 - W$. We let $f : \pi_1(W) \to H_1(W)$ be the natural (Hurewicz) homomorphism and $\Delta = \ker f$.

DEFINITION: A group G is perfect if G is equal to its own commutator subgroup.

We can now state the special case we intend to prove here.

THEOREM 2. *If W is an open subset of S^4 which satisfies (1.1) and (1.2) and if Δ is a perfect group, then $W \cong S^4 - K$ for some locally flat 2-sphere $K \subset S^4$.*

In order to get some feeling for how strong the assumption of perfection of Δ is, recall that a classical knot in S^3 satisfies this condition if and only if its Alexander polynomial is trivial [1].

The proof of Theorem 2, like that of Theorem 1, is based on a result of Guilbault.

THEOREM 3. (Guilbault [3]) *If W is a connected open subset of S^4 such that*

(3.1) $\pi_1(W) \cong \mathbb{Z}$, *and*

(3.2) *W has one end ϵ with $\pi_1(\epsilon)$ stable and $\pi_1(\epsilon) \cong \mathbb{Z}$,*

then there exists a compact set $N \subset S^4$ such that $N \cong S^2 \times B^2$ and $N \cap W \cong \partial N \times [0,1) \cong S^2 \times S^1 \times [0,1)$.

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[1] The transcript of a talk presented by G. Venema.

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Theorem 3 differs in two ways from the theorem actually stated in [**3**]. First, the existence of N is not mentioned explicitly in the conclusion of [**3**, Theorem 4.3], but we are merely being specific about what is actually proved in [**3**]. Second, the hypothesis (3.1) replaces the apparently stronger hypothesis that W has the homotopy type of S^1. But a duality argument similar to that in [**5**] shows that (3.1) and (3.2) imply that the higher homotopy groups of W vanish and thus W has the homotopy type of S^1. (An alternative proof of this fact can be based on [**7**, Proposition 3.3]; see, e.g., [**4**, Theorem 3] or [**2**, Proposition 11.6C(1)].)

PROOF OF THEOREM 2: First note that $\pi_1(W)$ is finitely generated (because of the fact that $\pi_1(\epsilon)$ is finitely generated). This in turn implies that Δ is the normal closure of a finite set (namely the commutators of the generators of $\pi_1(W)$). Let $\ell_1, \ldots, \ell_n$ be locally flat PL embedded loops which represent this finite set. There is a collection $D_1, \ldots, D_n$ of disjoint locally flat PL disks such that $\ell_i = \partial D_i$. (Use finger moves to push any singularities off edges.) These disks determine natural framings for the loops. Using these framings, do surgery to W along the loops. Specifically, a regular neighborhood of ℓ_i is homeomorphic to $S^1 \times B^3$; remove the interior of such a regular neighborhood of each ℓ_i and glue in copies of $B^2 \times S^2$. The result is a new open set, called W_1, which has the same end as W. Since each of the surgeries corresponds to adding a connected summand of $S^2 \times S^2$ to S^4, we have $W_1 \subset M$ where M is homeomorphic to the connected sum of n copies of $S^2 \times S^2$. Furthermore, since the loops $\ell_1, \ldots, \ell_n$ normally generate Δ, we have that $\pi_1(W_1) \cong \mathbb{Z}$.

The second step is to do 2-dimensional surgery to M in order to get back to S^4 where we can apply Theorem 3. In order to do so we will take the natural $S^2 \vee S^2$'s which generate $H_2(M)$ and use [**2**] to re-embed them in W_1. Consider one loop ℓ_i. A regular neighborhood of ℓ_i has been replaced by a copy of $B^2 \times S^2$. Let A_i be the 2-sphere $(B^2 \times \{*\}) \cup D_i$ and let B_i be the 2-sphere $\{0\} \times S^2$. Notice that B_i misses Σ but that A_i likely intersects Σ. Since Δ is perfect, ℓ_i must bound a disk-with-handles D_i' in W such that each loop on D_i' represents an element of Δ. By general position we may also assume that $D_i' \cap \ell_j = \emptyset$ and, by piping, that $D_i' \cap D_j' = \emptyset$ for $i \neq j$. Form A_i' from A_i by replacing D_i with D_i'. We have then represented the homology class of the 2-sphere A_i by an embedded orientable surface A_i' which is disjoint from Σ. Consider a symplectic basis for $H_1(A_i')$. Each element of this basis is null-homotopic in W_1. Use singular disks representing one half of this basis to surger A_i' and replace it with a singular 2-sphere $A_i'' \subset W_1$.

We use λ to represent intersection numbers and μ to represent self-intersection numbers. (Note: it is important to remember that both are measured in $\mathbb{Z}[\pi_1(W_1)]$ — see [**2**, §1.7]). We claim that the family of singular 2-spheres $\{A_i'', B_j\}$ satisfies

$$\lambda(A_i'', B_j) = \delta_{ij}$$

$$\lambda(A_i'', A_j'') = \lambda(B_i, B_j) = \mu(A_i'') = \mu(B_j) = 0$$

for every i and j. Once that claim has been verified, we are finished because [**2**, Theorem 5.1A] allows us to replace each $A_i'' \cup B_i$ with a locally flat embedded $S^2 \vee S^2$ in W_1. A tubular neighborhoods of each $S^2 \vee S^2$ is then removed and replaced with a 4-ball. This makes M back into S^4 and changes W_1 into an open set $W_2 \subset S^4$ such that $\pi_1(W_2) \cong \mathbf{Z}$ and $S^4 - W_2 = M - W_1 = S^4 - W = \Sigma$. We then apply Theorem 3 to $W_2 \subset S^4$. The compact set N given by Theorem 3 only intersects W_2 in a collar of the end and both the 1- and 2-surgeries could be done outside this collar, so $N \subset S^4$ and $N \cap W \cong S^2 \times S^1 \times [0,1)$. The 2-sphere K in the conclusion of Theorem 2 is the core of N.

To prove the claim we show that the excess intersection points of $A_i'' \cap A_j''$ and $A_i'' \cap B_j$ can be paired off in such a way that each pair has a singular Whitney disk in W_1. Let x and y be a pair in the symplectic basis for A_i'. The curves x and y bound singular disks D_x and D_y, respectively, in W_1. Let us say that, in constructing A_i'', we added two copies of D_x to A_i'. The excess intersection points will arise because of points of $D_x \cap A_j''$ or $D_x \cap B_j$. These points are naturally paired since two copies of D_x were used in forming A_i''. A singular Whitney disk for such a pair is constructed from a thin disk following an arc in D_x from the intersection point over to the point $x \cap y$ together with a copy of D_y. ∎

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Whitehead Contractible n-manifolds for $n > 3$

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This paper is the summary of an expository talk given at the Sixth Annual Western Workshop on Geometric Topology held at Brigham Young University, Provo, Utah on July 27-29, 1989.

Introduction. The purpose of this note is to generalize a 3-dimensional construction of Whitehead [9] to obtain contractible open n-manifolds of dimension n > 3 that are not homeomorphic with Euclidean n-space. There are already many examples of such contractible manifolds of dimension n > 3 [1], [2], [3], [6]. Hence, one may wonder why such manifolds should be of interest. However, these manifolds are relatively easy to describe, and it is hoped that it will be possible to show that such manifolds cannot be covering spaces as was shown by Myers [5]in dimension 3 for Whitehead's contractible 3-manifold as well as for many other contractible 3-manifolds as described by McMillan [4]. (Added September 1989. The author has subsequently been able to show that these Whitehead contractible n-manifolds cannot non-trivially cover any manifold.)

LEMMA 1. *Let A, B be manifolds with boundary of the same dimension so that $A \subset \text{Int}\, B$ and B - $\text{Int}\, A$ is a manifold with boundary that is boundary incompressible; i.e., loops in the boundary of B - $\text{Int}\, A$ are essential in B - $\text{Int}\, A$ if and only if they are essential in the boundary of B - $\text{Int}\, A$. Then for any manifold M without boundary the pair $A \times M$, $B \times M$ has the same properties as the pari A, B; i.e., $A \times M$, $B \times$ M are manifolds with boundary of the same dimension so that $A \times M \subset \text{Int}\, B \times M$ and $(B \times M)$ - $\text{Int}\,(A \times M)$ is a manifold with boundary that is boundary incompressible.*

Proof. The boundary of B - $\text{Int}\, A$ equals $\text{Bd}\, A \cup \text{Bd}\, B$. The set $(B \times M)$ - $\text{Int}\,(A \times M)$ is the manifold $(B$ - $\text{Int}\, A) \times M$ whose boundary is $\text{Bd}\, A \times M \cup \text{Bd}\, B \times M$. Suppose γ is a loop in the boundary of $(B$ - $\text{Int}\, A) \times M$. Without loss of generality, we assume that γ lies in the set $\text{Bd}\, A \times M$. Since the fundamental group of the product is the product of the fundamental groups, we may assume that γ is equal to the product of loops α and β where α is a loop in $\text{Bd}\, A \times \{m\}$ and β is a loop in $\{a\} \times M$ where a and m are points in $\text{Bd}\, A$ and M, respectively. We also regard α, β as loops in $\text{Bd}\, A$ and M, respectively. If γ is trivial in $(B$ - $\text{Int}\, A) \times M$, then, by projection into M, we see that β is trivial in M. By projection into

B - Int A, we see that α is trivial in B - Int A. By boundary incompressibility of B - Int A, α is trivial in Bd A. Hence, γ is trivial in Bd $A \times M$, and we see that $(B \times M)$ - (Int $A \times M$) is boundary incompressible.

DEFINITION 2. A *solid n-torus* is a space homeomorphic to $B^2 \times S_1^1 \times S_2^1 \times \cdots \times S_{n-2}^1$ where B^2 is a 2-cell and each S_i^1 is homeomorphic to the 1-sphere S^1.

DEFINITION 3. A *3 -dimensional Whitehead link* is a solid 3-torus T_0^3 embedded in the interior of a solid 3-torus T^3 so that T_0^3 contracts in T^3 and T^3 - Int T_0^3 is a boundary incompressible 3-manifold with boundary. An example of a 3-dimensional Whitehead link is shown in Figure 1.

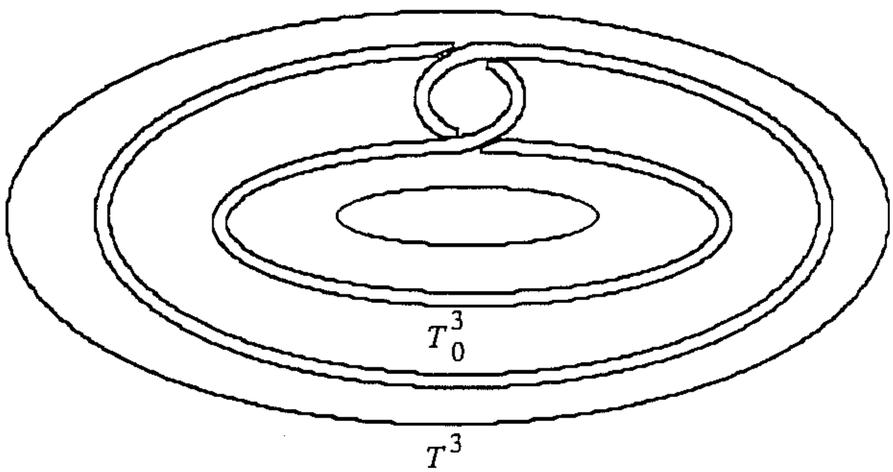

Figure 1

It is easy to see that T_0^3 contracts in T^3. It is well-known [8, Lemma 4.1] that the 3-manifold with boundary T^3 - Int T_0^3 is boundary incompressible.

We wish to describe an *n-dimensional Whitehead Link*, n > 3, by which we mean a solid n-torus T_0^n embedded in the interior of a solid n-torus T^n so that T_0^n contracts in T^n and T^n - Int T_0^n is a boundary incompressible n-manifold with boundary.

LEMMA 4. *There exist n-dimensional Whitehead links for $n > 3$.*

Proof. We show the existence of an n-dimensional Whitehead link by induction.

Let $T_0^k \subset T^k$, $k \geq 3$,be a k-dimensional Whitehead Link. We define $T^{k+1} = T^k \times S^1$. We set $T_1^{k+1} = T_0^k \times S^1$. Since T_0^k is a k-torus, we assume that $T_0^k = B^2 \times S_1^1 \times S_2^1 \times \cdots \times S_{k-2}^1$. Hence, we have that the solid $(k+1)$-torus $T_1^{k+1} = B^2 \times S_1^1 \times S_2^1 \times \cdots \times S_{k-2}^1 \times S^1$. Let P: $T_1^{k+1} \to B^2 \times S^1$ be the projection onto the first and last factors of T^{k+1}. Let T be a 3-dimensional Whitehead Link in $B^2 \times S^1$. We set $T_0^{k+1} = P^{-1}(T)$. We now show that T_0^{k+1} is a $(k+1)$-dimensional Whitehead Link in T^{k+1}.

Step 1. It is easy to verify that T_0^{k+1} is a solid $(k+1)$-torus embedded in the interior of the solid k-torus T^{k+1}.

Step 2. By Lemma 7.1 T^{k+1} - Int T_1^{k+1} and T_1^{k+1} - Int T_0^{k+1} are boundary incompressible $(k+1)$-manifolds with boundary. It follows that T^{k+1} - Int T_0^{k+1} is a boundary incompressible $(k+1)$-manifold with boundary.

Step 3. Let f_t be a contraction of the Whitehead link T in $B^2 \times S^1$, this induces a homotopy $F_t : T_0^{k+1} \to T^{k+1}$ so that F_0 is the inclusion map and F_1 maps into $T_0^k \times \{s\}$ where $s \in S^1$. But T_0^k is contractible in T^k; hence, any set in $T_0^k \times \{s\}$ is contractible in T^{k+1}. Therefore, T_0^{k+1} is contractible in T^{k+1}.

THEOREM 5. *For each $n \geq 3$ there is a Whitehead contractible n-manifold* $W^n = \bigcup_{i=0}^{\infty} T_i$ *where* T_i *is an n-dimensional Whitehead link in* T_{i+1}.

Proof. We simply form W^n as the direct limit of solid n-tori T_i so that T_i is an n-dimensional Whitehead link in T_{i+1}. It is now easy to check that W^n does, in fact, satisfy all the conditions for a Whitehead manifold.

McMillan [10] has shown that there are uncountably many contractible 3-manifolds each of which is the ascending union of 3-tori T_i so that T_i is a 3-dimensional Whitehead link in T_{i+1}.

QUESTION 6. *Are there uncountably many contractible n-manifolds each of which is the ascending union of n-tori* T_i *so that* T_i *is an n-dimensional Whitehead link in* T_{i+1}*?*

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PROBLEMS ABOUT FINITE DIMENSIONAL MANIFOLDS

Robert J. Daverman

What follows amounts, by and large, to an annotated combination of several lists I have been hoarding, expanding, polishing the last few years. It is highly personalized -- the title topic is far too extensive to allow treatment of all its various components, so I have not even tried. Instead, the combination identifies questions mainly in the areas of manifold structure theory, decomposition theory, and embedding theory. The more significant issues, and the one I prefer, tend to occur where at least two of these intersect, but admittedly several of the problems presented are light-hearted, localized, outside any overlap.

Before launching out into those areas named above, however, and mindful that the effort undoubtedly will invite disputation, I cannot resist stealing the opportunity to restate some of the oldest, most famous problems of this subject. Occasional reiteration spreads awareness, and this occasion seems timely, which is justification enough. Accordingly, well-versed readers should not expect to discover new material in the opening list of "Venerable Conjectures"; either they should skip it entirely or they can scan it critically for glaring omissions or wharever. Any other readers will benefit, I trust, by finding such a list in one convenient place.

The bibliography is intended as another convenience. Extensive but by no means complete, it is devised mainly to offer recent entry points to the literature.

At inception this project involved a host of mathematicians. Late one Oregon summer night during the 1987 Western Workshop in Geometric Topology, several people, including Mladen Bestvina, Phil Bowers, Bob Edwards, Fred Tinsley, David Wright (their names would have been protected if they were truly innocent), set out to construct a list of lesser known, intriguing problems deserving of wider publicity. They all made suggestions, and I kept the record. The evening's discussion led directly to a number of the problems presented here, which at one time constituted a separate list, but which in my tinkering I eventually grouped under topic headings. (No one else deserves any blame for my rearrangements.) If a question had strong support that evening for inclusion in the collection of "not-famous-enough problems", or if it just had marginal support with no major opposition, it shows up here preceded by an asterisk.

Other problem sets about finite dimensional manifolds published within the past decade should be mentioned. Here are a few. The most famous is Kirby's list(s) of low dimensional problems [K1] [K2]; the first installment is a bit old, but the second, put together after the 1982 conference of four-manifolds, includes a thorough update. Thurston [Th] has set forth some fundamental open problems about 3-manifolds and Kleinian groups. Much to my surprise, I could find no major collection focused on knot theory questions, although many such appear in Kirby's lists, and information arrived at press time about an extensive collection of braid theory problems edited by Morton [Mor]. Donaldson [Do] has raised some key 4-dimensional matters. In a more algebraic vein, Hsiang [Hs] has surveyed geqmetric applications of K-theory.

Finally, an acknowledgement of indebtedness to Mladen Bestvina, Marshall Cohen, Jim Henderson, Larry Husch, Dale Rolfsen, and Tom Thickstun for helpful comments and suggestions.

VENERABLE CONJECTURES

V1: Poincaré Conjecture.

V2: Thurston's Geometrization Conjecture. The interior of every compact 3-manifold has a canonical decomposition into pieces with geometric structure, in other words, into pieces with structure determined by a complete, locally homogeneous Riemannian metric. See [Th]. Of relatively recent vintage, this conjecture probably does not qualify as "venerable"; nevertheless, its boldness and large-scale repercussions have endowed it with stature clearly sufficient to support its inclusion on any list of important topological problems. It fits here in part by virtue of being stronger than the Poincaré Conjecture. A closely related formulation posits that every closed orientable 3-manifold can be expressed as a connected sum of pieces which are either hyperbolic, Seifert fibered, or Haken (i.e., contains some incompressible surface and each PL 2-sphere bounds a 3-ball there).

*V3: Hilbert-Smith Conjecture. No p-adic group can act effectively on a manifold. Equivalently, no compact manifold M admits a self-homeomorphism h such that (i) each orbit $\{h^n(x)\}$ has small diameter in M and (ii) $\{h^n | n \in Z\}$ is a relatively compact subgroup of the group of all homeomorphism $M \to M$.

V4: PL Schoenflies Conjecture. Every PL embedding of the $(n-1)$-sphere in R^n is PL standard, or equivalently, has image bounding a PL n-ball. The difficulty is 4-dimensional: if true for $n=4$ then the conjecture is true for all n.

V5. There is no topologically standard but smoothly exotic 4-sphere. This is the 4-dimensional Poincaré Conjecture in the smooth category, and an affirmative answer implies the truth of #V4. In broader terms Donaldson [Do] has asked which homotopy types of closed 1-connected 4-manifolds contain distinct smooth structures; specifically, do there exist homotopy equivalent but smoothly inequivalent manifolds of this type such that the positive part of the intersection form on 2-dimensional homology is even-dimensional?

V6: A problem of Hopf. Given a closed, orientable manifold M, is every (absolute) degree one map $f: M \to M$ a homotopy equivalence? Hausmann [Ha] has split this problem into component questions and has provided strong partial results: (1) must f induce fundamental group isomorphisms? and if so, (2) must f induce isomorphisms of $H_*(M; \mathbb{Z}\pi)$?

V6'. Hopf's problem led to the concept of Hopfian group, namely, a group in which every self-epimorphism is 1-1 . Does every compact 3-manifold have Hopfian fundamental group? Yes, if Thurston's Geometrization Conjecture is valid [He].

V7: Whitehead Conjecture [Wh]. Every subcomplex of an aspherical 2-complex is itself aspherical.

V7'. If K is a subcomplex of a contractible 2-complex, is $\pi_1(K)$ locally indicable (i.e., every nontrivial, finitely generated subgroup admits a surjective homomorphism to $\mathbb{Z}$; groups with this property are sometimes called locally $\mathbb{Z}$-representible). As mentioned in Howie's useful survey [Ho], an affirmative answer implies the Whitehead Conjecture.

V8: Borel Rigidity Conjecture. Every homotopy equivalence $N \to M$ between closed, aspherical manifolds is homotopic to a homeomorphism. Evidence in favor of this rigidity has been accumulating; see for example the work of Farrell-Hsiang [FH] and Farrell-Jones [FJ]. More generally, Ferry, Rothenburg and Weinberger [FRW] conjecture: every homotopy equivalence between aspherical manifolds which is a homeomorphism over a neighbornood of ∞ is homotopic to a homeomorphism.

V9: Zeeman Conjecture [Z]. If X is a contractible finite 2-complex, then $X \times I$ is collapsible. This is viewed as unlikely, because it is stronger than the Poincaré Conjecture. Indeed, when restricted to special spines (where all vertex links are circles with either 0, 2 or 3 additional radii) of homology 3-cells, it is equivalent to the Poincaré Conjecture [GR]. Cohen [Co] introduced a related notion, saying an complex X is q-collapsible provided $X \times I^q$ is collapsible, and he showed (among other things) that all contractible n-complexes X are 2n-collapsible. Best possible results concerning q-collapsibility are yet to be achieved, but Berstein, Cohen, and Connelly [BCC] have examples in all but very low dimensions (suspensions of nonsimply connected homology cells) for which the minimal q is approximately that of the complex.

V10: Codimension 1 manifold factor problem (generalized Moore problem). If $X \times Y$ is a manifold, is $X \times R^1$ a manifold? The earliest formulations of this problem, calling for X to be the image of S^3 under a cell-like map (see the decomposition section for a definition), date back at least to the early 1960s; see [Da4] for a partial chronology. In the presence of the manifold hypothesis on $X \times Y$, Quinn's obstruction theory [Q3] ensures the existence of a cell-like map from some manifold onto $X \times R^1$. When $X \times R^1$ has dimension at least 5, the question is just whether it has the following Disjoint Disks Property: any two maps of B^2 into $X \times R^1$ can be approximated, arbitrarily closely, by maps having disjoint images. No comparably simple test detects whether a 4-dimensional $X \times R^1$ is a manifold. Since $X \times R^2$ does have the Disjoint Disks Property mentioned above, Edwards' Cell-like Approximation Theorem [Ed] attests it is a manifold.

*V11: Resolution Problem. Does every generalized n-manifold X, $n \geq 4$, admit a cell-like resolution? That is, does there exist a cell-like map of some n-manifold M onto X? In one sense this has been answered -- Quinn [Q3] showed such a resolution exists iff a certain integer-valued obstruction $i(X)=1$ -- but in another sense it remains unsettled because no one knows whether $i(X)$ ever assumes a different value. A large measure of its significance is attached to the consequent characterization of topological manifolds: a metric space X is an n-manifold ($n \geq 5$) iff X is a finite dimensional, locally contractible, $H_*(X,X-x) \cong H_*(R^n,R^n-0)$ for all $x \in X$ (i.e., X is a generalized n-manifold), X has the Disjoint Disks Property, and $i(X)=1$. Is the final condition necessary?

V12: Kervaire Conjecture (also known as the Kervaire-Laudenbach Conjecture). If A is a group for which the normalizer of some element r in the free product $A*Z$ is $A*Z$

itself, then A is trivial. The main difficulty occurs in the case of an infinite simple group A. See Howie's survey [Ho] again for connections to other more obviously topological problems.

MANIFOLD AND GENERALIZED MANIFOLD STRUCTURE PROBLEMS

A generalized n-manifold is a finite dimensional, locally compact, locally contractible metric space X with $H_*(X,X-x) \cong H_*(R^n,R^n-0)$ for all $x \in X$. As Problems V10 and V11 suggest, the central problems are (1) whether every generalized manifold X is a factor of some manifold $X \times Y$ and (2) whether $X \times R^1$ is always a manifold.

Implications of homogeneity have not been fully determined, neither for distinguishing generalized manifolds from genuine ones nor for distinguishing locally flat embeddings of codimension one manifolds from wild embeddings.

M1. Does there exist a homogeneous compact absolute retract of dimension $2<n<\infty$? Bing and Borsuk [BB] showed that every homogeneous compact ANR (= absolute neighborhood retract) of dimension $n<3$ is a topological manifold.

M2. (Homogeneus ENRs versus generalized manifolds) If X is a homogeneous, locally compact ENR (= finite dimensional ANR), is X a generalized manifold? According to Bredon [Bre] (see alternatively Bryant [Bry]), it is provided $H_*(X,X-x;Z)$ is finitely generated for some (and, hence, for every) point $x \in X$.

M2'. Does every compact ENR X contain a point x_0 such that $H_*(X,X-x_0)$ is finitely generated?

M3. Is every homogeneous generalized manifold necessarily a genuine manifold? No if the 3-dimensional Poincaré Conjecture is false [Ja], but otherwise unknown.

M4. Do all finite dimensional H-spaces have the homotopy type of a closed manifold? Cappell and Weinberger [CW], who attribute the original question to Browder, have recent results.

M5. If M is a compact manifold, is the group Homeo(M) of all self-homeomorhisms an ANR? Ferry [Fe1] proved Homeo(M) is an ANR when M is a compact Hilbert cube manifold.

M6. Is every closed, aspherical 3-manifold virtually Haken (have a finite-sheeted cover by a Haken manifold)? Even stronger, does it have a finite sheeted cover by a manifold with infinite first homology?

*M7. Is every contractible 3-manifold W that covers a closed 3-manifold necessarily homeomorphic to R^3 ? Here one should presume W contains no fake 3-cells (i.e., no compact, contractible 3-manifolds other than 3-cells). Elementary cardinality arguments indicate some contractible 3-manifolds cannot be universal covers of any compact one, and Myers has identified specific examples, including Whitehead's contractible 3-manifold, that cannot do so. Davis's higher dimensional examples [Dv], by contrast, indicate this is a uniquely 3-dimensional problem.

*M8: Local connectedness of limit sets of conformal actions on S^3 . A group G of homeomorphisms of the 2-sphere is called a discrete convergence group if every sequence of distinct elements from G has a subsequence g_j for which there are points $x,y\in S^2$ with $g_j \to x$ uniformly on compact subsets of $S^2-\{y\}$ while $g_j^{-1} \to y$ uniformly on compact subsets of $S^2-\{x\}$ (or, equivalently, G acts properly discontinuously on $S^2\times S^2\times S^2-\{\text{distinct triples } (x,y,z)\}$). Its limit set $L(G)$ is the set of all scuh points x . If $L(G)$ is connected, must it be locally connected?

M9 (Bestvina). Must a $K(G,1)$ manifold M , where G is finitely generated, have only a finite number of ends? What if M is covered by R^n ?

M10 (M. Davis). Must the Euler characteristic (when nonvanishing) of a closed, aspherical 2n-manifold have the same sign as $(-1)^n$?

M11. Under what conditions does a closed manifold cover itself? cover itself both regularly and cyclically? Are the two classes different?

M12. Does there exist an aspherical homology sphere of dimension at least 4?

*M13: Simplified free surface problem in high dimensions – see also E1. Suppose W is a contractible n-manifold such that, for every compact $C \subset W$, there exists an essential map $S^{n-1} \to W-C$. Is W topologically equivalent to R^n ?

The Lusternik–Schnirelmann category of a polyhedron P, written $\mathrm{cat}(P)$, is the least integer k for which P can be covered by k open sets, each contractible in P. See Montejano's surveys [Mo1] [Mo2] for a splendid array of problems on this and related topics. Here are two eye-catching ones.

*M14. Does $\mathrm{cat}(M \times S^r) = \mathrm{cat}(M) + 1$? Singhof [Si] has answered this affirmatively for closed PL manifolds where $\mathrm{cat}(P)$ is fairly large compared to $\dim P$.

M15. If M is a closed PL manifold, does $\mathrm{cat}(M\text{-point}) = \mathrm{cat}(M) - 1$?

M16 (Ulam – problem #68 in The Scottish Book [Ma]). If M is a compact manifold with boundary in R^n for which every $(n-1)$-dimensional hyperplane H that meets M in more than a point has $H \cap \partial M$ an $(n-1)$-sphere, is M convex?

M17 (Borsuk). Can every bounded $S \subset R^n$ be partitioned into $(n+1)$-subsets S_i such that $\mathrm{diam} S_i < \mathrm{diam} S$? What about for finite S ?

M18. If X is a compact, n-dimensional space having a strongly convex metric without ramifications, is X an n-cell? (For definitions see Rolfsen's work [Ro], which solves the case $n=3$.) What if X is a generalized manifold with boundary? In that case is $X-\partial X$ homogeneous?

M19. Is there a complex dominated by a 2-complex but not homotopy equivalent to a 2-complex?

M20. Is every finitely presented perfect group (perfect = trivial abelianization) the normal closure of a single element?

DECOMPOSITION PROBLEMS

A decomposition G of a space X is a partition of X; it is upper semicontinuous if each $g \in G$ is compact and for every open set $U \supset g$ there exists another open set $V \supset g$ such that all $g' \in G$ intersecting V are contained in U. Associated with G is an obvious decomposition map $\pi: X \to X/G$ sending $x \in X$ to the unique $g \in G$ containing x; here X/G has the quotient topology.

The study of upper semicontinuous decompositions of a space X coincides with the study of proper closed mappings defined on X, but the emphasis is much different. Decomposition theory stresses, or aims to achieve, understanding of the image spaces through information about the decomposition elements.

All decompositions mentioned in this part are understood to be upper semicontinuous.

A compact subset C of an ANR is cell-like if it contracts in every preassigned neighborhood of itself, a property invariant under embeddings in ANRs; equivalently, C is cell-like if it has the shape of a point. A decomposition (a map) is cell-like if each of its elements (point inverses) is cell-like. A decomposition G of a compact metric space X is shrinkable iff for each $\varepsilon > 0$ there exists a homeomorphism $H: X \to X$ such that diam $H(g) < \varepsilon$ for all $g \in G$ and $d(\pi, \pi H) < \varepsilon$, where d is a metric on X/G; a convenient phrasing stems from the theorem (cf. [Da6, p.23]) showing G to be shrinkable iff $\pi: X \to X/G$ can be approximated, arbitrarily closely, by homeomorphisms. All elements in a shrinkable decomposition of an n-manifold are both cell-like and, better, cellular (i.e., can be expressed as the intersection of a decreasing sequence of n-cells).

The initial questions concern conditions precluding a decomposition (or a map) from raising dimension.

D1. The cell-like dimension-raising map problem for n=4,5,6. Dranišnikov [Dr] has described a cell-like map defined on a 3-dimensional metric compactum and having infinite dimensional image; this example automatically gives rise to another such map defined on S^7. On the other hand, Kozlowski-Walsh [KW] showed no such map can be defined on any 3-manifold. What can happen between these bounds is still open, although Mitchell-Repovš-Ščepin [MRS] have characterized the finite dimensional cell-like images of 4-manifolds in terms of a disjoint homological disk triples property. See also the surveys by Dranišnikov-Ščepin [DrS] and, more recently, Mitchell-Repovš [MR].

D2. Can a cell-like map defined on R^n have infinite dimensional image if all point-inverses are contractible? absolute retracts? cells? starlike sets? 1-dimensional compacta?

D3. If G is a usc decomposition of a compact space X into simple closed curves, is $\dim(X/G) \leq \dim X$?

D4. Could there be a decomposition G of an n-manifold M into closed connected manifolds (of varying dimensions) with $\dim(M/G) > n$?

D5 (Edwards). Can an open map $M \to X$ defined on a compact manifold and having 1-dimensional solenoids as point inverses ever raise dimension?

D6: The resolution problem for generalized 3-manifolds. Assuming the truth of the 3-dimensional Poincaré Conjecture, does every generalized 3-manifold X have a cell-like resolution? Does $X \times R^1$ have such a resolution? Independent of the Poincaré Conjecture, is X the cell-like image of a "Jakobsche" 3-manifold (i.e., an inverse limit of a sequence of 3-manifolds connected by cell-like bonding maps, as in [Ja])? Thickstun [Tk] verified this for X having 0-dimensional nonmanifold set.

D6': Thickstun's Full Blow-up Conjecture [Tk]. A compact homology n-manifold X is the conservative, strongly acyclic, hereditarily π_1-injective image of a compact n-manifold if for each $x \in X$ there exist a compact, orientable n-manifold M_x and a map $(M_x, \partial M_x) \to (X, X-\{x\})$ inducing an isomorphism on n-dimensional Čech homology. (Terminology: a homology n-manifold a finite-dimensional, locally compact metric space for which

$H_*(X,X-x) \cong H_*(R^n,R^n-0)$; by way of contrast, a generalized n-manifold is an ANR homology n-manifold. A map is conservative if its restriction to the preimage of the manifold set is an embedding; it is hereditarily π_1-injective if its restriction to the preimage of any connected open set induces an injection of fundamental groups; it is strongly acyclic if for each neighborhood U of a point preimage $f^{-1}(x)$ there exists another neighborhood V of $f^{-1}(x)$ such that inclusion induces the trivial homomorphism $H_*(V) \to H_*(U)$.) Thickstun avers [Tk] this may be an overly optimistic conjecture, since it implies the resolution conjecture for generalized n-manifolds and, therefore, the 3-dimensional Poincaré conjecture as well. He adds that according to M. H. Freedman the 4-dimensional case implies 4-dimensional topological surgery can be done in the same sense it is done in higher dimensions.

D7: The Approximation Problem for 3- and 4-manifolds. Which cell-like maps $p:M \to X$ from a manifold onto a finite-dimensional space can be approximated by homeomorphisms? Is it sufficient to know that, given any two disjoint, tame 2-cells $B_1, B_2 \subset M$, there are maps $\mu_i : B_i \to X$ approximating $p|B_i$ with $\mu_1(B_1) \cap \mu_2(B_2) = \emptyset$? The question carries a degree of credability because for $n \geq 5$ the condition is equivalent to X having the Disjoint Disks Property, which yields an affirmative answer [Ed].

Next, some problems about shrinkability of cellular decompositions of manifolds. The 3-dimensional version of each has been solved, all but D12 affirmatively.

*D8. Is each decomposition of R^n involving countably many starlike-equivalent sets shrinkable? A compact set $X \subset R^n$ is starlike if it contains a point x_0 such that every linear ray emanating from x_0 meets X in an interval, and X is starlike-equivalent if it can be transformed to a starlike set via an ambient homeomorphism. Denman and Starbird [DeS] have established shrinkability for $n=3$.

D9. Let $f:S^n \to X$ be a map such that if $f^{-1}f(x) \neq x$, then $f^{-1}f(x)$ is a standardly embedded n-cell. Can f be approximated by homeomorphisms? Same question when there are countably many nondegenerate $f^{-1}f(x)$, all standardly embedded $(n-2)$-cells. Although closely related, these are not formally equivalent. See [Ev] [SW] concerning $n=3$.

D10. Suppose G is a usc decomposition of n-space such that each $g \in G$ has arbitrarily small neighborhoods whose frontiers are $(n-1)$-spheres missing the nondegenerate elements of G? Is G shrinkable? What if the neighborhoods are Euclidean patches? Woodruff [Wo] developed the low dimensional result.

D11. Suppose $A \subset R^n$ is an n-dimensional annulus. Is there a parameterization of A as a product $S^{n-1} \times I$ for which the associated decomposition into points and the fiber arcs is shrinkable? Daverman-Eaton [DE] did this when $n=3$; work by Ancel-McMillan [AM] and Cannon-Daverman [CD] combines with Quinn's [Q2] homotopy-theoretic characterization of locally flat 3-spheres in R^4 to take care of $A \subset R^4$ as well.

D12. Is a countable, cell-like decomposition G of R^n shrinkable if every nondegenrate $g \in G$ lies in some affine $(n-1)$-hyperplane? If all nondegenerate elements live in one of two predetermined hyperplanes, Bing [Bi] produced a remarkable 3-dimensional counterexample while Wright [Wr2] established shrinkability for $n \geq 5$, but the matter is unsolved for $n=4$.

The rich variety of nonshrinkable decompositions of R^3 is not matched in higher dimensions; a plausible explanation is that descriptions of unusual 3-dimensional examples rely in unreproducable fashion on real world visualization experience. The next two problems point to 3-dimensional constructions lacking higher dimensional analogs.

D13. Consider any sequence $\{C(i)\}$ of nondegenerate cellular subsets of $R^{n \geq 4}$. Does there exist a nonshrinkable, cellular decomposition of R^n whose nondegenerate elements form a null sequence $\{g(i)\}$ with $g(i)$ homeomorphic to $C(i)$? Starbird's 3-dimensional construction [St] prompts the question

D14. Is there a nonshrinkable decomposition of n-space into points and straight line segments? Into convex sets? Armentrout

[Ar] provided a 3-dimensional example, and later Eaton [Ea] demonstrated the nonshrinkability of an older example developed by Bing.

Presented next are some uniquely 4-dimensional issues. Most are relatively unpredictable in that, like the second half of D12, higher/lower dimensional analogs transmit conflicting information.

D15. If X is the cell-like image of a 3-manifold M, does X embed in $M \times R$? More technically, if G is a cell-like decomposition of R^3, regarded as $R^3 \times 0 \subset R^4$, and if G^* denotes the trivial extension of G (i.e., G^* consists of the elements from G and the singletons from $R^4 - (R^3 \times 0)$), is R^4/G^* topologically R^4? This must be true if V10 has an affirmative answer.

D16. If X is a cellular subset of 4-space and G is a cell-like decomposition of X such that $\dim(X/G) \leq 1$, is the trivial extension of G over 4-space shrinkable? What if X is an arc? No to the latter when $n=3$ [RW] and yes to the former when $n \geq 5$ [Da2].

D17. Is each simple decomposition of R^4 shrinkable? Here one starts with a collection $\{N_i\}$ of compact n-manifolds with boundary in R^n, with $N_{i+1} \subset \mathrm{Int} N_i$, and studies the decomposition consisting of singletons and the components of $\cap N_i$. It is called simple if each component C_i of each N_i contains a pair of disjoint n-cells B_1, B_2 such that every component C' of N_{i+1} in C_i lies in either B_1 or B_2. The remarkable nonshrinkable decomposition of Bing [Bi] mentioned in D12 is simple, whereas the Cell-like Approximation Theorem of Edwards quickly reveals shrinkability when $n>4$ [Da6, p. 185].

D18. If $f: S^4 \to S^4$ is a map which is 1-1 over the complement of some Cantor set $K \subset S^4$, is f cell-like? What if f is 1-1 over the complement of a noncompact 0-dimensional set? Yes by work of McMillan [MM] for $n=3$, but counterexamples exist for $n>4$ [Da3].

D19. Can every cellular map $\theta : P \to Q$ between finite 4-complexes be approximated by homeomorphisms? Henderson [Hn1] [Hn2] produced approximations in the 3-dimensional case and counterexamples in higher dimensions.

Finite dimensional compact metric spaces X, Y are CE equivalent if they are related through a finite sequence

$$X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots X_m = Y ,$$

where " $X_i \leftrightarrow X_{i+1}$ " requires the existence of a cell-like surjection of one of the spaces onto the other. In short, the definition is satisfied iff some compactum Z admits cell-like, surjective mappings onto both X and Y. Ferry [Fe2] shattered a suspicion that CE equivalences might behave like simple homotopy equivalences; he also made repeated remarks suggesting a closer connection if one restricts to LC^1 spaces -- see D22 below.

D20. If X, Y are n-dimensional, LC^{n-1} compacta that are shape equivalent, are they CE equivalent? Daverman-Venema [DV1] have taken care of the always-difficult $n=1$ case.

D21 (Ferry). If X, Y are shape equivalent LC^k compacta, are they UV^k equivalent? Here one seeks a compactum Z as a source for surjective UV^k mappings onto X, Y, where " UV^k " means each point preimage has the shape of an i-connected object, $i \in \{0, 1, \ldots, k\}$.

D22. If X, Y are CE equivalent, LC^1 compacta, are they related through a finite sequence as in the definition of CE equivalence above where, in addition, all intermediate spaces X_i are LC^1 ? What happens for homotopy equivalent but simple homotopy inequivalent polyhedra X, Y ? The relationship does hold for LC^0 spaces [DV2].

D23 (Kozlowski). Suppose X is the inverse limit of a sequence of homotopy equivalences $S^2 \leftarrow S^2$. Is X CE equivalent to S^2 ?

D24. Let $K \subset R^n$ denote a k-cell. Under what conditions can K be squeezed to a (k-1)-cell, in the sense that there is a map $f:R^n \to R^n$ for which $f|K$ is conjugate to the "vertical" projection $B^k \to B^{k-1}$ while $f|R^n-K$ is a homeomorphism onto $R^n-f(K)$. What if K is cellular? What if each Cantor set in K is tame? Bass [Ba] provides a useful sufficient condition and raises several other appealing questions.

D25. Given a cell-like map $f:M \to X$ of an n-manifold onto a finite dimensional space, can f be approximated by a new cell-like map $F:M \to X$ such that each $F^{-1}(x)$ is 1-dimensional? Specifically, can this be done when $n \in \{3,4,5\}$?

D26. Is there a decomposition of R^n into k-cells $(k>0)$? Into copies of some fixed compact absolute retract ($\neq$ point) ? Cf. [Jo] [WW].

D27. Is there a decomposition of B^n into simple closed curves? of a compact contractible space? of a cell-like set?

D28 (Bestvina-Edwards). Does there exist a cell-like, non-contractible compactum whose suspension is contractible?

Standard Notation: M is an (n+k)-manifold; G is a usc decomposition of M into closed connected n-manifolds; B is the decomposition space M/G ; and $p:M \to B$ is the decomposition map. For convenience assume both M and all the elements of G are orientable.

Due to similarities imposed on the set of point preimages, one can regard the study of these maps $p:M \to B$ as somewhat comparable to the study of cell-like maps. At another level, when all point preimages are topologically the same, one can strive for the much more regular sorts of conclusions suggested by the theory of fibrations and/or locally trivial bundle maps.

D29. Is B an ANR? What if the elements of G are pairwise homeomorphic?

D30. Is B finite-dimensional? (It deserves emphasis here that if the elements of G are not required to be genuine manifolds but merely to be of that shape, a fairly common hypothesis in this topic, the product of S^n with a Dranišnikov dimension-raising cell-like decomposition of S^k quickly provides negative solutions.) What if the elements of G are simple closed curves?

D31. For which integers n and k is there a usc decomposition of S^{n+k} into n-spheres? into n-tori? into fixed products of spheres? into closed n-manifolds? Does R^{n+k} ever admit a decomposition into closed n-manifolds $(n>0)$?

D32. When n and k are both odd, does every closed $(n+k)$-manifold M admitting a decomposition into closed n-manifolds have Euler characteristic zero?

D33. If G is a usc decomposition of an $(n+k)$-manifold M into n-spheres, where $2<n+1<k<2n+2$, is M/G a generalized k-manifold? What if into homology n-spheres? Investigations when $k<n+1$ and $k=n+1$ are detailed in [DW] and [Sn], respectively.

D34. In case $k=3$, is the set of points at which B fails to be a generalized 3-manifold locally finite?

D35. If $k=3$, $n=1$, and the degeneracy set $K(B)$ of local 1-winding functions is empty (i.e., the 1-dimensional cohomology sheaf of $p:M \to B$ is Hausdorff), is B a generalized 3-manifold?

D36. If $k=1$ and all elements of G are 2-sided in M, must M have the homotopy type of a closed n-manifold?

D37. If W is a compact $(n+1)$-manifold with $\partial W \neq \emptyset$ and the inclusion $N \to W$ of some component N of ∂W is a homotopy equivalence, does W admit a decomposition into closed n-manifolds? What if the kernel of the induced π_1-homomorphism is simple (but contains no finitely generated perfect group)?

D38. When $n=3$ and $k=1$ does there exist a decomposition G of a connected M containing homotopy inequivalent elements? Information from [Da5] surrounds this 4-dimensional matter, comparable to D15–D19.

D39. Does there exist a compact 5-manifold W having boundary components M_0 and M_1, where $\pi_1(M_0) \cong 1$ and $\pi_1(M_1) \cong A_5$, the alternating group on 5 symbols, such that W admits a decomposition G into closed 4-manifolds (with $M_0, M_1 \in G$).

Daverman-Tinsley [DT] locate W when $H_*(M_1) \cong H_*(S^4)$ but not when $\pi_1(M_1)$ is an arbitrary finitely presented perfect group.

D40. Given a closed manifold N, does some $(n+k)$-manifold M admit a decomposition into copies of N such that $p:M \to B$ is not an approximate fibration? Are there other examples besides those with homology sphere factors and those that regularly, cyclically cover themselves? Is there a 2-manifold example N with negative Euler characteristic?

D41. For which n-manifolds N and integers k does the hypothesis that all elements of G are copies of N imply $p:M \to B$ is an approximate fibration? What if $\pi_1(N)$ is finite and $k=2$? What if N is covered by the n-sphere? What if N is hyperbolic? What if all $g \in G$ are required to be locally flat in M?

D42. If $k=2m$, $n=2m+1$, and $p:M \to B$ is a PL map from a PL $(n+k)$-manifold M to a simplicial complex B such that $H_j(p^{-1}(b)) \cong 0$ whenever $0<j<n$, is B a generalized manifold?

and Lay have an unpublished construction), and otherwise it is still open.

E11. Let $\lambda: X \to M$ denote a closed embedding of a generalized n-manifold X in a genuine (n+1)-manifold M. Can λ be approximated by 1-LCC embeddings? Yes for $n \geq 4$ (see [Da5, p.283] - key ideas are due to Cannon, Bryant, and Lacher [CBL]); what about for $n=3$? What if X is a generalized n-manifold with boundary? Ancel discusses this and related problems in [An].

E12. Which homology n-spheres K bound acyclic (n+1)-manifolds N such that $\pi_1(K) \to \pi_1(N)$ is an isomorphism? Is there a homology 4-sphere example?

E13. Let X be a cell-like subset of R^n. Does R^n contain an arc α with $R^n - \alpha$ homeomorphic to $R^n - X$? For $n \geq 6$ R^n has a 1-dimensional compact subset A with $R^n - A \approx R^n - X$ [Ne].

E14. Can there exist a codimension 3 cell D in R^n $(n>5)$ such that all 2-cells in D are wildly embedded in R^n but each arc (each Cantor set) there is tame? This question calls for new embedding technology, since existing examples [Da1] in which all 2-cells are wild essentially exploit the presence there of Cantor sets wildly embedded in the ambient manifold.

E15. Can every n-dimensional compact absolute retract be embedded in R^{2n} ?

E16. Can every S^n-like continuum be embedded in R^{2n} ? A metric space X is S^k-like if there exist ε-maps $X \to S^k$ for every $\varepsilon > 0$.

E17. Does S^4 contain a 2-sphere Σ, possibly wildly embedded, such that $S^4 - \Sigma$ is topologically $S^1 \times R^3$ but not smoothly so?

E18. (M. Brown) If a wedge $A \vee B \subset R^3$ is cellular, is A cellular?

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Problem Session

1. (Venema) If $\Sigma \subset S^2 \times S^2$ is a locally flat 2-sphere and $\pi_1(S^2 \times S^2 - \Sigma)$ is infinite cyclic, must Σ be flat; i.e., does Σ bound a 3-ball?

2. (Bowers) Does every orientation preserving self homeomorphism of the plane have a square root?

3. (Wright) Give an example of a specific contractible n-manifold $n > 3$ that does not cover a compact n-manifold. Do the Whitehead contractible n-manifolds cover a contractible n-manifold? (Added before publication: These questions have been answered.)

4. (Tinsley) What is the "simplest" presentation of a 3-dimensional knot group which abelianizes with perfect kernel? In particular, may this group be chosen to have a single defining relator? (Hempel suggests looking at untwisted doubles.)

5. (Walsh) If an ANR has the local homology of $\boldsymbol{R}^n$, must it be finite dimensional?

6. (Walsh) Is there an usc decomposition of S^4 into circles or shape circles?

7. (Bestvina) For every q does there exist $n = n(q)$ such that every map f from the n-torus to a q-dimensional space (e.g. $\boldsymbol{R}^q$) has a point preimage $f^{-1}(pt)$ such that the inclusion induced homomorphism $\check{H}_1(f^{-1}(pt)) \rightarrow H_1(T^n)$ is non-trivial (integer coefficients)?

8. (Cannon) Give a truly elementary proof of the Sullivan-Rodin theorem on the rigidity of the hexagonal circle packing in the plane.

9. (Cannon) Give an argument which verifies Gromov's assertion that most finitely presented groups are negatively curved.

10. (Cannon) Determine simple criteria that can be used to determine whether a sequence of shinglings of the plane is a conformal sequence of shinglings.

11. (Cannon) Give a simple proof that the dodecahedral reflection group creates a natural sequence of tilings of S^2 that is conformal.

12. (Cannon) Is ever closed 3-manifold with negatively curved fundamental group hyperbolic?

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