PROCEEDINGS

Fifth Annual Western Workshop in Geometric Topology

June 16-18 1988

Department of Mathmatics
The Colorado College
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The Fifth Annual Western Workshop in Geometric Topology was held at The Colorado College in Colorado Springs, Colorado on June 16-18, 1988. The participants were:

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Mladen Bestvina	University of California, Berkeley
James Cannon	Brigham Young University
Robert Daverman	University of Tennessee
Robert Edwards	University of California, Los Angles
Dennis Garity	Oregon State University
Craig Guilbault	University of Tennessee
John Hempel	Rice University
James Henderson	Colorado College
Kerry Jones	Rice University
Ning Lu	Rice University
Zoran Nevajdic	University of Tennessee
Paul Place	Rice University
Dale Rohm	Vanderbilt University
David Snyder	University of Tennessee
Frederick Tinsley	Colorado College
David Wright	Brigham Young University

These proceedings contain the notes of two one-hour talks given by the principal speaker John Hempel, summaries of talks given by other participants, and a problem list compiled at the end of the workshop. The success of the conference was due in large part to funding provided by the National Science Foundation (DMS-8802424) and The Colorado College. Both have supported these workshops in the past and we wish to express our gratitude for their continued support.

Jim Henderson
Fred Tinsley

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"The lattice of 3-manifolds as branched covers over a universal knot"

by John Hempel

(Notes compiled by Fred Tinsley)

1. Introduction

The classification of closed, orientable, irreducible 3-manifolds initially focuses on the fundamental group. Although some exceedingly difficult questions remain, such manifolds with finite fundamental group are reasonably well understood. Among those manifolds with infinite fundamental group, the Haken manifolds are also well understood.

Definition: A closed, orientable, irreducible 3-manifold which contains an incompressible surface is called a *Haken manifold*.

Of current interest is a larger class of 3-manifolds.

Definition: A closed, orientable 3-manifold which admits a finite cover by a Haken manifold is called a *virtually Haken manifold*.

Conjecture: A closed, orientable, irreducible 3-manifold which contains an immersed incompressible surface is virtually Haken.

There is a good deal of hope that many results for Haken manifolds can be extended to virtually Haken manifolds. In light of the above, one attack is to study finite group actions on Haken manifolds.

Relevance to the classification theorem is given by:

Conjecture: All closed, orientable, irreducible manifolds with infinite fundamental group are virtually Haken.

2. Properties of the Fundamental Group

Definition: A group G is *$\mathbb{Z}$-representable* if it admits a surjective homomorphism to $\mathbb{Z}$.

Definition: A group G is *virtually $\mathbb{Z}$-representable* if it has a subgroup of finite index which is $\mathbb{Z}$-representable.

<u>Theorem</u>: If a closed, orientable 3-manifold M is irreducible and has a $\mathbb{Z}$-representable fundamental group, then M is Haken.

Sketch of proof: Since S^1 is aspherical, map M to S^1 realizing the the homomorphism of $\pi_1(M) \to \mathbb{Z} \approx \pi_1(S^1)$. Put the map in general position with respect to a point. The inverse image of that point will be a 2-manifold which can be altered to be incompressible.

<u>Corollary</u>: If a closed, orientable 3-manifold M is irreducible and has a virtually $\mathbb{Z}$-representable fundamental group, then M is virtually Haken.

The converse to the corollary is conjectured to be true.

3. Branched Covers of Closed 3-Manifolds

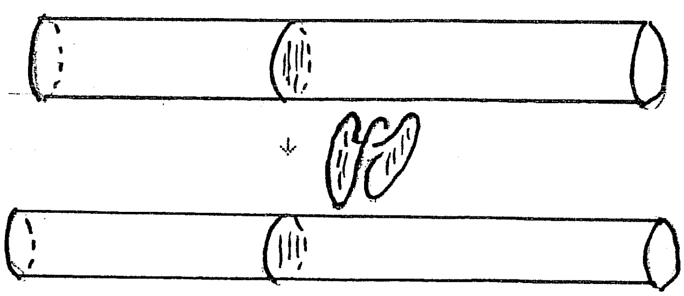
<u>Definition</u>: A map of 3-manifolds $p:\tilde{M} \to M$ of degree d is a *branched covering over a link L in M* if

$$p|: (M - p^{-1}(L)) \to M-L \text{ is a covering}$$

and

$p|: p^{-1}(L) \to L$ is a covering locally nice near $p^{-1}(L)$, ie, locally, meridian is mapped to meridian via $z \to z^n$ (see sketch below). The integer n is called the branching index for this component of $p^{-1}(L)$.

$p^{-1}(L)$



L

These n's may vary from component to component; however, when counted with appropriate multiplicity, they must sum to d, the degree of the branched cover.

<u>Definition</u>: A branched cover is called a *true cover* if all n's are equal to one.

<u>Definition</u>: A link in S^3 is *universal* if every closed, orientable 3-manifold is a branched cover over that link.

Examples of universal links include non-torus 2-bridge links (eg, figure eight knot), Boromean rings, and the Whitehead link. Since it is Seifert-fibered, the trefoil link is not universal, not even for Seifert fiberings.

<u>Definition</u>: The *monodromy* of a branched covering, $p:\tilde{M} \to M$ is the obvious homomorphism $\varphi: \pi_1(M - L) \to S_d$ where S_d is the symmetric group on d letters and φ describes the action of $\pi_1(M - L)$ on a fiber $p^{-1}(x_\emptyset)$.

3. Lattice of Branched Covers Over a Manifold:

The covers of M branched over L form a lattice. The order relation is: $\tilde{M}_1 \geq \tilde{M}_2$ if there is a branching map q so that the diagram of branched covers commutes,

$$\begin{tikzcd}(\tilde{M}_1,\tilde{L}_1) \arrow[dr, "p_1"'] & & (\tilde{M}_2,\tilde{L}_2) \arrow[ll, "q"'] \arrow[dl, "p_2"] \\ & (M\ ,\ L) & \end{tikzcd}$$

where $\tilde{L}_i = p_1^{-1}(L_i)$ $(i=1,2)$. This is equivalent to the lattice of subgroups of finite index in $\pi_1(M-L)$ ordered by reverse inclusion. From now on we use $N = M-L$.

The supremum of two branched covers $p_i: (\tilde{M}_i,\tilde{L}_i) \to (M,L)$ for $i=1,2$ is the covering corresponding to the unbranched covering, $p: \tilde{N} \to N$ satisfying $p_*(\pi_1(\tilde{N}_1)) = p_{1*}(\pi_1(\tilde{N}_1,\tilde{x}_1)) \cap p_{2*}(\pi_1(\tilde{N}_2,\tilde{x}_2))$ (recall $N = M-L$ and $\tilde{N}_i = (\tilde{M}_i - \tilde{L}_i)$.

Let $G = \pi_1(M-L)$ and $\varphi_i: G \to S_{d_i}$ be the monodromies of the two branched covers. Then there is a natural map $\varphi_1 \times \varphi_2: G \to S_{d_1 \cdot d_2}$ which although not in general transitive does determine all the pullbacks of p_1 and p_2. Let $\tilde{M}$ be a pullback and J^* a component of $p^{-1}(J)$ in the diagram:

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, "p"] & \\ \tilde{M}_1 \arrow[dr, "p_1"'] & & \tilde{M}_2 \arrow[dl, "p_2"] \\ & M & \end{tikzcd}$$

<u>Fact</u>: The branching index of p at $J*$ is equal to the least common multiple of the branching index of p_1 at $q_1(J*)$ and the branching index of p_2 at $q_2(J*)$.

<u>Corollary</u>: If each branch index of p_1 divides the corresponding branch index of p_2, then q_2 is a true cover.

This yields two results about the branched covering space in terms of data about the base space.

<u>Corollary</u>: Suppose $p_1 : \tilde{M}_1 \to M$ is a finite-sheeted, branched covering and $\pi_1(M)$ is virtually $\mathbb{Z}$-representable, then $\pi_1(\tilde{M}_1)$ is virtually $\mathbb{Z}$-representable, ie, each member of the lattice above M is also virtually $\mathbb{Z}$-representable.

Proof: Let $p_2 : \tilde{M}_2 \to M$ be the finite-sheeted true cover corresponding to a $\mathbb{Z}$-representable subgroup of finite index in $\pi_1(M)$. Let $\tilde{M}$ be the pullback.

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, "p"] & \\ \tilde{M}_1 \arrow[dr, "p_1"'] & & \tilde{M}_2 \arrow[dl, "p_2"] \\ & M & \end{tikzcd}$$

Now, p_2 has all branching indices equal to 1 so by the previous corollary q_1 is also a true cover. Also, q_2 is a finite-sheeted branched cover so $\pi_1(\tilde{M})$ contains a $\mathbb{Z}$-representable subgroup of finite index and, thus, is virtually $\mathbb{Z}$-representable. But $q_{1\#}(\pi_1(\tilde{M}))$ has finite index in $\pi_1(\tilde{M}_1)$.

Similar arguments also yield:

<u>Corollary</u>: If $p : \tilde{M} \to M$ is a finite sheeted branched cover, M is virtually Haken, and $\tilde{M}$ is irreducible, then $\tilde{M}$ is virtually Haken.

4. Branched Covers over S^3:

Since all closed, orientable 3-manifolds are branched covers of S^3, it is natural to investigate criteria which insure that the covering space is $\mathbb{Z}$-representable. However, $\pi_1(S^3)$ is trivial and certainly not $\mathbb{Z}$-representable.

<u>Example</u>: Consider Higman's group

$$G \equiv \langle a_i^{-1} a_{i+1} a_i = a_{i+1}^2 \mid i = 1, \ldots, 4 \rangle$$

G *has no proper subgroup of finite index*. However, G is not the fundamental group of a 3-manifold.

<u>Definition</u>: A finitely presented group G is *resentable* if it is the fundamental group of a closed 3-manifold and contains no proper subgroups of finite index.

<u>Question</u>: Are there any resentable groups?

The following shows the relevance to the question of which 3-manifold groups are virtually $\mathbb{Z}$-representable.

<u>Involution Theorem</u>: Let M be an orientable, closed 3-manifold. If M admits an orientation-reversing involution, τ, and $o(\pi_1(M)) > 2$ (or if $o(\pi_1(M))=2$ and $\mathrm{Fix}(\tau)$ is not equal to $\mathbb{RP}^2$), then $\pi_1(M)$ is either virtually $\mathbb{Z}$-representable or resentable.

5. The Involution Theorem and Branched Coverings

The following, then, is a natural question.

<u>Question</u>: Given a manifold-link pair (M,L), an orientation-reversing involution τ: $(M,L) \to (M,L)$, and a finite-sheeted branched covering $p: (\tilde{M},\tilde{L}) \to (M,L)$, when can τ be lifted to an involution and the involution theorem applied?

The following three facts yield one set of criteria. Recall that for a branched covering $p: (\tilde{M},\tilde{L}) \to (M,L)$ there is an associated unbranched covering $p| : \tilde{N} \to N$ where $\tilde{N} = \tilde{M} - \tilde{L}$ and $N = M - L$.

<u>Fact I</u>: For $\tau|: N \to N$ to lift to $\tilde{\tau}: \tilde{N} \to \tilde{N}$ with $\tilde{\tau}^2 = \mathrm{id}$ it must be the case that:

i. $\tau_{\#}$ must be invariant up to inner automorphism on $p_{\#}(\pi_1(\tilde{N}))$. This is needed to lift τ to a map of $\tilde{N}$.

ii. If $\mathrm{Fix}(\tau) \subset J$, a component of L, then a lift of τ need not have order two. To get this, an odd branch index is usually needed over J for τ must reverse orientation on J and, thus, "rotate" a normal disk, D, to J through a fixed point of τ. Then ∂D can be written as a path product $\mu = \alpha\beta$ where $\beta = \tau \circ \alpha$. Now suppose a component, X, of $p^{-1}(\partial D)$ is invariant under a lift, $\tilde{\tau}$, of τ. A path in X from $\tilde{x}_{\emptyset}$ to $\tilde{\tau}(\tilde{x}_{\emptyset})$ would be of the form $\alpha_1\beta_1 \cdot \cdot \cdot \alpha_k\beta_k\alpha_{k+1}$ where $p\circ\alpha_i = \alpha$ and $p\circ\beta_j = \beta$. If $\tilde{\tau}^2 = \mathrm{id}$, then $\deg p|X = 2k+1$, and is odd.

Note conversely that if some branching index over J is odd and if the cover $\tilde{N} \to N$ is regular, then the above argument can be reversed to find some lift of τ of order two if a lift exists at all. This cannot hold in general: the lens space L(5,2) is the double branched cover of S^3 branched over the figure eight knot. The orientation reversing involution of S^3 leaving the figure eight knot invariant lifts to an order four map of L(5,2). Of course, L(5,2) does not admit an orientation reversing involution.

iii. To extend $\tilde{\tau}$ to all of $\tilde{M}$, some equivariance is needed (eg, branching indicies are permuted and must match up).

<u>Fact II</u>: If $M = S^3$ and L either has a knotted component or for some component J of L $p^{-1}(J)$ is not connected, then $\pi_1(\tilde{M}) \neq 1$. We say that such a cover is *generic*.

Proof: Smith theory and the Smith conjecture.

<u>Fact III</u>: If $M = S^3$ and the cover is regular, then $\pi_1(M)$ is not resentable. (Shalen)

The following basic theorem results. First, a definition is necessary.

<u>Definition</u>: Let L be a link in S^3 with components J_1, J_2, . . ., J_k and let $q = (q_1, q_2, \ . \ . \ . \ , q_k)$, $q_i \in \mathbb{Z}^+$. A branched (over L) covering is *divisible by q* if each branching index of p at each component of $p^{-1}(J_i)$ is divisible by q_i. Also, q is *τ-invariant* if $q_i = q_j$ whenever $\tau(J_i) = J_j$.

<u>Theorem</u>: If there exists an orientation reversing involution τ: $(S^3,L) \to (S^3,L)$ where L has components $(J_1,J_2, \ . \ . \ . \ J_k)$, $q = (q_1,q_2, \ . \ . \ . \ ,q_k)$ is τ-invariant (in case $\mathrm{Fix}(\tau) \subset J_i$, then q_i must be odd), and p_1: $\tilde{M}_1 \to S^3$ is branched over L, generic, and divisible by q, then $\pi_1(\tilde{M}_1)$ is virtually $\mathbb{Z}$-representable.

Proof: There is a special regular covering p_q: $\tilde{M}_q \to S^3$ branched over L which is a $q_1q_2 \ldots q_k$-sheeted cover whose associated unbranched cover has covering transformations isomorphic to $\mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \ . \ . \ . \ \oplus \mathbb{Z}_{q_k}$.
Consider the pullback diagram:

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "r_q"'] \arrow[dr, "r_1"] \arrow[dd, "p"] & \\ \tilde{M}_q \arrow[dr, "p_q"'] & & \tilde{M}_1 \arrow[dl, "p_1"] \\ & S^3 & \end{tikzcd}$$

Previous work yields that r_1 is a true cover. So $\pi_1(\tilde{M}_1)$ is virtually $\mathbb{Z}$-representable if and only if $\pi_1(\tilde{M})$ is; so it suffices to show that $\pi_1(\tilde{M}_q)$ is virtually $\mathbb{Z}$-representable.

Sketch of proof: This follows from the involution theorem. Fact I and its converse are used to lift τ to an involution of $\tilde{M}_q$. Fact II is used to show $o(\pi_1(\tilde{M}_q))>2$. Finally, Fact III is used to show $\pi_1(\tilde{M}_q)$ is not resentable.

<u>Corollary</u>: For L equal to the Figure Eight Knot any cover M with all branching indices divisible by $q > 2$ has $\pi_1(M)$ virtually $\mathbb{Z}$-representable.

Proof: For q odd this follows from the theorem. The case $q = 4$ requires an explicit construction.

6. General 3-manifold Data

<u>Question</u>: In general, to which 3-manifolds does the corollary apply?

Unfortunately, a priori, not many. However, the Figure Eight Knot complement is a bundle over S^1 with fiber, F, a punctured torus.

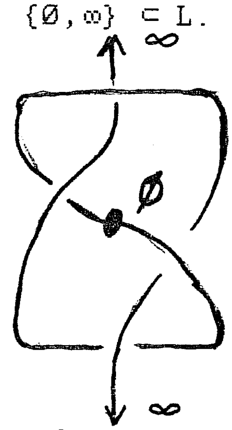
<u>Definition</u>: The *wrapping number* of a bundle is the number of components of $p^{-1}(F)$. This is also called the *wrapping number of the associated branch covering.*

The following fact is helpful in studying covers branched over any fibered knot.

<u>Fact</u>: Suppose $p: \tilde{N} \to N$ is a bundle. The wrapping number of the associated branched covering divides all branch indices.

Now, the Figure Eight Knot has an embedding in

$S^3 = R^3 \cup \infty$ so that the involution $\tau: S^3 \to S^3$ given by $\tau(x) = -x$ has $\mathrm{Fix}(\tau) = \{\emptyset, \infty\} \subset L$.



Thus, it follows easliy that:

<u>Theorem</u>: Any covering $p: \tilde{M} \to S^3$ branched over the Figure Eight Knot with wrapping number > 2 has $\pi_1(\tilde{M})$ virtually $\mathbb{Z}$-representable.

7. Concluding Comments and Questions

Many obstacles remain in the way of this promising approach to classification of 3-manifolds. Of particular interest is:

Question: Can one detect from the monodromy of a branched covering whether $\tilde{M}$ is Seifert fibered? whether $\pi_1(\tilde{M})$ is finite?

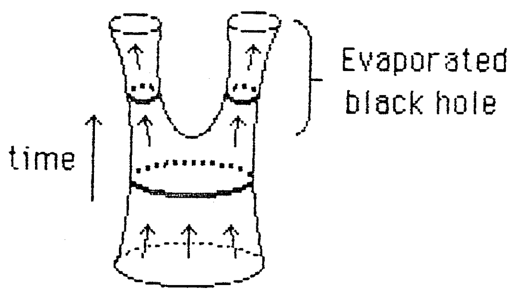
The branched covering representation of a 3-manifold is highly non-unique. In particular, there is a degree 720 covering $p: S^3 \to S^3$ branched over the Figure Eight Knot with branching indices 1's, 2's, and 4's and with $p^{-1}(L)$ containing the Figure Eight Knot. This raises obvious general questions.

Question: What can be said about the degree of non-uniqueness of representation of 3-manifolds by branched coverings? What about the minimal periodicity of representation?

Two applications of topology to physics

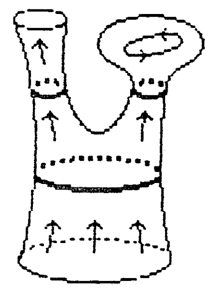
by Fredric D. Ancel

I. Stephen Hawking has predicted that black holes can "evaporate" and disappear from the universe. From the point of view of the universe as a 4-manifold with a preferred time direction, one possible explanation of the disappearance of a black hole is as an increase in the number of components in the cross-sections of the universe transverse to the time direction as time increases. (See Figure 1.)



the universe
(also $\mathbb{C}P^2$-(3 points))

Figure 1



$\mathbb{C}P^2 \# \mathbb{R}P^4$-(2 points)

Figure 2

The universe is a spacetime; i.e., a smooth 4-manifold with a Lorentz metric (a semi-Riemannian metric with signature: $-+++$). John Friedman of the Physics Department at the University of Wisconsin, Milwaukee asked for explicit examples of spacetimes which exhibit such behavior. In other words, he asked for specific examples of spacetimes in which the number of components in the cross-sections transverse to the time direction increases with increasing time?

It is simple to discover such examples based on the observation that a smooth manifold admits a Lorentz metric if it admits a non-zero vector field. (The vectors point in the direction of increasing time.) This observation reduces the question to an exercise in using the Poincaré-Hopf Index Theorem.

$\mathbb{C}P^2$ has Euler characteristic 3. Hence, it admits a vector field with 3 zeroes which can be chosen to be one source and two sinks. (Both sources and sinks have index +1 in dimension 4.) Thus, $\mathbb{C}P^2$-(3 points) has a Lorentz metric in which the number of components in the cross-sections transverse to the time direction changes from 1 to 2. Furthermore, a typical section with 2 components separates $\mathbb{C}P^2$-(3 points) into three noncompact pieces. (See Figure 1 again.)

$\mathbb{R}P^4$ has Euler characteristic 1 and, therefore, admits a vector field with one source (and some closed trajectories). Hence, $\mathbb{C}P^2 \# \mathbb{R}P^4$ has a vector field with one source and one sink. Thus, $\mathbb{C}P^2 \# \mathbb{R}P^4$-(2 points) has a Lorentz metric in which the number of components in the cross-sections transverse to the time direction changes from 1 to 2. Furthermore, a typical section with 2 components separates $\mathbb{C}P^2 \# \mathbb{R}P^4$-(2 points) into two noncompact pieces and one *compact* piece containing *closed* time-like particle paths. (See Figure 2.)

II. At the 1988 Spring Topology Conference in Gainesville, Florida, Otto Laback, an Austrian physicist, posed the following question. Given that we can directly observe only certain subsets of $\mathbb{R}^n$ (such as smoothly embedded 1-manifolds corresponding to particle paths), what possible topologies on $\mathbb{R}^n$ are compatible with the usual topology on physically observable subsets? To make this more precise, for a collection $\mathcal{S}$ of subsets of $\mathbb{R}^n$, let

$$\mathcal{T}_{\mathcal{S}} = \{ U \subset \mathbb{R}^n : U \cap S \text{ is a relatively open subset of } S \text{ for each } S \in \mathcal{S} \}$$
$= $ the largest topology on $\mathbb{R}^n$ which induces the standard topology on each element of $\mathcal{S}$,

and let

$$\mathcal{H}_{\mathcal{S}} = \text{the homeomorphism group of } \mathbb{R}^n \text{ with the topology } \mathcal{T}_{\mathcal{S}}.$$

Then we reformulate Laback's question as follows:

Question. For which collections $\mathcal{S}$ of subsets of $\mathbb{R}^n$ is $\mathcal{T}_{\mathcal{S}}$ the standard topology on $\mathbb{R}^n$, and is $\mathcal{H}_{\mathcal{S}}$ the standard homeomorphism group of $\mathbb{R}^n$?

We answer this question for two different choices of $\mathcal{S}$.

Theorem 1. *If $\mathcal{S}$ is a collection of subsets of $\mathbb{R}^n$ which contains all C^1 embedded 1-manifolds, then $\mathcal{T}_{\mathcal{S}}$ is the standard topology on $\mathbb{R}^n$ and $\mathcal{H}_{\mathcal{S}}$ is the standard homeomorphism group of $\mathbb{R}^n$.*

A subset S of $\mathbb{R}^n$ is a *smooth set* if for each $p \in S$, there is a neighborhood U of p in $\mathbb{R}^n$, there is an $r \geq 1$, and there is a C^{r+1} map $f : U \to \mathbb{R}$ such that $f^{-1}(0) = S \cap U$ and f has a non-zero partial derivative of order $\leq r$ at each point of U. For example, every C^2 embedded submanifold of $\mathbb{R}^n$ is a smooth set, and the zero set of every non-zero polynomial is a smooth set.

Theorem 2. *If $\mathcal{S}$ is the collection of all smooth subsets of $\mathbb{R}^n$, then $\mathcal{T}_{\mathcal{S}}$ is strictly larger that the standard topology on $\mathbb{R}^n$, and $\mathcal{H}_{\mathcal{S}}$ neither contains nor is contained in the standard homeomorphism group of $\mathbb{R}^n$.*

The following lemma is the key to the proof of Theorem 2.

Lemma 1. *There is a tame arc A in $\mathbb{R}^n$ with endpoint 0 such that for every smooth subset S of $\mathbb{R}^n$, if $0 \in S$, then $0 \notin \mathrm{cl}(S \cap (A-\{0\}))$.*

Then $A-\{0\}$ is a closed subset of $\mathbb{R}^n$ with respect to the topology $\mathcal{T}_{\mathcal{S}}$, but $A-\{0\}$ is not closed in the standard topology on $\mathbb{R}^n$. Furthermore, the standard homeomorphism of $\mathbb{R}^n$ which carries a straight line segment onto A is not continuous under the topology $\mathcal{T}_{\mathcal{S}}$, and, hence, is not an element of $\mathcal{H}_{\mathcal{S}}$. (A slight strengthening of this lemma is used to produce an element of $\mathcal{H}_{\mathcal{S}}$ which is not a standard homeomorphism.)

To produce the arc A, we use the following notation. Set $\omega = \{0,1,2,\cdots\}$. For $a = (a_1,\cdots,a_n) \in \omega^n$, set

$\|a\| = \Sigma a_i$,

$a! = \Pi(a_i!)$,

$x^a = \Pi x_i^{a_i}$ for $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$,

$$f^{(a)}(p) = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{a_n} f(p) \qquad \text{for } f : U \to \mathbb{R},\ U \text{ an open}$$

subset of $\mathbb{R}^n$, and $p \in U$.

Let $r \geq 1$, and let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is *of class C^r* if for each $a \in \omega^n$ with $\|a\| \leq r$, $f^{(a)}(p)$ exists and is continuous at every $p \in U$. Let $C^r(U)$ denote the collection of all functions from U to $\mathbb{R}$ which are of class C^r.

For $r \geq 1$, U an open subset of $\mathbb{R}^n$, $f \in C^r(U)$, and $p \in U$, the *degree r Taylor polynomial of f at p* is

$$T_p^r f(x) = \sum_{\substack{a \in \omega^n \\ \|a\| \leq r}} \frac{1}{a!} f^{(a)}(p)\, x^a \qquad \text{for } x \in \mathbb{R}^n.$$

We can now state

A Version of Taylor's Theorem. *Let* $r \geq 1$, *let* U *be an open subset of* $\mathbb{R}^n$, *let* $f \in C^{r+1}(U)$, *and let* $p \in U$. *If* $x \in \mathbb{R}^n$ *such that* U *contains the straight line segment from* p *to* $p+x$, *then there is a* $\theta \in (0,1)$ *such that*

$$f(p+x) = T_p^r f(x) + \sum_{\substack{a \in \omega^n \\ \|a\| = r+1}} \frac{1}{a!} f^{(a)}(p+\theta x)\, x^a .$$

We observe that a subset S of $\mathbb{R}^n$ is a smooth if for each $p \in S$, there is a neighborhood U of p in $\mathbb{R}^n$, there is an $r \geq 1$, and there is an $f \in C^{r+1}(U)$ such that $f^{-1}(0) = S \cap U$ and $T^r{}_q f \neq 0$ for each $q \in U$.

To prove Lemma 1, we impose a linear order $<$ on ω^n as follows. For $a, b \in \omega^n$, we declare that $a < b$ if either

$\|a\| < \|b\|$

or

$\|a\| = \|b\|$ and there is a k such that $1 \leq k \leq n$, $a_i = b_i$ for $1 \leq i < k$, and $a_k < b_k$.

To prove Lemma 1, we also need

Lemma 2. *There is an embedding* $\varphi = (\varphi_1, \cdots, \varphi_n) : [0,1] \to \mathbb{R}^n$ *such that if* $a, b \in \omega^n$ *and* $b < a$, *then*

$$\lim_{t \to 0} \frac{\varphi^a(t)}{\varphi^b(t)} = 0 .$$

(*Here* $\varphi^a(t) = \Pi(\varphi_i(t))^{a_i}$ *for* $a = (a_1, \cdots, a_n) \in \omega^n$.)

Proof of Lemma 2. First define $\psi : [0,1] \to [0,1]$ by $\psi(0) = 0$ and $\psi(t) = \ln 2/(\ln 2 - \ln t)$ for $0 < t \leq 1$. Then L'Hospital's Rule implies that for any $r \geq 0$, $t/(\psi(t))^r \to 0$ as $t \to 0$. For $1 \leq i \leq n$, set $\psi_i = \psi \circ \cdots \circ \psi$ (the i-fold composition of ψ with itself). Then for any $r \geq 0$, $\psi_i(t)/(\psi_{i+1}(t))^r \to 0$ as $t \to 0$, for $1 \leq i \leq n$. Finally, set $\varphi_i(t) = t\psi_i(t)$ for $1 \leq i \leq n$. □

Proof of Lemma 1. Set $A = \varphi[0,1]$. Suppose S is a smooth subset of $\mathbb{R}^n$ and $0 \in S$. Then there is a neighborhood U of 0 in $\mathbb{R}^n$, there is an $r \geq 1$, and there is an $f \in C^{r+1}(U)$ such that $f^{-1}(0) = S \cap U$ and $T^r{}_p f \neq 0$ for each $p \in U$. We shall show that $0 \notin \mathrm{cl}(S \cap (A - \{0\}))$. For assume $S \cap (A - \{0\})$ contains a sequence that converges to 0. We shall argue that $T^r{}_0 f = 0$, and thereby reach a contradiction.

By our assumption, there is a sequence $\{t_i\}$ in $(0,1]$ converging to 0 such that for each $i \geq 1$, $\varphi(t_i) \in S \cap U$ and U contains the straight line segment from 0 to $\varphi(t_i)$. According to Taylor's Theorem, for every $i \geq 1$, there is a $\theta_i \in (0,1)$ such that

$$0 = f(\varphi(t_i)) = T_0^r f(\varphi(t_i)) + \sum_{\substack{a \in \omega^n \\ \|a\| = r+1}} \frac{1}{a!} f^{(a)}(\theta_i \varphi(t_i))\, \varphi^a(t_i) .$$

$f^{(0)}(0) = f(0) = 0$, because $0 \in S$. Now suppose $b \in \omega^n$, $\|b\| \leq r$ and $f^{(a)}(0) = 0$ for every $a \in \omega^n$ such that $a < b$. We shall argue that $f^{(b)}(0) = 0$. It will then follow that $T^r{}_0 f = 0$.

For $i \geq 1$: if $0 \leq s \leq r$, set $x_{s,i} = 0$; and if $s = r+1$, set $x_{s,i} = \theta_i \varphi(t_i)$. Then the above Taylor formula becomes

$$0 = \sum_{\substack{a \in \omega^n \\ \|a\| \leq r+1 \\ b \leq a}} \frac{1}{a!} f^{(a)}(x_{\|a\|,i})\, \varphi^a(t_i) .$$

Divide this equation by $\varphi^b(t_i)$ and let $i \to \infty$. Then according to Lemma 2, we are left with $(1/b!) f^{(b)}(0) = 0$. □

Bounding the complexity of simplicial group actions on trees.

Mladen Bestvina and Mark Feighn

The first author gave a talk on the theorem stated below. This result generalizes theorems of Grushko and Dunwoody in combinatorial group theory, the discussion of which, as well a motivation for our theorem, can be found in excellent articles Peter G. Scott-C.T.C. Wall , Topological methods in group theory, in Homological Group Theory , ed. C.T.C. Wall, London Math. Soc. Lecture Notes 36 (1979) 137-203 and M.J. Dunwoody, The accessibility of finitely presented groups, Inv. Math. 81 (1985) 449-457.

A group action on a tree is *hyperbolic* if some pair of axes of hyperbolic elements of the group intersect in a compact set (and, in particular, high powers of the elements generate a free subgroup of rank two). A group is *small* if it admits no hyperbolic action. A group action on a tree is *minimal* if there is no proper invariant subtree, and is *reduced* if the quotient graph does not contain a valence two vertex whose stabilizer is equal to the stabilizer of an incident edge.

THEOREM. For every finitely presented group G there exists an integer $\mathfrak{C}(G)$ with the following property. The quotient graph associated to a minimal and reduced action of G on a tree with small edge stabilizers has less than $\mathfrak{C}(G)$ vertices and edges.

The theorem fails for finitely generated groups.

MULTI-LAYERED MANIFOLD DECOMPOSITIONS OF 3-MANIFOLDS

by Robert J. Daverman

The work described here was done jointly with Luis Montejano.

An upper semicontinuous decomposition G of a 3-manifold M is called a <u>multi-layered manifold decomposition</u> if each $g \in G$ is a closed, connected (usually, orientable) 1- or 2-manifold, and G includes both types.

Let T, T' be solid tori and B a 3-cell with $T \subset B \subset T'$ and T unknotted in B. The key constuction is an usc decomposition G of $T' \backslash \mathrm{Int} T$ into simple closed curves. This leads to several examples.

1. E^3 admits a multi-layered manifold decomposition.

2. Every solenoid can be embedded in S^3 so its complement admits a multi-layered manifold decomposition.

3. (Consequence of the construction - a better result follows from the Structure Theorem below.) Suppose M_1 and M_2 are 3-manifolds admitting usc decompositions into simple closed curves. Then $M_1 \# M_2$ admits a multi-layered manifold decomposition.

MAIN RESULT

Structure Theorem. Suppose the connected, orientable 3-manifold M^3 admits a multi-layered manifold decomposition G. Then M^3 contains a locally finite collection $\{C_i\}$ of pairwise disjoint objects, where C_i is either (1) a closed, connected 2-manifold for which $\chi(C_i)=0$, (2) $T\#T$ (where T denotes a solid torus), or (3) $T\#RP^3$, such that $M^3\setminus\cup C_i$ admits a usc decomposition into simple closed curves.

Remarks: all three possibilities can be realized. Proof details involve analysis of 1-winding functions for the (G_δ) subcollection of scc's. Other analysis in the argument reveals that every $g\in G$ satisfies $\chi(g)\geq 0$.

APPLICATIONS

1) Suppose M^3 is a closed, irreducible, orientable 3-manifold that contains no incompressible tori and G is a multi-layered manifold decomposition of M^3. Then M^3 admits a decomposition either into 2-manifolds or into 1-manifolds.

2) Suppose M^3 is a closed, irreducible 3-manifold with $\pi_1(M^3)$ finite and G is a multi-layered manifold decomposition of M^3. Then M^3 admits a decomposition into 1-manifolds.

3) E^3 is the only contractible 3-manifold admitting a multi-layered manifold decomposition; similarly, S^3 is the only homotopy 3-sphere admitting one.

4) Suppose that for $i=1,2$ G_i is a multi-layered manifold decomposition of the connected, orientable 3-manifold M_i. Then

$M_1 \# M_2$ admits a multi-layered manifold decomposition.

5) If G is a multi-layered manifold decomposition of S^3, then S^3/G is a cactoid.

QUESTION. Which non-compact M^3 that admit 2-manifold decompositions also admit multi-layered ones? Are there any others besides line bundles over surface S where $\chi(S)=0$?

SOME QUESTIONS ABOUT SYMMETRIC PRODUCTS

Dennis J. Garity

Definitions and History.

The n-fold symmetric product of a space X, X(n), is the subspace of the hyperspace of all closed subspaces of X, 2^X, consisting of all sets of cardinality less than or equal to n. The topology on X(n) is the Vietoris finite topology. A basis for this topology can be described as follows. Let $U_1, U_2, \ldots, U_k$ be sets in X. Let

$$<U_1, \ldots U_k> = \{A \subset X(n) \mid A \subset \bigcup_{i=1}^{k} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each i } \}.$$

A basis for the topology on X(n) is $\{<U_1, \ldots U_k> \mid \text{each } U_i \text{ is open in } X\}$. If X is metrizable via a metric d, this topology is equivalent to the topology induced by the Hausdorff metric. This metric $\tilde{d}$ is given by

$$\tilde{d}\,(A, B) = \max\left\{ \sup_{a \in A} \{\inf_{b \in B} \{d(a,b)\}\},\ \sup_{b \in B} \{\inf_{a \in A} \{d(a,b)\}\} \right\}.$$

For Hausdorff spaces, the symmetric product X(n) can also be viewed as a quotient space of the n-fold topological product of X, X^n. Define an equivalence relation G on X^n by $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Then $X(n) \cong X^n / G$.

Symmetric Products were introduced by Borsuk and Ulam [B-U] in 1931. They were interested in using symmetric products to learn more about hyperspaces, since the collection $\{X(n) \mid n \in \mathbb{Z}_+\}$ is dense in 2^X. They showed that local connectedness, separability, compactness and arcwise connectedness are invariants under the operation of taking symmetric products. They also showed that for n = 1, 2 or 3, I(n) is homeomorphic to I^n, that $S^1(2)$ is homeomorphic to the Mobius band, and that I(n) is not homeomorphic to a subset of $\mathbb{R}^n$ for $n \geq 4$.

In 1952, R. Bott [B] showed that $S^1(3) \doteq S^3$. In 1954, V. Ganea [G] showed that the dimension of X(n) is equal to the dimension of X^n for X separable metric, that symmetric products preserve contractibility and local contractibility for Hausdorff spaces, and that symmetric products preserve compact metric finite dimensional ARs and ANRs. In 1954, S. Liao shoed that $S^2(2)$ is homeomorphic to $\mathbb{C}P^2$. R. Molski [M] in 1956 showed that the symmetric square of a closed 2-manifold is a closed 4-manifold, that $I^2(n)$ is not homeomorphic to any subset of R^{2n} and that for $n \geq 3$, $I^n(2)$ is not homeomorphic with any subset of $\mathbb{R}^{2n}$.

R. Schori [S] in 1969 showed that $I^m(n)$ contains I^m as a factor for $m = \infty, 1, 2, \ldots$. He also showed that $I^m(2)$ is homeomorphic to $C(\mathbb{R}P^{m-1}) \times I^m$. J. Jaworowski [J] in 1971 showed that symmetric products of compact ANRs are ANRs. Kodama, Spież and Watanabe [KSW] in 1978 proved that if $Sh(X) \geq Sh(Y)$ then $Sh(X(n)) \geq Sh(Y(n))$. In 1980, C. Wagner [W] showed that T(2) is a nontrivial bundle over T with fiber S^2, where $T = S^1 \times S^1$.

V. Fedorchuk [F] in 1981 investigated symmetric products from a functorial point of view and showed that symmetric products of Q manifolds are Q manifolds. A. Dranishnikov [D] in 1984 showed that there was a non AR (non ANR) whose symmetric square is an AR (ANR).

Questions.

1. Which 4-manifolds arise as symmetric products of 2-manifolds?
2. Can $I^m(n)$ be described in terms of familiar spaces?
3. Is local homological connectedness or local homotopical connectedness preserved by symmetric products?
4. Are noncompact ANRs preserved by symmetric products?

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A 4-Dimensional Open Collar Theorem

Craig R. Guilbault

A natural question to ask about a manifold M^m with boundary is whether it has the simplest possible structure, i.e., whether or not M^m is homeomorphic to $\partial M \times [0,1)$. It was shown by Whitehead in 1937 that it is not sufficient to assume that $\partial M \subset M$ is a homotopy equivalence. In 1969, Siebenmann's "Open Collar Theorem" showed that, with the additional assumption that M^m have a "stable end" with the correct fundamental group, the desired conclusion holds whenever $m \geq 5$. The main result discussed here is an extension of The Open Collar Theorem to dimension 4, provided $\pi_1(M^m)$ falls into a certain class of groups.

The necessary fundamental group restrictions are the following. Let $\mathbf{F}$ denote the set of all Freedman groups (groups for which Freedman's disk embedding theorem can be proved), and let $\mathbf{S}$ denote the set of all groups G such that Wh(G) (the Whitehead group of G) is trivial. The set of allowable groups for our theorem is $\mathbf{A} = \mathbf{F} \cap \mathbf{S}$.

THEOREM 1 (4-Dimensional Open Collar Theorem)
Let M be a 4-manifold with $\pi_1(M) \in \mathbf{A}$. Then M is homeomorphic to $\partial M \times [0,1)$ iff each of the following each of the following conditions holds:

(a) $\partial M \subset M$ is a homotopy equivalence.

(b) π_1 is stable at infinity with $\pi_1(\infty) \to \pi_1(M)$ an isomorphism.

An easy consequence is the following;

COROLLARY 2 A 4-manifold M with $\pi_1(M) \in \mathbf{A}$ is homeomorphic to $\partial M\times[0,1)$ iff the two are proper homotopy equivalent.

Theorem 1 is proved by building a proper h-cobordism between M and $\partial M\times[0,1)$, and then applying the 5-dimensional proper s-cobordism theorem (see [F-Q]). The main tool is engulfing (including radial engulfing) along with an application of Siebenmann's (higher dimensional) Open Collar Theorem to help get the necessary homotopy conditions. Details can be found in [Gu_1] or [Gu_2].

As an application of Theorem 1 we are able to solve a problem concerning embeddings of 2-spheres in S^4. An embedded k-sphere $\Sigma^k \subset S^n$ is said to be "weakly flat" provided its complement is homeomorphic to the complement of the standard k-sphere in S^n. An especially interesting case occurs when k=n-2. Results by Daverman (n=3) and Hollingsworth and Rushing (n≥5) characterize weakly flat (n-2)-spheres in S^n as those which satisfy a certain embedding condition (Σ^{n-2} is globally 1-alg), along with $S^n-\Sigma^{n-2}$ being homotopy equivalent to a circle. The latter is needed to prevent knotting. We are able to extend this characterization to include 2-spheres in S^4. The theorem is as follows;

THEOREM 3 A 2-sphere $\Sigma^2 \subset S^4$ is weakly flat iff it is globally 1-alg and $S^4-\Sigma$ is homotopy equivalent to S^1.

The "global 1-alg condition referred to above is defined in the following way. An embedded k-sphere $\Sigma^k \subset S^n$ is <u>globally 1-alg</u> provided each neighborhood U of Σ contains a neighborhood V of Σ such that loops which are null-homologous in $V-\Sigma$ are contractible in $U-\Sigma$.

Proof of Theorem 3 is accomplished by removing the interior of a regular neighborhood of a generator of the fundamental group of $S^4-\Sigma$. Theorem 1 is then employed to verify that what is left is simply an open collar. The 1-alg condition is used to verify condition (b) of Theorem 1. Again details can be found in [Gu_1] or [Gu_2].

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Hyperbolic Structures on Branched Covers Over the Figure-Eight Knot

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Recall Thurston's construction of hyperbolic metrics on the figure-8 knot complement, using the fact that the figure-8 knot complement may be written as the union of two tetrahedra with vertices deleted: Ideal hyperbolic tetrahedra are parametrized by complex numbers in the upper half-plane, and the conditions for two tetrahedra, parametrized by w and z, to yield an (incomplete) hyperbolic structure on the figure-8 knot complement are polynomial equations in w and z. Some of these structures may be completed to induce hyperbolic structures on certain branched covers of spaces obtained by Dehn surgery along the figure-8 knot (namely those branched covers which have all their branching indices equal) and the conditions for this to happen are also polynomial equations in w and z.

Branched covers over the figure-8 knot (including all closed, orientable 3-mfds. by universality) can be subjected to an extension of the same analysis: let M be a branched cover over K, the figure-8 knot, with monodromys $\phi:\Pi_1(S^3 - K)\to S_d$ and branched covering $p:M\to S^3$. Then, $M - p^{-1}(K)$ may be written as the union of $2d$ ideal tetrahedra, parametrized by $w_1, \ldots, w_d, z_1, \ldots, z_d$. Conditions for hyperbolic structures to be induced by this decomposition on $\tilde{M}$, a branched cover over M with branching index n_j at all components of the pre-image in $\tilde{M}$ of K_j, the jth component of $p^{-1}(K)$, are as follows (writing permutations as operators on the right):

$$(1-w_j)(1-w_{j\phi(\bar{c})})(1-z_{j\phi(a\bar{c})})(1-z_{j\phi(a)})z_j w_{j\phi(a\bar{c})}=(1-z_j)(1-w_{j\phi(a\bar{c})}) \quad \forall j=1,\ldots,d$$

$$z_j w_j z_{j\phi(\bar{a}b)} w_{j\phi(b\bar{a})}(1-z_{j\phi(b)})(1-w_{j\phi(\bar{a})})=z_{j\phi(b)} w_{j(\bar{a})} \quad \forall j=1,\ldots,d$$

$$e^{2\pi i/n_j} = \prod_{k=1}^{o(\phi(a)\text{ on }j)} \frac{z_{j\phi(a^k b\bar{a}c)} w_{j\phi(a^k b\bar{a})}(1-z_{j\phi(a^k b)})}{z_{j\phi(a^k b)}} \quad \forall j=1,\ldots,d$$

(but note that only q of these are distinct, where q=# cycles in $\phi(a)$)

Where a,b,c refer to specific generators for $\pi_1(S^3 - K) = \langle a,b,c : b=ca\bar{c},\ c=[a,\bar{b}]\rangle$

Note that a complete hyperbolic structure corresponds to a solution where none of the w_j's and z_j's are real, while a complete hyperbolic foliation corresponds to a solution where all of the w_j's and z_j's are real. Note also that we may look for hyperbolic structures on M itself by setting all n_j's to 1.

Define a <u>tetrahedral hyperbolic structure</u> to be a complete hyperbolic structure that arises in this fashion. Note that whether or not a given structure is tetrahedral may depend on the particular branched covering representation for M (and thus $\widetilde{M}$).

Prop. 1: if M has any branching index equal to 1, then M has no tetrahedral hyperbolic structures.

Prop. 2: if $p^{-1}(K)$ is connected, then M has a complete hyperbolic structure if M has a tetrahedral hyperbolic structure.

One necessary condition for a given hyperbolic structure to be tetrahedral is that the geodesics in the free hemotopy classes of distinct components of the branch locus not intersect. If this condition is violated, we have:

Prop. 3: if two distinct components of the branch locus of (hyperbolic) M are freely homotopic to intersecting geodosics, then M contains an injectively immersed closed surface of genus ≥ 2.

Question: are there branched covers over K that have hyperbolic structures, but not tetrahedral hyperbolic structures?

Surface Mapping Class Groups
Heegaard Splittings of S^3 and Homology Spheres

Ning Lu

Let $\mathcal{M}_g$ be the mapping class group of the closed orientable surface of genus g, we prove that $\mathcal{M}_g$ is generated by three elements in general. They are <u>the linear cutting</u>

$L = [a_1b_1, b_1, a_2, b_2, \ldots, a_g, b_g]$,

<u>the normal cutting</u>

$N = [x\bar{a}_2b_1, \bar{a}_1, \bar{a}_1xb_2, \bar{a}_2, a_3, \ldots, b_g]$,

where $x = [a_1,b_1]\,[a_2,b_2]$, and

<u>the transport</u>

$T = [a_g, b_g, a_1, b_1, \ldots, a_{g-1}, b_{g-1}]$,

where, as a convention we denote

$f = [f(a_1), f(b_1), \ldots, f(a_g), f(b_g)]$

by the fact that, the mapping class f is determined by them. In particular, when g=1, we have T=1, and $\mathcal{M}_1 = \langle L_1N \mid N^6=1,\ NL=\bar{L}N^2 \rangle$. And when g=2, we have $T=N^3$, and $\mathcal{M}_2 = \langle L_1N \mid N^6=1,\ (LN)^5=1,\ (L\bar{N})^5(\bar{N}L)^5=1$

$L \leftrightarrow N^2L\bar{N}^2,\ L \leftrightarrow N^3L\bar{N}^3,\ N^3 \leftrightarrow (L\bar{N}LNL)^4 \rangle$

and an interesting consideration is that, denote $D_i = N^iL\bar{N}^i$, i=0, 1, ..., 5, then we have

i) All are Dehn twists, and any five of them form a family of Lickorish generators.

ii) $N = D_0D_1D_2D_3D_4 = D_1D_2D_3D_4D_5 = \ldots = D_5D_0D_1D_2D_3$

iii) Let $\nu \in \mathrm{Aut}(\mathcal{M}_2)$, $\nu(f) \underset{=}{\text{def}} Nf\bar{N}$, then $\nu(D_i) = D_{i+1}$.

And which induces us a possibly nice and new field of the group theory, <u>the balanced group theory</u>. (A group G is balanced, if it admits an element $a \vee G$, and a (periodic) automorphism $\nu \in \mathrm{Aut}(G)$, such that the family $\{\nu^n(a) \mid n \in \mathbb{Z}\}$ generates the group G. e.g. $\mathcal{M}_2$ is balanced) .

<u>Question</u>: What kind of groups is balanced? And an easy combination is that also $\mathcal{M}_1$ is balanced, since $N=LNL\bar{N}$ when $g=1$.

The most important application of the new generators is that the Heegaard splittings of the 3- sphere S^3 may be well classified and explicity described whose associated mapping classes form the semiproduct $\mathcal{L}\cdot\mathcal{N}\cdot\mathcal{M}$ of the subgroups $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$.

Where $\mathcal{N}=$ T, N^3, P^2, PN^2P is contained in the subset of mapping classes which can be extended to both handle bodies,

$\mathcal{L}=$ T^iLT^i i=0, 1, ..., g-1 and

$\mathcal{M}=$ T^iMT^i i=0, 1, ..., g-1

are free abelian subgroups of rank g, where $M = \bar{N}LN$.

As a consequence of our result, we show easily that any homology sphere is associated by some mapping class from the Torelli subgroup of the mapping class group $\mathcal{M}g$. And at the same time, more properties are investigated.

A PRELIMINARY REPORT ON CLOSED PRE-IMAGES OF C-SPACES

Yasunao Hattori, Dale M. Rohm* and Kohzo Yamada

For the purposes of this discussion, all spaces are assumed to be metric, although results do hold for more general topological spaces. The reader is referred to [R1] and [R2] for statements of the definitions and an introduction to the topics given in this note. Two types of questions will be considered, the first being

Question 1. When is the product of two weakly infinite-dimensional (in the sense of Alexandroff) spaces weakly infinite-dimensional?

Although R. Pol showed that this is not always the case [P2], the factors of his counter-example are not compact. It is unknown whether the product of two compact weakly infinite-dimensional spaces must always be weakly infinite-dimensional. Shortly thereafter, E. Pol showed that the product of two weakly infinite-dimensional spaces does not need to be weakly infinite-dimensional even when one factor is zero-dimensional [P].

A space is said to be a C-space if it has the covering property C as defined by Addis and Gresham [AG]. Every countable-dimensional space has property C and every C-space is weakly infinite-dimensional, however R. Pol has constructed a C-space which is not countable-dimensional [P1]. It is not known whether every weakly infinite-dimensional space must have property C. In particular, the weakly infinite-dimensional factors of the strongly infinite-dimensional products mentioned above are actually C-spaces. It is known that the product of two C-spaces, with one factor compact, is again a C-space [R2].

The other type of question to be considered is

Question 2. What can be said about the domain of a closed map with weakly infinite-dimensional image?

Answers to this question can be thought of as results on dimension-lowering maps for the infinite-dimensional dimension theories discussed above. As such, these answers complement, and provide partial converses to, the results on dimension-raising maps given in [G] and [GR].

In particular, Hattori and Yamada have generalized the product theorem for C-spaces, in so doing answering [R1, Question 2] and [R2, Question 2], to obtain the following infinite-dimensional version of a classical dimension theory Hurewicz-Morita theorem [E, Theorems 1.12.4 and 4.3.6].

Theorem 1. Let $f: X \to Y$ be closed mapping onto a C-space Y, then the domain X is a C-space if and only if $f^{-1}(y)$ is a C-space for each $y \in Y$.

Using a characterization of weak infinite-dimensionality in terms of open covers the second author has shown

Theorem 2. The product of a C-space with a compact weakly infinite-dimensional space is weakly infinite-dimensional.

This immediately generalizes by the techniques of Hattori and Yamada to give

Theorem 3. Let $f: X \to Y$ be closed mapping onto a C-space Y, then the domain X is weakly infinite-dimensional if and only if $f^{-1}(y)$ is weakly infinite-dimensional for each $y \in Y$.

With all spaces metric, the authors do not know whether the assumption of property C in Theorems 2 and 3 can be weakened to weak infinite-dimensionality. The following corollary is singled out as a special case of separate interest.

Corollary. A compact space X is weakly infinite-dimensional if and only if $X \times I$ is weakly infinite-dimensional.

Because of the similarities between these results and the Dowker Conjecture the following question is raised.

Question 3. If $X \times I$ is weakly infinite-dimensnional then must X be a C-space. If so for every compact space X, then every compact weakly infinite-dimensional space is a C-space.

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The K–Theory of Eilenberg-Maclane Complexes Revisited

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The focus of the talk is the "K–theoretic component" of the construction by A. Dranišnikov [**Dr**] of an infinite dimensional compact metric space having integral cohomological dimension three. Briefly, Dranišnikov combines a construction from [**Wa**] that produces compacta having finite cohomological dimension with a "K–theoretic" invariant that detects that the dimension is strictly larger than the cohomological dimension and, therefore, is infinite. There are two sources for the K–theory computations; namely, [**AH**] or [**BM**]. The talk is based on [**AH**].

The theory needed is complex K–theory, denoted by $k(X,A)$. Reduced groups are denoted by $\tilde{k}(X,A)$ and the groups with coefficients $\mathbf{Z}_p$ are denoted by $\tilde{k}(X,A;\mathbf{Z}_p)$. The specific fact used in [**Dr**] is the following:

$$\tilde{k}^*(K(\bigoplus_1^r \mathbf{Z}, n); \mathbf{Z}_p) \simeq 0,\ n \geq 3, \text{ p a prime}$$

where $K(G,n)$ denotes an Eilenberg-Maclane complex associated to an abelian group G. That is, $K(G,n)$ is a CW-complex with $\pi_n(K(G,n)) \simeq G$ and $\pi_j(K(G,n)) \simeq 0$ for $j \neq n$. Since $K(\bigoplus_1^r \mathbf{Z}, n) \simeq \prod_1^r K(\mathbf{Z},n)$, the Kunneth formula reduces the computation to showing that

$$\tilde{k}^*(K(\mathbf{Z},n);\mathbf{Z}_p) \simeq 0,\ n \geq 3, \text{ p a prime}.$$

The first step in the proof is to analyze the universal coefficient theorem

$$0 \to \tilde{k}^*(X) \otimes \mathbf{Z}_p \to \tilde{k}^*(X;\mathbf{Z}_p) \to Tor(\tilde{k}^*(X), \mathbf{Z}_p) \to 0$$

and reduce the computation to showing that $\tilde{k}^*(K(\mathbf{Z},n))$ is p-torsion free and p-divisible for $n \geq 3$.

The second step which is the "heart of the proof" is to establish that

$$k^*(K(\mathbf{Z},n)) \simeq k^*(K(\mathbf{Q},n)),\ n \geq 3,$$

the isomorphism being induced by the "natural" homomorphism $\mathbf{Z} \to \mathbf{Z} \otimes \mathbf{Q} \simeq \mathbf{Q}$. A brief outline of this step is given below.

The third step is to establish that $\widetilde{H}^*(K(\mathbf{Q},n);\mathbf{Z})$ is torsion free and divisible, i. e., a $\mathbf{Q}$ -module. This is carried out by induction on n. For $n=1$, a standard model for $K(\mathbf{Q},1)$ is the mapping telescope of the sequence of maps $\alpha_n : S^1 \to S^1$ where α_n has degree n. Using this model, which is a 2-dimensional complex, it is immediate that $H^k(K(\mathbf{Q},1);\mathbf{Z}) \simeq 0$, for $k \geq 3$. It is an easy exercise to show that $K(\mathbf{Q},1)$ admits no essential maps to S^1 and, thus, $H^1(K(\mathbf{Q},1);\mathbf{Z}) \simeq 0$. Further analyses shows that $H^2(K(\mathbf{Q},1);\mathbf{Z}) \simeq \lim^1\{X_i\}$, where X_i is the mapping telescope of $\alpha_1, \alpha_2, \cdots, \alpha_i$. Finally, using the explicit description of the group $\lim^1\{X_i\}$, it is easy to establish that the group is both torsion free and divisible. For $n \geq 2$, analyses using the Serre spectral sequence associated to the fibration

$$K(\mathbf{Q}, n-1) \hookrightarrow E \to K(\mathbf{Q}, n)$$

(be reminded that E is contractible) and induction complete the proof.

The fourth and final step is to use the spectral sequence whose E_2-terms are the groups $H^*(X; k^*(\text{point}))$ and which converges to $k^*(X)$ and establish that, for a space X with $H^*(X; \mathbf{Z})$ torsion free, $k^*(X) \simeq H^{**}(X; \mathbf{Z})$ as abelian groups. Recall that $k^*(X) = k^0(X) \oplus k^1(X)$ and $H^{**}(X; \mathbf{Z}) = H^{\text{ev}}(X; \mathbf{Z}) \oplus H^{\text{od}}(X; \mathbf{Z})$, where $H^{\text{ev}}(X; \mathbf{Z})$ is the group of formal power series having coefficients in the groups $H^{2i}(X; \mathbf{Z})$ and $H^{\text{od}}(X; \mathbf{Z})$ is the group of formal power series having coefficients in the groups $H^{2i+1}(X; \mathbf{Z})$. In view of step 3, this applies to $K(\mathbf{Q}, n)$ and it follows that $k^*(K(\mathbf{Q}, n))$ is divisible and torsion free.

As promised, a brief outline of step 2, namely, that

$$(\dagger) \qquad k^*(K(\mathbf{Z}, n)) \simeq k^*(K(\mathbf{Q}, n)),\ n \geq 3,$$

follows. The short exact sequence $\mathbf{Z} \hookrightarrow \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ gives rise to a sequence of fibrations:

$$\begin{gathered}
\mathbf{Q}/\mathbf{Z} \hookrightarrow K(\mathbf{Z}, 1) \to K(\mathbf{Q}, 1) \\
K(\mathbf{Q}/\mathbf{Z}, 1) \hookrightarrow K(\mathbf{Z}, 2) \to K(\mathbf{Q}, 2) \\
K(\mathbf{Q}/\mathbf{Z}, 2) \hookrightarrow K(\mathbf{Z}, 3) \to K(\mathbf{Q}, 3) \\
\vdots \\
K(\mathbf{Q}/\mathbf{Z}, n-1) \hookrightarrow K(\mathbf{Z}, n) \to K(\mathbf{Q}, n) \\
\vdots
\end{gathered}$$

In particular, the isomorphism in $(\dagger)$ follows (via a Vietoris type theorem) from

$$(\ddagger) \qquad \tilde{k}(K(\mathbf{Q}/\mathbf{Z}, n)) \simeq 0,\ n \geq 2.$$

The computation in $(\ddagger)$ is a consequence of $\mathbf{Q}/\mathbf{Z}$ being a countable torsion group. A countable torsion group is the union of an increasing sequence of finite subgroups and, thus, $K(\mathbf{Q}/\mathbf{Z}, n)$ can be represented as the increasing union of $K(\pi, n)$'s where π is a finite group. The groups $k(K(\mathbf{Q}/\mathbf{Z}, n))$ are determined by the inverse sequence of groups $k(K(\pi, n))$ and $(\ddagger)$ can be extracted once it is known that

$$\tilde{k}(K(\pi, n)) \simeq 0, \text{ for } \pi \text{ a finite abelian group and } n \geq 2.$$

Of course it is this last computation that forms the core of [**AH**] and the reader is directed there for the details. The starting point is the explicit computation in [**At**] of the groups $k(K(\pi, 1))$, for π a finite abelian group. The Serre spectral sequence of the fibration $K(\pi, 1) \hookrightarrow E \to K(\pi, 2)$ is used, along with profinite analyses, to deduce from the explicit computation of $k(K(\pi, 1))$ that $\tilde{k}(K(\pi, 2)) \simeq 0$. In turn, the same spectral sequence applied to $K(\pi, n-1) \hookrightarrow E \to K(\pi, n)$ and induction establish (much more easily) that $\tilde{k}(K(\pi, n)) \simeq 0$ for $n \geq 3$ as well.

References

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A STICKY ARC IN S^n (n>3)

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This paper is the summary of an expository talk given at the topology conference at Colorado College, Colorado Springs, Colorado on June 16-18,1988.

1. Introduction. If X and Y are subsets of a metric space S, we say that X can be *slipped* off Y in S if for each $\varepsilon > 0$ there is an ε-homeomorphism $h:S \to S$ such that $h(X) \cap Y = \emptyset$; otherwise, we say X *cannot be slipped* off Y. Results of Armentrout [A] and McMillian [M_2] show that if A and B are arcs in E^3, then A can be slipped off B. Previous results of the author [W] show that in E^n, $n > 3$, there exist cellular arcs A and B such that A cannot be slipped off B. The examples depend heavily on a result of McMillian [M_1]. In this note we show that we can choose A and B so that they have only an endpoint in common. Furthermore, we show the existence of a single arc A in E^n so that A cannot be slipped off itself.

McMillan constructed two disjoint continua in S^3 which we denote by X_+ and X_-. The set $X_+ = \bigcap_{i=1}^{\infty} H_i$ and $X_- = \bigcap_{i=1}^{\infty} K_i$ where H_i and K_i are disjoint unknotted cubes with two handles. The set $H_i \cup K_i$ has the property that it is not I -equivalent to $H \cup K$ where H and K are disjoint unknotted cubes with two handles in S^3 which can be separated by a 2-sphere. This observation leads to the fact that certain 4-manifolds cannot be embedded in S^4. Such 4-manifolds M_i can be obtained from a 4-ball by attaching two more disjoint 4-balls along H_i and K_i, respectively, via the identity mapping. The continuum X_- is cell-like. The continuum X_+ is not cell-like, but the suspension $\sum X_+$ is PL cellular in $\sum S^3 = S^4$.

Let G denote the upper semi-continuous decomposition of S^4 whose non-degenerate elements are $\sum X_+$ and and the disjoint copies of X_- found in the levels of $\sum X_-$. The decomposition G is shrinkable [E-M],[P-E]. Let $f:S^4 \to S^4$ be a map whose only non-degenerate point inverses are the non-degenerate elements of G. Let $p = f(\sum X_+)$ and $J = f(\sum X_+ \cup \sum X_-)$. Then J is a simple closed curve in S^4. Let A, B be arcs in J such that $A \cap B = \{p\}$.

THEOREM 1. The arcs A and B are cellular sets such that A cannot be slipped off B.

THEOREM 2. The arc $A \cup B$ is a cellular set that cannot be slipped off itself.

The fact that the arcs are cellular follows because given neighborhoods U and V of p and the arc, respectively, there exists a homeomorphism of S^4, fixed outside V that takes V into U.

Proof of Theorem 1. Let q be an element of J - $(A \cup B)$. Let A' and B' be the arcs in J with endpoints p,q containing the arcs A and B, respectively. By techniques in [W], there is a homeomorphism $h:S^4 \to S^4$ such that $h(A') \cap B' = \{q\}$ and h fixes points on a neighborhood of q. The arc $h(A') \cup B'$ has a neighborhood which is homeomorphic to M_i, for some i. But M_i does not embed in S^4, a contradiction.

Proof of Theorem 2. If $A \cup B$ can be slipped off itself, then A can be slipped off B. This contradicts Theorem 1.

Theorems 1 and 2 remain vailid for $n > 4$. The modifications are the same as used by McMillan [M_2]. [

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[A] S. Armentrout, Decompositions of E^3 with a compact 0-dimensional set of nondegenerate elements, Trans. Amer. Math Soc. 123 (1966), 165-177.

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[M_1] D. R. McMillan, Jr., An arc in a PL n-manifold with no neighborhood that embeds in S^n, n≥4., Michigan Math. J. 25 (1978), 29-43.

[M_2] ______________, Jr., An arc in a PL n-manifold with no neighborhood that embeds in S^n, n≥4., Michigan Math. J. 25 (1978), 29-43.

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Problem Session

Saturday, June 18 at 11:30 am

1. Are irreducible 3-manifolds with infinite fundamental group virtually Haken?

2. Is there a resentable 3-manifold group?

3. Does there exist an infinite 3-manifold group which contains a surface group? If so, is it virtually Haken?

4. Is every hyperbolic 3-manifold virtually Z-representable?

5. Given a branched cover, is a hyperbolic structure necessarily tetrahedral with respect to that branched cover?

6. If G is an usc decomposition of a 4-manifold into circles with no degeneracy, is the decomposition space a generalized 3-manifold?

7. Does there exist an usc decomposition of the n-cell (compact,contractible set) ((cell-like set)) into circles?

8. Recall that if $U_1, U_2, \ldots, U_n$ form a cover of T^n, then for some j, $\mathrm{Im}(H_1(U_j) \dashrightarrow H_1(T^n)) \neq \emptyset$. Is it possible to identify all $\{\mathrm{Im}(H_1(U_j)) \mid 1<j<n\}$? Suppose $\bar{U}_1, \ldots, \bar{U}_n$ is a shrinking of $U_1, \ldots, U_n$; is it possible to identify all relations $\mathrm{Im}(H_1(\bar{U}_j)) \subset \mathrm{Im}(H_1(U_j)$?

9. Is Dranishnikov's example strongly infinite dimensional?

10. Which 4-manifolds are symmetric products?

11. What local homology or homotopy properties are preserved under symmetric products?

12. Are the Mazur and standard link I-equivalent by a smooth surface?

13. Characterize all spaces whose product with any 0-dimensional space is weakly infinite dimensional. Is the image of every cell-like dimension-raising map strongly infinite dimensional?

14. If XxI is weakly infinite dimensional, then must X have property C?

15. Can a 2-sphere in S^3 be weakly flat but not PL weakly flat?

16. Given an usc decomposition of M^{n+k} into orientable, closed n-manifolds, what is the largest k for which M/G is always a generalized n-manifold?

17. Is there a decomposition of S^3 into circles?

18. Under what conditions can you put a wild end on a manifold?

19. If you have a degree 1 map from a closed n-manifold to itself, must it be a homotopy equivalence?

20. Find invariants to detect non-I equivalence for smooth, PL or locally flat categories.

21. Define a syllabus to be a motivated, briefly annotated, and directed reading list, shorter than and at a more elementary level than the usual survey article.

    Continued mathematical productivity often requires transition from field to field. Wide-spread, short, current syllabi could greatly aid such transitions.

    Describe the appropriate format for such syllabi. Give sample syllabi for your field. Cite examples of such syllabi in the published literature.

    Should there be a journal devoted to such syllabi? Would "study guide" be a better term?