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The Third Annual Western Regional Mini-conference in Geometric Topology was held at The Colorado College, Colorado Springs, Colorado on June 12-14, 1986. The participants were:

Phillip Bowers	Florida State University
James Cannon	Brigham Young University
Dennis Garity	Oregon State University
James Henderson	Colorado College
Jack Lamoreaux	Brigham Young University
Terry Lay	Idaho State University
Frederick Tinsley	Colorado College
John Walsh	Univ. of Calif., Riverside
David Wright	Brigham Young University

These proceedings contain the notes of two one-hour talks given by the invited speaker, John Walsh, and summaries of one hour talks by other participants.

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Jim Henderson
Fred Tinsley

TABLE OF CONTENTS

John J. Walsh, <u>Applications of Sheaf Theory to Geometric Topology</u>	1
James W. Cannon, <u>Solvgroups Are Not Almost Convex</u>	8
Dennis J. Garity, <u>Menger Spaces and Inverse Limits</u>	9
James P. Henderson, <u>Menger Spaces and σ</u>	14
Terry L. Lay, <u>On Cellular Decompositions of the Hilbert Cube</u>	18
Frederick Tinsley, <u>Acyclic Maps of 4-manifolds</u>	22
David G. Wright, <u>Bing-Whithead Decompositions of E^3</u>	25
<u>PROBLEM SESSION QUESTIONS</u>	30

Defn: A sheaf is associated to each presheaf, S , on X :

a. stalks: $x \in X \quad S_x = \text{dir lim}_{x \in U} \{S(U)\}$

b. total space: $\bigcup_{x \in X} S_x = S$

c. topology: $U^{\text{open}} \subset X; \alpha \in S(U)$

$$U_\alpha = \{x \in X : \alpha|_U \in S_x\}$$

d. $p : S \rightarrow X$, given by $p(S_x) = x$, is a local homeomorphism. However, S is seldom T_2

Consider the sheaves associated with the examples given above of presheaves:

(1¹) homology sheaves H_k

$$(H_k)_x = \text{stalk} = \text{dir lim}_{x \in U} H_k(X, X - U)$$

$$= H_k(X, X - x)$$

(2¹) cohomology sheaves of $f : X \rightarrow Y : H_k(f)$

$$f \text{ proper: } (H^k(f))_y = \text{stalk} = \text{dir lim}_{y \in U} H^k(f^{-1}(U))$$

$$= H^k(f^{-1}(y))$$

(3¹) $X \times G$ The discrete topology on G yields the constant sheaves.
 \downarrow
 X

Defn: The complete presheaf, S , associated with a sheaf, Γ_S, S , is given by the sections:

$$\Gamma_S(U) = \{s : U \rightarrow S \mid p \circ s(x) = x\}$$

$$\begin{array}{ccc} & \Gamma_S & \\ U & \xrightarrow{\quad} & \Gamma_S(U) \\ \downarrow & & \downarrow i_* \\ V & \xrightarrow{\quad} & \Gamma_S(V) \\ & \Gamma_S & \end{array}$$

where $i_*(s) = s|_V$.

Applications of Sheaf Theory to Geometric Topology

John J. Walsh

(From notes compiled by Fred Tinsley)

0. Preliminaries:

X , topological space.

Defn: Presheaves on X are contravariant functors S :

$$\left\{ \begin{array}{c} \text{open subsets of } X \\ \text{inclusions} \end{array} \right\} \xrightarrow[S]{} \left\{ \begin{array}{c} \text{abelian groups} \\ \text{homomorphisms} \end{array} \right\}$$

Examples:

1. homology presheaves H_k on X

U, V open in X

$$U \longrightarrow H_k(X, X - U)$$

$$\bigcup H_k \quad \downarrow i_*$$

$$V \longrightarrow H_k(X, X - V)$$

2. cohomology presheaves H^k of a map $f: X \longrightarrow Y$

U, V open in Y

$$U \longrightarrow H^k(f^{-1}(U))$$

$$\bigcup H^k \quad \downarrow i^*$$

$$V \longrightarrow H^k(f^{-1}(V))$$

3. Constant presheaves on X ; G abelian

U, V open in X

$$S(U) = G$$

$$V \subset U \quad S(U) \xrightarrow{\text{id}} S(V)$$

The trivial presheaf on X is the constant presheaf on X with $G = 0$.

The composition Presheaves \rightarrow Sheaves \rightarrow Complete presheaves assigns a complete presheaf to each presheaf:

$$S \xrightarrow[\text{limits}]{\text{direct}} S \xrightarrow{\text{sections}} \Gamma_S$$

or

$$S(U) \rightarrow \Gamma_S(U)$$

$$\alpha \rightarrow s_\alpha(x) = \alpha_x$$

However, important information may be lost. Consider the Hopf map $h:S^3 \rightarrow S^2$ versus the projection $\pi: S^2 \times S^1 \rightarrow S^2$. Note that

$$H^1[h](S^2) = H^1(S^3) = 0$$

$$H^1[\pi](S^2) = H^1(S^2 \times S^1) = \mathbb{Z}.$$

However, the corresponding sheaves are identical.

$$H^1[h] = H^1[\pi] = S^2 \times \mathbb{Z}, \text{ the constant sheaf.}$$

1. Applications

The first application is to local connectivity of decomposition spaces. In 1957 Smale proved:

Theorem: Suppose $f:X \rightarrow Y$ is a map of compact metric spaces with $X \in LC^k$ and $f^{-1}(y)$ a k -connected ANR (each $y \in Y$). Then $Y \in LC^k$.

Work in the 1960's and 1970's weakened the hypothesis of "triviality of fiber homotopy" to "adequate aligning of fibers." Also, the ANR hypothesis was replaced by a shape version.

The proofs intermingled homotopy and homology theory. Sheaf theory allows a purely homological version.

The second application is to cohomological dimension theory. Let $\dim(X)$ and $\dim_{\mathbb{Z}}(X)$ denote the dimension and cohomological dimension of X , respectively. The classical theorem states:

Theorem: If $f: X \dashrightarrow Y$ is a proper surjection and cardinality $(f^{-1}(y)) \leq n + 1$, $y \in Y$, then $\dim(Y) \leq \dim(X) + n$.

Addendum: f as above. If $\dim(X) < \infty$ and cardinality $(f^{-1}(y)) < \infty$, then Y is countable dimensional.

Sheaf theory yields a cohomological version.

Theorem: If $f: X \dashrightarrow Y$ is a proper surjection with

- i) Y complete
- ii) $\dim_{\mathbb{Z}}(X) < \infty$
- iii) $H^*(f^{-1}(y))$ finitely generated, $y \in Y$.
- iv) There is an n such that
 - $\text{rank}(H^k(f^{-1}(y))) \leq n, y \in Y, k \geq 0$
 - $\text{cardinality}(\text{Tor}(H^k(f^{-1}(y)))) \leq n, y \in Y, k \geq 0.$

Then $\dim_{\mathbb{Z}}(Y) < \infty$

Addendum: f as above. If $f^{-1}(y) \in \text{ANR}$, $y \in Y$, then Y is cohomologically countable dimensional.

The third application is to the study of homology manifolds, motivated from two problems posed by Borsuk.

1. Is a finite dimensional compact, connected, and homogeneous AR always a point?
2. Are compact, connected, finite dimensional, homogeneous ANR's always homology manifolds?

Bryant gives a partial answer to (2):

Theorem: If X satisfies hypotheses of (2) and if $H_*(X, X - x)$ is finitely generated at some x , then X is a homology manifold.

Bryant's proof uses the fact that ANR's have mapping cylinder neighborhoods.

To see that the homology condition is necessary, let $f_i: S_i^1 \rightarrow S_i^1$ be of degree 2 and X be the one point compactification of:

$$\bigcup_{i=1}^{\infty} \text{Map}(f_i)$$

Then $H_*(X, X - x)$ is not finitely generated.

A slightly stronger theorem avoids the deep mapping cylinder structure theorem.

Theorem: Let X be a compact, connected, homologically locally connected metric space with $\dim_Z(X) = n$ and $H_*(X, X - x) = H_*(X, X - y)$ finitely generated for all $x, y \in X$. Then X is a homology manifold.

Proof: The proof depends on the two results:

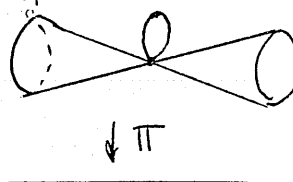
1. **Key Tool:** (Bredon) If the homology sheaves of X are locally constant with finitely generated stalks, then X is a homology manifold.

Defn: A presheaf S on X is locally finitely generated provided for all $x \in U$ open, there V open $\subset U$, $x \in V \subset U$, such that $\text{im}\{(S(U) \rightarrow S(V))\}$ is finitely generated.

Example: Let C be a Cantor set and $\pi: C \times S^1 \rightarrow C$ be projection on the first factor. Then the presheaf $H^1[\pi]$ is locally constant, but the induced sheaf, $H^1[\pi]$ is not locally finitely generated.

2. **Lemma:** (Dydak-Walsh, Theorem 1) Suppose G is a finitely generated abelian group and X is completely metrizable. If S is a locally finitely generated pre-sheaf with $S_x = G$ for all $x \in X$, then S is locally constant on a dense open subset of X .

Example: Consider



Note that $H^1[\pi]$ has mutually isomorphic stalks but is not locally constant.

Together, the Key Tool and the Lemma yield the desired theorem, for X is a homology manifold over a dense open set $W \subset X$ and for $x \in W$, excision yields $H_*(X, X - x) = H_*(W, W - x)$.

The proof of the Lemma appears in [Dydak-Walsh]. We sketch the proof of the Key Tool.

Key Tool: Suppose X is a compact, connected, homologically locally connected metric space with $\dim_{\mathbb{Z}}(X) = n$, $H_*(X, X - x) = H_*(X, X - y)$ for all $x, y \in X$, and the homology sheaves are locally constant with finitely generated stalks. Then X is a homology manifold.

Pf: Let $G_k[\mathbb{Z}]$ denote the stalk.

To show:

$$G_k[\mathbb{Z}] = \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$$

Now, $G_k[F] = H_k(X, X - pt; F)$, F the coefficient ring. The homology sheaf on X naturally yields the spectral sequence. $E_2^{p,q}[F]$, $-\infty < p, q < \infty$, with $E_2^{p,q}[F] = H_C^p(G_{-q}(F))$. Also, $E_2^{p,q}[F] = 0$ for $q > -r$ and $p > m$ so $E_2^{m,-r}[F] = E_{\infty}^{m,-r}[F] = H_{r-m}^C(X; F)$.

Take $F = Q$ and $\dim_Q X = m$. Then WLOG $H_C^m(X; Q) \neq 0$. Let r be the smallest integer with $G_k[Q] = 0$ for $k < r$ (or $r=m$). By the above, $H_C^m(X; G_k[Q]) = H_{k-m}^C(X, Q)$ ($k \leq r$). Also, $H_{k-m}^C(X; Q) = \begin{cases} 0 & \text{for } k < m \\ Q & \text{for } k = m \end{cases}$

So, for $k = r$, $0 \neq H_C^m(X; G_k[Q]) = H_{k-m}^C(X; Q)$. Thus, $k \geq m$. But $r \leq m$, so $r = k = m$. In particular, $G_m[Q] \neq 0$. In fact,

$$H_C^m(X; G_m[Q]) = H_0^C(X, Q) = Q \quad (k=m). \text{ Also,}$$

$$H_C^m(X; G_m[Q]) = H_C^m(X; Q) \times G_m(Q) \text{ by the universal coefficient theorem.}$$

$$\text{Thus, } H_C^m(X; Q) \times G_m(Q) = Q.$$

So $G_m(Q) = Q$ and X is a rational homology manifold.

Consider the exact sequence

$$0 \longrightarrow G_k[\mathbb{Z}] \times Q \longrightarrow G_k[Q] \longrightarrow G_k[\mathbb{Z}] *_{\text{tor}} Q \longrightarrow 0$$

It follows that the free part of $G_k[\mathbb{Z}]$ is that of a homology manifold.

Repeat the above argument with $F = \mathbb{Z}/t\mathbb{Z}$, t prime, to see that $G_k[\mathbb{Z}]$ has no torsion.

Solvgroups Are Not Almost Convex

by

J.W. Cannon, W.J. Floyd, M.A. Grayson, and W.P. Thurston

This is a summary of an expository talk given by J.W. Cannon at the topology conference at The Colorado College, Colorado Springs, Colorado on June 12-14, 1986.

We show that no cocompact discrete group based on solv-geometry, Sol, is almost convex. Almost-convexity is a metric property satisfied by all cocompact hyperbolic groups, all Euclidean groups, all free products with amalgamation of finite groups, all HNN extensions of finite groups, and all small cancellation groups. Intuition suggests that it should be satisfied by those cocompact groups based on geometries whose metric balls are convex. Therefore the property is likely to apply to braid groups, mapping class groups, complex hyperbolic groups, groups of higher rank symmetric spaces whose factors have convex metric balls, etc. It is likely to apply to nilgroups as well, whose metric balls, though not convex, are almost convex.

Our result shows how clearly the combinatorial structure of a geometric group mirrors the properties of the geometry on which it is based: the metric balls in Sol are highly nonconvex and nonsimply connected. Our result has significance in the study of 3-manifolds and their groups. W.P. Thurston has conjectured that each low dimensional manifold (dimension ≤ 3) admits a unique geometric structure. Thus any package of decision algorithms designed to compute within the fundamental groups of low-dimensional manifolds and orbifolds must be able to deal with the groups from each of the standard geometries.

Menger Spaces and Inverse Limits

A talk given by Dennis J. Garity at the Geometric Topology Conference at Colorado College on June 12, 1986.

This represents joint work with David G. Wright.

In 1984, M. Bestvina [Be] characterized the Menger universal n -dimensional compactum μ_n as follows.

Theorem A space X is homeomorphic to μ_n if and only if X satisfies the following properties:

1. X is compact and n -dimensional,
2. X is C^{n-1} ,
3. X is LC^{n-1} , and
4. X satisfies $DD^n P$.

Using this characterization, Bestvina showed that the various constructions in the literature of compact universal n -dimensional spaces ([Mg], [Lf], [Pa]) all yield μ_n . In addition, Bestvina showed that each μ_n is homogeneous. Prior to this result, there had been characterizations only of μ_0 (the Cantor set) and μ_1 (the universal curve) [An].

Using Bestvina's characterization, it is possible to identify certain inverse sequences that have μ_n as inverse limit. This leads to the construction of models of μ_n in the Hilbert Cube. These models can be described by putting restrictions on the coordinates of points in the Hilbert Cube.

There are a number of results in the literature giving conditions which imply that the inverse limit of LC^n compacta is itself LC^n . Z. Cerin [Ce] shows that the inverse limit is LC^n if and only if the inverse sequence is strongly n -e-movable. L. McAuley and E. Robinson [M,R] show that the inverse limit is LC^n if each bonding map is UV^n .

For the examples we are interested in, we need conditions that yield both C^{n-1} and LC^{n-1} . Conditions 2 and 3 in the next Theorem are sufficient for this purpose.

Theorem 1

Let $\{X_i, p_i\}$ be an inverse sequence of LC^{n-1} n -dimensional compacta, satisfying the following conditions.

1. For each i and map $f: B^n \rightarrow X_i$ there exists $j > i$ and maps $h_1, h_2: B^n \rightarrow X_j$ with $h_1(B^n) \cap h_2(B^n) = \emptyset$ and $p_{ij} \circ h_e = f$ for $e = 1, 2$.
2. X_1 is C^{n-1} .
3. There is a constant c so that for each map $f: B^{K+1} \rightarrow X_i$, $K \leq n-1$, and for each map $g: S^K \rightarrow X_{i+1}$ with $p_{i+1} \circ g = f|_{S^K}$, there is an extension $h: B^{K+1} \rightarrow X_{i+1}$ with $p_{m,i+1} \circ h$ within $\frac{c}{2^{i+1}}$ of $p_{m,i} \circ g$ for each $m \leq i$.

Then $X = \varprojlim \{X_i, p_i\}$ is homeomorphic to μ_n .

Theorem 2

Fix $n \geq 0$. Let $P_i \subset I^1$, $i \geq n$, be a sequence of compact n -dimensional LC^{n-1} spaces so that

- a. $P_i \times \left\{0, \frac{1}{2^{i+1}}\right\} \subset P_{i+1}$ and $P_{i+1} \subset P_i \times I_{i+1}$,
- b. P_n is C^{n-1} , and
- c. For each map $f: B^{K+1} \rightarrow P_i \times I_{i+1}$ ($K \leq n-1$) with $f(S^K) \subset P_{i+1}$, there is a map $g: B^{K+1} \rightarrow P_{i+1}$ with $d(f, g) < \frac{1}{2^{i+1}}$ and with $f|_{S^K} = g|_{S^K}$.

Let $X = \bigcap_{i=n}^{\infty} (P_i \times Q_{i+1})$. Then $X \cong \mu_n$.

The proof of Theorem 2 uses the conditions in Theorem 1.

We now construct a specific model satisfying the conditions in Theorem 2. Again, fix $n \geq 0$.

For $X = I^i$ or Q , let

$$X_* \equiv \left\{ x \in X \mid \text{for each choice of } n+1 \text{ coordinates } x_{m_1}, \dots, x_{m_{n+1}} \text{ of } x \text{ with } m_1 < \dots < m_{n+1}, \text{ at least one of the coordinates is dyadic of order } \leq m_{n+1} \right\}$$

Let $P_i = I_*^i$.

For $n = 1$, the one-dimensional polyhedra P_1 , P_2 , and P_3 are illustrated in Figure 1.

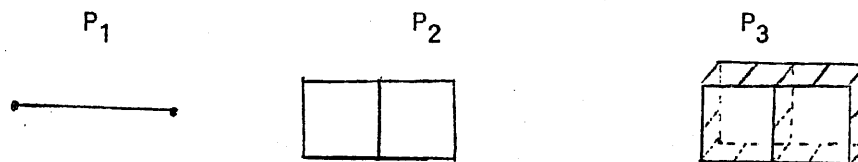


Figure 1

When $n = 0$, $P_i = \prod_{j=1}^i \left\{ 0, \frac{1}{2^j} \right\}$, the corner points of the i -cell I^i .

Let $X_n = \bigcap_{i=n}^{\infty} (P_i \times Q_{i+1})$. Then X_n is Q_* . Note that X_0 is the Cantor Set consisting of the corner points of the Hilbert Cube. In Theorem 3 below, we show that $X_n \cong \mu_n$. Before proving this theorem, we provide an alternate description of P_i that is easier to work with.

Fix n . Let $P'_n = I^n$. Let P'_n be viewed as a cell complex consisting of rectilinear n -cells with sides of length $\frac{1}{2^{n+1}}$ by subdividing each factor I_i of I^n into subintervals of lengths $\frac{1}{2^{n+1}}$. Let A_n be the $(n-1)$ skeleton of this cell complex.

Define $P'_{n+1} \subset I^{n+1}$ as

$$A_n \times I_{n+1} \cup P'_n \times \left\{0, \frac{1}{2^{n+1}}\right\}.$$

Note that P'_{n+1} can be viewed as a cell complex consisting of rectilinear n -cells with sides of length $\frac{1}{2^{n+2}}$ by subdividing each factor I_i of I^{n+1} into subintervals of length $\frac{1}{2^{n+2}}$.

Inductively assume $P'_j \subset I^j$ has been defined so that P'_j can be viewed as a cell complex consisting of rectilinear n -cells with sides of length $\frac{1}{2^{j+1}}$ by subdividing each factor I_i of I^j into subintervals of length $\frac{1}{2^{j+1}}$.

Let A_j be the $n-1$ skeleton of this cell complex. Define $P'_{j+1} \subset I^{j+1}$ as

$$A_j \times I_{j+1} \cup P'_j \times \left\{0, \frac{1}{2^{j+1}}\right\}.$$

Lemma 1 For each i , $P_i = P'_i$.

Theorem 3 $X_n \cong \mu_n$.

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MENGER SPACES AND σ

James P. Henderson

This paper is a summary of a talk given at the Geometric Topology conference held at The Colorado College on June 12-14, 1986.

In "Menger Spaces and Inverse Limits", these proceedings, Dennis Garity outlines a procedure for constructing n -dimensional Menger spaces X_n , $n \geq 1$, in the Hilbert cube with the property that X_n is contained in X_{n+1} . Using his notation and descriptions, it is possible to show that $X = \bigcup X_n$ is homeomorphic to σ . Recall that σ may be viewed as the set of points in Hilbert space having at most finitely many nonzero coordinates. In order to obtain the desired goal, we will show that X satisfies the following characterization [He]:

X is a σ -manifold if and only if:

- (1) X is an ANR
- (2) X is the countable union of finite dimensional compacta
- (3) Each compact subset of X is a strong Z -set in X
- (4) For each integer k , mapping $f: \mathbb{R}^k \rightarrow X$, and $\varepsilon: X \rightarrow (0,1)$, there is an injection $f': \mathbb{R}^k \rightarrow X$ with $d(f(x), f'(x)) < \varepsilon(x)$.

The last property is referred to as the Euclidean injection property (EIP). Condition (3) means that if A is a compact subset of X , for each open cover \mathcal{U} of X and sequence of mappings $\alpha_1, \alpha_2, \dots$ of Q into X , there are \mathcal{U} -approximations β_1, β_2, \dots such that $U(\beta_i(Q): 1 \leq i < \infty)$ misses a neighborhood of A [B,B,M,W]. Condition (2) is satisfied since each X_n is a compact, finite dimensional set. The space X will be shown to satisfy the other conditions through a sequence of results.

The first lemma involves approximating mappings of R^k into Q by mappings into X . Throughout the remainder, by a basic open set V in Q we will mean an open set in Q of the form

$$V = (\prod V_i) \times Q_{n+1} \text{ where } V_i \text{ is a connected, open set in } I_i.$$

Lemma 1 Let V be a basic open subset of Q . For $f: R^k \rightarrow V$ and $\varepsilon: R^k \rightarrow (0,1)$, there is a mapping $f': R^k \rightarrow D$ with

$d(f(x), f'(x)) < \varepsilon(x)$ where D is the set of points in V having at most finitely many nonzero coordinates.

We now turn to the problem of showing that X is an ANR. According to Dugundji [Du], it suffices to show that given any open cover \mathcal{U} of X , there is an open cover \mathcal{V} of X such that given any simplicial complex K , any partial realization of K in \mathcal{V} extends to a full realization of K in \mathcal{U} . A partial realization of K in \mathcal{U} is a mapping $h: L \rightarrow X$ in which L is a subcomplex of K containing every vertex of K and such that the sets $h(|Ls|)$ refine \mathcal{U} where s is a simplex of K . A full realization of K in \mathcal{U} is a partial realization of K in \mathcal{U} where $K=L$. The following proposition is an easy consequence of this characterization of ANR's.

Proposition 2 X is an ANR.

Proposition 3 follows directly from lemma 1.

Proposition 3 X satisfies the EIP

The final necessary result is that each compact subset of X be a strong Z -set. This can be accomplished by first showing that each compact subset of X is a Z -set in Q , and then getting the stronger property in X . Recall that a closed subset A of an ANR X is a Z -set if the relative homology groups $H_*(U, U-A; Z) = 0$ for each open set U in X and A is 1-LCC embedded in X .

Proposition 4 X_n is a Z -set in Q

Corollary 5 Each compact subset C of X is a Z -set in Q .

Proposition 6 Every compact subset of X is a strong Z -set in X .

Since X satisfies the characterization theorem, X is a σ -manifold. We have not shown that X is homeomorphic to σ . However, a σ -manifold may be factored as $|K| \times \sigma$ where K is a countable, locally finite simplicial complex [Ch]. It follows from Lemma 1 that $\Pi_n(X) = 0$ for all n , so $\Pi_n(K) = 0$ for all n , and K is contractible. Thus X is contractible and homeomorphic to σ since they have the same homotopy type [Ch].

Theorem 7 X is homeomorphic to σ

It should be noted that a more general result follows from the proofs of the above results. The following theorem is immediate.

Theorem 8 Let $X=UX_n$, where each X_n is a compact, finite dimensional Z -set in Q , with X containing the set of all points in Q having at most finitely many nonzero coordinates. Then X is homeomorphic to σ .

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ON CELLULAR DECOMPOSITIONS OF THE HILBERT CUBE

TERRY L. LAY

This paper is the summary of an expository talk given at the topology conference at The Colorado College, Colorado Springs, Colorado on June 12-14, 1986.

Dennis Garity, in his thesis work and later, collaborating with R. J. Daverman, defined and then investigated the notion of a cell-like decomposition of E^n (S^n) being of intrinsic dimension k . What this essentially means is that when $\pi: E^n \rightarrow E^n/G$ is the canonical quotient mapping, then any sufficiently close cell-like approximation f to π has the property that $f(N_\pi)$ has dimension $\leq k$ and for some f this dimension is exactly k . (We are assuming that E^n/G is finite dimensional.) To illustrate this property Daverman and Garity [DG1, DG2] produced $(n-2)$ - and then $(n-1)$ -dimensional cellular decompositions of E^n . Although this is an extremely pathological phenomenon, they were able to show that in each case, the corresponding decomposition space $E^n/G \times R^1$ was a manifold.

Each of the decompositions above were constructed with the aid of a defining sequence. (See [CD] or [DW] for discussions of defining sequences.) Work of the second author [L1,L2] indicated a possible program for adapting a defining sequence for a finite dimensional decomposition in order to obtain a similar structure in the Hilbert cube Q or more generally in a Q -manifold M^Q . This program has succeeded in producing infinite dimensional versions of the finite dimensional decompositions of Cannon-Daverman [CD] and Daverman-Walsh [DW1]. The current investigation has as its goal to produce appropriate infinite dimensional versions of the Daverman-Garity decompositions.

The notion of the image of the non-degeneracy having dimension k is inappropriate in the Q -manifold setting since any finite dimensional subset of such a space has infinite codimension and thus the decomposition space is automatically a manifold [DW2]. Conversations with John Walsh have produced the following definitions which will govern the current program.

Definition 1. A subset B of a Q -manifold M^Q has codimension $\geq k$ if for each closed subset A of B , each open set U in M^Q and each $q < k$, the homology module $H_q(U, U \setminus A)$ is trivial. B is said to have codimension equal to k if it has codimension $\geq k$ but does not have codimension $\geq k+1$.

Definition 2. A cellular decomposition G of a Q -manifold M^n is intrinsically of codimension $\leq k$ if each sufficiently close cell-like approximation f to the quotient map $\pi: M \rightarrow M/G$ has $f(N_\pi)$ of codimension $\leq k$.

It appears that the defining sequence for the $(n-2)$ -dimensional decomposition in [DG1] will program quite nicely into a cellular, intrinsically codimension 2 decomposition of Q . Moreover, the defining sequence will exhibit those properties discussed in [L2] which will insure that $Q/G \times I$ will be a Q -manifold. A much more hazy issue is the $(n-1)$ -dimensional decomposition. The construction in [DG2] uses a particular linking pattern which has no obvious counterpart in the infinite dimensional setting. The only visible obstruction, however, is this difficulty with the linking and we are optimistic that the program will go through in this setting as well.

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Acyclic Maps of 4-manifolds

by

F. C. Tinsley

Intro:

Daverman constructed acyclic decompositions of special homology n -spheres ($n \geq 5$) [Da, Ex. 3]. Daverman and Tinsley showed how to construct similar decompositions in 5-manifolds with fundamental groups containing finitely generated perfect subgroups. In light of recent developments, we investigate similar constructions in dimension four.

A compactum, A , in an ANR, X , is strongly \mathbb{Z} -acyclic if for each U open in X with $A \subset U$, there is a V open in X with $A \subset V \subset U$ and the inclusion-induced $i_*: H_*(V; \mathbb{Z}) \rightarrow H_*(U; \mathbb{Z})$ the zero homomorphism. A map $f: X \rightarrow Y$ of ANR's is acyclic if $f^{-1}(y)$ is strongly \mathbb{Z} -acyclic for each $y \in f(X)$. Let $\mathcal{E}Map(f)$, the extended mapping cylinder of f , be the mapping cylinder of f with a collar attached to Y . More precisely, $\mathcal{E}Map(f) = X \times [-1, 0] \cup_f Y \times [0, 1]$ where $f': X \times \{0\} \rightarrow Y \times \{0\}$ by $f'(x, 0) = (f(x), 0)$.

A common source of such acyclic maps is the decomposition of a manifold. Let G be an USC decomposition of a manifold, M , into strongly \mathbb{Z} -acyclic compacta. If M/G is an ANR, then the decomposition map $f: M \rightarrow M/G$ is an acyclic map of ANR's.

1. Examples

Ex. 1.1: Let M^n be a non-simply connected homology n -sphere ($n \geq 3$). Let B^n be a flat n -cell in M^n . Then $b^n = \text{clos}(M^n - B^n)$ is a non-simply connected homology n -cell. Let G be the decomposition of M^n into points and the single, non-degenerate set, b^n . Then $f: M^n \rightarrow M^n/G \cong S^n$ is an acyclic map of manifolds.

Note that $E = \mathcal{E}Map(f)$ is not an $(n+1)$ manifold at $f(b^n) \times \{0\}$.

For the next example, we need:

Lemma 1.2.1: Let M^n ($n \geq 4$) be a homology n -sphere. Then there is a locally flat embedding, $t: h^{n-1} \rightarrow M^n$, where h^{n-1} is a homology $(n-1)$ -sphere and $t_\#: \pi_1(h^{n-1}) \rightarrow \pi_1(M^n)$ is a surjection.

Pf: h^{n-1} is obtained as the boundary of a topological regular neighborhood of an acyclic 2-complex in M^n . This requires a good deal of effort, particularly for $n = 4$.

Ex. 1.2: Let M^n be as in 1.1 ($n \geq 4$). Now, 1.2.1 yields $h^{n-1} \times [0,1] \subset M^n$ with inclusion inducing surjection on π_1 . Choose a Cantor set $C \subset (0,1)$. Let G be the decomposition of M^n into points and non-degenerate sets $\{b^{n-1} \times c \mid c \in C\}$ (here, b^{n-1} is the complement of a flat, open $(n-1)$ -cell in h^{n-1}). Then $f: M^n \rightarrow M^n/G$ is an acyclic map of ANR's.

For $n \geq 4$, $E = \mathcal{E}Map(f)$ is a 5-manifold. For $n \geq 5$, $M^n/G \cong S^n$. However, for $n = 4$, M^n/G is not a manifold at $f(b^{n-1} \times c)$ for $c \in C$.

2. Theorems for $n = 4$

Theorem 2.1: Let M^4 be a homology 4-sphere. There exists an acyclic map $f: M^4 \rightarrow S^4$ with $\mathcal{E}Map(f)$ a 5-manifold.

Pf: Let $h^3 \times [0,1] \subset M^4$ and $b^3 \subset h^3$ be as in Ex. 1.2. Let $h^4 = \text{clos}(M^4 - b^3 \times [\frac{1}{3}, \frac{2}{3}])$. Let K be a 2-spine of b^3 . Let G be the decomposition of M^4 into points and non-degenerate sets h^4 and $\{K \times s \mid s \in (\frac{1}{3}, \frac{2}{3})\}$.

Theorem 2.2: Every homology 4-sphere laminates to S^4 (in the sense of Da-Ti).

3. Unresolved questions

The acyclic map of 2.1 is one to one over the complement of a 1-dimensional set.

Question 3.1: In Theorem 2.1, can f be replaced by a map which is one to one over the complement of a zero dimensional set?

A possibly related question is:

Question 3.2: Is there a wild Cantor set in R^4 which is defined by contractible manifolds (objects)?

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BING-WHITEHEAD DECOMPOSITIONS OF E^3

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This paper is the summary of an expository talk given at the topology conference at Colorado College, Colorado Springs, Colorado on June 12-14, 1986.

1. Introduction

Let G be an upper semi-continuous decomposition of E^3 consisting of points and components of ΠM_i where M_0 is solid torus and M_{i+1} is obtained from M_i either by the *Bing Construction* (placing two solid tori in each component as in the construction of the Bing Cantor set--see Figure 1) or by the *Whitehead Construction* (placing a Whitehead link in each component--see Figure 2). If the sequence M_i has only a finite number of Whitehead constructions, then the decomposition is shrinkable by an argument due to R. H. Bing. If the sequence M_i has only finitely many Bing constructions, then the nondegenerate elements are not 1-LCC. Hence, the decomposition is not shrinkable. Let n_1 be the number of consecutive Bing constructions placed in M_0 before the first Whitehead construction. In general let n_i be the number of consecutive Bing constructions between the $(i-1)^{st}$ and i^{th} Whitehead constructions in the sequence M_i . Of course n_i could equal zero if there are consecutive Whitehead constructions.

Recently F. D. Ancel communicated to me that such decompositions are shrinkable if and only if $\sum_{i=1}^{\infty} n_i / 2^i$ diverges. This paper outlines a proof of this fact.

2. Divergence implies shrinkability

Definition 2.1. Let $R_1, R_2, \dots, R_k, B_1, B_2, \dots, B_k$ be disjoint meridional

disks in a solid torus T . Let $R = \bigcup A_i$ and $B = \bigcup B_j$. We say that (R, B) is a *k-interlacing collection of meridional disks* if each component of $T - (R \cup B)$ has exactly one R_i and one B_j in its closure. We think of the disks in R and B as being colored red and blue respectively.

Definition 2.2. Let R and B be disjoint sets and T a solid torus. We say that (R, B) is a *k-interlacing* for T if there are subsets R' and B' of R and B respectively so that (R', B') is a *k-interlacing collection of meridional disks*, but it is impossible to find such subsets that form a $(k+1)$ -interlacing collection of meridional disks.

Lemma 2.3. If $k > 0$ and (R, B) is a *k-interlacing* for a solid torus T so that each of $R \cap T$ and $B \cap T$ is the union of finitely many disjoint meridional disks of T , then it is possible to put a Whitehead link T' in T so that (R, B) is a $(2k-1)$ interlacing of T' and each of $R \cap T$ and $B \cap T$ is the union of finitely many disjoint meridional disks of T .

Lemma 2.4. If $k > 0$ and (R, B) is a *k-interlacing* of a solid torus T so that each of $R \cap T$ and $B \cap T$ is the union of finitely many disjoint meridional disks of T , then it is possible to put two solid tori T_1 and T_2 in T as in the Bing Construction so that (R, B) is a $(k-1)$ -interlacing of each T_i and so that each of $R \cap T_i$ and $B \cap T_i$ is the union of finitely many disjoint meridional disks of T_i .

Note: We say (R, B) is a *0-interlacing* of T in case T misses either R or B . With this definition Lemma 2.4 makes sense for $k=1$. Also it is clear that if (R, B) is a *0-interlacing* of T then (R, B) is a *0-interlacing* for any solid torus contained in T .

Proof of divergence implies shrinkability.

Suppose $\sum_{i=1}^{\infty} n_i / 2^i$ diverges. We show how to construct a homeomorphism h of E^3 , fixed outside M_0 so that the components of $h(M_r)$ are small for some integer r . We may assume without loss of generality that the B^2 factor of

M_0 is small. Let (R, B) be a k -interlacing of M_0 so that each of $R \cap M_0$ and $B \cap M_0$ is the union of finitely many disjoint meridional disks of M_0 and so that any connected subset of M_0 that misses R or B is small.

Choose n so that the partial sum $\sum_{i=1}^n n_i/2^i$ is larger than $k/2$. We now choose the homeomorphism h so that $h(M_{i+1})$ is embedded in $h(M_i)$ as in the above lemmas and remark through the n^{th} Whitehead construction. Let M_r be the set obtained with the n^{th} Whitehead construction. Then (R, B) is an m -interlacing of each component of M_r where m equals

$\max\left[\left(2^{n+1}\left(k/2 - \sum_{i=1}^n n_i/2^i - \sum_{i=1}^n 1/2^{i+1}\right)\right), \text{zero}\right]$. But n was chosen so that m is equal to zero. Hence, if M_r is obtained as the n^{th} Whitehead construction, then the components of $h(M_r)$ are small.

The theorem now follows by applying the Bing shrinking criterion and the above argument applied to components of M_i .

3. Shrinkability implies divergence

Definition 3.1. Let H be a properly embedded disk with holes in a solid torus T so that the inclusion map is 1-essential; i.e., the inclusion map on the boundary of H cannot be extended to a map of H into the boundary of T . We call H a *meridional disk with holes* for the solid torus T .

Note: The non-trivial boundary components of H must be meridional simple closed curves of T whose algebraic sum is ± 1 .

Definition 3.2. We define a *k-interlacing collection of meridional disks with holes* by replacing *meridional disks* in Definition 2.1 by *meridional disks with holes*.

Definition 3.3. Let R and B be disjoint sets and T a solid torus. We say that (R, B) is a *k-interlacing* for T if there are subsets R' and B' of R and B respectively so that (R', B') is a k -interlacing collection of meridional disks with holes, but it is impossible to find such subsets that form a $(k+1)$ -interlacing collection of meridional disks with holes.

Note: This generalizes Definition 2.2.

Lemma 3.4. Suppose R and B are disjoint 2-manifolds properly embedded in a solid torus T so that (R, B) is a k -interlacing for T . If T' is a Whitehead link in T that is in general position with respect to $R \cup B$, then (R, B) is a k' interlacing for T' where $k' \geq 2k - 1$.

Lemma 3.5. Suppose R and B are disjoint 2-manifolds properly embedded in a solid torus T so that (R, B) is a k -interlacing for T . If T_1 and T_2 are embedded in T by the Bing construction and are in general position with respect to $R \cup B$, then (R, B) is a k' interlacing for either T_1 or T_2 where $k' \geq k-1$.

The proofs of the above lemmas require looking at the universal cover of T and an understanding of how an I -essential disk with holes in the universal cover meets the lifts of T' and $T_1 \cup T_2$.

Proof of shrinkability implies divergence.

Suppose that $\sum_{i=1}^{\infty} n_i / 2^i$ converges. Let k be a positive integer so that $k/2$ is greater than $\sum_{i=1}^{\infty} n_i / 2^i + 1$. Let (R, B) be a k -interlacing for M_0 so that each of $R \cap T$ and $B \cap T$ is the union of finitely many disjoint meridional disks of M_0 . We further suppose that $R \cup B$ is in general position with each M_i for $i \geq 1$. Let M_n be the set obtained with the n^{th} Whitehead construction. Then by the above lemmas (R, B) is an m -interlacing for some component of M_n where m is greater than or equal to $2^{n+1} (k/2 - \sum_{i=1}^n n_i / 2^i - \sum_{i=n+1}^{\infty} 1/2^{i+1})$. But k was chosen so that this number is positive for any choice of n . Hence for all i some component of M_i must be large enough to meet both R and B . But this contradicts the fact that the diameter of the components of M_i tend to zero as i gets large. This contradiction arose from the

supposition that $\sum_{i=1}^{\infty} n_i/2^i$ converges. Therefore we conclude that $\sum_{i=1}^{\infty} n_i/2^i$ diverges.

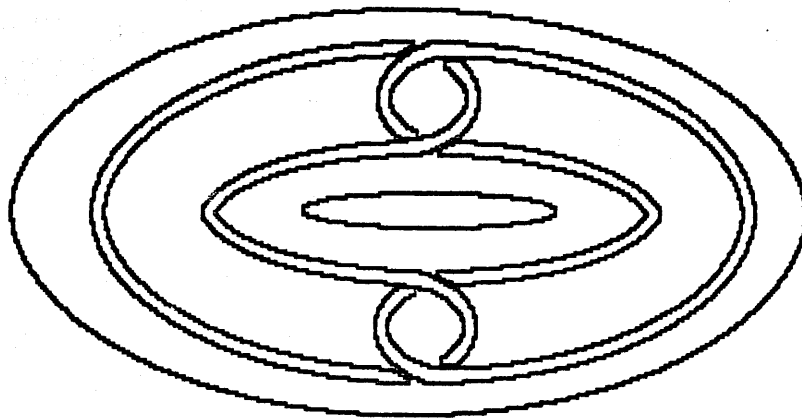


Figure 1

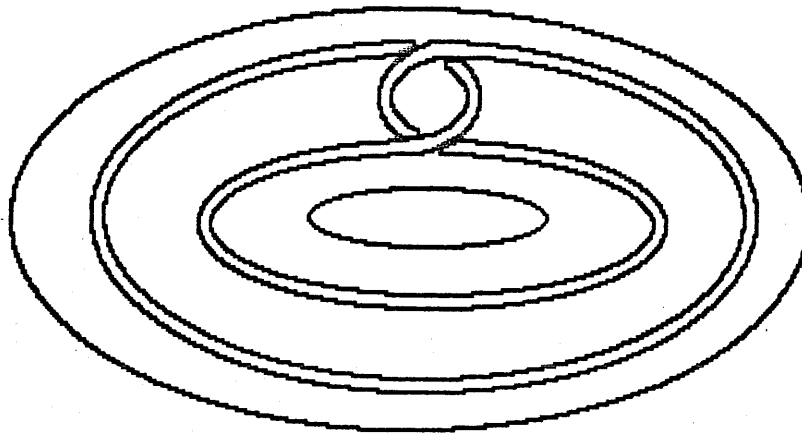


Figure 2

QUESTIONS POSED BY PARTICIPANTS AT THE
1986 GEOMETRIC TOPOLOGY CONFERENCE
THE COLORADO COLLEGE

1. Is there a wild Cantor set in E^3 such that no wild sub-Cantor set is open, closed, and definable by solid tori?
2. Find a Cantor set so that any handlebody description has an unbounded number of handles.
3. Is every strongly homogeneously embedded Cantor set tame in E^3 ?
4. Is there a wild Cantor set in R^4 defined by contractible manifolds (objects)?
5. Does the Garity-Wright construction of Menger sets in the Hilbert cube show homogeneity of those sets?
6. Can the spine of a Mazur 4-manifold be pushed off itself by a homeomorphism fixing the boundary?
7. Are the Daverman Cantor sets sticky?
8. Is there a map $f: B^n \dashrightarrow X$ compact with $\dim(f(C)) = \infty$ for every non-degenerate continuum C ?
9. Let X be a locally compact ANR such that $H_*(X, X-x) = 0$ for all $x \in X$. Is $X \times I^2$ a Q -manifold? Does X satisfy the disjoint Cech carriers property? Does X contain a 2-dimensional closed subset?
10. A space, X , has property C_n if for every sequence of covers U_1, U_2, \dots with each U_i having cardinality n , there is a sequence W_1, W_2, \dots where each W_i is a pairwise disjoint collection of open sets refining U_i and $\bigcup W_i$ covers X . Is there a space, X , with property C_2 but not property C_3 ? If so, then property $C \neq \text{WID}$.
11. Let G be an U.S.C. decomposition of a locally compact ANR X into ANR's. To what extent is it true that X/G is homologically locally connected?
12. Does every homology sphere h^n bound an $(n+1)$ -dimensional homology cell with inclusion inducing an injection of fundamental groups?
13. Is every finitely presented perfect group the normal closure of a single element?
14. Is there a locally indicable finitely presented group which embeds in its own wild group?
15. Does every compact ANR have a point at which the local homology is finitely generated?

16. Let f be a cellular map of polyhedra, $f:P \rightarrow Q$, with $\dim(P) = 4$. Is f approximable by homeomorphisms?
17. Is every cocompact discrete nil group almost convex?
18. Suppose $X_1 \subset X_2 \subset X_3 \subset \dots \subset Q$ where X_i is n -dimensional Menger space and X_i is a Z -set in X_{i+1} . Is $\bigcup_{i=1}^{\infty} X_i = \sigma$?
19. Is every strongly homogeneous 2-sphere in R^3 tame?
20. Is there a finite dimensional homogeneous ANR homology manifold that is not a manifold?
21. If an arc in E^n can be instantly isotoped off itself, is the arc tame? Q instead of E^n ?