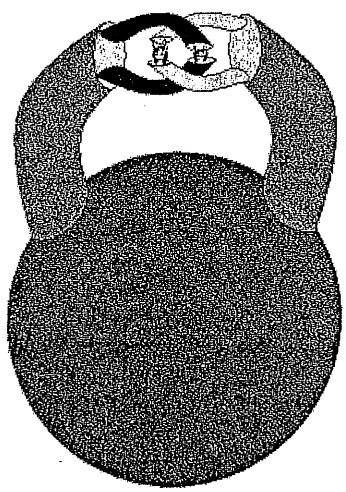
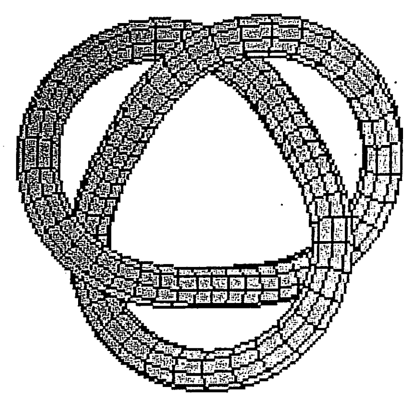
Proceedings
of the
Eighteenth Annual Workshop
in
Geometric Topology





Hosted by the Mathematics Department
at
Oregon State University

June 21-23, 2001

Dedication:

This volume of the Proceedings of the Workshop in Geometric Topology is dedicated to the memory of Yaki Sternfeld (1944-2001). Yaki attended a number of previous workshops and was valued as a good friend and stimulating mathematical colleague by many workshop participants and the organizers. His warmth, humor noble spirit and mathematical insight are sorely missed. We strive to honor Yaki's memory through this dedication.

The Eighteenth Annual Workshop in Geometric Topology was hosted by Oregon State University and was held at Corvallis, Oregon on June 21-23, 2001. There were 3 main talks by the principal speaker, Abigail Thompson and there were 15 contributed talks. There were thirty two participants. Of these, six were graduate students. A list of past and future workshop locations and principal speakers is included below.

Year	Location	Principal Speaker
1984	Brigham Young University	--
1985	Colorado College	Robert Daverman
1986	Colorado College	John Walsh
1987	Oregon State University	Robert Edwards
1988	Colorado College	John Hempel
1989	Brigham Young University	John Luecke
1990	Oregon State University	Robert Daverman
1991	University of Wisconsin - Milwaukee	Andrew Casson
1992	Colorado College	Mladen Bestvina
1993	Oregon State University	John Bryant
1994	Brigham Young University	Michael Davis
1995	University of Wisconsin - Milwaukee	Shmuel Weinberger
1996	Colorado College	Michael Freedman
1997	Oregon State University	James Cannon
1998	Brigham Young University	Steve Ferry
1999	University of Wisconsin - Milwaukee	Robert Edwards
2000	Colorado College	Robert Gompf
2001	Oregon State University	Abigail Thompson
2002	Calvin College	Alexander Dranishnikov
2003	Brigham Young University	Martin Bridson

Conference Proceedings have been produced for each workshop except the second.

These proceedings contain a summary of the three one-hour talks delivered by the principal speaker, Abigail Thomson. Summaries of talks given by some of the other participants are also included. The success of the workshop was helped by generous funding from the Oregon State University College of Science and Mathematics Department and the National Science Foundation.

Editors: Dennis Garity, Fredric Ancel, Craig Guilbault, Frederick Tinsley, David Wright, and Gerard Venema

Participants:

Fredric Ancel	University of Wisconsin - Milwaukee
Kathy Andrist	Utah Valley State College
Bill Bogley	Oregon State University
Stoyu Barov	University of Alabama
Tony Bedenikovic	Bradley University
Paul Britton *	University of Wisconsin - Milwaukee
Nicolay Brodskiy *	University of Saskatchewan
Robert Daverman	University of Tennessee
Tadek Dobrowolski	Pittsburg State University
Christine Escher	Oregon State University
Paul Fabel	Mississippi State University
Jonathon Fassett *	Oregon State University
Hanspeter Fischer	University of Wisconsin - Milwaukee
Dennis Garity	Oregon State University
Troy Goodsell	Brigham Young University
Craig Guilbault	University of Wisconsin - Milwaukee
Denise Halverson	Brigham Young University
Ivan Ivanšić	University of Zagreb
Seong Kun Kim *	Oregon State University
Terry Lay	Idaho State University
Margaret May *	University of Wisconsin – Milwaukee
Chris Nord *	Oregon State University
Philip Schapiro	Langston University
Richard Schori	Oregon State University
Tom Thickston	Southwest Texas State University
Abigail Thompson	University of California at Davis
Frederick Tinsley	The Colorado College
Gerard Venema	Calvin College
Julia Wilson	SUNY Fredonia
Bobby Winters	Pittsburg State University
David Wright	Brigham Young University
Matjaž Željko	University of Ljubljana

* Students

Conference Schedule

Thursday		Friday		Saturday	
8:15	Refreshments	8:15	Refreshments	8:15	Refreshments
9:00	Abby Thompson	9:00	Abby Thompson	9:00	Abby Thompson
10:00	Break	10:00	Break	10:00	Break
10:15	Ric Ancel	10:15	Bob Daverman	10:15	Fred Tinsley
10:40	Stoyu Barov	10:40	Tadeusz Dobrowolski	10:40	Craig Guilbault
11:05	Nicolay Brodskiy	11:05	Paul Fabel	11:05	Problem Session
11:30		11:30	Denise Halverson	11:25	Conference Ends
11:50	Lunch	11:50	Lunch		
2:00	David Wright	2:00	Ivan Ivansic		
2:25	Matjaz Zeljko	2:25	Leonard Rubin		
2:50	Hanspeter Fischer	2:50	Philip Schapiro		
3:10	Break	3:10	Break		
6:30	Conference Dinner	7:30-9:30	Dessert		

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** Principal Speaker

CURVES IN RP^2 AND A THEOREM OF FABRICIUS-BJERRE

ABIGAIL THOMPSON

1. Introduction

This paper is a brief exposition of notes from a talk given at Oregon State University in June 2001. Details of the proofs and further applications can be found in [6].

Let K be a smooth immersed curve in the plane. Fabricius-Bjerre [2] found the following relation among the double tangent lines, crossings, and inflections points for a "generic" K:

$$T_1 - T_2 = C + (1/2)I$$

where T_1 and T_2 are the number of exterior and interior double tangent lines of K, C is the number of crossings, and I is the number of inflection points. A series of papers followed. Halpern [4] re-proved the theorem and obtained some additional formulas using analytic techniques. Banchoff [1] proved an analogue of the theorem for piece-wise linear planar curves, using deformation methods. Fabricius-Bjerre himself gave a variant of the theorem for curves with cusps [3]. Weiner [7] generalized the theorem to closed curves lying on a 2-sphere. Finally Pignoni [5] generalized the theorem to curves in real projective space, but his theorem depends, both in the statement and in theproof, on the selection of a base point for the curve.

We will discuss two main results. The first is a generalization of the theorem in [3] to RP^2, with no basepoint requirement. The difficulties encountered are due to the problems in distinguishing between two "sides" of a closed geodesic in RP^2. These are overcome by a careful attention to the natural metric on the space.

The main results are tied together by the observation that, in the cusped version of the original theorem, the quantities in the formula are naturally dual to each other in RP^2. This leads to the second main result, which is a dual formula for "generic" curves in RP^2. By considering a plane curve as lying in a small disk in RP^2, this specializes

Date: August 3, 2003.

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to a new formula for "generic" smooth curves in the plane. This new formula has the interesting property that it makes delicate geometric distinctions between topologically very similar planar curves.

The outline of the paper is as follows: in Section 2 we state the generalization of [3] to curves in RP^2, and give a brief sketch of the proof. In Section 3 we describe the duality between terms of the formula. In Section 4 we state the dual formulation, and give a corollary for planar curves.

2. A Fabricius-Bjerre Formula for RP^2

Let RP^2 be endowed with the standard metric, inherited from the round 2-sphere of radius one. With this metric, a simple closed geodesic in RP^2 has length π. Most of the figures will use a standard disk model for RP^2, in which the boundary of the disk twice covers a closed geodesic. Let K be an oriented immersed curve in RP^2, which is smooth except for cusps of type 1, that is, cusps at which locally the two branches of K coming into the cusp are on opposite sides of the tangent geodesic. Assume that tangent lines at crossings of K are neither parallel nor perpendicular, that the tangent line through an inflection point or at a cusp is everywhere else transverse to K, that a geodesic goes through at most two tangent points or cusps of K, that no crossings occur at inflection points, and that a line normal to K at one point is tangent to K at at most one point and everywhere else transverse to K. We will call such a K *generic*. We will need some definitions.

Definitions.
Let τ_p be the tangent geodesic to K at p, with orientation induced by K.

Let a_p, the *antipodal point to $p*, be the point on τ_p a distance $\pi/2$ from p.

τ_p is divided by p and a_p into two pieces. Let ${\tau_p}^+$ be the segment from p to a_p and ${\tau_p}^-$ the segment from a_p to p. At cusp points ${\tau_p}^+$ and ${\tau_p}^-$ are not well-defined.

Let ν_p be the normal geodesic to K at p.

Let c_p (which lies on ν_p) be the center of curvature of K at p.

We orient ν_p so that the length of the (oriented) segment from p to c_p is less than the length of the segment from c_p to p. This orientation is well-defined except at cusps and inflection points.

There is a natural duality from RP^2 to itself. Under this duality simple closed geodesics in RP^2 are sent to points and vice versa. This duality is most easily described by passing to the 2-sphere S which is the double cover of RP^2; in this view a simple closed geodesic in RP^2 lifts to a great circle on S. If this great circle is called the equator, the dual point in RP^2 is the image of the north and south poles.

Under this duality the image of K is a *dual knot* K'. To describe K' we need only observe that a point on K comes equipped with a tangent geodesic, τ_p. The *dual point to* p, called p', is the point dual to the tangent geodesic τ_p.

Another useful description is that p' is the point a distance $\pi/2$ along the normal geodesic to K at p. Notice that $\nu_p = \nu'_p$ and $c(p) = c(p')$.

An ordered pair of points (p, q) on K is an *antipodal pair* if $q = a_p$, or if p is a cusp point and q is a distance $\pi/2$ from p.

Suppose (p, q) is an antipodal pair, with p not a cusp point of K. Let C be the geodesic dual to c_p. We impose the additional requirement for genericity that τ_q should be neither τ_p nor C. C and τ_p intersect at q and divide RP^2 into two regions, R_1 and R_2. One of the regions, say R_1, contains c_p. The geodesic τ_q lies in one of the two regions. An antipodal pair (p, q) is of type 1 if τ_q lies in R_1, type 2 if τ_q lies in R_2. Let A_1 be the number of type 1 antipodal pairs of K, A_2 the number of type 2. Distinguishing the two types of pairs in which p is a cusp point is similar.

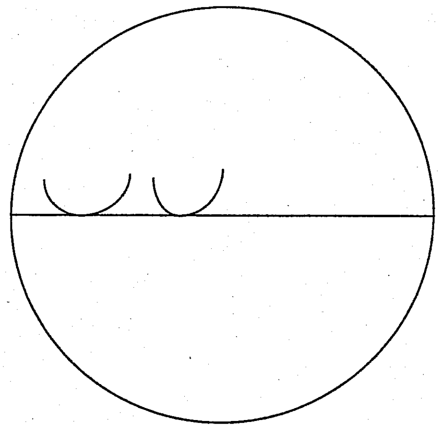
T is a *double-supporting geodesic* of K if T is either a double tangent geodesic, a tangent geodesic through a cusp or a geodesic through two cusps. The two tangent or cusp points of K divide T into two segments, one of which has length less than $\pi/2$. We distinguish two types of double supporting geodesics, depending on whether the two points of K lie on the same side of this segment (type 1) or opposite sides (type 2). Let T_1 be the number of double supporting geodesics of K of type 1, T_2 the number of type 2.

The tangent lines at a crossing of K define four angles, two of which, α and β, are less than $\pi/2$. In a small neighborhood of a crossing there are four segments of K. The crossing is of type 1 if one of these segments lies in α and another in β, type 2 if two lie in α or two lie in β. Let C_1 be the number of type 1 crossings of K, C_2 the number of type 2.

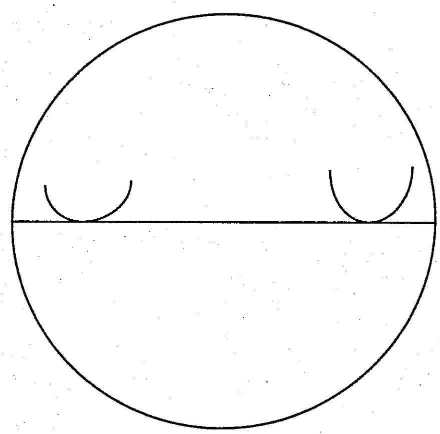
Let I be the number of inflection points of K.

Let U be the number of (type 1) cusps of K.

We are now ready to state the first main theorem, which is a generalization of the main theorem of [2] to the projective plane. We note specifically that, unlike [5], we do not need to choose a base-point for K.



Type 1 double supporting geodesic



Type 2 double supporting geodesic

FIGURE 1

Theorem 1. *Let K be a generic singular curve in RP^2 with type 1 cusps. Then*

$$T_1 - T_2 = C_1 + C_2 + (1/2)I + U - (1/2)A_1 + (1/2)A_2$$

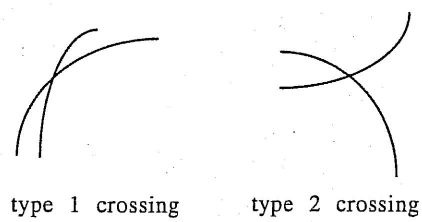


FIGURE 2

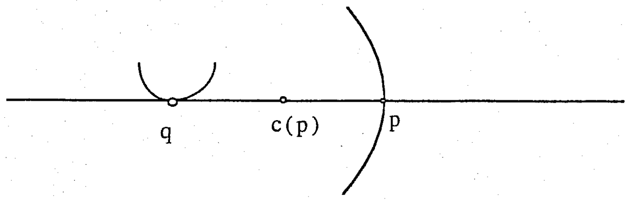
Sketch of Proof:
The proof proceeds as in [3], with some caution being required at antipodal pairs. We choose a starting point p on K. Let ${M_p}^+$ be the number of times K intersects ${\tau_p}^+$, ${M_p}^-$ be the number of times K intersects ${\tau_p}^-$, and $M_p = {M_p}^+ - {M_p}^-$. We keep track of how M_p changes as we traverse the knot. Double-supporting lines, crossings, cusps and inflection points all behave as in [3]. The contribution of an antipodal pair depends on the type, reflected in the sign difference in the formula.

3. DUALITY IN RP^2

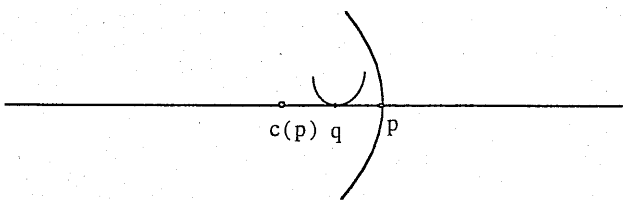
We describe the dual relations between crossings and double tangencies, cusps and inflection points, and antipodal points and normal-tangent pairs (defined below).

Definitions.
The points p and c_p divide ν_p into two pieces, ${\nu_p}^+$ from p to c_p and ${\nu_p}^-$ from c_p to p. An ordered pair of points (p, q) on K is a *normal-tangent pair* if $\tau_q = \nu_p$. A normal-tangent pair (p, q) is of type 1 if q lies on ${\nu_p}^-$, type 2 if q lies on ${\nu_p}^+$. Let N_1 be the number of type 1 normal-tangent pairs of K, N_2 the number of type 2 (Figure 3).

Proposition 2. *Let K be a generic curve in RP^2, with dual curve K'. Let $i = 1, 2$. Then:*
1) A crossing of type i in K is dual to a double tangent line of type i in K'.
2) A cusp in K is dual to an inflection point in K'.
3) An antipodal pair of type i in K is dual to a normal-tangent pair of type i in K'.
As the dual of K' is again K, these correspondences work in both directions.



(p,q) is a type 1 normal-tangent pair



(p,q) is a type 2 normal-tangent pair

FIGURE 3

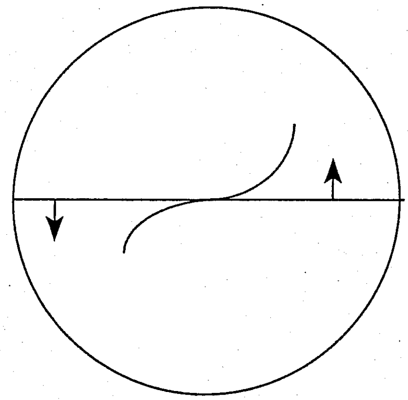
The proof is left to the reader.

This correspondence breaks down slightly when we consider double supporting lines between cusps and tangents, or cusps and cusps, which are dual to quantities involving inflection points of the curves. In order to incorporate curves with inflection points we need to add *inflection geodesics* to our picture of K.

Let p be an inflection point of K, with tangent geodesic τ_p. Endow $\tau_p - p$ with a normal direction at each point (except the inflection point) by the convention shown in Figure 4.

Definition.
Call this the *inflection geodesic to K at p.*

For crossings between K and an inflection geodesic τ_p, or between two inflection geodesics, the piece of τ_p in the neighborhood of the crossing should be construed as bending slightly towards its normal direction for the purposes of classifying the crossing type. A point on the inflection geodesic has center of curvature a distance $\pi/2$ in the normal direction.



inflection geodesic with normal direction

FIGURE 4

Definition.
Let $\overline{K}$ be K together with all its inflection geodesics, with crossings and normal tangencies counted as described above.

If K is a generic curve with dual K', then double supporting geodesics in K involving cusp points correspond to crossings in $\overline{K'}$ involving inflection geodesics, and an antipodal pair (p, q) with p a cusp point will correspond to a normal-tangent pair (p', q') with p' a point on an inflection geodesic in $\overline{K'}$.

4. A Dual Formula

The simplest version of the dual theorem applies to curves with no inflection points.

Theorem 3. *Let K be a generic singular curve in RP^2 with type 1 cusps and no inflection points. Then*

$$C_1 - C_2 = T_1 + T_2 + (1/2)U - (1/2)N_1 + (1/2)N_2$$

Since inflection points are dual to cusps, we also have:

Corollary 4. *Let K be the dual in RP^2 of a smooth singular curve. Then Theorem 3 holds for K.*

Sketch of proof (of Theorem 3):

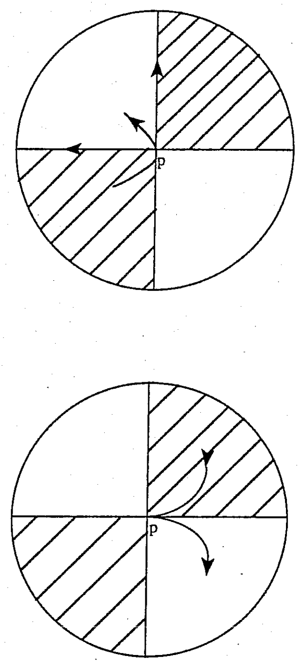


FIGURE 5

The theorem can either be proved directly, via duality on the formula, or by considering the dual of the tangent geodesic in the original argument, essentially constructing the dual argument. Since the second approach allows a direct proof for curves in the plane, we describe it briefly.

The geodesics τ_p and ν_p intersect in a single point (at p) and divide RP^2 into two regions. We first define the *tangent-normal frame* F_p to K at p as follows: F_p is the union of τ_p and ν_p together with a black-and-white coloring of the two regions of RP^2. We color them by the convention that if we think of τ_p and ν_p at P as being analogous to the standard $x-$ and $y-$ axiis, the region corresponding to the quadrants where x and y have the same sign is colored white, the complementary region black. The frame and its coloring are well-defined (this is one place where inflection points in K would cause some difficulty). At cusps, the orientations of τ_p and ν_p *both* reverse, with the happy effect that the coloring of the normal-tangent frame is well-defined as we pass through a cusp point (notice that this is not true if we allow type 2 cusps) (Figure 5).

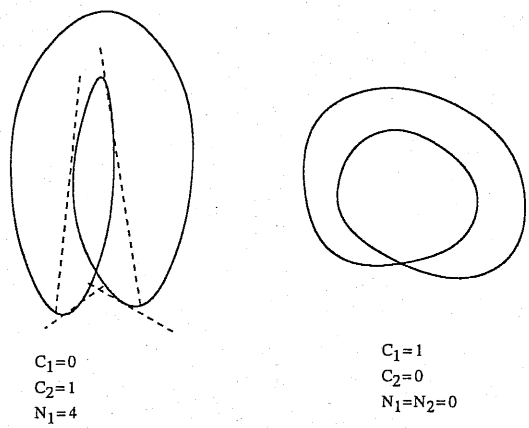


FIGURE 6

At a given point p on K, we define W_p to be the number of geodesics through p and tangent to K which lie in the white region and B_p to be the number of geodesics through p and tangent to K which lie in the black region. Let $V_p = B_p - W_p$. The proof proceeds by tracking the changes in V_p as we traverse K. V_p, B_p and W_p are the natural duals to M_p, M_p^+ and M_p^-.

Theorem 5. *Let K be a generic singular curve in RP^2 with type 1 cusps. Then for $\overline{K}$,*

$$C_1 - C_2 = T_1 + T_2 + (1/2)U + I - (1/2)N_1 + (1/2)N_2$$

With the addition of the inflection geodesics, we can adapt our original argument to prove:

Theorem 6. *Let K be a generic singular curve in RP^2 with type 1 cusps. Then for $\overline{K}$,*

$$T_1 - T_2 = C_1 + C_2 + (1/2)I + U - (1/2)A_1 + (1/2)A_2$$

And finally for K a curve with no cusps, inflection points, or antipodal pairs (or for a smooth immersed curve in R^2 with no inflection points), we can combine these results to obtain:

Corollary 7.

$$4T_1 - 4C_1 = N_1 - N_2$$
$$4T_2 + 4C_2 = N_1 - N_2$$

Note that for the two curves shown in Figure 6, we obviously have the values $T_1 = 1$, $T_2 = 0$. For the right-hand curve, $C_1 = 1$ and $C_2 = 0$, while for the left, $C_1 = 0$ and $C_2 = 1$. By observation, the right curve has no normal-tangent pairs, and the two equations in corollary 7 are easily seen to be satisfied. Applying corollary 7 to the left-hand curve, however, we obtain

$$4 = N_1 - N_2$$

and we can locate four normal-tangent pairs of type 1 (Figure 6).

References

[1] Banchoff, Thomas F. *Global geometry of polygons. I: The theorem of Fabricius-Bjerre* Proc. Amer. Math. Soc. 45 (1974), 237–241.

[2] Fabricius-Bjerre, Fr. *On the double tangents of plane curves* Math. Scand 11 (1962), 113–116.

[3] Fabricius-Bjerre, Fr. *A relation between the numers of singular points and singular lines of a plane closed curve* Math. Scand 40 (1977), 20-24.

[4] Halpern, Benjamin *Global theorems for closed plane curves* Bull. Amer. Math. Soc. 76 1970 96–100.

[5] Pignoni, Roberto *Integral relations for pointed curves in a real projective plane* Geom. Dedicata 45 (1993), no. 3, 263–287.

[6] Thompson, *Global invariants for curves in the projective plane*, in preparation.

[7] Weiner, Joel L. *A spherical Fabricius-Bjerre formula with applications to closed space curves* Math. Scand. 61 (1987), no. 2, 286–291.

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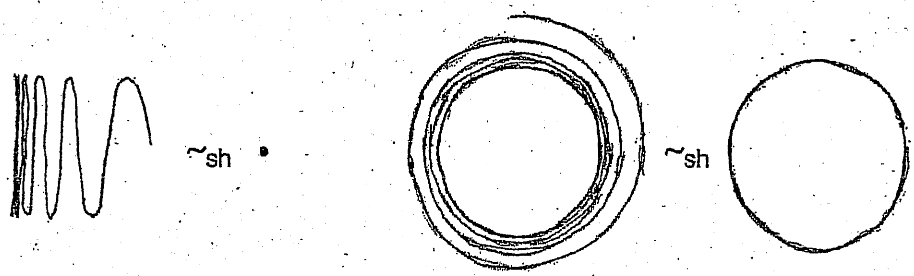
On the Cell-like Equivalence of CAT(0) Group Boundaries

by F. Ancel, C. Guilbault[1] and J. Wilson

Roughly speaking, two metric compacta X and Y are *shape equivalent,* denoted $X \sim_{sh} Y$, if when X and Y are embedded in the Hilbert cube $[0,1]^{\infty}$, they have homotopy equivalent neighborhood sequences mod index shifts. (See [C] page 39, for example, for the precise definition.)

Examples.



Definition. A metric compactum is a *cell-like set* if it is shape equivalent to a point. A function $f : X \to Y$ between metric compact is *cell-like* if $f^{-1}(y)$ is a cell-like set for every $y \in Y$.

Definition. Two finite dimensional metric compacta X and Y are *cell-like equivalent,* denoted $X \sim_{ce} Y$, if there is a finite "zigzag" sequence of finite dimensional metric compacta and cell-like maps

$$\begin{tikzcd}[column sep=small]
& Z_1 \arrow[dl] \arrow[dr] & & Z_3 \arrow[dl] \arrow[dr] & & \arrow[dl] \cdots \arrow[dr] & & Z_{2n+1} \arrow[dl] \arrow[dr] & \\
X & & Z_2 & & Z_4 & & Z_{2n} & & Y
\end{tikzcd}$$

joining X to Y.

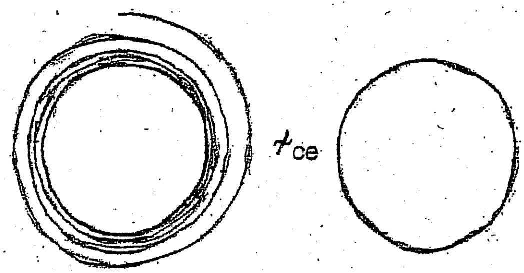
Fact: [Sh] For finite dimensional metric compacta, $X \sim_{ce} Y \Rightarrow X \sim_{sh} Y$.

Remark. This implication is false without the assumption of finite dimensionality for X, Y and all the Z_n's in the definition of "cell-like equivalent". [T]

[1] The second author wishes to acknowledge the support of the National Science Foundation.

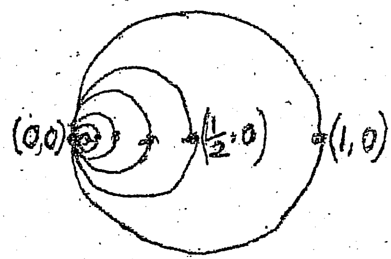
Also, the converse of this implication is false:

Example. [F]



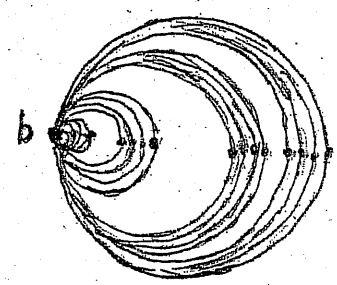
Definition. If p and q are distinct points in the Euclidean plane $\mathbb{R}^2$, let [p,q] denote the straight line segment joining them and let C[p,q] denote the unique circle with diameter [p,q]. The *Hawaiian earring* is the space

$$\mathcal{H} = \bigcup_{n \geq 1} C[(0,0),(1/n,0)]$$



More generally, if X is any compact totally disconnected infinite subset of the x axis $\mathbb{R} \times \{0\}$ and $b \in X$, any space homeomorphic to

$$\mathcal{H}(X,b) = \bigcup \{ C[b,p] : p \in X \}$$



is called a *generalized Hawaiian earring.*

Lemma 1. If X is any compact totally disconnected infinite subset of the x axis and $b \in X$, then $\mathcal{H}(X,b) \sim_{ce} \mathcal{H}$.

Proof. $\mathcal{H}(X,b) \swarrow \Sigma(X) \swarrow C(X) \cup J \searrow \mathcal{H}$,

where:

- $\Sigma(X)$ denotes the suspension of X and the cell-like map $\Sigma(X) \to \mathcal{H}(X,b)$ is the quotient map obtained from $\Sigma(X)$ by crushing the suspension arc through the point b to a point.

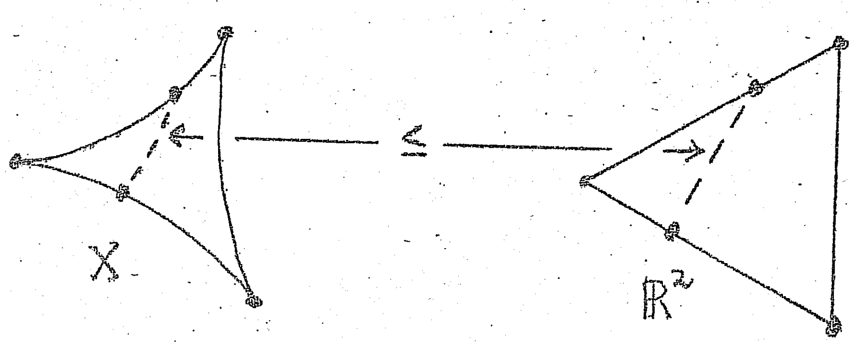
- J is the shortest closed interval in the x axis containing the set X, C(X) is the cone over the set X from a vertex point lying above the x axis in $\mathbb{R}^2$, and the cell-like map $C(X) \cup J \to \Sigma(X)$ is the quotient map obtained from $C(X) \cup J$ by crushing J to a point.
- The cell-like map frm $C(X) \cup J \to \mathcal{H}$ is the quotient map obtained from $C(X) \cup J$ by crushing C(X) to a point. (The resulting quotient space is the wedge of a decreasing sequence of circles. This space is homeomorphic to $\mathcal{H}$.) □

Results like this are found in [DV].

Geometric group theory originated in the work of M. Dehn who in 1912 proved that the fundamental group of a surface of genus ≥ 2 has a solvable word problem. His proof is geometric. It exploits the fact that the universal cover of a surface of genus ≥ 2 can be identified with the hyperbolic plane so that the fundamental group of the surface acts by isometry on the hyperbolic plane. In the 1980's, M. Gromov achieved a sweeping generalization of Dehn's work by introducing and studying hyperbolic groups. These are groups that act by isometry on metric spaces which have negative curvature in a very general sense. A further generalization to groups that act by isometry on non-positively curved (or CAT(0)) metric spaces has led to the study of CAT(0) groups.

Definition. X is a *CAT(0) space* if:

- X is a *geodesic* metric space. (For any two points p and q ∈ X, if d = d(p,q), then there is an embedding $e : [0,d] \to X$ such that e(0) = p, e(d) = q, and $d(e(s),e(t)) = |s-t|$ for all s, t ∈ [0,d].)
- X is a *proper* metric space. (For every p ∈ X and every r > 0, the closed metric ball $B_r(p) = \{ q \in X : d(p,q) \leq r \}$ is compact.)
- Distances in geodesic triangles in X are dominated by distances in comparison triangles in $\mathbb{R}^2$.

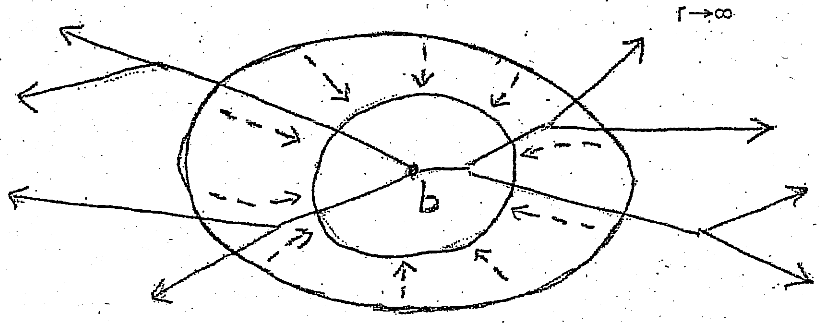


The concept of a CAT(0) space comes from the work of A. Alexandrov in the 1950's, although the terminology is Gromov's. The book [BH] is a comprehensive source.

Definition. A space X is *hyperbolic* if it as a proper geodesic metric space and there is a $\delta > 0$ such that all geodesic triangles in X are δ-slim. A geodesic triangle T in a metric space X is *δ-slim* if every non-vertex point p of T is distance $< \delta$ from a point on one of the two sides of T that doesn't contain p.

Definition. Let X be a CAT(0) space and choose a basepoint $b \in X$. Then X has a *visual boundary* denoted ∂X:

$$\partial X = \{\text{ geodesic rays in X emanating from b }\} = \varprojlim_{r \to \infty} \partial B_r(b).$$



Remark. We regard geodesic rays as functions from $[0,\infty)$ into X, and we put the compact-open topology on ∂X. The CAT(0) property implies that ∂X is independent of the choice of basepoint. Alternatively, the CAT(0) property implies that for $0 < r < s$, a geodesic joining b to a point of $\partial B_s(b)$ intersects $\partial B_r(b)$ at a unique point. This observation gives rise to the "geodesic retraction" $g_{s,r}$: $\partial B_s(b) \to \partial B_r(b)$. Hence, we obtain an inverse system $\{ \partial B_r(b), g_{r,s} \}$ which has an inverse limit $\varprojlim_{r \to \infty} \partial B_r(b)$. This inverse limit is also homeomorphic to ∂X.

Remark. The visual boundary of a hyperbolic space is defined similarly. See [BH], page 427, for details.

Definition. A group G is a *CAT(0) group* if G acts geometrically on a CAT(0) space. The action of a group G on a metric space X is *geometric* if the action is:

- properly discontinuous (For every compact subset C of X, $\{ g \in G : C \cap g(C) \neq \emptyset \}$ is finite.)
- cocompact (The orbit space X/G is compact.), and
- by isometry.

Definition. A group is *hyperbolic* if it acts geometrically on a hyperbolic space.

Remark. If a CAT(0) (or hyperbolic) group G acts geometrically on a CAT(0) (or hyperbolic) space X, then the action of G on X extends to an action of G on ∂X (because the elements of G are isometries of X, and isometries of X carry geodesic rays to geodesic rays).

Definition. If a CAT(0) (or hyperbolic) group G acts geometrically on a CAT(0) (or hyperbolic) space X, then the visual boundary ∂X of X is called a *boundary* of G.

Remark. There is an intimate connection between certain algebraic properties of a CAT(0) group G and certain topological properties of the boundaries of G. For example:

- If ∂G is any boundary of G, then $\text{c-dim}_{\mathbb{Z}}\, G = \dim(\partial G) + 1$. [BM], [B]
- If G is a one-ended CAT(0) group, ∂G is any boundary of G, and ∂G has a global cut point, then G contains an infinite torsion subgroup that fixes the global cut point. [Sw]

Theorem. (Gromov) If G is a hyperbolic group, then all the boundaries of G are equivariantly homeomorphic. ([G], page 189.)

Remark. The use of "equivariant" makes sense here because the action of G on X extends to an action of G on ∂X.

Example. The *Croke-Kleiner group* is the group

$$\Gamma = (\, a, b, c, d \mid [a,b] = [b,c] = [c,d] = 1 \,).$$

(In other words, Γ is the free group generated by a, b, c and d modulo the smallest normal subgroup containing the three commutators $[a,b] = aba^{-1}b^{-1}$, $[b,c] = bcb^{-1}c^{-1}$ and $[c,d] = cdc^{-1}d^{-1}$.) Γ is a CAT(0) group with non-homeomorphic boundaries. [CK]

Theorem. [B] If G is a CAT(0) group, then all the boundaries of G are shape equivalent.

Question. [B] Are all the boundaries of a given CAT(0) group cell-like equivalent?

Digression. Within the discipline of the philosophy of science, an oft quoted example of an apparent scientific law is the statement "All crows are black". (The question raised in conjunction with this law is whether a non-black non-crow (e.g., a white rabbit) should be regarded as supporting evidence for this law.) Regardless of whether this statement is truly a scientific law, there is an assertion which at this moment in time may be regarded as a truth:

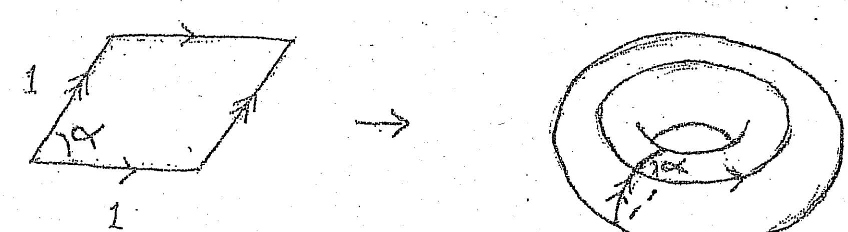
All known crows are black.

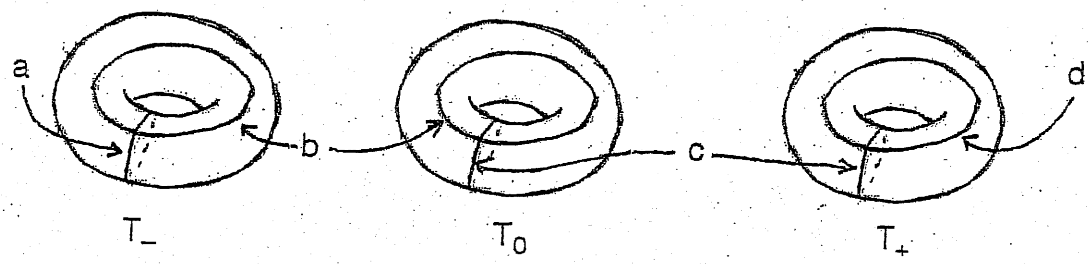
Encouraged by this example of a scientific truth, we state the principal result of this article.

Theorem 1. (F. Ancel, C. Guilbault, J. Wilson) All known boundaries of the Croke-Kleiner group are cell-like equivalent - - - to the Hawaiian earring $\mathcal{H}$.

More precisely: In [CK] an infinite family of CAT(0) spaces $X(\alpha)$, $0 < \alpha \leq \pi/2$, is described with the property that the Croke-Kleiner group Γ acts geometrically on each of them, and the their visual boundaries are not all homeomorphic. At the moment this is being written, the visual boundaries of the spaces $X(\alpha)$, $0 < \alpha \leq \pi/2$, are the only know boundaries of the Croke-Kleiner group Γ. We prove that for each $\alpha \in (0,\pi/2]$, the visual boundary of $X(\alpha)$ is cell-like equivalent to the Hawaiian earring $\mathcal{H}$.

Sketch of proof of Theorem 1. We first describe some basic features of the CAT(0) spaces $X(\alpha)$. Let $0 < \alpha \leq \pi/2$. $X(\alpha)$ is the universal cover of a space $Y(\alpha)$ that is the union of three 2-dimensional tori T_-, T_0, and T_+. Each of these tori is obtained by isometrically identifying opposite edges of a parallelogram in which each side has edge length 1 and the angles between the sides are α and $\pi - \alpha$. Thus, the fundamental group of each torus is generated by two closed geodesics of length 1 that intersect in a single point where the angle between them is α. To form the space $Y(\alpha)$ from the three tori, let b and c denote the two length 1 closed geodesic π_1-generators on T_0, identify b with one of the length 1 closed geodesic π_1-generators on T_-, and identify c with one of the length 1 closed geodesic π_1-generators on T_+. If we let a denote the other (non-identified) length 1 closed geodesic π_1-generator on T_-, and we let d denote the other (non-identified) length 1 closed geodesic π_1-generator on T_+, then clearly $\pi_1(Y(\alpha)) \approx \Gamma$.





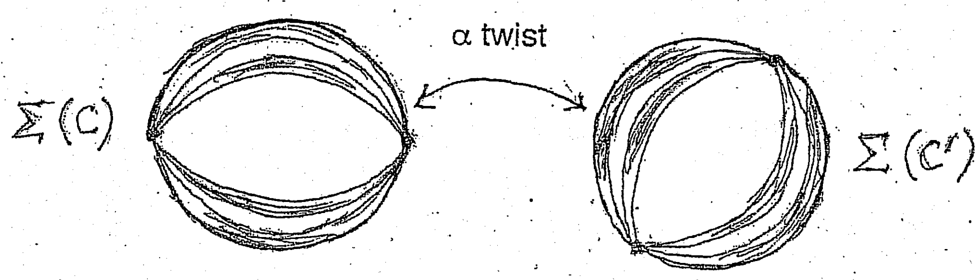
$$Y(\alpha) = T_- \underset{b}{\cup} T_0 \underset{c}{\cup} T_+$$

We let $X(\alpha)$ be the universal cover of $Y(\alpha)$, and we lift the geometry of $Y(\alpha)$ to $X(\alpha)$ by declaring the length of a path in $X(\alpha)$ to be the same as the length of its image in $Y(\alpha)$. This makes $X(\alpha)$ a CAT(0) space, and it makes Γ act geometrically on $X(\alpha)$.

It is useful to organize the structure of $X(\alpha)$ into "blocks". There are two types of blocks in $X(\alpha)$: –blocks and +blocks. Each component in $X(\alpha)$ of the inverse image of $T_- \underset{b}{\cup} T_0$ under the covering map $X(\alpha) \to Y(\alpha)$ is called a *–block,* and each component in $X(\alpha)$ of the inverse image of $T_0 \underset{c}{\cup} T_+$ under the covering map $X(\alpha) \to Y(\alpha)$ is called a *+block.* Thus, each –block is a universal cover of $T_- \underset{b}{\cup} T_0$ and each +block is a universal cover of $T_0 \underset{c}{\cup} T_+$. Since the spaces $T_- \underset{b}{\cup} T_0$ and $T_0 \underset{c}{\cup} T_+$ are homeomorphic to $S^1 \times 8$ (where 8 denotes a topological figure 8), then each block is homeomorphic to $\mathbb{R} \times \mathbb{T}$ (where $\mathbb{T}$ denotes an infinite 4-valent tree = the universal cover of 8).

In $X(\alpha)$, distinct –blocks are disjoint and distinct +blocks are disjoint. A –block and a +block may be disjoint, or they may intersect in a 2-dimensional plane (called a *wall*) that is a component of the inverse image of T_0 under the covering map $X(\alpha) \to Y(\alpha)$. It is helpful to encode the intersection pattern of the blocks in a 1-complex called the *nerve.* The vertices of the nerve correspond to the blocks in $X(\alpha)$, and two vertices of the nerve are connected by an edge if and only if the two corresponding blocks share a common wall. The nerve is a tree in which each vertex has $\aleph_0$ edges emanating from it. If the vertex at one end of an edge corresponds to a –block, then the vertex at the other end must correspond to a +block, and vice versa.

Since each block B is homeomorphic to $\mathbb{R} \times \mathbb{T}$, then its visual boundary ∂B is homeomorphic to the suspension of a Cantor set $\Sigma(C)$. The visual boundary of each block is embedded as a subset of the visual boundary $\partial X(\alpha)$ of $X(\alpha)$. If two blocks B and B′ share a common wall W (and, therefore, represent adjacent vertices in the nerve), then their visual boundaries $\partial B = \Sigma(C)$ and $\partial B' = \Sigma(C')$ intersect in a circle which is the visual boundary ∂W of that common wall. (C and C′ are Cantor sets.) The suspensions $\Sigma(C)$ and $\Sigma(C')$ are glued together along the common circle ∂W with a twist through the angle α. In other words, the suspension points of $\Sigma(C)$ appear as diametrically opposed *poles* on the circle ∂W, as do the suspension points of $\Sigma(C')$, and the angle between the north pole of $\Sigma(C)$ and the north pole of $\Sigma(C')$ is α. (α is also the angle between the south pole of $\Sigma(C)$ and the south pole of $\Sigma(C')$.) The union of the visual boundaries of all the blocks is a dense subset of the entire visual boundary of $X(\alpha)$.



By analyzing the positions and limit properties of the poles in $\partial X(\alpha)$, one can distinguish visual boundaries $\partial X(\alpha)$ and $\partial X(\beta)$ for certain values of α and β. The first result along these lines was:

Theorem. [CK] If $0 < \alpha < \pi/2$, then $\partial X(\alpha)$ is not homeomorphic to $\partial X(\pi/2)$.

Thus the Croke-Kleiner group Γ has at least two distinct boundaries.

More recently, we have established:

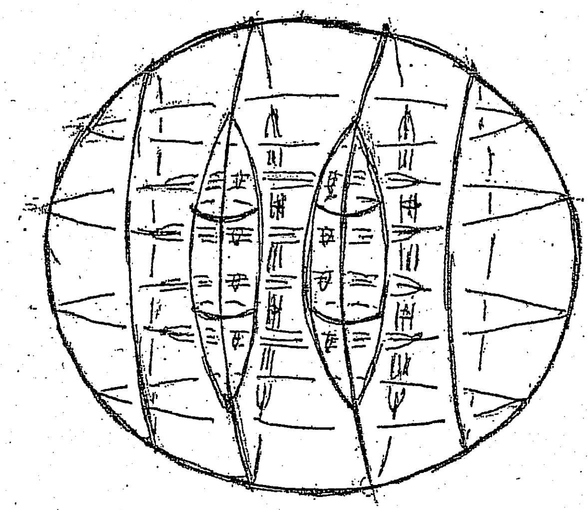
Theorem. [AW] If $0 < \alpha < \pi/2n \leq \beta \leq \pi/2$ for some positive interger n, then $\partial X(\alpha)$ is not homeomorphic to $\partial X(\beta)$.

Thus, the Croke-Kleiner group Γ has at least $\aleph_0$ distinct boundaries.

To prove that each visual boundary $\partial X(\alpha)$ is cell-like equivalent to the Hawaiian earring $\mathcal{H}$, it is useful to consider the inverse limit representation of $\partial X(\alpha)$. To this end, we choose a basepoint b of $X(\alpha)$. (For technical reasons, we choose b so that it doesn't lie in any wall of $X(\alpha)$.) For $r > 0$, let S_r denote the sphere of radius r centered at b in $X(\alpha)$. Then for $0 < r < r'$, there is a geodesic retraction from $S_{r'}$ to S_r. Choose an increasing sequence of positive real numbers converging to ∞: $0 < r_1 < r_2 < r_3 < \ldots$ For each $n \geq 1$, let $g_n : S_{r_{n+1}} \to S_{r_n}$ denote the geodesic retraction. Then $\partial X(\alpha)$ is homeomorphic to the inverse limit

$$\lim_{\substack{\longleftarrow \\ n\to\infty}} \{S_{r_n}, g_n\}.$$

Each S_r is a finite 1-complex that is a union of circles. Here is a picture of a typical S_r for $\alpha = \pi/2$ and $r = 2.13$.



To analyze the inverse system $\{S_{r_n}, g_n\}$, we expand to the inverse sequence

$$\begin{tikzcd}
S_{r_1} & T_1 \arrow[l] & S_{r_2} \arrow[l] \arrow[ll, bend right, "g_1"'] & T_2 \arrow[l] \arrow[ll, bend left, "h_1"] & S_{r_3} \arrow[l] \arrow[ll, bend right, "g_2"'] & T_3 \arrow[l] \arrow[ll, bend left, "h_2"] & \ldots \arrow[l] \arrow[ll, bend right, "g_3"']
\end{tikzcd}$$

by interpolating a second inverse sequence $\{T_n, h_n\}$. (Thus, $\lim\limits_{\substack{\longleftarrow \\ n\to\infty}} \{T_n, h_n\} \cong$

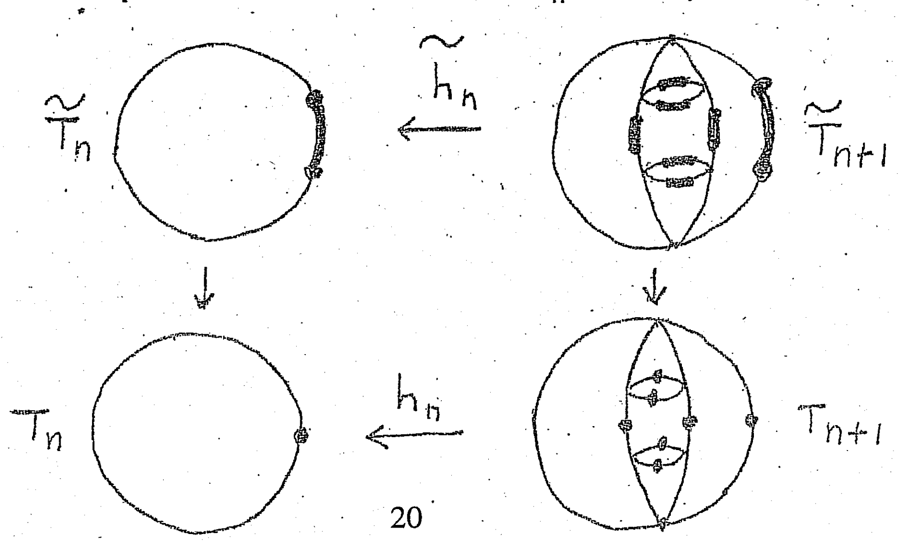
$\varprojlim_{n\to\infty}\{S_{r_n}, g_n\} \cong \partial X(\alpha)$.) Furthermore, we choose the inverse sequence $\{T_n, h_n\}$ so that it has the following properties. For each $n \geq 1$:

- T_n is a finite 1-complex.
- There is a finite subset A_n of T_n such that $T_n - A_n$ is contractible, and for each $x \in A_n$, $\varnothing \neq h_n^{-1}(x) \subset A_{n+1}$.
- $\lim_{n\to\infty}$ cardinality $(A_n) = \infty$.
- No point of A_n is an "essential vertex" of T_n. (In other words, each point of A_n has a neighborhood in T_n that is homeomorphic to $\mathbb{R}$.)
- h_n locally separates its image at each point of $h_n^{-1}(A_n)$. (In other words, for every $x \in h_n^{-1}(A_n)$, for each arc neighborhood U of $h(x)$ in T_n, there is an arc neighborhood V of x in T_{n+1} such that $h(V) \subset U$ and h maps the two components of $V - \{x\}$ into distinct components of $U - \{h_n(x)\}$.)

Now, to finish the proof of Theorem 1 (i.e., that $\partial X(\alpha) \sim_{ce} \mathcal{H}$), it suffices to establish:

Lemma 2. $\varprojlim_{n\to\infty}\{T_n, h_n\} \sim_{ce} \mathcal{H}$.

Outline of proof of Lemma 2. In each T_n, "blow up" each point of A_n to an arc.



If done carefully, this process "blows up" the entire inverse sequence $\{T_n, h_n\}$ to an inverse sequence $\{\tilde{T}_n, \tilde{h}_n\}$ with inverse limit $\tilde{T}_\infty$, giving rise to an infinite commutative diagram;

$$\begin{array}{ccccccccc}
\tilde{T}_1 & \xleftarrow{\tilde{h}_1} & \tilde{T}_2 & \xleftarrow{\tilde{h}_2} & \tilde{T}_3 & \xleftarrow{\tilde{h}_3} & & \cdots & \tilde{T}_\infty \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
T_1 & \xleftarrow[h_1]{} & T_2 & \xleftarrow[h_2]{} & T_3 & \xleftarrow[h_3]{} & & \cdots & \partial X(\alpha)
\end{array}$$

For each $n \geq 1$, $\tilde{T}_n$ is the union of finitely many arcs (the "blow ups" of the points of A_n) and a compact contractible set P_n (the closure of the inverse image of $T_n - A_n$ under the blow up map). Passing to the inverse limit, we see that $\tilde{T}_\infty$ is the union of a Cantor set's worth of arcs and a cell-like set P_∞. (P_∞ is the inverse limit of the sequence of contractible sets $\{P_n\}$.) Furthermore, since the blow up maps $\tilde{T}_n \to T_n$ are cell-like and converge to the map $\tilde{T}_\infty \to \partial X(\alpha)$, then the map $\tilde{T}_\infty \to \partial X(\alpha)$ is cell-like.

Now consider quotient space $\tilde{T}_\infty/P_\infty$. Clearly, $\tilde{T}_\infty/P_\infty$ is a Cantor set's worth of arcs with all of their endpoints identified to a single point. Because of the way that $\tilde{T}_\infty$ arises as an inverse limit, one can see that $\tilde{T}_\infty/P_\infty$ is, in fact, a generalized Hawaiian earring $\mathcal{H}(C,b)$ where C is a Cantor set. Also, since P_∞ is a cell-like set, then the quotient map $\tilde{T}_\infty \to \tilde{T}_\infty/P_\infty$ is cell-like. Thus, we have cell-like maps

$$\partial X(\alpha) \leftarrow \tilde{T}_\infty \to \tilde{T}_\infty/P_\infty \cong \mathcal{H}(C,b).$$

Hence, $\partial X(\alpha) \sim_{ce} \mathcal{H}(C,b)$.

In addition, Lemma 1 implies $\mathcal{H}(C,b) \sim_{ce} \mathcal{H}$. We conclude that $\partial X(\alpha) \sim_{ce} \mathcal{H}$. □

Questions.

Question 1) Are the known boundaries of the Croke-Kleiner group Γ (i.e., the $\partial X(\alpha)$'s for $0 < \alpha \leq \pi/2$) the only boundaries of Γ?

Question 2) Assuming the answer to Question 1 is "no": are all the boundaries of Γ cell-like equivalent to the Hawaiian earring $\mathcal{H}$?

Definition. Suppose that a group G acts on the finite dimensional metric compacta X and Y. We say that X and Y are *equivariantly cell-like equivalent* if there is a finite sequence $Z_1, Z_2, \ldots, Z_{2n+1}$ of finite dimensional metric compacta on which G acts and a "zigzag" sequence

$$\begin{tikzcd}
 & Z_1 \arrow[dl] \arrow[dr] & & Z_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Z_{2n+1} \arrow[dl] \arrow[dr] & \\
X & & Z_2 & & Z_4 & & Z_{2n} & & Y
\end{tikzcd}$$

of equivariant cell-like maps.

Recall: If a CAT(0) group G acts geometrically on a CAT(0) space X, then the action of G on X extends to an action of G on ∂X.

Question 3) Are any two boundaries of the Croke-Kleiner group Γ equivariantly cell-like equivalent?

Definition. Suppose that a CAT(0) group G acts geometrically on CAT(0) spaces X and Y. We say that ∂X and ∂Y are *equivariantly cell-like equivalent through boundaries of G* if there is a finite sequence $Z_1, Z_2, \ldots, Z_{2n+1}$ of CAT(0) spaces on which G acts geometrically and a "zigzag" sequence

$$\begin{tikzcd}
 & \partial Z_1 \arrow[dl] \arrow[dr] & & \partial Z_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \partial Z_{2n+1} \arrow[dl] \arrow[dr] & \\
\partial X & & \partial Z_2 & & \partial Z_4 & & \partial Z_{2n} & & \partial Y
\end{tikzcd}$$

of equivariant cell-like maps.

Question 4) Are any two boundaries of the Croke-Kleiner group Γ equivariantly cell-like equivalent through boundaries of Γ?

The next two questions are refinements of Question 4.

Question 5) If the Croke-Kleiner group Γ acts geometrically on the CAT(0) spaces X and Y, then does Γ act geometrically on a CAT(0) space Z so that there are equivariant cell-like maps

$$\begin{tikzcd} & \partial Z \arrow[dl] \arrow[dr] & \\ \partial X & & \partial Y \end{tikzcd}$$

Question 6) Is there a *maximal* Croke-Kleiner group boundary? In other words, does the Croke-Kleiner group Γ act geometrically on a CAT(0) space Z with the property that if Γ acts geometrically on any other CAT(0) space X, then there is an equivariant cell-like map $\partial Z \to \partial X$?

References.

[AW] F. Ancel and J. Wilson, Distinguishing Croke-Kleiner boundaries, in preparation.

[B] M. Bestvina, Local homology properties of boundaries of groups, *Michigan Math. J.* 43 (1996), 123-139.

[BM] M. Bestvina and G. Mess, The boundary of negatively curved groups, *Journal Amer. Math. Soc.* 4 (1991) 469-481.

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature,* Springer, 1999.

[C] T. Chapman, *Lectures on Hilbert Cube Manifolds,* Amer. Math. Soc. CBMS publications, number 28.

[CK] C. Croke and B. Kleiner, Spaces with nonpositive curvature and their ideal boundaries, *Topology* 39 (2000), 549-556.

[DV] R. Daverman and G. Venema, CE equivalence and shape equivalence of 1-dimensional compacta, *Top. Appl.* 26 (1987), 131-142.

[F] S. Ferry, Shape equivalence does not imply CE equivalence, *Proc. Amer. Math. Soc.* 80 (1980), 154-155.

[G] M. Gromov, Hyperbolic groups, in *Essays in Group Theory* (S. Gersten. ed.), Springer, MSRI Publ. 8 (1987), 75-263.

[Sh] R. Sher, Realizing cell-like maps in euclidean spaces, *Top. Appl.* 2 (1972), 75-89.

[Sw] E. Swenson, A cut point theorem for CAT(0) groups, *J. Diff. Geom.* 53 (1999), 327-358.

[T] J. Taylor, A counterexample in shape theory, *Bull. Amer. Math. Soc.* 81 (1975), 629-632.

CLASSIFYING STARLIKE BODIES

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ABSTRACT. We are interested in the structure of starlike bodies. Topological and smooth classifications of such bodies in the infinite-dimensional spaces are given. This involves an approximation of convex sets by smooth convex bodies. Some finite-dimensional examples are also discussed.

This is a preliminary report; the details will appear elswhere.

A closed subset A of a Banach space X is a starlike body if its interior $\operatorname{int} A$ is nonempty and there exists a point $x_0 \in \operatorname{int} A$ such that every ray emanating from x_0 meets ∂A, the boundary of A, at most once. With the use of suitable translation, we can always assume (and we will do so) that $x_0 = 0$ is the origin of X.

For a starlike body A, we define the characteristic cone of A as

$$ccA = \{x \in X | rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A as

$$\mu_A(x) = \inf\{\lambda > 0 | \frac{1}{\lambda} x \in A\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional μ_A is a continuous function which satisfies $\mu_A(rx) = r\mu_A(x)$ for every $r \geq 0$ and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X | \mu_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(\lambda x) = \lambda \psi(x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body. More generally, for a continuous function $\psi : X \to [0, \infty)$ such that $\psi_x(\lambda) = \psi(\lambda x)$, $\lambda > 0$, is increasing and $\sup \psi_x(\lambda) > \varepsilon$ for every $x \in X \setminus \psi^{-1}(0)$, the set $\psi^{-1}([0, \varepsilon])$ is a starlike body whose characteristic cone is $\psi^{-1}(0)$. Starlike bodies that are convex are called convex bodies. For a convex body U, ccU is always a convex set,

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but in general the characteristic cone of a starlike body is not convex. We will say that A is a C^p (or, real-analytic) smooth starlike body provided its Minkowski functional μ_A is C^p smooth (or, real-analytic) on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. Finally, two (smooth) starlike bodies A, B in a Banach space X are relatively homeomorphic (relatively diffeomorphic) if there exists a self-homeomorphism (diffeomorphism) $g : X \to X$ so that $g(A) = B$.

Starlike bodies often appear in nonlinear functional analysis as natural substitutes of convex bodies or in connection with polynomials, more precisely, for every n-homogeneous polynomial $P : X \to \mathbb{R}$ the set $\{x \in X | P(x) \leq c\}$, $c > 0$, is either a (real-analytic) starlike body or its complement is an interior of such a body (see [AD]). It is therefore reasonable to ask to what extent the geometrical properties of convex bodies are shared with the more general class of starlike bodies. In [AD] the question of whether James' theorem on the characterization of reflexivity (one of the deepest classical results of functional analysis) is true for starlike bodies was answered in the negative. In [AC] it was shown that the boundary of a smooth Lipschitz bounded starlike body in an infinite-dimensional Banach space is smoothly Lipschitz contractible; furthermore, the boundary is a smooth Lipschitz retract of the body. Here, we deal with the question as to what extent the known results on the topological classification of convex bodies can be generalized for the class of starlike bodies.

In [K], Klee gave a topological classification of the convex bodies of a Hilbert space. This result was generalized for every Banach space with the help of Bessaga's non-complete norm technique (see [BP]). To get a better insight in the history of the topological classification of convex bodies the reader should consult the papers by Stocker [S], Corson and Klee [CK], Bessaga and Klee [BK1, BK2], and Dobrowolski [Do1]. These results have recently been sharpened to obtain a full classification of the C^p smooth convex bodies in every Banach space [ADo]. In its most general form the result on the topological classification of (smooth) convex bodies reads as follows (see [ADo]); here, $p = 0, 1, 2, ..., \infty$, and "C^0 diffeomorphic" means just "homeomorphic".

Theorem 1. *Let U be a C^p convex body in a Banach space X.*

(a) *If ccU is a linear subspace of finite codimension (say $X = ccU \oplus Z$, with Z finite-dimensional), then U is C^p relatively diffeomorphic to $ccU + B_Z$, where B_Z is an*

Euclidean ball in Z.

(b) *If ccU is not a linear subspace or ccU is a linear subspace such that the quotient space X/ccU is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space (that is, $\{x \in X \mid x^*(x) \geq 0\}$, for some $x^* \in X^*$).* $\square$

Let us discuss to what extent this result can be generalized for (smooth) starlike bodies. The following simple example shows that the assertion (b) of Theorem 1 is not true for starlike bodies whose characteristic cones are not convex sets.

Example 1. Let $A = \{(x, y) \in \mathbb{R}^2 | x^2y^2 \leq 1\}$. It is clear that A is a (real-analytic) starlike body in $\mathbb{R}^2$, whose characteristic cone is the union of the coordinate axes. Hence A, having its boundary disconnected, cannot be relatively homeomorphic to a half-plane in $\mathbb{R}^2$. $\square$

However, every two (smooth) starlike bodies with the same characteristic cone are relatively homeomorphic (diffeomorphic). Though this fact is elementary, the proof of the smooth case must be done with some care. The real-analytic counterpart of this fact is unknown to us.

Proposition 1. *Let X be a Banach space, and let A_1, A_2 be C^p smooth starlike bodies such that $ccA_1 = ccA_2$. Then there exists a C^p diffeomorphism $g : X \to X$ such that $g(A_1) = A_2$, $g(\partial A_1) = \partial A_2$, and $g(0) = 0$. Moreover, $g(x) = \eta(x)x$, where $\eta : X \to [0, \infty)$, and hence g preserves the rays emanating from the origin.* $\square$

As said above, it is impossible to extend Theorem 1(b) to the class of starlike bodies. The variety of the characteristic cones of (unbounded) starlike bodies is enormous. If one wants to stick with the Bessaga-Klee classification scheme then the best result one can aim at is the assertion of Theorem 1 for the class of starlike bodies whose characteristic cones are convex sets.

We will state such a result, which relies on the following proposition, which might be of independent interest in the theory of smoothness in Banach spaces, and which implies that every closed convex cone in a separable Banach space can be regarded both as the characteristic cone of some C^∞ smooth convex body and as the set of zeros of a C^∞ smooth convex function. We say that a nonempty subset C of a Banach space X is a **cone** (resp., a cone over a set K) provided $[0,\infty)C = C$ (resp., $C = [0,\infty)K$). The cone C is proper if $C \neq X$.

Proposition 2. *For every proper closed convex set C in a separable Banach space X there exists a C^∞ smooth convex function $f : X \longrightarrow [0,\infty)$ so that $f^{-1}(0) = C$ and $f'(x) \neq 0$ for all $x \in X \setminus C$. Moreover, when C is a cone, $U = f^{-1}([0,1])$ is a C^∞ smooth convex body in X so that $ccU = C$.* □

Now we have arrived at the following generalization of Theorem 1.

Theorem 2. *Let A be a C^p starlike body in a separable Banach space X. Assume that ccA is a convex subset of X.*

(a) *If ccA is a linear subspace of finite codimension (say $X = ccU \oplus Z$, with Z finite-dimensional), then A is C^p relatively diffeomorphic to $ccA + B_Z$, where B_Z is a Euclidean ball in Z.*

(b) *If ccA is either not a linear subspace or else ccA is a linear subspace such that the quotient space X/ccA is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space.*

In the case $p = 0$, the assertions (a) and (b) hold for all Banach spaces X.

Proof. To obtain (a) it is enough to apply Proposition 1 for $A_1 = A$ and $A_2 = ccA + B_Z$.

To obtain (b) write $C = ccA$ for the closed convex cone of X. By Proposition 2, there exists a C^∞ smooth convex body U so that $ccU = C = ccA$. Then, by Proposition 1, the starlike bodies U and A are C^p relatively diffeomorphic. On the other hand, by the assumption, $ccU = C$ is either not a linear subspace or else is a linear subspace such that $\dim(X/C) = \infty$. Now, by Theorem 1(b), U is C^p relatively diffeomorphic to a closed half-space, and hence so is A.

In the case $p = 0$, it is easy to see that, for every closed convex cone $C \subset X$, the set $U = \overline{C + B}$, where B is the unit ball of X, is a closed convex body so that $C = ccU$. Hence, the above argument applies. □

It is natural to ask whether, for starlike bodies A and B with homeomorphic boundaries ∂A and ∂B, A and B are relatively homeomorphic.

The following theorem, answering this question in the affirmative, provides a full classification of starlike bodies in terms of the homotopy type of their boundaries in infinite-dimensional Banach spaces.

Theorem 3. *Let X be a Banach space and let A, B be starlike bodies in X with boundaries ∂A and ∂B. The following statements are equivalent:*

(1) *∂A has the same homotopy type as ∂B;*

(2) *∂A and ∂B are homeomorphic;*

(3) *A and B are relatively homeomorphic.*

The proof involves infinite-dimensional topology, see [BP]. The bodies A and B, and their boundaries ∂A and ∂B are so-called Hilbert manifolds. Since A and B are contractible, in fact, they are homeomorphic to X. Moreover, ∂A and ∂B are the so-called Z-sets in A and B, respectively. The fact that ∂A and ∂B have the same homotopy type implies they actually are homeomorphic. By the homeomorphism extension theorem for Z-sets, any homeomorphism $h : \partial A \to \partial B$ extends to a homeomorphism H of A onto B. Finally, it is easy to extend H to a self-homeomorphism of X.

Starlike bodies in a Banach space X are, in some sense, in one-to-one correspondence with closed subset K (open subsets U) of the unit sphere S of X. Let A be a starlike body in X. Let $r : X \setminus \{0\} \to S$ be the radial retraction. Clearly, $S(A) = r(ccA \setminus \{0\})$ is a closed subset of S such that $ccA = [0, \infty)S(A)$, the cone over $S(A)$, and $r(\partial A) = S \setminus S(A)$ is an open subset of S. As it is easily seen below, a closed subset K of S gives rise to a starlike body whose characteristic cone is the cone over K.

Proposition 3. *Let K be a closed subset of S. There exists a starlike body $A = A_K$ such that $S(A) = K$. If X is separable and C^p smooth, then we may require that the body A is C^p smooth as well.*

Proof. Take any continuous function $\lambda : S \to [0, 1]$ with $\lambda^{-1}(0) = K$. Define $\psi(x) = \|x\|\lambda(\frac{x}{\|x\|})$ for $x \neq 0$ and $\psi(0) = 0$. We see that $\psi : X \to [0, \infty)$ is a positively homogeneous continuous function with $\psi^{-1}(0) = [0, \infty)K$. It is enough to set $A = \psi^{-1}([0, 1])$.

In the smooth case, if X is C^p smooth, there exists a bounded C^p smooth starlike body whose charcteristic cone is $\{0\}$ [DGZ]. Let μ stand for the Minkowski functional of this body. Using the fact that X admits C^p smooth partitions of unity, one can find a continuous function $\lambda : X \to [0,1]$ which is C^p smooth off $\lambda^{-1}(0) = [0,\infty)K$. Define $\psi(x) = \mu(x)\lambda(\frac{x}{\mu(x)})$ for $x \neq 0$ and $\psi(0) = 0$. Clearly, $\psi : X \to [0,\infty)$ is a positively homogeneous continuous function which is C^p smooth off $\psi^{-1}(0) = [0,\infty)K$. Set $A = \psi^{-1}([0,1])$. □

Remark 1. The smooth assertion holds true if one replaces the separability assumption by the existence of C^p smooth partions of unity. □

In the proof of Proposition 3, instead of using the functional μ, we could have used a weak hilbertian norm ω on the separable space X, that is, a continuous norm of the form $\omega(x) = \|T(x)\|$ that is determined by an injective continuous linear operator $T : X \to \ell_2$. In such a case, ω is real-analytic off $\omega^{-1}(0)$. If K is a compact subset of S, then $K_0 = ([0,\infty)K) \cap B_\omega$, where B_ω is the unit ω-sphere, is also compact. Hence, $T(K_0)$ is compact in ℓ_2 and, by [Do2], there exists a continuous function $\lambda : B_\omega \to [0,1]$ that is real-analytic off $\lambda^{-1}(0) = K_0$.

Remark 2. Letting $\psi(x) = \omega(x)\lambda(\frac{x}{\omega(x)})$ for $x \neq 0$ and $\psi(0) = 0$, the set $A = \psi^{-1}([0,1])$ is a real-analytic starlike body with $ccA = [0,\infty)K$. As a consequence, in a separable Banach space, for every starlike body A with the locally compact ccA there exists a real-analytic starlike body A_0 with $ccA_0 = ccA$. □

We do not know whether this last statement holds for an arbitrary starlike body A. However, if ccA is weakly closed, then we can find a weak hilbertian norm ω so that ccA is ω-closed. We can then construct a continuous function $\lambda : B_\omega \to [0,1]$ that is C^∞ off $\lambda^{-1}(0) = ccA \cap B_\omega$. Since the characteristic cone of a weakly closed starlike body is weakly closed, we have the following:

Remark 3. For a starlike body A in a separable Banach space, which is closed in the weak topology, there exists a C^∞ starlike body A_0 with $ccA = ccA_0$. □

For a closed set $K \subset S$, all (smooth) starlike bodies of the form A_K are relatively (diffeomorphic) homeomorphic. As a consequence of Theorem 3, we have:

Corollary 1. *For two closed sets $K_1, K_2 \subset S$ in an infinite-dimensional Banach space X,* *the starlike bodies A_{K_1} and A_{K_2} are relatively homeomorphic if and only if the complements $S \setminus K_1$ and $S \setminus K_2$ have the same homotopy type.*

Proof. This is a consequence of Theorem 3 because the boundary of A_{K_i} is homeomorphic to $S \setminus K_i$, $i = 1, 2$. $\square$

It is unknown what necessary and sufficient conditions for K_i, $i = 1, 2$ one has to impose in order for their complements in S to have the same homotopy type. (Since the sphere S is homeomorphic to X, we can replace S by X.) If K is a Z-set in S (e.g., K is compact), then the complement of K is homeomorphic to S; hence, in such a case, A_K is relatively homeomorphic to the unit ball. If K_1 is a one-point set and K_2 is a small closed ball intersected with S, then K_1 is a Z-set, while B_2 is not a Z-set, but the complements of K_1 and K_2 have the same homotopy type (they are contractible), and therefore A_{K_1} and A_{K_2} are relatively homeomorphic (with the unit ball). The following simple example shows that the contractibility of K_1 and K_2 does not suffice to obtain the same homotopy type of their complements.

Example 2. Let $K_1 \subset S$ be a one point set and $K_2 = S \cap X_0$, where X_0 is a codimension 1 vector subspace of X. Then, K_1 and K_2 are contractible, but the complement of K_2 is disconnected, while the complement of K_1 is contractible (even homeomorphic to X). We see that A_{K_1} is relatively homeomorphic to the unit ball in X, while $ccA_{K_2} = X_0$ and, consequently, A_{K_2} is relatively homeomorphic to $X_0 \times [-1, 1]$, which, in turn, (having disconnected boundary in $X_0 \times \mathbb{R}$) is not homeomorphic to the unit ball in X. $\square$

Since, for a Z_σ-set Z (that is, Z is a countable union of Z-sets) in S, the spaces $S \setminus Z$ and S are homeomorphic, one can hope that if K_1 and K_2 have the same homotopy type modulo Z_σ-set, then the complements of K_i, $i = 1, 2$, have the same homotopy type. (Two closed sets P_1, P_2 are meant to have the same homotopy type modulo Z_σ-set if there are closed sets $P_i' \subset P_i$, $i = 1, 2$, such that P_i', $i = 1, 2$, have the same homotopy type and both $P_1 \setminus P_1'$ and $P_2 \setminus P_2'$ are Z_σ-sets.) This, however, is not the case because K_1 and K_2 of Example 2 have the same homotopy type modulo Z_σ-set.

The finite-dimensional case. Below we provide several examples showing that Corollary 1 cannot be extended in any reasonable way for the finite-dimensional space X.

Example 3. Let $S = S^1$ and B be the unit sphere and the unit ball in $X = \mathbb{R}^2$, respectively. Consider two compacta K_1 and K_2 in S; K_1 is a copy of an infinite convergent sequence space and K_2 is a copy of the Cantor set. Then, the bodies A_{K_1} and A_{K_2} (having their boundaries homeomorphic) are not homeomorphic.

To see this it suffices to notice that each A_{K_i} is homeomorphic to $B \setminus K_i$. It is then clear that any non-isolated point of K_1 has a basis of neighborhoods (in A_{K_1}) that can be chosen to be topologically different from any neighborhood of any point of K_2. We can obviously make those starlike bodies to be real-analytic, so an improvement in smoothness is not any help. $\square$

In higher dimensions, one can provide more regular examples.

Example 4. Let $S = S^2$ be the unit sphere in $X = \mathbb{R}^3$. Consider $C_1 = U_1 \cup U_2 \cup U_3$, where $U_1 = \{(x, y, z) \in S | |z| < 1/8\}$, $U_2 = \{(x, y, z) \in S | |z - 1| < 1/8\}$, and $U_3 = -U_2$, and $C_2 = U_1 \cup U_2 \cup U_3'$, where $U_3' = \{x, y, z) \in S | z - 1/2| < 1/8, y > 0\}$. Letting $K_i = S \setminus C_i$, $i = 1, 2$, we see that the boundaries of the starlike bodies A_{K_i} (being homeomorphic to C_i) are homeomorphic. However, there is no homeomorphism of A_{K_1} onto A_{K_2}. $\square$

In $\mathbb{R}^4$, we have the following example.

Example 5. Let $S = S^3$ be the unit sphere in $X = \mathbb{R}^4$. Let K be the (doubled) Fox-Artin arc in S, that is, K is a topological arc whose complement is a contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$, see [Ru, p. 68]. Then, for a starlike body $A = A_K$, ccA is the cone over an arc, therefore, it is contractible. Moreover, A_K is not homeomorphic to a half-space in $\mathbb{R}^4$ though both bodies have contractible boundaries. $\square$

In general, for every $n \geq 4$, the sphere $S = S^{n-1}$ in $X = \mathbb{R}^n$ contains an open contractible $(n-1)$-manifold U that is not homeomorphic to $\mathbb{R}^{n-1}$. One can take U to be the so-called Whitehead manifold. In each dimension, there are continuum many pairwise non-homeomorphic such objects. While the complement $S^3 \setminus U$ is a continuum that is not contractible, for $n > 4$, always one can pick U so that $S^{n-1} \setminus U$ is a contractible $(n-1)$-manifold. To see this, let M be a contractible $(n-1)$-manifold with non-simply connected boundary; the existence of M is due to N.H.A. Newman for $n > 5$ (see [G]), and due to B. Mazur and V. Poenaru for $n = 5$. Gluing together two copies of M along their boundaries we obtain the double space N, which is a topological copy of S^{n-1} (cf. [AG, p. 2, items (4) and (9)]). The complement of one copy of M in N is just the interior of the other copy, which yields a requested manifold U. Since U is not simply connected at infinity, U is not homeomorphic to $\mathbb{R}^{n-1}$; moreover, the manifold U, being the interior of a contractible manifold, is itself contractible.

Example 5. Write $K = S \setminus U$. Any starlike body A_K in $\mathbb{R}^n$, $n > 4$, has both ccA_K and ∂A_K contractible. However, A_K is not homeomorphic to a half-space in $\mathbb{R}^n$. □

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References

[AG] F.D. Ancel and C.R. Guilbault, *Compact contractible n-manifolds have arc of spines ($n > 5$)*, Pacific J. Math. **168** (1995), 1–10.

[AC] D. Azagra and M. Cepedello, *Smooth Lipschitz retractions of starlike bodies onto their boundaries in infinite-dimensional Banach spaces*, Bull. London Math. Soc. (to appear).

[AD] D. Azagra and R. Deville, *James' theorem fails for starlike bodies*, J. Funct. Anal. **180** (2001), 328–346.

[ADo] D. Azagra and T. Dobrowolski, *Smooth negligibility of compact sets in infinite-dimensional Banach spaces, with applications*, Math. Ann. **312** (1998), 445–463.

[BK1] C. Bessaga and V. L. Klee, *Two topological properties of topological linear spaces*, Israel J. Math. **2** (1964), 211–220.

[BK2] C. Bessaga and V. L. Klee, *Every non-normable Fréchet space is homeomorphic with all of its closed convex bodies*, Math. Ann. **163** (1966), 161–166.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Polish Scientific Publishers, Warszawa, 1975.

[CK] H. H. Corson and V. L. Klee, *Topological classification of convex sets*, Proc. Symp. Pure Math. 7, Convexity, Amer. Math. Soc., Providence, R. I. (1963), 37–51.

[DGZ] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces*, vol. 64, Pitman Monographies and Surveys in Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1993.

[Do1] T. Dobrowolski, *Relative classification of smooth convex bodies*, Bull. Acad. Polon. Sci., Sér. Sci. Math. **25** (1977), 309–312.

[Do2] T. Dobrowolski, *Smooth and R-analytic negligibility of subsets and extension of homeomorphism in Banach spaces*, Studia Math. **65** (1979), 115–139.

[G] C.R. Guilbault, *Some compact contractible manifold containing disjoint spines*, Topology **34** (1995), 99–108.

[K] V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. **74** (1953), 10–43.

[Ru] T. B. Rushing, *Topological Embeddings*, Academic Press, New York and London, 1973.

[S] J. J. Stoker, *Unbounded convex point sets*, Amer. J. Math. **62** (1940), 165–179.

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The fundamental group of a visual boundary versus the fundamental group at infinity

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There is a natural homomorphism from the fundamental group of the boundary of any non-positively curved geodesic space to its fundamental group at infinity. We will show that this homomorphism is an isomorphism in case the boundary admits a universal covering space, and that it is injective in case the boundary is one-dimensional.

1. Definitions

A metric space is called *proper* if all of its closed metric balls are compact. A *geodesic space* is a metric space in which any two points lie in a geodesic, i.e. a subset that is isometric to an interval of the real line in its usual metric.

A proper geodesic space X is said to be *non-positively curved* if any two points on the sides of a geodesic triangle in X are no further apart than their corresponding points on a reference triangle in Euclidean 2-space.

The *boundary* of a non-positively curved geodesic space X, denoted by bdy X, is defined to be the set of all geodesic rays emanating from a fixed base point x_0 endowed with the compact-open topology. (This definition is independent of the choice of x_0 [1, Proposition II.8.8].) Examples of such boundaries include the Sierpinski gasket and the one-dimensional Menger space [2].

While bdy X has a well-defined fundamental group $\pi_1(\text{bdy } X, \omega)$ based at a geodesic ray $\omega : [0,\infty) \to X$ with $\omega(0) = x_0$, there is also the notion of a *fundamental group at infinity* of X based at ω, denoted by $\pi_1^\infty(X,\omega)$. It is defined to be the limit of the inverse system whose terms are the fundamental groups of complements of compact subsets of X and whose bonds are induced by inclusion. Since the sequence of closed metric balls $B(k) = \{x \in X \mid d(x,x_0) \leq k\}$ is cofinal in the system of compact subsets of X, we get

$$\pi_1^\infty(X,\omega)$$
$$= \varprojlim \left(\pi_1(X \setminus B(1),\omega(2)) \xleftarrow{i_2} \pi_1(X \setminus B(2),\omega(3)) \xleftarrow{i_3} \pi_1(X \setminus B(3),\omega(4)) \xleftarrow{i_4} \cdots\right),$$

where i_k is defined to be the the composition of the inclusion induced homomorphism $incl_{k\#} : \pi_1(X \setminus B(k), \omega(k+1)) \to \pi_1(X \setminus B(k-1), \omega(k+1))$ and the isomorphism $s_k : \pi_1(X \setminus B(k-1), \omega(k+1)) \xrightarrow{\sim} \pi_1(X \setminus B(k-1), \omega(k))$, which "slides" the base point from $\omega(k+1)$ to $\omega(k)$ along ω.

For the remainder of this note let us fix a non-positively curved geodesic space X with base point x_0 and a geodesic ray ω emanating from x_0. We shall be interested in the relationship between $\pi_1(\operatorname{bdy} X, \omega)$ and $\pi_1^\infty(X, \omega)$.

2. The natural homomorphism

LEMMA 1. *There is a natural homomorphism* $\varphi : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1^\infty(X, \omega)$.

PROOF. Denoting by $[x_0, x]$ the (unique) geodesic in X from x_0 to x, we define a geodesic retraction map $r_k : S(k) \to S(k-1)$ by $x \mapsto [x_0, x] \cap S(k-1)$. Similarly we define $r_k' : X \setminus B(k-1) \to S(k-1)$ by $x \mapsto [x_0, x] \cap S(k-1)$. This allows us to write

$$\operatorname{bdy} X = \varprojlim \left(S(1) \xleftarrow{r_2} S(2) \xleftarrow{r_3} S(3) \xleftarrow{r_4} \cdots \right),$$

where we now interpret a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x_0$ as the sequence $(\gamma(1), \gamma(2), \cdots)$. Notice that the diagram

$$\begin{array}{ccccccc}
 & & \pi_1(X \setminus B(1), \omega(2)) & \xleftarrow{i_2} & \pi_1(X \setminus B(2), \omega(3)) & \xleftarrow{i_3} & \cdots \\
 & \swarrow r'_{2\#} & & \nwarrow incl_\# & & \swarrow r'_{3\#} & \\
\pi_1(S(1), \omega(1)) & & \xleftarrow{r_{2\#}} & & \pi_1(S(2), \omega(2)) & \xleftarrow{r_{3\#}} & \cdots
\end{array}$$

commutes. Hence its top row is pro-equivalent to its bottom row. Therefore the limit of the top inverse sequence, which defines $\pi_1^\infty(X, \omega)$, agrees with that of the bottom one. We obtain

$$\pi_1^\infty(X, \omega) = \varprojlim \left(\pi_1(S(1), \omega(1)) \xleftarrow{r_{2\#}} \pi_1(S(2), \omega(2)) \xleftarrow{r_{3\#}} \pi_1(S(3), \omega(3)) \xleftarrow{r_{4\#}} \cdots \right). \quad (1)$$

The inverse limit projections $q_k : \operatorname{bdy} X \to S(k)$ clearly induce homomorphisms $q_{k\#} : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1(S(k), \omega(k))$ that commute with the homomorphisms $r_{k\#}$. We therefore get an induced homomorphism $\varphi : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1^\infty(X, \omega)$ defined by $[\alpha] \to ([\alpha_1], [\alpha_2], \cdots)$, where for a map $\alpha : (S^1, *) \to (\operatorname{bdy} X, \omega)$ we put $\alpha_k = q_k \circ \alpha$. //

3. Coincidence with the first shape group

The typical examples of non-positively curved geodesic spaces have the structure of certain locally finite simplicial complexes whose simplices are isometric to simplices in a complete simply connected Riemannian manifold of some constant sectional curvature. In such spaces, metric spheres are ANRs. It is therefore not very restrictive to make the following

General Assumption. *Each $(S(k), \omega(k))$ has the homotopy type of a pointed ANR.*

Consequently, the projection

$$(\operatorname{bdy} X, \omega) \xrightarrow{(q_k)} \Big((S(1), \omega(1)) \xleftarrow{r_2} (S(2), \omega(2)) \xleftarrow{r_3} (S(3), \omega(3)) \xleftarrow{r_4} \cdots\Big)$$

induces an HPol$_*$-expansion in the sense of shape theory [8]. It follows from (1) that the fundamental group at infinity $\pi_1^\infty(X, \omega)$ coincides with the first shape group of $(\operatorname{bdy} X, \omega)$, which we will denote by $\check{\pi}_1(\operatorname{bdy} X, \omega)$. We record

Lemma 2. $\pi_1^\infty(X, \omega) = \check{\pi}_1(\operatorname{bdy} X, \omega)$. //

4. Boundaries with universal covers

Theorem 1. *If bdy X admits a universal covering space, then the natural homomorphism $\varphi : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1^\infty(X, \omega)$ is an isomorhism.*

We recall the definition of the pointed Čech system of a pointed compact metric space (Z, z) from [6] and [8]:

Consider the collection $\mathcal{C}$ of finite open covers $\mathcal{U}$ of Z which contain exactly one element $v(\mathcal{U}) \in \mathcal{U}$ with $z \in v(\mathcal{U})$. Then $\mathcal{C}$ is naturally directed by refinement. Denote by $(N(\mathcal{U}), v(\mathcal{U}))$ a geometric realization of the pointed nerve of $\mathcal{U}$, i.e. of the abstract simplicial complex $\{\Delta \mid \emptyset \neq \Delta \subseteq \mathcal{U}, \bigcap \Delta \neq \emptyset\}$ with distinguished vertex $v(\mathcal{U})$. For every $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ such that $\mathcal{V}$ refines $\mathcal{U}$, choose a pointed simplicial map $p_{\mathcal{U}\mathcal{V}} : (N(\mathcal{V}), v(\mathcal{V})) \to (N(\mathcal{U}), v(\mathcal{U}))$ with the property that the vertex corresponding to an element $V \in \mathcal{V}$ gets mapped to a vertex corresponding to an element $U \in \mathcal{U}$ with $V \subseteq U$. (Any assignment on the vertices which is induced by the refinement property will extend linearly.) Then $p_{\mathcal{U}\mathcal{V}}$ is unique up to pointed homotopy and we denote its pointed homotopy class by $[p_{\mathcal{U}\mathcal{V}}]$. For each $\mathcal{U} \in \mathcal{C}$ choose a pointed map $p_{\mathcal{U}} : (Z, z) \to (N(\mathcal{U}), v(\mathcal{U}))$ such that $p_{\mathcal{U}}^{-1}(St(U, N(\mathcal{U})) \subseteq U$ for all $U \in \mathcal{U}$, where $St(U, N(\mathcal{U}))$ denotes the open star of the vertex of $N(\mathcal{U})$ which corresponds to U. (For example, define $p_{\mathcal{U}}$ based on a partition of unity subordinated to $\mathcal{U}$.) Again, such a map $p_{\mathcal{U}}$ is unique up to pointed homotopy and we denote its pointed homotopy class by $[p_{\mathcal{U}}]$. Then $[p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{V}}] = [p_{\mathcal{U}}]$, and $(Z, z) \xrightarrow{([p_{\mathcal{U}}])} ((N(\mathcal{U}), v(\mathcal{U})), [p_{\mathcal{U}\mathcal{V}}], \mathcal{C})$ is an HPol$_*$-expansion, so that

$$\check{\pi}_1(Z, z) = \varprojlim\, (\pi_1(N(\mathcal{U}), v(\mathcal{U})), p_{\mathcal{U}\mathcal{V}\#}, \mathcal{C}). \tag{2}$$

A proof of the following lemma can be found in Section 2 of [3]:

LEMMA 3. *Let $\mathcal{V} \in \mathcal{C}$. Suppose every element of $\mathcal{V}$ is connected and every loop which lies in the union of any two elements of $\mathcal{V}$ contracts in Z, then the homomorphism $p_{\mathcal{V}\#} : \pi_1(Z, z) \to \pi_1(N(\mathcal{V}), v(\mathcal{V}))$ is an isomorphism.*

To prove Theorem 1, we let $(Z, z) = (\operatorname{bdy} X, \omega)$. In view of (2) and Lemma 2, it suffices now to show that for every element $\mathcal{U} \in \mathcal{C}$ there is an element $\mathcal{V} \in \mathcal{C}$ such that $\mathcal{V}$ refines $\mathcal{U}$ and $p_{\mathcal{V}\#} : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1(N(\mathcal{V}), v(\mathcal{V}))$ is an isomorphism. Since by assumption, bdy X is a connected, locally path connected, semi-locally simply connected, compact metric space, every $\mathcal{U} \in \mathcal{C}$ can easily be refined by an element $\mathcal{V} \in \mathcal{C}$ that satisfies the requirements of Lemma 3.

5. One–Dimensional Boundaries

THEOREM 2. *If* bdy X *is one-dimensional, then the natural homomorphism* $\varphi : \pi_1(\operatorname{bdy} X, \omega) \to \pi_1^\infty(X, \omega)$ *is injective.*

PROOF. Suppose (P_k, p_k) is any sequence of pointed compact metric spaces having the homotopy type of pointed ANRs, and $f_{k-1,k} : (P_k, p_k) \to (P_{k-1}, p_{k-1})$ are continuous maps such that

$$(\operatorname{bdy} X, \omega) = \varprojlim \left((P_1, p_1) \xleftarrow{f_{1,2}} (P_2, p_2) \xleftarrow{f_{2,3}} (P_3, p_3) \xleftarrow{f_{3,4}} \cdots \right).$$

Then the projections $f_k : (\operatorname{bdy} X, \omega) \to (P_k, p_k)$ induce a canonical homomorphism

$$\psi : \pi_1(\operatorname{bdy} X, \omega) \to G = \varprojlim \left(\pi_1(P_1, p_1) \xleftarrow{f_{1,2\#}} \pi_1(P_2, p_2) \xleftarrow{f_{2,3\#}} \pi_1(P_3, p_3) \xleftarrow{f_{3,4\#}} \cdots \right),$$

defined by $\psi([\alpha]) = ([f_1 \circ \alpha], [f_2 \circ \alpha], [f_3 \circ \alpha], \cdots)$. Since

$$(\operatorname{bdy} X, \omega) \xrightarrow{(f_k)} \left((P_1, p_1) \xleftarrow{f_{1,2}} (P_2, p_2) \xleftarrow{f_{2,3}} (P_3, p_3) \xleftarrow{f_{3,4}} \cdots \right)$$

is another HPol$_*$–expansion, there is an isomorphism $i : \check{\pi}_1(\operatorname{bdy} X, \omega) \to G$ such that $i \circ \varphi = \psi$. The assertion of the theorem will follow if we choose the sequence $((P_k, p_k), f_{k-1,k})$ such that $\psi : \pi_1(\operatorname{bdy} X, \omega) \to G$ is injective. This can be done using any one of the following three theorems. //

THEOREM. [4] *Let Z be the one-dimensional Menger space, obtained by intersecting the standard nested sequence (P_k) of three-dimensional handlebodies. Fix a point $z \in Z$. Then the canonical homomorphism*

$$\psi : \pi_1(Z,z) \to \varprojlim \left(\pi_1(P_1,z) \overset{incl_\#}{\longleftarrow} \pi_1(P_2,z) \overset{incl_\#}{\longleftarrow} \pi_1(P_3,z) \overset{incl_\#}{\longleftarrow} \cdots \right)$$

is injective.

REMARK. This theorem suffices to finish the proof of Theorem 2, since every one-dimensional compact metric space Y embeds in the one-dimensional Menger space Z so that the induced homomorphism on fundamental groups $\pi_1(Y,y) \to \pi_1(Z,z)$ is injective:

$$\begin{array}{ccc} \pi_1(Y,y) & \overset{incl_\#}{\longrightarrow} & \pi_1(Z,z) \\ \downarrow & \circlearrowleft & \downarrow \\ \check{\pi}_1(Y,y) & \overset{incl_\#}{\longrightarrow} & \check{\pi}_1(Z,z) \end{array}$$

THEOREM. [7] *Let Z be the limit of an inverse sequence $P_1 \overset{f_2}{\longleftarrow} P_2 \overset{f_3}{\longleftarrow} P_3 \overset{f_4}{\longleftarrow} \cdots$ of one-dimensional compact polyhedra and $z = (p_n) \in Z$. Then the canonical homomorphism*

$$\psi : \pi_1(Z,z) \to \varprojlim \left(\pi_1(P_1,p_1) \overset{f_{2\#}}{\longleftarrow} \pi_1(P_2,p_2) \overset{f_{3\#}}{\longleftarrow} \pi_1(P_3,p_3) \overset{f_{4\#}}{\longleftarrow} \cdots \right)$$

is injective.

REMARK. This theorem suffices to finish the proof of Theorem 2, because every one-dimensional compact metric space is the limit of an inverse sequence of one-dimensional compact polyhedra.

THEOREM. [5] *Let Z be a one-dimensional, compact, connected metric space, and $z \in Z$. Then Z can be embedded in three-dimensional Euclidean space such that there exists a sequence $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$ of handlebodies with $\bigcap P_k = Z$ and such that the canonical homomorphism*

$$\psi : \pi_1(Z,z) \to \varprojlim \left(\pi_1(P_1,z) \overset{incl_\#}{\longleftarrow} \pi_1(P_2,z) \overset{incl_\#}{\longleftarrow} \pi_1(P_3,z) \overset{incl_\#}{\longleftarrow} \cdots \right)$$

is injective.

References

1. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der math. Wissenschaften **319**, Springer Verlag (1999).

2. N. Benakli, *Polyèdres Hyperboliques, Passaga du Local au Global.* Thèse, Univ. de Paris Sud (1992).

3. J. Cannon, *Geometric Group Theory.* Pre-Print.

4. M. Curtis and M. Fort, *Singular homology of one-dimensional spaces.* Ann. Math. **69** (1959) 309–313.

5. J. Cannon and G. Conner, *On the fundamental groups of one-dimensional spaces.* Pre-print.

6. J. Dydak and J. Segal, *Shape Theory: An Introduction.* Lecture Notes in Mathematics **688**, Springer Verlag (1978).

7. K. Eda and K.Kawamura, *The fundamental groups of 1-dimensional spaces.* Topology and its Applications **87** (1998) 163–172.

8. S. Mardešić and J. Segal, *Shape theory: the inverse limit approach.* North-Holland Mathematical Library **26**, North-Holland Publishing Company (1982).

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ENDS OF MANIFOLDS: RECENT PROGRESS

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ABSTRACT. In this note we describe some recent work on ends of manifolds. In particular, we discuss progress on two different approaches to generalizing Siebenmann's thesis to include manifolds with non-stable fundamental groups at infinity.

1. INTRODUCTION

In this note we discuss some of our recent work on ends of manifolds. For simplicity we focus our attention on one-ended open manifolds.

- A manifold M^n is *open* if it is noncompact and has no boundary.
- A subset V of M^n is a *neighborhood of infinity* if $\overline{M^n - V}$ is compact.
- M^n is *one-ended* if each neighborhood of infinity contains a connected neighborhood of infinity.

Example 1. *$\mathbb{R}^n$ is an open n-manifold for all $n \geq 1$. If $n \geq 2$, then $\mathbb{R}^n$ is one-ended.*

Example 2. *If P^n is a closed connected manifold, then $P^n \times \mathbb{R}^k$ is an open manifold for all $n \geq 1$. $P^n \times \mathbb{R}^k$ is one-ended iff $k \geq 2$.*

Example 3. *Let P^n be a compact manifold with non-empty connected boundary. Then int (P^n) is a one-ended open manifold.*

A natural question to ask about open manifolds is the following.

Question. When is an open n-manifold M^n just the interior of a compact manifold with boundary?

Equivalent Question. When does M^n contain an "open collar" neighborhood of infinity? (V is an *open collar* if $V \approx \partial V \times [0,1)$.

These questions were answered (in high dimensions) by Siebenmann in his 1965 Ph.D. thesis.

Theorem 1.1 (see [Si]). *A one ended open n-manifold M^n ($n \geq 6$) contains an open collar neighborhood of infinity if and only if each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is stable at infinity, and*

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(3) $\sigma_\infty(M^n) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$ *is trivial.*

In the above theorem:

- *inward tame* means $\forall$ neighborhood V of infinity, $\exists$ homotopy $H : V \times [0,1] \to V$ such that $H_0 = id$ and $\overline{H_1(V)}$ is compact.
- π_1 *stable at infinity* means $\exists$ a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of neighborhoods of infinity with $\bigcap V_i = \emptyset$ and inclusion induced homomorphisms all isomorphisms:
$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots .$$
- Condition 3) ensures that the V_i's have finite homotopy type.

Remark. Siebenmann's Theorem (and variations due to Quinn) have been extremely important in manifold topology—especially embedding theory. In other situations—for example the study of universal covering spaces—the hypotheses are too strong. Thus, it has been asked:

Question. Are there versions of Siebenmann's Theorem that apply to a more general class of manifolds?

The main goal of this note is to discuss two different (but related) programs for generalizing Siebenmann's Theorem.

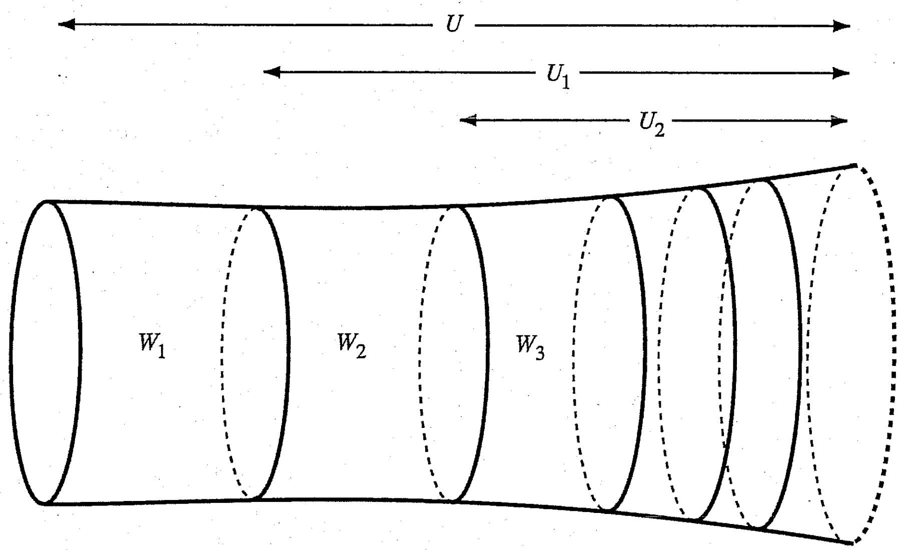
2. Generalizing Siebenmann's Thesis: Approach #1

We begin by generalizing the notion of an open collar to that of a "pseudo-collar". We then seek conditions that imply that a given open n-manifold contains a pseudo-collar neighborhood of infinity.

- A manifold U with compact boundary is a *homotopy collar* if $\partial U \hookrightarrow U$ is a homotopy equivalence.
- If, in addition, U contains arbitrarily small homotopy collar neighborhoods of infinity, we call U a *pseudo-collar.*

One nice aspect of a pseudo-collar structure is that it may be decomposed into a countable union of compact "one-sided h-cobordisms". (A *one-sided h-cobordism* can be deformation retracted onto one of its boundary components, but not necessarily onto the other.) These cobordisms have been the object of frequent study. See for example [DT] and Sections 11.1 and 11.2 of [FQ]. Given a pseudo-collar U and a cofinal sequence $U_1 \supset U_2 \supset U_3 \supset \cdots$ of homotopy collar neighborhoods of infinity, let $W_i = U_{i-1} - int(U_i)$. Then each $(W_i, \partial U_{i-1}, \partial U_i)$ is a one-sided h-cobordism. See Figure 1 for a schematic picture.

Example 4. *A particularly interesting collection of pseudo-collarable (but not collarable) open n-manifolds are the exotic universal covering spaces constructed by M. Davis in* [Da].



So far, our best theorem for ensuring pseudo-collarability in an open n-manifold is:

Theorem 2.1 (see [Gu1]). *A one-ended open n-manifold M^n ($n \geq 7$) is pseudo-collarable provided each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is perfectly semistable at infinity,*
(3) $\sigma_\infty(M^n) \in \varprojlim \left\{ \widetilde{K}_0\pi_1(M^n \backslash A) \mid A \overset{cpt.}{\subset} M^n \right\}$ *is zero, and*
(4) *π_2 is semistable at infinity.*

In this theorem:

- π_1 *semistable at infinity* means $\exists$ a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of neighborhoods of infinity with $\bigcap V_i = \emptyset$ and inclusion induced homomorphisms all surjective:
$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots.$$
- *perfectly semistable* means that, in addition, we can arrange that $\ker(\lambda_i)$ is perfect for each i,
- requiring that $\sigma_\infty(M^n) = 0$ ensures that the V_i's have finite homotopy types, and
- π_2 *semistable at infinity* means what you think it does...

Remark. Conditions 1)-3) are also necessary. Hence, the following question is natural.

Question. Can Condition 4) be eliminated from Theorem 2.1?

Another intriguing open problem is:

Question. Does condition 2) follow from Condition 1)?

Combining the above two questions we arrive at:

Big Question #1. Do conditions 1) and 3) suffice?

3. Generalizing Siebenmann's Thesis: Approach #2

Instead of viewing Theorem 1.1 as detecting open collar neighborhoods of infinity, one may view it as answering the question: "When can an open manifold be compactified to a manifold with boundary by adding a boundary $(n-1)$-manifold?". Taking this point of view, our second approach to generalizing Theorem 1.1 is to look for compactifications which permit a less rigid sort of boundary (a "$\mathcal{Z}$-boundary").

- A closed subset A of a compact ANR Y is a *$\mathcal{Z}$-set* if, for every open set U of Y, $U\backslash A \hookrightarrow U$ is a homotopy equivalence.
- A compactification $\widehat{X}$ of a space X is a *$\mathcal{Z}$-compactification* if $\widehat{X}\backslash X$ is a $\mathcal{Z}$-set in $\widehat{X}$. In this case, we call $\widehat{X}\backslash X$ a *$\mathcal{Z}$-boundary* for X.

Example 5. *If P^n is a manifold with boundary, then any closed subset of ∂P^n is a $\mathcal{Z}$-set in P^n.*

Example 6. *Adding a manifold boundary to an open manifold is a (particularly nice) $\mathcal{Z}$-compactification.*

Example 7. *Davis' exotic universal covering spaces admit $\mathcal{Z}$-compactifications—but not manifold compactifications.*

Example 8. *If P^n is a closed aspherical manifold with CAT(0) or word hyperbolic fundamental group, then $\widetilde{P}^n$ admits a $\mathcal{Z}$-compactification.*

Question. Under what conditions does a one-ended open manifold admit a $\mathcal{Z}$-compactification?

For the case of Hilbert cube manifolds, this question was answered by the following theorem.

Theorem 3.1 (see CS). *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification iff each of the following is satisfied.*

a) *X is inward tame at infinity.*

b) $\sigma_\infty(X) \in \varprojlim \left\{ \widetilde{K}_0\pi_1(X\backslash A) \mid A \overset{cpt.}{\subset} X \right\}$ *is zero.*

c) $\tau_\infty(X) \in \varprojlim{}^1 \left\{ Wh\pi_1(X\backslash A) \mid A \overset{cpt.}{\subset} X \right\}$ *is zero.*

Corollary 3.2. *For any locally compact ANR Y, the above conditions are necessary and sufficient for $Y \times [0,1]^\infty$ to be $\mathcal{Z}$-compactifiable.*

This corollary raises a natural question first posed by Chapman and Siebenmann.

Question. Are these conditions sufficient for the ANR Y itself to be $\mathcal{Z}$-compactifiable?

In [Gu2] we answered this question in the negative. The counterexample is a 2-dimensional polyhedron, but not a manifold. We consider the following to be an important open problem.

Big Question #2. Does an open manifold satisfying conditions a)-c) admit a $\mathcal{Z}$-compactification?

4. Recent Progress

In this section we describe some recent progress on some of the questions raised in the previous two sections. Proofs will be contained in a pair of papers that are currently in progress.

The first new result reveals a connection between conditions 1) and 2) of Theorem 2.1. Note, however, that it does not imply that condition 1) implies condition 2).

Theorem 4.1. *Let M^n be a one-ended open n-manifold. If M^n is inward tame at infinity, then π_1 is semistable at infinity.*

The second new result is related to "Big Question #2". Although it does not settle the problem, it provides the best possible "stabilized" answer to that question.

Theorem 4.2. *Let M^n be a one-ended open n-manifold ($n \geq 5$). Then $M^n \times [0,1]$ admits a $\mathcal{Z}$-compactification (in fact $M^n \times [0,1]$ is a "missing boundary manifold") if and only if M^n satisfies a)-c) of Theorem 3.1.*

Remark. In [Fe], Ferry has shown that if a k-dimensional polyhedron K satisfies a)-c), then $K \times [0,1]^{2k+5}$ admits a $\mathcal{Z}$-compactification. Previously, in [OB], O'Brien showed that $[0,1]^3$ suffices if K is a one-ended open manifold.

References

[CS] T.A. Chapman and L.C. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. **137** (1976), 171-208.

[DT] R. J. Daverman and F. Tinsley, *Controls on the plus construction*, Michigan Math. J. **43** (1996), 389-416.

[Da] M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117**(1983) 293-325.

[Fe] S. Ferry, *Stable compactifications of polyhedra,* Michigan Math. J. **47** (2000), 287–294.

[FQ] M.H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, New Jersey, 1990.

[Gu1] C.R. Guilbault, *Manifolds with non-stable fundamental groups at infinity*, Geom. & Top. **4** (2000), 537-579.

[Gu2] C.R. Guilbault, *A non-$\mathcal{Z}$-compactifiable polyhedron whose product with the Hilbert cube is $\mathcal{Z}$-compactifiable*, Fund. Math. **168** (2001), 165-197.

[OB] G. O'Brien, *The missing boundary problem for smooth manifolds of dimension greater than or equal to six,* Topology Appl. **16** (1983), 303–324.

[Si] L.C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Ph.D. thesis, Princeton University, 1965.

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ON THE EXISTENCE OF EXTENSION DIMENSION

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ABSTRACT. We prove the existence of extension dimension for all stratifiable spaces of a fixed bounded weight with respect to the class of all simplicial complexes. Since the class of stratifiable spaces contains the class of metrizable spaces, the result applies to metrizable spaces.

The starting notion is that of absolute extensor. Let X be a topological space, K be a simplicial complex, and $|K|$ be its associated polyhedron. The notation $K \in \mathrm{AE}(X)$ (or $|K| \in \mathrm{AE}(X)$) means that for every closed subspace A of X and map $f : A \to |K|$, there exists a map $F : X \to |K|$ which is an extension of f. Two other notations for $K \in \mathrm{AE}(X)$ are $X \tau K$ and $\dim X \leq K$.

Let $\mathcal{S}$ be a class of simplicial complexes and $\mathcal{C}$ a class of spaces. Let K, K' be simplicial complexes. The origin of the whole theory is the following postulation:

If it is true that for all $X \in \mathcal{C}$, $|K| \in \mathrm{AE}(X)$ implies that $|K'| \in \mathrm{AE}(X)$, then we write $K \leq K'$.

This defines a preorder among simplicial complexes (see [DD]). One specifies $K \sim K'$ if and only if $K \leq K'$ and $K' \leq K$; then $\sim$ is an equivalence relation on the class of simplicial complexes. An equivalence class under this relation is called an *extension type,* or more precisely, a $(\mathcal{C}, \mathcal{S})$-extension type, since it depends on the considered classes $\mathcal{C}$ and $\mathcal{S}$. We then write $\dim X \leq [K]$ to mean that $\dim X \leq K'$ for all $K' \in [K]$. Denote by $\mathrm{ET}(\mathcal{C}, \mathcal{S})$ the class of all extension types. Then the above relation $\leq$ induces a partial order on $\mathrm{ET}(\mathcal{C}, \mathcal{S})$. So, for a given space $X \in \mathcal{C}$ we may ask if there is a minimal element in the following class of extension types:

$$(*) \qquad \{[L] \in \mathrm{ET}(\mathcal{C}, \mathcal{S}) \mid \dim X \leq [L]\}.$$

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If there is a minimal element $[K]$, then it is called the *extension dimension* of X relative to $(\mathcal{C}, \mathcal{S})$ and is denoted by $\text{ext-dim}_{(\mathcal{C},\mathcal{S})}(X) = [K]$. If $\mathcal{S}$ is the class of all (simplicial) polyhedra, then for $X \in \mathcal{C}$ we simply say extension dimension of X and write $\text{ext-dim}(X) = [K]$.

In [Dr] A.N. Dranishnikov considered the existence of extension dimension for compact Hausdorff spaces. Later he and J. Dydak [DD] went further in this direction. They defined a class of spaces named ext-spaces. Their proof that extension dimension exists for ext-spaces has a gap, so we sought a tool that would allow us to approach the problem from a different direction. The content of this contributed talk was the presentation of the main steps of the proof that $\text{ext-dim}_{(\mathcal{C}_\alpha,\mathcal{S})}$ exists where $\mathcal{C}_\alpha$ is the class of all stratifiable spaces of weight $\leq \alpha$ and $\mathcal{S}$ is the class of all polyhedra.

The notion of stratifiable space was introduced in 1966 by J. Ceder [Ce] as a generalization of metrizable spaces. The class of stratifiable spaces lies between the classes of paracompact and metrizable spaces. Since this class of spaces is not widely used, we give the definition and list some important facts.

1. Definition. *A T_1-space X is stratifiable provided there is a function (a stratification) assigning to each open subset U of X a sequence (U_n) of open subsets of X such that*

(S1) $\overline{U}_n \subset U$ *for each* n,

(S2) $\bigcup_{n=1}^{\infty} U_n = U$, *and*

(S3) $U \subset V$ *implies* $U_n \subset V_n$ *for each* n.

Some important facts are:

(a) Stratifiable spaces are hereditarily paracompact.

(b) The trace of a stratification on a subspace is again a stratification.

(c) Any countable product of stratifiable spaces is stratifiable.

(d) Every CW-complex is stratifiable.

(e) CW-complexes are absolute neighborhood extensors for stratifiable spaces.

Assume for the sequel that every class $\mathcal{C}$ of spaces under study has the property that polyhedra are absolute neighborhood extensors for all its elements. In a series of preceeding papers the class of all CW-complexes was considered instead of the class of all simplicial complexes. In our approach we find simplicial complexes more convenient. Since every CW-complex is homotopy equivalent to a polyhedron we may use simplicial complexes instead of CW-complexes.

Homotopy types and extension types of polyhedra (or CW-complexes) are related. Namely, if simplicial complexes are homotopy equivalent, i.e., the associated polyhedra are homotopy equivalent, then they have the same extension type. The converse is not true, as a simple example shows: S^n and $S^n \vee S^{n+1}$ have the same extension type with respect to any class of paracompacta, but have different homotopy types.

In order to decide whether (*) has a minimal element for a considered class of spaces $\mathcal{C}$ and the class of all simplicial complexes one may notice that the wedge, or one point union, of simplicial complexes is a helpful construction since if $K \vee K' \in AE(X)$, then obviously $K \in AE(X)$ and $K' \in AE(X)$, or $K \vee K' \leq K$ and $K \vee K' \leq K'$. But finding a minimal element in (*) causes us to require the opposite implication: if $K \in AE(X)$ and $K' \in AE(X)$ then $K \vee K' \in AE(X)$, and not only for two summands but for a collection, in other words we need a so-called wedge theorem.

For which class of spaces will a wedge theorem apply? We define X to be a *dd-space* if X has the property that $|K| \in AE(X)$ for every contractible simplicial complex K, i.e., a simplicial complex whose polyhedron $|K|$ is contractible. One may easily detect that the following classes of spaces are dd-spaces.

2. Lemma. *If X is either compact Hausdorff or stratifiable, then X is a dd-space.*

Let us state the following wedge theorem needed later.

3. Theorem. *Let X be a dd-space and $\{K_\alpha | \alpha \in \Gamma\}$ be a collection of simplicial complexes. Put $K = \bigvee_v K_\alpha$, where say v is a vertex common to each K_α. Suppose that for each $\alpha \in \Gamma$, $|K_\alpha| \in \mathrm{AE}(X)$. Then $|K| \in \mathrm{AE}(X)$. Conversely, for any space X, if $|K| \in \mathrm{AE}(X)$, then $|K_\alpha| \in \mathrm{AE}(X)$ for all $\alpha \in \Gamma$.*

As a consequence of this, the wedge theorem holds for stratifiable or compact Hausdorff spaces.

If $f : X \to |K|$ is a map, where K is a simplicial complex, then there is a minimal subcomplex $L \subseteq K$ such that $f(X) \subseteq |L|$. What is the cardinal number $\operatorname{card} L$ of the set of vertices of L? It turns out that $\operatorname{card} L$ is bounded above by the weight of the space X, $\operatorname{card} L \leq \operatorname{wt} X$. But if we wish to extend a map from a closed subset $A \subset X$ to X, then we need to engulf L by a subcomplex of K, say F, such that $|F| \in AE(X)$, i.e., L is engulfed by an absolute extensor F for X. Related to this is the question, when is $|K| \in AE(X)$ if K is a union of subcomplexes which are absolute extensors for X? These questions led us to introduce two notions related to infinite cardinals. Here are their definitions, where I denotes the unit segment.

4. Definition. *Let α be an infinite cardinal and α_1 the first cardinal with $\alpha_1 > \alpha$. We define an increasing well-ordered collection $\{\tau_\sigma | \sigma < \alpha_1\}$ of cardinals as follows. Put $\tau_0 = \operatorname{card} Y_0$ where Y_0 is the collection of all subsets of $I^\alpha \times I^\alpha$. Note,*

(1) $\alpha \leq \tau_0$.

Now suppose $0 < \beta < \alpha_1$ and we have defined τ_σ for all $\sigma < \beta$.

(2) *If β is a limit ordinal, then let $\tau_\beta = \sup\{\tau_\sigma | \sigma < \beta\}$.*

(3) *If $\beta = \sigma + 1$, then put $\tau_\beta = \operatorname{card} Y_{\sigma+1}$, where $Y_{\sigma+1}$ is the collection of all subsets of $I^\alpha \times I^{\tau_\sigma}$.*

Finally we define

$$\operatorname{excd}(\alpha) = \sup\{\tau_\sigma | \sigma < \alpha_1\}$$

and call it the **extension cardinal** *of α.*

5. Definition. *Let α be an infinite cardinal. Suppose that X is a space having the property that for every simplicial complex K with $|K| \in \mathrm{AE}(X)$, there exists a collection $\mathcal{F}$ of subcomplexes of K so that:*

(a) *For each subcomplex L of K with $\operatorname{card} L \leq \alpha$, there exists $F \in \mathcal{F}$ with $L \subset F$;*

(b) $|F| \in \mathrm{AE}(X)$ *for all $F \in \mathcal{F}$;*

(c) $\operatorname{card} F \leq \operatorname{excd}(\alpha)$ *for all $F \in \mathcal{F}$, and*

(d) $K = \cup \mathcal{F}$.

Then we shall say that X is an ext$_\alpha$-**space**.

One may ask the reason for the powers of the unit segment in the definition of the extension cardinal. Let us recall that the Tychonoff cube I^α is universal for the class of

Tychonoff spaces of weight $\leq \alpha$ and that $|L|$ embeds into $I^{\operatorname{card} L}$. Having these notions formulated one can use a transfinite induction argument to prove that there is a wide class of ext_α-spaces, namely the following theorem holds:

6. Theorem. *Let α be an infinite cardinal and X be a space such that*

(a) $\operatorname{wt} X \leq \alpha$, *and*

(b) X *is Tychonoff.*

Then X is an ext_α-space.

Narrowing slightly the class of spaces in the previous theorem to the class of stratifiable spaces, we have the following statement which gives a sufficient condition for a simplicial complex to be an $AE(X)$ if it is the union of subcomplexes each of which is an $AE(X)$.

7. Lemma. *Let α be an infinite cardinal, X be a stratifiable space with $\operatorname{wt}(X) \leq \alpha$, K be a simplicial complex, and $\mathcal{F}$ be a collection of subcomplexes of K such that*

(a) *for every subcomplex L of K with $\operatorname{card} L \leq \alpha$, there exists $F \in \mathcal{F}$ such that $L \subset F$;*

(b) $|F| \in \mathrm{AE}(X)$ *for all $F \in \mathcal{F}$, and*

(c) $K = \bigcup \mathcal{F}$.

Then $|K| \in \mathrm{AE}(X)$.

8. Theorem. *Let α be an infinite cardinal. Put $\mathcal{S}$ equal the class of all simplicial complexes and $\mathcal{C}_\alpha$ the class of stratifiable spaces of weight $\leq \alpha$. Then for each $X \in \mathcal{C}_\alpha$, the class*

$$\{[L] \in \mathrm{ET}(\mathcal{C}_\alpha, \mathcal{S}) \mid \dim X \leq L\}$$

has a minimal element.

Proof. Let $\beta = \operatorname{excd}(\alpha)$. Choose a set $\mathcal{S}_0 \subset \mathcal{S}$ such that for every $L \in \mathcal{S}$ with $\operatorname{card} L \leq \beta$, there is an $L' \in \mathcal{S}_0$ with L' isomorphic to L. We shall show that the minimum is represented by

$$K = \bigvee \{L \mid \dim X \leq L, L \in \mathcal{S}_0\}.$$

Since X is a dd-space, Theorem 3 shows that $|K| \in \mathrm{AE}(X)$, or $\dim X \leq K$. We have to show that for all L in $\mathcal{S}$ if $\dim X \leq L$, then for $Y \in \mathcal{C}_\alpha$, $\dim Y \leq K$ implies $\dim Y \leq L$.

Now by Theorem 6, X is an ext_α-space so by Definition 5, there is a collection $\mathcal{F}$ of subcomplexes of L, so that $L = \bigcup \mathcal{F}$, and for each $F \in \mathcal{F}$, $|F| \in \mathrm{AE}(X)$, and $\operatorname{card} F \leq \beta$. Then by the definition of K, every $F \in \mathcal{F}$ is, up to isomorphism, a summand of K. Thus by Theorem 3, we have $|F| \in \mathrm{AE}(Y)$ for all $F \in \mathcal{F}$. By Lemma 7 applied to Y, one has that $|L| \in \mathrm{AE}(Y)$, or $\dim Y \leq L$. $\square$

References

[Ce] J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math **11** (1961), 105–125.

[DD] A. Dranishnikov and J. Dydak, *Extension dimension and extension types*, Trudy Math. Steklov Inst. **212** (1996), 61–94.

[Dr] A. N. Dranishnikov, *Eilenberg-Borsuk theorem for maps into arbitrary complexes*, Math. Sbornik **185** (1994), 81–90.

[IR] I. Ivanšić and L. R. Rubin, *Extension dimension of metrizable spaces*, Top. Appl. (submitted).

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$\mathbb{Z}/p^\infty$-ACYCLIC RESOLUTIONS FOR METRIZABLE COMPACTA

LEONARD R. RUBIN AND PHILIP J. SCHAPIRO

ABSTRACT. We shall prove a G-acyclic resolution theorem for $\dim_G$, cohomological dimension modulo the group $G = \mathbb{Z}/p^\infty$, in the class of metrizable compacta. This means that, given a metrizable compactum X such that $\dim_{\mathbb{Z}/p^\infty} X \leq n$ ($n \geq 2$), there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:

(a) π is $\mathbb{Z}/p^\infty$-acyclic,
(b) $\dim Z \leq n+1$, and
(c) $\dim_{\mathbb{Z}/p^\infty} Z \leq n$.

To say that a map π is G-acyclic, for an abelian group G, means that each fiber $\pi^{-1}(x)$ of π is G-acyclic, i.e., that all the reduced Čech cohomology groups of $\pi^{-1}(x)$ modulo the group G are trivial.

The Edwards-Walsh resolution theorem, the first resolution theorem for cohomological dimension, was proved in [Wa] (see also [Ed]). It states that if X is a metrizable compactum and $\dim_{\mathbb{Z}} X \leq n$ ($n \geq 0$), then there exists a metrizable compactum Z with $\dim Z \leq n$ and a surjective cell-like map $\pi : Z \to X$. This result, in conjunction with Dranishnikov's work ([Dr1]) showing that in the class of metrizable compacta, $\dim_{\mathbb{Z}}$ is distinct from dim, was a key ingredient for proving that cell-like maps could raise dimension (see [Ru1] for background). For the reader seeking fundamentals on the theory of cohomological dimension, $\dim_G$, the references [Ku], [Dr3], [Dy], and [Sh] could be helpful.

Now a map is cell-like provided that each of its fibers is cell-like, or, equivalently, has the shape of a point ([MS1]). Every cell-like compactum has trivial reduced Čech cohomology with respect to any abelian group G. This means that for every abelian group G, every cell-like map is G-acyclic, i.e., all its fibers have trivial reduced Čech cohomology with respect to the group G. Moreover, when a Hausdorff compactum or metrizable space X has $\dim X \leq n$, then also $\dim_{\mathbb{Z}} X \leq n$.

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Key words and phrases. G-acyclic resolution, dimension, cohomological dimension, cell-like map, shape of a point, inverse sequence, Edwards-Walsh resolution, simplicial complex, CW-complex, Moore space, Čech cohomology.

With these ideas in mind, one may ask, what kind of parallel resolution theorems can be obtained under the assumption that $\dim_G X \leq n$, where G is an abelian group different from $\mathbb{Z}$? It turns out that it is not possible always to have cell-like resolutions as in the Edwards-Walsh theorem, nor can one even require in such propositions that $\dim Z \leq n$ be true (see [KY2]). So, what kind of resolution theorems can we expect? The main result of this paper exemplifies the situation. Let $\mathbb{P}$ denote the set of prime numbers.

1.1. Theorem. *For each $p \in \mathbb{P}$, $n \geq 2$, and metrizable compactum X with $\dim_{\mathbb{Z}/p^\infty} X \leq n$, there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is $\mathbb{Z}/p^\infty$-acyclic,*

(b) *$\dim Z \leq n+1$, and*

(c) *$\dim_{\mathbb{Z}/p^\infty} Z \leq n$.*

This and the Edwards-Walsh theorem are special cases of the following conjecture:

1.2. Conjecture. *Let G be an abelian group and X be a metrizable compactum with $\dim_G X \leq n$ $(n \geq 2)$. Then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*

(b) *$\dim Z \leq n+1$, and*

(c) *$\dim_G Z \leq n$.*

Let us mention that the Edwards-Walsh theorem has been generalized to the class of arbitrary metrizable spaces by Rubin and Schapiro ([RS1]) and to the class of arbitrary compact Hausdorff spaces by Mardešić and Rubin ([MR]). Conjecture 1.2 was proved by Dranishnikov ([Dr2]) for the group $\mathbb{Z}/p$, where p is an arbitrary prime number, but with the stronger outcome that $\dim Z \leq n$. Later, Koyama and Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for that of compact Hausdorff spaces.

In their work [KY2], Koyama and Yokoi have made a substantial amount of progress in the resolution theory of metrizable compacta, that is, towards proving Conjecture 1.2. Their method relies heavily on the existence of Edwards-Walsh resolutions, which had been studied by Dydak and Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. The definition of an Edwards-Walsh resolution can be found in [KY2], but we shall not use it herein.

To overcome a flaw in the proof of Lemma 4.4 of [DW], Koyama and Yokoi proved the existence of Edwards-Walsh resolutions for some groups G, but under a stronger set of assumptions on G than had been thought necessary in [DW]. It is still not known if these stronger assumptions are needed to insure the existence of the resolutions. Nevertheless, Koyama and Yokoi were able to prove substantial G-acyclic resolution theorems. Let us state two of the important theorems from [KY2] (Theorems 4.9 and 4.12, respectively), which greatly influenced the direction of the work in this paper.

1.3. Theorem. *Conjecture 1.2 is true for every torsion free abelian group G.*

1.4. Theorem. *Let G be an arbitrary abelian group and X be a metrizable compactum with $\dim_G X \leq n$, $n \geq 2$. Then there exists a surjective G-acyclic map $\pi : Z \to X$ from a metrizable compactum Z where $\dim Z \leq n+2$ and $\dim_G Z \leq n+1$.*

In case G is a torsion group, they prove (Theorem 4.11) that conjecture 1.2 holds, but without part (c). Of course Theorem 1.4 falls short of providing a positive solution of Conjecture 1.2. We observed that one of the main reasons for the relative weakness of this theorem was that Koyama and Yokoi proved it by an indirect technique, a type of "finesse." Their approach depends heavily on the Bockstein basis theorem and the Bockstein inequalities (see [Ku]), instead of the more direct method, involving Edwards-Walsh resolutions, used to prove Theorem 1.3.

We want to point out that Theorem 1.3 includes as a corollary, and therefore redeems, the $\mathbb{Q}$-resolution theorem of Dranishnikov ([Dr4]). The Koyama and Yokoi proof shows that in the proof of Theorem 3.2 of [Dr4], the statement that $\alpha_m \circ \omega_m$ is an Edwards-Walsh resolution over $\tau_m^{(n+1)}$ is not true. This was a subtle point; to fully understand it, the interested reader may examine the text immediately following the proof of Fact 1 of the proof of Theorem 3.1 in [KY2]. Getting around the barrier naturally led to a quite complicated construction.

Our proof of Theorem 1.1 will be direct, using extensions which are different from

Edwards-Walsh resolutions. We shall employ the technique of inverse sequences both to represent our given space X and to determine the resolving space Z. The map $\pi : Z \to X$ will be obtained in a standard, yet complicated manner similar to that used in [Wa]. The full text can be found in the preprint [RS2].

References

[Dr1] A. Dranishnikov, *On P. S. Aleksandrov's problem*, Mat. Sb. **135 (4)** (1988), 551–557.

[Dr2] A. Dranishnikov, *On homological dimension modulo p*, Math. USSR Sbornik **60, no. 2** (1988), 413–425.

[Dr3] A. Dranishnikov, *Homological dimension theory*, Russian Math. Surveys **43(4)** (1988), 11–63.

[Dr4] A. Dranishnikov, *Rational homology manifolds and rational resolutions*, Topology and its Appls. **94** (1999), 75–86.

[Dy] J. Dydak, *Cohomological dimension theory*, preprint.

[DW] J. Dydak and J. Walsh, *Complexes that arise in cohomological dimension theory: a unified approach*, J. London Math. Soc. **(2) (48), no. 2** (1993), 329–347.

[Ed] R. D. Edwards, *A theorem and question related to cohomological dimension and cell-like maps*, Notices of the Amer. Math. Soc. **25** (1978), A-259.

[KY1] A. Koyama and K. Yokoi, *A unified approach of characterizations and resolutions for cohomological dimension modulo p*, Tsukuba J. Math. **18(2)** (1994), 247–282.

[KY2] A. Koyama and K. Yokoi, *Cohomological dimension and acyclic resolutions*, Topology and its Appls., to appear.

[Ku] W. I. Kuz'minov, *Homological Dimension Theory*, Russian Math. Surveys **23** (1968), 1–45.

[MR] S. Mardešić and L. Rubin, *Cell-like mappings and nonmetrizable compacta of finite cohomological dimension*, Trans. Amer. Math. Soc. **313** (1989), 53–79.

[MS1] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[MS2] S. Mardešić and J. Segal, *Stability of almost commutative inverse systems of compacta*, Topology and its Appls. **31** (1989), 285–299.

[Ru1] L. Rubin, *Cell-like maps, dimension and cohomological dimension: a survey*, Banach Center Publications **18** (1986), 371–376.

[RS1] L. Rubin and P. Schapiro, *Cell-like maps onto non-compact spaces of finite cohomological dimension*, Topology and its Appls. **27** (1987), 221–244.

[RS2] L. Rubin and P. Schapiro, *$\mathbb{Z}/p^\infty$-acyclic resolutions for metrizable compacta*, preprint.

[Sh] E. Shchepin, *Arithmetic of dimension theory*, Russian Math. Surv. **53** (1998), 975–1069.

[Wa] J. Walsh, *Shape Theory and Geometric Topology*, Lecture Notes in Mathematics, volume 870, Springer Verlag, Berlin, 1981, pp. 105–118.

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Neighborhoods of crumpled manifold boundaries

by F. C. Tinsley

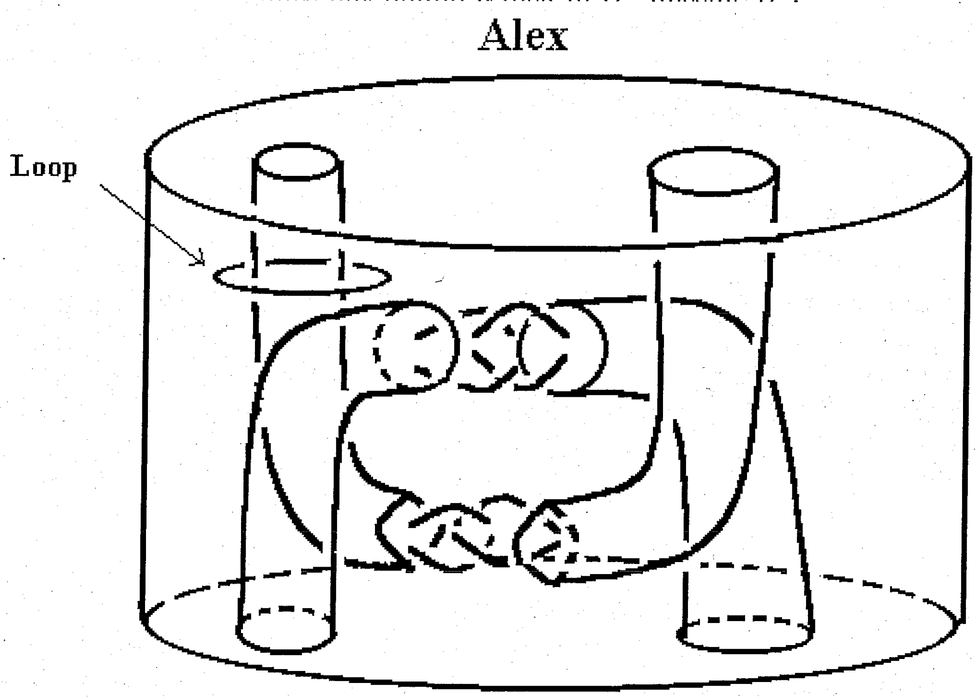
I. Setting: Let S^n denote the n-sphere and let $h : S^{n-1} \to S^n$ be an embedding. Identify S^{n-1} with $h(S^{n-1})$. In other words, assume S^{n-1} sits in S^n. Then, S^{n-1} separates S^n into two components. Denote the closure of one of these components by C^n. We call C^n a *crumpled cube*. Observe that $bdy(C^n) = S^{n-1}$ naturally.

EXAMPLE 1: Think of S^{n-1} in $\mathbb{R}^n$ as the set of points a distance one from the origin and S^n as the one-point compactification of $\mathbb{R}^n$. Then the crumpled cube containing the origin is, in fact, a cell.

DEFINITION 1: S^{n-1} is *tame in* C^n if C^n is a cell. Otherwise, S^{n-1} is *wild in* C^n.

We also may refer to a cell as a *trivial* crumpled cube.

EXAMPLE 2: The most famous example of a wild sphere is Alexander's Horned Sphere ($n = 3$) shown below in $\mathbf{R}^3$. S^2 is wild in its bounded component, C^3. Specifically, its interior, $C^3 \backslash S^2$, is not simply connected and so cannot be the interior of a cell. The loop labeled by an arrow does not bound a disk in C^3 missing S^2.



II. Some History: The study of wild embeddings of spheres has a rich history extending back more than half a century. In particular, myriads of necessary and sufficient conditions for a crumpled cube to be a cell have been developed. Morton Brown developed the following characterization that is valid in all dimensions:

PROPOSITION 1: A crumpled cube, C^n, is a cell if and only if S^{n-1} is *collared* in C^n, ie, if and only if there is an embedding $c : S^{n-1} \times [0,1] \to C^n$ with $c_0|S^{n-1}$ the identity.

A second, highly useful characterization was developed. This result is due to R. H. Bing in dimension 3, Frank Quinn in dimensions 4 and 5, and Bob Daverman in dimensions 6 and higher.

PROPOSITION 2: A crumpled cube, C^n, is a cell if and only if S^{n-1} has *1-LCC complement* in C^n, ie, given any $x \in S^{n-1}$ and $\epsilon > 0$ there is a $\delta > 0$ so that loops in $N_\delta(x, C^n) \backslash S^{n-1}$ bound 2-disks in $N_\epsilon(x, C^n) \backslash S^{n-1}$.

In short, C^n is a cell if and only if small loops near S^{n-1} and missing S^{n-1} bound small disks near S^{n-1} and missing S^{n-1}.

One obviously necessary condition for tameness still remains a candidate for also being a sufficient condition. S^{n-1} is *free* in C^n if for each $\epsilon > 0$ there is a map $f_\epsilon : S^{n-1} \to C^n \backslash S^{n-1}$ with $d(x, f(x)) < \epsilon$ for all $x \in S^{n-1}$.

FREE SPHERE QUESTION: Suppose S^{n-1} is free in C^n. Is C^n a cell?

III. Strategy of Investigation:

For $n = 3$, the free surface question is an extremely difficult and well-known unsolved problem. Also, Bob Daverman has developed methods for "inflating" examples of crumpled cubes from dimension n to dimension $n+1$. These facts suggest that the answer to the question is YES for $n = 3$ and NO for $n > 3$. From 1985-1995, Daverman and I constructed many new, intrinsically high-dimensional examples of non-trivial crumpled cubes. This research focuses on whether our new knowledge has anything to say about the free surface question in high dimensions.

We begin with what may be an easier question. For $n = 3$ it is well known that if S^2 is free in C^3, then $C^3 \backslash S^2$ is homeomorphic to an open 3-cell. The proof relies on the Sphere Theorem, an intrinsically 3-dimensional result.

POSSIBLY EASIER QUESTION: Suppose $n > 3$ and S^{n-1} is free in C^n, then is $C^n \backslash S^{n-1}$ homeomorphic to an open (n-1)-cell?

IV. Some Progress:

We focus for a moment on a small loop, s, in $C^n \backslash S^{n-1}$ near S^{n-1} and categorize what s may bound in order of increasing nastiness.

1. s bounds a *small disk* in $C^n \backslash S^{n-1}$ near S^{n-1}.
2. s bounds a *large disk* in $C^n \backslash S^{n-1}$ near S^{n-1}.
3. s bounds a *half-open annulus* properly embedded in $C^n \backslash S^{n-1}$ near S^{n-1}.
4. s bounds a *small disk with a Cantor set's worth of holes* properly embedded in $C^n \backslash S^{n-1}$ near S^{n-1}.

Comments:

1. This is the 1-LCC complement condition referred to above. So, S^{n-1} is tame in C^n.

2. This condition implies that $C^n \backslash S^{n-1}$ is homeomorphic to the interior of an n-disk. It often is described by saying $C^n \backslash S^{n-1}$ *is 1-LC at infinity.*

3. Alternatively, we may say that s can be "tubed to infinity". In our setting, this is equivalent to $E^n \backslash S^{n-1}$ being *outward tame*, ie, closed subsets of $E^n \backslash S^{n-1}$ near S^{n-1} can be homotoped in $E^n \backslash S^{n-1}$ arbirarily close to S^{n-1}. (See the article by Craig Guilbault in these proceedings.)

4. In general, this is the most we can hope for. However, a bit more is true. Any loop, s, can be made to bound a 2-disk in C^n so that the preimage in this disk of its intersection with S^{n-1} is a compact 0-dimensional set.

We show that crumpled cubes that belong to our category 3 and have S^{n-1} free in C^n also have Euclidean interior.

PROPOSITION 3: Suppose S^{n-1} is free in C^n and given any neighborhood U of S^{n-1} in C^n there is a neighborhood V of S^{n-1} in C^n such that any loop s in V can be tubed to infinity in U. Then $C^n \backslash S^{n-1} \cong \mathbb{R}^n$.

Proof: We need to show that loops close to S^{n-1} in $C^n \backslash S^{n-1}$ bound disks close to S^{n-1}

in $C^n \backslash S^{n-1}$ (the 1-LC at ∞ condition).

To this end, let U be an arbitrary neighborhood of S^{n-1} in C^n. Let V be the neighborhood S^{n-1} in C^n satisfying the hypthesis of this theorem. Let s be an arbitrary loop in $V \backslash S^{n-1}$. Then, s bounds a half-open annulus, $A \cong S^1 \times [0, \infty)$, entirely contained in U and properly embedded in $C^n \backslash S^{n-1}$.

Let $\epsilon > 0$ be a small positive number and, by freeness, let $f : S^{n-1} \longrightarrow C^n \backslash S^{n-1}$ be an ϵ-map. If ϵ is small enough, then $f(S^{n-1}) \subset V$, $f(S^{n-1})$ will separate s from S^{n-1}, and f will be a degree one map. Assume that f is in general position with respect to itself and A. Since A is embedded, $f^{-1}(A)$ is a finite union of simple closed curves in S^{n-1}, say $s_1, s_2, \cdots, s_m$. Finally, for at least one j, $f|s_j : S^1 \longrightarrow A$ is essential at the π_1 level since f is of degree one (we the details of this argument to the reader).

Let $p : \tilde{U} \longrightarrow U$ be the universal covering space of U. Since S^{n-1} is simply connected, the map f lifts to a map $\tilde{f} : S^{n-1} \longrightarrow \tilde{U}$. Consider a component, $\tilde{A}$, of $p^{-1}(A)$. Now, $\tilde{A}$ must be either a half-plane or again an annulus. But, $\tilde{A}$ cannot be a half-plane because, on the one hand, $f|s_j$ is essential and would lift to a line but, on the other hand, $f|s_j$ must lift to a loop since $s_j \subset S^{n-1}$ and f lifts. Thus, $\tilde{A}$ is a half-open annulus.

Then $f|\tilde{A} : \tilde{A} \longrightarrow A$ is a k to 1 map for some positive integer k. We argue that $k = 1$. Let $\alpha = * \times [0, \infty)$ be an arc in A running from $* \in s$ out to S^{n-1}. Without loss of generality, assume f and α are in general position so that their intersection is a finite number of points. Using orientations on S^{n-1}, C^n, and α, we may assign a $+1$ or a -1 to each point of intersection. Since f is of degree one, the sum of these must be $+1$ or -1. However, since $f|s_i$ lifts to $\tilde{A}$ for each i, $1 \leq i \leq m$, the sum must be congruent to 0 mod k. The only possibility is for $k = 1$.

V. Closing:

What makes our category 3 tractible is that the annulus, A, is a has an abelian fundamental group ($\mathbb{Z}$). As a result, study of the covering spaces of A is straightforward. The situation in general is considerably more complicated. We are led to understanding the intersections between f and disks with more than one hole. These objects have free fundamental groups with more than one generator and, thus, have a plethora of covering spaces.

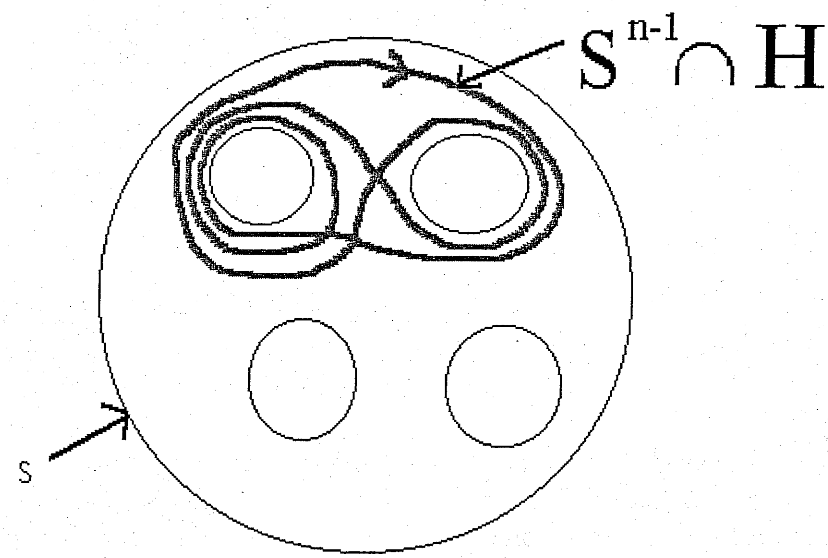
The possible advantage to this complexity is that it may make aid in finding a counterexample. We close with a specific intersection pattern that would allow us possibly to use the our constructions referred to above. We abbreviate the commutator of two group elements, a and b, by $[a, b]$, ie, $a^{-1}b^{-1}ab = [a, b]$.

FINAL QUESTION: Let G be Higman's group presented with four generators and four relators as

$$\langle a_i | a_i = [a_i, a_{i+1}], a_5 = a_1 \ 1 \leq i \leq 4 \rangle$$

Is there a non-trivial crumpled cube C^n with S^{n-1} free in C^n, a loop s bounding a disk with four holes H, and a map $f : S^{n-1} \to C^n \backslash S^{n-1}$ whose intersection pattern with H yields Higman's group?

We illustrate H with only the first of the four relators, $a_1 = [a_1, a_2]$. There would be three similar curves relating the other consecutive pairs of holes:



Genus of a Cantor set

Matjaž Željko *

August 3, 2003

Abstract

We define a genus of a Cantor set as a minimal number of the maximal number of handles over all possible defining sequences for it. The relationship between the local and the global genus is studied for genus 0 and 1. The criterion for estimating local genus is proved along with the example of Cantor set having prescribed genus. It is shown that some condition similar to 1-ULC implies local genus equal to 0.

Keywords: Cantor set, defining sequence, genus, 1-ULC

AMS classification: 57M30

1 Introduction

We will consider Cantor sets embedded in 3-dimensional Euclidean space $\mathbb{E}^3$. A defining sequence for a Cantor set $X \subset \mathbb{E}^3$ is a sequence (M_i) of compact 3-manifolds M_i with boundary such that each M_i consists of disjoint cubes with handles, $M_{i+1} \subset \operatorname{Int} M_i$ for each i and $X = \bigcap_i M_i$.

Armentrout [?] proved that every Cantor set has a defining sequence. In fact every Cantor set has many nonequivalent (see [?] for definition) defining sequences and in general there is no canonical way to choose one. One approach is to compress unnecessary handles in the given defining sequence for a Cantor set. A class for which this process terminates is characterized by some property similar to 1-ULC (see [?] for details). But in general this process is infinite so the "incompressible" defining sequence may not exist. Hence we look at the minimal number of the maximal number of handles over all possible defining sequences for it and take the defining sequence for which this number is minimal. Unfortunately this sequence need not to be canonical, but the minimal number (i.e. *the genus*) itself has some interesting properties.

Using different terminology Babich [?] actually proved that the genus of a wild scrawny (see [?] for definition) Cantor set is at least 2.

2 The genus

Let M be a cube with handles. We denote the number of handles of M by $g(M)$. For disjoint union of cubes with handles $M = \bigsqcup_{\lambda \in \Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda);\ \lambda \in \Lambda\}$.

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Let (M_i) be a defining sequence for a Cantor set $X \subset \mathbb{E}^3$. For any subset $A \subset X$ we denote by M_i^A an union of these components of M_i which intersect A. Define

$$\begin{aligned} g_A(X;(M_i)) &= \sup\{g(M_i^A);\ i \geq 0\} \quad \text{and} \\ g_A(X) &= \inf\{g_A(X;(M_i));\ (M_i) \in \mathcal{D}(X)\}. \end{aligned}$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A.* For $A = X$ we call the number $g_X(X)$ *the genus of the Cantor set X* and denote it simply by $g(X)$. For any point $x \in X$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

As a trivial consequence of the definition one can prove

Lemma 1 *Genus of a Cantor set is a monotone function. Precisely:*

1. *For $A \subset B \subset X$ where X is a Cantor set we have $g_A(X) \leq g_B(X)$.*
2. *For $A \subset X \subset Y$ where X is a closed subset of a Cantor set Y we have $g_A(X) \leq g_A(Y)$.*

By standard construction of Antoine necklace $\mathcal{A}$ we know $g(\mathcal{A}) \leq 1$. As Cantor set $\mathcal{A}$ is wild we conclude $g(\mathcal{A}) = 1$. So there exist a Cantor set of genus 1. We call such Cantor sets *toroidal.*

Using the result of A. Babich [?] one can prove that there exists a Cantor set of genus 2. We will extend the theorem [?, Theorem 2] to obtain a criterion for estimating the local genus and thus constructing a Cantor set of arbitrary genus.

3 Genus 0

By theorem of Bing [?] we know that the Cantor set $X \subset \mathbb{E}^3$ is tame if and only if $g(X) = 0$. By theorem of Osborne [?, Theorem 4] we know that the Cantor set $X \subset \mathbb{E}^3$ is tame if and only if $g_x(X) = 0$ for every point $x \in X$.

Theorem 2 *Let x be an arbitrary point of a Cantor set $X \subset \mathbb{E}^3$. If for every $\varepsilon > 0$ exists $\delta > 0$ such that for every mapping $f\colon S^1 \to \operatorname{Int} B(x,\delta) \setminus X$ exists such map $F\colon B^2 \to \operatorname{Int} B(x,\varepsilon) \setminus X$ that $F|_{S^1} = f$ then $g_x(X) = 0$.*

REMARK. The reader may note that the hypothesis of this theorem is not enough for the Cantor set X to be locally tame at x. However if the hypothesis of the theorem is satisfied for every $x \in X$ we obtain the well known 1-ULC taming theorem due to R. H. Bing [?].

4 The existence of a Cantor set of arbitrary genus

Let Γ be a tree having $r+1$ nodes. For $k \in \{2,3,\ldots,r\}$ we denote by $G(\Gamma,r,k)$ the number of nodes of Γ whose degree is at most k. We define

$$G(r,k) = \inf\{G(\Gamma);\ \Gamma \text{ is a tree with } r+1 \text{ nodes}\}.$$

One can estimate

$$\lceil r+1-\tfrac{1}{k}(r-1)\rceil \leq G(r,k) \leq r+1,$$

where $\lceil x \rceil$ denotes the least integer not less than given $x \in \mathbb{R}$ (for example $\lceil \pi \rceil = 4$).
REMARK. For $k = 2$ we have $G(r, 2) \geq \lceil \frac{r+3}{2} \rceil$ and for $k = r$ we have $G(r, r) \geq \lceil r + \frac{1}{r} \rceil = r + 1$.
Using the following criterion we can estimate the lower bound for local genus of a Cantor set.

Theorem 3 *Let $X \subset \mathbb{E}^3$ be a Cantor set and $x_0 \in X$ its arbitrary point. Let there exist a 3-disk B and 2-disks $D_1, \ldots, D_r$ such that*

1. *For every disk D_i we have $D_i \cap X = \operatorname{Int} D_i \cap X = \{x_0\}$.*
2. *For distinct pair of disks D_i in D_j we have $D_i \cap D_j = \{x_0\}$.*
3. *The point x_0 lies in the interior of B and $\operatorname{Fr} D_i \cap B = \emptyset$ for every disk D_i.*
4. *If there exist planar compact surface in $B \setminus X$ which boundary components lie in $(D_1 \cup \cdots \cup D_r) \cap \operatorname{Fr} B$ then this surface has at least $k + 1$ bundary components.*

Then $g_{x_0}(X) \geq G(r, k)$.

PROOF. We will prove that every cube with handles $N \subset \operatorname{Int} B$ such that $x_0 \in N$ and $\operatorname{Fr} N \cap X = \emptyset$, has at least $G(r, k)$ handles. We may assume that D_i intersects $\operatorname{Fr} N$ transversally (shortly $D_i \pitchfork \operatorname{Fr} N$) and that $\operatorname{Fr} N$ has minimal genus. We may also assume that among all cubes with $g(\operatorname{Fr} N)$ handles N minimizes the number of components of $\operatorname{Fr} N \cap (D_1 \cup \cdots \cup D_r)$.

Fix disk D_i. The intersection $D_i \cap \operatorname{Fr} N$ has at least one component and each of them bounds a disk in $\operatorname{Int} D_i$. If some of such disks in $\operatorname{Int} D_i$ does not contain x_0 we pick the innermost one and denote it by E. (Disk E need not to be unique.) The loop $\operatorname{Fr} E$ bounds a disk $E^* \subset \operatorname{Fr} N$ as otherwise N could be compressed along E and hence $g(\operatorname{Fr} N)$ would decrease. So we can replace E by E^* in order to decrease the number of components in $\operatorname{Fr} N \cap D_i$.

Therefore the components of $D_i \cap \operatorname{Fr} N$ are nested and each of them bounds a disk containing x_0. The number of components is odd as $x_0 \in D_i \cap N$ and $\operatorname{Fr} D_i \cap N = \emptyset$. If $D_i \cap \operatorname{Fr} N$ has at least three components there exist consecutive two of them which bound an annulus $A \subset D_i$ such that $A \cap \operatorname{Fr} N = \operatorname{Fr} A$ and $A \subset N$. Now we cut N along A to obtain the manifold N^* which has at most two components. As $\chi(A) = 0$ we have $\chi(\operatorname{Fr} N) = \chi(\operatorname{Fr} N^*)$. If N^* has two components we dispose that one which does not contain x_0. Therefore $g(\operatorname{Fr} N^*) \leq g(\operatorname{Fr} N)$ and the number of components of $\operatorname{Fr} N^* \cap D_i$ is less than the number of components of $\operatorname{Fr} N \cap D_i$. We repeat the procedure until there is only one component of $\operatorname{Fr} N \cap D_i$ left. The remaining component (say η_i) separates $\operatorname{Fr} N$ as D_i separates N.

So there are exactly $r + 1$ components of $\operatorname{Fr} N \setminus (\eta_1 \cup \cdots \cup \eta_r)$. Let us denote their closures by $K_1, \ldots, K_{r+1}$. For every i the compact surface K_i is either nonplanar having at least one boundary component or planar having $k+1$ boundary components. The surface K_i cannot be a disk with less than k holes as otherwise one can attach onto it appropriate annuli in D_i bound by η_i and $\operatorname{Fr} B \cap D_i$ to obtain a planar surface in $B \setminus X$ having at most k boundary components (and all of them are contained in $(D_1 \cup \cdots \cup D_r) \cap \operatorname{Fr} B$).

Finally we construct a graph Γ related to the components of $\operatorname{Fr} N \setminus (\eta_1 \cup \cdots \cup \eta_r)$. The nodes of Γ shall be $\{K_1, \ldots, K_{r+1}\}$. The nodes K_i and K_j are connected in Γ if and only if $K_i \cap K_j \neq \emptyset$. The graph Γ is a tree as each of $\eta_1, \ldots, \eta_r$ separates $\operatorname{Fr} N$. The tree Γ has at least $G(r,k)$ nodes of degree at most k so there are at least $G(r,k)$ nonplanar components in $\{K_1, \ldots, K_{r+1}\}$. Hence $g(\operatorname{Fr} N) \geq G(r,k)$. ■

REMARK. It is easier to check the last condition in the statement of the theorem when k is small but we get the most out of this criterion for $k = r$ as we have $G(r,r) = r+1$.

Theorem 4 *For every number $r \in \mathbb{N} \cup \{0, \infty\}$ there exist a Cantor set $X \subset \mathbb{E}^3$ such that $g(X) = r$.*

PROOF. For the sake of simplicity we replace $\mathbb{E}^3$ by S^3. We know that every tame Cantor set has genus 0 and for example the Antoinone necklace has genus 1. Therefore we may assume $2 \leq r < \infty$.

Fix arbitrary point $x_0 \in S^3$. We will construct a defining sequence (M_i) for the Cantor set X. Let M_1 be a cube with r handles containing x_0 in its interior. The manifold M_2 shall have $5r+1$ components. One of them (denoted by M_2^0) is a cube with r handles containing x_0 in its interior. We link each handle of M_2^0 by a chain of five tori and this chain is spread along the core of some of the handles in M_1. Now we construct the manifold M_3. The components of M_3 which lie in toroidal components of M_2 for a chain of linked tori (use the Antoine construction) and there are $5r+1$ components of M_3 in M_2^0 embedded in the same way as M_2 is embedded in M_1. Repeat the procedure inductively.

(See figure ?? for details. There are only two "legs" of X drawn in the figure, the remaining $r-2$ ones are supposed to be in the dotted part in the middle.)

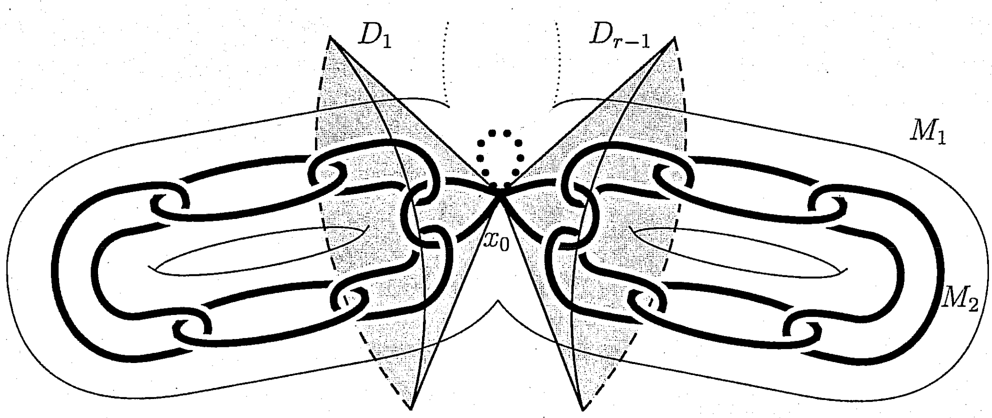

Figure 1: Defining sequence for a Cantor set of genus r, $r \geq 2$

By construction it is clear that $g(X) \leq r$. Using the $r-1$ disks $D_1, \ldots, D_{r-1}$ and the criterion ?? we will prove that $g_{x_0}(X) \geq r$.

We have to prove that there does not exist a planar surface $F \subset \operatorname{Int} B \setminus X$ which has r boundary components $\gamma_1, \ldots, \gamma_r$ such that $\gamma_i \subset D_i$ and γ_i is parallel to $\operatorname{Fr} D_i$ in D_i. Assume to the contrary: let such F exist.

Simple connected curves γ_i bounds disks $E_i \subset \operatorname{Int} D_i$ and $x_0 \in \operatorname{Int} E_i$ for every i. By attaching disks E_i to the surface F we obtain a singular sphere Σ. As there are $r+1$ "legs" of Cantor set joining in x_0 but only r "peaks" in Σ there exist a point $a \in X$ close to x_0 such that $\mathrm{lk}_{\mathbb{Z}_2}(\Sigma, a) = 1$ (i.e. singular sphere Σ winds around a). Let A be the "leg" of X which contains a. Therefore A is a Cantor set obviously homeomorphic to the Antoine necklace. The singular sphere Σ can be modified near x_0 so that it lies in $S^3 \setminus A$. (One has just to space out the peaks of Σ near x_0.) Let $f\colon S^2 \to \Sigma$ be a continuos map representing Σ. Let

$$h\colon \pi_2(S^3 \setminus A) \to H_2(S^3 \setminus A; \mathbb{Z})$$

be a Hurewicz homomorphism and

$$m\colon H_2(S^3 \setminus A; \mathbb{Z}) \to H_2(S^3 \setminus A; \mathbb{Z}_2)$$

be a map induced by homomorphism $\operatorname{mod} 2\colon \mathbb{Z} \to \mathbb{Z}_2$. Kernel of a map h is a subgroup of $\pi_2(S^3 \setminus A)$ which we denote be N. If $[f] \in N$ then also $mh([f]) = 0 \in H_2(S^3 \setminus A; \mathbb{Z}_2)$ but this contradicts $\mathrm{lk}_{\mathbb{Z}_2}(\Sigma, a) = 1$. Hence $[f] \notin N$. Using the sphere theorem we replace f by a nonsingular sphere $g\colon S^2 \to S^3 \setminus X$. As $[g] \neq 0 \in \pi_2(S^3 \setminus X)$ the sphere $g(S^2)$ winds around at least one point of A, but not around all of them. Therefore some two points of A can be separated by sphere in $S^3 \setminus A$. But it is well known that this is impossible. Hence by theorem ?? we have $g_{x_0}(X) \geq r$ and therefore $g(X) = r$.

Finally we prove the case $r = \infty$. Let X_r be a Cantor set of genus $r \in \mathbb{N}$. One can take a disjoint union of X_r's converging to the point (say x_∞). Therefore $X = \bigsqcup_r X_r$ is a Cantor set and $g_{x_\infty}(X) = \infty = g(X)$. ■

Remark. The Cantor set in the previous theorem does not have simply connected complement (except for $r = 0$). It is interesting to note that using the same construction one can exhibit a Cantor of arbitrary genus with simply connected complement. We just have to replace the building block: instead of Antoine necklace we use Bing-Whitehead Cantor set as its complement is simply connected (see [?] for details). The proof itself is almost the same: for the final contradiction we refer to [?, Paragraph 5] as Bing-Whitehead Cantor set can be separated by spheres but not with arbitrarily small ones.

Let $X \subset \mathbb{E}^3$ be a Cantor set. From ?? we see that $g_x(X) \leq g(X)$ for every point $x \in X$. The author believes that the following conjecture may not be true in general:

Conjecture 1 *For every Cantor set X there exist a point $x \in X$ such that $g_x(X) = g(X)$.*

The conjecture may be restated as

Conjecture 2 *Let $g_x(X) \leq r$ for every point x of a Cantor set X. Then $g(X) \leq r$.*

For $r = 0$, however, this is true [?]. We will prove this conjecture for $r = 1$ under some additional technical hypothesis.

5 Local genus vs. global genus

Let $X \subset \mathbb{E}^3$ be a Cantor set. We say that the Cantor set X is *splittable* if there exist a 2-sphere S in the complement of X which separates some two points of X. For a splittable Cantor set we may define $\mu(X) = \inf\{\mathrm{diam}(S);\ S \in \mathcal{S}\}$ where $\mathcal{S}$ is a set of separating

2-spheres for X. If a Cantor set X is not splittable we set $\mu(X) = \infty$. The number $\mu(X)$ is called *the lower bound of splitability.*

The number $\mu(X)$ certainly depends on embedding $X \hookrightarrow \mathbb{E}^3$. One can prove that for equivalently embedded (see [?] for definition) Cantor sets X and X' we have

$$\begin{aligned} \mu(X) = 0 &\quad \text{if and only if} \quad \mu(X') = 0, \\ \mu(X) > 0 &\quad \text{if and only if} \quad \mu(X') > 0, \\ \mu(X) = \infty &\quad \text{if and only if} \quad \mu(X') = \infty. \end{aligned}$$

Obviously $\mu(X) = 0$ for a tame Cantor set X. One can easily construct a wild Cantor set X such that $\mu(X) = 0$. As the Antoine necklace $\mathcal{A}$ is not splittable we have $\mu(\mathcal{A}) = \infty$. Finally there exist a wild cantor set with positive lower bound of splitability (see [?, page. 361] for more details).

Lemma 5 *Let $\mu(X) > 0$ for a given Cantor set $X \subset \mathbb{E}^3$. Let M and N be two solid tori in $\mathbb{E}^3$ such that $\operatorname{Fr} M \pitchfork \operatorname{Fr} N$, $X \subset M \cup N \setminus (\operatorname{Fr} M \cup \operatorname{Fr} N)$ and $\operatorname{diam}(M \cup N) < \mu(X)$. Then for every $\eta > 0$ there exist (at most) two disjoint solid tori whose interiors cover X and each of them lies entirely in $\{x \in \mathbb{E}^3;\ \operatorname{dist}(x, M) < \eta\}$ or $\{x \in \mathbb{E}^3;\ \operatorname{dist}(x, N) < \eta\}$.*

Lemma 6 *Let $\mu(X) > 0$ for a given Cantor set $X \subset \mathbb{E}^3$. Then for every solid torus $T \subset \mathbb{E}^3$ and every 3-disk $B \subset \mathbb{E}^3$, such that $X \subset \operatorname{Int} T \setminus B$, $B \not\subset T$, $\operatorname{Fr} B \pitchfork \operatorname{Fr} T$ and $\operatorname{diam}(T \cup B) < \mu(X)$, there exists a solid torus $T' \subset T \setminus B$ which contains X in its interior.*

Now we can state the main theorem for Cantor sets having local genus equal to 1.

Theorem 7 *Let $\mu(X) > 0$ for a given Cantor set $X \subset \mathbb{E}^3$. If $g_x(X) = 1$ for every point $x \in X$ then $g(X) = 1$.*

PROOF. Denote $\mu(X)$ simply by μ and fix $\varepsilon > 0$. We will find a finite collection of disjoint small tori whose interiors cover X.

Using the assumption that $g_x(X) = 1$ for every point x of a compact set X there exist a finite collection $\mathcal{T} = \{T_i\}_{i=1}^m$ of tori such that $\operatorname{diam}(T_i) < \min\{\varepsilon, < \frac{1}{2}\mu\}$ and $\operatorname{Fr} T_i \cap X = \emptyset$ for every $i = 1, 2, \ldots, m$. We may also assume that boundaries of these tori intersect transversally.

We assign the number $c(\mathcal{T}) = \sum_{1 \le i < j \le m} c_{i,j}$ to the cover $\mathcal{T}$ where

$$c_{i,j} = \begin{cases} 0, & \text{if } \operatorname{Fr} T_i \cap \operatorname{Fr} T_j = \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

If $c_{i,j} = 0$ for every i and j the tori are disjoint and $\mathcal{T}$ is the collection we are looking for. Otherwise we define

$$\eta := \min\{\tfrac{\varepsilon}{2(m-1)}, \min\{\operatorname{dist}(T_i, T_j);\ T_i \cap T_j = \emptyset\}\}$$

and pick the least pair of indexes (i, j), $i < j$, such that $c_{i,j} = 1$. Using the lemma ?? for the pair of tori $M := T_i$ and $N := T_j$ with control η we replace the tori T_i in T_j with disjoint T_i' in T_j' to obtain a new cover $\mathcal{T}'$. The number η was chosen appropriately to assure that for every $k \neq i, j$ we have: $T_i' \cap T_k = \emptyset$ if $T_i \cap T_k = \emptyset$ and $T_j' \cap T_k = \emptyset$ if

$T_j \cap T_k = \emptyset$. Therefore $c(\mathcal{T}') < c(\mathcal{T})$ and we repeat the procedure with new cover $\mathcal{T}'$. The diameters of tori T_i' in T_j' has increased at most by $\frac{\varepsilon}{2(m-1)}$. The procedure must stop after at most $\frac{m(m-1)}{2}$ steps so the diameters of components increase at most to 2ε as every torus is involved in the procedure at most $m-1$ times. ■

As a trivial consequence of the preceding theorem we obtain

Corollary 8 *Let $X \subset \mathbb{E}^3$ be a nonsplittable Cantor set. If $g_x(X) = 1$ for every point $x \in X$ then $g(X) = 1$.*

We say that the Cantor set X is *locally nonsplittable*, if for every point $x \in X$ there exists a neighbourhood $U \subset \mathbb{E}^3$ of x such that $X \cap U$ is a nonsplittable Cantor set. Therefore

Corollary 9 *Every locally nonsplittable and locally toroidal Cantor set is toroidal.*

6 Genus of an union of Cantor sets

If the Cantor sets X and Y are disjoint we have $g(X \cup Y) = \max\{g(X), g(Y)\}$. A tame Cantor set behaves nicely with respect to the genus as we have

Theorem 10 *Let $X \subset \mathbb{E}^3$ be a tame Cantor set. Then $g(X \cup Y) = g(Y)$ for every Cantor set $Y \subset \mathbb{E}^3$.*

Theorem 11 *Let $X, Y \subset \mathbb{E}^3$ be Cantor sets. If $X \cap Y \subset \mathrm{T}(X) \cap \mathrm{T}(Y)$, then $g(X \cup Y) = \max\{g(X), g(Y)\}$.*

Theorem 12 *Let $X, Y \subset \mathbb{E}^3$ be nondisjoint Cantor sets and $a \in X \cap Y$ such point that there exist a 3-disk B and a 2-disk $D \subset B$ such that*

1. *$a \in \operatorname{Int} B$, $\operatorname{Fr} D = D \cap \operatorname{Fr} B$, $D \cap (X \cup Y) = \{a\}$ and*
2. *we have $X \cap B \subset B_X \cup \{a\}$ and $Y \cap B \subset B_Y \cup \{a\}$ where B_X and B_Y are the components of $B \setminus D$.*

Then $g_a(X \cup Y) = g_a(X) + g_a(Y)$.

REMARK. Using the preceding theorem one can alternatively prove the existence of the Cantor set of given genus.

Summarizing the above theorems one may conjecture:

Conjecture 3 *For arbitrary Cantor sets $X, Y \subset \mathbb{E}^3$ we have*

$$\max\{g(X), g(Y)\} \leq g(X \cup Y) \leq g(X) + g(Y). \tag{1}$$

Using (??) we easily prove the left inequality above. But the right inequality above is not true in general. We will briefly explain the defining sequences for such Cantor sets.

Let X and Y be a self-similar Cantor sets be given by defining sequences (M_i) in (N_i) which are symmetric with respect to $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$ (see figure ??). The plane $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$ contains equators of all 3-disks.

We have $X \cap Y \subset \mathbb{E}^2 \times \{0\}$ hence the (Cantor) set $X \cap Y$ is tame. Obviously $g(X) = g(Y) = 1$ and one can prove that $g_a(X \cup Y) = 3$ for every $a \in X \cap Y$.

Hence the new conjecture is

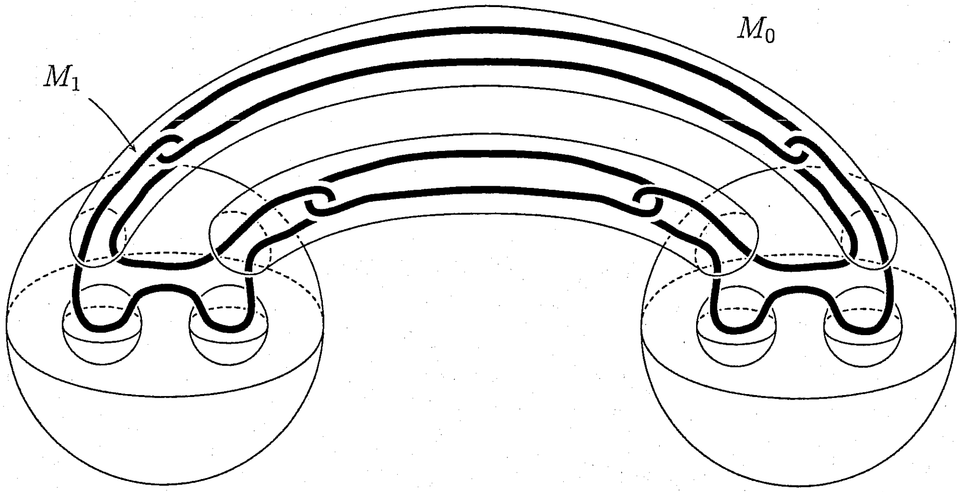

Figure 2: Example of $g(X \cup Y) = g(X) + g(Y) + 1$

Conjecture 4 *If the intersection of Cantor sets $X \subset \mathbb{E}^3$ and $Y \subset \mathbb{E}^3$ is a tame (Cantor) set, we have*

$$g(X \cup Y) \leq g(X) + g(Y) + 1.$$

The author believes that in general genus of the union of Cantor is not related to $g(X) + g(Y)$, more precisely

Conjecture 5 *For every $r \in \mathbb{N}$ there exist Cantor sets X and Y, such that*

$$g(X \cup Y) \geq g(X) + g(Y) + r.$$

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References

[1] S. Armentrout, *Decompositions of* $\mathbf{E}^3$ *with a compact* 0*-dimensional set of nondegenerate elements*, Trans. Amer. Math. Soc. **123** (1966), 165–177.

[2] A. Babich, *Scrawny Cantor sets are not definable by tori*, Proc. of the Amer. Math. Soc. **115** (1992), 829–836.

[3] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. **56** (1952), 354–362.

[4] R. H. Bing, *Tame Cantor sets in* $\mathbf{E}^3$, Pacific J. Math. **11** (1961), 435–446.

[5] R. P. Osborne, *Embedding Cantor sets in a manifold, Part I: Tame Cantor sets in* $\mathbb{E}^n$, Michigan Math. J. **13** (1966), 57–63.

[6] T. B. Rushing, *Topological Embeddings*, Academic Press, New York and London, 1973.

[7] R. B. Sher, *Concerning wild Cantor sets in* $\mathbf{E}^3$, Proc. of the Amer. Math. Soc. **19** (1968), 1195–1200.

[8] R. Skora, *Cantor sets in* S^3 *with simply connected complements*, Topol. Appl. **24** (1986), 181–188.

[9] D. G. Wright, *Bing-Whitehead Cantor sets*, Fund. Math. **132** (1989), 105–116.

[10] M. Željko, *On defining sequences for Cantor sets*, Topol. Appl. **113** (2001), 321–325

Problem Session

Matjaž Želko:
For all known rigid Cantor Sets $X \subset E^3$, we have $\pi_1(E^3 \setminus X) \neq 0$. Does there exist a rigid Cantor Set in E^3 with simply connected complement?
Note: A subset A of E^n is *rigid* if and only if for any homeomorphism $f : E^n \to E^n$, with $f(A) = A$, we have $f|_A = id_A$.

David Wright:
Find a manifold M (closed, orientable, dimension n) so that each homeomorphism is isotopic to the identity.

Tadek Dobrowolski:
Find necessary and sufficient conditions on closed subsets $A, B \subset \ell_2$ such that $\ell_2 \setminus A$ is homeomorphic to $\ell_2 \setminus B$.
Define A and B to have the same homotopy type mod Z_σ if for some closed sets A' in A and B' in B, where $A \setminus A'$ and $B \setminus B'$ are Z_σ sets in ℓ_2, A' and B' have the same homotopy type.
Are $\ell_2 \setminus A$ and $\ell_2 \setminus B$ homeomorphic if (and only if) A and B have the same homotopy type mod Z_σ?

Tom Thickstun:
Conjecture: (Brin-Thickstun) If a noncompact 3-manifold contains a singular, proper essential annulus, then it contains an embedded one.
Definition: A singular proper essential annulus in U^3 is a proper map $f : S^1 \times [0, 1) \to U$ such that $f(S^1 \times \{0\})$ is not null homotopic

Craig Guilbault:
Does there exist a homology sphere Σ^n ($n \neq 3$) that admits a map $f : S^n \to \Sigma^n$ such that the degree of f is not 0? Equivalently, does there exist a homology sphere Σ^n ($n \neq 3$) such that the universal cover $\widehat{\Sigma^n}$ is a rational homology sphere?
Notes:
(1) In dimension 3, the covering projection $p : S^3 \to P^3$ where P^3 is the Poincare homology sphere has degree 120.
(2) By an Euler characteristic argument, examples cannot exist for n even.

(More problems on next page)

Ric Ancel:

Introduction: The paper "Boundaries of Nonpositively Curved Groups of the Form $G \times Z^n$ " by Philip Bowers and Kim Ruane (Glasgow Math J. 38 (1996), 177-189) presents two distinct geometric group actions of the group $F_2 \times Z$ (where F_2 is the free group of rank 2) on the CAT(0) space $T \times R$ (where T is the Cayley graph of F_2 = an infinite 4-valent tree). One of the actions, $\cdot$, is the product of the natural action of F_2 on T with the natural action of Z on R. The other action, $*$, is determined as follows. Let a and b be the generators of F_2, and let 1 be the generator of Z. For $(t, r) \in T \times R$:

$$a * (t, r) = a \cdot (t, r) = (at, r)$$

$$b * (t, r) = (bt, r + 2)$$

$$1 * (t, r) = 1 \cdot (t, r) = (t, r + 1)$$

The visual boundary, $\partial(T \times R)$ of $T \times R$ is a suspension of a Cantor set. The actions $\cdot$ and $*$ naturally extend to actions on $\partial(T \times R)$. Bowers and Ruane show there is no equivariant homeomorphism from $(\partial(T \times R), \cdot)$ to $(\partial(T \times R), *)$. However, it is not difficult to find an action $\circ$ of $F_2 \times Z$ on $\partial(T \times R)$ and equivariant cell like maps $(\partial(T \times R), \circ) \to (\partial(T \times R), \cdot)$. and $(\partial(T \times R), \circ) \to (\partial(T \times R), *)$. In other words, $(\partial(T \times R), \cdot)$ and $(\partial(T \times R), *)$ are equivariantly cell like equivalent. But it is not clear whether the action $\circ$ of $F_2 \times Z$ on $\partial(T \times R)$ extends to a geometric action of $F_2 \times Z$ on a CAT(0) space.

Question: Is there a geometric action $\#$ of $F_2 \times Z$ on a CAT(0) space X and equivariant cell like maps $(\partial X, \#) \to (\partial(T \times R), \cdot)$ and $(\partial X, \#) \to (\partial(T \times R), *)$? In other words, are $(\partial(T \times R), \cdot)$ and $(\partial(T \times R), *)$ equivariantly cell like equivalent through the boundary restriction of a geometric action of $F_2 \times Z$ on a CAT(0) space?