

Proceedings

Seventeenth Annual Workshop in Geometric Topology

Hosted by the Colorado College
June 15-17, 2000

The Seventeenth Annual Geometric Topology Workshop

Co-sponsored by the NSF and Colorado College

The Seventeenth Annual Workshop in Geometric Topology was hosted by Colorado College and held in Colorado Springs, CO on June 15-17, 2000. As always, the participants included a mixture of graduate students and professional mathematicians. Professor Robert Gompf of the University of Texas-Austin was the principle speaker.

A list of past principle speakers follows:

Past Principle Speakers

Year	Hosting Institution	Principle Speaker
1984	Brigham Young University	None
1985	Colorado College	Robert Daverman
1986	Colorado College	John Walsh
1987	Oregon State University	Robert Edwards
1988	Colorado College	John Hempel
1989	Brigham Young University	John Luecke
1990	Oregon State University	Robert Daverman
1991	University of Wisconsin-Milwaukee	Andrew Casson
1992	Colorado College	Mladen Bestvina
1993	Oregon State University	John Bryant
1994	Brigham Young University	Mike Davis
1995	University of Wisconsin-Milwaukee	Shmuel Weinberger
1996	Colorado College	Michael Freedman
1997	Oregon State University	James Cannon
1998	Brigham Young University	Steve Ferry
1999	University of Wisconsin-Milwaukee	Robert Edwards
2000	Colorado College	Robert Gompf

These conference proceedings contain a summary of the three one-hour talks given by Professor Gompf and summaries of talks given by other participants. In addition, they contain summaries of a few talks given at the 15th and 16th annual workshops hosted by Brigham Young University and University of Wisconsin-Milwaukee respectively.

Editor: Frederick Tinsley of Colorado College

The Seventeenth Annual Geometric Topology Workshop
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Colorado Springs, CO: June 15-17, 2000

Participant List

Name	Affiliation
Emina Alibegovic	Univ of Utah
Kathy Andrist*	Utah Valley St College
Bob Daverman	Univ of Tennessee
Jan Dijkstra	Univ of Alabama
Tadek Dobrowolsi	Pittsburgh St Univ
Bill Eaton	Univ of Texas
Bob Edwards	UCLA
Paul Fabel	Mississippi St Univ
Carla Farsi	Univ of Colorado
Hanspeter Fischer	BYU
David Gay	Univ of Arizona
Troy Goodsell	McMurray Univ
Sumanta Guha	UW-Milwaukee
Denise Halverson	BYU
Vo Thanh Liem	Univ of Alabama
Philippe Mazaud	Idaho St Univ
Jung S. Rno	Univ of Cincinnati
Lenny Rubin	Univ of Oklahoma
P. J. Schapiro	Langston Univ
Tom Thickstun	SW Texas St Univ
Mat Timm	Bradley Univ
Jerry Venema	Calvin College
Roger Vogler	BYU
Julia Wilson	UW-Milwaukee
Organizers	
Ric Ancel	UW-Milwaukee
Dennis Garity	Oregon St Univ
Craig Guilbault	UW-Milwaukee
Fred Tinsley	Colorado College
David Wright	BYU
Principal Speaker	
Bob Gompf	Univ of Texas

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¹ Presented at the 16th Geometric Topology Workshop; Milwaukee, WI; June, 1999

² Presented at the 15th Geometric Topology Workshop; Park City, UT; June, 1998

³ Presented at the 15th Geometric Topology Workshop; Park City, UT; June, 1998

⁴ Presented at the 15th Geometric Topology Workshop; Park City, UT; June, 1998

The Topology of Symplectic Manifolds

Robert E. Gompf

1. Introduction

The purpose of this article is twofold: First, we provide an informal introduction to symplectic structures from a topological viewpoint. Second, we address the question of whether symplectic manifolds can ultimately be described as purely topological objects. We sketch work that will appear in [G2], pointing towards an affirmative answer to the question. The first section of the present article motivates and defines symplectic structures, and then discusses obstructions to their existence. In Section 2, we focus on a particular topological construction of symplectic structures, and in Section 3 we see that the construction is likely to be universal in the sense of realizing a dense subset of all symplectic structures on any given manifold. This would lead to a complete topological characterization of those manifolds that admit symplectic structures, and to a reinterpretation of a dense set of symplectic structures on a given manifold as a certain set that should ultimately be describable by purely topological means. Further details will appear in [G2]; see also [GS] for a discussion of the 4-dimensional case. For additional reading on symplectic topology, see e.g., [McS]. In this article, manifolds will always be assumed to be smooth, closed and oriented.

1.1. Why study symplectic manifolds?

While symplectic structures naturally arise in diverse contexts such as Hamiltonian mechanics and algebraic geometry, we focus on a topological application: the classification problem for simply connected 4-manifolds. The most direct approach to a classification problem is to begin by writing down examples. The main classical source of examples of simply connected 4-manifolds was complex surfaces (complex manifolds of complex dimension 2, hence real dimension 4). These can be constructed, for example, by writing down collections of homogeneous polynomials in $n + 1$ complex variables. The common zero locus then specifies a well-defined subset of projective space $\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}$ modulo multiplication by nonzero complex scalars. If this subset happens to be a manifold, it will automatically be complex. Complex surfaces arising in this manner are called *algebraic* surfaces. Many examples of simply connected algebraic surfaces are known — for example, any generic collection of $n - 2$ homogeneous polynomials will determine a

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simply connected algebraic surface in \mathbb{CP}^n (as will some nongeneric collections of more than $n - 2$ polynomials).

Once we have examples of simply connected 4-manifolds, we can construct many more by the *connected sum* operation: We remove the interior of a 4-ball from each of the 4-manifolds X_1 and X_2 , and glue along the resulting boundary 3-spheres so that the new manifold $X_1 \# X_2$ inherits the same orientation from each summand. This can be thought of as an unnecessary complication, however. We would like to restrict attention to those 4-manifolds that cannot be split as connected sums. Unfortunately, it is still unknown whether every 4-manifold homeomorphic to the 4-sphere is actually diffeomorphic to it, and it is even possible that *every* 4-manifold could split off arbitrarily many nontrivial summands homeomorphic to S^4 . Thus, we define a 4-manifold X to be *irreducible* if for every (smooth) decomposition $X \approx X_1 \# X_2$, one summand X_i must be homeomorphic (but not necessarily diffeomorphic) to S^4 .

We can now begin a brief history of the classification problem for simply connected, irreducible 4-manifolds. In the 1970's, virtually nothing was known. While there were many examples of simply connected complex surfaces, these could in general not be distinguished from each other (up to diffeomorphism) or shown to be irreducible. In fact, it was possible that a complete list of irreducible, simply connected 4-manifolds could be given by S^4 , $\pm\mathbb{CP}^2$ (the complex projective plane with both orientations), $S^2 \times S^2$ ($= \mathbb{CP}^1 \times \mathbb{CP}^1$) and $\pm K3$ (where $K3$ denotes the zero locus of a generic quartic polynomial in \mathbb{CP}^3). Furthermore, it was unknown whether $K3$ could split as $X \# S^2 \times S^2$ for some unknown manifold X . In the 1980's, the situation began to change dramatically, due to techniques pioneered by Donaldson using gauge theory. While it now seems likely (in light of Freedman's breakthrough for topological 4-manifolds) that any simply connected (smooth) 4-manifold is *homeomorphic* to a connected sum of manifolds from the above list, the diffeomorphism classification is much more complicated. Our present knowledge about simply connected complex surfaces can be summed up as follows:

- There are many diffeomorphism types (sometimes infinitely many within a homeomorphism type).
- They are irreducible (when minimal).

Minimality is a technical condition that causes no essential difficulties here — any complex surface X can be “blown down” to a minimal complex surface Y , and then X is diffeomorphic to a connected sum of Y with copies of $-\mathbb{CP}^2$. The $K3$ -surface, for example, is minimal and hence irreducible.

This breakthrough in understanding complex surfaces highlighted our ignorance regarding a related question: Are all simply connected, irreducible 4-manifolds ($\neq S^4$) complex? By the end of the 1980's, no counterexamples were known. An affirmative answer would have reduced the classification problem to that of understanding complex surfaces, much as the study of oriented surfaces can be reduced to that of complex curves (Riemann surfaces). In 1990, the question was answered in the negative: Infinitely many irreducible 4-manifolds homeomorphic to the $K3$ -surface were produced, and shown not

to admit complex structures with either orientation [GM]. Subsequently, many other families of counterexamples have been constructed. (See [GS] for a recent survey.) However, the underlying beauty of the question suggested looking for a generalization. It had long been known that every simply connected complex surface is algebraic (after deformation of the complex structure). But every algebraic manifold inherits a *Kähler* structure, i.e., a symplectic structure compatible with its complex structure. (See 1.2-3 for definitions.) Thus, we could generalize to the following question: Are all simply connected, irreducible 4-manifolds ($\neq S^4$) *symplectic*? Work in the early 1990's showed this to be a reasonable question. In fact:

- There are many diffeomorphism types of symplectic, noncomplex 4-manifolds [G1]. For example, the exotic $K3$ -surfaces of [GM] are symplectic. Dropping the simple connectivity hypothesis, we find that every finitely presented group is realized as the fundamental group of a symplectic 4-manifold, whereas fundamental groups of Kähler manifolds and complex surfaces are quite restricted.
- Minimal, simply connected, symplectic 4-manifolds are irreducible (Kotschick [K], after Taubes [T]), at least when $b_2^+ \neq 1$.

(The discussion of symplectic minimality is parallel to that of the holomorphic version discussed above. For b_2^+ , see 1.3.) At present, there are only a few known methods for constructing simply connected, irreducible, noncomplex 4-manifolds. These are highly restricted cut-and-paste constructions. (More general cut-and-paste constructions seem to invariably result in connected sums of simple manifolds.) These restricted operations can be shown to preserve symplectic structures under reasonable hypotheses [G1], [S]. Could it be that the only way to build an irreducible manifold is by equipping it with a symplectic structure? In fact, the answer is no: In 1996, Szabó [Sz] produced simply connected, irreducible 4-manifolds admitting no symplectic structures, by applying these operations under more general hypotheses. Subsequently, Fintushel and Stern [FS] generalized the method to produce an abundance of such examples. At present, there seems to be no reasonable question of this sort to ask to shed light on the structure of arbitrary simply connected, irreducible 4-manifolds.

In summary, we are left with the following classes of simply connected, irreducible 4-manifolds (up to diffeomorphism):

$$\emptyset \subset \{\text{complex}\} \subset \{\text{symplectic}\} \subset \{\text{arbitrary}\} .$$

We have seen that each class contains many elements not in the previous one — in fact, there seems to be a sense in which “most” elements of a given class lie outside the previous one. At present, there seems to be little hope of classifying arbitrary simply connected, irreducible 4-manifolds, so we might hope to simplify the problem by restricting to one of the other classes. Complex surfaces, on the other hand, are difficult for a topologist to study. There is little hope of cutting and pasting, due to the rigid nature of holomorphic functions, so one must resort to methods of algebraic geometry. Symplectic manifolds, however, are accessible by topological methods. The main constructions of symplectic, noncomplex manifolds are of a cut-and-paste nature (e.g., [G1], [S]). In Sections 2 and 3

we will discuss a different topological construction, motivated by fiber bundles, that (at least in dimension 4) provides a complete topological characterization of those manifolds admitting symplectic structures. Thus, one can consider the diffeomorphism classification of simply connected, irreducible, symplectic 4-manifolds as a purely topological problem that may be more accessible than the original classification problem for smooth 4-manifolds.

1.2. Symplectic structures

Definition 1.1. A *symplectic manifold* is a $2n$ -manifold X together with a *symplectic form* ω on X , i.e., a differential 2-form that is closed ($d\omega = 0$) and nondegenerate.

Here, nondegeneracy has its usual meaning in the context of bilinear forms: For any nonzero $v \in T_x X$ there is a vector $w \in T_x X$ such that $\omega(v, w) \neq 0$. An equivalent condition is that the top exterior power ω^n of ω should be nowhere zero, i.e., a volume form on X . (This indicates why X must have even dimension.) The volume form ω^n determines an orientation on X ; we will always use this orientation when considering X as an oriented manifold. For example, we will see that \mathbb{CP}^2 admits a symplectic structure while $-\mathbb{CP}^2$ does not.

It is instructive to compare the above definition with Riemannian geometry. If ω were symmetric rather than skew-symmetric, nondegeneracy would imply that ω was a Riemannian or Lorentzian metric. The condition that $d\omega = 0$ can be compared with requiring a Riemannian metric to have constant curvature. In each case, the relevant partial differential equation guarantees the absence of local structure — two Riemannian n -manifolds with the same constant curvature are locally identical (i.e., any two points have isometric neighborhoods), and the same holds for symplectic $2n$ -manifolds (any two points have symplectomorphic neighborhoods). Thus, symplectic structures can be thought of as skew-symmetric analogs of constant curvature metrics. In the Riemannian case, constant curvature allows a classification theory, which reduces to a study of discrete groups of isometries of Euclidean, hyperbolic or spherical space. One might hope to similarly reduce the study of symplectic manifolds to a topological or combinatorial problem. One cannot hope to generalize the Riemannian theory directly, since there is no symplectic analog of geodesics, and since the classification problem is already difficult for simply connected symplectic manifolds. We will use a different approach to this problem in Section 3.

1.3. Obstructions to constructing symplectic structures

The study of which manifolds admit symplectic structures has two directions: existence and nonexistence. We now discuss the three known sources of obstructions to existence, and defer the discussion of constructing symplectic manifolds to the next section.

To obtain the first obstruction, note that since a symplectic form is closed, it determines a cohomology class $[\omega] \in H_{dR}^2(X) \cong H^2(X; \mathbb{R})$. Nondegeneracy implies that the top exterior power $[\omega]^n = [\omega^n] \in H_{dR}^{2n}(X) \cong \mathbb{R}$ (for X connected) is positive relative to the

given orientation on X . Thus, a symplectic structure cannot exist unless there is a class $\alpha \in H^2(X; \mathbb{R})$ with $\alpha^n > 0$.

Examples. S^{2n} admits no symplectic structure for $n > 1$, since $H^2(S^{2n}; \mathbb{R}) = 0$. $S^2 \times S^{2n-2}$ admits no symplectic structure for $n > 2$, for although $H^2(S^2 \times S^{2n-2}; \mathbb{R}) \cong \mathbb{R}$, the generator α has $\alpha \wedge \alpha = 0$. Similarly, $-\mathbb{CP}^2$ admits no symplectic structure since the generator of $H^2(-\mathbb{CP}^2; \mathbb{R}) \cong \mathbb{R}$ has negative square.

For the second source of obstructions, we forget the closure condition on ω , and consider arbitrary nondegenerate 2-forms on X . Such a 2-form reduces the structure group of the tangent bundle TX from $GL(2n, \mathbb{R})$ to the subgroup $Sp(2n)$ consisting of isomorphisms of \mathbb{R}^{2n} preserving the standard symplectic form $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. (This corresponds to the reduction to $O(n) \subset GL(n, \mathbb{R})$ in the Riemannian case.) The group $Sp(2n)$ is noncompact, but it deformation retracts onto its maximal compact subgroup $U(n) \subset GL(n, \mathbb{C})$ (where we identify \mathbb{R}^{2n} with \mathbb{C}^n in the obvious way). Since the latter inclusion is also a homotopy equivalence, the homotopy classification of nondegenerate 2-forms on X is equivalent to the homotopy classification of *almost-complex structures*, i.e., complex vector bundle structures on TX . This is a classical problem in obstruction theory. For example, a homotopy class of nondegenerate 2-forms inherits Chern classes from the corresponding homotopy class of almost-complex structures.

Examples. $\mathbb{CP}^2 \# \mathbb{CP}^2$ admits no symplectic structure, even though it has classes $\alpha \in H^2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{R})$ with $\alpha \wedge \alpha > 0$, because it admits no almost-complex structure. In fact, standard characteristic class theory shows that such a structure would have a Chern class with $c_1^2 = 2\chi + 3\sigma = 14$ (where χ is the Euler characteristic and σ is the signature of the wedge product pairing on H^2), but a routine computation shows that no integral cohomology class has square 14. More generally, a 4-manifold X cannot admit an almost-complex structure unless the invariant $\frac{1}{2}(\chi + \sigma) = 1 - b_1(X) + b_2^+(X)$ is even, where b_2^+ is the dimension of a maximal positive definite subspace of H^2 under the wedge product. In contrast, $S^3 \times S^1$ admits a complex structure as $\mathbb{C}^2 - \{0\}$ modulo multiplication by 2, and this automatically determines an almost-complex structure on $S^3 \times S^1$. Thus, this manifold admits nondegenerate 2-forms. Such forms cannot be closed, however, since $H^2(S^3 \times S^1; \mathbb{R}) = 0$.

We will find it useful to link symplectic structures more explicitly with almost-complex structures. First note that the latter structures can be specified by choosing the effect of multiplication by i on each tangent space. Thus, an almost-complex structure can be thought of as a linear bundle isomorphism $J : TX \rightarrow TX$ (covering id_X) such that $J^2 = -\text{id}_{TX}$. It is routine to verify that such an isomorphism actually does specify a complex structure; we require the induced orientation to agree with the given one on TX .

Definition 1.2. A 2-form ω *tames* an almost-complex structure J if for any nonzero tangent vector v we have $\omega(v, Jv) > 0$. If, in addition, $\omega(Jv, Jw) = \omega(v, w)$ for any two tangent vectors v, w lying in the same tangent space, we say that ω and J are *compatible*.

Thus, ω tames J if it is a positive area form on each complex line (in the complex orientation). The compatibility condition, that J preserves ω (i.e., $J \in \text{Sp}(TX)$), corresponds to orthogonality of J in the Riemannian case. A compatible pair (ω, J) determines a Hermitian structure on TX via the metric $g(v, w) = \omega(v, Jw)$. For a fixed nondegenerate form on X , the spaces of tamed and compatible almost-complex structures are nonempty and contractible (e.g., [McS]), so either condition exhibits the above correspondence of homotopy classes. In the remaining sections, we will make extensive use of the following

Observations. (1) If a 2-form tames some almost-complex structure, it is obviously nondegenerate. Hence, a closed, taming form is automatically symplectic.

(2) If $\omega_1, \dots, \omega_k$ tame a fixed J , then any convex combination $\sum_{i=1}^k t_i \omega_i$ (all $t_i \geq 0$, $\sum t_i = 1$) will also tame J .

(3) The taming condition is *open*, i.e., preserved under sufficiently small perturbations of ω and J .

To verify the last observation, note that the taming condition $\omega(v, Jv) > 0$ is satisfied provided that it holds for vectors v in the unit sphere bundle $\Sigma \subset TX$ (given by any preassigned metric). Since Σ is compact, taming implies that $\omega(v, Jv)$ is bounded below by a positive constant on Σ , so it will remain positive under small perturbations of ω and J . Note that compatibility is not an open condition. For this reason, we will mainly use the taming condition in subsequent sections, although compatibility appears more commonly in the literature.

Examples. While every symplectic manifold (X, ω) has a compatible almost-complex structure J , this latter structure may not come from a complex structure on X . (For J to be a complex structure on X , it must be locally identical to \mathbb{C}^n , which is equivalent to requiring J to satisfy a certain partial differential equation.) If J actually is a complex structure on X , the triple (X, J, ω) is called a *Kähler manifold*. A standard example of this is \mathbb{CP}^n , which inherits both J and ω in simple ways from \mathbb{C}^{n+1} . (Restrict the standard $\omega = \sum_{i=0}^n dx_i \wedge dy_i$ from \mathbb{C}^{n+1} to S^{2n+1} , then note that its projection to \mathbb{CP}^n is well-defined by $U(1)$ -invariance and the fact that all tangent vectors projecting to 0 pair trivially with TS^{2n+1} .) Since a complex submanifold of a Kähler manifold is Kähler, it follows immediately that any algebraic manifold is Kähler. (More generally, if ω tames J and $Y \subset X$ is J -holomorphic, i.e., each $T_y Y \subset T_y X$ is a J -complex subspace, then $\omega|_Y$ tames $J|_Y$.)

The third and final known source of obstructions to the existence of symplectic structures is the Seiberg-Witten invariants (from gauge theory) on 4-manifolds [T] (cf. also [GS], [K]). An important example is that minimal, simply connected symplectic 4-manifolds with $b_2^+ \neq 1$ must be irreducible. Similarly, if an arbitrary symplectic 4-manifold has a connected sum splitting, the wedge product pairing on H^2 must be negative definite for all but one summand. The Seiberg-Witten invariants are much more subtle than the previously discussed invariants, and will not be needed in the subsequent sections.

Example. $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ has no symplectic structure, since the pairing on $H^2(\mathbb{CP}^2)$ is not negative definite. However, it clearly has a cohomology class α with $\alpha \wedge \alpha > 0$, and can be shown to admit an almost-complex structure. Similarly, a connected sum of 3 copies of \mathbb{CP}^2 with 19 copies of $-\mathbb{CP}^2$ does not admit a symplectic structure, but it is actually homeomorphic (although clearly not diffeomorphic) to a Kähler manifold. The homeomorphism is covered by an isomorphism of tangent bundles. This shows that the obstructions from Seiberg-Witten theory are more subtle than the homotopy-theoretic ones discussed previously.

2. Constructing symplectic structures

We turn to the construction of symplectic manifolds. Historically, the first examples of (compact) symplectic manifolds were the Kähler manifolds, obtained largely by algebrogeometric methods. We will consider in detail the first construction of symplectic manifolds admitting no Kähler structure. (For other topological constructions, see e.g., [G1], [Mc], [S].) We will then generalize the construction into a form suitable for the applications in Section 3.

2.1. Symplectic forms on bundles

The original construction of symplectic, nonKähler manifolds, due to Thurston [Th] (see also [McS]), consists of finding a symplectic form on the total space of a fiber bundle. The basic method is quite simple, and reminiscent of techniques previously introduced into complex analysis by Grauert. We state the simplest version of Thurston's theorem, in which the fibers are 2-dimensional.

Theorem 2.1. [Th] *Let $f : X^{2n} \rightarrow Y^{2n-2}$ be a bundle map, with X connected, Y symplectic and $[f^{-1}(y)] \neq 0 \in H_2(X; \mathbb{R})$. Then X admits a symplectic structure.*

Recall that all manifolds are assumed to be compact and oriented. Thus, the fibers $f^{-1}(y)$ are all closed, oriented surfaces and homologous, so the homological condition makes sense and is independent of y . To see that this condition is necessary, consider the bundle map $f : S^3 \times S^1 \rightarrow S^2$ obtained by projecting to S^3 and applying the Hopf fibration. The theorem generalizes to bundles with higher dimensional fibers. In that case one also needs the fibers to be symplectic and the transition functions to be symplectomorphisms. These additional conditions are automatically satisfied when the fibers have dimension 2, since a symplectic form on a surface is the same as an area form.

Example. It is easy to construct a torus bundle over the torus whose total space X has $b_1(X) = 3$. For example, begin with a torus bundle over S^1 whose monodromy is a Dehn twist, then cross with S^1 . This example clearly has a section, so $[f^{-1}(y)] \neq 0$, and Theorem 2.1 provides a symplectic structure on X . However, it is a basic fact that the odd-degree Betti numbers of a Kähler manifold must be even, so X is not even homotopy equivalent to a Kähler manifold. This example from Thurston's paper was also known to Kodaira. It can also be seen as the quotient of $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ by the discrete

group of symplectomorphisms generated by unit translations along the x_1, y_1 and x_2 axes and the map $(x_1, y_1, x_2, y_2) \mapsto (x_1 + y_1, y_1, x_2, y_2 + 1)$.

2.2. Symplectic forms induced by J -holomorphic maps

Although Theorem 2.1 has a simple direct proof, we will proceed by the alternate method of generalizing the theorem and supplying a proof of the generalization that is not significantly harder than the original proof. To motivate the generalization, first recall that the symplectic manifold Y in Theorem 2.1 automatically has a compatible almost-complex structure J_Y . We can easily construct an almost-complex structure J on X for which f is J -holomorphic, i.e., $df \circ J = J_Y \circ df$. (That is, each $df_x : T_x X \rightarrow T_{f(x)} Y$ is complex linear.) For example, choose a metric on X and let $H \subset TX$ be the subbundle of orthogonal complements to the fibers of f . Clearly, $df|_H : H \rightarrow TY$ is an isomorphism on each fiber; let $J|_H$ be the pullback of J_Y . Define J on the tangent spaces to the fibers of f to be $\frac{\pi}{2}$ counterclockwise rotation (using the metric and preimage orientation and the fact that these spaces are 2-dimensional). J is now uniquely determined on TX by linearity, and f is J -holomorphic by construction. Thus, the hypotheses of Theorem 2.1 are hiding almost-complex structures on X and Y making f J -holomorphic. Once we observe this, we find that the bundle structure is completely unnecessary! We obtain the following theorem:

Theorem 2.2. *Let $f : X \rightarrow Y$ be a J -holomorphic map of almost-complex manifolds. Let ω_Y be a symplectic form on Y taming J_Y . Fix a class $c \in H_{dR}^2(X)$. Suppose that for each $y \in Y$, $f^{-1}(y)$ has a neighborhood W_y with a closed 2-form η_y such that $[\eta_y] = c|_{W_y} \in H_{dR}^2(W_y)$, and such that η_y tames $J|_{\ker df_x}$ for each $x \in W_y$. Then X admits a symplectic structure.*

Note that $\ker df_x$ is a J -complex subspace of $T_x X$ (since f is J -holomorphic); the taming condition means $\eta_y(v, Jv) > 0$ for each nonzero $v \in \ker df_x$.

To motivate the remaining hypotheses of Theorem 2.2, we show that it implies Theorem 2.1. This is essentially the first part of Thurston's proof. We leave it as an exercise to state and deduce the analog of Theorem 2.1 for bundles with higher dimensional fibers.

Proof of Theorem 2.1. We assume the hypotheses of Theorem 2.1 and deduce those of Theorem 2.2; the conclusion follows. We have already obtained the first two sentences of Theorem 2.2. We may assume the fibers $f^{-1}(y)$ are connected, by passing to a finite cover of Y if necessary. Let c be any class for which $\langle c, f^{-1}(y) \rangle = 1$; such classes exist since $[f^{-1}(y)] \neq 0$ in $H_2(X; \mathbb{R})$ and $H_{dR}^2(X)$ is dual to this space. For each $y \in Y$, let D_y be an open disk containing y and let $W_y = f^{-1}(D_y) \approx D_y \times f^{-1}(y)$. Choose an area form on $f^{-1}(y)$ with area 1 (and inducing the preimage orientation on $f^{-1}(y)$), and let η_y be the pullback of this form to W_y via the projection $W_y \rightarrow f^{-1}(y)$. Since $H_2(W_y)$ is generated by $[f^{-1}(y)]$, the equalities $\langle \eta_y, f^{-1}(y) \rangle = 1 = \langle c, f^{-1}(y) \rangle$ show that $[\eta_y] = c|_{W_y} \in H_{dR}^2(W_y)$. Since f is a bundle map, $\ker df_x$ is the tangent plane to the fiber at x , so the required taming condition is just the obvious statement that η_y tames J when restricted to each fiber of f in W_y . \square

Proof of Theorem 2.2. The proof follows Thurston's paper, except for two deviations where we exploit the almost-complex structures. The first step is to splice together the locally defined forms η_y into a global closed form η on X satisfying the corresponding hypotheses that $[\eta] = c$ and η tames $J| \ker df_x$ for all $x \in X$. Unfortunately, splicing the forms in the obvious way by a partition of unity destroys the closure condition, so we use a trick: Fix a representative ζ of the deRham class $c = [\zeta]$. Now for each $y \in Y$ we have $\eta_y = \zeta + d\alpha_y$ on W_y , for some 1-form α_y on W_y (since $[\eta_y] = c|_{W_y}$). We splice the forms η_y by splicing the 1-forms α_y as follows. Let $\{\rho_i\}$ be a partition of unity on Y , subordinate to a sufficiently fine cover. Pull back by f to obtain the corresponding partition of unity $\{\rho_i \circ f\}$ on X , and let $\eta = \zeta + d \sum_i (\rho_i \circ f) \alpha_{y_i}$, where the sum splices the forms α_y in the usual way via $\{\rho_i \circ f\}$. Clearly, η is a closed 2-form on X with $[\eta] = c$. To verify the taming condition, we carry out the differentiation to obtain $\eta = \zeta + \sum_i (\rho_i \circ f) d\alpha_{y_i} + \sum_i (d\rho_i \circ df) \wedge \alpha_{y_i}$. The last term clearly vanishes when applied to a pair of vectors in $\ker df_x$, so on $\ker df_x$ we have $\eta = \zeta + \sum_i (\rho_i \circ f) d\alpha_{y_i} = \sum_i (\rho_i \circ f) \eta_{y_i}$. This is a convex combination of forms taming $J| \ker df_x$, so it tames as required (by Observation 2 of 1.3). Note how the almost-complex structures guide the construction here — an arbitrary convex combination of symplectic forms need not be symplectic, e.g., any symplectic form ω satisfies $-\omega + \omega = 0$, but $\pm\omega$ are both symplectic for the same oriented manifold if the dimension is divisible by 4.

As in Thurston's proof, we now wish to show that the closed form $\omega_t = t\eta + f^*\omega_Y$ on X is symplectic for sufficiently small $t > 0$. By Observation 1 of 1.3, it suffices to show that ω_t tames J for small $t > 0$, so we only need to verify that $\omega_t(v, Jv) > 0$ on the unit tangent bundle $\Sigma \subset TX$. But

$$\begin{aligned} \omega_t(v, Jv) &= t\eta(v, Jv) + \omega_Y(df(v), df(Jv)) \\ &= t\eta(v, Jv) + \omega_Y(df(v), J_Y df(v)) , \end{aligned}$$

where the last line uses J -holomorphicity of df ($df \circ J = J_Y \circ df$). Since ω_Y tames J_Y , the last term is ≥ 0 , with equality if and only if $v \in \ker df$. On the other hand, η tames J on $\ker df$, so by openness of the taming condition (Observation 3 of 1.3), $\eta(v, Jv) > 0$ for v in some neighborhood U of the subset $\Sigma \cap \ker df$ in Σ . Thus, $\omega_t(v, Jv) > 0$ for all $t > 0$ when $v \in U$. But $\Sigma - U$ is compact, so on $\Sigma - U$, $\eta(v, Jv)$ is bounded and the last displayed term is bounded below by a *positive* constant (since it is positive away from $\ker df$). It is now clear that for sufficiently small $t > 0$, $\omega_t(v, Jv) > 0$ for all $v \in \Sigma$, as required. \square

3. Characterizing symplectic manifolds

Fiber bundles form an interesting but relatively small class of manifolds. We wish to find more general structures to which Theorem 2.2 can be applied. We ultimately define structures with sufficient generality that they can probably be found in any symplectic manifold, providing our desired topological characterization of manifolds admitting symplectic forms. Such a structure determines an essentially unique symplectic form, and one should be able to realize a dense subset of all symplectic forms in this manner. This

could lead to a purely topological way of understanding the set of symplectic structures on any given manifold.

3.1. Lefschetz pencils

We begin by considering what topological structure can be found on an algebraic surface X . By definition, X is a holomorphic submanifold of \mathbb{CP}^N for some N . Let $A \subset \mathbb{CP}^N$ be a generic linear subspace of complex codimension 2 (so it is a copy of \mathbb{CP}^{N-2} cut out by two homogeneous linear equations $p_0(z) = p_1(z) = 0$). Then A intersects X transversely in a finite set B called the *base locus*. The set of all hyperplanes through A is parametrized by \mathbb{CP}^1 . (They are given by the equations $y_0 p_0(z) + y_1 p_1(z) = 0$, for $(y_0, y_1) \in \mathbb{C}^2 - \{0\}$ up to scale.) These hyperplanes intersect X in a family of (possibly singular) complex curves $\{F_y \mid y \in \mathbb{CP}^1\}$. Since the hyperplanes fill \mathbb{CP}^N and any two intersect precisely in A , we have $\bigcup_{y \in \mathbb{CP}^1} F_y = X$ and $F_y \cap F_{y'} = B$ for $y \neq y'$. The canonical map $\mathbb{CP}^N - A \rightarrow \mathbb{CP}^1$ induced by the hyperplanes restricts to a holomorphic map $f : X - B \rightarrow \mathbb{CP}^1$ determined by the condition $f^{-1}(y) = F_y - B$. Since A intersects X transversely, each F_y is smooth near B , and f can be locally identified with projectivization $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{CP}^1$ there. (In fact, the hyperplanes restrict to the complex lines through 0 on the tangent plane to X at each $b \in B$.) Since A is generic, so is the function f . This means f is the complex analog of a Morse function, i.e., its critical points are complex quadratic. The structure we have defined here is called a *Lefschetz pencil* on X , and can be generalized from holomorphic to smooth manifolds.

Definition 3.1. A *Lefschetz pencil* on a 4-manifold X is a finite *base locus* $B \subset X$ and a map $f : X - B \rightarrow \mathbb{CP}^1$ such that

- (1) each $b \in B$ has an orientation-preserving local coordinate map to $(\mathbb{C}^2, 0)$ under which f corresponds to projectivization $\mathbb{C}^2 - \{0\} \rightarrow \mathbb{CP}^1$, and
- (2) each critical point of f has an orientation-preserving local coordinate chart in which $f(z_1, z_2) = z_1^2 + z_2^2$ for some holomorphic local chart in \mathbb{CP}^1 .

Note that there is no analog of the Morse index, since $-z^2 = +(iz)^2$. In the literature, additional conditions are sometimes imposed. For example, after perturbing f we can assume that f is injective on the (finite) set of critical points. In addition, our algebraic prototype has the property that each component of $F_y - \{\text{critical points}\}$ intersects B (since its closure is a complex curve and hence homologically essential in the corresponding hyperplane); some version of this condition is needed for constructing symplectic structures (e.g., to rule out the torus bundle $f : S^3 \times S^1 \rightarrow S^2$).

Like Morse functions in the real-valued setting, Lefschetz pencils determine the topology of the underlying 4-manifolds. A useful way to exploit this is to “blow up” the base locus B , compactifying $X - B$ by one-point compactifying each fiber separately at each $b \in B$. This changes X by connected summing with a copy of $-\mathbb{CP}^2$ for each $b \in B$. We obtain a singular fibration $X \# k(-\mathbb{CP}^2) \rightarrow \mathbb{CP}^1$ called a *Lefschetz fibration*, characterized by having only complex quadratic critical points as above. Explicit

handle diagrams can be drawn for Lefschetz fibrations, using the fact that each critical point corresponds to a 2-handle. Alternatively, one can remove the critical values from $\mathbb{CP}^1 = S^2$, and delete the corresponding singular fibers from X , obtaining an honest fiber bundle over $S^* = S^2 - (\text{finite set})$. The monodromy around each critical value will be a right-handed Dehn twist of the fiber (assuming $f|_{\{\text{critical points}\}}$ is injective), and the monodromy representation $\pi_1(S^*) \rightarrow \text{Map}(F)$ (into the group of orientation-preserving diffeomorphisms of the fiber up to isotopy) will determine the Lefschetz fibration if the fiber has genus ≥ 2 . Thus, the study of Lefschetz fibrations reduces to a purely combinatorial problem about the mapping class group $\text{Map}(F)$. A similar reduction can be made for Lefschetz pencils, using diffeomorphisms of the fiber that fix a point and its tangent plane for each $b \in B$. (In this case, one must remove an extra point from S^* , around which the monodromy is nontrivial due to the twisted normal bundles of the exceptional spheres.) Lefschetz fibrations on 4-manifolds have recently become a particularly active area of research. For example, many Lefschetz fibrations have been directly constructed for which the underlying manifold X admits no complex structure. In particular, one can use monodromy representations to construct Lefschetz fibrations whose fundamental groups include all finitely presented groups [ABKP]. (Recall that most finitely presented groups cannot be realized by complex surfaces.) For a recent (but rapidly becoming outdated) survey of Lefschetz pencils and fibrations, see [GS].

Our construction of Lefschetz pencils on algebraic surfaces can be generalized to algebraic manifolds of any dimension. If we continue to require $\text{codim}_{\mathbb{C}} A = 2$, we obtain a map $f : X - B \rightarrow \mathbb{CP}^1$, where B is a submanifold of (complex) codimension 2. The map f will look like projectivization in the directions normal to B , and the critical points of f will be locally modeled by $f(z_1, \dots, z_n) = \sum_{i=1}^n z_i^2$. (These correspond to n -handles.) Such structures are still called Lefschetz pencils. They were first used by Lefschetz to study the topology of algebraic manifolds. (See [L].) One can analyze them using the monodromy representation as in the 4-dimensional case, although at present, little work has been done on this. For a further generalization, we can allow A to have complex codimension $k + 1 \geq 2$, and consider linear subspaces with codimension k containing A . We then obtain a map $f : X - B \rightarrow \mathbb{CP}^k$ with $\text{codim}_{\mathbb{C}} B = k + 1$. (For example, we can make B finite by setting $k = \dim_{\mathbb{C}} X - 1$. For larger k , B vanishes entirely.) The map f will still be projectivization on normal slices to the manifold B , but critical points will no longer be isolated and they may require higher degree terms in their local models. (For example, the $k = 2$ case can be thought of as a Lefschetz pencil of pencils. A single Lefschetz pencil has isolated critical points, but these will sweep out sheets as we vary through a pencil of pencils, and for some values of the parameter, quadratic critical points will coalesce to form those of higher-degree.) Algebraic geometers call structures of this more general form *linear systems*. In principle, one could try to analyze their topology via monodromies and induction on dimension. Note that if X has a linear system for a given k , then it has them for all smaller values of k : Simply compose f with the canonical

projection map $\mathbb{CP}^k - \{\text{pt.}\} \rightarrow \mathbb{CP}^{k-1}$. (This corresponds to choosing a new A containing the old one with codimension 1.) Thus, the information content of a linear system increases with k .

3.2. Hyperpencils

Linear systems $f : X - B \rightarrow \mathbb{CP}^k$ provide the sort of “fibrationlike” structure on a $2n$ -manifold X that allows us to construct symplectic forms by the method of Section 2. Our present goal is to carefully define such a structure in topological terms, in such a way as to guarantee the existence of symplectic forms. A plausible starting place would be the $k = n$ case, where B is empty and f is a sort of singular branched covering. However, it seems best to start with the weakest possible definition guaranteeing a symplectic structure, meaning we should use the smallest possible value of k . But if $k \leq n - 2$, the fibers will have (real) dimension > 2 , and theorems producing symplectic structures will require hypotheses guaranteeing that the fibers and transition functions will be symplectic. Thus, for a theorem without symplectic hypotheses, the optimal case seems to be $k = n - 1$, where generic fibers are surfaces and hence are automatically symplectic. We will call such a structure a *hyperpencil*, with the prefix indicating that k should be changed from 1 (for a pencil) to complex codimension 1. The definition is analogous to that of Lefschetz pencils. However, the critical points are necessarily more complicated, so we allow them to be modeled by *any* holomorphic function (provided that within each fiber they are isolated). In fact, the situation is not significantly complicated by taking the function to be just locally J -holomorphic with respect to *almost*-complex structures (subject to a certain technical condition that is automatically satisfied in the holomorphic case or when $n \leq 3$). We allow these almost-complex structures to be C^0 rather than smooth, both for convenience and to emphasize that their primary function is homotopy-theoretic in nature, controlling monodromies. (For example, if we allow orientation-reversing charts at critical points in our definition of Lefschetz pencils, so that some monodromies are given by *left*-handed Dehn twists, then we can construct such structures on manifolds admitting no symplectic forms; see e.g. [GS].) As a final generalization, we allow the almost-complex structure on \mathbb{CP}^{n-1} to be different for different points on a given fiber, by using locally defined almost-complex structures on the bundle $f^*T\mathbb{CP}^{n-1}$ rather than on $T\mathbb{CP}^{n-1}$ itself. (The reader can simplify the setup by pretending these are given by the standard holomorphic structure on \mathbb{CP}^{n-1} .) We require these structures to be compatible with the standard symplectic form $\omega_{\mathbb{CP}^{n-1}}$ on \mathbb{CP}^{n-1} (pulled back to a skew-symmetric pairing on $f^*T\mathbb{CP}^{n-1}$). We then obtain the following definition (which can, and probably should, be generalized even further).

Definition 3.2. A *hyperpencil* on a $2n$ -manifold X is a finite set $B \subset X$ and a map $f : X - B \rightarrow \mathbb{CP}^{n-1}$ such that

- (1) each $b \in B$ has an orientation-preserving local coordinate map to $(\mathbb{C}^n, 0)$ under which f corresponds to projectivization $\mathbb{C}^n - \{0\} \rightarrow \mathbb{CP}^{n-1}$,

- (2) each fiber $F_y = \text{cl } f^{-1}(y) \subset X$ contains only finitely many critical points of f , each locally modeled by a holomorphic map if $n \geq 4$, and each critical point has a neighborhood U with C^0 almost-complex structures on U and $f^*T\mathbb{CP}^{n-1}|_U$ for which the latter is compatible with $\omega_{\mathbb{CP}^{n-1}}$ and f is J -holomorphic, and
- (3) each component of $F_y - \{\text{critical points}\}$ intersects B .

The construction of Section 2 can be used to produce a symplectic structure on any manifold with a hyperpencil. In particular, any 4-manifold with a Lefschetz pencil (with $B \neq \emptyset$) admits a symplectic structure. (While a Lefschetz pencil need not satisfy condition (3) above, the condition is actually unnecessary in this case, provided $B \neq \emptyset$; see [GS]. However, without the condition, we lose both the control of $[\omega]$ and the uniqueness statement given below.) The construction allows us to control the cohomology class $[\omega]$. Recall that $H_{dR}^2(\mathbb{CP}^{n-1}) \cong \mathbb{R}$ has a canonical generator h , the *hyperplane class*, Poincaré dual to $[\mathbb{CP}^{n-2}]$. The class f^*h is defined on $X - B$, but $H_{dR}^2(X - B) \cong H_{dR}^2(X)$ for $n > 1$, so we can think of f^*h as a class in $H_{dR}^2(X)$ determined by the hyperpencil. (For $n = 1$, it is natural to identify f^*h with the Poincaré dual of $[B] \in H_0(X; \mathbb{R})$.) The construction allows us to arrange $[\omega] = f^*h$. The form ω is then completely determined by the construction, up to isotopy. (Two symplectic forms ω_0, ω_1 on X are *isotopic* if there is a diffeomorphism φ isotopic to the identity with $\varphi^*\omega_1 = \omega_0$. Thus, isotopic symplectic forms only differ by a deformation of X .) Furthermore, the isotopy class is unchanged if we deform the hyperpencil. (A deformation of hyperpencils should be roughly thought of as a bundle Z over a path connected parameter space S , whose fibers are hyperpencils. More precisely, we naturally generalize the definition of hyperpencil to this parametrized setting. For example, the base locus becomes a finite covering $B \rightarrow S$, and the local almost-complex structures at a critical point of X become continuous families of fiber-wise almost-complex structures defined near a point in Z .) More specifically, we obtain our main theorem:

Theorem 3.1. *A deformation class of hyperpencils uniquely determines an isotopy class of symplectic forms. This isotopy class is characterized as being the unique class containing representatives ω for which $[\omega] = f^*h \in H_{dR}^2(X)$ and ω tames a given hyperpencil in the deformation class.*

We say that ω *tames* a hyperpencil $f : X - B \rightarrow \mathbb{CP}^{n-1}$ if there is a C^0 almost-complex structure $J_{\mathbb{CP}^{n-1}}$ compatible with $\omega_{\mathbb{CP}^{n-1}}$ on $f^*T\mathbb{CP}^{n-1}$, such that each $x \in X$ has a neighborhood with a C^0 almost-complex structure tamed by ω and making f J -holomorphic. It can be shown that if ω tames f then there is a *global* almost-complex structure J on X with ω taming J and f J -holomorphic. Similarly, the local almost-complex structures in (2) of the definition of hyperpencils can be made global. In each case, the global structures (including $J_{\mathbb{CP}^{n-1}}$) can be arranged to be standard near B . Similar statements apply in the setting of deformations. See [G2] for details.

3.3. Proof of Theorem 3.1

We sketch the proof; for further details, see [G2]. To prove existence, we fix a hyperpencil $f : X - B \rightarrow \mathbb{CP}^{n-1}$ in the given deformation class, and establish the hypotheses of Theorem 2.2. As remarked in the previous paragraph, there are global structures J on X and $J_{\mathbb{CP}^{n-1}}$ on $f^*T\mathbb{CP}^{n-1}$, standard near B , making f J -holomorphic. Let $c = f^*h \in H_{dR}^2(X)$. For each $y \in \mathbb{CP}^{n-1}$, we construct the required W_y and η_y : At each critical point $x \in F_y$, $(T_x X, J)$ can be identified with \mathbb{C}^n . Thus, the standard linear symplectic form on \mathbb{C}^n tames J at x . Extend this to a closed 2-form η_y near x ; we can assume η_y tames J on this neighborhood by openness of the taming condition. Since F_y is a J -complex curve, $\eta_y|_{F_y}$ is an area form defined near the singular points of F_y . Extend this to an area form on all of F_y whose total area on each component F_i of $F_y - \{\text{critical points}\}$ is $\langle f^*h, c|_{F_i} \rangle$ (which is positive since it equals the number of points in $F_i \cap B$; cf. (3) of Definition 3.2). W_y and η_y can now be constructed in a manner analogous to the proof of Theorem 2.1, by pulling back $\eta_y|_{F_y}$ by a suitable map. By construction, $[\eta_y] = c|_{W_y}$. Also, η_y tames J on each $T_x X$ near critical points, and on $\ker df_x = T_x F_{f(x)}$ elsewhere (for W_y sufficiently small). (We have ignored minor technical difficulties arising, e.g., if F_y has nonconelike singularities.) A slight generalization of Theorem 2.2 now gives a symplectic form ω on $X - B$ taming J . (Note that $f : X - B \rightarrow \mathbb{CP}^{n-1}$ fails the hypotheses of Theorem 2.2 in that the domain is noncompact; this can be fixed by working relative to a standard symplectic form defined near B .) Unfortunately, ω is singular at B — it has the form $t\eta + f^*\omega_{\mathbb{CP}^{n-1}}$, and the second term is singular. Fortunately, we have an explicit description of ω and J near each $b \in B$ (on a neighborhood identified with a neighborhood of 0 in \mathbb{C}^n , with f given by projectivization). This local model shows that we can dilate X at b and glue in a symplectic ball to make ω smooth everywhere, without losing the taming condition. (This construction is essentially equivalent to blowing up B , applying Theorem 2.2 to the resulting singular fibration on a compact manifold, and blowing back down. However it bypasses some technical difficulties involving working with the blown-up points.) We now have the desired symplectic form. It obviously tames f via J , and $[\omega] = [t\eta + f^*\omega_{\mathbb{CP}^{n-1}}] = tc + f^*[\omega_{\mathbb{CP}^{n-1}}] = (t+1)f^*h$ (since $[\omega_{\mathbb{CP}^{n-1}}] = h$), so ω satisfies the required conditions after rescaling.

To prove uniqueness, we start with symplectic forms ω_0 and ω_1 on X , satisfying the two conditions in Theorem 3.1 with respect to deformation equivalent pencils f_0 and f_1 , respectively. We show that ω_0 and ω_1 are isotopic, completing the proof of the theorem. We can assume the deformation is parametrized by the interval $I = [0, 1]$. Each form ω_i is given to tame f_i , so as indicated at the end of the previous subsection, we can find a global C^0 almost-complex structure J_i on X making f_i J_i -holomorphic, and with ω_i taming J_i . Using the same paragraph (in the parametrized version rel $\{0, 1\}$ without a taming ω), we can extend J_0, J_1 to a continuous family J_t of almost-complex structures, $0 \leq t \leq 1$, with f_t J_t -holomorphic for some family $J_{t, \mathbb{CP}^{n-1}}$ on $f_t^*T\mathbb{CP}^{n-1}$. (This is the one place where Definition 3.2 requires the condition for $n \geq 4$, and compatibility with $\omega_{\mathbb{CP}^{n-1}}$ rather than taming.) For each hyperpencil f_t in the deformation and each

structure J_t , construct ω_t taming J_t as in the previous paragraph. While the resulting forms ω_t , $0 \leq t \leq 1$, will be *a priori* unrelated to each other, we can make the family smooth by a trick: By openness of the taming condition, each ω_t will tame each J_s in some neighborhood of $t \in I$. Thus, we can cover I by neighborhoods U_α on which a single ω_t tames each J_s , $s \in U_\alpha$. Using a partition of unity $\{\rho_\alpha\}$ on I subordinate to $\{U_\alpha\}$, splice together these forms ω_t . The resulting smooth family (still called ω_t) will consist of closed forms (since each ρ_α is constant on each fiber of the deformation), and each ω_t will tame the corresponding J_t (since it is a convex combination of taming forms). Thus, we have a smooth family of symplectic forms on X . Furthermore, $[\omega_t] = f_t^* h$ is independent of t (since we can assume B is fixed and invoke homotopy invariance of induced maps). The theorem now follows from:

Theorem 3.2 (Moser [M]). *Let ω_t , $0 \leq t \leq 1$, be a smooth family of symplectic forms on X , with $[\omega_t] \in H_{dR}^2(X)$ independent of t . Then there is an isotopy $\varphi_t : X \rightarrow X$ with $\varphi_0 = \text{id}_X$ and $\varphi_t^* \omega_t = \omega_0$.*

(The proof of Moser's Theorem is actually quite short. One simply writes down a suitable formula for a time-dependent vector field, then integrates to obtain φ_t .) \square

3.4. Characterization

We now turn to the question of how general the hyperpencil construction of symplectic structures is, addressing the topological characterization of symplectic manifolds. The answer lies in work of Donaldson [D] followed by Auroux [A], the roots of which go back to Kodaira in the holomorphic setting (the Kodaira Embedding Theorem, e.g., [GH]). If $\sigma_0, \dots, \sigma_k$ are sections of a complex line bundle $L \rightarrow X$, then because each fiber L_x is canonically \mathbb{C} up to (complex) scale, the vector $(\sigma_0(x), \dots, \sigma_k(x))$ in $(L_x)^{k+1}$ determines an element of \mathbb{C}^{k+1} up to scale. Thus, projectivizing gives a well-defined map $f = [\sigma_0 : \dots : \sigma_k] : X - B \rightarrow \mathbb{CP}^k$, where B is the common zero locus of $\sigma_0, \dots, \sigma_k$. Kodaira used this idea to characterize which complex manifolds are algebraic (i.e., embed holomorphically in \mathbb{CP}^N) in terms of line bundles: Given the existence of a suitable holomorphic line bundle L over a complex manifold X , one can obtain arbitrarily many holomorphic sections by taking sufficiently large tensor powers $L^{\otimes m}$ of the line bundle, eventually yielding an embedding $f : X \hookrightarrow \mathbb{CP}^N$. It automatically follows that X is Kähler with symplectic form $\omega = f^* \omega_{\mathbb{CP}^N}$ satisfying $[\omega] = c_1(L^{\otimes m}) = m c_1(L)$. Donaldson's contribution was to extend this idea to the symplectic setting. Starting with a symplectic manifold (X, ω) with ω *integral* (i.e., $[\omega] \in \text{Im}(H^2(X; \mathbb{Z}) \rightarrow H_{dR}^2(X))$), he chose a compatible almost-complex structure J , and arranged a line bundle $L \rightarrow X$ with Chern class $c_1(L) = [\omega]$ to be J -holomorphic in a suitable sense. Unfortunately, in this setting holomorphic sections rarely exist. However, by defining a suitable notion of "approximately holomorphic" sections and applying hard analysis on the line bundles $L^{\otimes m}$, Donaldson was able to construct a Lefschetz pencil $X - B \rightarrow \mathbb{CP}^1$ on any integral symplectic manifold. Subsequently, Auroux [A] generalized the method to construct a linear system $X - B \rightarrow \mathbb{CP}^k$ for $k = 2$, and he is currently extending his work to arbitrary k .

In particular, the case $k = n - 1$ should yield a hyperpencil (with particularly nice local properties) tamed by the original symplectic form ω , and with $f^*h = c_1(L^{\otimes m}) = m[\omega]$ for any given sufficiently large integer m . (It may be useful to slightly weaken the definition of a hyperpencil here.) This would complete the proof of the following conjecture, which currently seems to be established in dimensions ≤ 6 (by the above cases $k = 1, 2$.)

Conjecture 3.3. *For any integral symplectic manifold (X, ω) and sufficiently large $m \in \mathbb{Z}$, there is a hyperpencil on X for which the canonical isotopy class of symplectic forms contains $m\omega$.*

Now note that the nondegeneracy condition for symplectic forms is open, and $H^2(X; \mathbb{Q})$ is dense in $H_{dR}^2(X)$. Thus, rational symplectic forms on a manifold X are dense in the space of all symplectic forms. Furthermore, any rational cohomology class can be rescaled to an integral one. Thus, up to scale, the hyperpencil construction should give a dense subset of *all* symplectic forms on any given manifold. That is:

Proposition 3.4. *Let $\mathcal{P}(X)$ be the set of deformation classes of hyperpencils on a manifold X , $\mathcal{S}(X)$ be the set of isotopy classes of rational symplectic structures, and $\Omega : \mathcal{P}(X) \rightarrow \mathcal{S}(X)$ be the map given by Theorem 3.1. Suppose that all integral symplectic structures on X satisfy Conjecture 3.3. Then the induced map $\bar{\Omega} : \mathcal{P}(X) \rightarrow \mathcal{S}(X)/\mathbb{Q}_+$ is surjective. Equivalently, there is a surjection $\tilde{\Omega} : \mathcal{P}(X) \times \mathbb{Q}_+ \rightarrow \mathcal{S}(X)$, where $\tilde{\Omega}(f, q)$ is obtained from $\Omega(f)$ by rescaling so that $[\tilde{\Omega}(f, q)]$ is q times a primitive integral class. \square*

Corollary 3.5. *In dimensions where Conjecture 3.3 holds (e.g. dimensions ≤ 6), a manifold admits a symplectic structure if and only if it admits a hyperpencil. A 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil with $B \neq \emptyset$. \square*

(For the 4-dimensional version, see also [GS].) Thus, Conjecture 3.3 topologically characterizes those manifolds admitting symplectic structures. From there, to completely determine, in topological terms, the dense subset $\mathcal{S}(X)$ of the space of symplectic forms on X , we only need to identify the point preimages of $\tilde{\Omega}$ (which are the same as for $\bar{\Omega}$).

Conjecture 3.6. *The point preimages of $\tilde{\Omega}$ (or equivalently, $\bar{\Omega}$) can be given by a topologically defined equivalence relation on $\mathcal{P}(X)$.*

To do this, it may be useful to strengthen the definition of hyperpencils. The main evidence for this conjecture is that the theorems of Donaldson and Auroux come with uniqueness statements, up to “stabilization,” or multiplying m by large integers (taking tensor powers of the relevant line bundle). This suggests that one should be able to topologically define stabilization maps $\sigma_k : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $k \in \mathbb{Z}_+$, with $\sigma_1 = \text{id}_{\mathcal{P}(X)}$, $\sigma_k \circ \sigma_\ell = \sigma_{k\ell}$ and $\Omega \circ \sigma_k = k\Omega$, and that the required equivalence relation should be given by $f \sim g$ if and only if $\sigma_k(f) = \sigma_\ell(g)$ for some $k, \ell \in \mathbb{Z}_+$. However, these stabilization maps seem complicated, even in dimension 4.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, USA
 E-mail address: gompf@math.utexas.edu

On some classes of 8-dimensional manifolds.

Philippe Mazaud,

Western Topology Seminar, Colorado College, June 2000.

Summary of the talk:

We consider closed smooth manifolds M that accept an effective codimension-two action of the group $S^3 \times S^3$ (S^3 : unit-quaternions). The basic philosophy is to construct nice classes of spaces that “fiber” over a surface.

This study involves two distinct problems: that of an *equivariant* classification (determining the invariants that characterize the action, up to equivalence), and the question of the *topological* classification and characterization of these spaces (we may have distinct actions of $S^3 \times S^3$ on the same space). The first is completely solved (cf. [Maz1]): roughly, the invariants consist of (1) the orbit data, that is, the quotient space M^* (a 2-manifold or orbifold, possibly with boundary) along with a specified isotropy structure (each point in M^* has a “weight” assigned, given as a conjugation class of stabilizers for the corresponding orbit), and (2) an invariant that represents the obstruction to a “uniform” section to the action.

The second problem is the one we hoped to emphasize. Much remains to be done here. As a first step in this direction one decomposes these spaces into various manifold types, that are in some sense irreducible. These types are actually broad families of *equivariant* data, but one can show that they are in fact largely distinct topologically.

We single out two types here. The first consists of so-called Seifert-like manifolds. Equivariantly, all $S^3 \times S^3$ orbits are fully 6-dimensional, but a finite number have finite-cyclic isotropy; the quotient space is a closed orbifold. As it turns out, most of these spaces are indeed Seifert fiberings in the sense of Conner-Lee-Raymond (see [LLR] for instance): quotients of principal $S^3 \times S^3$ bundles over S^2 or \mathbb{R}^2 or \mathbb{H}^2 (the hyperbolic plane). In fact they may also be viewed as 4-manifold bundles over $S^2 \times S^2$, where the fiber is itself again a Seifert fibering modeled on principal T^2 bundles (these 4-dimensional Seifert fiberings are completely classified in [OR2]).

The other type is a family of simply-connected manifolds (denoted \mathcal{H}). Equivariantly the quotient \mathcal{H}^* is a disk; its interior consists of free orbits, its boundary is partitioned into finitely many vertices and edges. The vertices correspond to isolated T^2 -stabilized orbits (T^2 =maximal 2-torus in $S^3 \times S^3$), the edges’ interiors consist of S^1 -stabilized orbits, where the circles S^1 must sit in T^2 in certain prescribed ways. These manifolds may also be viewed as 4-manifold bundles over $S^2 \times S^2$ with toral structure group. $S^3 \times S^3$ acts on \mathcal{H} by bundle maps. Now, the homeomorphism type of the fiber can be read directly from the $S^3 \times S^3$ equivariant data. The topological question in this case has to do with whether distinct bundles (same fiber and base, different action of the T^2 structure group) are homeomorphically distinct (see [Maz2]).

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FIBERED TRANSVERSE LINKS AND FOUR-DIMENSIONAL SYMPLECTIC CONSTRUCTIONS

DAVID T. GAY

The results presented here are presented in detail with background and proof in [3]. For further background on symplectic and contact geometry the reader is referred to [2], [5] and [4].

An alternate title for this talk might have been “Trying to understand 4-dimensional symplectic bordisms”. Consider the following question: In a category where the morphisms are 4-dimensional symplectic bordisms, what should the objects be? A general answer would be that the objects should be pairs (M, \mathcal{G}) where M is an oriented 3-manifold and \mathcal{G} is a *germ* of a 4-dimensional symplectic neighborhood of M . By such a germ we mean an equivalence class of symplectic 4-manifolds containing M as a submanifold, where (X_1, ω_1) is equivalent to (X_2, ω_2) if there exist neighborhoods N_i of M in X_i and a symplectomorphism from N_1 to N_2 which restricts to the identity on M .

To understand such pairs (M, \mathcal{G}) we need to understand 3-dimensional data on M which encodes the 4-dimensional germ \mathcal{G} . There is a standard way to do this using *contact geometry*. We recall some basic definitions. If M is an oriented 3-manifold then:

1. a 1-form α on M is a *contact form* if $\alpha \wedge d\alpha$ is nowhere zero, and is said to be positive if $\alpha \wedge d\alpha > 0$ and negative if $\alpha \wedge d\alpha < 0$, and
2. a plane-field ξ on M is a (positive/negative) *contact structure* if $\xi = \ker \alpha$ for some (positive/negative) contact form α on M .

(Note that, if α is a contact form and f is any nonzero function on M , then $\ker \alpha = \ker f\alpha$, so that a contact structure can be thought of as a conformal class of contact forms.) Given a symplectic 4-manifold (X, ω) and a vector field V , we say that V is a *symplectic dilation* (resp. *contraction*) if $\mathcal{L}_V \omega = \omega$ (resp. if $\mathcal{L}_V \omega = -\omega$). This provides a connection with contact structures as follows: If M is an oriented 3-dimensional submanifold of (X, ω) and V is positively transverse to M , consider the 1-form $\alpha = \iota_V \omega|_M$. Then:

1. if V is a symplectic dilation then α is a positive contact form with $d\alpha = \omega|_M$ and
2. if V is a symplectic contraction then α is a negative contact form with $-d\alpha = \omega|_M$.

Now we can see the connection between contact forms and symplectic germs. A contact form α on an oriented 3-manifold M uniquely determines a germ $\mathcal{G}(\alpha)$ of a 4-dimensional symplectic neighborhood of M in the following sense: Firstly, there exists a symplectic 4-manifold (X, ω) containing M with a symplectic dilation (if α is positive) or contraction (if α is negative) V which is positively transverse to M such that $\iota_V \omega|_M = \alpha$. Secondly, any other symplectic 4-manifold containing M with

such a dilation or contraction represents the same germ along M . The standard example is $X = \mathbb{R} \times M$, $\omega = \pm d(e^{\pm t}\alpha)$ and $V = \partial_t$ (where \pm is $+$ or $-$ according to whether α is positive or negative). In particular, using this construction, we can say that, given (M, α) and a positive function f on M , there exists a symplectic bordism, topologically a product, from $(M, \mathcal{G}(\alpha))$ to $(M, \mathcal{G}(e^{\pm f}\alpha))$: the bordism is simply $X = \{(t, p) | 0 \leq t \leq f(p)\} \subset \mathbb{R} \times M$ with $\omega = \pm d(e^{\pm t}\alpha)$.

However, to get more topologically interesting bordisms we need to understand how to attach handles symplectically along knots in a contact 3-manifold (M, α) . There is a standard result due to Eliashberg [1] and Weinstein [6] which says that, if $K \subset M$ is a Legendrian knot (this means that $TK \subset \xi = \ker \alpha$), then we can attach a symplectic 2-handle along K provided we use the canonical framing, the framing determined by a vector field along K lying in ξ and transverse to K . This construction gives us a bordism from $(M, \mathcal{G}(\alpha))$ to $(M_K, \mathcal{G}(\alpha_K))$, where (M_K, α_K) is a contact 3-manifold resulting from surgery along K .

There is a basic limitation to these constructions, which is that the symplectic forms constructed are always exact. To get beyond this limitation we now generalize the constructions above. Suppose M is an oriented 3-manifold. Then a *contact pair* is a pair (α^+, α^-) of 1-forms defined on open subsets $M^\pm \subset M$ with $M = M^+ \cup M^-$, $\pm \alpha^\pm \wedge d\alpha^\pm > 0$ and $d\alpha^+ = -d\alpha^-$ on $M^0 = M^+ \cap M^-$. A *dilation-contraction pair* on a symplectic 4-manifold (X, ω) is a pair (V^+, V^-) of vector fields, V^+ a symplectic dilation and V^- a symplectic contraction, defined on open subsets $X^\pm \subset X$, with $\omega(V^+, V^-) = 0$. Now note that, if $M \subset X^+ \cup X^-$ with both V^+ and V^- positively transverse to M , then $\alpha^\pm = \iota_{V^\pm} \omega|_M$ defines a contact pair (α^+, α^-) on M . Using this idea we can state the following proposition:

Proposition 0.1. *A contact pair (α^+, α^-) uniquely determines a symplectic germ $\mathcal{G}(\alpha^+, \alpha^-)$ in the sense that:*

1. *there exists a symplectic 4-manifold (X, ω) with a dilation-contraction pair (V^+, V^-) containing M such that $\alpha^\pm = \iota_{V^\pm} \omega|_M$ and,*
2. *any other such (X, ω, V^+, V^-) represents the same germ along M .*

Now we will see that contact pairs have a natural connection to fibered links. Given a 3-manifold M with a contact pair (α^+, α^-) , notice the following: $\gamma = \pm d\alpha^\pm$ is a globally defined, closed, nondegenerate 2-form on M (not necessarily exact) and $\alpha^0 = \alpha^+ + \alpha^-$ is a closed 1-form on $M^0 = M^+ \cap M^-$ with $\alpha^0 \wedge \gamma > 0$. This suggests that we specialize to the case where $\alpha^0 = kdp$ for some constant k and some fibration $p: M^0 \rightarrow S^1$. This in turn suggests that we consider the case where $M \setminus M^0 = L$ is a link (and thus a fibered link), with $L = L^+ \cup L^-$ and $L^\pm = M^\pm \setminus M^0$.

Theorem 0.2. *In this case, if $K \subset L^\pm$ and (α^+, α^-) is well-behaved near K , then one can attach a symplectic 2-handle along K (with certain restrictions on the framing) to get a symplectic bordism from $(M, \mathcal{G}(\alpha^+, \alpha^-))$ to $(M_K, \mathcal{G}(\alpha_K^+, \alpha_K^-))$, where $(M_K, (\alpha_K^+, \alpha_K^-))$ is a 3-manifold with contact pair resulting from surgery along K . The intersection of the co-core of the handle with M_K is a knot in M_K^\mp , which is to say that if the handle was attached along a positive component of L , then the co-core will lie in M_K^- and that if the handle was attached along a negative component then the co-core will lie in M_K^+ .*

The precise definitions of the “well-behaved” condition and the framing restrictions are spelled out carefully in [3]. Here we will present the essential idea by means of an example.

Suppose that Σ is a compact surface with non-empty boundary. Then there exists a 1-form β on Σ such that $d\beta > 0$ and there exists a vector field V on Σ such that $\iota_V d\beta = \beta$ and such that V is transverse to $\partial\Sigma$. Furthermore β and V can be constructed to have particular controlled behavior near $\partial\Sigma$. First let $\partial\Sigma = \partial_+\Sigma \cup \partial_-\Sigma$, where $\partial_+\Sigma$ is the union of components of $\partial\Sigma$ on which V points out of Σ , while $\partial_-\Sigma$ is the union of components on which V points in to Σ . Then we can arrange that a collar on each component of $\partial\Sigma$ has the model $[a, b] \times S^1$, with corresponding coordinates (x, y) , such that:

- ∂_x points out of Σ ,
- $\beta = xdy$,
- $V = x\partial_x$, and
- $b > 0$ on $\partial_+\Sigma$ while $b < 0$ on $\partial_-\Sigma$.

Let Σ' denote the interior of Σ and let $M^0 = S^1 \times \Sigma'$ (we will use t as the coordinate for this S^1 factor). We will construct M as a Dehn-filling of M^0 . Choose two positive constants k^+ and k^- and let $\alpha^\pm = k^\pm dt \pm \beta$.

Now one can draw a picture looking in at $S^1 \times \partial_+\Sigma = S^1 \times \{b\} \times S^1$. Both $\xi^+ = \ker \alpha^+$ and $\xi^- = \ker \alpha^-$ will contain the vector field ∂_x , so that we can see what these contact structures look like just by looking at their slopes in the (y, t) -plane. For ξ^+ we compute that $dt/dy = -x/k^+$, a decreasing negative slope near $x = b$ limiting on $-b/k^+$, and for ξ^- we compute that $dt/dy = x/k^-$, an increasing positive slope near $x = b$ limiting on b/k^- . Similarly one can draw a picture looking in at $S^1 \times \partial_-\Sigma = S^1 \times \{b\} \times S^1$. Here, since $b < 0$, we get that ξ^+ has a decreasing positive (y, t) -slope near $x = b$, again limiting on $-b/k^+$, and that ξ^- has an increasing negative (y, t) -slope near $x = b$ limiting on b/k^- .

To construct M from M^0 we simply fill in the ends with (y, t) -slopes that match the slope of ξ^+ at the positive ends ($S^1 \times \partial_+\Sigma$) and the slope of ξ^- at the negative ends ($S^1 \times \partial_-\Sigma$). Then α^+ will extend across the fillings of the positive ends while α^- will extend across the fillings of the negative ends. Of course, the constants k^\pm and the values of b at each end should be chosen so that these slopes are rational.

Then we have a 3-manifold M with a contact pair (α^+, α^-) such that M^0 fibers over S^1 with fibration t , such that $\alpha^0 = (k^+ + k^-)dt$, and such that $M \setminus M^0$ is a fibered link L . The local model near each component of L is exactly the local model which is guaranteed by the “well-behaved” condition mentioned in the theorem above.

Finally, the framings which are allowed in the theorem can be seen explicitly in this example as follows: A framing translates into a (y, t) -slope in our pictures of the ends of M^0 . At the positive ends, the allowed framings are those that correspond to slopes more positive than the limiting slope b/k^- of ξ^- , while at the negative ends the allowed framings are those that correspond to positive slopes less than the slope $-b/k^+$ of ξ^+ . After the surgery associated to these framings, the contact structure which did not extend across the Dehn filling now does extend and the contact structure that did extend now does not.

This example can be generalized to the case where M^0 is a surface bundle over S^1 , and thus we are working in the world of open-book decompositions of 3-manifolds. The hope is that we can use this to understand the category of 3-manifolds with symplectic germs and 4-dimensional symplectic bordisms through a careful analysis of contact structures in relation to open-book decompositions of 3-manifolds.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, 617 NORTH SANTA RITA, TUCSON, ARIZONA 85721, USA

Current address: Nankai Institute of Mathematics, Nankai University, 94 Weijing Road, Nankai District, Tianjin, 300071, China

E-mail address: dtgay@math.arizona.edu

FAILURE OF SARDS THEOREM AND EXISTENCE OF STRANGE SMOOTH BUMP FUNCTIONS IN INFINITE DIMENSIONS

TADEUSZ DOBROWOLSKI

ABSTRACT. We apply weak bump functions to construct smooth surjections with derivatives of rank-1 and smooth bump functions whose derivatives are surjections in infinite-dimensional Banach spaces

Sard's theorem precludes the existence of a smooth surjection between Euclidean spaces whose derivative is everywhere nonsurjective. This phenomenon is, in general, not true in infinite-dimensions, as shown in [Ba] (see also references therein). We elaborate on the technique of constructing smooth surjections presented in [Ba], and we put this technique in the language of weak bump functions. We apply this same technique to construct certain peculiar smooth bump functions and to obtain an extension of a result from [AD].

1. Weak bump functions. Let $(X, \|\cdot\|)$ be a normed linear space and $B = \{x \in X \mid \|x\| \leq 1\}$ be its unit closed ball. Let us start with the following property that was isolated in [Ba]:

Definition 1. We say that a separating sequence of continuous functionals $(x_n^*) \subset X^*$ satisfies (B) if there exists a sequence $(x_n) \subset B$ such that $x_n^*(x_m) = \delta_{nm}$ and such that, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such for every $[k] = \{n_1 < n_2 < \dots < n_k\}$ we have $\|\pi^{[k]}Tx\|_\infty < \varepsilon$ for every $x \in B$, where $T : X \rightarrow \mathbb{R}^\infty$ is the operator given by $T(x) = (x_n^*(x))$, $\pi_{[k]}(y) = (y_{n_1}, y_{n_2}, \dots, y_{n_k})$ for $y = (y_n) \in \mathbb{R}^\infty$, and $\pi^{[k]}(y) = y - \pi_{[k]}(y)$.

Clearly, the operator T maps X into the space c_0 , and the condition spelled in the above definition is much stronger than TB is just a subset of the unit ball of c_0 . Recall that a function $\varphi : X \rightarrow [0, 1]$ is called a bump function if $\varphi^{-1}((0, 1])$ is a nonempty bounded set; to be precise such a function φ will be called a $\|\cdot\|$ -bump function. (For our considerations bump functions can be "identified" with functions $\psi : X \rightarrow \mathbb{R}$ with $X \setminus \psi^{-1}(0)$ nonempty and bounded.) Let b be a standard C^∞ bump function on c_0 , that is, $b(y) = \prod_{n=1}^\infty \lambda(y_n)$ for $y = (y_n) \in c_0$, where $\lambda : \mathbb{R} \rightarrow [0, 1]$ be a fixed C^∞ function such that $\lambda = 1$ on $[-1, 1]$ and $\lambda = 0$ when $|r| \geq 2$. It follows that b has support in the ball of radius 2 and $b = 1$ on the unit ball.

Proposition. *There exist a norm ω on X with $\omega(x) \leq \|x\|$ and a C^∞ ω -bump function $\varphi : (X, \omega) \rightarrow [0, 1]$ with the following properties:*

- (1) $\varphi(x) = 1$ if $\omega(x) \leq 1$ and $\varphi(x) = 0$ if $\omega(x) \geq 2$;
- (2) for every $n, m \in \mathbb{N}$ there exists $M_{nm} < \infty$ such that $\|\varphi^{(i)}(x)\| \leq M_{nm}$ for all $\|x\| \leq n$ and all $0 \leq i \leq m$;
- (3) B is not relatively ω -compact.

Proof. Define $\omega(x) = \|Tx\|_\infty$. Since $\omega(x_n - x_m) = 1$ for $n \neq m$ (for Tx_n is the standard unit vector of c_0), the ball B is not relatively ω -compact. Let $\varphi(x) = b(Tx)$, where b is the above bump function on c_0 . The estimate (2) is an easy consequence of the condition on T from Definition 1. \square

Notice that an arbitrary C^∞ ω -bump function can easily be modified in order to satisfy (1) and, at the same time, to maintain condition (2).

Definition 2. A norm ω will be called a (B) norm if, together with some C^∞ ω -bump function φ , the conditions (1)-(3) are satisfied. The function φ will be referred to as a **weak bump function**.

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Remark 1. Assume that there exists a C^∞ bump function $b : (\hat{X}, \omega) \rightarrow [0, 1]$, where (\hat{X}, ω) stands for the ω -completion of X . Then condition (2) (for $\varphi(x) = b(x)$, $x \in X$) is an easy consequence of the following stronger one:

(2') for every $m \in \mathbb{N}$ there exists $M_m < \infty$ such that $\omega(b^{(i)}(y)) \leq M_m$ for all y and $0 \leq i \leq m$.

Example 1. Let $X = \ell^p$, $p \geq 1$. Then, for every $n \in \mathbb{N}$ with $2n > p$, $\omega(x) = \omega_{2n}^p(x) = \|x\|_{2n}$, $x \in \ell^p$, is a (B) norm.

To see this it suffices to take $b(y) = \lambda(\|y\|_{2n}^{2n})$, where λ is the function described above the statement of Proposition. It is clear that, for the sequence of the standard unit vectors (e_n) , we have $\omega_{2n}^p(e_n - e_m) = 1$ for $n \neq m$.

2. An application to surjective maps. The following result is an abstraction on the technique described in [Ba]:

Theorem 1. Suppose that a Banach space $(X, \|\cdot\|)$ admits a (B) norm ω . Then for every separable Banach space Y there exists a C^∞ surjection $g : X \rightarrow Y$. In case (2') is satisfied, g can be viewed as a C^∞ surjection of (X, ω) onto Y . Furthermore, for every $x \in X$, the derivative $g'(x)$ is a rank-1 operator of X into Y .

Observe that it would be enough to construct a C^∞ surjection onto the space ℓ^1 because an arbitrary Banach space Y is an image of ℓ^1 under a continuous linear operator.

Proof. We will be working with the normed space (X, ω) ; we will use the symbol B_ω to denote the closed ball of this space. Since B is not relatively ω -compact there exists $\varepsilon > 0$, $0 < \varepsilon < \frac{1}{32}$, and a sequence $(z_n) \subset B$ such that $B_\omega(z_n, 4\varepsilon) \subset B_\omega$ and $\omega(z_n - z_m) \geq 8\varepsilon$ for $n \neq m$. Define $T_n : X \rightarrow X$ by $T_n x = z_n + \varepsilon x$. We have $T_n B_\omega = B_\omega(z_n, \varepsilon)$. Moreover, for a sequence $(n_1, n_2, n_3, \dots) \in \mathbb{N}^\mathbb{N}$,

$$T_{n_1} B_\omega \supset T_{n_1} T_{n_2} B_\omega \supset T_{n_1} T_{n_2} T_{n_3} B_\omega \supset \dots$$

is a nested sequence of ω -balls. Furthermore, we have $T_{n_1} T_{n_2} \dots T_{n_k} x = z_{n_1} + \varepsilon z_{n_2} + \dots + \varepsilon^{k-1} z_{n_k} + \varepsilon^k x$. It follows that the intersection $\bigcap_{k=1}^\infty T_{n_1} T_{n_2} \dots T_{n_k} B_\omega$ is nonempty in the completion (\hat{X}, ω) , and is precisely a one-point set, say $\{p\}$. However, since $\|z_n\| < 1$ and $p = \sum_{k=1}^\infty \varepsilon^{k-1} z_{n_k}$, we infer that $p \in X$ and $\|p\| \leq 1$. For $(n_1, n_2, \dots, n_k) \in \mathbb{N}^k$, define $\varphi_{n_1 n_2 \dots n_k} \varphi(T_{n_1} T_{n_2} \dots T_{n_k})^{-1}$, where φ is an ω -bump function of Definition 2. More precisely,

$$\varphi_{n_1 n_2 \dots n_k}(x) = \varphi\left(\frac{x - \sum_{j=1}^{k-1} \varepsilon^{j-1} z_{n_j}}{\varepsilon^k}\right).$$

For the i th derivative we have $\varphi_{n_1 n_2 \dots n_k}^{(i)}(x) \varepsilon^{-ki} \varphi^{(i)}\left(\frac{x - \sum_{j=1}^{k-1} \varepsilon^{j-1} z_{n_j}}{\varepsilon^k}\right)$. An application of (2) yields that there exists $1 \leq M_k < \infty$ such that

$$\|\varphi_{n_1 n_2 \dots n_k}^{(i)}(x)\| \leq \varepsilon^{-k^2} M_k$$

for every $\|x\| \leq k$ and $0 \leq i \leq k$. Pick a sequence $\{y_n\} = D$ that is dense in the unit ball of Y . Define $f_k : X \rightarrow Y$ by letting $f_k(x) = \sum_{(n_1, n_2, \dots, n_k) \in \mathbb{N}^k} \delta_k \varphi_{n_1 n_2 \dots n_k}(x) y_{n_k}$, where $\delta_k \leq 2^{-k} \varepsilon^{k^2} M_k^{-1}$ and $\delta_1 = 1$. Since the sets $T_{n_1} T_{n_2} \dots T_{n_k} (4B_\omega)$, $(n_1, n_2, \dots, n_k) \in \mathbb{N}^k$, are pairwise disjoint and the support of φ is contained in $2B_\omega$, each f_k is C^∞ . Moreover, for every $x \in T_{n_1} T_{n_2} \dots T_{n_k} (4B_\omega)$ and every $0 \leq i \leq k$, we have $f_k^{(i)}(x) = \delta_k \varphi_{n_1 n_2 \dots n_k}^{(i)}(x) y_{n_k}$. This implies

$$\|f_k^{(i)}(x)\| \leq 2^{-k} \text{ for } \|x\| \leq k$$

and every $0 \leq i \leq k$. Letting $f = \sum_{k=1}^\infty f_k$, it follows that $f : X \rightarrow Y$ is a C^∞ map. Moreover, f has support in B_ω , that is, $f(x) = 0$ for $\omega(x) \geq 1$. Our construction also assures that, for every $l \in \mathbb{N}$ and every $x \in X$, $\sum_{k=1}^l f_k'(x)$ is a rank-1 operator. It follows that $f'(x)$ is a rank-1 operator as well. Now we show that the image of f contains the unit ball of Y . For $x \in \partial T_{n_1} T_{n_2} \dots T_{n_k} B_\omega$, we have

$$f(x) = y_{n_1} + \delta_2 y_{n_2} + \dots + \delta_k y_{n_k}.$$

Fix $y \in Y$ with $\|y\| \leq 1$. Pick $y_{n_1} \in D$ so that

$$\|y - y_{n_1}\| \leq \delta_2,$$

next pick $y_{n_2} \in D$ so that

$$\|y - y_{n_1} - \delta_2 y_{n_2}\| \leq \delta_3$$

and so on; finally, pick $y_{n_k} \in D$ so that

$$\|y - y_{n_1} - \delta_2 y_{n_2} - \cdots - \delta_k y_{n_k}\| \leq \delta_{k+1}.$$

Now, for a so chosen sequence $(n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ and $x \in \partial T_{n_1} T_{n_2} \dots T_{n_k} B_\omega$,

$$\|y - f(x)\| = \|y - (y_{n_1} + \delta_2 y_{n_2} + \cdots \delta_k y_{n_k})\| \leq \delta_{k+1}.$$

Thus, for $p = \sum_{k=1}^{\infty} \varepsilon^{k-1} z_{n_k} \in \bigcap_{k=1}^{\infty} T_{n_1} T_{n_2} \dots T_{n_k} B_\omega$, we have $f(p) = y$. Since we have already observed that $p \in X$, the image of f contains the unit ball of Y . Finally, fixing any $z \in X$ with $\omega(z) = 1$, we let

$$g(x) = \sum_{n=1}^{\infty} n f\left(\frac{x - 4^n z}{n}\right).$$

Clearly, $x \rightarrow f(\frac{x - 4^n z}{n})$ have pairwise disjoint supports so that g is a C^∞ map. It is easy to see that $g(X) = Y$ and that $g'(x)$ is a rank-1 operator for every $x \in X$. In case condition (2') is satisfied $f : (X, \omega) \rightarrow Y$ is a C^∞ map; consequently, $g : (X, \omega) \rightarrow Y$ will be a required C^∞ surjection. \square

As observed in [Ba] every superreflexive Banach space admits a sequence requested in Definition 1; hence, superreflexive spaces admit (B) norms. By an application of Theorem 1, we obtain:

Corollary 1. *The assertion of Theorem 1 holds for an arbitrary separable Banach space Z that admits a noncompact continuous operator into a separable superreflexive Banach space X . In particular, this is true if $Z = X_W$; here W is a noncompact bounded balanced convex subset of X and X_W is the completion of the space whose norm is given by the Minkowski functional of W .*

Since the 'identity' operator of $C[0, 1]$ into $L^2[0, 1]$ is noncompact, Theorem 1 holds for $C[0, 1]$. Combining our Theorem 1 and Remark 1, we obtain:

Corollary 2. *For every $p \geq 1$ and $n \in \mathbb{N}$ with $2n > p$, the incomplete normed space (ℓ^p, ω_{2n}^p) admits a C^∞ surjection g onto every separable Banach space Y so that $g'(x)$ is a rank-1 operator for every x . \square*

Remark 2. The construction used in Theorem 1 provides a "uniform" closed copy A of the irrationals \mathbb{P} in (X, ω) . Since $g(A) = Y$, we see that in infinite dimensions it is possible to map \mathbb{P} , a zero-dimensional space, onto Y in a " C^∞ " way.

We wonder whether this pathology would remain true if we wanted to map \mathbb{P} onto Y in a "real-analytic" way. In general, real-analytic (polynomial) surjections exist in infinite dimensions:

Example 2. The map $(x_n) \rightarrow (x_n^3)$ is a polynomial sending ℓ^3 onto ℓ^1 .

Remark 3. If, in Definition 2, we replace " C^∞ " by " C^p ", $p \geq 1$, then the assertion of Theorem 1 holds true with C^∞ replaced by C^p .

Condition (2) is crucial for the technique described in Theorem 1 to work. The existence of a C^p bump function on X does not, in general, imply that such a function has the p th, or even the first, derivative bounded. (This can be equivalently phrased as follows: if φ is a C^1 bump function then φ' does not necessarily give rise to a bump function. If $\varphi'(X)$ is bounded in X^* then, for every $z \in X \setminus \bigcap_{x \in X} \ker \varphi'(x)$, the function $x \rightarrow \varphi'(x)(z)$ can easily be modified to a bump function on X .) The case of C^1 weak bump functions is simple. Namely, by Joseffson-Nissenzweig theorem, every infinite-dimensional separable Banach space X admits an injective noncompact continuous linear operator T in c_0 . On the other hand, c_0 admits a C^1 norm $\|\cdot\|$ (even, a C^∞ norm). Letting $\omega(x) = \|Tx\|$, one can easily see that a C^1 version of (2') holds. We can recover another main result of [Ba]:

Corollary 3. *Every infinite-dimensional separable Banach space X admits a C^1 (Lipschitz) surjection g onto an arbitrary separable Banach space Y for which $g'(x)$ is a rank-1 operator. \square*

On the other hand, no C^p bump function, $2 \leq p \leq \infty$, on c_0 has its second derivative bounded and the technique of Theorem 1 fails. Let us restate (after [Ba]) an intriguing problem:

Question. Is there a surjective C^p , $2 \leq p \leq \infty$ map of c_0 onto ℓ^2 ?

3. Strange bump functions. For a Banach space X , by $L_s^k(X)$ we denote the space of k -linear symmetric continuous operators of X^k into \mathbb{R} (hence, $L_s^1(X) = X^*$). Clearly, if X is finite-dimensional then for any smooth bump function b , the set $b^{(k)}(X)$ is a bounded subset of $L_s^k(X)$. As shown in [AD], this may dramatically change for some infinite-dimensional spaces X to the effect that b' may be surjective. We extend that result as follows:

Theorem 2. *Suppose that a separable Banach space X admits a (B) norm ω w/r to a suitable C^p weak bump function φ , $1 \leq p \leq \infty$. Then there exists another C^p weak bump function ψ on X such that $\psi'(X) = L_s^k(X)$ for every $1 \leq k \leq p$.*

Proof. For simplicity we only consider the case of $p = 2$; we will follow the lines of the proof of Theorem 1. Partition \mathbb{N} into two countable sets N_1 and N_2 . Hence, $(z_n) = (z_s)_{s \in N_1} \cup (z_t)_{t \in N_2}$. For sequences $(s_1, s_2, s_3, \dots) \in N_1^{\mathbb{N}}$ and $(t_1, t_2, t_3, \dots) \in N_2^{\mathbb{N}}$, we have $\bigcap_{k=1}^{\infty} T_{s_1} T_{s_2} \dots T_{s_k} B_{\omega} = \{p\}$ and $\bigcap_{k=1}^{\infty} T_{t_1} T_{t_2} \dots T_{t_k} B_{\omega} = \{q\}$, where $p = \sum_{k=1}^{\infty} \varepsilon^{k-1} z_{s_k} \in X$ and $q = \sum_{k=1}^{\infty} \varepsilon^{k-1} z_{t_k} \in X$, and $\|p\|, \|q\| \leq 1$. Define also $\varphi_{s_1 s_2 \dots s_k}$ and $\varphi_{t_1 t_2 \dots t_k}$ as in the proof of Theorem 1. An application of (2) yields that there exists $1 \leq M_k < \infty$ such that

$$\|\varphi_{s_1 s_2 \dots s_k}^{(i)}(x)\| \leq \varepsilon^{-2k} M_k \quad \text{and} \quad \|\varphi_{t_1 t_2 \dots t_k}^{(i)}(x)\| \leq \varepsilon^{-2k} M_k$$

for every $\|x\| \leq k$ and $0 \leq i \leq 2$. Pick sequences (x_s^*) and (l_t) in the unit balls of X^* and $L_s^2(X)$, respectively. Choose δ_k so that $0 < \delta_k \leq 2^{-k} \varepsilon^{-2k} M_k^{-1}$, $\delta_1 = 1$ and define

$$f_k^1(x) = \sum_{(s_1, s_2, \dots, s_k) \in N_1^k} \delta_k \varphi_{s_1 s_2 \dots s_k}(x) (x_{s_k}^*(x) + 1)$$

and

$$f_k^2(x) = \sum_{(t_1, t_2, \dots, t_k) \in N_2^k} \delta_k \varphi_{t_1 t_2 \dots t_k}(x) \frac{l_{t_k}(x, x) + 1}{2}.$$

It easily follows that $f = \sum_{i=1}^2 \sum_{k=1}^{\infty} f_k^i$ is a C^2 function supported by B_{ω} . For every $x \in \partial T_{s_1} T_{s_2} \dots T_{s_k} B_{\omega}$ and $x' \in \partial T_{t_1} T_{t_2} \dots T_{t_k} B_{\omega}$, we have $f'(x) = x_{s_1}^* + \delta_2 x_{s_2}^* + \dots \delta_k x_{s_k}^*$ and $f''(x') = l_{t_1} + \delta_2 l_{t_2} + \dots \delta_k l_{t_k}$. Now, for fixed x^* and l of the unit balls of X^* and $L_s^2(X)$, respectively, we find sequences $(s_1, s_2, s_3, \dots) \in N_1^{\mathbb{N}}$ and $(t_1, t_2, t_3, \dots) \in N_2^{\mathbb{N}}$, and $(x_{s_k}^*)$ and (l_{t_k}) subsequences of (x_s^*) and (l_t) , respectively, so that $\sum_{k=1}^{\infty} \delta_k x_{s_k}^* = x^*$ and $\sum_{k=1}^{\infty} \delta_k l_{t_k} = l$. It follows that for $p = \sum_{k=1}^{\infty} \varepsilon^{k-1} z_{s_k}$ and $q = \sum_{k=1}^{\infty} \varepsilon^{k-1} z_{t_k}$, we have $f'(p) = x^*$ and $f''(q) = l$. The function f is a weak bump function such that $f'(X)$ contains the unit ball in X^* and $f''(X)$ contains the unit ball in $L_s^2(X)$. Now, it suffices to define $\psi(x) = \sum_{n=1}^{\infty} n f(\frac{x - z_n}{\varepsilon})$. \square

We claim that the space c_0 does not admit a C^2 ω -bump function. If it did then, by Theorem 2, there would exist a C^2 function ψ on c_0 with $\psi'(c_0) = \ell^1$. This is however impossible because, by a result of [Ha], for any C^2 function μ on c_0 , μ' locally is a compact subset of ℓ^1 ; hence $\psi'(X)$ cannot be ℓ^1 . Here are a couple of observations in case $m = 1$. Firstly, a similar argument that has justified Corollary 3 yields:

Corollary 4. *Every infinite-dimensional separable Banach space X admits a C^1 weak bump function ψ with $\psi'(X) = X^*$. \square*

Secondly, the following fact strengthens a result of [AD]).

Corollary 5. *If an infinite-dimensional separable Banach space X admits a C^1 bump function then it also admits a bump function ψ for which $\psi'(X) = X^*$.*

Proof. Let ω be a C^∞ weak (B) norm on X used in Corollary 3. Since X admits a C^1 bump function, by a result of [DGZ, II, Proposition 5.1] there exists a functional $\mu : X \rightarrow [0, \infty)$ that is C^1 away from the origin and satisfies $\mu(tx) = t\mu(x)$ for all $t \geq 0$ and $x \in X$, and $\|x\| \leq \mu(x)$ for all $x \in X$. Then $A_\mu = \{x \in X | \mu(x) \leq 1\}$ is starlike body that is contained in the unit ball of X . Let H be a (radial) C^1 selfdiffeomorphism of X that sends A_μ onto the unit ω -ball. With the help of H we will modify the construction of f_k^1 (now denoted by f_k) from the proof of Theorem 2. Formally, $f_k(x) = \sum_{(s_1, s_2, \dots, s_k) \in N_1^k} \delta_k \bar{\varphi}_{s_1, s_2, \dots, s_k}(x) (x_{s_k}^*(x) + 1)$, where $\bar{\varphi}_{s_1, s_2, \dots, s_k}(x) = \varphi_{s_1, s_2, \dots, s_k}(H(x))$. We have $\|\bar{\varphi}'_{s_1, s_2, \dots, s_k}(x)\| \leq \varepsilon^{-2k} M_k \|H'(x)\|$ for every x with $\|H(x)\| \leq k$. It follows that locally $f = \sum_{k=1}^\infty f_k$ is a C^1 function. That same argument used in the proof of Theorem 2 works to show that for a given $x^* \in X^*$, $\|x^*\| \leq 1$, $f'(H^{-1}(p)) = x^*$. It is also clear that the support of f is contained in the unit ball of X . Finally, as in the proof of Theorem 2, we let $\psi(x) = \sum_{n=1}^\infty n f(\frac{x - z_n}{\varepsilon})$. \square

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DEPARTMENT OF MATHEMATICS, PITTSBURG STATE UNIVERSITY, PITTSBURG, KS 66762
E-mail address: tdobrowo@mail.pittstate.edu

A STRONGER LIMIT THEOREM IN EXTENSION THEORY

LEONARD R. RUBIN

ABSTRACT. This work describes an improvement to a limit theorem which has been proved by the author and P. J. Schapiro. In that result it was shown that for a given simplicial complex K , if an inverse sequence of metrizable spaces X_i each has the property that $X_i\tau|K|$, then it is true that $X\tau|K|$, where X is the limit of the sequence. The property that $X\tau|K|$ means that for each closed subset A of X and each map $f : A \rightarrow |K|$, there exists a map $F : X \rightarrow |K|$ which is an extension of f . This is the fundamental notion of extension theory.

The version put forth herein is stronger in that it places a requirement only on the bonding maps, but one which is necessarily true in case each $X_i\tau|K|$.

1. Introduction. The notion of extension theory is a generalization of dimension theories such as covering and cohomological dimensions; good sources for extension theory can be found in [DD] and [Sh]. Under the light shed by extension theory, it is frequently possible to obtain theorems which apply to dimension theory, but which are much more general. The limit theorem ([RS]) for inverse sequences of metrizable spaces in extension theory is such an example. We have proved [Ru2] a stronger version of that limit theorem.

Recall that if K is a CW-complex and X is a space, then $X\tau K$ means that for each closed subset A of X and map $f : A \rightarrow K$, there exists a map $F : X \rightarrow K$ which is an extension of F . This is the fundamental notion of extension theory.

For information about inverse sequences and their limits, one may consult [Du]. When K below is a simplicial complex, then $|K|$ will be given the weak topology determined by K .

The result in [RS], Theorem 3.1, goes this way:

1.1. Theorem. *Let K be a simplicial complex and $X = \lim \mathbf{X}$, where $\mathbf{X} = (X_i, p_{i+1}, \mathbb{N})$ is an inverse sequence of metrizable spaces X_i and $X_i\tau|K|$ for all $i \in \mathbb{N}$. Then $X\tau|K|$. \square*

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This eclipses the original limit theorem of Nagami for covering dimension \dim , Remark 27.9 of [Na2] (see also [Na1]). We state it here for convenience.

1.2. Proposition. *Let $n \in \mathbb{Z}$ and X be the limit of an inverse sequence of metrizable spaces X_i with $\dim X_i \leq n$ for all i . Then $\dim X \leq n$.*

The reason 1.1 is stronger is that, as is well-known, for a metrizable space X , $\dim X \leq n$ if and only if $X \tau S^n$.

In order to state the improved version, let us first give a definition taken from a notion previously introduced by A. Dranishnikov.

1.3. Definition. *Let $\mathbf{X} = (X_i, p_{i,i+1}, \mathbb{N})$ be an inverse sequence and K be a CW-complex. We shall write that $\mathbf{X} \tau K$ if for each $i \in \mathbb{N}$, closed subset A of X_i , and map $f : A \rightarrow K$, there exists $j \geq i$ and a map $g : X_j \rightarrow K$ such that $g(x) = f \circ p_{i,j}(x)$ for every $x \in p_{i,j}^{-1}(A)$.*

The next lemma is easy to prove.

1.4. Lemma. *Let $\mathbf{X} = (X_i, p_{i,i+1}, \mathbb{N})$ be an inverse sequence and K be a CW-complex. Then $\mathbf{X} \tau K$ if and only if for each $i \in \mathbb{N}$, closed subset A of X_i , and map $f : A \rightarrow K$, there exists $j \geq i$ such that for all $k \geq j$, there is a map $g : X_k \rightarrow K$ such that $g(x) = f \circ p_{i,k}(x)$ for every $x \in p_{i,k}^{-1}(A)$. \square*

We prove the following.

1.5. Theorem. *Let $\mathbf{X} = (X_i, p_{i,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces, K be a CW-complex such that $\mathbf{X} \tau K$, and $X = \lim \mathbf{X}$. Then $X \tau K$.*

Since every CW-complex K is homotopy equivalent to $|K_0|$, for some simplicial complex K_0 , and every CW-complex is an absolute neighborhood extensor for metrizable spaces, then Theorem 1.5 is equivalent to our main result,

1.6. Theorem. *Let $\mathbf{X} = (X_i, p_{i,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces X_i , K be a simplicial complex such that $\mathbf{X} \tau |K|$, and $X = \lim \mathbf{X}$. Then $X \tau |K|$.*

Surely Theorem 1.6 implies Theorem 1.1.

The following is a seemingly weaker theorem.

1.7. Theorem. Let $\mathbf{X} = (X_i, p_{i,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces X_i with surjective bonding maps $p_{i,i+1}$, K be a simplicial complex such that $\mathbf{X} \tau |K|$, and $X = \lim \mathbf{X}$. Then $X \tau |K|$.

But it is not weaker because of the ensuing fact.

1.8. Proposition. Theorem 1.7 implies Theorem 1.6.

Proof. Let \mathbf{X} be an inverse sequence as in Theorem 1.6. We begin a recursive process. Let $X_1^* = X_1$ and put $Y_1 = X_1^* \setminus p_{1,2}(X_2)$. There exists a metrizable space $X_2^* = X_2 \cup Z_2$ where X_2 is an open and closed subspace of X_2^* , and Z_2 is a discrete subspace of X_2^* having the same cardinality as Y_1 .

Define $p_{1,2}^* : X_2^* \rightarrow X_1^*$ so that $p_{1,2}^*|X_2 = p_{1,2}$ and $p_{1,2}^*(Z_2) = Y_1$. Such a procedure may be continued recursively resulting in an inverse sequence $\mathbf{X}^* = (X_i^*, p_{i,i+1}^*, \mathbb{N})$ of metrizable spaces so that for each $i \in \mathbb{N}$,

- (1) $p_{i,i+1}^*$ is surjective,
- (2) X_i is an open and closed subspace of X_i^* ,
- (3) $p_{i,i+1}^*|X_{i+1} = p_{i,i+1} : X_{i+1} \rightarrow X_i$, and,
- (4) $X_i^* \setminus X_i$ is a discrete subspace of X_i^* .

Using (2)–(4), along with the information $\mathbf{X} \tau |K|$, the reader will easily check that $\mathbf{X}^* \tau |K|$. Let $X^* = \lim \mathbf{X}^*$. By Theorem 1.7, $X^* \tau |K|$. Of course X^* is a metrizable space; one sees from (2) and (3) that X embeds as a closed subspace of X^* . So $X \tau |K|$. \square

Such a result, i.e., Theorem 1.7, for inverse systems, even for approximate inverse systems ([MR], [MW]), of compact spaces is true as has been proved in [Ru1].

2. Extension of Results.

In recent work, [Ma], S. Mardešić has been able to improve the previously stated limit results. To see what his work yields, let us make one definition.

2.1. Definition. A T_1 -space X is called **stratifiable** provided for each open subset U of X there has been assigned a sequence (U_n) of open subsets of X in such a manner that:

- (S1) $\overline{U_n} \subset U$,
- (S2) $\bigcup_{n=1}^{\infty} U_n = U$, and

(S3) $U \subset V$ implies that $U_n \subset V_n$.

It turns out that every stratifiable space is paracompact and perfectly normal and that stratifiable spaces are hereditarily stratifiable. In some sense, stratifiable spaces are a generalization of metrizable spaces. For example, it is true that the limit of an inverse sequence of stratifiable spaces is stratifiable. Polyhedra form an important class of stratifiable spaces (polyhedron means $|K|$ for a simplicial complex K with the weak topology).

The main theorem of [Ma] is,

2.2. Theorem. *Let $\mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ be an inverse sequence of stratifiable spaces, K be a CW-complex such that $\mathbf{X} \tau K$, and $X = \lim \mathbf{X}$. Then $X \tau K$.*

As in the previous work with metrizable spaces, one may also replace the CW-complex by a polyhedron. Thus we also have an equivalent theorem.

2.3. Theorem. *Let $\mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ be an inverse sequence of stratifiable spaces X_i with surjective bonding maps p_{ii+1} , K be a simplicial complex such that $\mathbf{X} \tau |K|$, and $X = \lim \mathbf{X}$. Then $X \tau |K|$.*

One interesting corollary of this work is the following.

2.4. Corollary. *Let $\mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ be an inverse sequence of polyhedra with $\dim X_i \leq n$ for each i and $X = \lim \mathbf{X}$. Then $\dim X \leq n$.*

In proving Theorem 2.3, the author introduces a new concept called a **filtered factorization**. This enables him to give a much more descriptive proof than had occurred in its predecessors. We shall not try to lay out in this exposition how the technique of filtered factorizations operates. We remark though, that it appears to apply to open sets U in the limit and how they relate to certain open sets in the coordinate spaces X_i . It seems to us that such a concept would be even more powerful if it could be stated without reference to the sets U , but rather strictly in terms of the coordinate spaces X_i (and of course the bonding maps in the sequence). With this in mind, there still could be some new, stronger results of this type to be found in future research.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, 601 ELM AVE., RM. 423, NORMAN,
OK 73019, USA

E-mail address: lrubin@ou.edu

Title: Some Remarks on Homogeneous Continua

by R. D. Edwards

Abstract: An important issue in topology (increasingly so, in my opinion) is to understand (and, ideally, to locally "classify") those continua which are homogeneous. Familiar examples include manifolds, both classical and Menger, and (conjecturally) ENR-homology manifolds having the DDP. Also of interest is the non-locally-connected side of the problem, particularly the following question (raised by K. Kuperberg around 1980, and perhaps by others earlier) :

Question: Does there exist a homogeneous continuum which is path-connected but not locally-(path-)connected? That is: Does there exist a homogeneous path-connected non-Peano continuum?

On Computing the Geometric Index

Kathryn B. Andrist and David G. Wright

October 19, 2000

Abstract

We present a simple method for computing the geometric index of a knot or a link in a solid torus.

1. INTRODUCTION AND STATEMENT OF RESULT

Throughout this paper we will be working in the piecewise-linear category. We consider a knot or link J that lies in the interior of a solid torus T with meridional disk D so that J pierces D at each point of $J \cap D$. The closure of $J - D$ is the union of arcs $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$, and C_1, C_2, \dots, C_n . For each $i, 1 \leq i \leq m$, the arc A_i begins and ends on one side of D while the arc B_i begins and ends on the other side of D and no meridional disk of T , which misses D , also misses $A_i \cup B_i$. For each $j, 1 \leq j \leq n$, the arc C_j begins and ends on opposite sides of D as in Figure 1. Under these conditions we conclude that the geometric index of J in T is equal to the number of points in $J \cap D$. This number is, in fact, $2m + n$.

2. DEFINITIONS

We begin by reviewing some basic definitions. For a manifold M , let $\text{int}M$ and ∂M denote the interior and boundary of M , respectively. For a set A , let $|A|$ denote the cardinality of A . A solid torus is a space homeomorphic to the product of a 2-dimensional disk and a circle. Recall that a *meridional disk* for a solid torus T is a disk D in T so that $D \cap \partial T = \partial D$ and ∂D does not separate ∂T . If J is a knot (a

On computing the geometric index

simple closed curve) in the interior of a solid torus T , the *geometric index* $N(J, T)$ of J in T is defined to be the minimum of $|J \cap D|$ where D ranges over all meridional disks [2]. By allowing J to be a link (finite disjoint union of simple closed curves), we can similarly define the geometric index of a link in T .

3. SOME LEMMAS

In this section we state some lemmas that will lead to the proof of our proposition on computing the geometric index. The sets J, D , and T are as in Section 1.

LEMMA 1. *If K is a meridional disk that misses D , then $|J \cap K| \geq |J \cap D|$.*

PROOF. For each $i, 1 \leq i \leq m$, either A_i or B_i must meet K at least twice. For each $j, 1 \leq j \leq n$, C_j must meet K at least once. So $|J \cap K| \geq 2m + n = |J \cap D|$. //

LEMMA 2. *Let k be a disk that lies in $\text{int}T$ with $k \cap D = \partial k$ and $\partial k \cap J = \emptyset$. Let d be the disk bounded by ∂k in the disk D . Then $|J \cap k| \geq |J \cap d|$.*

PROOF. Suppose, to the contrary, that $|J \cap k| < |J \cap d|$. Then the meridional disk $(D - d) \cup k$ meets J in fewer points than D . This disk can be pushed off D so that it still meets J in fewer points than D . But this contradicts Lemma 1. //

LEMMA 3. *Suppose K is a meridional disk in T so that $K \cap \partial T = \partial K$, $\partial K \cap \partial D = \emptyset$, K is in general position with respect to D , and $J \cap K \cap D = \emptyset$. Then $|J \cap K| \geq |J \cap D|$.*

PROOF. The proof is by induction on the number of components of $K \cap D$. If $K \cap D = \emptyset$, then it is true by Lemma 1. Now consider a simple closed curve component α of $K \cap D$ that is innermost on K . Then α bounds a disk k in K and a disk d in D . By Lemma 2 the (possibly) singular disk $(K - k) \cup d$ meets J in no more points than K . The singularities of $(K - k) \cup d$ consist of disjoint double curves that can be cut apart to give a non-singular disk K' that meets J in no more points than K [1]. By pushing K' slightly off D we obtain a disk that satisfies the hypothesis

of the Lemma and $D \cap K'$ has fewer components. Therefore, K' and hence K must meet J in at least as many points as D . //

4. THE MAIN THEOREM

THEOREM 1. *Let J be a knot or link that lies in the interior of a solid torus T with meridional disk D so that J pierces D at each point of $J \cap D$. The closure of $J - D$ is the union of arcs $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$, and C_1, C_2, \dots, C_n . For each $i, 1 \leq i \leq m$, the arc A_i begins and ends on one side of D while the arc B_i begins and ends on the other side of D and no meridional disk of T , which misses D , also misses $A_i \cup B_i$. For each $j, 1 \leq j \leq n$, the arc C_j begins and ends on opposite sides of D . Under these conditions we conclude that the geometric index of J in T is equal to the number of points in $J \cap D$. This number is, in fact, $2m + n$.*

PROOF. Let K be a meridional disk of T so that $|J \cap K|$ is minimal. Without loss of generality, we may assume that the disk K satisfies the hypothesis of Lemma 3. An application of Lemma 3 gives the result. //

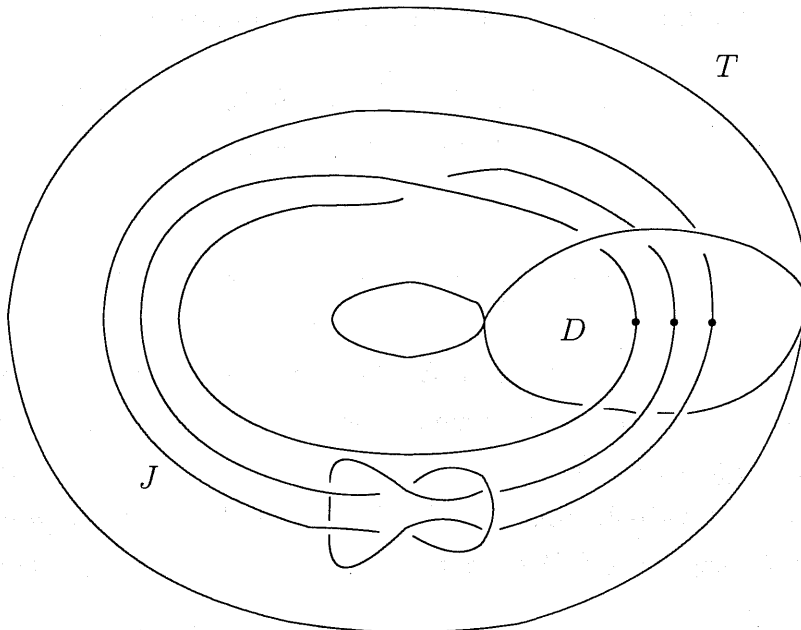


Figure 1

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DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
PROVO, UTAH 84602
U.S.A.

andrist@math.byu.edu

wright@math.byu.edu

ON RELATIVE CONNECTEDNESS

HANSPETER FISCHER AND DAVID G. WRIGHT

ABSTRACT. We introduce relative versions of connectedness where $A \subset B \subset Y$ are topological spaces. We define what it means for A to be *path-connected* in B , A to be *one-ended in B relative to Y* , A to be *simply connected* in B , and A to be *simply connected at infinity in B relative to Y* . When a group G acts on a space Y , we give conditions on G and Y that yield pairs of connected sets that have the relative versions of connectedness.

1. DEFINITIONS AND NOTATION

If W is a subset of a topological space X , we let $\text{int } W$ and ∂W denote the interior and boundary of W in X , respectively.

Definition 1.1. A topological space X is said to be *locally simply connected* if for each $x \in X$ and neighborhood U of x ; there is a neighborhood V of x with $V \subset U$ so that loops in V are inessential in U .

Let $A \subset B \subset Y$ where Y is a topological space and A, B are subspaces.

Definition 1.2. The set A is said to be *path-connected in B* if for each pair of points $p, q \in A$, there is a path from p to q which lies in B .

Definition 1.3. The set A is said to be *one-ended in B relative to Y* if A does not lie in a compact subset of Y and for each compact set $C \subset Y$, there is a bigger compact set $D \subset Y$ so that $A \setminus D$ is path-connected in $B \setminus C$. We say that Y is one-ended if $A = B = Y$ in the above definition.

Definition 1.4. The set A is said to be *simply connected in B* if each loop in A is inessential in B .

Definition 1.5. The set A is said to be *simply connected at infinity in B relative to Y* if A does not lie in a compact subset of Y and for each compact set $C \subset Y$, there is a bigger compact set $D \subset Y$ so that $A \setminus D$ is simply connected in $B \setminus C$. We say that Y is *simply connected at infinity* if $A = B = Y$ in the above definition.

Definition 1.6. If K is a set of homeomorphisms of a topological space S and $A \subset S$, then

$$K(A) = \bigcup \{k(A) | k \in K\}.$$

Definition 1.7. A group G of homeomorphisms of a topological space X is said to act *properly discontinuously* if for every compact $C \subset X$, $\{g \in G | g(C) \cap C \neq \emptyset\}$ is finite.

Definition 1.8. A group G of homeomorphisms of a topological space X is said to act *cocompactly* if there is a compact set $C \subset X$, so that $G(C) = X$.

2. GROUPS ACTING ON SPACES

For this section, let G be a group that acts properly discontinuously on the connected, locally path connected, locally compact, Hausdorff spaces X and Y , respectively. Assume further that G acts cocompactly on X . Observe that G is necessarily finitely generated.

Theorem 2.1. *For every compact set $A \subset Y$, there is a bigger compact set $B \subset Y$ such that $G(A)$ is path connected in $G(B)$.*

Proof. Let $E \subset X$ be compact such that $G(\text{int } E) = X$. Let a compact set $A \subset Y$ be given. Choose an open, path connected set $B_o \subset Y$ so that $A \subset B_o$, the closure B of B_o is compact, and $g_1(B_o) \cap g_2(B_o) \neq \emptyset$ whenever $g_1, g_2 \in G$ and $g_1(E) \cap g_2(E) \neq \emptyset$. (This is possible since $\{g \in G | g(E) \cap E \neq \emptyset\}$ is finite.) If $p, q \in G(A)$, then $p \in g_p(A)$ and $q \in g_q(A)$ for some $g_p, g_q \in G$. Pick points $p' \in g_p(E)$, $q' \in g_q(E)$, and a path $\gamma' : [0, 1] \rightarrow X$ from p' to q' . Choose a positive integer n such that for each $i \in \{1, 2, \dots, n\}$, there is a $g_i \in G$ such that $\gamma'([\frac{i-1}{n}, \frac{i}{n}]) \subset g_i(\text{int } E)$. Now we have $g_p(B_o) \cap g_1(B_o) \neq \emptyset, g_q(B_o) \cap g_n(B_o) \neq \emptyset$, and $g_i(B_o) \cap g_{i+1}(B_o) \neq \emptyset$ for all $i \in \{1, 2, \dots, n-1\}$. Since B_o is path connected, there is a path $\gamma : [0, 1] \rightarrow g_p(B_o) \cup g_1(B_o) \cup g_2(B_o) \cdots \cup g_n(B_o) \cup g_q(B_o) \subset G(B)$ such that $\gamma(0) = p$ and $\gamma(1) = q$. \square

Theorem 2.2. *Suppose X is one-ended. Then for every compact set $A \subset Y$ there is a bigger compact set $B \subset Y$ so that $G(A)$ is one-ended in $G(B)$ relative to Y .*

Proof. The proof is a *relative* version of Theorem 2.1. Here are the details. Let $E \subset X$ be compact such that $G(\text{int } E) = X$. Let a compact set $A \subset Y$ be given. Choose an open path connected set $B_o \subset Y$ so that $A \subset B_o$, the closure B of B_o is compact, and $g_1(B_o) \cap g_2(B_o) \neq \emptyset$ whenever $g_1, g_2 \in G$ and $g_1(E) \cap g_2(E) \neq \emptyset$.

Now let $C \subset Y$ be compact. Let C' be the compact subset of X given by $C' = \bigcup \{g(E) | g \in G, g(B) \cap C \neq \emptyset\}$. Let D' be a compact

subset of X so that $X \setminus D'$ is path-connected in $X \setminus C'$. Finally, define the compact subset $D = \bigcup \{g(B) | g \in G \text{ and } g(E) \cap D' \neq \emptyset\}$.

Let p and q be points in $G(A) \setminus D$, then there are g_p, g_q in G such that $p \in g_p(A)$ and $q \in g_q(A)$. Hence, $g_p(E) \cap D' = \emptyset$ and $g_q(E) \cap D' = \emptyset$. Pick points $p' \in g_p(\text{int } E)$ and $q' \in g_q(\text{int } E)$, and a path $\gamma' : [0, 1] \rightarrow X \setminus C' = X \setminus \bigcup \{g(E) | g \in G, g(B) \cap C \neq \emptyset\} \subset \bigcup \{g(\text{int } E) | g \in G, g(B) \cap C = \emptyset\}$ from p' to q' . Choose a positive integer n such that for each $i \in \{1, 2, \dots, n\}$, there is a $g_i \in G$ with $g_i(B) \cap C = \emptyset$ such that $\gamma'([(\frac{i-1}{n}, \frac{i}{n}]) \subset g_i(\text{int } E)$. Now we have $g_p(B_o) \cap g_1(B_o) \neq \emptyset, g_q(B_o) \cap g_n(B_o) \neq \emptyset$, and $g_i(B_o) \cap g_{i+1}(B_o) \neq \emptyset$ for all $i \in \{1, 2, \dots, n-1\}$. Furthermore, $g_p(B) \cap C = \emptyset, g_q(B) \cap C = \emptyset$, and $g_i(B) \cap C = \emptyset$. Since B_o is path connected, there is a path $\gamma : [0, 1] \rightarrow g_p(B_o) \cup g_1(B_o) \cup g_2(B_o) \cup \dots \cup g_n(B_o) \cup g_q(B_o) \subset G(B) \setminus C$ such that $\gamma(0) = p$ and $\gamma(1) = q$. \square

Lemma 2.3. *Suppose Z is a simply connected, locally simply connected, locally path connected, locally compact Hausdorff space. Then for every compact set $C \subset Z$ there is a compact set $D \subset Z$ such that $C \subset D$ and C is simply connected in D .*

Proof. Choose open subsets $V_0, V_1, \dots, V_k, U_0, U_1, \dots, U_k$ of Z such that $C \subset \bigcup \{U_i | i = 0, 1, \dots, k\}$, $U_i \subset V_i$ and loops in U_i contract in V_i for all i , and such that each V_i has compact closure. Choose a finite collection \mathcal{W} of open path connected subsets of Z such that $C \subset \bigcup \mathcal{W}$ and for each pair $W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 \neq \emptyset$, there is a U_i such that $W_1 \cup W_2 \subset U_i$. We justify the existence of such a collection \mathcal{W} as follows. Let C' be a compact subset so that $C \subset \text{int } C'$ and $C' \subset \bigcup \{U_i | i = 1, 2, \dots, k\}$. Let $\phi_i : C' \rightarrow [0, 1]$ for $i = 0, 1, \dots, k$ be a partition of unity dominated by $\{U_i\}$; i.e., each ϕ_i is continuous, $\phi_i^{-1}(0, 1] \subset U_i$, and $\sum_{i=0}^k \phi_i(x) = 1$ for each $x \in C'$. Define a map F from C' to a k -simplex $\sigma_k = \langle v_0, v_1, \dots, v_k \rangle$ by sending $x \in C'$ to $F(x) = \sum_{i=0}^k \phi_i(x) v_i$. Let \mathcal{W}' be a covering of σ_k by finitely many open sets so that for each pair of open sets $W'_1, W'_2 \in \mathcal{W}'$ with $W'_1 \cap W'_2 \neq \emptyset$, $W'_1 \cup W'_2$ lies in the open star S_i of v_i for some i . The path components of $\{F^{-1}(W') \cap \text{int } C' | W' \in \mathcal{W}'\}$ cover C and any two path components of this collection which meet must lie in some U_i since $F^{-1}(S_i) \subset U_i$ for each i . The collection \mathcal{W} is given by choosing finitely many of the path components of $\{F^{-1}(W') \cap \text{int } C' | W' \in \mathcal{W}'\}$.

Define a graph Γ as follows. For each $W \in \mathcal{W}$ take a vertex $v(W)$. Join two distinct vertices $v(W)$ and $v(W')$ by an edge $e(W, W')$ whenever $W \cap W' \neq \emptyset$. Choose a map $\mu : \Gamma \rightarrow Y$ such that $\mu(v(W)) \in W$ and $\mu(e(W, W')) \subset W \cup W'$ for all $W, W' \in \mathcal{W}$. Since Y is simply connected, there is a homotopy from μ to a constant map. Choose

the compact set D such that it contains the closure of each V_i and the image of this homotopy. A loop α in C can now be subdivided into finitely many paths α_i so that each α_i lies in an element $W_i \in \mathcal{W}$. If we connect the endpoints of each α_i to $\mu(v(W_i))$, we produce a bootstrap pattern between α and Γ whose loops lie alternately in a member of \mathcal{W} and in the union of two intersecting members of \mathcal{W} . This allows us to homotope α into $\mu(\Gamma)$ within D . From there we can contract it to a point within D . \square

In the proof of the next theorem we use the following notation. For a positive integer n define the set

$$\mathcal{G}(n) = \left\{ \left\{ \frac{i}{n} \right\} \times \left[\frac{j}{n}, \frac{j+1}{n} \right] \mid i \in \{0, 1, \dots, n\}, j \in \{0, 1, \dots, n-1\} \right\} \\ \cup \left\{ \left[\frac{i}{n}, \frac{i+1}{n} \right] \times \left\{ \frac{j}{n} \right\} \mid i \in \{0, 1, \dots, n-1\}, j \in \{0, 1, \dots, n\} \right\}$$

We also define $\mathcal{B}(n) = \{P \in \mathcal{G}(n) \mid P \subset \partial[0, 1]^2\}$, $\mathcal{I}(n) = \mathcal{G}(n) \setminus \mathcal{B}(n)$, and $\mathcal{D}(n) = \left\{ \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\} \times \left\{ \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}$. Furthermore, for $x \in [0, 1]^2$ we define $\mathcal{N}(x, n) = \{P \in \mathcal{G}(n) \mid x \in P\}$.

Theorem 2.4. *Suppose both X and Y are simply connected and that Y is locally simply connected. Then for every compact set $A \subset Y$ there is a compact set $B \subset Y$ such that $A \subset B$ and $G(A)$ is simply connected in $G(B)$ relative to Y .*

Proof. Let a compact set $A \subset Y$ be given and let $A_+ \subset Y$ be a compact set so that $A \subset \text{int } A_+$. We choose a path connected, open set $E_o \subset X$ such that the closure E of E_o is compact, $G(E_o) = X$, and in addition $g_1(E_o) \cap g_2(E_o) \neq \emptyset$ whenever $g_1, g_2 \in G$ and $g_1(A_+) \cap g_2(A_+) \neq \emptyset$. Choose an open, path connected set $F_o \subset Y$ so that the closure F of F_o is compact, $A_+ \subset F_o$, and $\bigcap \{g(F_o) \mid g \in S\} \neq \emptyset$ whenever $S \subset G$ and $\bigcap \{g(E) \mid g \in S\} \neq \emptyset$. Put $F_+ = \bigcup \{g(F) \mid g \in G, F \cap g'(F) \neq \emptyset \text{ and } g'(F) \cap g(F) \neq \emptyset \text{ for some } g' \in G\}$. By Lemma 2.3, there is a compact subset $B \subset Y$ such that $F_+ \subset B$ and F_+ is simply connected in B .

Let $\gamma : \partial[0, 1]^2 \rightarrow G(A)$ be a loop. Since $\gamma(\partial[0, 1]^2) \subset G(\text{int } A_+)$ there is a positive integer n such that for all $P \in \mathcal{B}(n)$ there is a $g_P \in G$ such that $\gamma(P) \subset g_P(\text{int } A_+)$. We now copy this loop in X ; i.e., we choose a loop $\gamma' : \partial[0, 1]^2 \rightarrow X$ with $\gamma'(P) \subset g_P(E_o)$ for all $P \in \mathcal{B}(n)$. Since X is simply connected, we may extend γ' to $f' : [0, 1]^2 \rightarrow X$. Since $G(E_o) = X$, there is a positive integer m ($m \geq 2$) such that for all $P \in \mathcal{G}(nm)$ there is a $g_P \in G$ such that $f'(P) \subset g_P(E_o)$.

We now extend γ to a map f from $\partial[0, 1]^2 \cup \mathcal{D}(nm)$ to $G(F_o)$ such that $f(x) \in \bigcap \{g_P(F_o) \mid P \in \mathcal{N}(x, nm)\}$ for all $x \in \mathcal{D}(nm)$. Next extend

f to a map from $\bigcup \mathcal{G}(nm)$ to $G(F_o)$ by sending $P \in \mathcal{I}(nm)$ into $g_P(F_o)$ in case $P \subset \text{int } [0, 1]^2$ and by sending P into $g_P(F_o) \cup g_Q(F_o)$ for some $Q \in \mathcal{B}(n)$ with $P \cap Q \neq \emptyset$ in case $P \not\subset \text{int } [0, 1]^2$.

Finally, since for all $i, j \in \{0, 1, \dots, nm-1\}$ there is a $P \in \mathcal{G}(nm)$ such that $f(\partial[\frac{i}{nm}, \frac{i+1}{nm}] \times [\frac{j}{nm}, \frac{j+1}{nm}]) \subset g_P(F_+)$, we can, by the choice of B , extend f to a map from $[0, 1]^2$ into $\bigcup \{g_P(B) | P \in \mathcal{G}(nm)\}$. Hence, γ is inessential in $G(B)$ relative to Y . \square

Theorem 2.5. *Suppose X is simply connected at infinity and Y is simply connected and locally simply connected. Then for every compact set $A \subset Y$ there is a compact set $B \subset Y$ so that $G(A)$ is simply connected at infinity in $G(B)$.*

Proof. We have to change the proof of Theorem 2.4 only slightly. The construction of B in that theorem did not use the simple connectivity of X . We use the same B and the same setup. Let $C \subset Y$ be compact. Let $C' = \bigcup \{g(E) | g \in G, g(B) \cap C \neq \emptyset\}$. Choose a compact set $D' \subset X$ so that $X \setminus D'$ is simply connected in $X \setminus C'$. Let D equal the compact set $C \cup \bigcup \{g(A_+) | g \in G, g(E) \cap D' \neq \emptyset\}$. Now if $\gamma : \partial[0, 1]^2 \rightarrow G(A) \setminus D$ is a loop, then γ lies in $\bigcup \{g(\text{int } A_+) | g \in G, g(E) \cap D' = \emptyset\}$.

As in the proof of Theorem 2.4, we *copy* this loop in X . We do this by first choosing a positive integer n so that for all $P \in \mathcal{B}(n)$ there is a $g_P \in G$ such that $\gamma(P) \subset g_P(\text{int } A_+)$. Notice that for each $P \in \mathcal{B}(n)$, we have $g_P(E_o) \cap D' = \emptyset$. We choose a loop $\gamma' : \partial[0, 1]^2 \rightarrow X \setminus D'$ with $\gamma'(P) \subset g_P(E_o)$. Then γ' extends to $f' : [0, 1]^2 \rightarrow X \setminus C' = X \setminus \bigcup \{g(E) | g \in G, g(B) \cap C \neq \emptyset\} \subset X \setminus \bigcup \{g(E_o) | g \in G, g(B) \cap C \neq \emptyset\}$.

Since $G(E_o) = X$ there is a positive integer m ($m \geq 2$) such that for all $P \in \mathcal{G}(nm)$ there is a $g_P \in G$ such that $f'(P) \subset g_P(E_o)$. As in the proof of Theorem 2.4, we use $\mathcal{G}(nm)$ and the map f' to extend the loop γ . Recall from the proof of Theorem 2.4 that the map γ gets extended to $f : [0, 1]^2 \rightarrow \bigcup \{g_P(B) | P \in \mathcal{G}(nm)\}$ where $f'(P) \subset g_P(E_o)$. However, if $f'(P) \subset g_P(E_o)$, then $g_P(B) \cap C = \emptyset$; otherwise, $f'(P) \subset C'$. Therefore f maps into $G(B) \setminus C$. \square

The following two well-known theorems follow from Theorem 2.2 and Theorem 2.5. They justify the definitions of a group being *one-ended* and *simply connected at infinity*.

Theorem 2.6. *If a group G acts properly discontinuously and cocompactly on the connected, locally path connected, locally compact Hausdorff spaces X and Y , then X is one-ended if and only if Y is one-ended.*

Theorem 2.7. *If a group G acts properly discontinuously and cocompactly on the connected, simply connected, locally path connected, locally*

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simply connected, locally compact, Hausdorff spaces X and Y , then X is simply connected at infinity if and only if Y is simply connected at infinity.

HANSPETER FISCHER, DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602

E-mail address: fischer@math.byu.edu

DAVID G. WRIGHT, DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602

E-mail address: wright@math.byu.edu

2-GROUPS AND APPROXIMATE FIBRATIONS

ROBERT J. DAVERMAN

For the past ten years a central thrust of my research has involved a quest to identify homotopy types through which a proper map defined on an arbitrary manifold of a given dimension can be quickly recognized as an approximate fibration, simply because all point preimages have the specified homotopy type. Often the goal has been to present closed n -manifolds N which force proper maps $p : M \rightarrow B$ to be approximate fibrations, when M is an $(n + 2)$ -manifold and each $p^{-1}(b)$ has the homotopy type (or, more generally, the shape) of N . Such a manifold N is called a codimension-2 fibrator. A theme has been that, perhaps contrary to intuition, most manifolds are codimension-2 fibrators; that was even the explicit title of my talk at this conference back in 1993.

The work outlined here was done jointly with Yongkuk Kim, who just completed his Ph.D. at the University of Tennessee.

1. PRELIMINARIES

Let N^n be a closed manifold. A proper map $p : M \rightarrow B$ is N^n -like provided each fiber $p^{-1}(b)$ is shape equivalent to N^n . For simplicity, we shall assume that each fiber $p^{-1}(b)$ in an N^n -like map to be an ANR having the homotopy type of N^n .

A map $f : X \rightarrow Y$ between ANRs is called an *approximate fibration* if it satisfies an approximate version of the familiar Homotopy Lifting Property used to define the familiar class of fibrations. This class, introduced and studied by Coram and Duvall [2] [3], seems to carry all the same useful algebraic features held by fibrations. An n -manifold N is called a *codimension- k (orientable) fibrator* if every N -like map $p : M \rightarrow B$ from an (respectively, orientable) $(n + k)$ -manifold M onto a finite-dimensional space B is an approximate fibration (and B necessarily is an ANR).

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The issue here is codimension 2, and thus the basic objects under scrutiny are N^n -like maps $p : M \rightarrow B$ defined on $(n+2)$ -manifolds. When N^n is orientable, one speaks of the *continuity set* of p , namely, the collection C of all $b \in B$ for which there exist a neighborhood U of $p^{-1}(b)$ in M and a (shape) retraction $U \rightarrow p^{-1}(b)$ that restricts to a degree one mapping $p^{-1}(b') \rightarrow p^{-1}(b)$ for all b' with $p^{-1}(b') \subset U$. Independent of this orientability assumption, one can always speak of the *mod 2 continuity set* of p , namely the set C' where the same statement holds when degree is computed with \mathbb{Z}_2 -coefficients. The following fact, stemming from analysis by Coram and Duvall [4], is central to all investigations of codimension-2 fibrators.

Proposition 1.1. [5, Proposition 2.8] *In the setting just described, if M and N are orientable, then the space B is a 2-manifold and $D = B \setminus C$ is locally finite in B , where C represents the continuity set of p . Moreover, if either M or N is nonorientable, B is a 2-manifold with boundary (possibly empty) and $D' = (\text{int } B) \setminus C'$ is locally finite in B , where C' represents the mod 2 continuity set of p .*

Call a closed manifold N *Hopfian* if it is orientable and every degree one map $N \rightarrow N$ inducing a π_1 -isomorphism is a homotopy equivalence. Generally, call N *s-hopfian* if \tilde{N} is hopfian, where \tilde{N} is the covering space of N corresponding to $H = \bigcap_{i \in I} \{H_i : [\pi_1(N) : H_i] = 2\}$, when N is non-orientable, and N itself is hopfian otherwise; this is a variation on the *strongly hopfian* notion introduced by Kim [9].

A group Γ is said to be *hopfian* if every epimorphism $\Gamma \rightarrow \Gamma$ is an isomorphism; furthermore, Γ is said to be *hyperhopfian* if every homomorphism $\varphi : \Gamma \rightarrow \Gamma$ with $\varphi(\Gamma)$ normal and $\Gamma/\varphi(\Gamma)$ cyclic is an isomorphism (onto).

Important connections involving these hopfian concepts are spelled out in the results below:

Proposition 1.2. *Let N be a Hopfian n -manifold having hopfian fundamental group. Then every N -like map $p : M \rightarrow B$ defined on an $(n+2)$ -manifold M is an approximate fibration over its continuity set.*

Proposition 1.3. [10, Theorem 3.3] *Every closed s-hopfian manifold N such that $\pi_1(N)$ is hyperhopfian is a codimension-2 fibrator.*

2. MAIN RESULTS

Theorem 2.1. *Every closed n -manifold N whose fundamental group is an Abelian 2-group is a codimension-2 fibration.*

N. Chinen [1] proved that a closed manifold N is a codimension-2 fibration if $\pi_1(N)$ is a finite product of copies of any fixed cyclic 2-group, and he raised the question which is answered by Theorem 2.1. He also asked whether the same fibration conclusion would hold for manifolds whose fundamental groups are 2-groups.

Recent examples of mine [7] illustrate the need to concentrate on 2-groups. For each odd integer $m > 2$ there are closed manifolds $N_m \cong L_{m,q} \times S^3$ ($L_{m,q}$ denotes a Lens space) with $\pi_1(N_m)$ cyclic of order m and N_m not a codimension-2 fibration; by taking products with, say, other Lens spaces, one can produce codimension-2 nonfibrations having fundamental group isomorphic to any finite Abelian group except a 2-group.

Finiteness of $\pi_1(N)$, not merely $H_1(N)$, is imperative in Theorem 2.1, as the well-known nonfibration $RP^n \# RP^n$ reveals. The remaining results stated in this section involve $H_1(N)$ conditions leading to fibration properties.

Theorem 2.2. *Let N be a closed hopfian n -manifold such that $\pi_1(N)$ is hopfian and $H_1(N) = \mathbb{Z}_{2^t}$ for some t . Then N is a codimension-2 orientable fibration. If, in addition, the 2-1 cover \widetilde{N}_H of N is a codimension-2 orientable fibration, then N is a codimension-2 fibration.*

Theorem 2.3. *Let N be a closed n -manifold N for which $\pi_1(N)$ is finite and $H_1(N) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$. If N is orientable, it is a codimension-2 orientable fibration; if the cover \widetilde{N}_H of N corresponding to the commutator subgroup of $\pi_1(N)$ happens to be a codimension-2 orientable fibration, N is a codimension-2 fibration.*

Lemma 2.4. *If Γ is a finitely generated residually finite group having abelianization \mathbb{Z}_d , where the order of no element of Γ divides d , then Γ is hyperhopfian.*

Theorem 2.5. *Suppose N is a closed, s -hopfian n -manifold such that $H_1(N)$ is cyclic of order d and $\pi_1(N)$ is a residually finite group, no element of which has order dividing d . Then N is a codimension-2 fibration. In particular, every aspherical manifold N with residually finite $\pi_1(N)$ and finite cyclic $H_1(N)$ is a codimension-2 fibration.*

Cyclicity of $H_1(N)$ is critical to Theorem 2.5, which does not hold for $H_1(N)$ an arbitrary 2-group, not even for a direct sum of copies of \mathbb{Z}_2 . We have an example sharpening the contrast between 2.3 and 2.5.

Example. A closed n -manifold N , $n > 4$, which fails to be a codimension-2 fibrator but $H_1(N) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\pi_1(N)$ is torsion free.

3. TECHNICAL ASPECTS

Lemma 3.1. *Suppose N is a closed n -manifold such that \widetilde{N}_H is a codimension-2 orientable fibrator. Then every N -like map $p : M \rightarrow B$ defined on an $(n+2)$ -manifold M is an approximate fibration over its mod 2 continuity set C' .*

Lemma 3.2. *Let N be a closed n -manifold with finite $\pi_1(N)$ and $p : M \rightarrow B$ a proper N -like map defined on an $(n+2)$ -manifold. If N is orientable, or if p is an approximate fibration over its mod 2 continuity set C' , then ∂B is empty.*

A cyclic decomposition of a finitely generated Abelian group A is a representation $A \cong C_1 \oplus \cdots \oplus C_i \oplus \cdots \oplus C_k$, where each C_i is cyclic. For convenience, we shall always assume these to be arranged in nondecreasing order, i.e., $|C_i| \leq |C_{i+1}|$ for all i . When we write $C_1 \oplus \cdots \oplus \widehat{C}_i \oplus \cdots \oplus C_k$, we mean the direct sum of all C_j 's except C_i .

Lemma 3.3. *Let N be a closed n -manifold such that $H_1(N)$ is a 2-group, with cyclic decomposition $C_1 \oplus \cdots \oplus C_k$, and let $p : M \rightarrow \mathbb{R}^2$ be a proper, N -like map defined on an $(n+2)$ -manifold that restricts to an approximate fibration over $\mathbb{R}^2 \setminus \text{origin}$. Then $H_1(M \setminus p^{-1}(\text{origin}))$ is isomorphic either to $\mathbb{Z} \oplus H_1(N)$ or to $\mathbb{Z} \oplus C_1 \oplus \cdots \oplus \widehat{C}_i \oplus \cdots \oplus C_k$. Moreover, C' , the mod 2 continuity set of p , equals \mathbb{R}^2 if and only if $H_1(M \setminus p^{-1}(\text{origin})) \cong \mathbb{Z} \oplus H_1(N)$; in case M and N are both orientable, $C = \mathbb{R}^2$ if and only if $H_1(M \setminus p^{-1}(\text{origin})) \cong \mathbb{Z} \oplus H_1(N)$.*

Lemma 3.4. *Suppose $C_1 \oplus \cdots \oplus C_k$ is a cyclic decomposition of an Abelian 2-group A , with $|C_1| \leq \cdots \leq |C_k|$. If $\kappa : A \rightarrow C_1 \oplus \cdots \oplus \widehat{C}_i \oplus \cdots \oplus C_k$ is an epimorphism with C_i cyclic of order 2^d , then there exists a unique $\xi \in \ker(\kappa)$ such that ξ has order 2 and $\xi = 2^{d-1} \cdot \xi'$, for some $\xi' \in A$.*

Proof. The key algebraic fact, this is proved by induction on k . Suppose $i = k$. First assume $|C_j| < 2^d$ for $j < k$. Take $\xi = \langle 0, \dots, 0, 2^{d-1} \rangle$. Here ξ must belong to $\ker(\kappa)$, for otherwise $C_1 \oplus \dots \oplus C_{k-1}$ would contain the cyclic subgroup $\kappa(C_k)$ of order 2^d , and ξ is divisible by $\langle 0, \dots, 0, 1 \rangle \in A$. Clearly, no other nonzero element of A is divisible by an element of order 2^{d-1} . Next, when $|C_{k-1}| = 2^d$, examine

$$A \xrightarrow{\kappa} C_1 \oplus \dots \oplus C_{k-1} \xrightarrow{\text{projection}} C_{k-1}$$

note that all elements of A have order dividing 2^d to obtain a splitting $A = A_1 \oplus C_{k-1} \cong A_1 \oplus C_k$, with $A_1 = \ker(\text{projection} \circ \kappa)$. Here $A_1 \cong C_1 \oplus \dots \oplus C_{k-1}$, by uniqueness of cyclic decompositions, and $\kappa(A_1) = C_1 \oplus \dots \oplus C_{k-2}$. Now induction applies.

When $i < k$ essentially the same reduction can be brought to bear on

$$A \xrightarrow{\kappa} C_1 \oplus \dots \oplus \widehat{C}_i \oplus \dots \oplus C_k \xrightarrow{\text{projection}} C_k.$$

□

Lemma 3.5. *Suppose N is a closed n -manifold such that $\pi_1(N)$ is an Abelian 2-group, and suppose $p : M \rightarrow \mathbb{R}^2$ is a proper N -like map, defined on an $(n+2)$ -manifold, which is an approximate fibration over $\mathbb{R}^2 \setminus \text{origin}$. Then $C' = \mathbb{R}^2$.*

All the pieces, algebraic and geometric, come together in 3.5; 3.4 is used to show $H_1(M \setminus p^{-1}(\text{origin})) \cong \mathbb{Z} \oplus H_1(N)$ by ruling out the other possibility allowed by 3.3. The chief result, Theorem 2.1, then follows from a combination of 3.5, 3.1, 3.2, and 1.1.

4. QUESTIONS

1. Is every closed n -manifold Hopfian? This is an old, important, and famous question originally raised by H. Hopf. As Hausmann has shown [8], the answer is affirmative when $n \leq 4$; he also constructed examples admitting self maps of nonzero degree, but not degree 1, which induce noninjective epimorphisms at the fundamental group level.

2. If $\pi_1(N)$ is a finite 2-group, must N be a codimension 2 fibrator?

3. If $\pi_1(N)$ is finite and $H_1(N)$ is a 2-group, must N be a codimension 2 fibrator?

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ADDITIVE SUBGROUPS IN BANACH SPACES THAT ARE NOT HOMOTOPICALLY TRIVIAL (AFTER J. GRABOWSKI)

TADEUSZ DOBROWOLSKI

ABSTRACT. A locally compact, locally connected, finite-dimensional group G is a Lie group (in particular, G is a manifold) - this is a complete solution to the 5th Hilbert problem obtained in the early fifties. For some time it was hoped that, for a closed subgroup G of a Hilbert space, its nice additive structure does the job of the local compactness and the finite-dimensionality in the above result to the effect that such a G is a manifold (finite or infinite-dimensional) provided it is locally connected. Recently, this conjecture has been refuted by R. Cauty [Ca], who exhibited a certain closed subgroup C in the space $L^2([0, 1]^2)$ and, using an involved and lengthy argument, showed that C was locally connected but not locally 1-connected (and consequently, carrying no manifold structure).

J. Grabowski [Gr] shrewdly observed that a quotient of the well-known group consisting of integer-valued elements of $L^2[0, 1]$ can be used as a building block to provide yet another example of such a group. We present and develop his idea.

1. INTRODUCTION

The only closed subgroups of \mathbb{R}^n are products of discrete groups (isomorphic to \mathbb{Z}^k , $0 \leq k \leq n$) by vector subspaces (isomorphic to \mathbb{R}^k , $0 \leq k \leq n$); equivalently, nondiscrete subgroups of \mathbb{R}^n are vector subspaces. To show how this picture dramatically changes in the infinite-dimensional case let us consider in $L^2[0, 1]$, the Hilbert space of all equivalence classes of measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ with $\int_0^1 |f(t)|^2 dt < \infty$, the following additive, closed subgroup

$$L_{\mathbb{Z}}^2 = \{f \in L^2[0, 1] \mid f(t) \in \mathbb{Z} \text{ a.e.}\}.$$

One easily observes the following facts.

Lemma 1. *The group $L_{\mathbb{Z}}^2$ does not contain any line (that is, $L_{\mathbb{Z}}^2$ is a line-free group). \square*

Lemma 2. *The group $L_{\mathbb{Z}}^2$ is contractible. More precisely, the map*

$$\Phi(f, t) = f \cdot \chi_{[0, t]}$$

contracts $L_{\mathbb{Z}}^2$ to a point, that is, $\Phi(L_{\mathbb{Z}}^2 \times [0, 1]) \subseteq L_{\mathbb{Z}}^2$, and Φ satisfies $\Phi(f, 0) = 0$, and $\Phi(f, 1) = f$ for every f and t . \square

Since every topological contractible group is also locally contractible, $L_{\mathbb{Z}}^2$ is locally contractible. Actually, $L_{\mathbb{Z}}^2$ is homeomorphic to $L^2[0, 1]$ (see [Do]).

2. SUBGROUPS WITH NONTRIVIAL FUNDAMENTAL GROUP

The following elementary example of a covering map provides homogeneous spaces with nontrivial fundamental group.

Lemma 3. *If a group G is locally path-connected and 1-connected, and $G' \subset G$ is a discrete subgroup of G , then the homogeneous space $G \rightarrow G/G'$ is a universal covering. In particular, $\pi_1(G/G') = G'$. \square*

Combining Lemmas 2 and 3, it is now clear that if G' is a nontrivial discrete subgroup of $L_{\mathbb{Z}}^2$, then $\Gamma = L_{\mathbb{Z}}^2/G'$ will be an additive group with nontrivial fundamental group. Furthermore, Γ will be locally homeomorphic to $L^2[0, 1]$. We will now show that the quotient $\Gamma = L_{\mathbb{Z}}^2/G'$ can be naturally realized as an additive subgroup of a Hilbert space. We wish to present this in a more general setting.

Let E be a Banach space. Let G be a nontrivial, line-free, closed (additive) subgroup of E . Pick any $g \in G$, $g \neq 0$. Form the quotient Banach space $E/\mathbb{R}g$ and the quotient group $G/\mathbb{Z}g$, and consider the canonical linear map

$$\kappa : E \rightarrow E/\mathbb{R}g$$

and its restriction

$$\kappa|_G : G \rightarrow G/\mathbb{Z}g,$$

a group homomorphism. Write $i : G \rightarrow E$ for the inclusion map.

Lemma 4. *There exists a group homomorphism*

$$e : G/\mathbb{Z}g \rightarrow E/\mathbb{R}g$$

satisfying $e \circ \kappa|_G = \kappa \circ i$. Moreover, e is a group-topological embedding of the quotient group $G/\mathbb{Z}g$ into a Banach space $E/\mathbb{R}g$. \square

To show that i is well-defined by the condition $e \circ \kappa|_G = \kappa \circ i$ use the fact that G is line-free, which in turn yields $G \cap \mathbb{R}g = G \cap \mathbb{Z}g$.

Proposition 5. *Let G be a nontrivial, line-free, closed subgroup of a Banach space E . Pick any $g \in G$, $g \neq 0$. Then the quotient group $G_0 = G/\mathbb{Z}g$ can be considered as an additive, closed subgroup of $E/\mathbb{R}g$, the latter can be identified with a closed subspace E_0 of codimension 1 in E . Moreover, we have that*

- (a) *the quotient map $G \rightarrow G_0 (= G/\mathbb{Z}g)$ is a covering map; in particular, G_0 is locally homeomorphic to G , and*
- (b) *if G is locally connected and 1-connected then G_0 is locally connected and $\pi_1(G_0) = \mathbb{Z}$. \square*

It follows that if a Banach space E contains a nontrivial, line-free, locally connected, 1-connected, closed subgroup, then it also contains a (nontrivial, line-free) locally connected, closed subgroup Γ that is homotopically nontrivial. In view of this, it would be interesting to know which infinite-dimensional Banach spaces contain a nontrivial, line free, closed subgroup that is locally connected and 1-connected. The group $L^2_{\mathbb{Z}}$ is an example of such a group in the Hilbert space $L^2[0, 1]$. It is likely that every infinite-dimensional Banach space contains a nontrivial, contractible, line-free, closed subgroup. Note that the group $L^p_{\mathbb{Z}}$ consisting of all integer-valued functions in the space $L^p[0, 1]$, $p \geq 1$, is also contractible. Since the sequential Hilbert space ℓ^2 is linearly isomorphic to $L^2[0, 1]$, ℓ^2 contains such a group as well. It would be interesting to find a nontrivial, contractible, closed subgroup of ℓ^p for every $p \geq 1$. Let us finish this discussion with the following fact.

Proposition 6. *For no distinct p and q , $p, q \geq 1$, the groups $L^p_{\mathbb{Z}}$ and $L^q_{\mathbb{Z}}$ are group-topological isomorphic.*

Proof. Suppose that $\Phi : L^p_{\mathbb{Z}} \rightarrow L^q_{\mathbb{Z}}$ establishes a group-topological isomorphism, where $p \neq q$. For a rational number $r = \frac{k}{l}$ and $f \in L^p_{\mathbb{Z}}$, let us write $\tilde{\Phi}(rf) = r\Phi(f)$. It is straightforward to check that $\tilde{\Phi}$ designed in this way is a well-defined continuous homomorphism of the group $\tilde{L}^p_{\mathbb{Z}}$ consisting of all rational combinations of elements of $L^p_{\mathbb{Z}}$ into the group $\tilde{L}^q_{\mathbb{Z}}$ defined in a similar way. Similarly, we can define a continuous homomorphism $\tilde{\Psi} : \tilde{L}^q_{\mathbb{Z}} \rightarrow \tilde{L}^p_{\mathbb{Z}}$ that extends $\Phi^{-1} : L^q_{\mathbb{Z}} \rightarrow L^p_{\mathbb{Z}}$. One easily checks that $\tilde{\Phi} \circ \tilde{\Psi} = \tilde{\Psi} \circ \tilde{\Phi} = \text{id}$. Since $\tilde{L}^p_{\mathbb{Z}}$ and $\tilde{L}^q_{\mathbb{Z}}$ is dense in $L^p[0, 1]$ and $L^q[0, 1]$, $\tilde{\Phi}$ and $\tilde{\Psi}$ extends to a continuous homomorphism $T : L^p[0, 1] \rightarrow L^q[0, 1]$ and $S : L^q[0, 1] \rightarrow L^p[0, 1]$, respectively. It is clear that T and S are linear operators that are inverse of each other. Since $L^p[0, 1]$ is not isomorphic to $L^q[0, 1]$, we arrive to a contradiction. \square

Let us return to our specific example of $L^2_{\mathbb{Z}}$. Formally, we can apply Proposition 5 to $E = L^2[0, 1]$, $G = L^2_{\mathbb{Z}}$, $g = \chi_{[0,1]}$. Then

$$E_0 = \{f \in L^2[0, 1] \mid \int_0^1 f(t)dt = 0\}$$

and

$$G_0 = \{f \in L^2_{\mathbb{Z}} \mid \int_0^1 f(t)dt = 0\}.$$

Since G is contractible, we end up with the following conclusion.

Corollary 7. *The subgroup G_0 of the Hilbert space E_0 above is locally path-connected (even, locally homeomorphic to $L^2[0, 1]$) and not 1-connected.*

Remark 8. In Proposition 5 the line $\mathbb{R}g$ can be replaced by any closed linear subspace L so that $L \cap G$ is discrete. Then, the quotient group $G_0 = G/L \cap G$ can be identified with a subgroup of the Banach space $E_0 = E/L$, and $G \rightarrow G_0$ is a covering map. If G is locally connected and 1-connected, then $\pi_1(G_0) = L \cap G$.

It is not difficult to see that any group H that is a (finite or infinite) direct sum of \mathbb{Z} can be obtained as $H = L \cap L^2_{\mathbb{Z}}$ for some closed linear subspace L of $L^2[0, 1]$. Consequently, we have

Corollary 9. *Let H be a group that is a direct sum of \mathbb{Z} . There exists a line-free, closed subgroup G of a Hilbert space E such that G is locally homeomorphic to E , and $\pi_1(G) = H$.*

3. SUBGROUPS THAT ARE LC^0 BUT NOT LC^1

Our aim is to recover the following Cauty's [Ca] result

Theorem 10. *There exists a closed subgroup Γ of a Hilbert space H such that Γ is locally path-connected and not locally 1-connected.*

For a sequence of Hilbert spaces $(H_n, \|\cdot\|_n)$, we let

$$\oplus_{\ell^2} H_n = \{f = (f_n) \in \prod H_n \mid \sum \|f_n\|_n^2 < \infty\}.$$

The space $H = \oplus_{\ell^2} H_n$ when equipped with the norm

$$\|f\| = \sqrt{\sum \|f_n\|_n^2}$$

is a Hilbert space; the space H is called the Hilbertian sum of H_n . For a sequence of additive groups $G_n \subseteq H_n$, $n \geq 1$, we similarly let

$$\oplus_{\ell^2} G_n = \{f = (f_n) \in \oplus_{\ell^2} H_n \mid f_n \in G_n \text{ for every } n\}.$$

It is clear that $\oplus_{\ell^2} G_n$ is an additive subgroup of H which is closed in H provided each G_n is closed in H_n .

Proof of Theorem 10. Consider our Hilbert space E_0 , and the subgroup G_0 from Corollary 7. Let H be the Hilbertian sum of $E_0 = H_n$, $n \geq 1$. For $n \geq 1$, we let $G_n = \frac{1}{n}G_0 \subseteq H_n = E_0$. We finally let

$$\Gamma = \oplus_{\ell^2} G_n \subseteq \oplus_{\ell^2} E_0 = H.$$

To see that Γ is locally path-connected let V be a neighborhood of $0 \in G_0$ that is path-connected within the ball $B_{\|\cdot\|}(\varepsilon)$, for some $\varepsilon > 0$. Then, evidently, for $\delta > 0$ with $B_{\|\cdot\|}(\delta) \subset V$, $B_{\|\cdot\|}(\delta)$ is path-connected within $B_{\|\cdot\|}(\varepsilon)$.

Now, we will show that Γ is not locally 1-connected. Let $\varphi : S^1 \rightarrow G_0$ be a nonextendable element of $\pi_1(G_0)$ such that $\max \|\varphi(t)\| \leq 1$. Then, for every n , the map $\frac{1}{n}\varphi : S^1 \rightarrow G_n$ is nonextendable in G_n . It follows that, for no n , $B_{\|\cdot\|}(\frac{1}{n})$ is 1-connected. \square

Remark 11. Not only that the ball $B_{\|\cdot\|}(\frac{1}{n})$ is not simply connected, but it contains loops that are not extendable in the whole Γ . Consequently, Γ is not semi-simply connected.

It is possible to adjust the use of Hilbertian sum to construct a closed subgroup Γ that is LC^0 and not LC^1 in an arbitrary infinite-dimensional Banach space E provided every infinite-dimensional Banach space contains a closed subgroup that is locally connected and not simply connected. As observed previously, the latter is doable if every infinite-dimensional Banach space contains a nontrivial, line-free, closed subgroup G that is locally connected and simply connected.

Question 1. Does every infinite-dimensional separable Banach space E contain a nontrivial, line-free, locally connected and simply connected, closed subgroup G ?

Obviously the method presented above is not applicable for construction groups that are LC^1 and not LC^2 .

Question 2. Does there exist an additive, closed subgroup in a Hilbert space that is LC^1 and not LC^2 ?

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DEPARTMENT OF MATHEMATICS, PITTSBURG STATE UNIVERSITY, PITTSBURG, KS 66762

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-950
WARSZAWA, POLAND

E-mail address: `tdobrowo@mail.pittstate.edu`

A counterexample to a question by Chapman and Siebenmann

Craig R. Guilbault

Department of Mathematical Sciences
University of Wisconsin-Milwaukee

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Abstract

We describe a locally compact polyhedron X with the property that $X \times [-1, 1]^\infty$ is \mathbb{Z} -compactifiable but X is not. This answers a question first posed by Chapman and Siebenmann in 1976.

This note outlines the main result of [Gu]. The interested reader should consult that paper for a more complete exposition.

A closed subset A of a compact ANR X is a \mathbb{Z} -set if either of the following is satisfied:

- There is a homotopy $H: X \times I \rightarrow X$ with $H_0 = \text{id}_X$ and $H_t(X) \cap A = \emptyset$ for all $t > 0$.
- For every open set U of X , $U \setminus A \hookrightarrow U$ is a homotopy equivalence.

Let Y be a noncompact ANR. A \mathbb{Z} -compactification of Y is a compact ANR $[\hat{Y}]$ containing Y as an open subset and having the property that $[\hat{Y}] - Y$ is a \mathbb{Z} -set in $[\hat{Y}]$. In this case we call $[\hat{Y}] - Y$ a \mathbb{Z} -boundary for Y and denote it $\partial_{\mathbb{Z}} Y$.

Note. Y may admit many different \mathbb{Z} -boundaries, hence $\partial_{\mathbb{Z}} Y$ is not well defined unless the \mathbb{Z} -compactification is specified.

We now review some important examples of Z -compactifications.

Example 1 Let M^n be a compact n -manifold with boundary. Then ∂M^n is a Z -set in M^n , so M^n is a Z -compactification of $\text{int}(M^n)$ with $\partial_Z(\text{int}(M^n)) = \partial M^n$.

Example 2 Let Y be a locally compact $CAT(0)$ ANR and let $S_\infty(0)$ denote the visual sphere of Y from some point $0 \in Y$. Then Y admits a Z -compactification with $S_\infty(0)$ as the Z -boundary.

Example 3 [Bestvina-Mess] Let Γ be a word hyperbolic group, and let $P(\Gamma)$ be a contractible Rips complex for G . Then $P(\Gamma)$ may be Z -compactified to $\hat{P}(\Gamma) = P(\Gamma) \cup \partial\Gamma$, where $\partial\Gamma$ is the Gromov boundary of Γ . See [BM].

Example 4 [Bestvina] Suppose K is a finite $K(G,1)$ where G is either a $CAT(0)$ or word hyperbolic group. If G is word hyperbolic, let ∂G be the (unique) Gromov boundary of G , otherwise, let ∂G be an arbitrary $CAT(0)$ boundary for G . Then, the universal cover $[K \setminus \tilde{K}]$ admits a Z -compactification with Z -boundary equal to ∂G . See [Be].

Let $Q = [-1,1]^\infty$ denote the Hilbert cube. A separable metric space, X , is a Hilbert cube manifold if each point has a neighborhood homeomorphic to an open subset of Q .

Example 5 In 1976 Chapman and Siebenmann [CS] gave necessary and sufficient conditions for a Hilbert cube manifold to be Z -compactifiable. This was probably the first place where Z -compactifications were explicitly studied. In particular, they proved:

Theorem 1 [Chapman and Siebenmann, 1976] *A Hilbert cube manifold X admits a Z -compactification iff each of the following is satisfied.*

- a) X is inward tame at infinity.
- b) $\sigma_\infty(X) \in \varprojlim \{ [K \setminus \text{tilde}] \pi_1(X \setminus A) \mid A \subset X \text{ compact} \}$ is zero.
- c) $\tau_\infty(X) \in \varprojlim^1 \{ Wh \pi_1(X \setminus A) \mid A \subset X \text{ compact} \}$ is zero.

Near the end of their paper, Chapman and Siebenmann observe that the notion of inward tameness and the definitions of σ_∞ and τ_∞ can also be applied to locally compact ANRs and that a Z -compactifiable ANR will necessarily satisfy a)-c). The key to these observations is a theorem by R.D. Edwards [Ed] which guarantees that if X is a locally compact ANR then $X \times Q$ is a Hilbert cube manifold. Chapman and Siebenmann then pose the following:

Question: If a locally compact ANR, X , satisfies a)-c), must it be Z -compactifiable?

Equivalent Question: If X is a locally compact ANR and $X \times Q$ is Z -compactifiable, must X be Z -compactifiable?

These questions were later included in *Open Problems in Infinite Dimensional Topology*-an appendix to Chapman's CBMS Lecture Notes, and in the 1979 and 1990 updates to that list (See [Ch], [Ge] and [We]). The following provides a negative answer to the above questions.

Proposition 2 *There exists a non- Z -compactifiable locally compact 2-dimensional polyhedron X such that $X \times Q$ admits a Z -compactification.*

The construction of X is not difficult and will be given below. Our proof that X satisfies

the desired properties is quite complicated and reader is referred to [Gu].

Construction of X :

Let $\Theta: S^1 \rightarrow S^1$ be the degree 1 map which wraps the unit circle around itself twice counterclockwise and then once back in the clockwise direction. Let K denote the mapping cylinder of Θ and $L, L' \subset K$ denote the domain and range ends of K , respectively.

Let $\{K_i\}_{i=1}^{\infty}$ be a disjoint collection of copies of K and let $L_i, L'_i \subset K_i$ be the corresponding copies of L and L' . For each i , let $h_i: L'_i \rightarrow L_{i+1}$ be a homeomorphism. Our example X is obtained by gluing the K_i 's together via the h_i 's; more precisely, $X = (\cup_{i=1}^{\infty} K_i) \setminus \text{diagup} \{x \sim h_i(x) \text{ for each } x \in L'_i \text{ and } i=1,2,3,\dots\}$.

Questions:

The following problems remain open.

1. (from the Infinite Dimensional Topology List) If Y is a locally compact polyhedron, when does Y admit a Z -compactification? (In other words, what additional conditions must be added to those of Chapman and Siebenmann in order to ensure Z -compactifiability?)
2. Are Chapman and Siebenmann's conditions sufficient if Y is a finite dimensional open manifold?
3. If J is a finite $K(\pi, 1)$, does its universal cover, $[J \setminus \text{tilde}]$ satisfy any (or all) of Chapman and Siebenmann's conditions? Is $[J \setminus \text{tilde}]$ Z -compactifiable? What if J is a closed aspherical manifold?

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Problem Session

1. Are all the boundaries of a Coxeter group cell-like equivalent?
2. Let X be a resolvable, generalized 4-manifold. What additional property would ensure that X is a manifold?

It is known that if Y is a 3-manifold that satisfies the *spherical simplicial approximation property* (maps of 2-spheres into Y can be approximated by maps whose images are homotopically tame), then $Y \times \mathbb{R}$ is resolvable. Conjecture: $Y \times \mathbb{R}$ is a 4-manifold.

3. Let n be a non-negative integer. A set $X \subset \mathbb{R}^2$ is an *n-point set* if $|X \cap L| = n$ for all lines $L \in \mathbb{H}$. A set $X \subset \mathbb{R}^2$ is a *partial n-point set* if $|X \cap L| \leq n$ for all lines $L \in \mathbb{H}$. It is known that 2-point sets have dimension 0, n-point sets can have dimension 1 for $n \geq 4$, and 3-point sets cannot contain non-trivial subcontinua. What can be said about the dimension of 3-point sets? It is also known that if a partial 2-point set has dimension 1, then it must contain an arc. Is this true for n-point sets for $n \geq 2$?

4. In S^3 , let ξ denote the field of planes orthogonal to the fibers of a Hopf fibration. Look at the flows of S^3 that preserve ξ . If K is a fibered knot in S^3 , is it possible to isotop K in S^3 so that the Hopf fibers become transverse to some such flow?

5. Do there exist finitely presented groups, Γ_1, Γ_2 , such that $\Gamma_1 \not\cong \Gamma_2$, $\Gamma'_1 \cong \Gamma_2$ and $\Gamma_1 \cong \Gamma'_2$, and Γ_i/Γ'_i is cyclic of order d_i ($i = 1, 2$) with d_1, d_2 relatively prime? If so, can Γ_i be chosen to be the fundamental group of a finite, spherical complex?