Proceedings

Sixteenth Annual Workshop in Geometric Topology

Hosted by the University of Wisconsin-Milwaukee
June 10-12, 1999

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Sixteenth Annual Workshop in Geometric Topology

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Principal Speaker: Robert D. Edwards, U.C.L.A.
Topic: *Cantor groups, their classifying spaces, and their actions on ENR's*

The Sixteenth Annual Workshop in Geometric Topology was held at the University of Wisconsin–Milwaukee on June 10-12, 1999. A list of all participants may be found on the following page. These proceedings contain a brief description of the three one hour talks given by principal speaker Robert D. Edwards as well as summaries of several of the 20 minute talks given by other participants. In accordance with tradition, the workshop concluded with a problem session. Several questions posed during this problem session are listed at the end of these proceedings.

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UWM CONFERENCE INTRODUCTION TO: THE ESSENTIALITY OF p-ADIC CANTOR GROUP CLASSIFYING SPACES, AND THE NONEXISTENCE OF FREE CANTOR GROUP ACTIONS ON ENR'S

ROBERT D. EDWARDS [1999.06.08]

ABSTRACT. It is known that for any *cantor group* (:= a topological group homeomorphic to the cantor set, i.e. an infinite second-countable profinite topological group), e.g. the p-adic integers, there is a classifying space which classifies *arbitrary* free actions by the group. [Free actions of cantor groups are more general than *principal* actions, i.e. *locally trivial* (= *locally sliceable*) free actions, for which e.g. the Milnor-join or the bar-construction principal bundle suffices.] Furthermore, just like their principal-action counterparts, these free-action classifying spaces have natural finite dimensional skeleta, which classify those free actions which in addition have finite dimensional quotient spaces. This paper shows (by analogy with the finite group case, if you wish) that each of these n-skeleta is homotopically *persistently* n-dimensional, that is, cannot be homotoped into a (substantially) lower dimensional subspace. (In the analogy with $\mathbb{Z}/2\mathbb{Z}$, the corresponding (classical) fact is that real-projective n-space $\mathbb{RP}^n$ cannot be homotoped (in say $\mathbb{RP}^\infty$) into any lower dimensional subspace, e.g. into $\mathbb{RP}^{n-1}$.) In the case of the p-adic integer group, this can be interpreted as saying that, although its classical (principal-action) cohomological dimension is 1, its free-action cohomological dimension is infinite.

This result has consequences with respect to the Hilbert-Smith Conjecture and its variations. The primary immediate corollary is: There is no free action by any cantor group on any ENR (= euclidean neighborhood retract; e.g. a manifold) (having finite dimensional orbit-quotient space).

SETTING THE SCENE: THE MOTIVATION

This paper is motivated by the following [sweeping and profound, in my humble opinion!]

Free-Set Z-Set Conjecture [for Cantor-Group Actions on ENR's]. *Suppose that $G \times E \to E$ is an action of a cantor group G on an ENR E. Then the free-set of the action is a homology-Z-set in E.*

Restricting attention to the free-action case leads to the

Conjellary. *There is no free action by any cantor group on any ENR.*

Proof. The maximal homology-Z-set (= "homology-Z-boundary") of an ENR is a codense G_δ subset (that is, its complement is a dense F_σ subset). □

Alternative, equivalent restatements of the the two preceding assertions are given at the end of this section.

Although weaker than the FSZS Conjecture, perhaps better known is the following

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Key words and phrases. cantor group actions, p-adic group actions, p-adic classifying spaces, Hilbert-Smith Conjecture.

Conjellary (The Hilbert-Smith Conjecture). *Any compact subgroup of the homeomorphism group of a connected manifold (or cell-complex) is a Lie group. Equivalently, no cantor group (e.g. the p-adic integers) can act effectively on a connected manifold (or cell-complex).*

Discussion. The FSZS Conjecture implies that the free set of any effective cantor group action would have to be empty. Now one applies theorems of Montgomery and Newman [refs?] to establish a (locally) uniform bound on orbit size, showing that in fact the action would have to factor through that of a finite group. Incidentally, one can relax *connected* to allow *finitely* many components. However, any cantor group acts effectively (but not freely!) on a countably-infinite set.

Here are the relevant definitions, and some examples.

Definition. A *cantor group* is a topological group whose underlying space is (homeomorphic to) the cantor set.

Such a group can be characterized as an *infinite second-countable profinite* group, since a cantor group is an inverse limit (i.e. *projective* limit) of *finite* groups. (This assertion amounts to/can be regarded as the 0-dimensional case of the fundamental Peter-Weyl Theorem, and is an enjoyable Exercise.)

Reminder. (Important!): Finite groups are *not* cantor groups.

Examples of cantor groups:

1. Any countably-infinite product of (nontrivial) finite groups. In this regard, for products of cyclic groups the notation $\mathbb{Z}^\infty/\boldsymbol{q}\mathbb{Z}^\infty$ is useful, where $\boldsymbol{q} = (q_1, q_2, \dots)$ is a given sequence of integers in $\mathbb{Z}_{\geq 2}$, and $\mathbb{Z}^\infty/\boldsymbol{q}\mathbb{Z}^\infty := \bigtimes_{k=1}^{\infty} \mathbb{Z}/q_k\mathbb{Z}$.
2. The p-adic integer group $\hat{\mathbb{Z}}_p := \operatorname{inv}\lim(\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \dots)$ (often called the p-adic integers, for short).

Cantor groups may be non-abelian (e.g. as in Example (1)), but, as we will see in time, the issues of concern in this paper quickly reduce to the abelian case, and then to the p-adic integer case. (But there's no sense making these restrictions until we get there.)

Definition. A *euclidean neighborhood retract* (= ENR) is any space embeddable as a subset of some euclidean space so as to be a retract of some neighborhood there.

Examples: Manifolds, finite-dimensional simplicial- and cell-complexes, the Mandelbrot set (conjecturally), and many more. We recall that an ENR can be characterized as a locally compact, locally connected finite dimensional separable metrizable space.

Definition. The *free set* of the action is $\{x \in E \mid G \to Gx \text{ is 1-to-1}\}$, that is, the subset of E upon which G acts freely.

Definition. A *homology-Z-set* in an ENR E is any subset of the (co)homological "boundary" of $E :=$ those points of E which are (co)homologically invisible in E, i.e., points $x \in E$ for which $H_*(E, E \neg \{x\}) = 0$ $(= H^*(E, E \neg \{x\}))$.

E.g., for a manifold, the (maximal) homology-Z-boundary is the ordinary boundary. By way of context and contrast, the (more common) notion of Z-*set* (= *homotopy-Z-set*) in an ENR means any subset which is *homotopically* invisible, that is, a subset $Z \subset E$ such that for any open $U \subset E$, the inclusion $U \setminus Z \hookrightarrow U$ is a homotopy-equivalence. (If we replace 'homotopy' here with 'homology,' we have a characterization of homology-Z-set.)

As suggested above, the Free-Set Z-set Conjecture is the proper form of (and stronger than) the classical *Hilbert-Smith Conjecture*, regarding which the problem for 50+ years has been to rule out effective actions on manifolds by the p-adic integers.

Examples of cantor group actions on ENR's.

1. To any cantor group $G = \operatorname{inv}\lim(G_1 \underset{\phi_2}{\twoheadleftarrow} G_2 \underset{\phi_3}{\twoheadleftarrow} G_3 \twoheadleftarrow \dots)$ is associated a contractible ENR (= ER) *cantor fan* $\mathrm{CF}(G)$ upon which G acts effectively, with the free set being the Z-set endpoint-set. $\mathrm{CF}(G)$ is the inverse limit of the mapping telescope sequence of the ϕ_k's (beginning say with the trivial homomorphism $\phi_1 \colon G_1 \to G_0 := \{1\}$), with Z-endpoint-set Z_G, which is a copy of G. See Figure (1) for the example where $G = \bigtimes_{k=2}^{\infty} \mathbb{Z}/k\mathbb{Z}$.
2. By appropriately ramifying the construction in (1), attaching feelers everywhere-densely to feelers, one can enlarge the cantor fan to a ER cantor superfan $\mathrm{CSF}(G)$, in which the free-Z-endpoint-set is a dense G_δ-subset of $\mathrm{CSF}(G)$. (Exercise.)
3. Homology Z-set example: Let G be a cantor group as in (1), and let K be a finite acyclic (but not-necessarily-contractible) complex. Then the cantor fan $\mathrm{CF}(G)$ can be "blown-up by K" over each non-endpoint, obtaining $\mathrm{CF}(G, K) := (\mathrm{CF}(G) \times K)/\{\{z\} \times K \mid z \in \mathrm{Z}_G\}$ (decomposition-space notation), which is an interesting G-ER. (To get off the ground wonderstanding it, recall that the suspension ΣK of K is contractible.)

To wrap up this introduction, we offer equivalent restatements of the two conjectures presented at the beginning of this section, recasting them in a somewhat broader, more positive form.

Free-Set Z-Set Conjecture [for Locally Compact Topological Group Actions on ENR's]. *Suppose that $G \times E \to E$ is an (arbitrary continuous) action of a locally compact topological group G on an ENR E. Then the union of the non-locally-connected orbits is a homology-Z-set in E, i.e., lies in the homology-Z-boundary of E.*

Restricting attention to the free-action case leads to the

Conjellary. *Suppose that $G \times E \to E$ is a free action of a locally compact topological group G on an ENR E. Then G is a Lie group.*

These conjectures are related to their initial versions by using the amazing result (= the "solution" of (the first half of) Hilbert's 5th problem, recalled in more detail below), that a locally compact topological group which is not locally connected (in particular, which is not a Lie group) must contain a cantor subgroup.

In this paper we address the central (initial) case for these conjectures/issues, namely the free-action (cantor group) case.

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A 1-DIMENSIONAL MENGER CURVE IN STRONG GENERAL POSITION

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ABSTRACT. We show how to place a Menger Curve in strong general position in the sense that there is a bound to the number of times that a hyperplane can intersect it.

1. INTRODUCTION

In 1970 Berkowitz and Roy [1] introduced the theory of strong general position for finite simplicial complexes. This was a generalization of standard general position for simplicial complexes in that it provided a bound on the number of simplexes that a hyperplane can intersect. More specifically they stated the following theorem.

Theorem 1.1. *If K is a complex and $f : K \to R^n$ is a semi-linear map and $\epsilon > 0$, then there is a semi-linear map $g : K \to R^n$ such that $d(g(v), f(v)) < \epsilon$ for each vertex v of K, and for each hyperplane H in R^n with $dimH < n$ and each $m < n - dimH$, the number x of pairwise disjoint simplexes of K of dimension not exceeding m whose images under g intersect H satisfies the following inequality*

$$x \leq \frac{(n - dimH)(1 + dimH)}{n - m - dimH}.$$

A proof of this theorem is given in [4] and several interesting applications appear in [3],[4],[6].

In this paper we will show how to generalize the techniques of strong general position to handle Menger curves. That is we will construct a 1-dimensional Menger curve in R^3 such that no line intersects the curve more than 4 times. We will also generalize this to higher dimensional Menger curves.

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2. Strong General Position

We will not go into the details of placing a simplicial complex in strong general position. Such details can be found in [4]. In summary, if all the coordinates of all the vertices of the complex are distinct and form an algebraically independent set then the complex is in strong general position. This means that it satisfies the bound given in Theorem 1.1. The fact that a simplicial complex with countably many vertices can be placed in strong general position by moving vertices less than any given ϵ and extending linearly to the rest of the complex is proven in [5]. Another important result proven in [5] that we will use is the following:

Theorem 2.1. *If K is a simplicial complex in strong general position in R^n and H is a hyperplane, then there exists a $\delta > 0$ such that the number x of δ neighborhoods of disjoint simplexes of dimension $m < n - dimH$ that H can intersect satisfies*

$$x \leq \frac{(n - dimH)(1 + dimH)}{n - m - dimH}.$$

3. Menger Curves

In this section we will show how to place Menger curves in strong general position. For a good reference on Menger curves and their properties see [2].

Let M_0 denote a 3-simplex in R^3 and let M_0^0 and M_0^1 denote its zero and one skeletons respectively. Let M_1 denote the second derived neighborhood of M_0^1. In general let M_n denote the second derived neighborhood of M_{n-1}^1. Note that $M_{n-1} \subset M_n$ for all n. Let $M = \cap_{i=0}^{\infty} M_i$. Then M is a Menger curve. Note that for any i we have $M \subset M_i$ where M_i is a regular neighborhood about the 1-skeleton of the previous stage.

This is a standard construction of a Menger curve. What is important in the construction is that at each stage we are building a regular neighborhood of the 1-skeleton of the previous stage and that the diameter of these regular neighborhoods is going to zero. Using second derived neighborhoods is a standard method. But alternatively we could have used finer barycentric subdivisions to construct arbitrarily small regular neighborhoods at any stage.

We will now give a brief outline of our strategy to place Menger curves in strong general position. Place the first stage in strong general position. Apply Theorem 2.1 to get the δ associated with this simplicial complex. Use this δ to determine how fine of barycentric subdivision to use when building the regular neighborhood about the 1-skeleton for the next stage. Then build a map that places that stage in strong general position without moving points outside of the δ neighborhoods. Repeat this process ad infinitum and compose the maps. We will show that the image of the Menger curve under this map has the desired property.

We will now discuss the details of this construction. Assume M_0 is in strong general position and let $\epsilon > 0$ be given. Let M_1 denote the second derived regular neighborhood about M_0^1. Let $\epsilon_1 = \frac{\epsilon}{2}$.

We first construct a map $f_1 : R^3 \to R^3$ that has the following properties:

1. $f_1(v) = v$ for all $v \in M_0^0$,
2. if $v \in M_1^0 \setminus M_0^0$ then $d(v, f_1(v)) < \epsilon_1$,
3. f_1 is linear on M_1,
4. $f_1(M_1)$ is in strong general position,
5. f_1 moves no point in M_1 by more than $\frac{\epsilon}{2}$,
6. for all $v \in M_1^0$, $f_1(v)$ is in the star of v relative to the simplicial complex M_1,
7. f_1 is fixed outside some neighborhood of M_0,
8. f_1 is a homeomorphism that can be realized by an ambient isotopy.

Since the simplicial complex $f_1(M_1)$ is in strong general position we can apply Theorem 2.1 to get a δ_1 such that no line in R^3 can intersect more than four δ_1 neighborhoods about disjoint 1-simplexes of $f_1(M_1)$. The purpose of the δ here, and throughout the rest of the construction, is the following. We want to restrict the number of times a line can intersect the image of the 1-skeleton of $f_1(M_1)$ after we have finished constructing all the maps. But subsequent maps subdivide and move these 1-simplexes repeatedly. So we use the fact that not only is there a restriction on the number of 1-simplexes in $f_1(M_1)$ that a line can intersect, there is also a restriction on the number of δ_1 neighborhoods about the 1-simplexes that a line can intersect. We will construct all subsequent maps so that no points in these 1-simplexes are ever moved out of the δ_1 neighborhood of that simplex. We will repeat this at all subsequent stages. In fact we will perform the construction such that the entire Menger curve lies in the δ neighborhoods and stays in the δ neighborhoods. These are the crucial facts in our final argument that the image of the Menger curve is in strong general position.

Let M_2 be a derived regular neighborhood about M_1^1 using a barycentric subdivision chosen so finely that the mesh of the image of M_2 under f_1 is less than $\frac{\delta_1}{4}$. Let ϵ_2 be less than the diameters of all simplexes in $f_1(M_2)$ and $\frac{\epsilon_1}{2}$. Note that ϵ_2 is also less than $\frac{\delta_1}{4}$ since it is smaller than the mesh of $f_1(M_2)$. We now construct a map $f_2 : R^3 \to R^3$ that satisfies the following properties:

1. $f_2(f_1(v)) = f_1(v)$ for all $v \in f_1(M_1^0)$,
2. if $v \in f_1(M_2^0) \setminus f_1(M_1^0)$ then $d(v, f_2(v)) < \epsilon_2$,
3. f_2 is linear on $f_1(M_2)$,
4. $f_2(f_1(M_2))$ is in strong general position,
5. f_2 moves no point in $f_1(M_2)$ by more than $\frac{\epsilon}{4}$,
6. for all $v \in f_1(M_2^0)$, $f_2(v)$ is in the star of v relative to the simplicial complex $f_1(M_2)$,
7. $f_2(f_1(M_2))$ lies within the union of the $\frac{\delta_1}{2}$ neighborhoods about the 1-simplexes in $f_1(M_1^1)$,
8. If x is a point on a one simplex of $f_1(M_1)$ then $d(x, f_2(x)) < \frac{\delta_1}{4}$
9. f_2 is fixed outside some neighborhood of $f_1(M_1)$,
10. f_2 is a homeomorphism that can be realized by an ambient isotopy.

Note that by property 7 there is still plenty of room within the δ_1 neighborhoods about $f_1(M_1^1)$ to construct the rest of the Menger curve and to keep its images under all subsequent maps within this neighborhood.

Now the simplicial complex $f_2(f_1(M_2))$ is in strong general position so there is a δ_2 associated to it by Theorem 2.1 such that no line can hit more than four δ_2 neighborhoods about 1-simplexes in $f_2(M_2)$.

In general let δ_{n-1} be the minimum of the δ associated to $f_{n-1} \circ \cdots \circ f_1(M_{n-1})$ and $\frac{\delta_{n-2}}{2}$. Choose M_n to be a derived regular neighborhood of M_{n-1}^1 chosen so that the mesh of its image under $f_{n-1} \circ \cdots \circ f_1$ is less than $\frac{\delta_{n-1}}{4}$. Let ϵ_n be less than the minimum of $\frac{\epsilon_{n-1}}{2}$ and the diameters of all simplexes in $f_{n-1} \circ \cdots \circ f_1(M_n)$ and thus less than $\frac{\delta_{n-1}}{4}$. Then construct a map $f_n : R^3 \to R^3$ that has the following properties:

1. $f_n(f_{n-1} \circ \cdots \circ f_1(v)) = f_{n-1} \circ \cdots \circ f_1(v)$ for all $v \in M_{n-1}^0$,
2. if $v \in f_{n-1} \circ \cdots \circ f_1(M_n^0) \setminus f_{n-1} \circ \cdots \circ f_1(M_{n-1}^0)$ then $d(v, f_n(v)) < \epsilon_n$,
3. f_n is linear on $f_{n-1} \circ \cdots \circ f_1(M_n)$.
4. $f_n \circ f_{n-1} \circ \cdots \circ f_1(M_n)$ is in strong general position,
5. f_n moves no point of $f_{n-1} \circ \cdots \circ f_1(M_n)$ by more than $\frac{\epsilon}{2^n}$,
6. for all $v \in f_{n-1} \circ \cdots \circ f_1(M_n^0)$, $f_n \circ \cdots \circ f_1(v)$ is in the star of v relative to the simplicial complex $f_{n-1} \circ \cdots \circ f_1(M_n)$,
7. $f_n \circ \cdots \circ f_1(M_n)$ lies within the union of δ_i neighborhoods about 1-simplexes of $f_i \circ \cdots \circ f_1(M_i^1)$ for all $i \in \{1, \cdots, n-1\}$,
8. no point on a one simplex $\sigma \in f_i \circ \cdots \circ f_1(M_i^1)$ is moved out of a δ_i neighborhood of σ by $f_n \circ \cdots \circ f_{i+1}$ for all $i \in \{1, \cdots, n-1\}$,
9. f_n is fixed outside some neighborhood of $f_{n-1} \circ \cdots \circ f_1(M_{n-1})$,
10. f_n is a homeomorphism that can be realized by an ambient isotopy.

Define $f : R^3 \to R^3$ to be the composite of the countable collection of maps $\{f_n\}$. That is $f = \cdots \circ f_n \circ \cdots \circ f_2 \circ f_1$. Let $M = \bigcap_{n=0}^{\infty} M_n$. Then M is a Menger curve.

The map f has the following properties:

1. f moves no point of M more than ϵ,
2. f is a homeomorphism that can be realized by an ambient isotopy,
3. f is fixed outside some neighborhood of M,
4. If $x \in \bigcup_{n=0}^{\infty} \mid M_n^1 \mid \subset f(M)$ and i is the first stage such that $x \in\mid M_i^1 \mid$ and σ_x is the 1-simplex in M_i^1 containing x then $f(x)$ lies within a δ_i-neighborhood of $f_i \circ \cdots \circ f_1(\sigma_x)$,
5. for each index i, $f(M)$ lies within the union of the δ_i neighborhoods about $f_i \circ \cdots \circ f_1(M_i^1)$.

We will now show that $f(M)$, the image of the Menger curve, is in strong general position in the sense that no line intersects $f(M)$ in more than four points. Assume that the line l intersects $f(M)$ in 5 points say $x_1, \cdots, x_5$. For each i either $x_i \in f(\mid \bigcup_{n=o}^{\infty} M_n^1 \mid)$ or x_i is in the closure of this set. That is either x_i is the image of a point on a 1-simplex of some stage or it is a limit point. If it is the image of a point on a 1-simplex of say the jth stage then by property 4 of f the point x_i is within the δ_j neighborhood about that 1-simplex, and similarly for all subsequent stages. If x_i is a limit point then by property 5 of the map f the point x_i lies within a δ_j neighborhood of some 1-simplex in $f_j \circ \cdots \circ f_1(M_j^1)$. Thus in either case the points x_i lies within δ_j neighborhoods about 1-simplexes of a simplicial complex which is in strong general position. By increasing the index j large enough we can assume that these are δ_j neighborhoods about disjoint 1-simplexes. But this contradicts Theorem 2.1. Thus there can be at most 4 points in the intersection of the line l and $f(M)$.

These same techniques can be applied in higher dimensions to form k-dimensional Menger curves in R^n such that the number of times a hyperplane H can intersect the curve is $\leq \frac{(n-dimH)(1+dimH)}{n-k-dimH}$. More details on these constructions can be found in [5].

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Hilbert's fifth problem: a survey

By Sören Illman

The following text was originally written for the Proceedings of the International Conference on Mathematics and Applications, dedicated to the 90th Anniversary of L.S. Pontryagin, Moscow, Russia, August 8–September 6, 1998, and it will appear in those Proceedings.

Of the twenty-three mathematical problems that Hilbert posed at the Second International Congress of Mathematicians in Paris 1900, see [10], the fifth problem is concerned with Lie's theory of transformation groups, and in a second latter part of the problem with what Hilbert calls "infinite groups," but which are not groups in the modern use of the term. The questions in this second part of the fifth problem concern functional equations and difference equations, and have for example connections with the work of N.H. Abel. These questions lie completely outside the theory of transformation groups, and we shall not discuss them here any further. We refer to [1] for the "state of the art," in 1989, in this area of research.

Recall that a topological transformation group consists of a topological group G, a topological space X, and a continuous action of G on X, that is a continuous map

(1) $$\Phi: G \times X \to X, \quad (g, x) \mapsto gx,$$

with the following two properties,

1) $ex = x$, for all $x \in X$, where e is the identity element in G, and
2) $g(g'x) = (gg')x$, for all $g, g' \in G$, and all $x \in X$.

A topological group G is a *Lie group* if G is a real analytic manifold and the multiplication $\mu: G \times G \to G$, $(g, g') \mapsto gg'$, is a real analytic map. It then follows, by using the real analytic implicit function theorem, that the map $\iota: G \to G$, $g \mapsto g^{-1}$, is real analytic. In the case when G is a Lie group, X is a real analytic manifold, and the action Φ of G on X is real analytic, we have a real analytic transformation group.

Most of the natural examples of group actions in geometry and many other parts of mathematics are real analytic group actions, i.e., real analytic transformation groups. We suggest that a real analytic transformation group be called a *Lie transformation group.*

In his fifth problem Hilbert asks the following. Let G be a locally euclidean topological group, and let M be a locally euclidean topological space, i.e., M is a topological manifold, and suppose that we are given a continuous action

(2) $$\Phi: G \times M \to M$$

of G on M. Is it then always possible to choose the local coordinates in G and M in such a way that the action Φ becomes real analytic? In other words, is it possible to give the topological manifolds G and M real analytic structures so that Φ is real analytic?

In his discussion of the fifth problem Hilbert also expresses the possibility that some assumption of differentiability is actually unavoidable for a positive answer to the question in (2). Hilbert mentiones the theorem, announced by Lie [16] but first proved by F. Schur [31], which says that any transitive C^2 transformation group can be made real analytic by means of suitable coordinate changes. This result can be considered to be the origin of Hilbert's fifth problem, cf. [30, p. 177–178].

Let us first discuss the special case, of Hilbert's question, where $M = G$. In this case the question is whether we can give G a real analytic structure such that the multiplication

$$\mu: G \times G \to G \tag{3}$$

is real analytic.

In this special case the answer to Hilbert's question is always *yes*. This affirmative answer is obtained by combining the result in Gleason [8], with the result in Montgomery–Zippin [21], and we can express the combined result as follows.

Theorem 1. *Every locally euclidean group is a Lie group.*

We say that a topological group G does not have small subgroups if there exists a neighborhood of the identity element which contains no other subgroup than the trivial subgroup $\{e\}$. It is easy to see from the structure of one-parameter subgroups of a Lie group that a Lie group does not have small subgroups, see [7, p. 193]. Gleason proves in [8] that every finite-dimensional, locally compact, topological group G that does not have small subgroups is a Lie group. In [21] Montgomery and Zippin prove, by inductively using the above result of Gleason, that a locally connected, finite-dimensional, locally compact topological group does not have small subgroups. Since a locally euclidean topological group is clearly both locally connected, locally compact, and finite-dimensional, we see that [8] and [21] together prove Theorem 1.

This affirmative result is often considered as the solution of Hilbert's fifth problem, but it should be noted that Hilbert's question is more general and is concerned with transformation groups, cf. also Montgomery [20, p. 185]. We refer to Montgomery [19, p. 442–443] for some interesting speculation, made in 1950, concerning the possible answers to Hilbert's general question in (2). An authoritative and very good discussion of the state, in 1955, of Hilbert's fifth problem is given in Montgomery–Zippin [23, Section 2.15].

Before the general result in Theorem 1 was proved by Gleason, Montgomery and Zippin, the result had been known in some special cases. It follows by von Neumann [24] that Theorem 1 holds when G is compact. For commutative groups Theorem 1 was proved by Pontryagin [29, Theorem 44].

Let us here also mention that it is proved in Pontryagin [29, Chapter IX] that each C^k group, $k \geq 3$, can be made into a real analytic group, i.e., into a Lie group. G. Birkhoff [4] proved that each C^1-group can be made into a Lie group.

Let us now return to Hilbert's general question whether it is possible to give G and M real analytic structures such that the group action in (2) becomes real analytic. We have already seen above that a locally euclidean group G can always be given a real analytic structure so that it becomes a Lie group, and moreover it follows by a well-known, very basic, theorem for Lie groups, see e.g. [9, Theorem II.2.6], that such a real analytic structure on G is strictly unique. Hence we can now assume that G is a Lie group, and that M is a topological manifold on which G acts by a continuous action Φ as in (2), and we are asking if M can be given a real analytic structure such that Φ becomes real analytic.

In [2] Bing constructs a continuous action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ that cannot be C^r smooth for any $r \geq 1$, and hence in particular it cannot be real analytic. If one in Bing's example instead considers the action to be on the one-point compactification S^3 of $\mathbb{R}^3$, one obtains a continuous action of $\mathbb{Z}_2$ on S^3, with the property that the fixed point set is a horned sphere in S^3. Montgomery–Zippin [22] modified the example of Bing to give an example of a continuous action of the circle S^1 on $\mathbb{R}^4$ that cannot be C^r smooth for any $r \geq 1$, and hence in particular it cannot be real analytic.

In both these examples, in [2] and [22], the action of the group $\mathbb{Z}_2$ and S^1, respectively, is not locally smooth, in the sense of [5, IV.1]. But there exist continuous locally smooth group actions that cannot be made smooth, and hence not either real analytic. For example, there exists a 12-dimensional, compact, smoothable manifold M, which admits a locally smooth effective action of S^1, but which does not admit a non-trivial smooth action of S^1 in any of the smooth structures on M, see Bredon [5, Corollary VI.9.6]. Thus we see that the answer to the general question in (2) is *no* in the case of topological continuous actions. One may in fact point out that the answer to the general question in (2) is no even for the trivial group $G = \{e\}$, since there exist topological manifolds that do not have any smooth structure, and hence also no real analytic structure. The first example of such a manifold was given by Kervaire [14].

In Montgomery–Zippin [23, p. 70] the following easy example of a C^∞ smooth action that cannot be real analytic is given. The group is the group of reals $\mathbb{R}$, and it acts in the plane by the map

$$\Phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,$$

where $\Phi(t, \mathrm{r}e^{i\varphi}) = e^{i\alpha(r)t} \cdot \mathrm{r}e^{i\varphi}$, for all $t \in \mathbb{R}$ and all $\mathrm{r}e^{i\varphi} \in \mathbb{R}^2$, $r \geq 0$ and $\varphi \in \mathbb{R}$. Here

$$\alpha\colon \mathbb{R} \to \mathbb{R}$$

is a C^∞ function such that

$$\alpha(x) = 0, \text{ for all } x \leq 1,$$

$$\alpha(x) = 1, \text{ for all } x \geq 2.$$

Clearly Φ is a C^∞ map, and Φ is an action of the group of reals $\mathbb{R}$ on $\mathbb{R}^2$. Note that for $0 \leq r \leq 1$ we have that

$$\Phi(t, \mathrm{r}e^{i\varphi}) = \mathrm{r}e^{i\varphi}, \quad \text{for all } \ t \in \mathbb{R}.$$

For $r \geq 2$ we have that

$$\Phi(t, \mathrm{r}e^{i\varphi}) = e^{it} \cdot \mathrm{r}e^{i\varphi}, \quad \text{for all } \ t \in \mathbb{R}.$$

Thus the action is the trivial action in the unit disk, and outside the open disk with radius 2 the action of $\mathbb{R}$ is by standard rotation in the plane.

This example of an action of $\mathbb{R}$ on $\mathbb{R}^2$ can however not be real analytic, in any real analytic structure on $\mathbb{R}^2$. This is because the action is the trivial action in the open unit disk $\mathring{D}^2$, and thus, if the action was real analytic it would have to be the trivial action in the whole plane, which is not the case.

If G is a compact Lie group the above kind of phenomena cannot occur. Every C^r smooth action, $1 \leq r \leq \infty$, of a compact Lie group G on a, second countable, C^r smooth manifold can be made into a real analytic action, see Matumoto–Shiota [18, Theorem 1.3]. The technique of the proof here is the same as in Palais [27], and this in turn is an equivariant version of Whitney's proof [33] of the fact that every, second countable, C^r smooth manifold can be given a real analytic structure, compatible with given C^r smooth structure, $1 \leq r \leq \infty$.

How about the general case with actions of an arbitrary Lie group G? We saw in the elementary example above that there exist C^∞ smooth actions that cannot be real analytic. However the following result holds, see Illman [12].

Theorem 2. *Let G be a Lie group which acts on a C^r manifold M by a C^r smooth Cartan action, $1 \leq r \leq \infty$. Then there exists a real analytic structure β on M, compatible with the given C^r smooth structure on M, such that the action of G on M_β is real analytic.*

In Theorem 2 the manifold M is not assumed to be second-countable, or even to be paracompact, but we wish to stress that this great generality is not an essential point. The main interest of Theorem 2 is, of course, in cases where M is second countable.

Recall that an action of G on M is said to be *Cartan* if each point x in M has a compact neighborhood A such that the set

$$G_{[A]} = \{g \in G \mid gA \cap A \neq \emptyset\}$$

is a compact subset of G, see [26]. An action of G on M is *proper* if for every compact subset A of M we have that the set $G_{[A]}$ is compact. Thus every proper action is Cartan, and Theorem 2 holds in particular for proper actions. There exist smooth actions of Lie groups that are Cartan but not proper, such actions have non-Hausdorff orbit spaces.

In the case when G is a discrete group the proper actions are the *properly discontinuous* actions, which have been the object of much research.

Theorem 2 answers Hilbert's question concerning which group actions can be made real analytic. Furthermore the answer is best possible since, as we have seen above, there exist smooth, in fact C^∞ smooth, non-Cartan actions of Lie groups that cannot be made real analytic.

Concerning the proof of Theorem 2 let us here note that we are assuming that G is an arbitrary Lie group, and hence G need not be a linear Lie group. (The first example of a connected Lie group which is not a linear Lie group is given in Birkhoff [3].) Therefore one cannot in general imbed the G-manifold M as a G-invariant subset of some finite-dimensional linear representation space for G, and hence it is not possible to use some equivariant version of Whitney's method [33], as was done in [18], in the proof of Theorem 2. Instead we use a maximality argument, involving the use of Zorn's lemma, for the global part of the proof of Theorem 2. This argument is analogous to the one used by W. Koch and D. Puppe in [15], in a non-equivariant situation. For the local technical part of the proof, one first of all needs to use the well-known result that in a paracompact smooth Cartan G-manifold there exists a smooth slice at each point, see Palais [26, Proposition

2.2.2]. (This result is extended to the non-paracompact case in [12, Proposition 1.3], but as we already mentioned this generality is not a main issue.) An important role in the technical part of the proof of Theorem 2 is played by the result on approximations of smooth slices proved in [12, Lemma 6.1].

Let us here also use the opportunity to correct a mistake in [12]. Lemma 2.3 in [12] is not correct as stated, and this was pointed out to me by Sarah Packman (a Graduate Student at Berkeley) [25]. In the proof of [12, Lemma 2.3] I refer to Lemma 2.2.8 in Hirsch [11], which is stated there only for the C^r case, $r < \infty$, but by mistake I used it in the C^∞ case, where it fails to hold. Since Lemma 2.3 of [12] is used in the proof of the main result of [12], it is important to correct the mistake, and Sarah Packman also inquired how one could do this.

The easiest and best way to correct the mistake in [12, Lemma 2.3] is simply to use another topology on the C^∞ function spaces. Instead of using the *strong* (also called the *Whitney*) topology, defined in Hirsch [11, Section 2.1 and 2.4] and in Mather [17, Section 2], on the C^∞ function spaces, one should use the topology defined in Cerf [6, Definition I.4.3.1]. We name this topology the *very-strong Whitney topology* on $C^\infty(M,N)$, it is called the 'very strong topology' in [28, p. 59]. The fact that Lemma 2.3 in [12] is valid for the very-strong Whitney topology follows by [6, p. 273].

The very-strong Whitney topology ought in fact to be the natural choice for a topology on $C^\infty(M,N)$. It is the topology which gives the right means for expressing Whitney's result concerning approximation of C^∞ maps by real analytic maps. We recall here that the strong (or Whitney) topology on $C^\infty(M,N)$, given in [11] and [17], is just the union of all strong C^r topologies on $C^\infty(M,N)$, where $1 \leq r < \infty$. Whitney's result, see [32, Lemma 6], concerning approximation of a C^∞ map by a real analytic map involves approximation of partial derivatives of increasingly high order as one approaches infinity, and this phenomenon is captured by the very-strong Whitney topology, but *not* by the strong topology.

The only other change required in [12], in order for the proof of the main result to run exactly as in [12, Section 7], is that [12, Therem 2.1] should also be given using the very-strong Whitney topology. Theorem 2.1 in [12] is the result by T. Matumoto and M. Shiota [18, Theorem 1.2], which says that the set $C^\omega_K(M,N)$ of K-equivariant real analytic maps is dense in $C^\infty_K(M,N)$. Here K denotes a compact Lie group, and Matumoto and Shiota use the strong topology (they call it the Whitney topology), defined in [11] and [17]. It is not difficult to prove that [12, Theorem 2.1] also holds for the very-strong Whitney topology, but this fact does not seem to appear in the present literature. Hence we give a proof of it in [13], where we also give a more detailed discussion of the very-strong Whitney topology and its basic properties.

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A UNIVERSAL SPACE BASED ON THE SIERPIŃSKI TRIANGULAR CURVE

Ivan Ivanšić and Uroš Milutinović

In this talk[1] we restrict ourselves to Čech-Lebesgue covering dimension and to the class of metrizable spaces. The goal is to present a result which shows that the Sierpiński triangular curve (known also as Sierpiński gasket) is an object that can be used in constructing an n-dimensional separable metric space, denoted $L_n(3)$ that is universal for separable metrizable spaces of dimension $\leq n$, i.e. if X is a separable metric space of dimension $\leq n$, then X can be topologically embedded into $L_n(3)$.

In general, if $\mathcal{C}$ is a class of topological spaces, then Y is called universal for the class $\mathcal{C}$ if Y belongs to $\mathcal{C}$, and if every X from $\mathcal{C}$ can be topologically embedded into Y. In our result mentioned above, the class $\mathcal{C}$ is the class of all separable metric spaces of dimension $\leq n$, but we shall give examples of universal spaces for many other classes. Let us point it out that Sierpiński was the originator of the theory of universal spaces.

In this note we shall review both cases of n-dimensional universal spaces, separable and nonseparable. A short history of creation of dimension theory can be found in the introduction to [H-W]. [E] also contains many historical and bibliographic notes scattered throughout the book. We start from 1915 when W. Sierpiński [S1] defined his triangular curve which we denote $\Sigma(3)$ (for reasons that will become clear later on) and proved that (almost) every of its points is a ramification point. For our purposes, the following description of Sierpiński triangular curve is convenient (the usual description is in the xy-plane).

Let ϕ_1, ϕ_2 and ϕ_3 be homotheties with coefficients $1/2$ and centers e^1, e^2 and e^3, where e^1, e^2 and e^3 are the vertices of the standard 2-simplex Σ (see figure):

$$(\phi_i(x))_j = \begin{cases} (x_i+1)/2 & \text{if } j=i, \\ x_j/2 & \text{if } j \neq i, \end{cases}$$

for all $i, j \in \{1,2,3\}$.

If

$$\Sigma_n(3) = \bigcup_{(\lambda_1,\ldots,\lambda_n)\in\Lambda^n} \phi_{\lambda_1}\circ\cdots\circ\phi_{\lambda_n}(\Sigma), \qquad \Lambda = \{1,2,3\},$$

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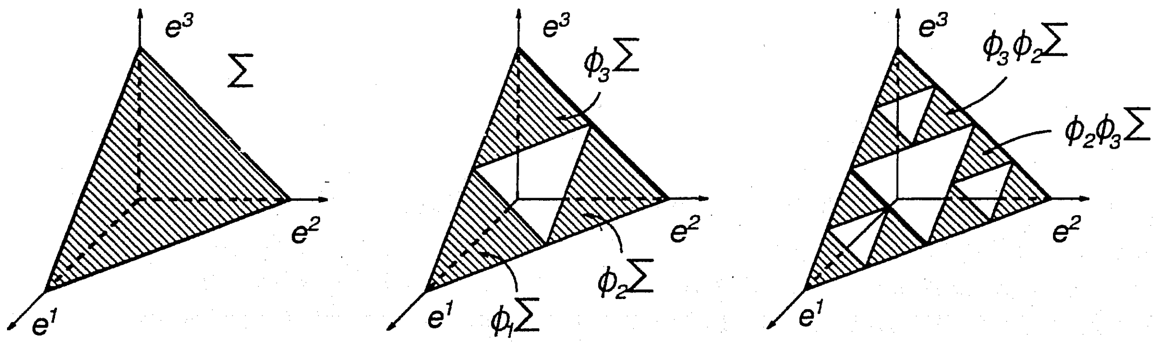


then the Sierpiński triangular curve is

$$\Sigma(3) = \bigcap_{n \in \mathbb{N}} \Sigma_n(3).$$

From this description one sees that the points of $\Sigma(3)$ which are not vertices of triangles at any stage are uniquely presented by a sequence of indices of ϕ's with terms in the set $\{1, 2, 3\}$, i.e. in a set of cardinality 3. The points of $\Sigma(3)$ which are vertices of triangles at some stage (except the first) have two presentations and these presentations are related in a particular way which will be described later. Such a description allows a generalization to a set Λ of infinite cardinality τ leading to a **generalized Sierpiński curve** which we denote $\Sigma(\tau)$. The space $\Sigma(\tau)$ is a subset of the Hilbert space $\ell_2(\tau) = \{(x_\lambda) \in \mathbb{R}^\tau \mid \sum_{\lambda \in \Lambda} x_\lambda^2 < \infty\}$ equipped with an inner-product-space structure. Consider "homotheties" $\phi_\lambda : \ell_2(\tau) \to \ell_2(\tau)$ defined analogously to above by

$$(\phi_\lambda(x))_\mu = \begin{cases} (x_\lambda + 1)/2 & \text{if } \mu = \lambda, \\ x_\mu/2 & \text{if } \mu \neq \lambda, \end{cases}$$

for all $\lambda, \mu \in \Lambda$.

Let

$$\sigma = \{(x_\lambda) \in \ell_2(\tau) \mid \sum_{\lambda \in \Lambda} x_\lambda = 1 \text{ and } 0 \leq x_\lambda \leq 1,\ \lambda \in \Lambda\}.$$

The closure of σ is called "the standard τ-simplex" and is denoted by Σ. This is also the closed convex hull of the set of unit vectors e^λ in $\ell_2(\tau)$ (i.e. $(e^\lambda)_\mu = \delta_{\mu\lambda}$). It can be easily shown that

$$\Sigma = \operatorname{cl} \sigma = \{(x_\lambda) \in \ell_2(\tau) \mid \sum_{\lambda \in \Lambda} x_\lambda \leq 1 \text{ and } 0 \leq x_\lambda \leq 1,\ \lambda \in \Lambda\}.$$

In order to define $\Sigma(\tau)$ one considers, analogously to $\Sigma(3)$, the sets

$$\Sigma_n(\tau) = \bigcup_{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n} \phi_{\lambda_1} \circ \cdots \circ \phi_{\lambda_n}(\Sigma),$$

and defines

$$\Sigma(\tau) = \bigcap_{n\in\mathbb{N}} \Sigma_n(\tau).$$

Besides $\Sigma(3)$, W. Sierpiński [S2] defined in 1916 his well known square curve M_1^2 (Sierpiński carpet) and proved that M_1^2 is universal for planar sets of dimension ≤ 1. In 1921 W. Sierpiński [S3] introduced dimension 0 and proved that the irrational numbers and the Cantor set are universal for all 0-dimensional metric compacta. The next move in this direction happened in 1926 when K. Menger [Mg-1] generalized Sierpiński's construction from $\mathbb{R}^2$ to $\mathbb{R}^3$ and proved that M_1^3 is universal for metric compacta of dimension ≤ 1. Menger also generalized the constructions of M_1^2 and M_1^3 to higher dimensions, introducing compacta M_n^m, $n = 1, \ldots, m-1$ (see, for example, [E] for the construction) and conjecturing that M_n^m is universal for the class of compacta in $\mathbb{R}^m$ of dimension $\leq n$. These were later named Menger compacta after him. Soon after Menger's mentioned result, in 1931, G. Nöbeling [Nö] discovered that the n-dimensional space $N_n^{2n+1} \subset \mathbb{R}^{2n+1}$, which consists of all points in $\mathbb{R}^{2n+1}$ that have at most n rational coordinates, is universal for all separable metrizable spaces of dimension $\leq n$. So, at this point one compares statements related to M_1^3 and N_1^3 and wonders whether M_n^{2n+1} is universal for all separable metrizable spaces of dimension $\leq n$? This was answered by S. Lefschetz [Lf] in 1931. We note that Menger's conjecture was proved in 1971 by M.A. Štanko [Š], 45 years after it was posed.

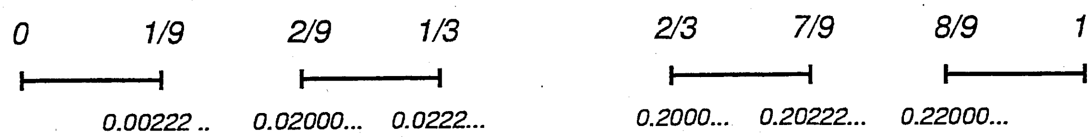
Let us now turn to nonseparable finite-dimensional metrizable spaces. Let τ be the weight of the space X, $\tau \geq \aleph_0$. Discovery of an n-dimensional universal nonseparable metric space slowly followed the separable case and used infinite-dimensional spaces. In 1947, C.H. Dowker [D] proved that the Hilbert space $\ell_2(\tau)$ is universal for metrizable spaces of weight τ. Hence, the question was around, are there n-dimensional universal spaces like in the separable case? The next infinite-dimensional object along these lines was discovered in 1957 by H.J. Kowalski [K] who introduced the so-called star space $S(\tau)$ and proved that a space of weight τ is metrizable if and only if it can be embedded into $S(\tau)^{\aleph_0}$, the countable product of star spaces. If we compare this result with Urysohn's metrization theorem, which in our context says that the Hilbert cube $I^{\aleph_0}$ is a universal space for all separable metrizable compacta, then we see that in both cases we have the countable power of a 1-dimensional space and that the role of the segment I in the separable case is played by the star space $S(\tau)$ in the nonseparable case.

The star space $S(\tau)$ is defined in the following way. Take an index set Λ of cardinality τ. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be the family of unit segments $I_\lambda = [0, 1]$, indexed by Λ. The space $S(\tau)$ is obtained by identifying the 0-points of the I_λ's and introducing the following metric:

$$d([x_\lambda], [y_\mu]) = \begin{cases} |x - y| & \text{if } \lambda = \mu, \\ |x| + |y| & \text{if } \lambda \neq \mu. \end{cases}$$

The weight of $S(\tau)$ is τ and its dimension is one.

Finally, J. Nagata constructed two n-dimensional universal spaces for n-dimensional metrizable spaces of weight τ. The first one [Ng-1] he found in the Hilbert space $\ell_2(\tau)$ but its description is complicated, while the second, which he discovered in 1963 [Ng-2], is a subset of Kowalski's space $S(\tau)^{\aleph_0}$. Denote by $K_n(\tau) \subset S(\tau)^{\aleph_0}$ the set of points in $S(\tau)^{\aleph_0}$ which have at most n nonzero rational coordinates. Then $K_n(\tau)$ is an n-dimensional universal space for all metrizable spaces of weight $\leq \tau$ and of dimension $\leq n$. Since both of Nagata's universal spaces are subsets of infinite-dimensional spaces he has asked [Ng-3] whether infinite-dimensionality can be avoided? This was answered in the affirmative by L.S. Lipscomb. In his dissertation of 1973 Lipscomb first solved the one-dimensional case, published in [L1], and then solved the n-dimensional case in 1975. Let us describe Lipscomb's n-dimensional universal space, denoted by $L_n(\tau)$.



We start from an inspiring construction of the segment from the "middle third" representation of the Cantor set. If we identify adjacent endpoints in the Cantor set we obtain a space homeomorphic to the unit segment. In the first step of this process, the points 1/3 and 2/3 are identified, in the second the points 1/9 and 2/9, 7/9 and 8/9, see illustration. Underneath those points are written their ternary representations without using the digit 1. One can see the rule that applies to representations of the identified points. First, after some place, a digit becomes periodic (same digit repeats up to infinity); second, the identified points have the same representation up to one step before the digits 0 and 2 are interchanged and the periodicity occurs. L.S. Lipscomb has applied this process to generalized Baire's 0-dimensional space of weight $\tau \geq \aleph_0$ and obtained a 1-dimensional space $J(\tau)$ which is then the main building block in building $L_n(\tau)$. For the sake of completeness we point out that generalized Baire's 0-dimensional space $N(\Lambda)$, $\tau = |\Lambda|$, is a countable product of discrete spaces Λ, i.e.

$$N(\Lambda) = \prod_{n \in \mathbb{N}} \Lambda_n, \quad \Lambda_n = \Lambda,$$

and $N(\Lambda)$ is a universal space for the class of 0-dimensional metrizable spaces of weight $\leq \tau$. Now, in $N(\Lambda)$ we identify points of the form

$$\lambda_1\lambda_2\lambda_3\ldots\lambda_{j-1}\lambda_j\mu_j\mu_j\mu_j\ldots$$
$$\lambda_1\lambda_2\lambda_3\ldots\lambda_{j-1}\mu_j\lambda_j\lambda_j\lambda_j\ldots$$

The obtained quotient space $J(\tau) = N(\Lambda)/\sim$ (see [L-1]) is metrizable, 1-dimensional and has weight τ. The classes which are doubletons are called the rational points of $J(\tau)$ and the classes which are singletons are called the irrational points of $J(\tau)$.

Now we can describe Lipscomb's n-dimensional universal space $L_n(\tau)$ for metrizable spaces of weight $\leq \tau$ and of dimension $\leq n$ (see [L-2]). This is the following subset of the $(n+1)$-st power of $J(\tau)$:

$$L_n(\tau) = \{(x_1, \ldots, x_{n+1}) \in J(\tau)^{n+1} \mid \text{at most } n \text{ coordinates are rational}\}.$$

Denote by $p : N(\Lambda) \to J(\tau)$ the quotient map and define $q : N(\Lambda) \to \Sigma(\tau)$ by

$$q(\lambda_1, \dots, \lambda_n \dots) = \bigcap_{n \in \mathbb{N}} \phi_{\lambda_1} \circ \dots \circ \phi_{\lambda_n}(\Sigma),$$

where ϕ_{λ_j} are the above homotheties and $\Sigma \subset \ell_2(\tau)$ is the standard τ-simplex. Then it is proved in [Mi] that $J(\tau)$ and $\Sigma(\tau)$ are homeomorphic, in fact there is a homeomorphism $\chi : J(\tau) \to \Sigma(\tau)$ such that $\chi \circ p = q$. The points that come from doubletons are called the rational points of $\Sigma(\tau)$ and the others are called the irrational points. This way the space

$$L_n(\aleph_0) = \{(x_1, \dots, x_{n+1}) \in \Sigma(\aleph_0)^{n+1} \mid \text{at most } n \text{ coordinates are rational}\}$$

is universal for separable metrizable spaces of dimension $\leq n$.

Since the definition of both $N(\Lambda)$ and Lipscomb's equivalence relation on $N(\Lambda)$ remain meaningful even in the case when Λ is finite a natural question in the separable case was raised: is it possible to replace $L_n(\aleph_0)$ by $L_n(k)$, k finite? The answer is yes if $k \geq 3$, proved in [I-M]. If $k = 2$ then $J(2)$ is homeomorphic to $[0,1]$ and this cannot be done (e.g. $L_1(2)$ is a planar set). But if $k = 3$ then (via the homeomorphism mentioned above) $J(3) \cong \Sigma(3)$, the Sierpiński triangular curve. The rational points of $\Sigma(3)$ are all vertices of triangles used in the constructing procedure except the vertices of the first triangle Σ — all the other points are irrational. This also holds for any $k \geq 3$, since $\Sigma(3)$ is naturally contained in $\Sigma(k)$, $k \geq 3$. Hence the space

$$L_n(3) = \{(x_1, \dots, x_{n+1}) \in \Sigma(3)^{n+1} \mid \text{at most } n \text{ coordinates are rational}\}$$

is a universal space for the class of separable metrizable spaces of dimension $\leq n$.

It seems to us that in this way the Sierpiński triangular curve has found a role in dimension theory that it started to look for some 75 years ago.

Comment on the proof. The experience from proving that $J(\tau)$ and $\Sigma(\tau)$ are homeomorphic gave geometric insight into $L_n(\tau)$ and a hint for proving the universality of $L_n(3)$. In proving the universality of $L_n(3)$ we start from a partition of an n-dimensional space X into $n+1$ 0-dimensional subspaces

$$X = X_1 \cup \dots \cup X_{n+1}$$

and apply Lipscomb's Lemma 4 of [L-2], getting slightly modified Lipscomb's decompositions. Then in several steps we modify the decompositions in a way so that it is possible to accomplish an indexing of modified decompositions yielding an embedding of X into $L_n(3)$.

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ACYCLIC RESOLUTIONS OF METRIZABLE SPACES

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ABSTRACT. Recent work of Koyama and Yokoi shows that for any Bockstein group G, and (metrizable) compactum X with $\dim_G X \leq n$, there exists a G-acyclic resolution $\pi : Z \to X$ from a compactum Z with $\dim Z \leq n+1$ and $\dim_G Z \leq n$. For arbitrary abelian groups their research provides a resolution from a compactum Z with $\dim Z \leq n+2$ and with $\dim_G Z \leq n+1$. This short exposition explains some of the history of the resolution theorems and expresses a direction of research which would allow for resolution theorems in case X comes from the class of arbitrary metrizable spaces instead of just compact metrizable spaces.

1. Introduction. Unless otherwise specified, space will mean metrizable space in this article. Map will mean continuous function.

The cohomological dimension of a space X over an abelian group G, $\dim_G X$, [DD], may be defined as follows. If $X = \emptyset$, then write $\dim_G X \leq -1$. If $X \neq \emptyset$ and n is a nonnegative integer, then $\dim_G X \leq n$ means that for every closed subset A of X and every map $f : A \to K$ where K is a CW-complex of type $K(G,n)$, there exists a map $F : X \to K$ which is an extension of f. We define $\dim_G X = \infty$ if no such n exists; otherwise, $\dim_G X = \min\{n \mid \dim_G X \leq n\}$.

A proper, surjective map $\pi : Z \to X$ is called G-acyclic if each of its fibers $\pi^{-1}(x)$ is G-acyclic. This means that $\tilde{\check{H}}^k(\pi^{-1}(x); G) \approx 0$ for all $x \in X$. One calls π cell-like if each $\pi^{-1}(x)$ has the shape of a point [MS].

1.1. FACT. *Every cell-like compactum is a continuum and is G-acyclic for all abelian groups G.* $\square$

2. Background–Resolution Theorems.

In this subject, a resolution theorem usually means a proposition showing that for a given space X which satisfies the property $\dim_G X \leq n$ for some abelian group G, there exists a proper surjective map $\pi : Z \to X$ from a space Z with control on the dimension $\dim Z$, perhaps control on $\dim_G Z$, and such that the fibers of π are either G-acyclic or

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are cell-like. Here we make a list of most of the known resolution theorems. The symbol n will stand for an element of $\mathbb{N}$.

2.1. Edwards-Walsh Theorem [Wa], 1981. *For each compactum X with $\dim_{\mathbb{Z}} X \leq n$, there exists a compactum Z with $\dim Z \leq n$ and a cell-like map $\pi : Z \to X$.*

2.2. Theorem [RS1], 1987. *For each metrizable space X with $\dim_{\mathbb{Z}} X \leq n$, there exists a metrizable space Z with $\dim Z \leq n$ and a cell-like map $\pi : Z \to X$.*

2.3. Theorem [Dr2], 1988. *Let p be a prime and X be a metrizable compactum with $\dim_{\mathbb{Z}/p} X \leq n$. Then there exists a metrizable compactum Z, $\dim Z \leq n$, and a $\mathbb{Z}/p$-acyclic map $\pi : Z \to X$.*

2.4. Theorem [MR], 1989. *For each compact Hausdorff space X with $\dim_{\mathbb{Z}} X \leq n$, there exists a compact Hausdorff space Z with $\dim Z \leq n$ and a cell-like map $\pi : Z \to X$.*

2.5. Theorem [KY1], 1994. *If X is either a metrizable space or a Hausdorff compactum and $G \in \{\mathbb{Z}, \mathbb{Z}/p\}$, then there exists a G-acyclic map $\pi : Z \to X$ where $\dim Z \leq n$ and Z is either metrizable or compact Hausdorff, respectively.*

2.6. Theorem [Dr3], 1996. *If X is a metrizable compactum and $\dim_{\mathbb{Q}} X \leq n$, then there exists a metrizable compactum Z with $\dim Z \leq n+1$ and a $\mathbb{Q}$-acyclic map $\pi : Z \to X$. If $n > 1$, then we may also conclude that $\dim_{\mathbb{Q}} Z \leq n$.*

Let us mention that in [ARS] one can find an alternate proof of 2.1.

3. Edwards–Walsh Complexes.

The first "Edwards–Walsh" complexes were constructed in [Wa]. Since that time, with the exception of [ARS], their use was instrumental in getting resolution theorems. Given an abelian group G and a simplicial complex L, an Edwards–Walsh resolution consists of a CW-complex EW (an Edwards–Walsh complex) and a map f:EW$\to |L|$ which satisfies a certain list of conditions, which one may find in [DW], but in a more refined form in [KY2]. Such resolutions provide a method for finding appropriate bonding maps for the inverse systems needed to define the space Z and the resolving map $\pi : Z \to X$ for X with the desired type of fibers and requisite dimension of Z.

Initial work on the subject of these complexes was done in [DW]. But Koyama and Yokoi [KY2] have found a problem in one of the main lemmas of [DW], which necessitated some changes in one of the definitions in [DW].

4. Koyama–Yokoi Resolution Theorems.

Having changed the definition from [DW] in a suitable manner (conditions (EW1),(EW2)), Koyama and Yokoi [KY2] have been able to prove the existence of Edwards–Walsh resolutions for a certain important class of abelian groups and with respect to arbitrary simplicial complexes. These have led to the theorems in [KY2] which we are now going to state. For the next, assume that X is a metrizable compactum, G is an abelian group, $n \in \mathbb{N}$, and $\dim_G X \leq n$. Also assume that p designates a prime number.

4.1. Koyama–Yokoi Theorems.

(a) *There exists a metrizable compactum Z with* $\dim Z \leq n+2$ *and* $\dim_G Z \leq n+1$ *and a G-acyclic map $\pi : Z \to X$.*

(b) *If $G \in \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}/p, \mathbb{Z}/p^\infty\}$ (the so-called Bockstein groups), then there exists a G-acyclic map $\pi : Z \to X$ where* $\dim Z \leq n+1$ *and* $\dim_G Z \leq n$.[1]

(c) *If G is torsion free, then there exists a metrizable compactum Z with* $\dim Z \leq n+1$ *and* $\dim_G Z \leq n$ *and a G-acyclic map $\pi : Z \to X$.*

(d) *If G is a torsion group, then there exists a metrizable compactum Z with* $\dim Z \leq n+1$ *and a G-acyclic map $\pi : Z \to X$.* □

5. Sums of Cyclic Groups.

There is a significant class (see [Fu], Ch. 3) of groups G which are canonical direct sums (called decompositions) of cyclic groups, say $\bigoplus_{i \in I} G_i$ with $G_i = \mathbb{Z}$ or $\mathbb{Z}/p_i^{k_i}$ for primes p_i and natural numbers k_i. It can be shown ([RS2]) that the cohomological dimension types relative to such groups G can be placed in one-to-one, order-preserving correspondence with the non-empty subsets of the set $\mathbb{P}$ of all primes (with the inclusion of the set $\mathbb{Z}$ as greatest element) under inclusion. For such a group G, let $\mathcal{P}(G)$ be $\{\mathbb{Z}\}$, if $\mathbb{Z}$ is a summand of the canonical decomposition; otherwise let $\mathcal{P}(G)$ be $\{p \mid \mathbb{Z}/p^k$ is a summand for some value of $k\}$.

[1] This result was obtained for $\mathbb{Q}$ in [Dr3], but the proof there is incorrect.

If $\mathcal{P}(G') \subset \mathcal{P}(G)$, then $\dim_{G'} X \leq \dim_G X \leq \dim_{\mathbb{Z}} X \leq \dim X$ for all metrizable spaces X. Moreover, $\dim_G X = \dim_{\mathbb{Z}} X$ or $\dim_G X = \sup\{\dim_{\mathbb{Z}/p} X \mid p \in \mathcal{P}(G)\}$, respectively, in the two cases described.

The authors have contemplated constructing n-dimensional acyclic resolutions for X and such G's by using induction to intersperse bonding maps for inverse sequences of complexes, extending the techniques described in [RS1], [RS2]. Observe that such resolutions are constructed in [RS2] by using the "replacement" groups: $\mathbb{Z}$ or $\bigoplus\{\mathbb{Z}/p \mid p \in \mathcal{P}(G)\}$, respectively, instead of G itself. This program for compact metrizable spaces had been carried out in [Dr1].

6. Replacement Groups.

The notion of constructing a resolution for a group G by using resolutions for "replacement" groups was used by Koyama and Yokoi ([KY2]) to construct resolutions for metric compacta relative to the group $\mathbb{Z}_{p^\infty}$, which is the direct limit of the sequence of groups $\mathbb{Z}/p^k$ under bonding maps which are multiplication by p. Thus, if $\dim_{\mathbb{Z}_{p^\infty}} X = n$, then by the Bockstein inequalities ([Ku]), $\dim_{\mathbb{Z}_p} X \in \{n, n+1\}$. In the case that this dimension is $n+1$, they obtain an $(n+1)$-dimensional resolution. They show that a resolution is $\mathbb{Z}_{p^\infty}$-acyclic if and only if it is $\mathbb{Z}/p$-acyclic.

The authors in [KY] developed their method for creating resolutions of metric compacta for groups satisfying conditions formulated by them and denoted by the symbol "(EW)". They use the Bockstein Inequalities and the Bockstein Basis Theorem to find suitable replacement groups for G.

The Bockstein Basis Theorem states that there exists a "field of coefficients", say $\mathcal{F}$, where $\mathcal{F} = \{\mathbb{Q}, \mathbb{Z}/p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)} \mid p \in \mathbb{P}\}$. The basis $\sigma(G)$ for a particular group G, a subset of $\mathcal{F}$, is determined as specified, e.g., in [Ku]. Finally $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$.

7. Conditions (EW).

The conditions (EW) assert that there is a group homomorphism $\alpha : \mathbb{Z} \to G$ such that,

(1) $\alpha \otimes \mathrm{id}_G : \mathbb{Z} \otimes G \to G \otimes G$, and

(2) $\alpha^* : \mathrm{Hom}[G, G] \to \mathrm{Hom}[\mathbb{Z}, G]$

are isomorphisms (here $\alpha^*(f) = f \circ \alpha$). Now the groups in the set $\{\mathbb{Q}, \mathbb{Z}/p, \mathbb{Z}_{(p)}, \mathbb{Z}_{(\mathcal{P})}\}$ for $p \in \mathbb{P}$ and $\mathcal{P} \subset \mathbb{P}$ satisfy (EW), but, for example, $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z} \oplus \mathbb{Z}/2$ do not.

Koyama and Yokoi create by induction a ladder of inverse sequences:

$$\begin{tikzcd}
EW(G', L_i) \arrow[d, "\omega_{i-1}"] & EW(G', L_i) \arrow[l, "\mathrm{id}"'] \\
L_{i-1} \arrow[d, "\phi_{i-1}"] & L_i \arrow[l, "g_{i-1\,i}"'] \arrow[u, "f_i"'] \arrow[d, "\phi_i"] \\
K_{i-1} & K_i \arrow[l, "f_{i-1\,i}"']
\end{tikzcd}$$

with $\lim(K_i, f_{i-1\,i}) = X$, $\lim(L_i, g_{i-1\,i}) = Z$, and the G'-acyclic map $\pi : Z \to X$. Here the $\mathrm{EW}(G', L_i)$ are Edwards–Walsh (combinatorial) resolutions for complexes L_i, and G' satisfies (EW). The diagram commutes only approximately.

8. Generalization to Metrizable Spaces. Our idea is to obtain as many of the previously stated results as we can where X is in the class of metrizable spaces, that is, it is not restricted to the class of metrizable compacta. Some of the needed techniques for dealing with the inverse sequences that arise in this area can be found in [MRS]. Others can be found in certain preprints of Millspaugh–Rubin and of Rubin–Schapiro. Still others have not yet been developed but should be attainable with some effort.

One of the barriers to an easy extension of the results is the fact that the Bockstein Basis Theorem ([Ku]) is known not to hold true for arbitrary metrizable spaces ([DRS]). Its application for metrizable compacta was much exploited in the work of [KY2]. It will be difficult to overcome its inapplicability for arbitrary metrizable X as we proceed with our ongoing research in this field.

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Spaces whose only finite sheeted covers are themselves: A survey

Mathew Timm

Question. Which spaces M that have the property that whenever $p : X \longrightarrow M$ is a finite sheeted connected covering it follows that X is homeomorphic to M.

With a little reflection it becomes clear that there are two ways in which a space M can satisfy this property: either, (1) M has no non-trivial finite sheeted covers, that is, whenever $p : X \longrightarrow M$ is a finite sheeted cover, it follows that the given covering projection p must be a homeomorphism or (2) M has a k-fold connected cover $p : X \longrightarrow M$ for some $k \geq 2$ and the total space of every connected finite sheeted cover $p : X \longrightarrow M$ is such that X is homeomorphic to M via a map that is perhaps distinct from p.

Note that any simply connected X, or more generally any X that has a fundamental group with no proper finite index subgroups, is an example of a space satisfying the first condition and spaces of the form $T^n = \overset{n}{\underset{k=1}{\times}} S^1$ and $T^n \times X$ are spaces that have non-trivial self-covers and satisfy the second condition. Thus, spaces that have the property that all their finite sheeted covers have total space homeomorphic to the base space are quite abundant.

§1. Terminology and H-connected Spaces

In this paper, a *space* M will be a compact metric space in the metric topology, usually with additional structure imposed on it. Unless it is specified to the contrary, it is assume that all spaces are connected.

Definition 1.2. Let M be a connected metric space. Then M is *trivially h-connected or H-connected*[Jungck, 1983], if whenever $p : X \longrightarrow M$ is a finite-to-1 covering projection from the connected metric space X onto M, it follows that p is a homeomorphism.

The reader is referred to the survey paper by Heath[1995]. It is a good source of information on what is known about the general question of when a compact connected metric space can be either the domain or range of an exactly k-to-1 function that has up to finitely many discontinuities. It also contains a list of interesting open problems and its bibliography is quite extensive.

For spaces with additional structure, in particular, manifolds, there are several situations that appear to be related to the notion of H-connectedness. The reader is referred to work by Daverman[1991,1993] and a sequence of papers from the late 80's through 1998 by, among others, Wright[1992] and Meyers[1988,1999] for more of this related work. This collection of work generates two problems.

Problem 1.5. Is every closed H-connected n-manifold a *codimension k fibrator.*

Problem 1.6. Which H-connected 3-manifolds cannot non-trivially cover any 3-manifold.

It is well known, see, e.g., Jaco[1983, §V], that the groups that can be a *3-manifold group*, that is, the fundamental group of a 3-manifold, must satisfy quite restrictive criteria. Three such interesting criteria that relate to the discussion at hand follow.

Theorem 1.12. The Scott-Shalen Theorem. If G is a finitely generated group that is a 3-manifold group then G is finitely presented.

Theorem 1.13. If G is a finitely generated abelian 3-manifold group then G is isomorphic to one of $1, \mathbb{Z}, \mathbb{Z}\times\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}_2$, or $\mathbb{Z}_p$ for some $p \geq 2$.

A group G is *residually finite* if the intersection of all its finite index subgroups is 1.

Theorem 1.14. Hempel[1987, 1.2] If G is the fundamental group of a compact 3-manifold whose prime factors are either virtually Haken or have infinite cyclic fundamental group, then G is residually finite.

It is conjectured that all compact 3-manifolds can be written as a connect sum in which the prime factors are as stated in Theorem 1.14. As consequence of this conjecture and Hempel's result, it is conjecture that all compact 3-manifolds have residually finite fundamental group. Therefore, since the fundamental group of an H-connected n-manifold has no proper finite index subgroups, an H-connected manifold M can have residually finite fundamental group if and only if $\pi_1(\mathrm{M}) = 1$. Thus, for 3-manifolds, the most interesting problem relating to H-connectedness is the existence problem. In the light of the preceding we state the the existence problem as a conjecture.

Conjecture 1.15. There do not exist any non-simply connected compact H-connected 3-manifolds.

§2. Spaces with a non-trivial self cover

For this section consider the more general situation where all that is assumed is that M has at least one non-trivial self-cover $p:M \to M$ Say that $|p| = k \geq 2$.

It is immediately clear that such an M has infinitely many finite-to-1 self coverings, namely the (nk)-to-1 coverings (M, p^n) where the covering projection $p^n : M \longrightarrow M$ is the composition of p with itself n times. Thus if M is a manifold or cell complex $\pi_1(M)$ is infinite and, in fact contains subgroups $K_n \cong \pi_1(M)$ such that $|\pi_1(M) : K_n| \geq\ n$ Accordingly, no finite group can be the fundamental group of a manifold that non-trivially covers itself.

Examples of spaces that have at least one non-trivial self-cover include the Mobius band M, the Klein bottle K, the n-tori $T^n = S^1 \times \ \ldots\ \times S^1$ (for which all the finite sheeted covers are self-covers) and the products $M \times T^n$ and $K \times T^n$.

If M is a compact n-manifold then ∂M is a closed $(n-1)$-manifold with finitely many components. Assume that $p : M \longrightarrow M$ is a non-trivial self-cover with $|p| = k \geq 2$. Clearly, ∂M is defined in terms of a local condition. Therefore, $(p \,|\, \partial K) : \partial K \longrightarrow \partial K$ is also a k-fold self-cover. The Classification Theorem for compact 2-manifolds, together with the fact that the Euler characteristic of a space with a nontrivial self-cover must be 0, imply that the only closed 2-manifolds that non-trivially cover themselves are the Kline bottle and the 2-torus. So, if a compact 3-manifold non-trivially covers itself, its boundary is either empty or a finite union of disjoint 2-tori and Klein bottles.

For compact 3-dimensional manifolds there are three theorems of Tollefson[1968] giving a partial classification of those compact 3-manifolds that non-trivially cover themselves. I think the first is particularly surprising and state it here. The notation $\mathbb{P}^3$ is used to denote real projective 3-space.

Theorem 2.2. A closed connected non-prime 3-manifold M non-trivially covers itself if and only if M is homeomorphic to $\mathbb{P}^3 \# \mathbb{P}^3$.

§3. h-connected spaces and hc-groups

Definition 3.1. A connected space X is *h-connected* (note the lower case h) if whenever $p : M \to X$ is a finite sheeted covering projection it follows that M is homeomorphic to X. A group G is an *hc-group* if every finite index subgroup of G is isomorphic to G. A space is *trivially h-connected* if and only if it is H-connected. An hc-group is a *trivially hc-group* if it has no proper finite index subgroups.

Note that any h-connected complex has an hc-group as its fundamental group. As there is an abundance of h-connected complexes, there is accordingly an abundance of non-trivially h-connected spaces and hc-groups. The n-tori T^n provide the most readily available infinite collection of h-connected spaces, and accordingly, $\pi_1(T^n) = \underset{k=1}{\overset{n}{\times}} \mathbb{Z}$ provides an infinite collection of hc-groups. It is also easy to see that for all $n \in \mathbb{N}$ there are $(n+4)$-manifolds that are non-trivially h-connected. For example, let G to be any non-trivial finitely presented (necessarily infinite non-abelain) group with no proper finite index subgroups. Let M be your favorite 4-manifold with $\pi_1(M) = G$. Then, the spaces $M \times T^n$ are, for all $n \in \mathbb{N} \cup \{0\}$, non-trivially h-connected (n+4)-manifolds and so, their fundamental groups $\pi_1(M \times T^n) = G \times (\underset{k=1}{\overset{n}{\times}} \mathbb{Z})$ are non-trivially hc-groups that are non-abelian and possess proper finite index subgroups. Finally note that if M is an h-connected manifold and N is a trivially h-connected manifold then $M \times N$ is h-connected.

As of this date, the topology that has been developed for such spaces is very dependent on the group theory of hc-groups. The main group theoretic results on hc-groups are those of Robinson and Timm[1998]. The result with the most obvious topological consequences follows. The *finite residual* of the group G is the normal subgroup

$$F = \bigcap \{H : H \leq G \text{ and } |G : H| < \infty\}.$$

A group G is *directly indecomposable* if whenever $G \cong H \times K$, it follows that $H = 1$ or $K = 1$. The notation $C_G(G')$ denotes the subgroup of G called the centralizer in G of G' and is defined by $C_G(G') = \{g \in G : gx = xg \text{ for all } x \in G'\}$.

Fact 3.3. If G is a finitely generated abelian hc-group then G is free abelian.

Theorem 3.4. If G is a finitely generated hc-group then, G/G' is free abelian, $(G')' = G'$, and G' is the finite residual of G.

There are interesting consequences. First, if M is an h-connected manifold with finitely generated π_1, then $H_1(X, \mathbb{Z}) \cong \bigoplus_{k=1}^{n} \mathbb{Z}$. Second, if H is a subgroup with finite index in a finitely generated hc-group G, then $G' \leq H$ and $H \lhd G$. Thus, there is the surprising result that every finite index subgroup in a finitely presented hc-group has abelian quotient and every finite sheeted covering of an h-connected m-manifold with finitely generated fundamental group is a regular abelian covering. Third, a finitely generated residually finite hc-group is free abelian. Thus, an h-connected manifold with finitely generated residually finite fundamental group has a finitely generated free abelian fundamental group.

Specializing to the case of compact non-trivially h-connected 3-manifolds we can say a bit more. First, if M is a compact h-connected 3-manifold, we know its boundary is a disjoint union of finitely many 2-tori or Klein bottles and from Fact 3.3 that it has non-trivial first homology. So, combining this with Theorem 3.4, $H_1(M)$ is free abelian and has a $\mathbb{Z}$ summand. Therefore, M has n-to-1 self-covers for all $n \in \mathbb{Z}$. In particular, M has $2k$-fold self-covers for all $k \in \mathbb{N}$. As the even index covers of the Klein bottle are the 2-torus, this implies that ∂M is a disjoint union of finitely many 2-tori. Second, since the only residually finite hc-groups are the free abelian ones, we are again forced to confront the Geometrization Conjecture and the existence question.

Conjecture 3.7. There exist no finitely presented non-abelian non-trivially hc-groups that are the fundamental group of a compact 3-manifold.

§4. *h*-connected 3-manifolds.

The results in this section are of the form "if G is an hc-group or M is an h-connected 3-manifold and some other reasonable condition holds, then G or $\pi_1(M)$ is free abelian." They indicate that it may be possible to prove that a finitely generated 3-manifold hc-group is free abelian (or a trivially hc-group) in a manner that is independent of the Geometrization Conjecture.

Reasonable group theoretic hypotheses to add are additional subgroup conditions, e.g., one could look for nilpotent or solvable finitely generated hc-groups. But, by Timm[1994] or Robinson and Timm[1998], it is know that finitely generated nilpotent hc-groups are free abelian and, more generally by Robinson and Timm[1998] it is known that finitely generated solvable hc-groups are free abelian.

For the topological considerations, we begin by looking at the usual places where one begins to look for examples in 3-manifold topology: hyperbolic spaces and knot complements.

First as an easy consequence of the Mostow Rigidity Theorem, e.g., Thurston[1982, 3.1] there is the following result.

Fact 4.1. If M is a hyperbolic 3-manifold with finite volume and π_1(M) is an hc-group, then M is trivially h-connected and so π_1(M) has no proper finite index subgroups.

Fact 4.2. If M is a knot complement of a tame knot K in S^3 and M is h-connected then K is the trivial knot and so M is the solid torus $S^1 \times D^2$.

Proof. As a consequence of Burns, Karrass, and Solitar[1987], knot complements have residually finite fundamental group. By Theorem 3.4 above, G$'$ is the finite residual and so π_1(M) is free abelian and equal to H_1(M). So $\pi_1(M) = \mathbb{Z}$. Therefore, by Dehn[1908], K is trivial. (See also Kirby[1997, Problem 1.12].)

Fact 4.3. If M is a Seifert fibered h-connected 3-manifold then π_1(M) is free abelian.

Fact 4.4. If M is an irreducible h-connected 3-manifold and π_1(M) has non-trivial center, then π_1(M) is free abelian.

The *Frattini subgroup* of a group G is the group F that is the intersection of all the maximal subgroups of G. It is know, e.g. Allenby, et al[1979], that if M is an orientable, compact, irreducible, sufficiently large 3-manifold then M must be a Seifert fibered 3-manifold and so the Frattini subgroup of π_1(M) is cyclic. So, by Fact 4.3, a non-trivially h-connected (and so necessarily orientable) compact, irreducible, sufficiently large 3-manifold would have free abelian fundamental group. As free abelian groups have trivial Frattini subgroups, this is a contradiction. So, we have the next fact. (See also Kirby[1997, Problem 3.33].)

Fact 4.5. If M is a non-trivially h-connected, compact, irreducible, sufficiently large 3-manifold then π_1(M) has trivial Frattini subgroup and, in fact, π_1(M) is free abelian.

Facts 4.4 and 4.5 suggest two interesting versions of Problem 3.7 to investigate. One or the other may be simpler to solve then the original problem.

Problem 4.6. Can it be shown that every non-trivially h-connected, irreducible, compact 3-manifold M has a fundamental group with non-trivial center.

Problem 4.7. Do there exist any finitely presented hc-groups with non-trivial Frattini subgroup.

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Problem Session

1. (Greg Conner) Suppose a Peano continuum X contains a sequence $\{l_i\}_{i=1}^{\infty}$ of homotopically nontrivial loops with the property that each l_i is homotopic to each l_j, and the l_i's converge to a point. Does it follow that $\pi_1(X)$ is uncountable?

2. (Eric Swenson) Let X be a noncompact $CAT(0)$ space and G a group acting geometrically (properly discontinuously and compactly by isometries) on X.
   (a) Does G have an element of infinite order?
   (b) Can G contain an infinite torsion subgroup?
   (The conjectures are: "yes" for (a), and "no" for (b).)

   Background. Bridson and Häfliger have shown that there are finitely many conjugacy classes of finite subgroups of G. Ballmann and Brin have shown that the answer to (a) is "yes" if X is a 2-dimensional complex. It is also known that the answer to (a) is "yes" if G fixes a point of ∂X.

3. (Steve Ferry) If X is a 3-dimensional ANR homology manifold, is $X \times \mathbb{R}^2$ a manifold?

4. (Craig Guilbault) Let M be a one-ended open manifold. If M is inward tame at infinity, must π_1 be semistable (also called Mittag-Leffler) at infinity? **Note.** M is *inward tame* at infinity if, for arbitrarily small neighborhoods N of infinity, there exist homotopies $H : N \times I \to N$ such that $H_0 = id_N$ and $H_1(N)$ has compact closure.

5. Let M be a closed aspherical manifold.
   (a) (Guilbault) Must the universal cover of M be inward tame at infinity?.
   (b) (Ferry) Must the universal cover of $M \times S^1$ be controlled inward tame at infinity?.

6. (Ric Ancel) Is there a $CAT(0)$ group with precisely n distinct boundaries, where $1 < n < 2^{\aleph_0}$?

7. (Raymond Mess) See following page.

Problem presented to the geometric topology conference by Raymond Mess of UWM and MSC TECHNOLOGIES INC.

First I will give some definitions.

Definition-Solid- A solid is defined as a bounded, connected and topology regular subset of Euclidean 3-space which is a 3-dimensional manifold with boundary.

Definition-Topology Regular-A point is topology regular if it equals closure of its interior, i.e. if A=k{int(A)}.

The question is given any non-convex solid can it be decomposed into a finite set of convex solids which through boolean combinations form the original non-convex solid.

A typical algorithm is one such as developed by Yong Se Kim currently at UWM and D. J. Wilde of Stanford . The algorithm uses convex hulls and set difference to represent a non-convex object by a boolean combination of convex components. It should be noted that non-convergent cases (the object does not decompose into a finite set of convex solids) exist for the present algorithms that I have seen. Two good papers to introduce the topic are **A Convex Decomposition Using Convex Hulls and Local Cause of Its Non-Convergence** in the Journal of Mechanical Design Sept 1992 by Wilde and Kim and **Volumetric Feature Recognition Using Convex Decomposition** by Kim in Advances in Feature Based Manufacturing , Elsevier 1994.