

Proceedings

Fourteenth Annual Workshop in Geometric Topology

Hosted by Oregon State University
June 5–7, 1997

The Fourteenth Annual Workshop in Geometric Topology was hosted by Oregon State University and was held at Corvallis, Oregon on June 5--7, 1997. There were forty-seven participants. Of these, eighteen were graduate students. A list of past workshop locations and principal speakers is included below.

Year	Location	Principal Speaker
1984	Brigham Young University	--
1985	Colorado College	Robert Daverman
1986	Colorado College	John Walsh
1987	Oregon State University	Robert Edwards
1988	Colorado College	John Hempel
1989	Brigham Young University	John Luecke
1990	Oregon State University	Robert Daverman
1991	University of Wisconsin - Milwaukee	Andrew Casson
1992	Colorado College	Mladen Bestvina
1993	Oregon State University	John Bryant
1994	Brigham Young University	Mike Davis
1995	University of Wisconsin - Milwaukee	Shmuel Weinberger
1996	Colorado College	Michael Freedman
1997	Oregon State University	James Cannon

Conference Proceedings have been produced for each workshop except the second.

These proceedings contain a summary of the three one-hour talks delivered by the principal speaker, James Cannon. Summaries of talks given by some of the other participants are also included. The success of the workshop was helped by generous funding from the Oregon State University College of Science and Mathematics Department and the National Science Foundation.

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Fredric Ancel
Craig Guilbault
Frederick Tinsley
David Wright

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Kathy Andrist	Brigham Young University
Julia Berglund	Oregon State University
Igor Biskup	Oregon State University
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James Cannon	Brigham Young University
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Greg Conner	Brigham Young University
Robert Daverman	University of Tennessee
Tadek Dobrowolski	Pittsburg State University
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Christine Escher	Oregon State University
Paul Fabel	Mississippi State University
Hanspeter Fischer	University of Wisconsin - Milwaukee
Eric Freden	Utah State University
Dennis Garity	Oregon State University
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Dubravko Ivansic	University of Oklahoma
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Sava Kristic	Portland State University
John Leahy	University of Oregon
Don Lawrence	Pomona College
David Radcliffe	University of Wisconsin - Milwaukee
Dusan Repovs	University of Ljubljana
Hal Sadofsky	University of Oregon
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Reinhard Schultz	University of California, Riverside
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* Speaker at Workshop

** Principal Speaker

Talks by J. W. Cannon *
at the 1997 workshop in Geometric Topology

Talk 1: Dehn's Algorithm

Coworkers on this project are Michael Shapiro, Oliver Goodwin, Robert Gilman, and Brian Bowditch.

Max Dehn defined the fundamental problems in combinatorial group theory, namely, the word problem, the conjugacy problem, and the isomorphism problem; and he solved each of these problems for the fundamental groups of the closed surfaces. His solution to the word problem for the surfaces of negative curvature involved what has come to be known as the Dehn algorithm: scan the word for the majority of either the fundamental relator or a trivial relator, replace it by its complementary minority; iterate until no such shortening operation is possible; then the word represents the identity in the group if and only if the reduced representative is the empty word.

In the late 1970's, the author showed that a version of Dehn's algorithm is valid for all negatively curved groups: suppose G is negatively curved (that is, Gromov word hyperbolic); then there is an integer N such that every nonempty word representing the identity in G can be shortened by means of some relator of length $\leq N$.

In the early 1980's, it was discovered (Gromov, Gersten, Short) that a group G is negatively curved if and only if it admits a Dehn's algorithm in the sense explained in the previous paragraph.

The author and his coworkers consider the following slightly more general problem. Build a computing machine that applies the Dehn algorithm in the sense of replacing long subwords of a certain class by shorter words, with a word declared as trivial if and only if the process results in the empty word. For what class of groups can the word problem be so solved?

The most obvious and immediate answer is the class of negatively curved groups, with proof suggested by the paragraphs above which recounted the history of the problem. This obvious answer is false for a subtle reason. In the historical algorithm, one is required

* This research was supported in part by an NSF research grant.

to replace subwords of group relators by complementary subwords. The machine, on the other hand, is allowed to step outside of the group and to use virtual relators, relators in some bigger graph which need not be the Cayley graph and whose edges need not be labelled by generators of the group in question.

One thereby gets an immediate expansion of the class of groups simply by considering the subgroups of those groups whose word problem is so solvable. Since negatively curved groups often contain subgroups which are not negatively curved, one immediately sees that the desired class of groups is larger than the class of negatively curved groups.

Are there even more groups in the class? Free Abelian groups of rank > 1 embed in no negatively curved group. The authors show that these groups admit such an algorithm. The proof is essentially a logarithm proof. It acts in one sheet of a Baumslag Solitar group graph. Similar proofs work with other virtually nilpotent groups.

Since the talk was given, Shapiro has reported the following: “We have an interesting negative result on Dehn’s algorithms in groups. Roughly it says that if a group contains two infinite commuting sets A and B and one of them has exponential (even superpolynomial) growth, then the group does not have a Dehn’s algorithm. In particular, this says exactly which of the 8 geometries do not admit Dehn’s algorithms, likewise no graph manifold with a (non nil) Seifert fibered piece has a Dehn’s algorithm.”

The exact nature of the class of groups admitting a Dehn algorithm remains unresolved.

Talk 2: Flattening an Orange Peel into the Plane

Here is a famous open problem: Suppose C is a polyhedral convex body in Euclidean 3-dimensional space. Is it possible to cut the boundary of C along finitely many straight line segments in such a way that the boundary remains connected and can be flattened out in the plane without self-overlapping?

Hyam Rubinstein and the author have begun to study a related problem. Suppose that S is a polyhedral, tiled 2-sphere. Cut S into polyhedral disks by cutting along edges. Under what conditions is it possible to then develop these pieces into the plane in such a way that the intrinsic conformal structure of the surface is not distorted too terribly? In particular, at least locally, nearby pieces should stay nearby; individual pieces should not be too distorted; both sides of the same cut should stay relatively close to one another; (etc.?).

The problem is essentially that of a Riemann mapping problem, performed combinatorially. The hope is to apply results to the problem of showing that a negatively curved group with 2-sphere at infinity is Kleinian.

The problem looks difficult.

Talk 3: The combinatorial structure of the Hawaiian earring group.

Interesting related work is being developed by Bogley, Sieradski, and Zastrow. The earliest work on the Hawaiian earring group was published by Higman and Griffiths.

Greg Conner and the author are trying to understand the fundamental groups of those spaces whose groups do not naturally yield to analysis by covering space theory. The simplest of those spaces is the Hawaiian earring: the planar compact metric wedge of a null sequence of circles. The most famous classical results about this group are the following:

- (1) The group is uncountable and is not free.
- (2) Two copies of the cone over the Hawaiian earring can be wedged together in such a way that the wedge is not contractible (has uncountable fundamental group and first homology group).

The authors give a description by means of transfinite words on a countable alphabet. The description generalizes naturally to that of what the authors call a big free group. Group elements of a big free group are represented by transfinite words on a given alphabet of arbitrary cardinality. The big free groups are groups with additional structure, namely, an additional infinite multiplicative structure. The big free groups are a generalization of the finitely generated free groups. Each group element is represented by a unique reduced transfinite word.

The big free groups can also be described as fundamental groups of generalized Hawaiian earrings, provided that the fundamental group is suitably generalized to allow loops based not on the standard circle but on generalized circles of arbitrary size. The authors define the generalized Hawaiian earrings, the big fundamental group, and show that the big free group of arbitrary cardinality C is the big fundamental group of the generalized Hawaiian earring with C loops.

Finally, the authors extend the theory to study the fundamental groups of other complicated 1-dimensional spaces such as the Sierpinski curve and Menger curve. Again elements are represented by generalized words, there is a unique reduced representative for each el-

ement, and there is a cancellation theory which parallels the case of the big free group but involves R. L. Moore's theorem on cellular upper semicontinuous decompositions of the plane.

The work currently appears in three preprints in preparation.

THE FUNDAMENTAL GROUPS OF ONE-DIMENSIONAL SPACES VIA ELEMENTARY TOPOLOGY.

G.R. CONNER

ABSTRACT. This article is meant to describe a talk given at Oregon State University as part of the Western Workshop in Geometric Topology during the summer of 1997. All of the work described herein is joint with J.W. Cannon.

1. INTRODUCTION

First I would like to point out others who have worked in this area in a somewhat chronological order: G. Higman, H.B. Griffiths, J.W. Morgan together with I. Morrison, and B. deSmit all worked in the area of proving that the fundamental group of the Hawaiian earring is not a free group([Hi, Gri, MM, dS]), while A. Zastrow, A.J. Sieradski, and W.A. Bogley have studied fundamental groups of one-dimensional spaces and generalizations([Z4, S, BS1]). This article describes one facet of the work contained in a series of three papers([CC1],[CC2],[CC3]) authored by the the current author and J.W. Cannon. Cannon described many other facets in his series of talks given at this conference.

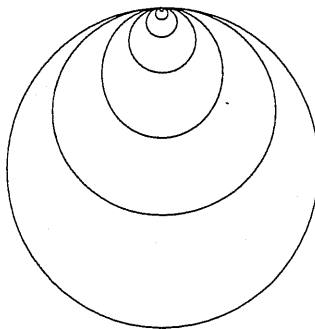


FIGURE 1. Hawaiian Earring

Let H denote the Hawaiian earring – the union of planar circles c_i of radius $1/i$ for each natural number i , each tangent to the x -axis at

the origin, 0 . A well-known example in topology, H is compact, and one-dimensional, however its fundamental group is uncountable and not a free group. It is relatively easy to see why the Hawaiian earring group, $G = \pi_1(H)$, is uncountable.

Why is it that G is not free? It is not for the obvious reasons that first come to mind. Since G *locally free* (each of its finitely generated subgroups is free), G has no relations in the classical sense. The reason that G is not free is that it is too “big” to have a free basis. More precisely, any generating set for G must contain “extraneous” generators. To see this, first note that, since G is not free, any generating set for G must satisfy some nontrivial relation among finitely many of the generators. However, since G is locally free, these finitely many generators generate a free subgroup of G and thus any such relation can only define a relation among elements of a free group.

2. FUNDAMENTAL GROUPS OF ONE-DIMENSIONAL SPACES

How does one prove that $\pi_1(H)$ is not free? There are several methods, but here we will discuss the one that most easily generalizes. The next result appears in [CC1].

Theorem 2.1. *If $\Phi : G \longrightarrow A$ is a homomorphism to a free-abelian group then $\phi(G)$ has finite rank.*

Assuming, for the moment, that the theorem is true we can now show that G is not free. Clearly, G cannot be free of infinite rank since if it were, it would have an infinite rank free abelian quotient contradicting Theorem 2.1. However, G cannot be free of finite rank since it has uncountably many elements.

One might ask: “What property of H makes its fundamental group contradict the intuitive notion that one-dimensional spaces should have free fundamental groups?” The answer is that the classical tools which are used to study the fundamental groups of one-dimensional spaces require the space in question to have certain local finiteness properties before they yield useful information. For instance, the fundamental group of H cannot be studied by considering its action on its universal cover, since H does not have a universal cover ! Furthermore, applying the Siefert-van Kampen theorem to H in the same manner which we apply it to prove that the fundamental group of a graph is free, reveals, in this case, only that $G = G * \mathbb{Z}$.

There are many examples of topologically important one-dimensional spaces whose fundamental groups which are not well-understood (at least by us). Examples include the Sierpinski curve, S , and the Menger

curve, M . It is surprising that no portion of the following conjecture (to our knowledge) has been solved.

Conjecture 2.1.1. No two of the following four groups $\pi_1(S)$, $\pi_1(M)$, $\pi_1(H)$, $\pi_1(H \vee_o H)$ are isomorphic.

Definition 2.2. We say that the topological space X is *semilocally simply connected at x* if there is a neighborhood U_x of x so that the image of $\pi_1(U_x, x)$ in $\pi_1(X, x)$ is trivial. Furthermore we say that X is *semilocally simply connected* if it is semilocally simply connected at each of its points. We recall from elementary covering space theory that a path connected Hausdorff space has a universal cover if and only if it is semilocally simply connected. Informally, a space is semilocally simply connected if “small” loops are contractible in the whole space.

The following appears in [CC2].

Theorem 2.3. *If X is a connected, locally path connected, one-dimensional, second countable metric space then the following are equivalent:*

1. $\pi_1(X)$ is free.
2. $\pi_1(X)$ is countable.
3. X is locally simply connected.
4. X admits a universal cover.

It is obvious that neither the Sierpinski nor the Menger curve is semilocally simply connected, thus their fundamental groups are both uncountable and not free.

We will now discuss one of the main tools used in proving this theorem. We need a definition first.

Definition 2.4 (Infinitely Divisible). If h is a nonidentity element of the group H , we say that h is *infinitely divisible* if there are infinitely many integers e for which there is an element h_e of H so that $h = (h_e)^e$.

Theorem 2.5. *Let X be a topological space, let $f : \pi(X, x_0) \rightarrow L$ be a homomorphism to the group L , $U_1 \supseteq U_2 \supseteq \dots$ be a countable local basis for X at x_0 , and P_i be the image of the natural map of $\pi(U_i, x_0)$ into $\pi(X, x_0)$. Then*

1. *If L is countable then the sequence $f(P_1) \supseteq f(P_2) \supseteq \dots$ is eventually constant.*
2. *If L is abelian with no infinitely divisible elements then $\bigcap_{i \in \mathbb{N}} f(P_i) = \{0_L\}$.*
3. *If L is countable abelian with no infinitely divisible elements then $f(P_i) = \{0_L\}$ for some $i \in \mathbb{N}$.*

Later on we will give a proof of the second part of the theorem. Assuming Theorem 2.5 we can now give a proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose $f : G \rightarrow A$ is a homomorphism from the Hawaiian earring group to free abelian group. With out loss of generality, we may assume f is surjective, and that A has countably infinite rank. Now for each j , let U_j be an open neighborhood of 0 in H which contains all but the outermost j circles c_i . Since free abelian groups have infinitely divisible elements we may apply part 3 of Theorem 2.5 to obtain that $f(P_i) = 0_A$ for some i . Thus $\ker f \supset P_i$.

Since G/P_i is a free group of rank i by the Seifert-van Kampen theorem, any free abelian factor group of G/P_i has finite rank. Thus, $A = G/\ker(f)$ has finite rank.

Another insightful question one might ask is, "What special property of one-dimensional spaces is used to prove the results Theorem 2.3?"

Definition 2.6 (Homotopically Hausdorff). We say that the topological space X is *homotopically Hausdorff at the point* $x_0 \in X$ if for every $g \in \pi(X, x_0) - \{1\}$ there is an open neighborhood of x_0 which contains no path representing g . We say that X is *homotopically Hausdorff* if X is homotopically Hausdorff at x_0 for every $x_0 \in X$.

The notation *homotopically Hausdorff* is motivated by the fact that the space of homotopy classes of based paths $\Omega(X, x_0)$ of a space X based at $x_0 \in X$ is Hausdorff if and only if X is homotopically Hausdorff.

The fact that one-dimensional spaces are homotopically Hausdorff follows immediately from the following lemma which appears in [CC2].

Lemma 2.7. *If X is a one-dimensional topological space and p is a closed path in X based at x_0 then p is homotopic to a unique (up to reparameterization) closed path based at x_0 which is either constant (if p is null-homotopic) or has no proper null-homotopic subpaths. Furthermore, the image of the homotopy can be chosen to be contained in the image of p .*

Possibly the most convincing way to convince oneself that being homotopically Hausdorff is a desirable property is to consider a space which does not have this property.

Example 2.7.1 (The doubled cone over the Hawaiian earring). Consider $C = H \times [0, 1]/H \times \{1\}$, the cone over the Hawaiian Earring. Let 0 , the basepoint of C , be the point $((0, 0), 0)$. Then the doubled cone over the Hawaiian earring is the space $D = C \vee_0 C$, in other words two copies of the cone C glued together at the basepoint.

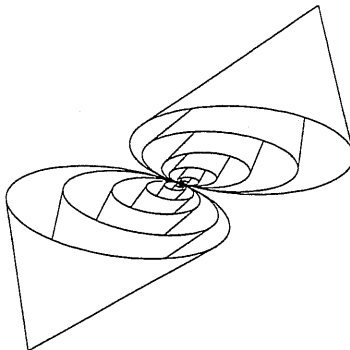


FIGURE 2. The doubled cone over the Hawaiian earring

We now mention a number of results from [CC1]. Note that \hat{C} is contractible. However D is not contractible since its fundamental group is uncountable! If we were to attach the two copies of θ with an arc rather than identifying them, the resulting space would be contractible. D is particularly misbehaved since there is one loop (namely going around the first copy of c_1 then the second, then the first copy of c_2 then the second, etc) which is homotopic into any neighborhood of the basepoint and yet is not nullhomotopic. Thus, this space is not homotopically Hausdorff. Note that both copies of each of these circles c_i is nullhomotopic. Thus, the fundamental group of D is carried by (infinite) products of curves whose homotopy classes are all trivial. We normally call a group which generated by products of one particular element as a *cyclic* group. Thus one could describe the fundamental group of D as an uncountable cyclic group generated by the identity element! It is evident from Theorem 2.5 that the fundamental group of D has no nontrivial countable free abelian factor groups and thus this group cannot embed in an inverse limit of finite rank free groups.

We propose the following conjecture.

Conjecture 2.7.2. The fundamental group of a connected, locally path connected space which is not homotopically Hausdorff is uncountable.

We now mention another result from [CC2].

Theorem 2.8. *If X is a compact, connected, locally path connected, metric space which is homotopically Hausdorff then the following are equivalent:*

1. $\pi(X)$ is countable

2. $\pi(X)$ is finitely generated
3. $\pi(X)$ is finitely presented
4. X has a universal cover.

Another interesting result which appears in [CC2] is

Theorem 2.9. *Let X be a compact, one-dimensional, connected metric space, and $x_0 \in X$. Then $\pi_1(X, x_0)$ embeds in an inverse limit of finite rank free groups.*

Corollary 2.10. *The group $\pi_1(X, x_0)$ is locally free, residually free, and residually finite.*

REFERENCES

- [BS1] W.A. Bogley, A.J. Sieradski, *Weighted Combinatorial Group Theory and Homotopy in Metric Complexes*, preprint.
- [BS2] W.A. Bogley, A.J. Sieradski, *Universal Path Spaces*, preprint.
- [CC1] J.W. Cannon, G.R. Conner, *The Combinatorial Structure of the Hawaiian Earring Group*, preprint Brigham Young University (1997).
- [CC2] J.W. Cannon, G.R. Conner, *The Big Fundamental Group, Big Hawaiian Earrings and the Big Free Groups*, preprint Brigham Young University (1997).
- [CC3] J.W. Cannon, G.R. Conner, *The Fundamental Groups of One-dimensional Spaces*, preprint Brigham Young University (1997).
- [dS] B. deSmit, *The fundamental group of the Hawaiian earring is not free.*, Internat. J. Algebra Comput. 2, no. 1, 33–37 (1992).
- [Gri] H.B. Griffiths, *Infinite Products of Semi-groups and Local Connectivity*, Proc. London Math. Soc. (3), 455–456 (1956).
- [Hi] G. Higman, *Unrestricted Free Products, and Varieties of Topological Groups*, Journ. London Math. Soc. 27, 73–81 (1952).
- [MM] J.W. Morgan, I. Morrison, *A Van Kampen Theorem for Weak Joins*, Proc. London Math. Soc, 53, #3, 562–576 (1986).
- [RLM] R.L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), 416–428.
- [S] A.J. Sieradski, *OMEGA-GROUPS: Group Theory for Wild Topology*, preprint.
- [Z1] A. Zastrow, *All subsets of \mathbb{R}^2 are $K(\pi, 1)$* , preprint.
- [Z2] A. Zastrow, *The Non-abelian Specker-Group is free*, preprint.
- [Z3] A. Zastrow, *Construction of an infinitely generated group that is not a free product of surface groups and abelian groups but which acts freely on an \mathbb{R} -tree*, to appear Proc. R. Soc. Edinb.
- [Z4] A. Zastrow, *The second van-Kampen theorem for topological spaces*, Topology and its Applications 59 (1994), 201–232.

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CAUTY'S CLASS OF COMPACTA DETERMINED BY HOMOLOGY, AND APPLICATIONS

TADEUSZ DOBROWOLSKI

1. INTRODUCTION

Let (Q, s) be a pair consisting of the Hilbert cube $Q = [0, 1]^\infty$ and its pseudo-interior $s = (0, 1)^\infty$. In [Ca₂], R. Cauty makes use of compacta in Q whose complements satisfy certain homological properties. It occurred to us that those general properties might find some other applications so that it is reasonable to formally distinguish a class of compacta satisfying those properties. The class we are going to describe is meant to be topological; hence, if an element $A \subseteq Q$ belongs to the class (that is, has certain properties with respect to the pair (Q, s)), and h is a homeomorphism of (Q, s) onto (Q', s') then $h(A)$ belongs to the class (that is, has the same properties with respect to (Q', s')).

1.1. Definition. A class of compacta $\mathcal{C}(Q)$ of Q will be called a Cauty class if

- (i) $Q \in \mathcal{C}(Q)$;
- (ii) if $A \in \mathcal{C}(Q)$, then $A \cap s$ is dense in A ;
- (iii) if $A \in \mathcal{C}(Q)$ and $P \subseteq Q$ is a subcube with $(\text{int } P) \cap A \neq \emptyset$, then $A \cap P$ contains an element of $\mathcal{C}(P)$;
- (iv) if (Q, s) is of the form $(Q_1 \times Q_2, s_1 \times s_2)$ and $A \in \mathcal{C}(Q)$ with $(\forall x \in Q_1) A_x \subsetneq Q_2$, then $(\exists x_0 \in Q_1 \setminus s_1)$ such that A_{x_0} contains an element of $\mathcal{C}(Q_2)$;
- (v) if $A \in \mathcal{C}(Q)$ and $A_1, A_2 \subsetneq A$ are compacta such that $A = A_1 \cup A_2$, then $A_1 \cap A_2$ contains an element of $\mathcal{C}(Q)$.

By a subcube P of Q we mean a set of the form $[a_1, b_1] \times \cdots \times [a_i, b_i] \times [0, 1] \times [0, 1] \times \cdots \subseteq Q$, where $a_j < b_j$, $1 \leq j \leq i$. The symbol A_x appearing in (iv) denotes $(\{x\} \times Q_2) \cap A$; we identify $\{x\} \times Q_2$ with Q_2 .

1.2. *Remark.* Condition (v) is equivalent to

(v') if $A \in \mathcal{C}(Q)$ and $B \subset A$ is a compactum such that $A \setminus B$ is disconnected, then B contains an element of $\mathcal{C}(Q)$.

Let $A \subset Q$ be a compactum. Following Cauty [Ca₂], we say that A is an irreducible barrier for a nontrivial element $\alpha \in H_n(Q \setminus A)$, $n \geq 0$, if for every compactum $B \subsetneq A$, α as an element of $H_n(Q \setminus B)$ is trivial. (Here, $H_*(X)$ denote the singular homology groups reduced at $n = 0$ so that $H_0(X) = 0$ iff X is path connected.)

Combining [Ca, Lemmas 2-4] we obtain

1.3. **Theorem.** *The class*

$$\text{HLG} = \{A \subseteq Q \mid (\exists n \geq 0) (A \text{ is an irreducible barrier for some } \alpha \in H_n(Q \setminus A))\} \cup \{Q\}$$

is Cauty. \square

2. APPLICATIONS

In this section we present an abstraction on the main result of Cauty [Ca₂].

Let L be a subset of a metric space X . Suppose that there are sigma-closed sets $E, M \subset X$ with $E \subset L \subset M$, and let $M = \bigcup_{k=1}^{\infty} M_k$, where each M_k is a closed subset of X .

2.1. **Definition.** A map $\psi : Z \rightarrow M_k$ is **locally decomposable** if for every z there exist an open neighborhood V of z and maps ψ^j , $1 \leq j \leq k$, defined on V such that

(d₁) for every $z \in V$, $\psi(z) \in L$ iff $\psi^j(z) \in L$, $1 \leq j \leq k$; and

(d₂) if $C \subset V$ is connected, $\psi^j(C) \cap E = \emptyset$, and $\psi^j(C) \cap L \neq \emptyset$ for some j , then $\psi^j(C) \subseteq L$.

We say that ψ is decomposed through (ψ^1, \dots, ψ^k) .

Our abstraction on Cauty's main result [Ca₂] reads as follows. Below A^ω denotes the countable product of A , and $W(Q, s) = \{(x_i) \in Q^\omega \mid x_i \in s \text{ for finitely many } i\}$.

2.2. **Theorem.** *There does not exist a map $\varphi = (\varphi_i) : Q^\omega \rightarrow X^\omega$ with*

$$\varphi^{-1}(L^\omega) = W(Q, s)$$

so that each map $\varphi_i|\varphi_i^{-1}(M_n)$ is locally decomposable.

Let T be a countable infinite set. Every filter F on T determines a function space which can be identified with $c_F = \{f \in \mathbb{R}^T \mid (\forall \varepsilon > 0)(\exists A \in F)(\forall t \in A)(|f(t)| \leq \varepsilon)\} \subset \mathbb{R}^T$, where \mathbb{R}^T is considered with the product topology. Let $s_F = \{f \in \mathbb{R}^T \mid (\exists A \in F)(\forall t \in A)(f(t) = 0)\}$. Cauty [Ca₂] has shown that, for an arbitrary filter F constructed in [LvMP], there are sets E and $M = \bigcup_{k=1}^{\infty} M_k$, $E \subset s_F \subset M$, as postulated at the beginning of this section so that any map $\psi : Z \rightarrow M_k$ is locally decomposable. Hence, we obtain

2.3. Corollary. *There does not exist a map $\varphi : Q^\omega \rightarrow (\mathbb{R}^T)^\omega$ with $\varphi^{-1}(s_F^\omega) = W(Q, s)$. \square*

This version of 2.2 allowed Cauty to refute a conjecture that the Borel type of c_F determines its topological type because it was known that if the conjecture held, then there would exist a map $\varphi : Q^\omega \rightarrow (\mathbb{R}^T)^\omega$ with $\varphi^{-1}(s_F^\omega) = W(Q, s)$.

Let X be a Banach space and $C \subset X$ a linearly independent Cantor set. Write $F = \text{span}(A)$ and $M = \text{span}(C)$. In [Ca₁,] Cauty was able to show to represent $M = \bigcup_{k=1}^{\infty} M_k$ in such a way that any map $\psi : Z \rightarrow M_k$ was locally decomposable (the decomposition is trivial, that is, such ψ satisfies 2.1(d₂)). Hence, in this case, 2.2 takes the following form

2.4. Corollary. *There does not exist a map $\varphi : Q^\omega \rightarrow X^\omega$ with $\varphi^{-1}((\text{span}(A)^\omega) = W(Q, s)$. \square*

In a similar way as above, this result allowed Cauty [Ca₁] to conclude that the Borel type of $(\text{span}(A))^\omega$ does not determine its topological type. Actually Cauty's proof of 2.4 (see [Ca₁]) does not involve a class described in 1.1.

3. PROOF OF THEOREM 2.2

Following closely the argument of Cauty [Ca₂], let us present a sketch of the proof of 2.2.

Let $Q^\omega = Q_0 \times Q_1 \times Q_2 \times \cdots \supset s_0 \times s_1 \times s_2 \times \cdots$, where $(Q_r, s_r) = (Q, s)$, $r \geq 0$. For a point $(y_0, y_1, \dots, y_{r_p}) \in Q_0 \times Q_1 \times \cdots \times Q_{r_p}$, we let $\mathbf{y}^{r_p} = (y_0, \dots, y_{r_p})$, and

$$Q^p = Q(\mathbf{y}^{r_p}) = \{\mathbf{y}^{r_p}\} \times Q_{r_p+1} \times Q_{r_p+2} \times \cdots \supset s^p = s(\mathbf{y}^{r_p}) = s(Q^p),$$

where $s(P)$ always stands for the psuedo-interior of P .

Suppose that φ does exist. Inductively, we will find sequences of integers $\{r_p\}_{p=0}^{\infty}$ with $1 \leq r_0 < r_1 < r_2 < \dots$, points $y_r \in Q_r$, $r \geq 0$, and sets $A_p \subseteq Q^p$ such that

- (1) $A_p \in \mathcal{C}(Q^p)$;
- (2) $A_p \subseteq A_{p-1}$ (let $A_{-1} = Q^\omega$);
- (3) $\varphi_p(A_p) \subset L$; and
- (4) $y_{r_p} \in Q_{r_p} \setminus s_{r_p}$.

Once this is done, we see that the point $y = (y_0, y_1, \dots) \in \bigcap_{p=1}^{\infty} A_p$; hence, by (3), $\varphi(y) \in L^\omega$. However, (4) yields $y \in Q^\omega \setminus W(Q, s)$, a contradiction.

Inductive Construction:

1^0 The case $p = 0$. Use the facts that $s^\omega \subset W(Q, s)$, $L \subset M$, s^ω has the Baire property, and that s^ω is dense in Q^ω to find an integer $k_0 \geq 0$ and a nonempty open set $V \subseteq Q^\omega$ with $\varphi_0(V) \subset M_{k_0}$. There exists an integer $t_0 \geq 0$ and points $y_r \in Q_r \setminus s_r$, $0 \leq r \leq t_0$ such that $Q(y^{t_0}) \subset V$. Since $\varphi_0|V$ is locally decomposable, we can assume that on $Q(y^{t_0})$ the map φ decomposes through $(\varphi_0^1, \varphi_0^2, \dots, \varphi_0^{n_0})$ so that the requirements of 2.1 hold. Select, if possible, points $y_r \in Q_r$, $t_0 < r \leq t_1$, with $\varphi_0^1(Q(y^{t_1})) \subset L$. If possible, continue in this manner selecting points $y_r \in Q_r$, $t_1 < r \leq t_{n_0}$ ($t_0 < t_1 < t_2 < \dots < t_{n_0}$), so that

$$\varphi_0^j(Q(y^{t_j})) \subset L \text{ for } 1 \leq j \leq n_0.$$

Finally let $r_0 = t_{n_0} + 1$, and set $A_0 = Q^0$ and $y_{r_0} \in Q_{r_0} \setminus s_{r_0}$ to be arbitrary. By the linearity of L and 2.1(d₁), condition (3) holds.

If the above is not doable, we arrive to a situation that either the selection on the stage $j = 1$ is impossible to make, or there exists j_0 , $1 < j_0 \leq n_0$ so that

- (i) $\varphi_0^j(Q(y^{t_j})) \subset L$, $1 \leq j < j_0$, and
- (ii) with $t = t_{j_0-1}$, for no choice in $y_{t+1}, \dots, y_u \in Q$, $\varphi_0^{j_0}(Q(y^u)) \subset L$.

If the above happens, consider y^t (with $t = t_0$ if the selection on stage $j = 1$ was impossible). Let

$$Q' = Q(y^t) \supset s' = s(Q') \text{ and } \varphi'_0 = \varphi_0^{j_0}|_{Q'}.$$

In order to obtain A_0 and r_0 in this case, we first find $B_j \in \mathcal{C}(P_j)$, where $P_j \supset P_{j-1} \supset \dots \supset P_0 = Q'$ is a sequence of subcubes such that $\varphi_0^j(B_j) \subset L$ (which can be viewed as a counterpart of $\varphi_0^j(Q(\mathbf{y}^{t_j})) \subset L$ above).

Since $E \subset L$, by (ii), $\text{int}_{Q'}(\varphi_0')^{-1}(E) = \emptyset$. Since $C = Q' \setminus (\varphi_0')^{-1}(E)$ is a G_δ -set, it follows that $C \cap s' \neq \emptyset$.

Claim 1. C is disconnected.

Proof. If C were connected, $\varphi_0'(C)$ would be also connected. For a point $x \in C \cap s' \subset W(Q, s)$, $\varphi_0(x) \in L$, and consequently $\varphi_0'(x) \in L$ by 2.1(d₁). Since $\varphi_0'(C) \cap E = \emptyset$, by 2.1(d₂), $\varphi_0'(C) \subset L$. Finally, $\varphi_0'(Q') = \varphi_0'(C) \cup \varphi_0'(Q' \setminus C) \subset L \cup E \subseteq L$, which contradicts (ii). \square

Since C is dense in Q' , we obtain

Claim 2. There are compacta $A_1, A_2 \subsetneq Q'$ such that $B = A_1 \cap A_2 \subset Q' \setminus C$. \square

Since $B \subset Q' \setminus C$, it follows that $\varphi_0^{j_0}(B) = \varphi_0'(B) \subset E \subset L$. Moreover, by 1.1(v), we can assume that $B \in \mathcal{C}(Q')$ (because $Q' \in \mathcal{C}(Q')$). So, the initial step is done: $P_0 = Q'$, and $B_0 = B$.

Claim 3. Writing $\varphi_1' = \varphi_0^{j_0+1}|_{B_0}$, there exists a subcube $P_1 \subset P_0$ and a compactum $B_1 \subset B_0$ with $\varphi_1'(B_1) \subset L$ and $B_1 \in \mathcal{C}(P_1)$.

Proof. If $\text{int}_{B_0}(\varphi_1')^{-1}(L) \neq \emptyset$, then $\varphi_1'(U_1 \cap B_0) \subset L$ for some open set $U_1 \subseteq P_0$. Pick a subcube $P_1 \subset U_1$ so that $(\text{int } P_1) \cap B_0 \neq \emptyset$. Let $B_1 = B_0 \cap P_1$. we have $B_1 \subset B_0$, and hence $\varphi_1'(B_1) \subset L$. Since $B_0 \in \mathcal{C}(P_0)$, by 1.1(iii), we additionally can require that $B_1 \in \mathcal{C}(P_1)$.

Assume $\text{int}_{B_0}(\varphi_1')^{-1}(L) = \emptyset$. We let $P_1 = P_0$. From 1.1(ii), we infer that $B_0 \cap s(P_0)$ is dense in B_0 ; moreover, by our assumption, $\text{int}_{B_0}(\varphi_1')^{-1}(E) = \emptyset$. Using the same argument as the one proceeding Claim 1 and Claim 1 itself, we conclude that $C = B_0 \setminus (\varphi_1')^{-1}(E)$ is disconnected. Using the fact that C is dense in B_0 and applying Claim 2, we find compact $A_1, A_2 \subsetneq B_0$ such that $B_1 = A_1 \cap A_2 \subset (\varphi_1')^{-1}(E)$. Summarizing, we have $\varphi_0^{j_0+1}(B_1) = \varphi_1'(B_1) \subset E \subset L$, and we additionally can require that $B_1 \in \mathcal{C}(P_1)$ by an application of 1.1(v). \square

Applying inductively Claim 3 $l = n_0 - j_0$ times, we arrive to a subcube $P_l \subset Q(\mathbf{y}^t)$ and a compactum $B_l \in \mathcal{C}(P_l)$ such that $\varphi_0^j(B_l) \subset L$, $1 \leq j \leq n_0$. Represent

$$P_l = \{\mathbf{y}^t\} \times R_{t+1} \times \cdots \times R_{r_0-1} \times Q_{r_0} \times Q_{r_0+1} \times \cdots,$$

where R_s is a subcube of Q_s , $t+1 \leq s \leq r_0-1$ and $r_0 > t$.

Claim 4. *There are $y_{t+1}, \dots, y_{r_0-1} \in Q$ and $y_{r_0} \in Q \setminus s$ such that $B_l \cap Q(\mathbf{y}^{r_0})$ contains $A_0 \in \mathcal{C}(Q^0)$, $Q^0 = Q(\mathbf{y}^{r_0})$, that satisfies (1)-(4).*

Proof. Let $Q_1 = \{\mathbf{y}^t\} \times R_{t+1} \cdots \times R_{r_0-1} \times Q_{r_0} \supset s_1 = s(Q_1)$ and $Q_2 = Q_{r_0+1} \times Q_{r_0+2} \times \cdots \supset s_2 = s(Q_2)$. Recall that for $x \in Q_1$ and $A \subseteq Q_1 \times Q_2$, $A_x = (\{x\} \times Q_2) \cap A$. Our assumption (ii) implies that, for no $x \in Q_1$, $\{x\} \times Q_2 \subset (B_l)_x$ because $\varphi_0^j(B_l) \subset L$ for all $1 \leq j \leq n_0$. According to 1.1(iv), there exists $x_0 \in Q_1 \setminus s(Q_1)$ such that $(B_l)_{x_0}$ contains an element of $\mathcal{C}(Q_2)$. It is clear that $\mathbf{y}^{r_0} = x_0$ is as required. \square

This concludes the first step of induction. In the text which follows we will show how this first step can be adjusted for the case of $p = 1$.

2^o The case $p = 1$. Consider $Q^0 = Q(\mathbf{y}^{r_0}) \supset s^0 = s(Q^0)$, and $A_0 \in \mathcal{C}(Q^0)$. By 1.1(ii), $A_0 \cap s^0$ is dense in A_0 . Use the facts that $s^0 \subset W(Q, s)$, $L \subset M$, $\varphi_1(W(Q, s)) \subset L$, $A_0 \cap s^0$ has the Baire property, and that s^0 is dense in Q^0 to find an integer $k \geq 1$ and a nonempty open set $V \subseteq Q^0$ with $\varphi_1(V \cap A_0) \subset M_k$. Let P_0 be a subcube of Q^0 such that $P_0 \subseteq V$ with $(\text{int } P_0) \cap A_0 \neq \emptyset$. We can assume that $\varphi_1|_{P_0}$ decomposes through $(\varphi_1^1, \varphi_1^2, \dots, \varphi_1^{n_1})$ as described in 2.1.

First we take the case where $A_0 = Q^0$. There exists an integer $t_0 > r_0$ and $y_r \in Q_r \setminus s_r$, $r_0 \leq r \leq t_0$ such that $Q(\mathbf{y}^{t_0}) \subset P_0$. Start a selection process in order to find $y_r \in Q_r$, $t_0 < r \leq t_{n_1} = t$ so that $\varphi_1^j(Q(\mathbf{y}^t)) \subset L$, $1 \leq j \leq n_1$. If this is doable, let $r_1 = t+1$ and $A_1 = Q^1 = Q(\mathbf{y}^{t+1})$, where $y_{r_1} \in Q \setminus s$ is arbitrary. Otherwise, such a selection is impossible, and there exists j_0 , $1 < j_0 \leq n_1$ so that

(i) $\varphi_1^j(Q(\mathbf{y}^{t_j})) \subset L$, $1 \leq j < j_0$, and

(ii) with $t = t_{j_0-1}$, for no choice in $y_{t+1}, \dots, y_u \in Q$, $\varphi_1^{j_0}(Q(\mathbf{y}^u)) \subset L$.

This same argument as in 1⁰ shows that there exists a subcube $P_l \subset Q(y^t)$ and a compactum $B_l \in \mathcal{C}(P_l)$ such that $\varphi_1^j(B_l) \subset L$, $1 < j \leq n_1$. Now, an application of Claim 4 easily yields $A_1 \in \mathcal{C}(Q^1)$, where $Q^1 = Q(y^{r_1})$ for some $r_1 > r_0$.

Suppose now that $A_0 \subsetneq Q^0$. By 1.1(iii), there exists $B_0 \in \mathcal{C}(P_0)$, $B_0 \subseteq A_0 \cap P_0$. We have

$$P_0 = \{y^{r_0}\} \times W_{r_0+1} \times W_{r_0+2} \times \cdots \times W_{t_0} \times W_{t_0+1} \times \cdots,$$

where W_s is a subcube of Q_s for $s \geq r_0 + 1$, and $W_s = Q_s$ for $s > t_0$ ($t_0 > r_0$). Applying Claim 3 n_1 times, beginning with $P_0 \supset B_0$ and the map $\varphi_1^1|_{B_0}$, we arrive to a subcube $P_{n_1} \subset P_0$, and a compactum $B_{n_1} \in \mathcal{C}(P_{n_1})$ with $\varphi_1^j(B_{n_1}) \subset L$, $1 \leq j \leq n_1$. The subcube is of the form

$$P_{n_1} = \{y^{r_0}\} \times R_{r_0+1} \times \cdots \times R_{r_1-1} \times Q_{r_1} \times \cdots,$$

where R_s is a subcube of W_s , $r_0 + 1 \leq s \leq r_1 - 1$ and $r_1 > r_0$. Now an application of Claim 4 provides $y_{r_0+1}, \dots, y_{r_1-1} \in Q$, $y_{r_1} \in Q \setminus s$, and $A_1 \in \mathcal{C}(Q^1)$ that satisfies (1)-(4), where $Q^1 = Q(y^{r_1})$. (We are eligible to apply Claim 4 because for no $x \in \{y^{r_0}\} \times R_{r_0+1} \times \cdots \times R_{r_1-1} \times Q_{r_1}$, $\{x\} \times Q_{r_1} \times Q_{r_1+1} \times \cdots \subset (B_{n_1})_x$; this follows from condition (ii) of 1⁰ and from the inclusions $(B_{n_1})_x \subset (B_0)_x \subset (A_0)_x$). This concludes the case of $p = 1$. It is clear now, how one can make an inductive step. \square

4. THE CLASS OF WILD COMPACTA

We think that it would be interesting to exhibit some more Cauty classes. It is likely that such classes can be determined by replacing in 1.3 homology means by homotopy ones. One possibility is to consider the class of wild sets, which can be thought as a class transversal to that consisting of Z -sets.

Recall that a compactum $A \subset Q$ is a Z -set if for every nonempty open set (equivalently, for every open subcube) $U \subset Q$, $U \setminus A \hookrightarrow U$ is a homotopy equivalence. Write $\mathcal{Z}(Q)$ for the class of Z -sets in Q . A compactum A will be referred to as wild, written $A \in \mathcal{W}(Q)$, if $A \notin \mathcal{Z}(Q)$. The class $\mathcal{W}(Q)$ may give rise to a Cauty class; a motivation for this comes from the following reformulation of a result by Wong [Wo] which resembles 1.1(iv).

4.1. Proposition. *Let (Q, s) be of the form $(Q_1 \times Q_2, s_1 \times s_2)$. If $A \in \mathcal{W}(Q)$ and, for every $x \in Q_1$, then there exists $x_0 \in Q_1 \setminus s_1$ such that $A_{x_0} \in \mathcal{W}(Q_2)$.*

Evidently the class $\mathcal{W}(Q)$ does not have the localization property of 1.1(iii). Therefore it is reasonable to consider the following class $\mathcal{W}_s(Q)$ of strong wild sets. A compactum $A \in \mathcal{W}_s(Q)$ if, for every subcube $P \subseteq Q$ with $(\text{int } P) \cap A \neq \emptyset$, $A \cap P \in \mathcal{W}(P)$.

Question 1. Is $\mathcal{W}_s(Q)$ a Cauty class?

It is not difficult to verify that for $A \in \mathcal{W}_s(Q)$, $A \cap S$ is dense in A ; hence, 1.1(ii) is satisfied. This together with 4.1 shows that an affirmative answer to Question 1 will follow from affirmative answers to the following questions (we think that these questions are interesting in their own).

Question 2. Does every $A \in \mathcal{W}(Q)$ contain an element $B \in \mathcal{W}_s(Q)$?

Question 3. Let $A \in \mathcal{W}_s(Q)$ and let B be a compactum which disconnects A . Is $B \in \mathcal{W}(Q)$?

It seems that another possibility of obtaining a Cauty class is to replace the class HLG of 1.3 by the class of HTP defined analogously as HLG but with the use of irreducible barriers w/r to homotopy. A compactum $A \subset Q$ is an irreducible barrier (w/r to homotopy) for $\alpha \in \pi_n(Q \setminus A)$, $\alpha \neq 0$, if for every compactum $B \subsetneq A$ the element α is trivial in $\pi_n(Q \setminus B)$. It is not clear to us whether the class HTP satisfies (iii) and (iv) of 1.1. It can be easily checked that 1.1(ii) holds for the class HTP. The following fact yields 1.1(v).

4.2. Proposition. *Let $A \subset Q$ be an irreducible barrier for $0 \neq \alpha \in \pi_n(Q \setminus A)$ and $A_1, A_2 \subsetneq A$ be compacta with $A_1 \cup A_2 = A$. Then $\pi_{n+1}(Q \setminus (A_1 \cap A_2)) \neq 0$.*

Proof. We will only take the case $n = 0$. Represent α by points x and y lying in different components of $Q \setminus A$. Since A_1 does not disconnect Q , there is a path γ_1 in $Q \setminus A_1$ joining x and y . Similarly, there is a path γ_2 joining x and y in $Q \setminus A_2$. The paths γ_1 and $-\gamma_2$ determine a 1-chain c . Using the Mayer-Vietoris exact sequence for the pair $(Q \setminus A_1, Q \setminus A_2)$, we infer that c is not a boundary of any 2-chain in $Q \setminus (A_1 \cap A_2)$. Consequently, the obvious natural map $S^1 \mapsto Q \setminus (A_1 \cap A_2)$ determined by c is not null homotopic in $Q \setminus (A_1 \cap A_2)$. \square

REFERENCES

- [Ca₁] R. Cauty, *Sur l'universalité des produits de rétractes absolus*, Bull. Acad. Polon. Sci. **44** (1996), 453–456.
- [Ca₂] R. Cauty, *La classe borélienne ne détermine pas le type topologique de $C_p(X)$* , preprint.
- [LvMP] D. Lutzer, J. van Mill and R. Pol, *Descriptive complexity of function spaces*, Trans. Amer. Math. Soc. **291** (1985), 121–128.
- [Wo] R. Y. Wong, *On homeomorphisms of infinite-dimensional bundles. I*, Trans. Amer. Math. Soc. **191** (1974), 245–259.

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Homogeneity of Boundaries of Right Angled Coxeter Groups having Manifold Nerves*

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Abstract

Given a right angled Coxeter group Γ we form the Vinberg-Davis complex \mathcal{A} . This contractible complex supports a $CAT(0)$ metric such that Γ acts geometrically on \mathcal{A} . The space of geodesic rays in \mathcal{A} , emanating from a fixed base point, is called the visual boundary of Γ . We will show that this boundary is homogeneous if the nerve N of Γ is a connected closed orientable PL-manifold. This will be done by showing that the boundary is homeomorphic to the Jakobsche space $X(|N|, \{|N|\})$.

1. DEFINITIONS AND NOTATION

Let V be a finite set and $m : V \times V \longrightarrow \{\infty\} \cup \{1, 2, 3, 4, \dots\}$ a function with the property that $m(u, v) = 1$ if and only if $u = v$ and $m(u, v) = m(v, u)$ for all $u, v \in V$. Then the group $\Gamma = \langle V \mid (uv)^{m(u,v)} = 1 \text{ for all } u, v \in V \rangle$ defined in terms of generators and relators is called a *Coxeter group*. The pair (Γ, V) is called a *Coxeter system*. If moreover $m(u, v) \in \{\infty, 1, 2\}$ for all $u, v \in V$ then (Γ, V) is called *right angled*. Let us fix a right angled Coxeter system (Γ, V) .

We call the abstract simplicial complex $N(\Gamma, V) = \{\emptyset \neq S \subseteq V \mid \langle S \rangle \text{ is finite}\}$ the *nerve* of Γ (where $\langle S \rangle$ is the subgroup of Γ generated by S) and metrize its geometric realization $|N|$ as a piecewise spherical all right complex (i.e. each simplex is given the angle metric of its corresponding standard simplex : the convex hull of standard basis vectors in Euclidean space.) The resulting path length metric on $|N|$ will be denoted by α .

The *Vinberg-Davis complex* \mathcal{A} [D1] is built as follows: The cone $Q = x_0 * |N'|$ is the *fundamental chamber* with *panels* $\{Q_v = |star(v, N')| \mid v \in V\}$ in its base (N' denotes the first barycentric subdivision of N .) We give Γ the discrete topology and put $\mathcal{A} = \Gamma \times Q / \sim$ with $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}h \in \langle v \mid x \in Q_v \rangle$. Note that \mathcal{A} is contractible [D1, Corollary 10.3].

Next, we *cubify* the complex \mathcal{A} : Let $\sigma \in N$. We identify $|x_0\sigma'|$ with the cube $[0, 1]^\sigma$ as follows. The cone point x_0 corresponds to 0 and the barycenter of a face $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ of σ to $e_{i_1} + e_{i_2} + \dots + e_{i_k}$, where e_1, e_2, \dots is the standard basis for Euclidean space. The

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cubical complex Q and subsequently the complex \mathcal{A} are then given the induced path length metric.

Since N is a flag complex (i.e. N contains all finite sets all whose proper subsets belong to N), this turns \mathcal{A} into a $CAT(0)$ geodesic space, cf. [G, pp. 120-122], [D2, §3] and [ADG]. Now, Γ acts *geometrically* on \mathcal{A} , that is, properly discontinuously by isometries with compact quotient Q .

For $x, y \in \mathcal{A}$ we denote the unique geodesic from x to y by $[x, y]$. As usual the *length* of an element $g \in \Gamma$ is the minimal number of generators from V needed to express g . The *visual boundary* of Γ is defined to be

$$bdy \Gamma = \varprojlim (M_0 \xleftarrow{r^1} M_1 \xleftarrow{r^2} M_2 \xleftarrow{r^3} M_3 \xleftarrow{r^4} \dots), \text{ where}$$

$$M_k = bdy \bigcup \{gQ \mid length(g) \leq k\} \text{ and}$$

$$\begin{aligned} r_k : M_k &\longrightarrow M_{k-1} \\ x &\longmapsto [x, x_0] \cap M_{k-1}. \end{aligned}$$

This definition of $bdy \Gamma$ is equivalent to taking the inverse limit of concentric metric spheres and geodesic retraction (i.e. the space of geodesic rays emanating from a fixed base point with the compact open topology), which in turn is independent of the choice of base point [Dr, Appendix].

2. HOMOGENEITY OF THE BOUNDARY

A topological space X is called *p-homogeneous* if given any two collections $\{x_1, x_2, \dots, x_p\}$ and $\{y_1, y_2, \dots, y_p\}$ of p distinct points in X there is a homeomorphism $h : X \longrightarrow X$ such that $h(x_i) = y_i$ for all i . We shall need the following two results:

THEOREM 1. [J] *Let $L_0 \xrightarrow{\alpha^1} L_1 \xrightarrow{\alpha^2} L_2 \xrightarrow{\alpha^3} L_3 \xrightarrow{\alpha^4} \dots$ be an inverse sequence of connected closed orientable manifolds and \mathcal{D}_k finite collections of disjoint collared disks in L_k such that*

- (a) *each L_k is a connected sum of finitely many copies of L_0 ,*
- (b) *each α_{k+1} is a homeomorphism over the set $L_k \setminus \{int D \mid D \in \mathcal{D}_k\}$,*
- (c) *each $\alpha_{k+1}^{-1}(D)$ ($D \in \mathcal{D}_k$) is homeomorphic to a copy of L_0 with the interior of a collared disk removed,*
- (d) *$\{\alpha_{j+1} \circ \alpha_{j+2} \circ \dots \circ \alpha_i(D) \mid D \in \mathcal{D}_i, i \geq j\}$ is null and dense in L_j for all j ,*
- (e) *$\alpha_{j+1} \circ \alpha_{j+2} \circ \dots \circ \alpha_i(D) \cap bdy D' = \emptyset$ for all $D \in \mathcal{D}_i, D' \in \mathcal{D}_j, i > j$.*

Then

$$\varprojlim (L_0 \xrightarrow{\alpha^1} L_1 \xrightarrow{\alpha^2} L_2 \xrightarrow{\alpha^3} L_3 \xrightarrow{\alpha^4} \dots)$$

is p-homogeneous for every positive integer p and depends on L_0 only.¹

¹This space is denoted by $X(L_0, \{L_0\})$ in [J]. Note also, that we did not state Jakobsche's result in its full generality.

THEOREM 2. [B, Theorem 2] For every two finite sequences

$$X_0 \xleftarrow{s_1} X_1 \xleftarrow{s_2} X_2 \xleftarrow{s_3} \dots X_{k-1} \xleftarrow{s_k} X_k \text{ and } X_0 \xleftarrow{t_1} X_1 \xleftarrow{t_2} X_2 \xleftarrow{t_3} \dots X_{k-1} \xleftarrow{t_k} X_k \xleftarrow{t_{k+1}} X_{k+1}$$

of maps between compact metric spaces there is a positive real number

$$a(s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k, t_{k+1})$$

such that whenever two inverse sequences

$$Y_0 \xleftarrow{\alpha_1} Y_1 \xleftarrow{\alpha_2} Y_2 \xleftarrow{\alpha_3} Y_3 \xleftarrow{\alpha_4} \dots \text{ and } Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2 \xleftarrow{\beta_3} Y_3 \xleftarrow{\beta_4} \dots$$

have the property that $d(\alpha_k, \beta_k) \leq a(\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \beta_1, \beta_2, \dots, \beta_{k-1}, \beta_k)$ for all $k \geq 2$, then

$$\varprojlim (Y_0 \xleftarrow{\alpha_1} Y_1 \xleftarrow{\alpha_2} Y_2 \xleftarrow{\alpha_3} Y_3 \xleftarrow{\alpha_4} \dots) \simeq \varprojlim (Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2 \xleftarrow{\beta_3} Y_3 \xleftarrow{\beta_4} \dots).$$

THEOREM 3. If a right angled Coxeter group has a nerve which is a connected closed orientable PL-manifold, then it has a boundary which is p -homogeneous for every positive integer p .

SKETCH OF PROOF.

For $g \in \Gamma$ we put

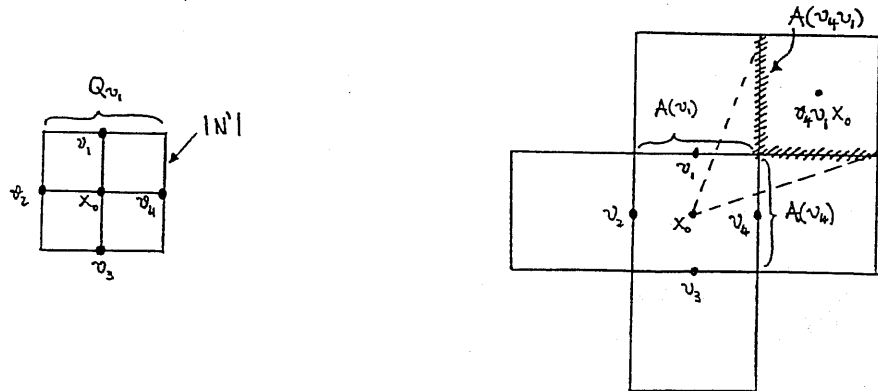
$$A(g) = \bigcup \{gQ_v \mid \text{length}(gv) < \text{length}(g)\}.$$

Then each $A(g)$ is a PL-disk and

$$M_k = M_{k-1} \#_{A(g)} \{g|N| \mid \text{length}(g) = k\}$$

(the connected sum $M_{k-1} \#_{A(g_1)g_1|N|} \#_{A(g_2)g_2|N|} \# \dots \#_{A(g_s)g_s|N|}$ which is independent of the order in which we list the elements $\{g \in \Gamma \mid \text{length}(g) = k\} = \{g_1, g_2, \dots, g_s\}$) [D1].

The inverse sequence $M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} M_3 \xleftarrow{f_4} \dots$ satisfies all but two conditions of Theorem 1: The collections $\{A(g) \mid \text{length}(g) = k\}$, being the natural candidates for the \mathcal{D}_k 's, are not disjoint and, worse yet, condition (e) does not hold in the simplest of examples (e.g. when $V = \{v_1, v_2, v_3, v_4\}$ and $\Gamma = \langle V \mid v_i^2 = 1, (v_i v_j)^2 = 1 \text{ for all } i \neq j \bmod 2 \rangle$). Then \mathcal{A} is just Euclidean 2-space, tessellated by squares. Obviously, condition (e) is violated heavily. See figure below.)



REMARK. Any compact PL-manifold is a nerve of some right angled Coxeter group. In fact, if N is a finite abstract simplicial flag complex (e.g. a barycentric subdivision of a finite complex) with vertex set V , then $\Gamma = \langle V \mid v^2 = (uv)^2 = 1 \text{ for all } v \in V \text{ and } \{u, v\} \in N \rangle$ is a right angled Coxeter system whose nerve is N [D1, Lemma 11.3].

REFERENCES

- [ADG] F. Ancel, M. Davis and C. Guilbault, *CAT(0) reflection manifolds*. Studies in Advanced Mathematics **2** (1997) (Part 1) 441-445.
- [B] M. Brown, *Some applications of an approximation theorem for inverse limits*. Proc. AMS **11** (1960) 478-483.
- [D1] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*. Ann. of Math. (2) **117** (1983), 293-325.
- [D2] M. Davis, *Non-positive curvature and reflection groups*. Proceedings of the 11th Annual Workshop in Geometric Topology, BYU, 1994.
- [DJ] M. Davis and T. Januskiewicz, *Hyperbolization of polyhedra*. J. Differential Geometry **34** (1991), 347-388.
- [Dr] A. Dranishnikov, *Boundaries and cohomological dimension of Coxeter groups*. preprint 1994.
- [G] M. Gromov, *Hyperbolic Groups*. Essays in Group Theory (S.M. Gersten, ed) M.S.R.I. **8** (1987) 75-264.
- [J] W. Jakobsche, *Homogeneous cohomology manifolds which are inverse limits*. Fund. Math. **137** (1991) 81-95.

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Excerpts from

Properties of convergence groups and spaces

Eric M. Freden

The 20 minute talk given in Corvallis summarized various group theoretic consequences of a convergence group action; these are included in section 3 of the cited paper. The entire text appears in the AMS journal *Conformal Geometry and Dynamics*, volume 1, (1997), pp. 13-23, with web address

<http://www.ams.org/ecgd/home-1997.html>

The following excerpt comes from the introduction of the above paper.

Introduction

Two generalizations of Möbius groups were introduced in 1986. Gromov defined the geometric notion of negatively curved (word hyperbolic) groups and spaces [Gr]. At the same time, Gehring and Martin gave a simple topological condition (the convergence property) that all Möbius groups obey [GM1]. While the former subject has seen great activity in recent years, the so-called convergence groups have only recently assumed a prominent role. That negatively curved groups display convergence properties is found in [B1] [F] [MS] [T1] [W]. More recently, Bowditch has shown a partial converse [B2].

The aim of this paper is to illustrate the considerable restrictions that accompany the convergence action of a (nonelementary) group G on a compact Hausdorff space X . Under mild hypotheses, the limit set of G must be metrizable with cardinality 2^{\aleph_0} . There is some control on the type and number of normal subgroups. Endomorphisms of G with finite kernel are automorphisms. There is a class of finitely generated convergence groups with solvable word problem.

PROJECTIONS OF COMPACTA

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INTRODUCTION

This note deals with imitating convex n -dimensional compacta in R^n with lower dimensional compactum. By imitating we mean finding a lower dimensional set that has the same projections in every direction. This is a major topic in the first authors dissertation. This note establishes the restrictions on the dimension that a subset may have if it is to imitate an n -simplex in R^n . We will also state the general case for arbitrary convex n -dimensional compacta. This general result is established in the dissertation with some expanded results to appear elsewhere.

DEFINITIONS

We say that two subsets X and Y of R^n have the same *projection* in the direction of a line l if and only if every line that is parallel to l which intersects X also intersects Y and vice versa. We can generalize this definition by using j -dimensional planes, denoted j -planes, rather than just lines. That is, we say that the two subsets X and Y of R^n have the same j -projection in the direction of a j -plane H if and only if every j -plane that is parallel to H that intersects X also intersects Y and vice versa. By two j -planes being parallel we mean that one is a translate of the other.

IMITATING SIMPLICES

Theorem. *There exists an i -dimensional subset C of the n -simplex, Δ_n , that has the same j -projections as Δ_n if and only if $i + j \geq n - 1$.*

Proof. To show the necessity portion of the proof we will show that the set C must contain the $n - j - 1$ skeleton of Δ_n . Then it would follow that the dimension i of C is greater than or equal to $n - j - 1$ for it contains a subset of that dimension. Consequently we have $i + j \geq n - 1$.

Let p be a point of the $n - j - 1$ -skeleton of Δ_n . Let h be a linear homeomorphism that sends Δ_n to the standard n -simplex with vertices e_0, \dots, e_n and which sends the k -face containing p to the k -simplex spanned by e_1, \dots, e_{k+1} . Let H' be the $(n - k - 1)$ -plane containing $h(p)$ which is parallel to the subspace spanned by e_{k+2}, \dots, e_n . The plane H' meets the standard simplex only in $h(p)$ so $H = h^{-1}(H')$ is an $n - j - 1$ -plane that hits p and misses the rest of Δ_n . Consequently p must be in the set C if C is to have the same j -projections as Δ_n . Since p was an arbitrary point of the $n - j - 1$ -skeleton the entire $n - j - 1$ -skeleton of Δ_n must be in C as claimed.

To prove sufficiency we will show how to construct an i -dimensional set that has the same j -projections as Δ_n where $i = n - k - 1$. To do this we will first prove the following lemma.

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Lemma. For a given n we can find an $(n-2)$ -dimensional subset C of Δ_n that has the same projections in every direction as Δ_n .

Proof of Lemma. Take a line segment from the barycenter of Δ_n to each of the barycenters of the $(n-1)$ -faces of Δ_n . Call the union of these line segments Y . Notice that no three of the endpoints or vertices of Y are collinear since we are using barycenters. Consequently we can take a small regular neighborhood N of Y such that a line meets Δ_n if and only if it meets $\Delta_n - N$. This guarantees that Δ_n and $\Delta_n - N$ have the same projections in every direction. This is the first stage of our construction. We continue by taking a triangulation of $\Delta_n - Y$ and repeating the above process to each of the n -simplices in this triangulation. If we repeat this process ad infinitum in such a manner that the mesh of the triangulations tends to zero and take the intersection we get an $(n-2)$ -dimensional subset of Δ_n . To see this notice that each stage retracts onto the $(n-2)$ -skeleton of the previous stage and that these retractions get small, in the sense that points are moved less, the farther we go in stages. Also, by construction, any line that hits Δ_n also hits the intersections of these sets. So this intersection is our desired set. Notice that it contains the $(n-2)$ -skeleton as required.

Now that we have this Lemma established we can finish the proof of the Theorem. To do this we will show how to construct an i -dimensional subset M_n^i of Δ_n such that every j -plane that intersects Δ_n intersects M_n^i .

We begin this construction by taking the $i+2$ simplices of Δ_n and noting that these lie in $(i+2)$ -planes. We apply the Lemma to each of these simplices. This gives an i -dimensional subset that has the same projections as the original simplex. By this we mean projections by lines lying in the corresponding $(i+2)$ -plane. Doing this for all of the $i+2$ simplices gives us an i -dimensional subset of Δ_n that contains the i -skeleton of Δ_n . Call this M_n^i .

To show that Δ_n and M_n^i have the j -projections we need to show that every j -plane that intersects Δ_n also intersects M_n^i . So let J be a j -plane that intersects Δ_n . If J meets the i -skeleton of Δ_n we are done for this skeleton lies in M_n^i . In fact if J intersects the $(i+2)$ -skeleton we are also done. To see this, assume that J intersects the $(i+2)$ -skeleton in a point x of an $(i+2)$ -simplex. Consider the $(i+2)$ -plane determined by this simplex. Since by assumption $i+j = n-1$ we know that the intersection of J and the plane is at least a line because $j+i+2 > n$. So we have a line in J that lies in the $(i+2)$ -plane and intersects the $(i+2)$ -simplex in the point x . But by the proof of the Lemma this line must also hit M_n^i .

So we will be done if we can show that every j -plane that intersects Δ_n must intersect the $(i+2)$ -skeleton of Δ_n .

Assume that J is a j -plane that intersects Δ_n . Let x be a point in the intersection. Assume x is in a k -simplex. As noted above if $k \leq i+2$ we are done. So assume $k > i+2$. We claim that J must also hit the boundary of the k -simplex. To see this note that $j+k > j+i+2 > n$. But this means that the intersection of J with the corresponding k -plane is at least a line. But a line in a k -plane which intersects the interior of a k -simplex in the plane must also hit the boundary of the k -simplex. Thus J must also hit a $(k-1)$ -simplex or, in other words, J must intersect the $(k-1)$ -skeleton of Δ_n . If $k-1 = i+2$ then we are done. If not then $k-1 > i+2$ and we repeat the same argument. We do this until we see that J intersects the $(i+2)$ -skeleton and we are done.

What we have shown is that any j -plane that intersects Δ_n intersects the $(i+2)$ -skeleton and thus intersects the i -dimensional set M_n^i . This finishes the proof of the theorem.

IMITATING CONVEX COMPACTA

The main idea of the above section was to show that a set of a certain dimension, namely the $n-j-1$ -skeleton, must be in any subset of Δ_n that imitates Δ_n . This gives a lower bound on the dimension of

imitating sets. We then showed how to finish constructing the imitating set in such a manner that the dimension was not raised.

The idea for the general case is similar but the details must be generalized. Instead of showing that the skeleton of the simplex must be in the imitating set we can show that faces of the convex n -dimensional compactum must be in the imitating set. That is we have the following result.

Theorem. *If K is a convex compactum of dimension n in R^n and C is a compact subset of K that has the same j -projections in every direction as K then the dimension of C is greater than or equal to $n-j-1$.*

Covers of Aspherical Manifolds with Geometric Fundamental Groups

FREDRIC D. ANCEL

CRAIG R. GUILBAULT

September 1, 1997

ABSTRACT. In this note we describe results which imply that closed aspherical n -manifolds ($n > 4$) having isomorphic fundamental groups which are either word hyperbolic or $CAT(0)$ have homeomorphic universal covers. This may be viewed as progress towards a weak version of the Borel Conjecture.

1. INTRODUCTION

One of the most famous open problems in geometric topology is the following:

The Borel Conjecture. If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then they are homeomorphic.

Since a solution to the Borel Conjecture has been so illusive, we suggest the following.

A Weak Borel Conjecture. If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then their universal covers are homeomorphic.

In this note we describe results which imply some special cases of the latter conjecture. In particular, we have:

Theorem 1. *Let P^n and Q^n be closed aspherical n -manifolds ($n > 4$) with isomorphic fundamental groups. If this group is word hyperbolic or $CAT(0)$, then P^n and Q^n have homeomorphic universal covers.*

This result is obtained by combining results from geometric group theory (see Lemma 3) with the following:

Theorem 2. *Suppose M^n and N^n are contractible open n -manifolds ($n > 4$) which admit Z -compactifications having homeomorphic Z -boundaries. Then M^n and N^n are homeomorphic.*

Definitions of the above terminology and descriptions of the relevant geometric group theory are contained in Section 2. A quick sketch of the proof of Theorem 2 is given in Section 3. For a thorough presentation of this work the reader should consult [AG].

2. DEFINITIONS AND EXAMPLES

A closed subset A of a compact ANR X is a \mathcal{Z} -set if any of the following equivalent conditions is satisfied:

- There is a homotopy $H : X \times I \rightarrow X$ with $H_0 = id_X$ and $H_t(X) \cap A = \emptyset$ for all $t > 0$.
- For every $\varepsilon > 0$ there is an ε -homotopy $K : X \times I \rightarrow X$ with $K_0 = id_X$ and $K_1(X) \subset X \setminus A$.
- For every $\varepsilon > 0$ there is a map $f : X \rightarrow X$ which is ε -close to the identity with $f(X) \subset X \setminus A$.
- For every open set U of X , $U \setminus A \hookrightarrow U$ is a homotopy equivalence.

Let Y be a noncompact ANR. A \mathcal{Z} -compactification of Y is a compact ANR \widehat{Y} containing Y as an open subset and having the property that $\widehat{Y} - Y$ is a \mathcal{Z} -set in \widehat{Y} . In this case we call $\widehat{Y} - Y$ a \mathcal{Z} -boundary for Y and denote it $\partial_{\mathcal{Z}}Y$. Note that Y may admit many different \mathcal{Z} -boundaries, hence $\partial_{\mathcal{Z}}Y$ is not well defined unless the \mathcal{Z} -compactification is specified.

For our purposes, the key facts from geometric group theory may be summarized in the following result borrowed from [Be] (see Lemma 1.4). A more thorough development including other useful references may be found in Section 3 of [AG].

Lemma 3. *Let P be an aspherical manifold with word hyperbolic or $CAT(0)$ fundamental group. Then the universal cover of P admits a \mathcal{Z} -compactification. Moreover, if Q is another aspherical manifold having fundamental group isomorphic to that of P , then the universal coverings of P and Q admit \mathcal{Z} -compactifications with homeomorphic \mathcal{Z} -boundaries.*

3. SKETCH OF THE PROOF OF THEOREM 2

The first major ingredient in the proof of Theorem 2 is a gluing theorem which is interesting in its own right.

Theorem 4. *Let \widehat{M}^n and \widehat{N}^n be \mathcal{Z} -compactifications of open n -manifolds ($n > 4$) and $h : \partial_{\mathcal{Z}}\widehat{M}^n \rightarrow \partial_{\mathcal{Z}}\widehat{N}^n$ be a homeomorphism. Then $\widehat{M}^n \cup_h \widehat{N}^n$ is an n -manifold. If M^n and N^n are contractible, then $\widehat{M}^n \cup_h \widehat{N}^n \approx S^n$.*

The proof of the above result is rather technical. The idea is to show that $\widehat{M}^n \cup_h \widehat{N}^n$ is a resolvable ANR homology manifold which satisfies the disjoint disks property. It then follows from Edwards' Cell-Like Approximation Theorem that $\widehat{M}^n \cup_h \widehat{N}^n$ is a manifold.

The last part of Theorem 4 is obtained by an application of the high dimensional Poincaré Conjecture.

To prove Theorem 2, identify $\widehat{M}^n \cup_h \widehat{N}^n$ with S^n viewed as the boundary of an $(n+1)$ -ball, B^{n+1} ; let Z denote the common copy of $\partial_Z \widehat{M}^n = \partial_Z \widehat{N}^n$ in S^n ; and let $W^{n+1} = B^{n+1} \setminus Z$ (see Figure 1). Using properties of Z -sets, one may show that (W^{n+1}, M^n, N^n) is a proper h -cobordism. Next, using the algebraic machinery developed in [Si] and [CS] it can be shown that the inclusion $M^n \hookrightarrow W^{n+1}$ is an (infinite) simple homotopy equivalence. An application of Siebenmann's proper s -cobordism theorem then guarantees that $W^{n+1} \approx M^n \times [0, 1]$, completing the proof.

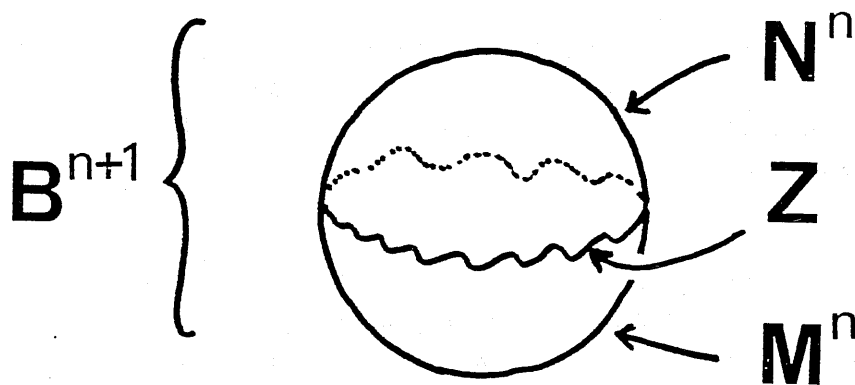


Figure 1.

REFERENCES

- [AG] F.D. Ancel and C.R. Guilbault, *Z-compactifications of open manifolds*, preprint.
- [Be] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J., **43** (1996), 123-139.
- [CS] T.A. Chapman and L.C. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. **137** (1976), 171-208.
- [Si] L.C. Siebenmann, *Infinite simple homotopy types*, Indag. Math. **32** (1970), 479-495.

Suspensions and Spheres

by

James P. Henderson

Given an n -sphere S^n , it is well known that for each pair of distinct points x and y in S^n , there is a compact space $L(x,y)$ such that S^n is homeomorphic to the suspension of $L(x,y)$, with the suspension points being sent to x and y . While $L(x,y)$ can be chosen to be an $(n-1)$ -sphere in each case, $L(x,y)$ need not even be a manifold as shown by the Double Suspension Theorem. R.J. Daverman asked whether the converse of this property of spheres holds:

Question: If X is a finite dimensional compact metric space containing at least two points with the property that for each distinct pair of points x and y in X , there is a compact metric space $L(x,y)$ such that X is homeomorphic to $\Sigma L(x,y)$, with the suspension points corresponding to x and y , must X be homeomorphic to S^n for some n ?

The first observation to be made is that X must be homogeneous. For given any two points in X , there is a homeomorphism taking $\Sigma L(x,y)$ to itself, interchanging the suspension points. This then gives the desired homeomorphism of X onto itself taking x to y . Also note that X will be locally cone-like, with each suspension point having a cone neighborhood.

A second point regards $L(x,y)$. If this set is in fact empty for any pair of distinct points x and y , then it follows that X must be S^0 .

If $L(x,y)$ is not empty but finite, then in fact X be S^1 . Away from the suspension points, neighborhoods of points will look like cones over a two point set, with homogeneity then forcing $L(x,y)$ to consist of two points. It then follows that X will be homeomorphic to S^1 .

The remaining possibility is that $L(x,y)$ must be infinite for each pair of distinct points. In this case, we'll add the assumption that for some pair x, y that $L(x,y)$ is a polyhedron. It then follows from homogeneity that X must be a manifold. Since X is homeomorphic to the suspension of $L(x,y)$, X is also a polyhedron. Take a point p in the interior of a top dimensional simplex σ^n in a triangulation of X . It now follows that all points of X will have a neighborhood homeomorphic to R^n as does p .

We now can conclude that $L(x,y)$ must be a generalized $(n-1)$ -manifold since $\Sigma L(x,y) - \{x,y\}$ is homeomorphic to $L(x,y) \times R^1$, an open subset of the n -manifold X . By considering an open neighborhood of the point x with the structure of an open cone over $L(x,y)$, it is possible to show that in fact $L(x,y)$ and S^{n-1} have the same homotopy type.

Using this information, it is relatively straightforward to see that X is a simply connected manifold with the same homology as S^n . Hence for $n \geq 4$, it follows from the Poincare conjecture that X must in fact be homeomorphic to S^n .

On the other hand, for $n=2$ or $n=3$, we can use the fact that $L(x,y)$ must be not just a generalized $(n-1)$ manifold, but an actual $(n-1)$ manifold and hence S^1 or S^2 . Thus it follows that X will be either S^2 or S^3 .

The question that remains regards whether the polyhedral hypothesis can be dropped. The space X has been shown to be locally cone-like and homogeneous. If one can show that in fact X has a CS stratification [2], then the homogeneity will force X to be a manifold, and the program can proceed as above. However, Handel [1] has given an example of a locally cone-like space which does not have the desired stratification. Thus we are led to the following conjecture:

Conjecture: If a finite dimensional metric space X is locally cone-like and homogeneous, then it must have a CS stratification, and hence be a manifold.

As was pointed out after this presentation in Oregon, an example providing a counterexample to this conjecture would also show that the Bing-Borsuk conjecture that a homogeneous ANR must be a manifold would also be false.

References

1. Michael Handel, A Resolution of Stratification Conjectures Concerning CS Sets, *Topology*, Vol 17 (1978) pp 167-175.
2. L.C. Siebenmann, Deformations of Homeomorphisms on Stratified Sets, *Comment. Math. Helv.* Vol 47 (1972), pp 123-163.

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A Relative Identity Property for Two-Complex Pairs

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I. Setting

Suppose X is the two-complex modeled on the group presentation $P = \langle x : r \rangle$, and we add 2-cells to X (i.e., relators to P) to obtain the two-complex Y modeled on $Q = \langle x : r, s \rangle$. Then the fundamental group of X is the group presented by P , call it G . Similarly, $\pi_1(Y)$ is the group presented by Q , call it H . Notice that the inclusion of X into Y induces a surjection of fundamental groups whose kernel is L , the normal subgroup of G generated by the set of words s .

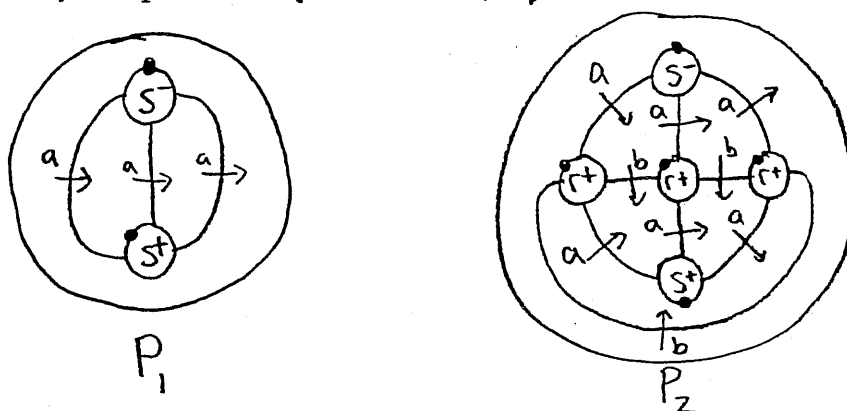
The relative identity property for such a pair of two-complexes (Y, X) determines a strong relationship between the 2nd homotopy groups of both spaces. We define this identity property via spherical pictures.

Spherical pictures, defined over a group presentation, are combinatorial descriptions of maps from the 2-sphere into the two-complex associated to the presentation. It is well-known that any element of π_2 can be represented by a spherical picture.

Example 1: Consider the two-complex pair

$$P = \langle a, b : [a, b] \rangle \text{ and } Q = \langle a, b : [a, b], a^3 \rangle.$$

Following our notation X is the torus, and Y is the torus with an additional 2-cell glued on to the 1-skeleton according to the relator $s = a^3$. As the torus, $\pi_2(X)$ is trivial. However, $\pi_2(Y)$ is generated (as $\mathbb{Z}H$ -module) by 2 elements, represented by the pictures: (note $r = [a, b]$)



A *spray* on a picture is a sequence of transverse paths $\{\gamma_i\}$ connecting the global basepoint of the picture to the basepoint of each interior disk Δ_i . The path to disk Δ_i , γ_i , determines a word, ω_i , in the ambient free group according to the arcs traversed along its route.

Now, suppose P is a spherical picture defined on the presentation

$$Q = \langle x : r, s \rangle$$

Given a spray on P , each disk Δ_i gets three important labels:

$$t_i \in r \cup s$$

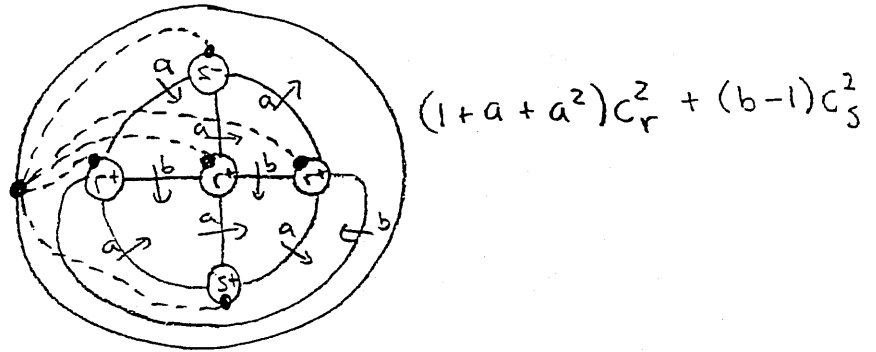
$$\varepsilon_i = \pm 1$$

$$\omega_i \in F(x)$$

Moreover, the words ω_i determine elements of the group H , h_i , that are independent of the choice of spray. A spray is used to map a picture (which represents an element of π_2) to an element of the second chain group of the universal cover.

Example 1 (cont):

The spray below allows us to compute the homological image of P_2 in $C_2(\tilde{Y}) = ZHc_r^2 \oplus ZHc_s^2$.



Definition: The pair (Q, P) (or (Y, X)) has the *identity property* if every spherical picture over Q has a pairing of its s-disks ($i \leftrightarrow j$) such that:

$$t_i = t_j; \quad \varepsilon_i \neq \varepsilon_j; \quad h_i = h_j.$$

In other words, the disks in any pair have the same relator label, but opposite sign, and the words associated to their respective paths from the global basepoint determine the same element of H .

Remarks:

1. This definition is a combinatorial description of asking the following composition to be trivial:

$$\pi_2(Y) \rightarrow C_2(\tilde{Y}) = \bigoplus_{r \in r} ZHc_r^2 \oplus \bigoplus_{s \in s} ZHc_s^2 \rightarrow \bigoplus_{s \in s} ZHc_s^2$$

where the first map is the standard injection of π_2 into the second chain group of the universal cover, a free ZH-module with basis in 1-1 correspondence with the relators of Q . The second map is projection onto the s-coordinates. The identity property asks that all elements of $\pi_2(Y)$ be homologically trivial, modulo the r 2-cells.

2. This identity property first arose in a paper of Mike Dyer's in which he outlined a possible strategy for the construction of a counterexample to Whitehead's Asphericity Conjecture. Using this property, Dyer translates homotopic requirements of Howie's infinite counterexample construction to homological requirements for such a construction.

Examples:

2. The pair of Example 1 does not have the identity property. Since $\pi_2(Y)$ is generated by the two pictures shown, it is true that any spherical picture over Q has a pairing of its s-disks satisfying the first two conditions of the definition. The third condition fails, however.

3. If the inclusion induced map on π_2 is surjective then (Y, X) has the identity property. Indeed, any spherical picture over Q is equivalent to one without s-disks in view of surjectivity. Thus, a pairing of the s-disks must exist. For instance, (Y, X) has the identity property if Y is aspherical.

4. If X is the one-skeleton of Y , then we recover the (absolute) identity property which detects asphericity of a two-complex:

$(Y, Y^{(1)})$ has the identity property $\Leftrightarrow Y$ has the identity property
 $\Leftrightarrow Y$ is aspherical.

Looking for non-trivial examples, the following group-theoretic characterization of the identity property proves useful.

Theorem (MPH): Let $P = \langle x : r \rangle$, $Q = \langle x : r, s \rangle$, and $Z = \langle x : s \rangle$. Let R be the normal subgroup of $F(x)$ generated by the set r , and S be the normal subgroup of $F(x)$ generated by s . Then (Q, P) has the identity property if and only if

$$i: \pi_2(Z) \rightarrow \pi_2(Y) \text{ is trivial and } R \cap S \subset [S, RS].$$

Example 5: Let $P = \langle x : [u, v]^n \rangle$, and $s = \{u, v\}$ (assume $n \geq 1$).

Here $R = \langle\langle [u, v]^n \rangle\rangle_F$ and $S = \langle\langle u, v \rangle\rangle_F$. Thus $R \subset S$ and $R \subset [S, S]$ from which the group-theoretic condition of the theorem holds. If we choose u

and v carefully (e.g, let $Z = \langle x: u, v \rangle$ be any two-relator aspherical presentation), then (Q, P) has the identity property.

We note that $0 \neq \pi_2(X) \xrightarrow{0} \pi_2(Y)$ if $n \geq 2$, a necessary initial condition in Dyer's counterexample construction. Admittedly, though, this example is not delicate enough as the new space Y is not Cockcroft.

Closing Remarks:

Examples of two-complex pairs with the identity property have been hard to come by. A good portion of my thesis has been devoted to a weakened version of this property, a generalized identity property. This generalized identity property is defined via a presentation of the "relative" relation module associated to the pair (Y, X) . It turns out that this generalized version is analogous to the weakening of asphericity (π_2 trivial) to combinatorial asphericity (π_2 generated by dipoles) in the absolute setting. Indeed, we recover combinatorial asphericity from the generalized identity property with the case that X is the one-skeleton of Y . In my thesis, notions of relative dipoles and relative combinatorial asphericity are developed.

FINITE-VOLUME HYPERBOLIC 4-MANIFOLDS AND THEIR ENDS

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Recall that the Poincaré upper half-space model of hyperbolic n -space is $\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, t) \in \mathbb{R}^n \mid t > 0\}$ with the metric given by $ds^2 = \frac{dx_1^2 + \dots + dx_{n-1}^2 + dt^2}{t^2}$. The *boundary at infinity* of a set $S \subseteq \mathbb{H}^n$ is the set of all points in $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$ that are in the (Euclidean) closure of S . In the upper-half-space model hyperbolic hyperplanes are either Euclidean half-spheres or Euclidean half-planes orthogonal to $\partial\mathbb{H}^n$ and they are uniquely determined by their own boundaries at infinity, which are Euclidean $(n-2)$ -spheres or $(n-2)$ -planes in $\mathbb{R}^{n-1} \cup \{\infty\}$. We will say that the hyperplane is *based*, respectively, on a sphere or a plane. (In our case $n = 4$, so the hyperplanes will be based on 2-spheres and 2-planes in \mathbb{R}^3 .)

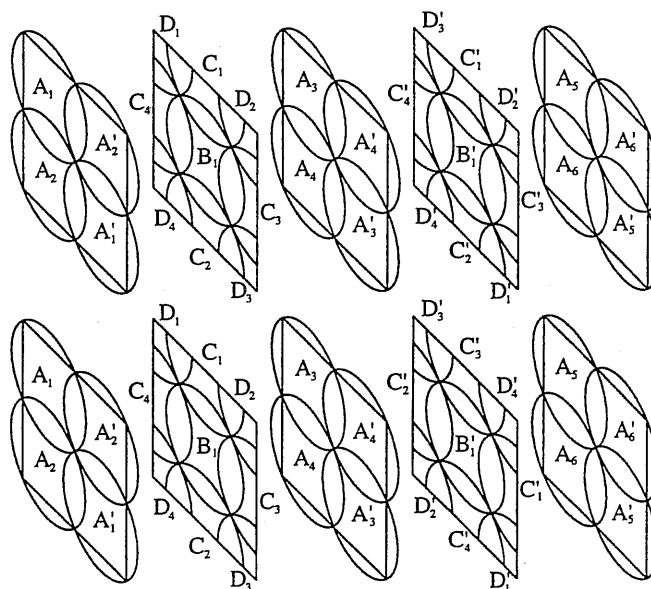
A hyperbolic n -manifold \mathbb{H}^n/G is the quotient of \mathbb{H}^n by the action of a discrete torsion-free group $G \subset \text{Isom}(\mathbb{H}^n)$. Similarly, a Euclidean n -manifold is the quotient of \mathbb{R}^n by a discrete group of isometries of \mathbb{R}^n .

It is known (see [A]) that a complete, hyperbolic, geometrically finite n -manifold has finitely many ends. If the manifold has finite volume, then all of the ends are standard cusp ends, that is, they are of the form $E \times [0, \infty)$, where E is a closed flat manifold.

We construct a hyperbolic 4-manifold by means of side-pairings of a hyperbolic 4-polyhedron P . Our P is the intersection of the half-space determined by hyperplanes based on the planes bounding the rectangular box $R = [-2, 2] \times [-2, 2] \times [-2\sqrt{2}, 2\sqrt{2}] \subset \mathbb{R}^3$ and on the 30 spheres depicted in Figure 1. The upper part of Figure 1 shows the intersections of these spheres with the five parallel planes that contain their centers. These are the five planes with constant z -coordinates $-2\sqrt{2}$, $-\sqrt{2}$, 0 , $\sqrt{2}$, $2\sqrt{2}$ going from left to right. All spheres are of radius $\sqrt{2}$. For the spheres we choose the half-spaces whose boundary at infinity is unbounded in \mathbb{R}^3 , for the planes the half-spaces so that the intersection of their boundaries at infinity is the rectangular box R . The polyhedron P is defined as the intersection of those half-spaces.

Next, two side-pairings, Φ_1 and Φ_2 , are defined on the polyhedron P . (See [I1] for the exact isometries pairing of the sides.) The sides that are paired are labeled S and S' in Figure 1. The sides of the rectangular box are paired by translations. By checking the conditions of Poincaré's polyhedron theorem (see [E-P] and [R])

This note summarizes some of the results from the author's PhD thesis. I would like to thank my advisor, Dr. Boris Apanasov, for many helpful conversations and suggestions concerning these results.


 FIGURE 2. Side-pairings Φ_1 (top) and Φ_2 (bottom)

we see that by identifying the sides of P that are paired, we arrive at noncompact hyperbolic manifolds M_1 and M_2 . They have 7 and 8 ends, respectively. Each end corresponds to an equivalence class under the side-pairing of vertices at infinity. The vertices at infinity and their equivalence classes are depicted in Figure 2.

Now form a polyhedron Q from n copies of P : attach the second P by identifying its side based on $z = -\sqrt{2}$ to the side of the first P based on $z = \sqrt{2}$, attach the third P to the second one in the same manner and so on. In \mathbb{R}^3 , the boxes with their spheres are simply strung linearly together in the z -direction. Define a side-pairing on the polyhedron Q by choosing either the side-pairing Φ_1 or Φ_2 on each copy of P . (There are many ways in which this can be done.) The side $z = -\sqrt{2}$ of the first P and the side $z = \sqrt{2}$ of the n -th P are paired by translation.

It turns out (Poincaré's polyhedron theorem again) that every of the side-pairings on Q defined above yields a hyperbolic manifold. One can show that it is obtained from n copies of either M_1 or M_2 by cutting each along an embedded totally geodesic submanifold and gluing each of the resulting two boundary components to a boundary component of another cut-open M_1 or M_2 . (We are stringing together n copies of cut-open M_1 or M_2 's, each with two boundary components, in a circular fashion.) If k copies of P had the pairing Φ_1 and $n - k$ copies of P had the pairing Φ_2 defined on them, it may be seen that the resulting manifold has $4 + 4n - k$ ends. We have complete freedom of choice for k so we can get manifolds with anywhere from $4 + 3n$ to $4 + 4n$ ends. Clearly, no two of those can be homeomorphic so we get

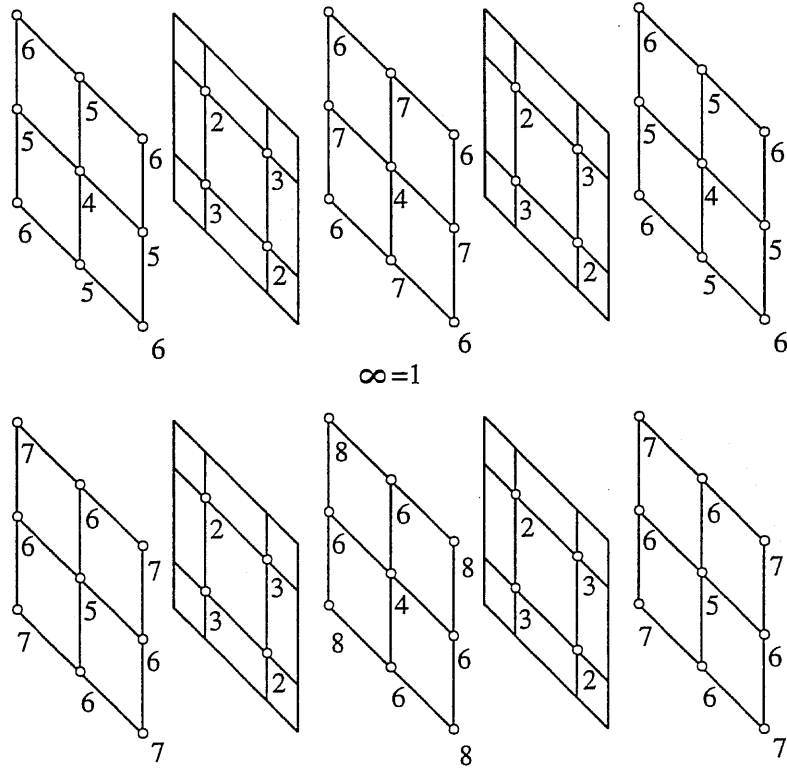


FIGURE 2. Cycles of ideal vertices for Φ_1 (top) and Φ_2 (bottom). In both cases, the vertex ∞ is labeled by 1

Theorem 1. *Given any number N , there exist more than N nonhomeomorphic, noncompact, complete, hyperbolic 4-manifolds of finite volume that share the same fundamental polyhedron in \mathbb{H}^4 . In particular, they have the same volume.*

The last statement relates to a theorem of Wang (see [W]) that says that for every constant $c > 0$ there are only finitely many complete nonhomeomorphic hyperbolic n -manifolds with volume $< c$, where $n \geq 4$. We just showed, for $n = 4$, that there is no bound on the number of manifolds with the same volume.

For $n = 3$ the set of volumes is a well-ordered (infinite) set, but still only finitely many manifolds may have the same volume. The number of manifolds with the same volume is again unbounded, as follows from theorems analogous to theorem 1 that have been established by Wielenberg (see [W1]) for the noncompact case, and by Apanasov and Gutsul ([A-G]) for the compact one.

A by-product of the above construction is

Theorem 2. *The set of all volumes of hyperbolic 4-manifolds contains the even multiples of $4\pi^2/3$.*

This comes from the Gauss-Bonnet formula for hyperbolic 4-manifolds, $\text{Vol}(M) = 4\pi^2/3 \cdot \chi(M)$, where $\chi(M)$ denotes the Euler characteristic of M . One shows that $\chi(M_1) = \chi(M_2) = 2$, so $\text{Vol}(P) = \text{Vol}(M_1) = 2 \cdot 4\pi^2/3$, which implies that $\text{Vol}(Q) = 2n \cdot 4\pi^2/3$. Hence, the volume of any of the manifolds constructed from Q is also $2n \cdot 4\pi^2/3$.

A stronger result, that the set of volumes is the positive integral multiples of $4\pi^2/3$ was obtained in [R-T], but the proof there was not constructive.

We now turn our attention to the ends of noncompact hyperbolic manifolds.

For a 3-dimensional manifold M , the ends are of form $E \times [0, \infty)$ where E is either a torus or a Klein bottle. Each of these is an S^1 -bundle over the circle, so $E \times [0, \infty)$ may be thought of as a punctured disc bundle over a circle. Hence, each end of M may be seen as a complement of a circle inside a solid torus or a "solid Klein bottle". Then $M \cup \{\text{circles}\}$ is a compact manifold, so M is a complement of some link inside a compact 3-manifold N . (Many examples are known where $N = S^3$ (see [Wi2] or [T]), and it is also known that almost all such N 's can be given a hyperbolic structure (see [T]).

We want to adress the same problem for noncompact hyperbolic 4-manifolds M : When can M be embedded as a complement of a 2-dimensional manifold B tamely contained in some compact 4-manifold N ?

It turns out, if such an embedding is possible, for every end $E \times [0, \infty)$ of M we must have that E is a circle bundle over B . Now we have the following

Theorem 3. *Let E be a compact flat n -manifold, $E = \mathbb{R}^n/G$, where G is a discrete subgroup of $\text{Isom } \mathbb{R}^n$. Then E is an S^1 -bundle over some base manifold B if and only if there exists an element $f \in G$ so that*

- (i) $\langle f \rangle$ is a normal subgroup of G
- (ii) For every $g \in G$, $g^k \in \langle f \rangle$ implies $g \in \langle f \rangle$. (Note that this is equivalent to $G/\langle f \rangle$ being torsion-free.

Applying this criterion to the ten (see [Wo] or [H-W]) 3-dimensional Euclidean manifolds, we find out that there are six that are S^1 -bundles over a manifold B , and four that are not (all the nonorientable ones are S^1 -bundles). A result of Niemershiem ([N]) asserts that for every Euclidean 3-manifold E , one can find a finite-volume hyperbolic 4-manifold M so that $E \times [0, \infty)$ is an end of M . This coupled with the observation just before the theorem tells us that there exist noncompact hyperbolic 4-manifolds that cannot be embedded as a complement of a surface inside a compact 4-manifold.

On the positive side, however, there is a theorem from [A-F] asserting that every hyperbolic n -manifold is finitely covered by a hyperbolic n -manifold all of whose ends are of the simplest form, $T^{n-1} \times [0, \infty)$, where T^{n-1} is the $(n-1)$ -torus. Since $T^{n-1} \times [0, \infty) = T^{n-2} \times S^1 \times [0, \infty) = T^{n-2} \times (\text{punctured disc})$, the covering manifold is a complement of $(n-2)$ -tori inside some compact n -manifold. This works in particular for $n = 4$.

For those hyperbolic 4-manifolds all of whose ends are among the desirable six ends, we can show that the surface B must be a Euclidean surface, i.e. B is either a torus or a Klein bottle.

These considerations generalize to almost all higher-dimensional hyperbolic manifolds: if a hyperbolic n -manifold M is a complement of a codimension-2 manifold B inside some compact n -manifold N , then the Euclidean manifolds appearing in the ends of M must be S^1 -bundles and B must itself be a Euclidean manifold. There is one dimension, 5, for which the proofs do not go through.

More details on previous results may be found in [I1] and [I2]. Both preprints can be obtained from the author and [I1] is also available on the Web,

<http://www.msri.org/MSRI-preprints/online/1997-052.html>.

REFERENCES

- [A] B.N. Apanasov, *Discrete groups in space and uniformization problems*, Kluwer Academic Press, 1991.
- [A-G] B.N. Apanasov and I.S. Gutsul, *Greatly symmetric totally godesic surfaces and closed hyperbolic 3-manifolds which share a fundamental polyhedron*, Topology '90.
- [E-P] D.B.A. Epstein and C. Petronio, *An exposition of Poincaré's polyhedron theorem*, L'Ensign. Math. **40** (1994), 113–170.
- [H-W] W. Hantzsche and H. Wendt, *Dreidimensionale euklidische Raumformen*, Math. Ann. **110** (1935), 593–611.
- [I1] D. Ivanišić, *Finite-volume hyperbolic 4-manifolds that share a fundamental polyhedron*, to appear in Jour. of Diff. Geom. and Appl.
- [I2] D. Ivanišić, *Embedability of noncompact hyperbolic manifolds as complements of codimension-2 submanifolds*, preprint.
- [N] B. Nimershiem, *All flat three-manifolds appear as cusps of hyperbolic four-manifolds*, preprint.
- [R] J. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer-Verlag, 1994.
- [R-T] J. Ratcliffe and S. Tschantz, *The volume spectrum of hyperbolic 4-manifolds*, preprint.
- [T] W. Thurston, *The geometry and topology of three-manifolds*, Princeton University, 1979.
- [W] H. C. Wang, *Topics on totally discontinuous groups*, Symmetric spaces, Pure Appl. Math. Vol. 8, Marcel-Dekker 1972, pp. 460–485.
- [Wo] J.A. Wolf, *Spaces of constant curvature*, McGraw-Hill, 1967.
- [Wi1] N. Wielenberg, *Hyperbolic 3-manifolds which share a fundamental polyhedron*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference (I.Kra and B. Maskit, eds.), Ann. of Math. Studies 97, Princeton Univ. Press 1981, pp. 505–513.
- [Wi2] N. Wielenberg, *The structure of certain subgroups of the Picard group*, Math. Proc. Cambridge Phil. Soc. **84** (1978), 427–436.

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$K(\pi, 1)$ S COMING FROM THE OUTER SPACE

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Any selected list of most important discrete groups in mathematics would certainly include $Aut(F_n)$, the automorphism groups of finitely generated free groups. Moreover, many members of the same list are in close relationship with $Aut(F_n)$. First, $GL_n(\mathbf{Z})$, being the automorphism group of a free abelian group, is an analogue and a quotient of $Aut(F_n)$. Some other members of our imaginary list occur as stabilizers of various objects associated with the action of $Aut(F_n)$ on F_n . (Note that in the case of $GL_n(\mathbf{Z})$, there is not much to say about stabilizers.) Here are some examples, in which we assume $\{a_1, \dots, a_n\}$ is a basis of F_n and (w) denotes the conjugacy class of an element w of F_n .

- (1) Braid groups B_n and pure braid groups P_n :

$$\begin{aligned} B_n &= Stab\{(a_1), \dots, (a_n)\} \cap Stab\{a_1 \cdots a_n\} \\ P_n &= Stab\{a_1 \cdots a_n\} \cap Stab\{(a_1)\} \cdots \cap Stab\{(a_n)\} \end{aligned}$$

- (2) The string group G_n (group of motions of n circles in the 3-space) and the pure string group C_n :

$$\begin{aligned} G_n &= Stab\{(a_1), (a_1^{-1}), \dots, (a_n), (a_n^{-1})\} \\ C_n &= Stab\{(a_1)\} \cap \cdots \cap Stab\{(a_n)\} \end{aligned}$$

- (3) Mapping class groups of orientable surfaces. In the simplest case when the surface has genus $n/2$ and one puncture, its mapping class group is the quotient of the group

$$Stab\{[a_1, a_2] \cdots [a_{n-1}, a_n]\}$$

modulo its infinite cyclic center.

The above characterizations are quite useful because many facts about the groups in question can be obtained by generalizing and refining the methods used for $Aut(F_n)$. In particular, the Outer Space of Culler and Vogtmann [3] is a rich source of interesting spaces on which the stabilizers act in a most desirable way. Let us denote by X_n the Culler-Vogtmann complex, a spine of the Outer Space. It is contractible, and $Aut(F_n)$ acts simplicially on it with finite vertex stabilizers and finite quotient.

Theorem. *Let H be the intersection of finitely many subgroups of $Aut(F_n)$, each of which is the stabilizer of a finite set of (a) elements of F_n , (b) conjugacy classes in F_n , or (c) finitely generated subgroups of F_n . Then there is a contractible subcomplex X_H of X_n on which H acts with finite stabilizers and finite quotient.*

This generalization, at least in the case when H is the stabilizer of a finite set of elements, has already been noticed in [3]. Cohomological information directly implied by

the theorem (type VFL, bounds on virtual cohomological dimension) has already been known for braid and mapping class groups. The purpose of my talk is to indicate that in some cases it is possible and worthwhile to study X_H in greater detail in order to obtain more cohomological information about stabilizers.

Firstly, the subcomplex X_H is not uniquely determined, and the existence of the best representative is not clear in general. Sometimes, however, concrete work does lead to a natural and efficient X_H . We can show, for example, that for all groups mentioned in (1)–(3) above, the dimension of X_H equals $vcd(H)$. In the case of mapping class groups, X_H turns out to be the same as the complex of curve systems studied by Harer [4]. When H is a braid group, our complex X_H gives an $(n-1)$ -dimensional Eilenberg-MacLane complex $H \setminus X_H$ for B_n whose combinatorics can be (but has not been!) studied in detail.

Particularly interesting is the case of string groups. For $H = C_n$, we obtain an Eilenberg-MacLane complex $K_n = H \setminus X_H$ with a toric structure.

Theorem. *K_n is covered by a finite family of tori which is closed under intersection.*

Not many groups are there which have a $K(\pi, 1)$ with this property. We wonder if the property implies quadratic isoperimetric inequality, presently unknown for C_n .

The family of tori for K_n can be described in full detail, so that we can compute the cohomology ring of C_n .

Theorem. (a) *The cohomology ring $H^*(C_n)$ is generated by 1-dimensional classes ξ_{ij} , and all relations among these generators are consequences of the relations*

$$\xi_{ij}^2 = 0, \quad \xi_{ij}\xi_{ji} = 0, \quad \xi_{ij}\xi_{ik} + \xi_{ij}\xi_{jk} + \xi_{kj}\xi_{ik} = 0.$$

(b) *Each $H^q(C_n)$ is free abelian, with basis in bijection with the set of rooted forests with q edges and n vertices.* (c) *The Poincaré series of C_n is the polynomial $(1 + nt)^{n-1}$.*

Part (a) was conjectured, and proved for $n \leq 3$, by Brownstein and Lee [2]. Note the striking similarity with Arnol'd's presentation of the cohomology ring of the pure braid groups [1].

References

- [1] V. I. Arnol'd, *Cohomology of the group of died braids*, Math. Zametki, **5**(1969), 227–231.
- [2] A. Brownstein and R. Lee, *Cohomology of the group of motions of n strings in 3-space*, Contemp. Math. **150**, 51–61.
- [3] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Inventiones Math. **84** (1986), 91–119.
- [4] J. Harer, *Stability of homology of the mapping class groups of orientable surfaces*, Annals of Math. **121**(1985) 215–49.

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On suspensions of noncontractible compacta of trivial shape

In 1904 Poincaré constructed the first example of a homological 3-sphere with a nontrivial fundamental group. The complement of an open 3-ball in this space is an acyclic finite noncontractible polyhedron P . It follows by the Mayer-Vietoris sequence and the van Kampen theorem that the suspension ΣP of this polyhedron is an acyclic space with the trivial fundamental group. It follows by the Hurewicz theorem that the suspension ΣP has all homotopy groups trivial and is hence a contractible space. Complex P is an acyclic noncontractible compactum. Every cell-like space is acyclic in Čech cohomology and every contractible compactum

is clearly cell-like. So there is a natural question: Does there exist a noncontractible cell-like compactum whose suspension is contractible? (Bestvina-Edwards, Problem D28 in van Mill-Reed *Open Problems in Topology*). Related to this open problem we have proved (in a joint paper with U. H. Karimov) the following result: *There exists a noncontractible cohomologically locally connected (clc) 2-dimensional compact metric space X of trivial shape whose reduced suspension is an absolute retract.*

A finiteness theorem for planes in 3-manifolds of finite genus at infinity

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September 23, 1997

Summary

A map $f : X \rightarrow Y$ is said to be *proper* if $f^{-1}(K)$ is compact for every K compact in Y .

Let M be a 3-manifold. A *plane* in M is a 2-submanifold that is homeomorphic to \mathbf{R}^2 . Let P be a plane in M . We say that P is *nontrivial* in M

1. if $P \cap K$ is compact for every compact $K \subset M$ and
2. if there is a compact $K_P \subset M$ such that there is no proper map $H : P \times [0, 1] \rightarrow M$ with $H(P, 0) = 1_P$ and $H(P, 1) \cap K_P = \emptyset$.

A 3-manifold M is said to be of finite genus at infinity if there is a non-negative integer g such that if $K \subset M$ is compact there is a compact 3-manifold M_K with $K \subset M_K - \partial M_K$ such that ∂M_K is a connected 2-manifold of genus g . We say that g is the *genus* of M if g is the smallest integer with this property.

Let N be a noncompact 3-manifold such that for every compact $K \subset N$ there is a closed 3-ball B_K such that $K \subset \text{Int } B_K$ and $B_K \cap P$ is a single disk for each component P of ∂N . Then we call N a *nearnode*.

The manifold $\mathbf{R}^2 \times I$ is an example of a nearnode, but is by no means the only one.

Nearnodes are interesting because, in spite of the fact that they are contractible, they can contain a large number of nontrivial planes.

Theorem 1 *There is a nearnode N such that ∂N has two components and N contains a family \mathcal{F} of planes such that every element of \mathcal{F} is nontrivial in N , no two elements of \mathcal{F} intersect, no two elements of \mathcal{F} are parallel, and \mathcal{F} has the cardinality of the Cantor set.*

This result illustrates that one cannot hope for a finiteness theorem for planes that is just a naive restatement of the Haken-Kneser Finiteness Theorem. However, we can obtain the following result:

Theorem 2 *Suppose that M is a noncompact, irreducible, orientable connected 3-manifold that has one end and is of genus $g \geq 2$ at infinity. Let U be a 3-manifold in M such that the inclusion map from U into M is proper and such that ∂U is the disjoint union of a finite number of planes. If $\#(\partial U) > 2g - 2$, then at least one component N of U is a nearnode such that ∂N has two components.*

Open Problems

J. Cannon

1. Is there a Dehn algorithm in the extended sense for every 3-manifold fundamental group?
2. Is the fundamental group of the Hawaiian Earring the same as the fundamental group of the union of two Hawaiian Earrings attached at opposite endpoints of an arc? Is the fundamental group of the Sierpinski carpet the same as the fundamental group of the Sierpinski curve?
3. Let C be a convex polyhedron in E^3 . Can the boundary of C be cut into a single disc that can be laid down in the plane without self overlapping?

R. Daverman

1. Let N^3 be a closed orientable hyperbolic three manifold. let $f : N \rightarrow N$ be such that $f_{\#}(\pi_1(N))$ has finite index in $\pi_1(N)$. Does f have degree not equal to 0 imply that f is a homotopy equivalence?
2. Suppose $p : M^n \rightarrow X$ is a cell-like map defined on an n -manifold M with empty boundary, and $\epsilon > 0$. Does there exist a UV^k map $g : X \rightarrow M$, $k = \lfloor \frac{n-3}{2} \rfloor$, such that $p \circ g$ is within ϵ of the identity on X ?

Dusan Repovš

X is Cantor homogeneous if for each pair of Cantor sets in X , there is a homeomorphism of X that takes one of the Cantor Sets onto the other. Hanna Patkowska studied this in dimensions one and two. Is there a 3-dimensional compact metric space X such that X is Cantor homogeneous? Such an example can't be LC^1 .

F. Ancel

If Γ is a $CAT(0)$ Coxeter group with a manifold nerve, does Γ have a unique boundary?

S. Bleiler

Given a two generator link of unknots that has a 2 generator presentation with one generator represented by a meridian, does this imply that the link is 2-bridge? Note that if the link has H-genus 2, the answer is yes. If the answer is yes, then for satellites, 2 generators implies H-genus 2.

Y. Sternfeld

Let $H = \ell_2$ and let $B = \{x \in H \mid \|x\| \leq 1\}$ and $S = \{x \in H \mid \|x\| = 1\}$.

1. Is there a homeomorphism f from B onto S such that both f and f^{-1} are Lipschitz functions?
2. Equivalently, is B Lipschitz homogeneous?