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NOTES ON ENDS OF HYPERBOLIC 3-MANIFOLDS

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Lecture 1. The central theme of these notes is a question of Marden [M]: Do all hyperbolic 3-manifolds with finitely-generated fundamental group have product ends?

An end is said to have a product structure (or to be tame) if it is homeomorphic to $(\partial \text{ end}) \times \mathbb{R}_+$. Marden's question may be reformulated as follows: Is every hyperbolic 3-manifold M with finitely-generated fundamental group homeomorphic to the interior of a compact 3-manifold?

By work of Canary [C], the affirmative answer to Marden's question for a finitely-generated Kleinian group $G \cong \pi_1(M)$ implies Ahlfors' conjecture [A] for G: The limit set Λ_G is either the whole 2-sphere at infinity or has Lebesgue measure zero.

Note that hyperbolic 3-manifolds with finitely-generated fundamental group have finitely many ends. (This is not true in general for dimensions greater than three.) To prove this, recall

Scott core theorem [S]. *Let M be a hyperbolic 3-manifold with finitely-generated fundamental group. Then there exists a compact, codimension zero submanifold C of M such that the inclusion map $C \hookrightarrow M$ induces an isomorphism of fundamental groups.*

In particular, $\pi_1(M)$ is finitely presented, and $H_2(M;\mathbb{Z}) \cong H_2(\pi_1(M);\mathbb{Z})$ is finitely generated. Let $(H_1^{l.f.}(M)$ denote the homology of locally-finite chains on M. By Poincaré duality $H_1^{l.f.}(M) \cong H^2(M)$ is also finitely generated. Considering lines exiting through various ends of M shows that k ends of M provide $(k-1)$ linearly independent elements of $H_1^{l.f.}(M)$, hence M has finitely many ends.

The classes of groups for which Marden's conjecture has been verified are:

(1) *Fuchsian groups (2-dimensional case): $G \subset PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.*

A fundamental domain for the action of G on $\mathbb{H}^3$ (Poincaré model) is an "apple core", bounded by a finite number of totally geodesic planes orthogonal to the

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standard $\mathbb{H}^2 \subset \mathbb{H}^3$. It is easy to visualize a product structure on $\mathbb{H}^3/G$:

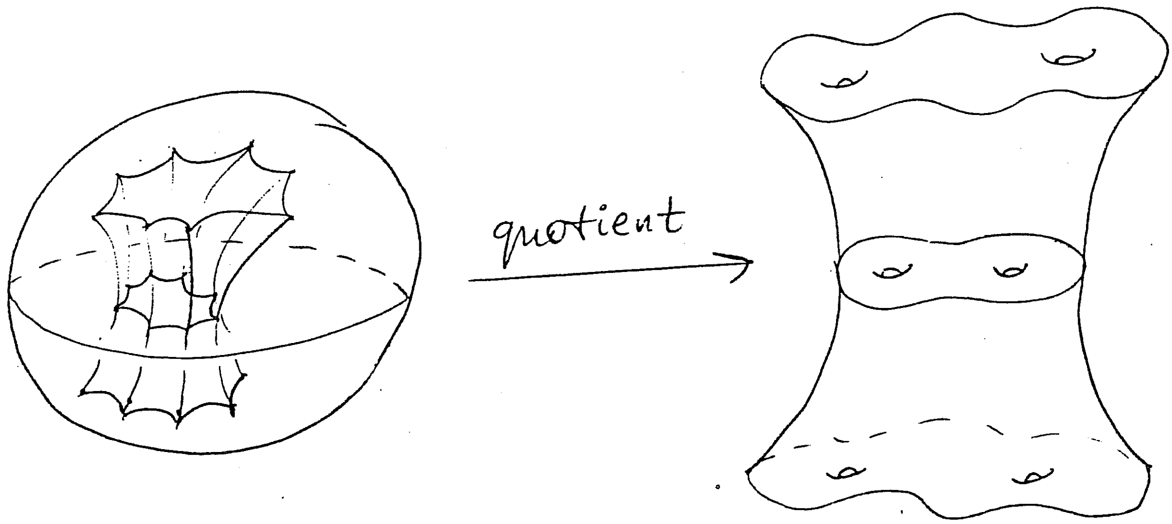


FIGURE 1.1

(2) *Quasi-Fuchsian groups.* The limit sets of these groups are quasi-circles; fundamental domains have more bites taken out of the "apple core" but are still finite-sided convex polyhedra.
(3) *Certain limits of geometrically-finite groups analyzed by Thurston [Th].*
(4) *Indecomposable groups (Bonahon [B]).* Recall that group G is called indecomposable if $G \ncong \mathbb{Z}$, and $G \cong A * B$ implies one of the groups A and B is trivial.

Notice that if $\pi_1(M)$ is indecomposable, Scott core C of M has a special feature: $\overline{M - C}$ is a homotopy product $(\partial C) \times \mathbb{R}_+$. To prove this, first observe that ∂C is incompressible in C. We may assume, using induction, that $M - C$ is connected. Suppose the homomorphism induced by inclusion $\pi_1(\partial C) \to \pi_1(C)$ is not injective. By the loop theorem and Dehn's lemma there exists an embedded disk $(D, \partial D) \subset (C, \partial C)$ with ∂D essential in ∂C. The disk D cannot be separating, since it would provide a free decomposition of $\pi_1(C) \cong \pi_1(M)$. In the non-separating case, since $\pi_1(M) \ncong \mathbb{Z}$, we can take a band sum of two parallel copies of D to get a separating disk $D' \subset C$ with $\partial D'$ still essential in ∂C. This contradicts the assumption that $\pi_1(\partial C) \to \pi_1(C)$ is not injective.

By Seifert-van Kampen's theorem $\pi_1(M)$ is the pushout:

$$\begin{array}{ccc} \pi_1(\partial C) & \xrightarrow{\alpha} & \pi_1(C) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ \pi_1(\overline{M \setminus C}) & \longrightarrow & \pi_1(M) \end{array}$$

Since α is injective and β is an isomorphism, γ is also an isomorphism. This shows that $\overline{M - C}$ is a homotopy product $(\partial C) \times \mathbb{R}_+$.

This is not necessarily the case when $\pi_1(M)$ is decomposable. For example, let M be a genus two handlebody. Then both the standard wedge of two circles and a knotted one are Scott cores of M.

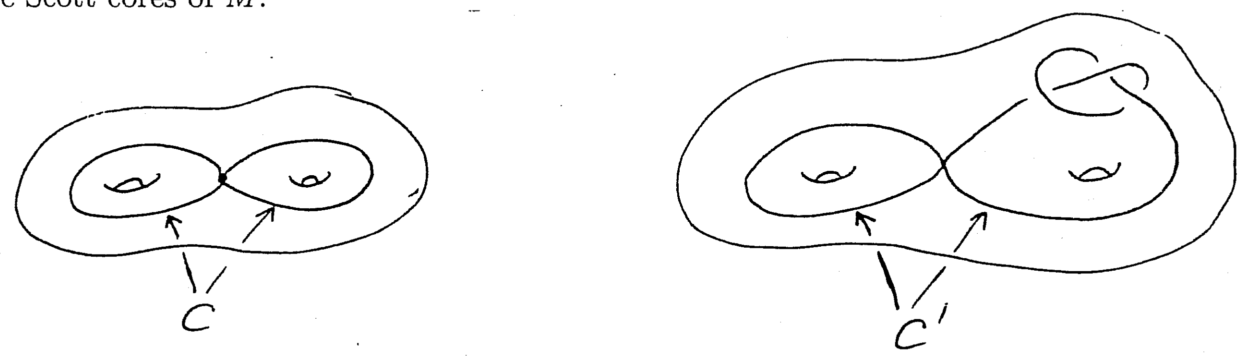


FIGURE 1.2

The idea of Thurston and Bonahon in the indecomposable case—to find a sequence of pleated surfaces, homeomorphic to ∂C, exiting the end of M—does not work in the decomposable case for a knotted core as in the example above. One would like to find a "natural" geometric core to avoid this problem.

We will now state two geometric reformulations of the condition that the ends of M are tame.

Fix a finitely-generated Kleinian group $G \cong \pi_1(M)$ and let $\mathcal{O}$ denote the orbit $G(x)$ of a point $x \in \mathbb{H}^3$.

Theorem 1.1. [F] *The following two conditions are equivalent:*

(a) *For any r there exists a (sufficiently large) R such that every arc in $(\mathbb{H}^3 \smallsetminus N_r(\mathcal{O}), \partial N_r(\mathcal{O}))$ can be deformed, relative to its ends, into $(N_R(\mathcal{O}) \smallsetminus N_r(\mathcal{O}), \partial N_r(\mathcal{O}))$, where N_r denotes the neighborhood of radius r.*

(b) *All ends of M are tame.*

To understand condition (a), note that for a simple closed curve K in S^3, knottedness is the failure of the relative fundamental group $\pi_1(S^3 \smallsetminus K, \partial)$ to be trivial. ¿From this perspective the negation of (a) means that the orbit of G is coarsely knotted at all size scales.

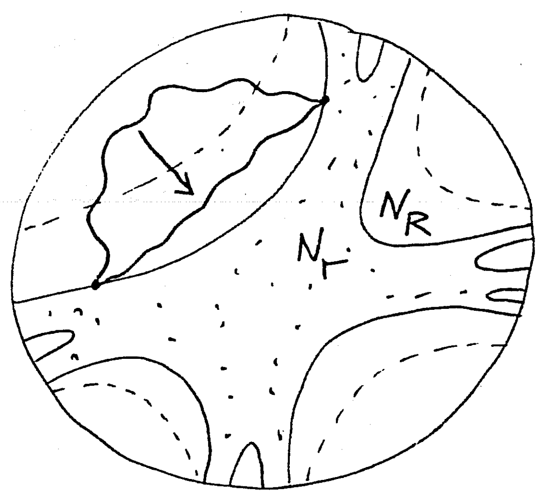


FIGURE 1.3

Before giving the proof of Theorem 1.1, we recall

Tucker's Theorem. [T] *Let M be a hyperbolic 3-manifold with finitely-generated fundamental group. Then all ends of M are tame if and only if $\pi_1(M \setminus K)$ is finitely generated for all compact submanifolds $K \subset M$.*

Proof of Theorem 1.1. Let $\bar{x}$ be the image of x in $M = \mathbb{H}^3/G$. Pick a basepoint $p \in \partial(M \setminus N_r(\bar{x}))$ and fix a preimage $\tilde{p} \in \partial(N_r(\mathcal{O}))$ of p. The fundamental group $\pi_1(M \setminus N_r(\bar{x}), p)$ may be interpreted as classes of paths in the universal cover, starting at $\tilde{p}$ and ending at various lifts of p. The multiplication is given by composition of paths, using covering translations. By Tucker's theorem M has product ends if and only if $\pi_1(M \setminus N_r(\bar{x}))$ is finitely generated for all r. This is equivalent to finding a set of generators in the universal cover within a bounded distance from $N_r(\mathcal{O})$. $\square$

Another sufficient condition for the existence of a product structure is the following "Elder Sibling Property".

Consider the case when the domain of discontinuity Ω_G is non-empty. Let $x \in \Omega_G$ and let $h \subset \mathbb{H}^3$ be a horoball based at x. Consider the orbit $G(h)$ in the upper-half space model where x is the point at infinity. Each horoball in the orbit meets only finitely many of its G-translates, and for h sufficiently large the union of the orbit $\cup G(h)$ is path-connected.

1.2. We say that G satisfies the **Elder Sibling Property** (ESP) if there exists a horoball h based at $x \in \Omega_G$ such that in the upper-half space model with x the point at infinity, each horoball in $G(h)$ of finite (Euclidean) size meets a larger or infinite one.

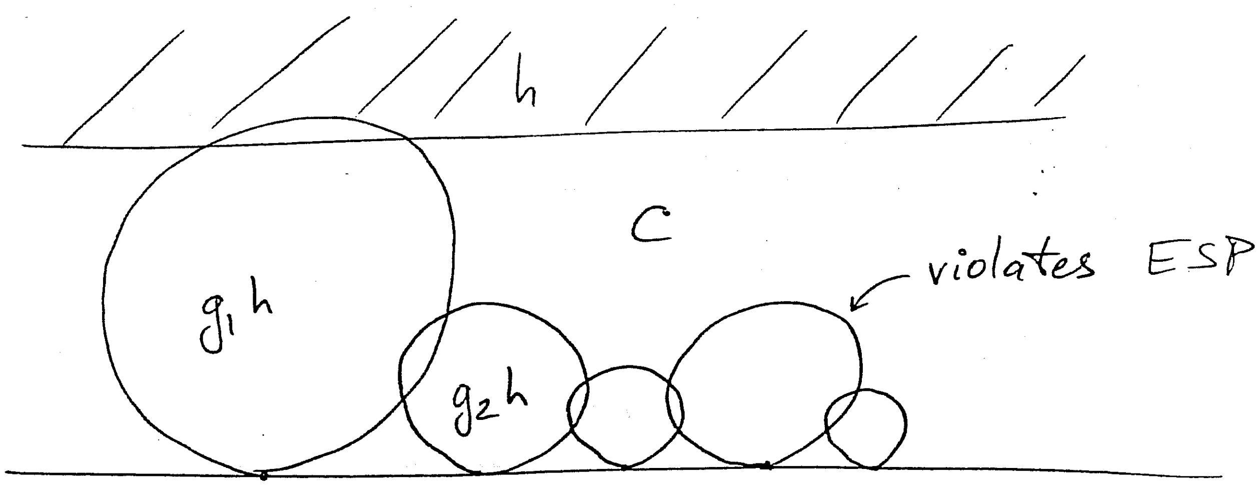


FIGURE 1.4

Let C denote $\mathbb{H}^3 \setminus (\cup G(h))$.

Theorem 1.3. [F] *Suppose $G \cong \pi_1(M)$ satisfies ESP. Then $\pi_1(C, \partial C)$ is trivial, and all ends of M are tame.*

The idea of the proof is to use Tucker's theorem with the image of h in M thought of as an almost compact set. In fact, h touches S^2_∞ at a point of discontinuity where its behavior is well understood.

Sketch of the proof. Consider the Morse function on C, given by the height in the upper-half space model. There are six types of critical points that may occur on ∂C—each one has one of three Morse indices $0, 1, 2$ and one of two signs $+$ or $-$, according to whether the interior of C lies above or below ∂C at the critical point. It follows from the geometry of C, that points of index $(+, 2)$ do not occur. The only possibility for $\pi_1(C, \partial C)$ to be non-trivial is that there are critical points of index $(+, 0)$, which introduce at these respective height levels new components of C, which are then connected in a non-trivial way at points of index $(-, 1)$. This possibility is eliminated by the ESP assumption, since each $(+, 0)$ point is canceled by a corresponding saddle point. $\square$

Lecture 2. *A new extension of the Kneser-Haken finiteness principle to manifolds with boundary.*

Let $G \cong \pi_1(M)$ be a free finitely-generated Kleinian group, for example the free group on two generators. Let x, y be a free basis of G. Similarly to representing an essential element of a fundamental group by a least-length geodesic, the chosen basis of G may be represented by a least length web (wedge of two circles) W. Let $\widetilde{W} \subset H^3$ denote the preimage of W in the universal cover.

Lemma 2.1. *$\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ is a free group.*

Proof. Let x be a vertex of $\widetilde{W}$, and let $v_1, \ldots, v_4$ denote the unit tangent vectors at x to the four edges of $\widetilde{W}$ containing x. It follows from the choice of W that $\sum_{i=1}^{4} v_i = 0$ —otherwise a first order variation would reduce the length of W.

Fix a point $p \in \mathbb{H}^3$ and let r denote the (hyperbolic) radius function on $\mathbb{H}^3$ centered at p. The restriction $r|_{\widetilde{W}}$ cannot have a local maximum at an interior point of an edge since W is a least length web, and it does not have one at a vertex by the "balanced" condition discussed above. It follows that the presentation of $\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ associated to r has no relations. $\square$

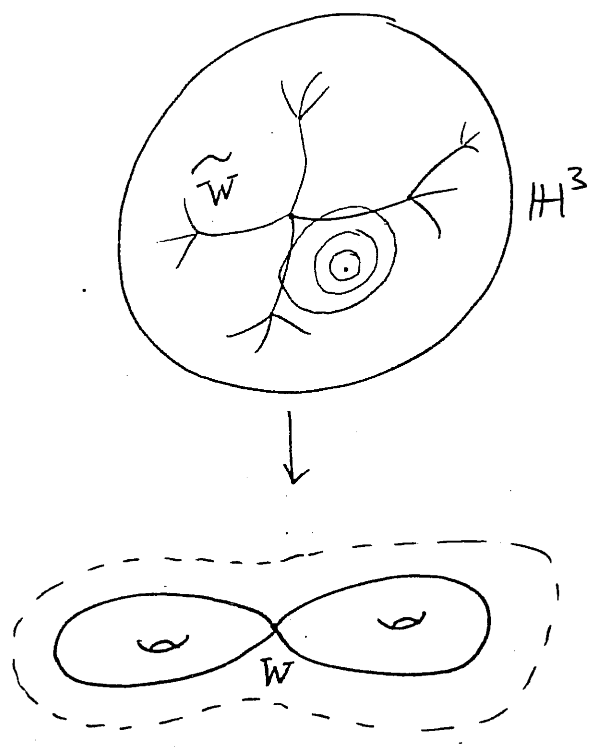


FIGURE 2.1

2.2. Examples of open 3-manifolds with free fundamental group on two generators may be built using an analogue of Whitehead's construction [Wh]. Let $(H_2)_i \hookrightarrow (H_2)_{i+1}$ be an inclusion of a genus two handlebody into the interior of another one, which is homotopic but not isotopic to the identity map.

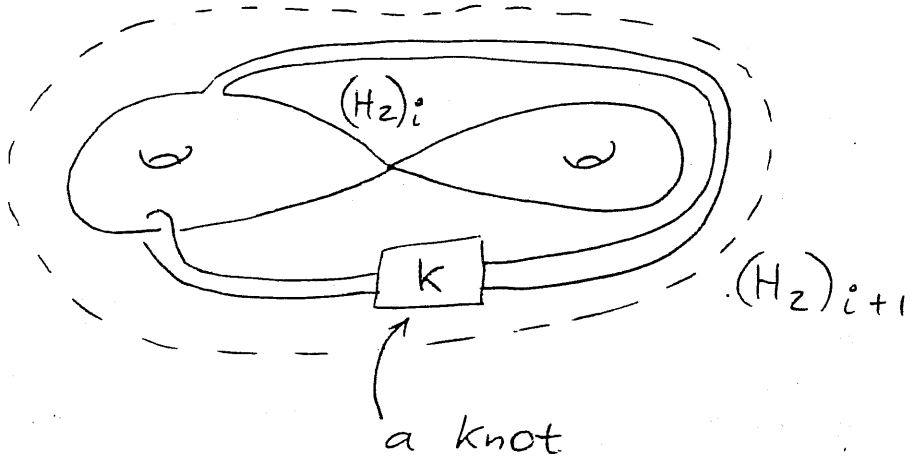


FIGURE 2.2

Let M denote the infinite nested union $\bigcup_{i=1}^{\infty}(H_2)_i$. For Marden's conjecture to hold, these manifolds should not admit a hyperbolic structure. This can be proved in many cases by finding a closed incompressible surface in $\mathbb{H}^3 \setminus \widetilde{W}$, showing that $\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ is not free. A technique for finding incompressible surfaces is described below. (See Theorem 2.6.)

It turns out that this method does not always work, since many 3-manifolds have locally free but not free fundamental group, and hence they do not contain incompressible surfaces. Recall that a group G is called *locally free* if every finitely-generated subgroup of G is free.

2.3. Example of a locally free but not free group. Let N be a thickening in $\mathbb{R}^3$ of the following infinite 2-complex:

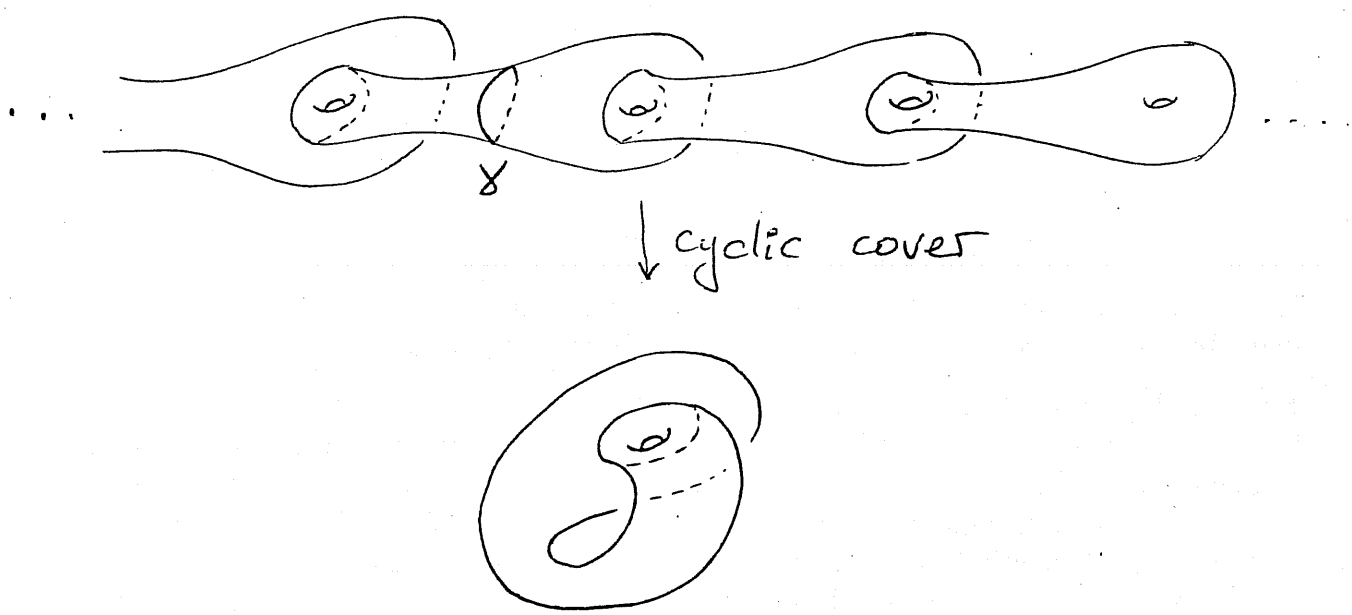


FIGURE 2.3

$\pi_1(N)$ is locally free, since the union of a finite number of consequent fundamental domains has free fundamental group, and these are joined along incompressible annuli. $\pi_1(N)$ is not free, since the loop γ bounds in N an infinite half-grope, showing that $[\gamma] \in (\pi_1(N))_\omega$ —the ω-term of the lower-central series. γ represents a non-trivial element of $\pi_1(N)$, since a null-homotopy would lie in a compact piece of N; on the other hand, the ω-term of free groups is trivial.

Question 2.4. Give a general geometric method for showing that a fundamental group is not free.

Lemma 2.5. *Let M be an irreducible non-compact 3-manifold without boundary. The following conditions are equivalent:*

(1) *$\pi_1(M)$ is locally free.*
(2) *M contains no closed incompressible surface.*
(3) *M is exhausted by handlebodies.*

We will now describe a technique for finding closed incompressible surfaces in non-compact 3-manifolds.

Theorem 2.6. [F-F] *Let C be a curve in the solid torus $M = D^2 \times S^1$ which is homotopic but not isotopic to the core $\{0\} \times S^1$ of M, and let $\widetilde{C} \subset D^2 \times \mathbb{R}$ denote its preimage in the universal cover. Then $(D^2 \times \mathbb{R}) \setminus \widetilde{C}$ contains a closed incompressible surface.*

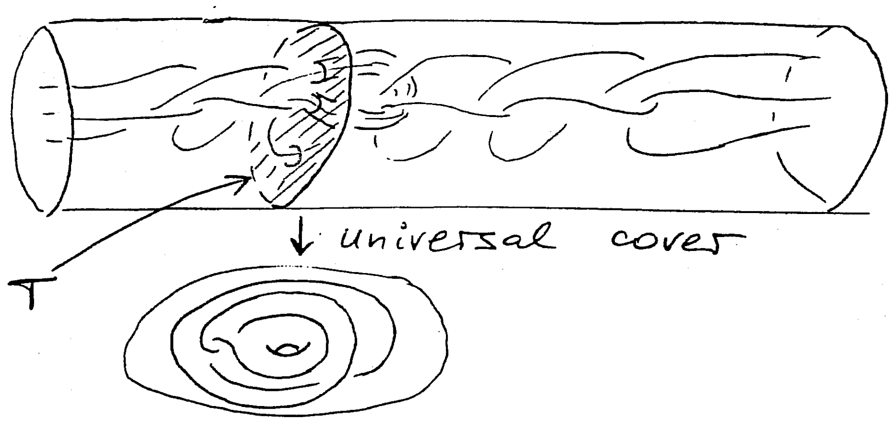


FIGURE 2.4

In this case, for a genus one handlebody, the existence of a hyperbolic structure is not an issue, but this is a good test case for finding incompressible surfaces.

Proof of Theorem 2.6. Let M_n denote the union of n consequent fundamental domains in $D^2 \times \mathbb{R}$, with $\widetilde{C}$ cut out. Notice that a surface which is incompressible in M_n is also incompressible in $D^2 \times \mathbb{R} \setminus \widetilde{C}$ since $\pi_1(M_n)$ injects into $\pi_1(D^2 \times \mathbb{R} \setminus \widetilde{C})$ as a factor in the free product with amalgamation. Let Σ_n denote the boundary of M_n —a genus three surface, pushed slightly into the interior of M_n. It will be shown that M_n is not a genus three handlebody for sufficiently large n. This will prove that Σ_n does not compress to a 2-sphere and will complete the proof of Theorem 2.6. □

Note that the argument given below gives n exponential in the complexity of the curve C in M. An easier argument was later found by C. Gordon and by A. Reid-D. Cooper-D. Long.

Let T denote the twice-punctured torus shown in FIGURE 2.4 with $\partial T \subset (D^2 \times \mathbb{R}) \setminus \widetilde{C}$. T and its translates provide $(n+1)$ non-parallel incompressible surfaces in M_n. If M_n was a genus three handlebody, this would give a contradiction for large n with the bounded case of Kneser-Haken finiteness theorem. (See Theorem 2.7 below.) $\square$

Recall the Kneser-Haken finiteness principle for a closed 3-manifold M: There exists an integer $c(M)$ which bounds from above the number of disjoint closed non-parallel incompressible surfaces in M. The idea of the proof is the observation that the intersection of a collection S of surfaces in normal form with each 3-simplex $\triangle$ in a triangulation of M provides at most six non-product regions of $\triangle \setminus S$ (FIGURE 2.5). The product regions of various tetrahedra glue up, modulo the homological correction term, to (surfaces) $\times\, I$.

This proof does not extend to the *bounded* case in the absence of the boundary-incompressibility assumption on surfaces: $S \cap \triangle$ may contain tunnels providing an uncontrolled number of non-product regions. (FIGURE 2.6.)

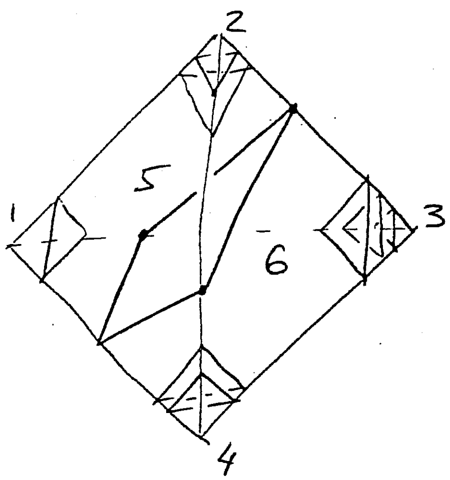


FIGURE 2.5

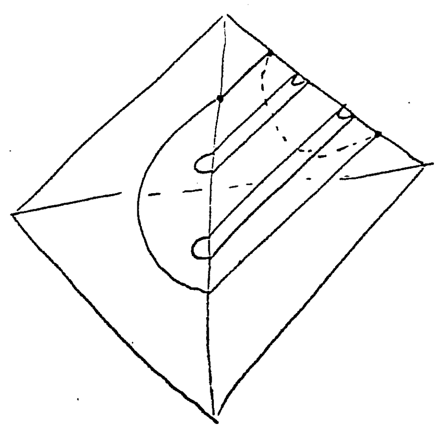


FIGURE 2.6

W. Sherman [Sh] exhibited arbitrarily many non-parallel incompressible surfaces in (genus two surface) $\times\, I$. His example has the property that the complexity of surfaces increases with their number.

Theorem 2.7. [F-F] *Let M be a compact 3-manifold with boundary and b an integer greater than zero. There is a constant $c(M,b)$ such that if $S_1, \dots, S_k$, $k > c$, is a collection of incompressible surfaces such that all Betti numbers, $b_1(S_i) < b$, $1 \le i \le k$, and no S_i, $1 \le i \le k$, is a boundary-parallel annulus or a boundary-parallel disk, then at least two members of S_i and S_j are parallel.*

Remarks. 1. The bound in the closed case is roughly $6 \cdot$ (number of tetrahedra in a triangulation of M), while in the bounded case Theorem 2.7 gives the bound which is exponential in the complexity of the data.

2. The proof of Theorem 2.6 does not apply to handlebodies of genus greater that one. The difference is that a free group with more than one generator has exponential growth, and the number of fundamental domains in the ball of radius R in the universal cover is comparable with the genus of the bounding surface.

Lecture 3. Before formulating a topological approach to the product-end conjecture, we first recall the observation (essentially contained in [C]) that in order to detect a wild end of a hyperbolic 3-manifold it suffices to check the infinite-generation condition in Tucker's theorem for just one of the "algebraically disk-busting" geodesics:

Lemma 3.1. *Let γ be a simple closed geodesic in an open hyperbolic 3-manifold M satisfying*

(1) $[\gamma] = 0 \in H_1(M; \mathbb{Z}_2)$,

(2) *no conjugate of γ lies in a factor of a non-trivial free product $\pi_1(M) \cong A * B$.*

If M has a wild end, then $\pi_1(M \smallsetminus \gamma)$ is infinitely generated.

Remarks. Note that the set of algebraically disk-busting (that is, satisfying condition (2) above) geodesics has full measure in the set of all geodesics ([K], [C]). If a geodesic γ is not simple, then the arguments below apply to any C^1-perturbation of γ.

The algebraically disk-busting condition for a curve γ in a *compact* 3-manifold N is equivalent to the condition that γ meets every essential disk in N.

Proof of Lemma 3.1. Suppose $\pi_1(M \smallsetminus \gamma)$ is finitely generated. We will show that the 2-fold branched cover $\widetilde{M}^\gamma$ of M with the branching locus γ has finitely-generated indecomposable fundamental group. By a theorem of Gromov and Thurston [G-Th] $\widetilde{M}^\gamma$ then admits a metric of pinched negative curvature, and Bonahon's proof [B] extends to this setting to show that $\widetilde{M}^\gamma$ has product ends. Since branched coverings preserve the structure of ends, the proof of Lemma 3.1 will be complete once we show that $\pi_1(\widetilde{M}^\gamma)$ is finitely generated and indecomposable.

By the relative Scott core theorem [Mc] there is a compact core $C \subset \overline{M \smallsetminus N(\gamma)}$ with $\partial N(\gamma) \subset C$, where $N(\gamma)$ denotes a closed tubular neighborhood of C. By assumption (1) γ bounds a surface Σ in M, and $\widetilde{M}^\gamma$ is constructed by gluing two copies of $\overline{M \smallsetminus N(\Sigma)}$ to obtain a double cover $\widetilde{M}^2$ of $\overline{M \smallsetminus N(\gamma)}$ and then regluing back $N(\gamma)$. Cores are functorial under these operations:

$$\begin{tikzcd} \widetilde{C}^\gamma \arrow[r, hook] & \widetilde{M}^\gamma \\ \widetilde{C}^2 \arrow[u, hook] \arrow[r, hook] \arrow[d] & \widetilde{M}^2 \arrow[u, hook] \arrow[d] \\ C \arrow[r, hook] & \overline{M \smallsetminus N(\gamma)} \end{tikzcd}$$

Now the problem is reduced to the compact setting, and as in [C] the equivariant Dehn's Lemma and Sphere Theorem, combined with the assumptions (1), (2) on γ, show that $\pi_1(\widetilde{C}^\gamma) \cong \pi_1(\widetilde{M}^\gamma)$ is indecomposable. $\square$

Topological Conjecture 3.2. *Let M be an open 3-manifold with finitely-generated fundamental group and with the universal cover $\widetilde{M} \cong \mathbb{R}^3$, and let $\gamma \subset M$ be a simple closed curve with $\pi_1(M \setminus \gamma)$ infinitely generated. Then $(\widetilde{M}, \pi^{-1}(\gamma)) \not\cong (\mathbb{R}^3$, standard countable collection of lines) where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection.*

A model for the standard collection above is given by the vertical lines in $\mathbb{R}^3$ passing through points of the integral lattice in a horizontal plane.

This topological conjecture implies Marden's and Ahlfors' conjectures, since for M hyperbolic the preimage in $\mathbb{H}^3$ of a simple closed geodesic γ is standard by Morse theory.

Some evidence for the conjecture is provided by the fact that while in the higher-dimensional case there are various restrictions that one may impose on an end, such as π_1-stability at infinity or vanishing of K_0-obstruction, the known conditions in dimension three that restrict wildness already imply the existence of a product structure.

3.3. Various examples of wild ends may be produced by using Whitehead's construction [Wh]. (See also [S-T].)

Let $M = \bigcup_{i=1}^{\infty} T_i$ where $T_i \cong S^1 \times D^2$ and the embedding $T_i \hookrightarrow T_{i+1}$ is shown in FIGURE 3.1.

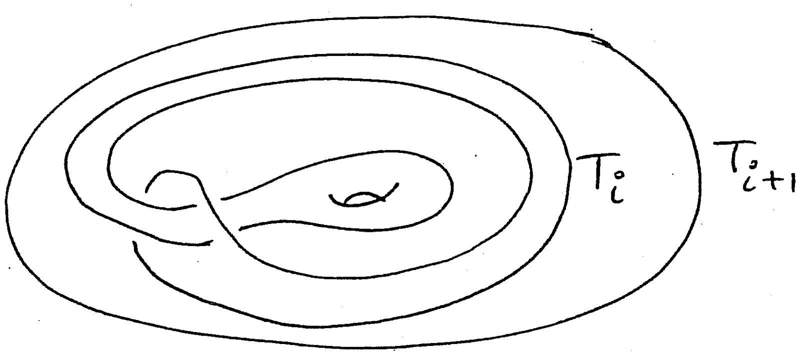
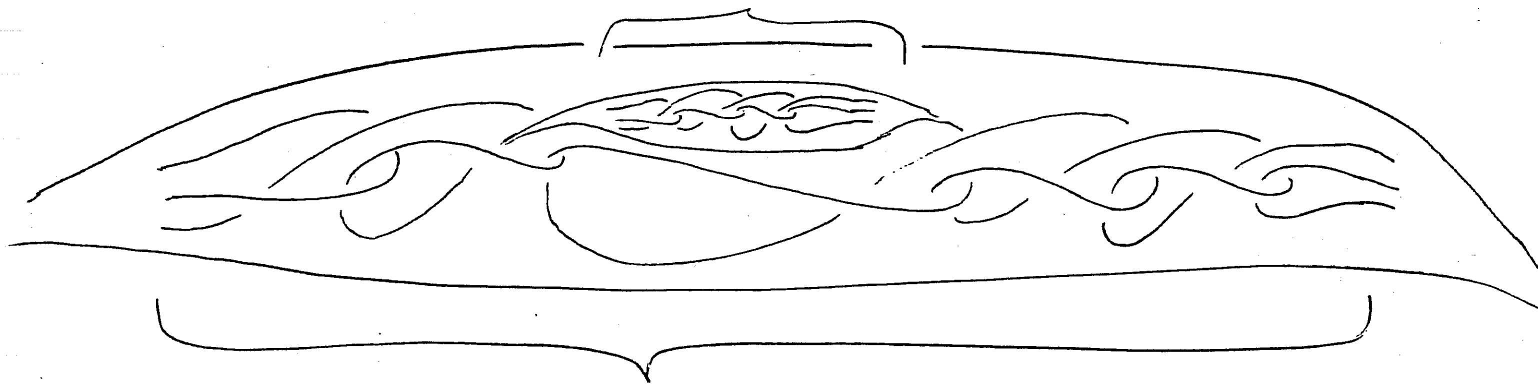


FIGURE 3.1

This 3-manifold has the properties that $\pi_1(M) \cong \mathbb{Z}$ and the universal cover $\widetilde{M} \cong \mathbb{R}^3$, but $M \not\cong S^1 \times \mathbb{R}^2$. Let γ denote the core curve of T_1. An algorithm for drawing the preimage of γ in the universal cover is given in FIGURE 3.2.

3 fundamental domains of $\widetilde{T}_1$



5 fundamental domains of $\widetilde{T}_1$ and $\widetilde{T}_2$

FIGURE 3.2

Let $\widetilde{T}_i$ denote the preimage of T_i in $\widetilde{M}$. One starts by drawing three fundamental domains of $\widetilde{T}_1$. The second step is to draw five fundamental domains of $\widetilde{T}_2$ and extend the existing picture for $\widetilde{T}_1$ by one (rather stretched) domain to the left and to the right inside $\widetilde{T}_2$. The next step is seven fundamental domains of $\widetilde{T}_3$ and an extension of $\widetilde{T}_2$ and then $\widetilde{T}_1$ to seven domains, each inside $\widetilde{T}_3 \ldots$. Note that each step of the algorithm does not change the result of the previous step, hence providing an exhausting picture for the universal cover $\widetilde{M}$. It follows that $\pi_1(\mathbb{R}^3 \setminus \widetilde{\gamma})$ is infinitely generated, and the conjecture is true in this example.

In some cases the fact that a collection of lines in $\mathbb{R}^3$ is not standard may be detected, as in Lecture 2, by finding an incompressible surface in the complement, showing that the fundamental group of the complement is not free. However, there is again a possibility of a locally free but not free group.

Example 3.4. Let M be the infinite-nested union of genus two handlebodies, $M = \bigcup_{i=1}^{\infty}(H_2)_i$, where the inclusion $(H_2)_i \hookrightarrow (H_2)_{i+1}$ is shown in FIGURE 3.3.

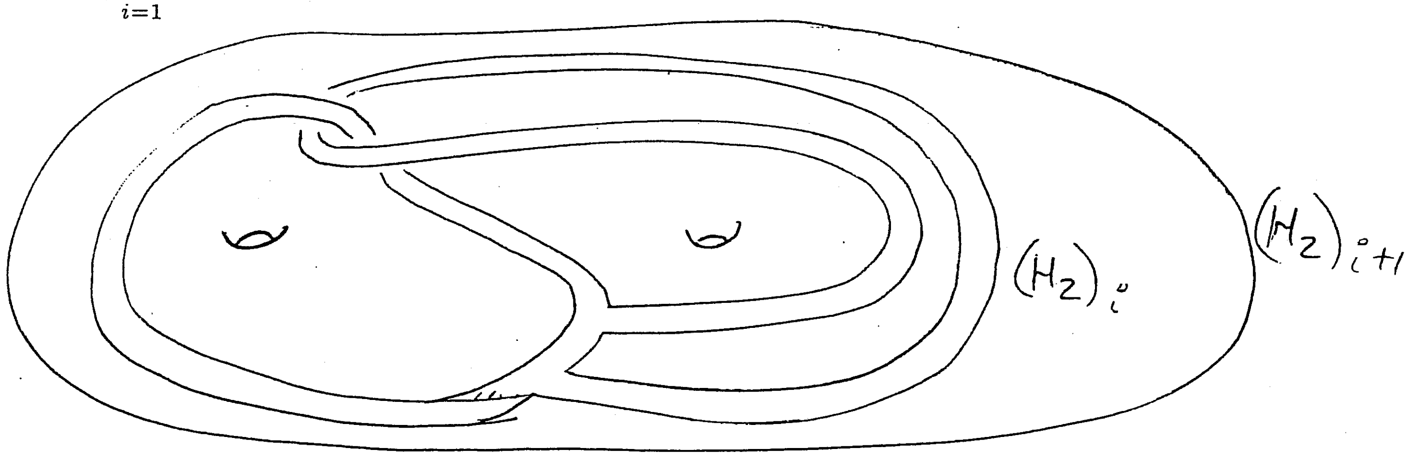


FIGURE 3.3

The map of fundamental groups $\mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z} * \mathbb{Z}$ induced by inclusion sends the first factor isomorphically onto the first factor, and the second factor to the identity element, hence $\pi_1(M) \cong \mathbb{Z}$. Let γ denote a curve representing the sum of the two standard generators of $\pi_1((H_2)_1)$.

By M. Brown's criterion of exhaustion by balls [Br] the universal cover $\widetilde{M}$ is $\mathbb{R}^3$. The complement of $\widetilde{\gamma}$ in $\widetilde{M}$ may be represented as a nested union of handlebodies, hence by Lemma 2.5, $\pi_1(\widetilde{M} \smallsetminus \widetilde{\gamma})$ is locally free. The fact that it is not a free group may be established in this example by a rather special argument: by Alexander duality its Abelianization is isomorphic to the integers.

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Hyperbolic Dehn Surgery and Convergence of Kleinian Groups

Timothy D. Comar

Let N be a compact, orientable three-manifold with boundary containing finitely many tori. One may obtain new three-manifolds from N by gluing solid tori to the boundary tori of N. This process is called Dehn filling. If N admits a hyperbolic structure and the boundary of N consists entirely of tori, then Thurston's Hyperbolic Dehn Surgery Theorem [14, 3] states that most manifolds obtained from N by Dehn filling admit hyperbolic structures. Thurston's theorem also implies that any finite volume hyperbolic three-manifold is the geometric limit of a sequence of closed hyperbolic three-manifolds.

We generalize Thurston's theorem to the case where the boundary of the initial manifold N may contain non-toroidal components in addition to the tori. We assume that M is a geometrically finite hyperbolic three-manifold without rank one cusps that is homeomorphic to the interior of N. If the boundary of N contains non-toroidal components, M will have infinite volume. We show that most Dehn fillings of N admit hyperbolic structures that are geometrically near M. We use this theorem to develop a general recipe for constructing a convergent sequence $\{\rho_n : \Gamma \to \mathbf{PSL(2,C)}\}$ of discrete, faithful geometrically finite representations of a group Γ such that the algebraic and geometric limit groups are geometrically finite and that the algebraic limit group is a proper subgroup of the geometric limit group.

We now recall some basic definitions that are used in the discussion of our results.

Definition 1 *A* ***Kleinian group*** Γ *is a nontrivial, discrete subgroup of* $\mathbf{PSL(2,C)} = Isom^+(\mathbf{H}^3)$, *where* $\mathbf{PSL(2,C)}$ *the group of* 2×2 *matrices with complex coefficients and determinant* 1 *modulo the equivalence relation* $A \sim -A$.

The Riemann sphere $S^2_\infty = \mathbf{C} \cup \{\infty\}$ is the conformal boundary of hyperbolic space. Elements of $\mathbf{PSL(2,C)}$ act conformally on S^2_∞ as Möbius transformations and on $\mathbf{H}^3$ by isometries. The action of a Kleinian group Γ on S^2_∞ partitions S^2_∞ into two sets, $\Omega(\Gamma)$ and $\Lambda(\Gamma)$. The set $\Omega(\Gamma)$, which is called the *domain of discontinuity*, is the open region of S^2_∞ on which Γ acts properly discontinuously, and $\Lambda(\Gamma)$, called the *limit set*, is the complement of $\Omega(\Gamma)$ in S^2_∞.

Our approach to studying Kleinian groups is primarily through three-dimensional hyperbolic geometry. We say that a compact, orientable three-manifold N is *hyperbolizable*, if there is a faithful representation

$$\rho : \pi_1(N) \to \mathbf{PSL(2, C)}$$

onto a torsion-free Kleinian group $\rho(\pi_1(N)) = \Gamma$ such that $int(N)$ is homeomorphic to $\mathbf{H}^3/\Gamma$. Let $M = \mathbf{H}^3/\Gamma$. We introduce a submanifold $C(M)$ of M called the *convex core.* The convex core $C(M)$ care the important topological and geometric properties of M. For example, M is homeomorphic to the interior of a regular neighborhood of $C(M)$. Moreover, $C(M)$ is smallest convex submanifold of M, which is homotopy equivalent to M. We define the convex core as follows. Let $\mathcal{C}(\Lambda(\Gamma))$ be the convex hull of $\Lambda(\Gamma)$ in $\mathbf{H}^3 \cup S^2_\infty$. Then $C(M)$ is defined by $C(M) = (\mathcal{C}(\Lambda(\Gamma)) \cap \mathbf{H}^3)/\Gamma$. The Kleinian group Γ is to be *geometrically finite* if the regular neighborhood of $C(M)$ of radius of 1 in M has finite volume.

We now describe the following topologies on spaces of Kleinian groups. Let G be a finitely generated group. The *algebraic topology* on the representation space $\mathrm{Hom}(G, \mathbf{PSL(2, C)})$ is the compact-open topology on this space. We say that a sequence $\{\Gamma_n\}$ of Kleinian groups converges *geometrically* to a Kleinian group Γ if the following conditions hold:

1. If $\gamma \in \Gamma$, then there is a sequence $\gamma_n \in \Gamma_n$ converging to γ.

2. If $\{\Gamma_{n_k}\}$ is a subsequence of $\{\Gamma_n\}$ and $\gamma_{n_k} \in \Gamma_{n_k}$ converges to γ, then $\gamma \in \Gamma$.

If $\{\Gamma_n\}$ is a sequence of torsion-free Kleinian groups converging geometrically to a torsion-free Kleinian group Γ, the sequence $\{\mathbf{H}^3/\Gamma_n\}$ of hyperbolic three-manifolds converges geometrically to $\mathbf{H}^3/\Gamma$ in the sense that, as $n \to \infty$, larger and larger (compact) portions of $\mathbf{H}^3/\Gamma_n$ are closer and closer to being isometric to portions of $\mathbf{H}^3/\Gamma$ (see [2]). A sequence $\{\rho_n : G \to \mathbf{PSL(2, C)}\}$ of representations converges *strongly* to $\rho : G \to \mathbf{PSL(2, C)}$ if it converges algebraically to ρ and the sequence of image groups $\{\rho_n(G)\}$ converges geometrically to $\rho(G)$.

Dehn filling is the process of gluing solid tori to toroidal boundary components of a three-manifold. Let N be a compact, oriented, differentiable three-manifold whose boundary ∂N contains k pairwise disjoint tori T_i, for $1 \leq i \leq k$. On each torus T_i, choose a meridian-longitude basis (m_i, l_i) for $\pi_1(T_i)$. Assume that m_i and l_i are represented by simple closed curves, also denoted by m_i and l_i, which meet transversely in a single point such that $T_i - (m_i \cup l_i)$ is homeomorphic to an open disc. Let $V = S^1 \times D^2$ be a standard solid torus with meridian $m = \{z\} \times \partial D^2$. Define

$$\mathcal{P}_* = \{(p, q) \in \mathbf{Z} \times \mathbf{Z} \mid \gcd(p, q) = 1\} \cup \{\infty\},$$

where $\gcd(p, q)$ denotes the greatest common divisor of the integers p and q. The three-manifold $N(d_1, d_2, ..., d_k)$ is obtained from N by Dehn filling with coefficients $d_1, d_2, \ldots, d_k \in \mathcal{P}_*$ if whenever $d_i = (p_i, q_i)$, then a copy of V is glued to along T_i via an orientation reversing homeomorphism $g_i : \partial V \to T_i$ such that $g_{i*}(m) = p_i m_i + q_i l_i$ and whenever $d_i = \infty$, then the torus T_i is removed.

Our main result is the following:

Theorem 1 *Let N be a compact, orientable three-manifold with incompressible tori T_i, $1 \leq i \leq k$, contained in ∂N. Let $\rho : \pi_1(N) \to \Gamma \subset$ $\mathbf{PSL(2, C)}$ be an isomorphism onto a geometrically finite Kleinian group Γ such that int(N) is homeomorphic to $M = \mathbf{H}^3/\Gamma$. Furthermore, assume that each parabolic element of $\rho(\pi_1(N))$ is conjugate into one of the rank two parabolic subgroups $\rho(\pi_1(T_i))$, $1 \leq i \leq k$. Then the following statements hold.*

1. *There is a neighborhood U of $(\infty, \infty, \ldots, \infty)$ in $(\mathbf{R}^2 \cup \{\infty\})^k$ such that if $\mathbf{d}$ is contained in $U \cap \mathcal{P}_*^k$, there is a representation $\phi(\mathbf{d}) : \pi_1(N) \to$ $\mathbf{PSL(2, C)}$, whose image is a geometrically finite Kleinian group and whose maximal parabolic subgroups are conjugate to one of the rank two subgroups $\phi(\mathbf{d})(\pi_1(T_i))$, where $d_i = \infty$. Moreover, there is an orientation preserving homeomorphism $h_{\phi(\mathbf{d})}$: int$(N(\mathbf{d})) \to M(\mathbf{d}) =$ $\mathbf{H}^3/\phi(\pi_1(N))$ such that $(h_{\phi(\mathbf{d})} \circ j_{\mathbf{d}})_* = \phi(\mathbf{d})$, where $j_{\mathbf{d}}$: int$(N) \to$ $N(\mathbf{d})$ is the natural inclusion map.*

2. *If $\{\mathbf{d_n}\}$ is a sequence of elements in $(\mathcal{P}_*)^k$ converging to $(\infty, \infty, \ldots, \infty)$, then there is a sequence $\{\phi(\mathbf{d_n}) : \pi_1(N) \to \mathbf{PSL(2, C)}\}$ converging strongly to ρ.*

This result can be used to construct examples of sequences of representations of geometrically finite groups with specified properties. A special case of this construction is stated below as Theorem 2. Theorem 1 has also been used by Anderson and Canary [1] to construct examples of sequences of Kleinian groups in which the quotient manifold of the algebraic limit group is not homeomorphic to any manifold in the corresponding sequence of quotient manifolds. We recall that a homotopically non-trivial, simple closed curve in a 3-manifold is said to be *primitive* if it is not homotopic to a multiple of any other simple closed curve.

Theorem 2 *Let N be a compact, orientable, hyperbolizable three-manifold with nonempty boundary ∂N such that ∂N contains no toroidal boundary components. Let δ be a primitive, simple, boundary parallel curve contained in $int(N)$. Then there exists a sequence*

$$\{\rho_n : \pi_1(N) \to \mathbf{PSL(2, C)}\}$$

of discrete faithful representations such that

1. *the complete hyperbolic manifold $\mathbf{H}^3/\rho_n(\pi_1(N))$ is homeomorphic to $int(N)$, for all n,*

2. *the sequence $\{\rho_n(\pi_1(N))\}$ converges geometrically to a Kleinian group Γ such that $\mathbf{H}^3/\Gamma$ is homeomorphic to $int(N - \delta)$, and*

3. *the sequence $\{\rho_n\}$ converges algebraically to a representation ρ such that*
   $\mathbf{H}^3/\rho(\pi_1(N))$ is homeomorphic to $int(N)$.

This theorem enables us to construct sequences where the algebraic limit groups are properly contained in the geometric limit groups. Such examples have been studied in particular cases by Jørgensen [7], Thurston [14], Jørgensen and Marden [8], Kerckhoff and Thurston [9], Ohshika [13], Marden [11], and Hejhal [6]. Bonahon and Otal [4] have explicitly constructed examples of geometrically infinite hyperbolic three-manifolds with arbitrarily short geodesics. Implicit in their arguments is a similar construction of examples of sequences of representations of Kleinian groups with differing algebraic and geometric limits. We also obtain a generalization of Theorem 2 in which δ is allowed to be the disjoint union of finitely many curves.

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Finding the Unknot in Polynomial Time

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This project is joint research - in progress - with Keith Wolcott in pursuit of a fast and implementable algorithm to recognize the unknot from a diagram. All knots are in S^3, although extension of our techniques to other manifolds may be possible. Extension to links may also be possible.

Let's begin with a bit of history. An (theoretical) algorithm for distinguishing knots by means of their normal spanning surfaces was developed by Haken. Unfortunately, it is not implementable. The first step in the algorithm is to triangulate the knot complement. Furthermore, the process of then determining all surfaces in normal position requires exponential time. It is also worth recalling that attempts to solve this problem by purely two-dimensional means - e.g. Reidemeister moves - seem doomed to failure, as it seems that intermediate stages must always lead to increasingly complex diagrams in some cases, and no means for systematically changing one diagram into another has been found.

We propose an algorithmic method with the following desirable properties:

- ◇ Only diagram is needed as input - no triangulation
- ◇ Procedure uses fast, easily implemented graph-theoretic methods
- ◇ Algorithm requires only low-degree polynomial time

At present, we are considering the several versions of such an algorithm which we believe, based on the evidence of examples, will give the correct answer in most cases when implemented. On the theoretical side, we hope to refine the procedure to the point where we can prove it always works.

Let K be a knot. Our algorithm is based on simplifying the spanning surface, S_1, given by Seifert's algorithm. If S_1 is incompressible, the diagram represents a genuine knot. If it is compressible, we compress it and continue the process with the new surface, S_2. Since the genus of the spanning surface decreases after each compression, this process must terminate. The key, of course, is to efficiently find compressions of a spanning surface.

We will denote the complement of an open regular neighborhood of S_i by M_i. We initially take advantage of the fact that Seifert's surface is unknotted: M_1 is a handlebody. (Its boundary is divided by the knot into two surfaces, both homeomorphic to S_1, which we will refer to as opposite faces.) Finding a properly embedded disk in M_1 which misses the knot is easy. We first cut along a complete disk system to form a ball. The knot will then be represented by the edges of a graph whose vertices are copies of the disks along which we have cut. A compressing disk, if it exists, can be efficiently found by searching for cut vertices of this graph and using them to choose a new complete disk system which reduces the valence of the graph. Eventually, a compressing disk will be revealed as a pair of free vertices.

M_2 is constructed by first compressing M_1 and then attaching a two handle along the curve on the opposite face which corresponds to the boundary of the compressing disk. The resulting manifold is a standard handlebody with attached 2-handle. (The fact that the 1-skeleton is standard follows from the fact that a regular neighborhood of S_2 is a handlebody.) In general, $M_i, i \geq 2$, is a standard handlebody with attached 2-handles. At each step, if $\partial M_i \setminus K$ is compressible in M_i, the compressing disk may be isotoped into the 1-skeleton after suitable handle slides, and M_{i+1} constructed as above. The crucial problem, on which we are currently engaged, is to algorithmically identify a compressing disk in a handlebody with attached 2-handles and perform the necessary handle slides.

Simplicial 2-complexes in R^4

Abstract

Vyacheslav Krushkal

In 1933 van Kampen introduced an obstruction $o(K) \in H^{2n}_{Z/2}(K^*; Z)$ to piecewise-linear embeddability of an n-dimensional simplicial complex K into R^{2n}. The cohomology in question is $Z/2$-equivariant cohomology where $Z/2$ acts on the deleted product $K^* = K \times K - \Delta$ of a complex K by exchanging the factors of K^* and acts on the coefficients by multiplication with $(-1)^n$. Roughly the obstruction is defined by taking any P.L. immersion of K into R^{2n} and counting the algebraic intersection numbers of the images of disjoint top-dimensional simplices of K.

For $n \neq 2$ the vanishing of van Kampen's obstruction provides a necessary and sufficient condition for the P.L. embeddability of an n-complex K^n into R^{2n}. This result for $n = 1$ follows from the Kuratowski subgraph condition, and for $n \geq 3$ was proved by Shapiro and Wu using Whitney trick (see also [2] for a modern exposition of this theorem.) In a joint paper with M. Freedman and P. Teichner [2] a family of 2-dimensional simplicial complexes was constructed for which the obstruction is trivial but which do not admit an embedding into R^4.

This result raised the question of finding higher obstructions to P.L. embeddability of 2-complexes in R^4 which would be defined when van Kampen's obstruction vanishes. In fact, in the simplest relative case - for $(D^2 \sqcup D^2, S^1 \sqcup S^1)$ - van Kampen's obstruction coincides with the linking number which generalizes to Milnor's $\bar{\mu}$-invariants for links, providing evidence for the existence of higher van Kampen's obstructions.

In order to describe these higher obstructions, I will first give a reformulation of van Kampen's obstruction in terms of configuration spaces. A necessary condition for the existence of an embedding $K^n \hookrightarrow R^{2n}$ is the existence of a Z_2-equivariant map $K \times K - \Delta \longrightarrow R^{2n} \times R^{2n} - \Delta$. It follows easily from definitions that the obstruction to existence of this map coincides with van Kampen's obstruction $o(K) \in H^{2n}_{Z/2}(K \times K - \Delta; \pi_{2n-1}(S^{2n-1}))$.

If $o(K)$ vanishes one may ask for $n = 2$ whether there is an S_3-equivariant map $K \times K \times K - \Delta_3 \longrightarrow R^4 \times R^4 \times R^4 - \Delta_3$ where Δ_3 denotes the "big" diagonal of the triple product. The vanishing assumption on $o(K)$ implies that (rationally) the only obstruction $o_3(K)$ to the existence of this map lies in $H^6_{S_3}(K \times K \times K - \Delta_3; \pi_5(R^4 \times R^4 \times R^4 - \Delta_3))$. Inductively one defines the obstruction $o_m(K)$ of order m to the existence of an S_m-equivariant map of symmetric m-products, provided the obstructions of order less than m vanish. It follows from obstruction theory that $o_m(K)$ is not in general well defined and depends on the map of the 3-skeleton of $K \times \ldots \times K - \Delta_m$, however for the examples [2] mentioned above the set $o_m(K)$ does not contain zero and hence detects their non-embeddability into R^4. It follows from the description of the homotopy type of configuration spaces [1] that the obstructions $o_m(K)$ have an additional structure: they may be computed as Massey products of certain "basic" 3-dimensional cohomology classes of $K \times \ldots \times K - \Delta_m$.

An alternative approach to defining higher obstructions for 2-complexes may be taken by considering their 4-dimensional thickenings.

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Proper Knots in Thickened Surfaces

Abstract

A proper knot is a proper embedding of the real line into an open 3-manifold. Two such knots are equivalent if they are connected by a proper isotopy. In [CS2], Churchard and Spring raised the question of the existance of inequivalent proper knots which run between the opposite ends of a manifold of the type $F \times R^1$ where F is a closed surface which is not S^2. In this note we sketch a proof that all such proper knots are equivalent via a proper topological isotopy. The details appear in [N4] which has been submitted for publication in a journal. The classification of such kinds of proper knots via a smooth or piecewise linear isotopies remains an open question.

Ollie Nanyes

1 Introduction

1.1 Background

Recall that a map $f : X \to Y$ is *proper* if for all compact $C \subset Y$, $f^{-1}(C)$ is compact in X. Proper knot theory, introduced by Churchard and Spring in [CS1], deals with the proper embeddings of the real line into non-compact three manifolds. Two proper knots are equivalent if they are connected via a proper isotopy. Because the knots are non-compact, the isotopy need not be ambient, even when we are working in the smooth category. For example, it was shown in [CS1] that all smooth proper knots in R^3 are equivalent in the smooth proper isotopy category.

In [CS2], classification theorems for proper knots in the smooth category were investigated. It was shown that all smooth proper knots which run from a S^2 end to another end which has a suitable product structure (say, a closed surface end or a "ladder end") can be smoothly properly isotoped to a proper knot which meets at least one collar $S^2 \times *$ "near" the S^2 end. Hence a classification theorem for smooth (and therefore p. l.) proper knots in $S^2 \times R^1$ was obtained. Also, it was noted that it was unknown if there were any proper knots running between the opposite ends of $F^2 \times R$ which were not (smoothly) equivalent to a "product line" $* \times R^1$. (where F^2 is a closed surface that is not S^2). It follows from Proposition 2.2 of [CS2] that if such a non trivial proper knot existed it would have to be a proper knot that met each level surface $F^2 \times t$ transversely in at least three points. It was pointed out by the author in [N1] and [N2] that the existence of non trivial proper knots connecting the opposite ends of the infinite cylinder $D^2 \times R^1$ was unknown.

In this paper it is shown that, up to topological proper isotopy (and of course, orientation of the proper knots), all topological proper knots running between the opposite ends of $D^2 \times R^1$ are equivalent and that the $D^2 \times R^1$ result implies that all "opposite end connecting" piecewise linear proper knots in $F^2 \times R^1$ are equivalent via a topological proper isotopy (where F^2 is now a compact orientable p. l. surface, with or without boundary).

1.2 Sketch of Proof of the Main Theorem

Theorem 1.1 *If K is the image of a piecewise linear (p. l.)proper knot that runs between the opposite ends of $F^2 \times R^1$ then there is a topological proper isotopy which connects K to a proper knot whose image is of the form $* \times (-\infty, \infty)$. In other words, up to orientation, all p. l. proper knots running between the ends of $F^2 \times R^1$ are topologically properly equivalent.*

Sketch of the proof. The idea is as follows: we start by showing that all proper knots that run between the opposite ends of $D^2 \times R^1$ are equivalent via a topological proper isotopy. Figures 1 and 2 show how the proof goes. Note that there is no requirement for the initial proper embedding to meet any tameness condition and that the isotopy is not necessairly smooth or p. l., even when the proper knot is smooth or p. l.

Next, we show that all piecewise linear proper knots in $F^2 \times R^1$ which connect the opposite ends are topological equivalent to "fiber" proper knot $* \times R^1$. Let K be the image of an arbitrary piecewise linear proper knot f that runs between the opposite ends. We start by properly isotoping f to a proper knot f_1 whose image K' is in a kind of bridge position (see

[S] and figure 3): K' consists of a locally finite collection of vertical segments of the form $* \times [a, b]$ and of p. l. arcs which lie on level surfaces $F^2 \times s$ $(a, b, s \in R^1, * \in F^2)$.

We then construct, on a piece by piece basis, a properly embedded $D^2 \times R^1$ which contains K' and which intersects at least one level F^2 transversely in a meridional disk. f_1 is now isotoped within this $D^2 \times R^1$ to a new proper knot f_2 whose image K'' can be thought of as being the center line $0 \times (-\infty, \infty) \subset D^2 \times R^1$. But K'' hits a level surface $F^2 \times t$ transversely in one point. It then follows from Proposition 2.2 of [CS2] that f_2 is equivalent to a fiber proper knot whose image is $* \times R^1$ (one merely uses the product structure of the two ends to comb f_2 along the collar lines of the respective ends). See [N4] for details.

1.3 Questions

The question remains as to whether every proper knot that runs between the ends of M is p. l. equivalent to a trivial proper knot. This work does show however that any invariant which detects inequivalence of p. l. proper knots must take the p. l. structure of the isotopy into account.

1.4 Acknowledgements

In addition to thanking Ric Ancel and Bob Daverman, who told me to try the technique used in the $D^2 \times R^1$ case, I would also like to thank Bobby Winters, Mat Timm and Cameron Gordon for the time they spent discussing this problem with me.

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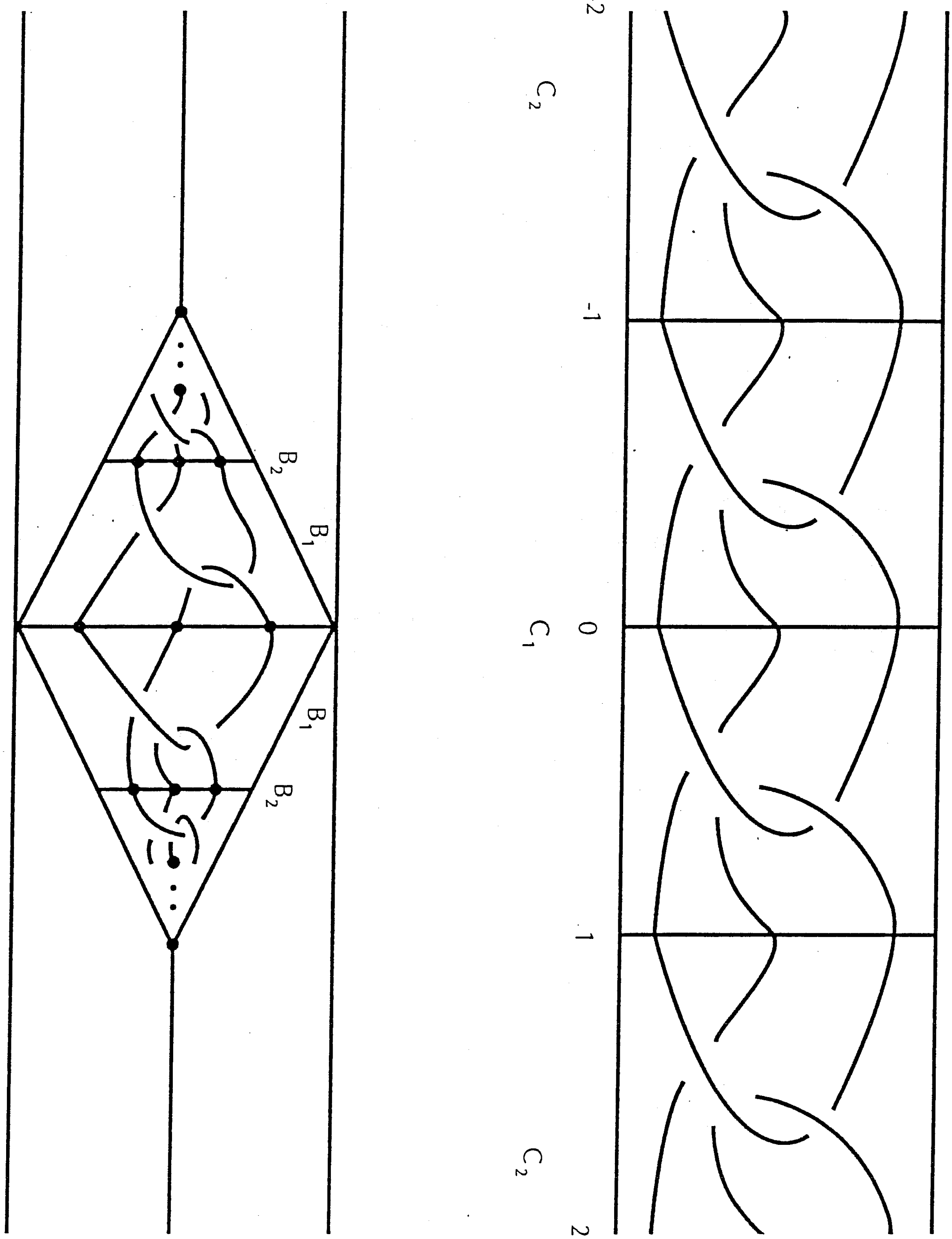
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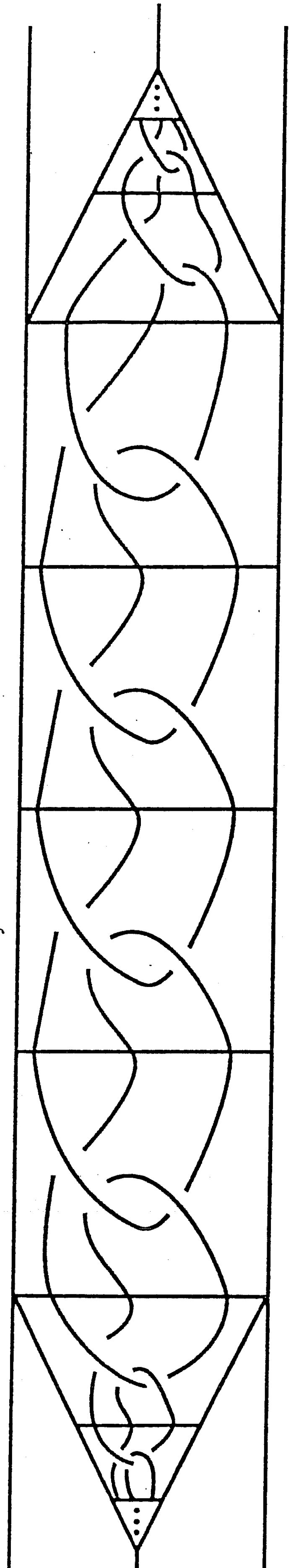
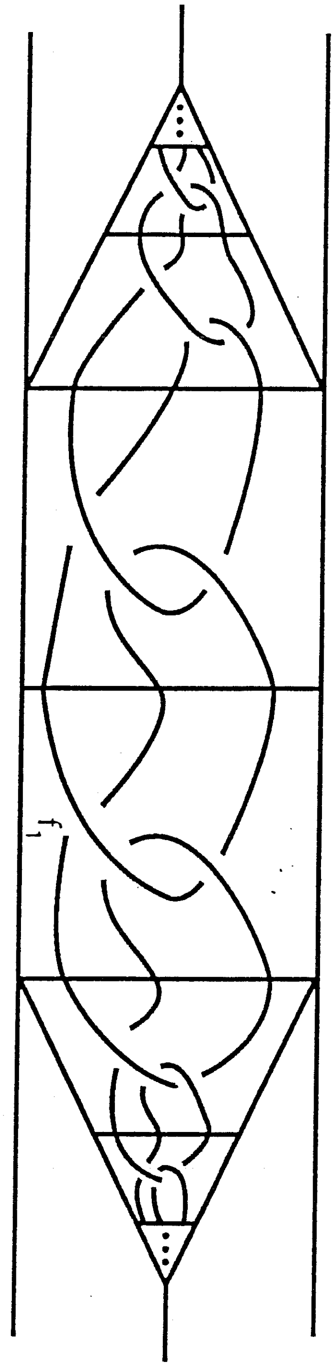


f_2



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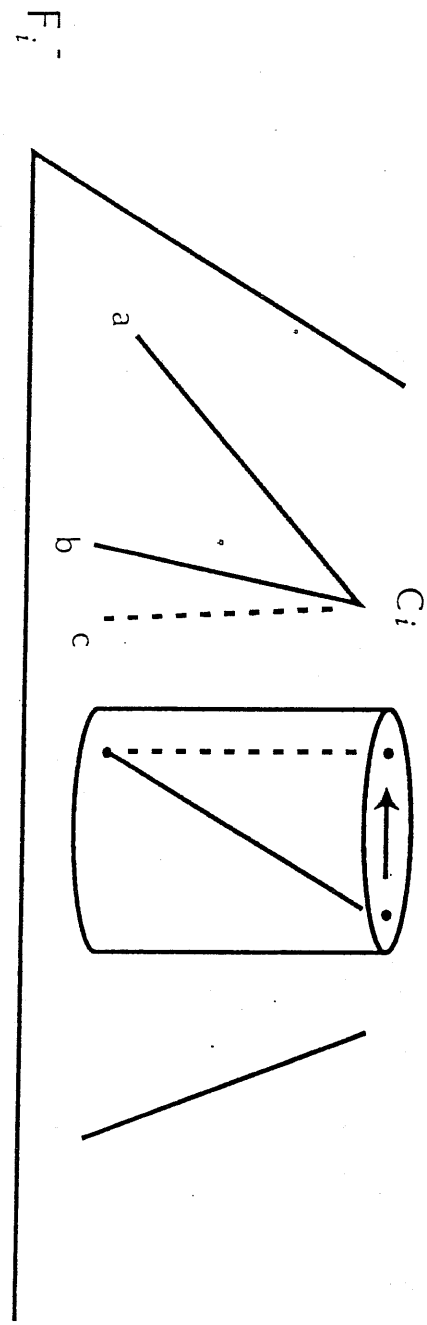
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On extensions of co-dimension one immersions

Christian Pappas

Abstract

Let $f : M \longrightarrow W$ be a co-dimension one immersion of orientable manifolds, where M is closed. An extension of f is an immersion $F : N \longrightarrow W$, where N is a manifold whose boundary ∂N is identified with M, such that the restriction $F|\partial N = f : M \longrightarrow W$. We construct all of the extensions of f, giving necessary and sufficient conditions for existence, as well as an understanding of the non-uniqueness; and describe the CW structure and diffeomorphism type of each.

Let M be a smooth, orientable connected closed m-manifold; W a smooth, orientable, connected $(m+1)$-manifold; and $f : M \longrightarrow W$ a smooth immersion with transverse intersections. An extension of f consists of a compact orientable $(m+1)$-manifold N (the extending manifold), whose boundary is identified with M, together with an immersion $F : N \longrightarrow W$ such that $F|_{\partial N} = f : M \longrightarrow W$. As $f(M)$ is two-sided, an extension lies locally to one side, called its direction. We will fix a normal vectorfield ν, and consider extensions whose direction agrees with that of a given normal vectorfield ν on $f(M)$. Two extensions are equivalent if they differ by precomposition with a diffeomorphism of the extending manifolds. It is a general fact that if there is an immersion of a manifold M into W that extends, then there exists immersions of M which have exactly k extensions for any natural number k. We describe below all of the inequivalent extensions of an immersion f (Theorem 2), as well as the CW structure and diffeomorphism type of each (Theorem 1). Ones of the same grading (see below) have the same Euler characteristic. In particular, extensions (with the same grading) of a circle in an orientable surface are diffeomorphic.

A necessary condition for f to extend is that $[f(M)] = 0 \in H_m(W, \mathbf{Z})$, the mth homology group of W; this will be assumed throughout. Let A_i denote the bounded components of $W``f(M)$, which we call rooms. The multiplicity of A_i, m_i, is the number of connect components of $F^{-1}(A_i)$. The grading of F is the minimum of the multiplicities m_i over all rooms A_i. This can be regarded as the number of (cut up) copies of W contained in N. If W is non-compact all extensions have grading 0.

The topology of an extending manifold, N, can be computed from $f(M)$ and W as follows. Let h be a Morse function on W. The critical points of h will be called ambient points. Now h can be locally perturbed so that it restricts to a Morse function, h', on $f(M)$ (i.e. hf is Morse on M). But note that its restriction, h'', to $F(N)$ is not a Morse function. Regardless, one can still obtain an analogous theory. A critical point, c, of $h' = h|f(M)$ will be called attaching (non-attaching) if the normal vector at c points in the (opposite) direction of the gradient flow.

Theorem 1 *Let M be a connected closed orientable m-manifold, W a connected orientable (m+1)-manifold. Let $f : M \longrightarrow W$ be an immersion. Fix a normal vectorfield ν on $f(M)$. Then all extensions of f in the direction ν have a CW structure whose cells correspond to the ambient points (with multiplicities, which only depends on the grading) and the attaching points. The dimension of the cell equaling the index of the critical point.*

If M is disconnected then the same statement holds as long as the extending manifold, N, is connected. The non-attaching points essentially prescribe the gluing maps.

To construct extensions of the immersion f, we sweep across W and build them up. We only need to consider the topological and intersection set changes, refering to the critical points of h, $h' = h|f(M)$ and h restricted to the double points, triple points, ... of the intersection locus of $f(M)$. The following describes exactly which critical points pose a possible obstruction (most do not), and a local condition for crossing them.

Let $x \in f(M)$; let P be a neighborhood of x in $F(N)$. Two points a and b of $f(M)$ contained in P are matched by F if there is an arc γ joining their pre-images in N such that $F(\gamma) \subset P$.

Theorem 2 *Let M be a connected closed orientable m-manifold, W a connected orientable $(m+1)$-manifold. Assume $[f(M)] = 0 \in H_m(W, \mathbf{Z})$. Let $f : M \longrightarrow W$ be an immersion, and ν a normal vectorfield on $f(M)$. As one sweeps across the level sets, one only needs to consider the following.*

The obstructions to constructing an extension are:

1. *0-handles of the intersection locus. One can be crossed (in a unique way) if and only if the sheets of $f(M)$ containing it are not matched.*

2. *Non-attaching points of index 1. One can be crossed (in a unique way) if and only if points of the descending 1-handles are matched.*

The following can always be crossed, but perhaps in more than one way, yielding inequivalent extensions.

1. *Non-attaching points of index 0. The number of ways to cross one equals the multiplicity of the room that lies below it.*

2. *Ambient points of index 1. The number of ways to cross one equals the number of ways to pair off the sheets meeting the descending 1-handle, which is $m(c)!$ (where $m(c)$ is the multiplicity of the critical point). Note that these may lead to redundancies.*

Eliashberg proved the remarkable fact that given two 3-manifolds N_1 and N_2 with the same boundary F, that there exists an immersion $f : F \longrightarrow \mathbf{R}^3$ which extends to both N_1 and N_2, and generalizations of this.

The outstanding questions of extensions involve "relating N_1 and N_2 via $f(S^2)$": When are N_1 and N_2 diffeomorphic? When do they have the same Betti numbers? ... The author gave obstructions to constructing a diffeomorphism between extensions of S^2 in R^3; and believes that anyone wishing to carry extensions further should begin by constructing and studying concrete examples of non-diffeomorphic extensions in order to enhance their intuition.

KNOTS, LINKS AND REPRESENTATION SHIFTS

Daniel S. Silver
Susan G. Williams

This paper is a summary of a 2-part talk presented by the authors at the Thirteenth Annual Western Workshop on Geometric Topology held in June 1996 at The Colorado College, Colorado Springs, CO. Most of the details can be found in [**SiWi 1**], [**SiWi 2**] and [**SiWi 3**].

Introduction. The group $\pi_1(S^3 - k)$ of a knot k contains an extraordinary amount of information. From combined results of W. Whitten [**Wh**] and M. Culler, C. McA. Gordon, J. Luecke and P.B. Shalen [**CuGoLuSh**] it is known that there are at most two distinct unoriented prime knots with isomorphic groups. Unfortunately, knot groups are generally difficult to use. Knot groups are usually described by presentations, and there is no practical algorithm to decide whether or not two knot groups are isomorphic.

In 1928 J.W. Alexander used homomorphisms (representations) of knot groups onto better understood groups in order to obtain topological invariants. Since then knot group representations have been used effectively by many others. The representations of a given knot group into a fixed finite group have the additional attraction that they are finite in number and so can be tabulated. R. Riley began such a program in [**Ri**].

We take a new approach, examining the representations of the commutator subgroup $K = [\pi_1(S^3 - k), \pi_1(S^3 - k)]$ into a fixed finite group Σ. Although $\mathrm{Hom}(K, \Sigma)$ is often infinite – in fact, uncountable – it has a rich structure that we can understand via symbolic dynamics. In this dynamical system the representations of the knot group $\pi_1(S^3 - k)$ appear (by restricting their domains) as special periodic points. However, the system contains other periodic points and often nonperiodic points, information that can be used to understand more about the structure of the knot exterior and its various covering spaces. The techniques, all algorithmic, apply equally well to links.

1. Representation shifts. Although here we emphasize applications to knot theory, the methods we describe apply in a wide variety of situations.

Definition. [**Si1**] An *augmented group system* (AGS) is a triple $\mathcal{G} = (G,\ \chi,\ x)$ consisting of a finitely presented group G, an epimorphism $\chi : G \to \mathbf{Z}$, and a distinguished element $x \in G$ such that $\chi(x) = 1$.

Two augmented group systems $(G_1,\ \chi_1,\ x_1)$ and $(G_2,\ \chi_2,\ x_2)$ are *equivalent* if there exists an isomorphism $f : G_1 \to G_2$ such that $f(x_1) = x_2$ and $\chi_1 = \chi_2 \circ f$. Equivalent augmented group systems are regarded as the same.

1991 *Mathematics Subject Classication.* Primary 57Q45; Secondary 54H20, 20E06, 20F05.

We can associate an AGS $\mathcal{G}_k$ to any oriented knot k as follows. Let $N(k) = k \times D^2$ be a tubular neighborhood of k. The closure of $S^3 - N(k)$ is the *exterior* of k, and we will denote it by $X(k)$. Let G be the fundamental group $\pi_1(X(k), *)$, where the basepoint is contained in the boundary $\partial X(k)$, and let $x \in G$ be the class of a meridian $m \subset \partial X(k)$ with orientation induced by that of k. Let χ be the abelianization homomorphism that maps x to 1. The uniqueness up to isotopy of tubular neighborhoods ensures that $\mathcal{G}_k$ is well defined.

An AGS $\mathcal{G}_l$ can be associated to any oriented link $l = l_1 \cup \ldots \cup l_d$ by the same general procedure as above. We let x be the class of a meridian of the component l_1. Since the abelianzation of $G = \pi_1(S^3 - l, *)$ is a free abelian group of rank d, there are many choices for χ when the link has more than one component. A natural choice for χ is the "total linking number homomorphism" that maps the class of every oriented meridian to 1.

Definition. [**SiWi1**] Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS, and let K be the kernel of χ. Assume that Σ is any finite group. The *representation shift* $\Phi_\Sigma(\mathcal{G})$ is the set of representations $\rho : K \to \Sigma$ together with the mapping $\sigma_x : \Phi_\Sigma(\mathcal{G}) \to \Phi_\Sigma(\mathcal{G})$ defined by

$$(\sigma_x \rho)(a) = \rho(x^{-1}ax) \quad \forall a \in K.$$

The mapping σ_x is a bijection with inverse $\sigma_{x^{-1}}$. In fact, if we define the topology on $\Phi_\Sigma(\mathcal{G})$ with basis

$$\mathcal{N}_{g_1,\ldots,g_n}(\rho) = \{\rho' \mid \rho'(g_i) = \rho(g_i),\ i = 1, \ldots, n\},$$

where $\rho \in \Phi_\Sigma(\mathcal{G})$ and $g_1, \ldots, g_n \in K$, then σ_x becomes a homeomorphism. Consequently, the pair $(\Phi_\Sigma(\mathcal{G}), \sigma_x)$ is a topological dynamical system (topological space + homeomorphism). We recall that two dynamical systems $(\Phi_1,\ \sigma_1)$ and $(\Phi_2,\ \sigma_2)$ are *topologically conjugate* if there exists a homeomorphism $h : \Phi_1 \to \Phi_2$ such that $\sigma_2 \circ h = h \circ \sigma_1$. Topologically conjugate dynamical systems are regarded as the same.

The main result of [**SiWi1**] is that the topological dynamical system $(\Phi_\Sigma(\mathcal{G}), \sigma_x)$ has the structure of a *shift of finite type*, a special sort of dynamical system that can be completely described by a finite graph Γ. The representations ρ correspond to the bi-infinite paths in Γ. Rather than repeat the proof of this result, we illustrate the algorithm for finding Γ.

Example. Consider the AGS associated to the knot 5_2 in figure 1. The Wirtinger algorithm [**BuZi**], [**Ro**] together with some obvious Tietze transformations produces the following presentation for the group G of the knot.

$$G = \langle x, a \mid x^{-1}a^2x \cdot a^{-2} \cdot x^{-1}ax \cdot x^{-2}a^{-2}x^2 \rangle.$$

The Reidemeister-Schreier method [**LySc**] gives us a presentation for K:

$$K = \langle a_i \mid a_{i+1}^2 a_i^{-2} a_{i+1} a_{i+2}^{-2} \rangle$$

Here $a_i = x^{-i} a x^i$ and the index i ranges over the integers. J.C. Hausmann and M. Kervaire have termed such a presentation *finite* **Z**-*dynamic* [**HaKe**]. We will think of the relator $a_{i+1}^2 a_i^{-2} a_{i+1} a_{i+2}^{-2}$ as a word $r(a_i, a_{i+1}, a_{i+2})$. Notice that each relator is gotten from $r(a_0, a_1, a_2)$ by "shifting" the subscripts of the generators involved.

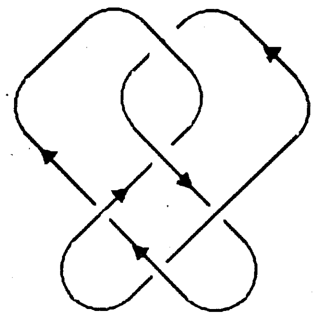

Figure 1

A representation $\rho : K \to \Sigma$ is a function ρ defined on the set $\{a_i\}$ such that $r(\rho(a_i), \rho(a_{i+1}), \rho(a_{i+2}))$ is trivial for all i. When Σ is the cyclic group $\mathbf{Z}/3$, for example, this condition reduces to $\rho(a_{i+2}) \equiv 2\rho(a_i)$ modulo 3. The sequence

$$\ldots\ 1\ 0\ 2\ 0\ 1\ \underline{0}\ 2\ 0\ 1\ 2\ 0\ \ldots$$

describes the representation $\rho : K \to \mathbf{Z}/3$ mapping a_0 to 0, a_1 to 2, etc. The sequence describing $\sigma_x(\rho)$ is simply

$$\ldots\ 1\ 0\ 2\ 0\ 1\ 0\ \underline{2}\ 0\ 1\ 2\ 0\ \ldots$$

Moreover, every representation $\rho : K \to \mathbf{Z}/3$ can be found from the directed graph Γ in figure 2. Any vertex can be regarded as an assignment of values for a_0 and a_1, while the vertex that follows is an assignment for a_1 and a_2, etc. The representation shift $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$ has exactly nine elements. Notice that the trivial representation, corresponding to the constant sequence of zeroes, is isolated in the space $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$.

$$\begin{tikzcd}
(0,0) \arrow[loop right] & (0,1) \arrow[r] & (1,0) \arrow[d] & (2,1) \arrow[r] & (1,1) \arrow[d] \\
 & (2,0) \arrow[u] & (0,2) \arrow[l] & (2,2) \arrow[u] & (1,2) \arrow[l]
\end{tikzcd}$$

Figure 2

Theorem. [**SiWi3**] Let $\mathcal{G}$ be any AGS and Σ a finite abelian group. Then the representation shift $\Phi_\Sigma(\mathcal{G})$ is finite if and only if the trivial representation is isolated. This will be the case whenever G has infinite cyclic abelianization.

The proof of the theorem makes use of a powerful structure theorem for Markov subgroups, shifts of finite type that are also abelian groups, proved by of B.P. Kitchens [**Ki**]. The theorem can also be proved using algebraic topological techniques of [**Mi**].

When $\mathbf{Z}/3$ is replaced by the symmetric group S_4 in the above example, the representation shift becomes uncountably infinite. Figure 3 contains a detail of the graph describing it. In figure 3 we see two circuits with a common vertex, and hence uncountably many bi-infinite paths. It follows that K contains uncountably many subgroups of index less than or equal to 4. Since the representation shift with $\Sigma = S_3$ can be shown to be finite, K contains uncountably many subgroups of index equal to 4. Further analysis using these techniques shows that K contains uncountably many subgroups of any index exceeding 3.

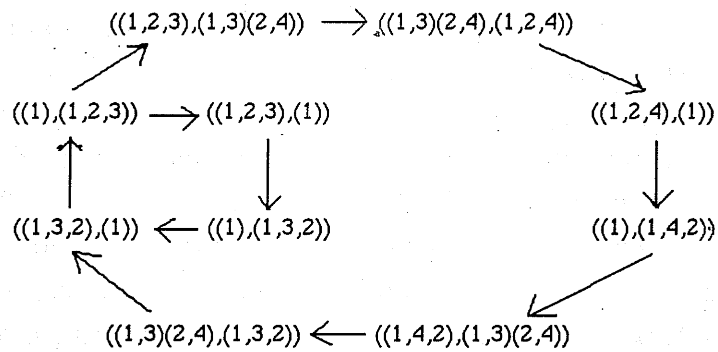

Figure 3

2. Periodic points. Assume that $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS and $\Phi_\Sigma(\mathcal{G})$ is an associated representation shift for some finite group Σ. Although the graph Γ that describes $\Phi_\Sigma(\mathcal{G})$ is not unique, it does contain invariants of $\Phi_\Sigma(\mathcal{G})$, which are necessarily invariants of $\mathcal{G}$.

Definition. A representation $\rho \in \Phi_\Sigma(\mathcal{G})$ is *periodic* if $\sigma_x^r \rho = \rho$ for some positive integer r. In this case, ρ has *period* r.

The representations of period r correspond to the closed paths in Γ having length r.

Example. A knot $k \subset S^3$ is *fibered* if the projection $\partial N(k) = k \times S^1 \to S^1$ extends to a locally trivial fibration $X(k) \to S^1$. By a theorem of J. Stallings (see Theorem 5.1 of [**BuZi**], for example) a knot k is fibered if and only if the commutator subgroup K of its group is finitely generated (and free). In this case, the representation shift $\Phi_\Sigma(\mathcal{G}_k)$ is finite for all finite groups Σ, and so every representation $\rho : K \to \Sigma$ is periodic.

Figure 4 displays the graph describing $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$, the representation shift of the trefoil 3_1. The representation shift $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ of the figure eight knot 4_1 is identical to $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$ (see figure 2). Notice that each representation shift contains nine elements. However, $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$ contains two points of period 2 and six of period 6 while $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ contains eight points of period 4. Since representation shifts with different numbers of periodic points for the same period clearly cannot be topologically conjugate, $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1}) \neq \Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$.

$$\begin{tikzcd} (0,0) \arrow[loop right] & (1,2) \arrow[r, shift left] & (2,1) \arrow[l, shift left] & (0,1) \arrow[r] & (1,1) \arrow[r] & (1,0) \arrow[d] \\ & & & (2,0) \arrow[u] & (2,2) \arrow[l] & (0,2) \arrow[l] \end{tikzcd}$$

Figure 4

The calculations above show that the trefoil and figure eight knots are different. However, they reveal much more. That there are two representations of period 2 in $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$ other than the trivial one is a consequence of the well-known fact that a trefoil knot diagram can be nontrivially tricolored in exactly six ways [**CrFo**], [**Pr**]; the absence of nontrivial period 2 representations in $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ indicates that the figure eight knot diagram can be tricolored only monochromatically. In general, the number of period 2 points of $\Phi_{\mathbf{Z}/n}(\mathcal{G}_l)$ for any oriented link l tells us how a diagram for l can be n-colored. The periodic points of period greater than 2 correspond to "generalized n-colorings." Details appear in [**SiWi2**].

Another invariant of $\Phi_\Sigma(\mathcal{G})$ that can be calculated from the graph Γ is topological entropy. The *topological entropy* of $\Phi_\Sigma(\mathcal{G})$ is equal to the logarithm of the spectral radius of the adjacency matrix of Γ. We will denote the topological entropy by $h_\Sigma(\mathcal{G})$. It depends only on the AGS $\mathcal{G}$ and target group Σ. In a sense, $h_\Sigma(\mathcal{G})$ is a measure of the average amount of information gained by taking one step in Γ. In the case of the AGS $\mathcal{G}_l$ associated to an oriented link l, the topological entropy is a numerical invariant of l.

In [**Lo**], [**Si2**] entropy invariants for knots were defined using Nielson-Thurston theory for surface homeomorphisms. The entropy invariants above are quite different. The invariants in [**Si2**] are nonzero for most fibered knots. However, if k is any fibered knot, then $\Phi_\Sigma(\mathcal{G}_k)$ is finite and so $h_\Sigma(\mathcal{G}_k) = 0$ for every finite group Σ. These new invariants seem to detect nonfibered knots. In fact, we make the following conjecture.

Conjecture. If k is a nonfibered knot, then $h_\Sigma(\mathcal{G}_k) > 0$ for some finite group Σ.

If the conjecture is true, then it would follow that the commutator subgroup K of the group of any nonfibered knot has uncountably many subgroups of index r whenever r is sufficiently large, a conclusion that we saw in the case of the knot 5_2.

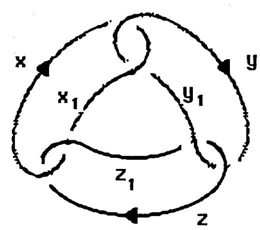

Figure 5

Example. Consider the 3-component link $l = 6^3_1$ as it appears in figure 5 with Wirtinger generators indicated. The group G of the link has presentation

$$\langle x,\ x_1,\ y,\ y_1,\ z,\ z_1 \mid y_1 z = zy,\ z_1 x = xz,\ x_1 y = yx,\ x_1 y_1 = y x_1,\ z y_1 = y_1 z_1\rangle.$$

Using the first three relators we eliminate the generators x_1, y_1 and z_1, obtaining

$$\langle x,\ y,\ z \mid xy^{-1}zyz^{-1}yx^{-1}y^{-1},\ zyz^{-1}xz^{-1}x^{-1}zy^{-1}\rangle.$$

The elements $x^{-1}y$ and $x^{-1}z$ vanish under the augmentation homomorphism χ. We use Tietze transformations to replace y and z by xa and xb (introducing new generators a and b). We can then rewrite the presentation for G as

$$\langle x,\ a,\ b,\ \mid a^{-2}b \cdot xab^{-1}ax^{-1},\ ab^{-2} \cdot x^{-1}ba^{-1}bx\rangle.$$

Let a_i and b_i denote $x^{-i}ax^i$ and $x^{-i}bx^i$, respectively. By the Reidemeister-Schreier method the following is a presentation for the kernel K of χ:

$$K = \langle a_i,\ b_i \mid a_{i+1}^{-2}b_{i+1}a_ib_i^{-1}a_i,\ a_ib_i^{-2}b_{i+1}a_{i+1}^{-1}b_{i+1}\rangle.$$

First we will determine $\Phi_{\mathbf{Z}/2}(\mathcal{G}_l)$. Any representation $\rho : K \to \mathbf{Z}/2$ factors through $\mathrm{Hom}(K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/2,\ \mathbf{Z}/2)$, where K^{ab} denotes the abelianization of K. We can get a presentation for $K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/2$ from that of K by allowing all of the generators to commute and by reducing exponents of relators modulo 2. The resulting presentation, expressed in additive notiation, is

$$\langle a_i,\ b_i \mid b_{i+1} = b_i,\ a_{i+1} = a_i\rangle.$$

Clearly, any homomorphism $\rho : K \to \mathbf{Z}/2$ is completely determined by the values $\rho(a_0)$ and $\rho(b_0)$. The representation shift $\Phi_{\mathbf{Z}/2}(\mathcal{G}_l)$ has exactly 4 elements, each a fixed point under σ_x. The topological entropy $h_{\mathbf{Z}/2}(\mathcal{G}_l)$ is zero.

Replacing $\mathbf{Z}/2$ by $\mathbf{Z}/3$ results in a very different representation shift. Consider the presentation for $K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/3$:

$$\langle a_i,\ b_i \mid a_i - a_{i+1} + b_i - b_{i+1}\rangle$$

The second relator does not appear because it is redundant. Selecting any fixed value for b_0, we can define a representation $\rho : K \to \mathbf{Z}/3$ by assigning arbitrary values for the a_i's; the corresponding values of b_i, $i \neq 0$, are uniquely determined. It follows that $\Phi_{\mathbf{Z}/3}(\mathcal{G}_l)$ is a Cartesian product of three full shifts on the symbols of $\mathbf{Z}/3$. (A *full shift* on a finite alphabet $\mathcal{A}$ consists of all sequences (α_i), $\alpha_i \in \mathcal{A}$, together with the shift mapping $\sigma : (\alpha_i) \mapsto (\alpha_i')$, where $\alpha_i' = \alpha_{i+1}$.) In this case the topological entropy $h_{\mathbf{Z}/3}(\mathcal{G}_l)$ is $\log 3$.

Let $\mathcal{G} = (G, \chi, x)$ be an AGS and let Σ be any finite group. By a theorem of [**SiWi1**] the topological entropy of a representation shift $\Phi_\Sigma(\mathcal{G})$ does not depend on the choice of distinguished generator x. Hence in the above example, the class x of the meridian of the first component of the link can be replaced by any element that maps to 1 under χ, and the resulting topological entropy will be unchanged. On the other hand, since G^{ab} is free abelian of rank 3 there are infinitely many choices for χ other than the total linking number homomorphism that we employed. Altering χ can change the entropy.

3. Other applications and directions for knots and links. In [**SiWi3**] we proved that for any knot k and finite group Σ, the points of period r in $\Phi_\Sigma(\mathcal{G}_k)$ are in one-to-one correspondence with representations of $\pi_1(\hat{X}_r)$, where $\hat{X}_r$ denotes the r-fold branched cyclic cover of k. Consequently, information, such as the following about the branched cyclic covers of k, is encoded in the symbolic dynamics of the representation shifts.

Theorem. [**SiWi3**] For any finite group Σ, the order of $\mathrm{Hom}(\pi_1\hat{X}_r, \Sigma)$ satisfies a linear recurrence relation.

Corollary. [**SiWi3**] (also proved by W.H. Stevens [**St**]) For any oriented knot k and finite abelian group Σ, there is a positive integer N such that

$$H_1(\hat{X}_{r+N}; \Sigma) \cong H_1(\hat{X}_r; \Sigma)$$

as $\Lambda/(x^r - 1)$-modules, where $\Lambda = \mathbf{Z}[x, x^{-1}]$).

The representation shift techniques used to prove the corollary provide an elementary algorithm for computing the period N. Moreover, they can be used to extend the conclusion of the theorem to orinted links. Details can be found in [**SiWi3**].

We conclude with a brief discussion of work in progress. Let $l = l_1 \cup \ldots \cup l_d$ be an oriented link with group $G = \pi_1(S^3 - l)$. Instead of choosing a homomorphism of G onto the integers, let $\mu : G \to \mathbf{Z}^d$ be the abelianization homomorphism that sends oriented meridians to the standard basis. Let K denote the kernel of μ. For any finite abelian group Σ, the set $\mathrm{Hom}(K, \Sigma)$, which is the same as $\mathrm{Hom}(K^{\mathrm{ab}}, \Sigma)$ admits d commuting automorphisms $\sigma_{x_1}, \ldots, \sigma_{x_d}$ corresponding to the various meridians of the link. The set of representations can be given a topology similar to that of a representation shift so that it becomes a *Z^d-shift of finite type* (see [**LiMa**]). Although $\mathrm{Hom}(K, \Sigma)$ cannot in general be described by a graph, its elements still have an appealing combinatorial description. Representations $\rho : K \to \Sigma$ correspond to labelings of the lattice $\mathbf{Z}^d$ by a finite alphabet $\mathcal{A}(= \Sigma^n)$ that satisfy a finite number of translation-invariant local conditions.

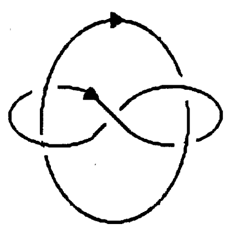

Figure 6

Example. Consider the 2-component link 5_1^2 oriented as as shown in figure 6. The abelianized kernel K^{ab} has presentation

$$\langle \xi_{i,j} \mid \xi_{i,j} - \xi_{i+1,j} - \xi_{i,j+1} + \xi_{i+1,j+1} \rangle,$$

where the indices i, j range over the integers. In this example the alphabet is simply Σ. Representations $\rho : K \to \Sigma$ correspond to Σ-labelings of the lattice $\mathbf{Z}^2$ such that in any 1×1 square the sum of the lower-left and upper-right labels is equal to the sum of the remaining two labels. Here is a particular representation $\rho : K \to \mathbf{Z}/3$.

$$\begin{array}{ccccccc} & & & \vdots & & & \\ \ldots & 0 & 1 & 2 & 1 & 0 & \ldots \\ \ldots & 2 & 0 & 1 & 0 & 2 & \ldots \\ \ldots & 1 & 2 & \underline{0} & 2 & 1 & \ldots \\ \ldots & 0 & 1 & 2 & 1 & 0 & \ldots \\ \ldots & 1 & 2 & 0 & 2 & 1 & \ldots \\ & & & \vdots & & & \end{array}$$

Here is the representation $\sigma_{x_1}\sigma_{x_2}^2(\rho)$.

$$\begin{array}{c} \vdots \\ \ldots\ 0\ 1\ 2\ \underline{1}\ 0\ \ldots \\ \ldots\ 2\ 0\ 1\ 0\ 2\ \ldots \\ \ldots\ 1\ 2\ 0\ 2\ 1\ \ldots \\ \ldots\ 0\ 1\ 2\ 1\ 0\ \ldots \\ \ldots\ 1\ 2\ 0\ 2\ 1\ \ldots \\ \vdots \end{array}$$

Numerical invariants such as *directional entropy* can be computed for such dynamical systems. Also interesting are the directions in which the $\mathbf{Z}^d$-shift, when restricted, is *expansive*. The latter are related to the geometric invariant of R. Bieri, B.H. Neumann and R. Strebel [**BiNeSt**]. We will discuss these topics in a forthcoming paper.

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Reidemeister Moves on 1-manifolds in $R \times D^2$

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December 19, 1996

Summary

Let L_1 and L_2 be 1-manifolds that are proper and properly embedded in $\mathbf{R} \times \mathbf{D}^2$. Let $J = [-1, 1]$ be the closed interval and let $p : \mathbf{R} \times \mathbf{D}^2 \to \mathbf{R} \times J$ be a projection map.

We say that L_1 and L_2 are *equivalent* if there is an isotopy $h_t : X \to X$ with $h_0 = 1_X$ and $h_1(L_1) = L_2$.

Let $T = \{\ldots < t_{-1} < t_0 < t_1 < \ldots\} \subset \mathbf{R}$. For every $i \in \mathbf{Z}$, let D_i be a closed disk in $(t_i, t_{i+1}) \times J \subset \mathbf{R} \times J$. Suppose that $C = \{C_{i,j} | i \in \mathbf{Z}$ and $1 \leq j \leq n_i\}$ is a set of disks such that $C_{i,j} \subset D_i - \partial D_i$ for $1 \leq j \leq n_i$ where $n_i \in \mathbf{N}$ for every $i \in \mathbf{Z}$. Suppose that, for every $i \in \mathbf{Z}$, $p(L_2) \cap ([t_i, t_{i+1}] \times J)$ is obtained from $p(L_1) \cap ([t_i, t_{i+1}] \times J)$ by Reidemeister moves and each Reidemeister move is contained in $C_{i,j}$ for some $1 \leq i \leq n_i$. Then we say that L_2 is obtained from L_1 by *a generalized countable Reidemeister move (GCR-move)*.

We say that L_1 is *GCR-equivalent to* L_2 if $p(L_2)$ is obtained from $p(L_1)$ by a finite number of GCR-moves and isotopies of $\mathbf{R} \times J$. Note that GCR-equivalence is an equivalence relation.

Main Theorem 1 *Let L_1 and L_2 be 1-manifolds that are proper in $\mathbf{R} \times \mathbf{D}^2$. Then L_1 is GCR-equivalent to L_2 iff L_1 is equivalent to L_2.*

A simple closed curve which does not link any line

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Introduction

In this note we show the construction of a simple closed curve in R^3 which does not link any line. By not linking any line we mean that it contracts in the complement of any line which misses the curve. The strategy will be to show that any line which misses the curve also misses a 3-ball containing the curve. From this it follows that the curve contracts in the complement of that line.

Opaque Cantor Sets

Before we construct our main example we will first consider the concept of opaque Cantor sets in the plane. This will motivate some of our methods. By opaque Cantor sets we mean Cantor sets which cannot be separated by a straight line. Another way to view this is that the projection of the Cantor set in any direction in the plane is an interval. This motivates the term opaque.

One way to construct such a Cantor set is the following. We start out by constructing the first step of the standard middle third Cantor set in the plane namely $C \times I \cap I \times C$, where C is the standard middle third Cantor set. This is sometimes called Cantor's Tartan. The first two stages are depicted below.

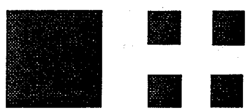

Figure 1

Now at each stage of the construction we throw in four additional squares as shown in Figure 2. This still gives a Cantor set but the additional squares prevents lines from separating the Cantor set.

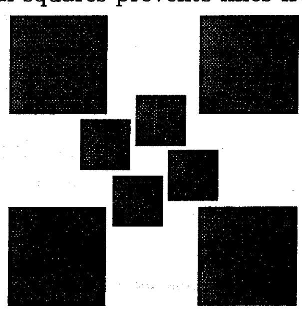

Figure 2

This method is not satisfactory for it requires the introduction of extra sets to gives us opaqueness, or informally to plug the gaps so that lines cannot pass through the gaps.

A more satisfying method is the following where we adjust the shape of the gaps themselves to prevent lines from passing through and separating.

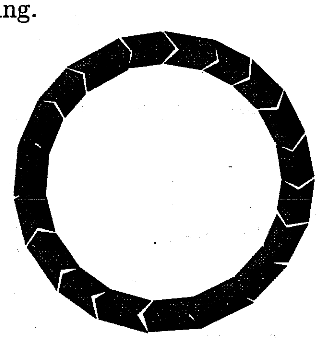

Figure 3

To begin this construction we take a disk and remove the interior of a smaller disk, called a core, to get an annulus. We then make jagged cuts to form gaps to break the annulus up into disks. We then repeat the process with each of these disks.

To be more precise on making these jagged cuts we consider radial lines through the annulus corresponding to where we want to make the cuts. We take three points on the portion of each of the line segments in the annulus. We then place these points in general position. Now no line segment can pass through each of the three points. Connect the new points with line segments and carefully thicken up these new line segments slightly to get the gaps through which no line can pass.

General position of points is what made this construction work. We will generalize such arguments for our main example.

Another feature of this construction that will have a very important analog in our main example is that any line that misses the Cantor set also misses the cores at each stage of the construction. This method prevents lines from passing through the core.

Construction of Main Example

Now for the construction of our main example. We will first briefly discuss how to construct our simple closed curve. Then we will come back and carefully discuss two of the points of the construction that will show that our curve has the desired properties.

We start with a solid torus T_0. We then take a small collar of the boundary in T_0. We will call the closure of the complement of this collar the *core* of the first stage and will denote it by T_0'.

We now coil another thin solid torus around this core lying in the collar. This torus will wrap around only once longitudinally in T_0 and many times meridionally. We want to also make sure that adjacent coils do not touch. We call this new solid torus T_1. We take a small collar in T_1 and create a core T_1' as before. Repeat this process.

This gives us a nested sequence of solid tori. Let $J = \cap T_i$. Then J is our simple closed curve. The fact that it is a simple closed curve follows from the fact that it is the intersection of a nested sequence of tori whose cross section diameters are going to zero, and they wrap only once longitudinally about the previous stage.

Now we need to discuss the two additional properties of the construction that we need to make the argument work.

Property 1: Both the first torus T_0 and its core T_0' do not link any line.

By a solid torus not linking any line we mean that any line which misses the torus also misses a three ball containing the torus. We can do this as illustrated below where the convex hull of the torus is the desired three ball. We also want to do this so that the core, T_0', also does not link any line.

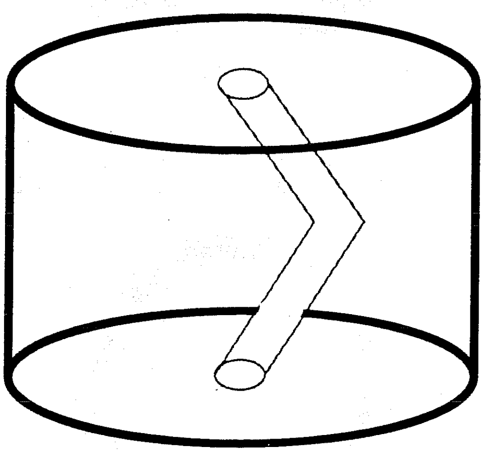

Figure 4

Property 2: Any line that misses T_i also misses the core, T_{i-1}', of the previous stage for all i. Even more strongly we will show that any line that misses T_i' also misses T_{i-1}'.

We will refer to this informally as T_i protecting the core T'_{i-1} from lines. This is the more delicate of the two properties.

Once we have this we will be done. To see that this finishes the argument note that if a line L misses J then it misses T_i for some i since the tori are nested. Then by Property 2 if L misses T_i it misses T'_{i-1}. So J is homotopic to a curve in T'_{i-1} which misses L. By an abuse of notation we will refer to this curve as J also. The strong version of property 2 then shows that L misses T'_{i-2} also. So once again we get a curve homotopic to J which we will also denote by J which lies in T'_{i-2}. Repeating this argument we get that L misses T'_0 and J is homotopic to a curve in T'_0. But by our construction of T'_0 we can take the convex hull of T'_0 to get a three ball missing L and containing J. From this it is clear that J contracts in the complement of L as desired.

Now we need to show how to establish Property 2. We will show how to do this for the first stage and the others will follow similarly. Take the solid T_0 and its collar as discussed in Property 1. Take a simple closed curve that wraps around the boundary of T_0 once longitudinally and many times meridionally as shown below.

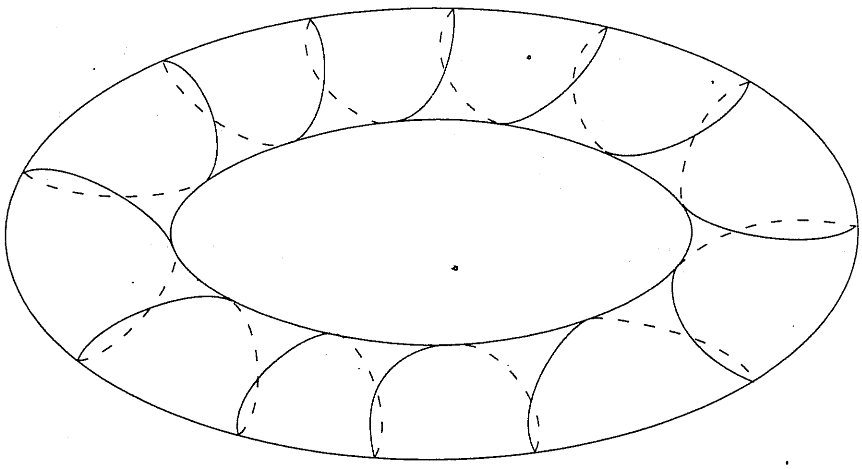

Figure 5.

We will use this as a guide to start to construct the gap between the coils of the desired solid torus T_1. Cross this simple closed curve with an interval to get an annulus, A_1, in the collar. If we thickened up this annulus and took the complement of it in the collar we would have a solid torus. We want to first discuss how to vary the shape of this annulus so that the gap will have the desired property that no straight lines may pass through it so as to satisfy Propery 2.

To do this we will use the following lemma:

Lemma: Given a complex in R^3 there exists a map which moves points less than any given ϵ such that any line in R^3 can intersect at most four of the 1-simplices of the complex.

This is a special case of what is known as *Strong General Position.* In its full generality strong general position gives conditions on the number of simplices of certain dimensions that a hyperplane can intersect. This was introduced by Berkowitz and Roy in [**1**].

To use this result we place five concentric simple closed curves in A_1 and incorporate these into a triangulation of A_1. We then place the resulting complex in strong general position. Now no line can pass through the annulus to its core for any line that meets the outer curve the annulus meets at least four edges of the triangulation before it meets the inner curve of the annulus. Thus the line cannot meet the inner curve of the annulus as claimed.

Now thicken up the annulus slightly so that no line can pass through the thickened version. We then take the complement of this thickened annulus in the collar. This gives us our solid torus T_1 which satisfies Property 2. Repeat this argument to construct the rest of the tori.

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Problem Session

1. (Dijkstra) If $h : X \to Y$ is a hereditary shape equivalence of complete spaces, do there exist compactifications C and D of X and Y respectively such that h extends to a hereditary shape equivalence $\bar{h} : C \to D$? What about cell-like instead of hereditary shape equivalence?

2. (Dijkstra) Do there exist dimension-raising hereditary shape equivalences other than from $ind = \omega$ to $ind = \omega + 1$? In particular, does there exist a countable λ such that for every $\alpha < \omega_1$, there is a hereditary shape equivalence from $ind = \lambda$ to $ind \geq \alpha$?

Notes: (Dijkstra):

This problem is connected to the question of whether countable dimensionality is conserved under hereditary shape equivalences.

3. (M. Bestvina and R. Edwards) Does there exist a non-identity map $f : \mathbb{R}^n \to R^n$ such that:

$$i) \text{ For all } q \geq 1,\ dist\,(f^q, iden) < 1, \text{ and}$$
$$ii) \text{ There exists } q_1 < q_2 < \cdots \text{ such that } dist\,(f^{q_k}, iden) \to 0 \text{ as } k \to \infty\ ?$$

Notes: (Edwards)

Even the special case where f is a homeomorphism is open. Also, the question could, of course, be asked with $\mathbb{R}^n$ replaced by an arbitrary manifold M (and in (i), 1 replaced by $\epsilon = \epsilon(M)$). However, the question could is morally local, and so the case $M = \mathbb{R}^n$ serves as the model.

The question is motivated by the Hilbert-Smith Conjecture. Namely, if it fails, e.g., if there is an effective action on $\mathbb{R}^n$ by a p-adic integer (Cantor) group with orbits of uniformly bounded diameter, then for $f : \mathbb{R}^n \to \mathbb{R}^n$ one would take the action by any non-zero group element (with appropriate rescaling to achieve the bound of 1, if necessary).

An interesting special case is where f is periodic (and bounded-close to the identity), solved by Newman. This was extended to the case where f is pointwise periodic, by Montgomery.

4. (Silver) We recall that a group is hopfian if it is not isomorphic to a proper quotient group. Also, a group is residually finite if the intersection of its normal subgroups of finite index is trivial.

Assume that K is both finitely generated and residually finite. A well-known theorem of A. Mal'cev states that K is hopfian. Another result, due to G. Baumslag, asserts that Aut(K), the automorphism group of K, is residually finite. The proofs of both results require that K be finitely generated only to be certain that the number of subgroups of K having a fixed finite index is finite.

Let K be the commutator subgroup of any classical knot group. It is known that K is residually finite. However, K can contain uncountably many subgroups of the same finite index. (This happens, for example, in the case of the knot 5_2. Possibly it happens whenever the knot is nonfibered.) We ask:

(i). Is K hopfian? (This question has been asked by W. Whitten and F. Gonzalez-Acuna.)

(ii). Is Aut(K) residually finite?

5. (Guilbault) Let M^n be a one-ended manifold without boundary.

(i) If M^n is inward tame, must π_1 be semistable at ∞ ?

(ii) Suppose N^n is a closed, aspherical manifold.

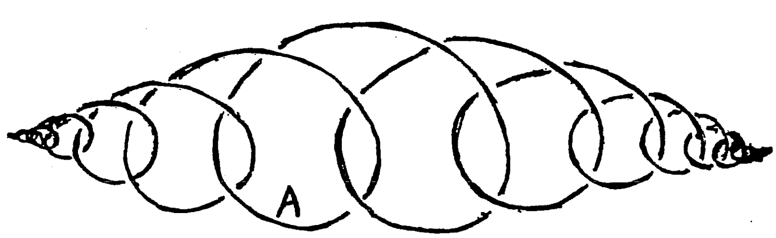
a. Must $\tilde{N}^n$ be inward tame?

b. Must $\tilde{N}^n$ be semistable at ∞?

5. (Daverman) Is there a closed aspherical manifold with non-Hopfian fundamental group?

6. (Ancel)

a. Let A be the Fox Artin arc in S^3 pictured here. Is $\pi_1(S^3 - A)$ indecomposable?



b. Are there infinitely (uncountably) many wild arcs $A_1, A_2, A_3, \cdots$ in S^3 such that $\pi_1\left(S^3 - A_i\right) \neq \pi_1\left(S^3 - A_j\right)$ for $i \neq j$?

c. Specifically, let B be the wild arc in the solid torus T pictured below. Suppose $k_i : T \to S^3$ is a knotted embedding such that $\pi_1\left(S^3 - e_i(T)\right) \neq \pi_1\left(S^3 - e_j(T)\right)$ for $i \neq j$. Is $\pi_1\left(S^3 - e_i(B)\right) \neq \pi_1\left(S^3 - e_j(B)\right)$ for $i \neq j$?