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UWM

The Twelfth Annual Workshop in Geometric Topology was held at the University of Wisconsin-Milwaukee on June 18-20, 1995. The participants were:

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Doug Bullock	University of Iowa
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These proceedings contain notes on the four one-hour talks given by principal speaker Shmuel Weinberger as well as summaries of several of the half-hour talks given by other participants. We wish to acknowledge the financial support provided by the National Science Foundation (DMS-9401185) and the University of Wisconsin-Milwaukee which contributed greatly to the success of this workshop.

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MILWAUKEE LECTURES ON CHARACTERISTIC CLASSES AND SINGULAR SPACES

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These notes are based on four talks I gave at the University of Wisconsin at Milwaukee during the summer of 1995. I have added little to the original oral presentation, and have also cut some discussions a bit. The first lecture was a quick run through of mainly pre-1970 ideas on characteristic classes. The middle two were devoted to some topological applications of intersection homology that stem from a deep connection between self dual sheaves, elliptic operators, surgery theory and intersection homology. Unfortunately, space did not permit me to develop these connections more fully. The parts unrelated to singular spaces[1] are, in any case, now documented in the Novikov conjecture literature. The last lecture was an introduction to the beautiful ideas related to toric varieties and their connections to lattice points and convex polytopes. This field has been very active recently, from points of view related to algebraic geometry, symplectic geometry, and pure combinatorics. I am not familiar with this literature and have certainly slighted them. Nonetheless, I felt that it was worthwhile to give a glimpse of some of this material to a group of topologists. I hope that these <u>rough</u> notes stimulate people to read the many more complete references, and perhaps to begin thinking about these ideas for themselves.

I would like to thank Ric Ancel and Craig Guilbault for their hospitality and excellent organization, the conference participants for putting up with a lot of material being thrust upon them too quickly and too superficially, and finally Sylvain Cappell and Julius Shaneson for explaining their beautiful work to me.

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[1]The cases of manifolds with boundary, $\mathbb{Z}_n$ manifolds, and orbifolds are now being explicated.

LECTURE ONE: Classical themes.

Nowadays there are characteristic classes for many different kinds of structures. There are characteristic classes associated to oriented and unoriented real complex or quaternionic vector bundles, PL block bundles, topological bundles, spherical fibrations, equivariant versions of these, for foliations, and fiber bundles. There are cohomology characteristic classes and homology characteristic classes, and they can live in an ordinary theory or a generalized (co)homology theory, or even a (co)sheaf)co(-homology theory. Indeed, almost all meaning has been leached out of this noun: what one does is assign an element of some group to some situation -- hopefully in a reasonably computable way and hopefully by some process that measures something interesting about the situation.

The characteristic classes that we will care about are generalizations to singular spaces of classes that were first defined in smooth situations, and usually by reference to the tangent bundle. Much of this lecture is just a review of book [MS].

A prototypical class is the EULER CLASS $e(\xi) \in H^n(X)$ defined for oriented n-dimensional vector bundles over X. It can be defined as the primary obstruction to finding a section of ξ over X -- if X is a manifold, it is Poincare dual to the set of zeroes of a generic vector field. (Transversality shows that this cycle is well defined.) Another definition comes from looking at the map in the Gysin sequence associated to the sphere bundle obtained (up to homotopy) by removing the 0-section. One can verify that it satisfies the following conditions that in fact characterize it:

(I) e is natural: $e(f^*(\xi)) = f^*e(\xi)$
(ii) $e(\xi\times\eta) = e(\xi)\times e(\eta)$
(iii) $e(\nu) = 1$, where ν is the normal bundle of $\mathbb{C}P^1 \subset \mathbb{C}P^2$.

Condition (i) alone restricts e enormously. Since every bundle is the pullback of the universal bundle over the Grassmanian ($\cong$ BU(n)), (i) implies that we are just talking about the cohomology of Grassmanians.

Already one sees that homology characteristic classes necessarily have a different flavor, since it is unlikely that the method of the universal example applies to them.

Note also that we could think of cohomology operations are just cohomology characteristic classes of cohomology classes -- an almost useless observation.

Condition (II) tells us what e is for a sum of line bundles. A general splitting principle asserts that cohomology (and K-theory) characteristic classes are determined by what they are for sums of line bundles: indeed, one merely has to verify that $H^*(G_n) \rightarrowtail H^*(G_1 \times \ldots \times G_1)$, which can be done in a number of ways, and (iii) is a normalization sufficient to determine e(universal line bundle).

It is trivial to see that these axioms directly imply that e vanishes if ξ has a nowhere vanishing section. (Hint: deduce $e(\xi \oplus \eta) = e(\xi)e(\eta)$ from the other axioms.)

A final definition can be given using the Pfaffian of the curvature for a connection on the bundle ξ. The details of this need not concern us. The power of characteristic classes come from the variety of definitions for certain of these classes, which underscore relations between different concepts and lead to effective computability of these invariants.

The classes of most interest to us are the Pontrjagin classes for an oriented vector bundle, and, in the last lecture, the Chern classes of a complex bundle. Indeed, as the Pontrjagin classes can be defined as the Chern classes of the complexification, logically we need only define the latter. (However, when we move on to PL manifolds which do not have vector bundles tangent to them, we will need some other idea.)

The CHERN CLASSES $c_i(\xi) \in H^{2i}(X)$, $i>0$, satisfy a similar set of axioms:

(0) Let $c = 1 + \Sigma c_i$

(I) c is natural: $c(f^*(\xi)) = f^*c(\xi)$

(ii) $c(\xi \times \eta) = c(\xi) \times c(\eta)$

(iii) $c_1(\nu) = 1$, where ν is the normal bundle of $\mathbb{C}P^1 \subset \mathbb{C}P^2$.

In fact, considering oriented 2-plane bundles = complex line bundles as principal S^1 bundles and the isomorphism $BS^1 \cong K(Z,2)$ one can easily see that c_1 defines an isomorphism between complex line bundles over X under $\otimes$ and $H^2(X ; Z)$.

A great deal of the beauty of the subject comes from the isomorphism $H^*(BU(n)) \cong Z[c_1, c_2, \ldots c_n]$. This will play little role for us. A consequence is that $c_i = 0$ for $i > \dim(\xi)$ and $c_i = e$ for $i = \dim(\xi)$. Naturality, these formulae, and STABILITY (the fact that $c(\xi \oplus \varepsilon) = c(\xi)$ if ε is trivial -- a statement false for e) also characterize the Chern classes. And finally, there is a curvature definition of these classes.

Rather than give axioms for PONTRJAGIN CLASSES, we will settle for the definition:

$$p_i(\xi) = c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X).$$

There are a number of ways in which these classes can be used. One way is to distinguish manifolds via the Pontrjagin classes of their tangent bundles. This is strong enough to distinguish some high dimensional lens spaces from one another; by considering sphere bundles over S^4 (produced by clutching using, say, $SO(3) \cong \mathbb{R}P^3$) and using the finiteness of the stable homotopy groups of spheres to see that there are only finitely many homotopy types among these, one finds infinitely many manifolds homotopy equivalent to $S^4 \times S^n$ distinguished by p_1. We will return to this theme in lecture 3.

Another type of application is to proving non-embedding and non-immersion results. This makes use of the vanishing above twice the dimension of the bundle (the 2 comes from complexification). If something embeds in low codimension then one has a low dimensional normal bundle. One can compute what the characteristic classes of the normal bundle must be using naturality and the product formula, so one might get obstructions in this manner.

This type of application does not generalize as widely. When we generalize rational Pontrjagin classes to PL bundles, it turns out that 3-dimensional bundles can have characteristic classes in arbitrarily high dimensions. For instance, rationally, $BPL_4 \cong K(Z,4) \times BSO$, where the $K(Z,4)$ is the Euler class. (It is also true that $BPL_3 \cong K(Z,4) \times BSO$ rationally, except that the homotopical term has a more unusual interpretation.)

The connection between characteristic classes and the cohomology of classifying spaces is the central point in the calculation of bordism groups (and that is one of the original motivations for considering characteristic classes with values in generalized cohomology theories). One of the main results of Thom is the following isomorphism of rings:

$$\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots.] \cong \mathbb{Q}[p_1, p_2, \ldots]^*$$

where Ω_* is the oriented bordism ring, graded by dimension, and an upper $*$ denotes dual. I should probably make explicit the dualization implicit in the graded setting when we form quantities like $\mathbb{Q}[p_1, p_2, \ldots]^*$. In dimension 4 we have the homomorphism p_1, in 8 we have two homomorphisms $p_1{}^2$ and p_2, in 12, there are $p_1{}^3$, $p_1 p_2$, and p_3. These polynomials in Pontrjagin classes which end up homogeneously graded are called Pontrjagin numbers, and are obstructions to bordism. Thom's result asserts that rationally they are the only obstructions. (The proof is well worth reading in Thom's original paper, where, for instance, transversality is developed in order to reduce calculations of bordism to stable homotopy theory, and where those calculations are done. The above formula follows from the Thom isomorphism theorem, Serre's work on rational homotopy groups of spheres and the calculation $H^*(BSO ; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]$ which follows from the result on the unitary group mentioned before.)

One important consequence of Thom's theorem is a formula of Hirzebruch (proven, he asserts, the day he saw Thom's Comptes Rendus note announcing the results on cobordism in the Fine Hall library). For this, let M^{4k} be a smooth oriented manifold. Then $H^{2k}(M)$ has a symmetric bilinear nonsingular inner product given by cup product:

$$\cup : H^{2k}(M) \otimes H^{2k}(M) \longrightarrow H^{4k}(M) \cong \mathbb{Q} \text{ (by the orientation)}$$

(Symmetry is because 2k is even, and non-singularity is part of Poincare duality.) Such quadratic forms can be diagonalized, and SIGN(M) is the signature of the quadratic form, i.e. the difference between the number of positive eigenvalues and the number of negative ones. In light of the calculation of bordism groups, the following "axioms" characterize signature:

(i) signature is bordism invariant
(ii) sign $(M \cup N) = \mathrm{sign}(M) + \mathrm{sign}(N)$
(iii) $\mathrm{sign}(M \times N) = \mathrm{sign}(M)\mathrm{sign}(N)$
(iv) $\mathrm{sign}(\mathbb{C}P^{2i}) = 1$.

(i) follows by consideration of the image in H^{2k} of any manifold bounding M; it will be self annihilating for U and by Poincare duality it is of half the dimension of $H^{2k}(M)$. (ii) and (iii) are straightforward cohomology calculations and the quadratic form arising in (iv) is just (1).

(i) and (ii) alone imply that in each dimension we can find a Pontrjagin number that equals the signature. Hirzebruch's formula is a systematic description. Let

$$L = 1 + L_1 + L_2 + \ldots \quad L_i \in H^{4i}(M)$$

be associated to the power series $\sqrt{z}/\tanh(\sqrt{z})$ by formally imagining that the complexified tangent bundle of M is expressed as a sum of line bundles and one then computes the product of the expressions $\sqrt{z}/\tanh(\sqrt{z})$ in these line bundles. Reorganize the product as a power series in symmetric functions of the line bundles and write these symmetric functions as Chern classes. We have

$$\mathrm{sign}(M) = \langle L(M), [M] \rangle \quad \text{(Hirzebruch Signature Theorem = HST)}$$

Here are the first few of the L's:

$$L_1 = 1/3\, p_1$$
$$L_2 = 1/45(7p_2 - p_1^2)$$
$$L_3 = 1/945(62p_3 - 13p_2p_1 + 2p_1^3)$$
$$L_4 = 1/14175(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4)$$

The Bernoulli numbers which enter the power series expansion of $\sqrt{z}/\tanh(\sqrt{z})$ thus enter topology through this formula. Number theorists, at least since the time of Kummer, have paid close attention to the numerator and denominators of these numbers, and for some purposes we must as well. We will also see in lecture 4 that these numbers must enter in characteristic class formulae because they are closely related to the Euler MacLaurin summation formula where such numbers also arise.

Milnor's detection of exotic spheres came from the 7 appearing in the numerator of L_2. Since Pontrjagin classes are integral cohomology classes, one deduces that for a 3-connected smooth 8-manifold which is parallelizable outside of a point, the signature must be divisible by 7. Building a 3-connected 8 manifold by "plumbing" according to a unimodular quadratic form of signature 8 produces a homotopy 7-sphere on the boundary which cannot be smoothly standard, for if it were, one could glue in a final disk and obtain a 3-connected smooth 8-manifold parallelizable outside of a point and signature 8.

There are a number of comments we should make regarding HST. The first is an important consequence: Signature is multiplicative for finite coverings (since the right hand side is). It is interesting an important that this is not true simply as a consequence of Poincare duality. (It though does turn out to be true for topological manifolds, unlike the integrality consequence discussed above.)

One way to see this is by writing down explicit 4-dimensional Poincare complexes (starting from quadratic forms over $Z[Z_p]$ for which multiplicativity does not hold, e.g. the 1×1 form (1) and doing an analogue of Whitehead's construction of simply connected 4-dimensional Poincare spaces with arbitrary unimodular quadratic form as its intersection pairing). There are other ways that come up naturally -- although their verifications are either quite tricky or require theorems (e.g. the Atiyah-Bott fixed point formula, the Atiyah-Singer G-signature theorem, or the Atiyah-Patodi-Singer index theorem for manifolds with boundary -- we'll have more to say about these later). If one takes cobordant lens spaces, which are cobordant over manifolds with isomorphic π_1 (One can use bordism theory to see when this happens; homotopy equivalent three dimensional lens spaces have this property). Then glue the ends together by a homotopy equivalence. If this homotopy equivalence were homotopic to a homeomorphism, then this Poincare space would be homotopy equivalent to a manifold. Nonmultiplicativity of signature sometimes obstructs this.

In fact, an extension of the multiplicativity of the signature always shows that the Poincare space is not homotopy equivalent to a manifold, and that therefore non-isometric lens spaces are always nonhomeomorphic.

If $G \times M \longrightarrow M$ is an orientation preserving action, then we can consider the quadratic form as a quadratic form with isometry. When we diagonalize the form, the positive and negative definite parts are preserved by the group action, and therefore we obtain a virtual representation

$G\text{-SIGN}(M) \in RO(G)$.

Equivalently, for $g \in G$, we can take the difference of the traces of g acting on the positive definite and negative definite pieces of H^{2k}. This gives a notion of $\text{SIGN}(g, M) \in \mathbb{R}$.

PROPOSITION: If the action of G on M is free, then G-sign(M) is a multiple of the regular representation; equivalently, for $g \neq e$, $\text{sign}(g,M) = 0$. ($\text{sign}(e,M) = \text{sign}(M)$.)

The relation between the two halves of the proposition is given by elementary representation theory. The relation to multiplicativity of the signature follows form the following fairly elementary proposition:

PROPOSITION: If G acts orientation preservingly on M, then M/G satisfies rational Poincare duality. $\text{sign}(M/G) = 1/|G| \, \Sigma \text{sign}(g,M)$

The proposition is proven using cobordism invariance of G-signature, and the isomorphism $\Omega_*(BG) \otimes \mathbb{Q} \cong \Omega_*(*) \otimes \mathbb{Q}$ (thinking of bordism as a homology theory by looking at cobordism classes of manifolds with maps into a given space) which holds true because all the group homology for a finite group is finite. $\Omega_*(BG)$ can be thought of a bordism of manifolds with free G-action, and the image of $\Omega_*(*)$ can be thought of those manifolds of the form $G \times N$ with action just on the first coordinate. For these manifolds either version of the proposition is obvious.

For the Poincare spaces we discussed it turns out that G-sign(M) is not a multiple of the regular representation.

Exercise: Show that these lens spaces are not even h-cobordant. Deduce that lens spaces do not become isomorphic after crossing with a circle. What about higher tori? (See lecture two.)

The work of Atiyah and Singer on indices of elliptic operators gave a proof of HST and these consequences that did not rely on bordism -- although it did rely on substantial analysis and K-theory (especially Bott periodicity) as a key topological tool. I cannot go that far afield to explain this circle of ideas, but let me just mention one consequence of this:

ATIYAH-SINGER G-SIGNATURE THEOREM: sign(g, M) can be computed by an explicit formula of the form $\langle \kappa(\nu) \cup L(F), [F] \rangle$ where $\kappa(\nu)$ is some explicit characteristic class of the equivariant normal bundle, and F is the fixed point set of the element g.

COROLLARY: If $M^g = \varnothing$, then sign(g,M) = 0.

which is a bit more refined than what we got out of HST. Later on we will return to a purely topological point of view on this corollary -- one which shows that it is true under much weaker conditions than smoothness. (I do not know if continuity of the action alone is enough.)

If M^g consists of isolated points, then the equivariant normal bundle at such a point is given by the differential dg acting on the tangent space at that point. To make the theorem more concrete, let me add the addendum:

ADDENDUM: The local contribution to sign(g, M) at a fixed point p is given by $((-1)^{n/2})/p \cdot \sum_j \prod_i \cot(\pi j a_i/2p)$, where the a_i are the rotation numbers of the action of g on the tangent space, and j runs over the odd numbers from 1 to 2p-1.

The difference of these numbers is the g-signature of the Poincare space constructed above, which can be used to distinguish any pair of lens spaces (this is the Franz independence lemma).

In lecture 4 we will need some algebro-geometric analogs of these theorems, the Hirzebruch Reimann-Roch theorem and Atiyah-Singer's equivariant generalization of it, but we need not discuss these here. Instead, let us go back to topology, and give our first extension of characteristic classes to a class of spaces that are somewhat more singular than smooth manifolds (the original domain of definition of these classes).

METHOD OF THOM-MILNOR-ROCHLIN: We will extend the definition of L-classes (and therefore rational Pontrjagin classes) to PL rational homology manifolds. As a corollary, Pontrjagin classes are rationally PL invariants, not just smooth invariants. (Of course this is true topologically as well by a celebrated theorem of Novikov, but that is a much deeper fact.)

We will need a basic result of Serre: Odd spheres are rationally Eilenberg Mac-Lane spaces. (This fact was also used by Thom in the course of his calculation of bordism.)

The basic point is the following formula that holds for F a submanifold with trivial normal bundle in M:

$$\mathrm{sign}(F) = \langle L(M), [F]\rangle$$

This formula is a consequence of the HST applied to F and the fact that since the normal bundle of F is trivial $L(M)|_F = L(F)$. Now, we can inversely define L(M) by insisting that the above formula be true for all closed sub-homology-manifolds with trivial normal bundle in M×Euclidean space. The point is that there are exactly enough sub-homology-manifolds with trivial normal bundle to fill rational homology in odd codimension by Serre's result (the submanifold is the transverse inverse image of a point in the lift of the dual cohomology class to a sphere), and by crossing with a suitable Euclidean space, every dimension can be take to be odd codimension! This is well defined by bordism invariance of signature.

Note: We have taken the Hirzebruch signature formula and made it into a tautology. It is not too difficult to change our definition from one involving manifolds to one involving PL block bundles and get our classes to lie in $H^*(BPL)$. Since we worked rationally none of the subtle divisibility arguments we discussed above hold.

Now let us turn to the bit of more integral characteristic class theory that is available for PL and Top. Indeed, in the next lecture, we will see that even for certain singular spaces, there are "integral" characteristic classes, which are homology classes rather than cohomology.

This method was invented by Sullivan who applied it to construct a class $\Delta(M) \in KO_n(M)\otimes\mathbb{Z}[1/2]$ for PL ($\mathbb{Q}$-homology-) manifolds, whose Pontrjagin character[2] is the Poincare dual of the L-class. A similar method gives information at 2 (see [MS].)

It is true, and we will sketch some of the technology necessary in the next lecture, that these classes can be defined for ANR $\mathbb{Q}$-homology manifolds, but they tend not to be orientations, even for $\mathbb{Z}$-homology manifolds. This means that their Pontrjagin characters will often not define $p_0 = 1$. Indeed, p_0 defined this way is Quinn's obstruction to resolving homology manifolds. (By definition, and simple naturality properties (with respect to open inclusions) these invariants do not change under CE maps.)

Embed M in some Euclidean space with codimension 4r and regular neighborhood (N, ∂N). We shall describe an element of $KO^{4r}(N, \partial N)$ which corresponds under (Spanier-Whitehead) duality (cf. [Sp]) to $\Delta(M)$. Sullivan proves a "Connor-Floyd type theorem" that shows that $KO^*(X)$ can be identified with $\Omega_*(X)\otimes_{\Omega(*)}\mathbb{Z}$ where $\Omega_*(X)$ is mod 4 graded and $\mathbb{Z}$ is viewed as a module over $\Omega(*)$ by signature. In light of this it is only necessary to give a pair of homomorphisms:

$$\Omega(N, \partial N) \longrightarrow \mathbb{Z}$$

$$\Omega(N, \partial N\,;\, \mathbb{Q}/\mathbb{Z}[1/2]) \longrightarrow \mathbb{Q}/\mathbb{Z}[1/2]$$

with good module properties with respect to taking products with closed oriented smooth manifolds, i.e. the homomorphism multiplies by the signature. For the first map, one takes a bordism class in (N,

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[2] The Pontrjagin character is the Chern character of the complexification. The Chern character is defined as Σe^{x_i} where the x_i are line bundles making up the given complex bundle, and the end result is reexpressed in terms of Chern classes by replacing the k-th symmetric function in the x's by c_k (as we did before in defining L-classes).

∂N).and takes the intersection with X and computes the signature. The second map goes the same way using "$\mathbb{Z}_N$-MANIFOLDS" (manifolds obtained by gluing n-tuples of boundary components together from some manifold with boundary) to represent the bordism with coefficients and taking mod n signatures of such spaces. (Note that $\mathbb{Q}/\mathbb{Z}[1/2]$ is the direct limit of $\mathbb{Z}_n$ for odd n.) A little effort shows that these signatures are well defined, and this combination of homomorphisms gives us our class.

Remark: The reason we had to use $\mathbb{Z}_n$ manifolds was because a cohomology class is not determined by its values on all cycles. The values on torsion cycles as well, though is enough to pin it down. A similar point, dealt with differently, arises in Atiyah's definition of a K_* class associated to an arbitrary elliptic operator on M.

The idea in brief is this: For each elliptic operator D on a compact smooth manifold M, not only is there an index (= dim ker D − dim ker D^*) $\in \mathbb{Z}$, but one can "twist" the operator by an arbitrary bundle over M. This gives one an element of $\mathrm{Hom}(K^*(M) : \mathbb{Z})$. Unfortunately, in general $K_*(M) \neq \mathrm{Hom}(K^*(M) : \mathbb{Z})$. Atiyah then notices that one has more: For each family of elliptic operators on M parametrised by any auxiliary space X one gets an index bundle in K(X). Thus, one can redo the construction to get a nice universal (in X) construction in $\mathrm{Hom}(K^*(X \times M) : K^*(X))$. This is enough to give the desired element of K-theory[3].

Remark: A $\mathbb{Z}_n$ manifold approach could probably be developed using subsequent work of Freed and Melrose, simplified by Higson. Subsequent work of Brown, Douglas, and Fillmore, and Kasparov shows that K-homology can be viewed as being a cycle theory based essentially on generalized elliptic operators. (Blackadar's book in the references is a good source for this.) If one associates to M its signature operator, then one guesses a relation between the signature operator class of Atiyah and Sullivan's class. The reader can consult [KM, PRW] for some more information.

[3] One can set X = the Spanier-Whitehead dual of M and look at the image of a suitable Bott element in $K(M \times DM)$ in $K(DM)$ to get the element in K-homology.

The Sullivan approach makes the following PROJECTION FORMULA quite transparent:

PROPOSITION: If $f:M \longrightarrow N$ is a fiber bundle (or PL block bundle) with no monodromy, then $f_*\Delta(M) = \mathrm{sign}(f^{-1}(p))*\Delta(N)$.

The reader should try to rewrite this in terms of Pontrjagin classes!

The proof is kind of obvious. After unwinding all the definitions, all that this formula is asserting is that $\mathrm{sign}(f^{-1}(C)) = \mathrm{sign}(f^{-1}(p))*\mathrm{sign}(C)$, i.e. multiplicativity of signature in fiber (or block) bundles. This is known to be true for bundles that have no monodromy.

In general, if there is monodromy, one knows there is a correction to the multiplicativity of signature (see [Atiyah 2]). This can be promoted to give the correction term for the projection formula.

To give a typical application of the projection formula, which foreshadows ideas to play a role in lecture four, we will sketch prove the following:

THEOREM [W]: Suppose S^1 acts on a manifold M with fixed set $F \neq \emptyset$. Let $\varphi: M \longrightarrow B\pi$ be an a map. Then

$$\varphi_*(L(M) \cap [M]) = \varphi_* i_*(L(F) \cap [F]) \in H_*(B\pi; \mathbb{Q}).$$

where the right hand side involves an implicit summation over all components of F tat have dimension $\equiv \dim(M) \bmod 4$.

We shall use a construction that has been named the BUBBLE QUOTIENT. $M//S$ is by definition M where all points on orbits that are ε away from the fixed set are identified with each other. If the fixed set were empty, this would be the usual quotient; in general one obtains a stratified space with a stratum that looks like a "bubble" around the fixed set.

The mapping cylinder of the map $M \longrightarrow M//S$ provides a cobordism in the sense of $\mathbb{Q}$-homology manifolds between M and the bubble around F. The bubble is a bundle over F with fiber a $\mathbb{Q}$-

homology $\mathbb{C}P^n$ where n is half the codimension. The projection formula proves the result for $* = 0$.

To prove the result in general one has to study the cobordism produced and check that one can still pair it with arbitrary elements of $H^*(B\pi\,;\mathbb{Q})$. Here is where $F\neq\emptyset$ and rational coefficients enter. (A similar argument appears in [BH].)

Remark: The L-class is the only genus for which the projection formula is true for arbitrary bundles with connected structural group. However, if the structural group is also finite dimensional then there are also elliptic genera that have this property. Presumably, there is a generalization of the theorem to elliptic genera. For the $\hat{A}$ genus the corresponding result is that of [BH].

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[4] These references include the main ideas referred to here, but do not suffice for all the stray comments. Some of those will be covered in later lectures. Conversely, in later lectures I will not repeat references already given here.

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LECTURE TWO: Intersection homology.

Many of the spaces that arise naturally in mathematics, while not themselves manifolds, are in fact manifolds with singularities or even stratified spaces. A STRATIFIED SPACE is a space X with a filtration $X = X_n \supset X_{n-1} \supset X_{n-2} \ldots \supset X_{-1} = \emptyset$ such that the "PURE STRATA", i.e. subspaces of the form $X_i - X_{i-1}$ are manifolds. It is also important to prescribe the way in which the strata are put together: one can get quite bizarre spaces by taking one point compactifications of manifolds. Many possibilities are discussed in the literature, but for our current purposes, we need not be precise. Sufficient for us would be that the spaces are polyhedra, and PL homogenous to the extent that any two points in the same component of a pure stratum can be moved to one another by a PL isotopy. More precisely, we find the context of Quinn's manifold[5] homotopically stratified spaces most suitable.

Examples abound. Manifolds with boundary are examples with two strata. Quotients of smooth or PL group actions are other examples: in the PL case the strata might not just be the fixed sets of various subgroups, unless one makes an additional hypothesis, like local linearity. Manifolds with distinguished submanifolds can be viewed as stratified spaces: there is no requirement that the stratification be topologically intrinsic to the space.

Each example of stratified space suggests its own problems. It turns out that there is a general characteristic class of stratified spaces that seems to be an optimal extension of L-classes (see [W1]). However, it usually does not live in a conventional homology or cohomology group. It lives in a "spectral cosheaf homology" group. In the next lecture we will discuss an example of this theory (although not all the relevant machinery). What tends to happen is that for stratified spaces having some sensible origin the cosheaf homology group also has a natural interpretation, and the characteristic class measures something basic and interesting.

In this lecture I would like to discuss intersection homology and some of its topological applications. (There have been many more applications to algebraic geometry and representation theory, see MacPherson's ICM talk for an older survey.) Intersection

[5]Actually, even more convenient would be to assume that the pure strata are ANR homology manifolds

homology provides us with a way of making certain nonmanifolds satisfy a variant of Poincare duality -- at some cost. Part of the cost is that the duality is not between homology groups. It is between intersection homology groups, which are not, say, homotopy invariant. On the other hand, they have many other wonderful properties that make them computable in cases of interest, and one can make a convincing case that they are the correct generalization of homology to singular algebraic varieties (see [ ]).

In any case, the idea is that because one gains Poincare duality for a larger class of spaces, one can mimic the Thom-Milnor-Rochlin idea from the previous lecture and define homology L-classes (and more) for these spaces. These classes, although defined using intersection homology, live in ordinary homology.

Now, ordinary homology is, of course, the homology of the chain complex built up out of all PL chains on X. If one only allows chains that are transverse to the singular set of X, i.e. that intersect any stratum of the singular set in a subset of the codimension of that stratum, then McCrory showed that, under mild hypotheses[6], one obtains a complex that computes the cohomology of X in the dual dimension. In a manifold, there is no difficulty in moving all chains to be transverse, and the identification of these groups is Poincare duality. The failure, then, of Poincare duality is then related to the difficulty in making chains transverse to the stratification. G

To circumvent this, Goresky and MacPherson suggested considering subchain complexes with cycles being somehow restricted in their intersections with the lower strata. One might for these, have enough transversality to get a good duality. Define a PERVERSITY function to be a non-decreasing function $p:\{2,3,....n\} \longrightarrow \mathbb{N}$ such that $p(2) = 0$ and $p(n+1) \leq p(n)+1$. There are two extreme perversities, the ZERO PERVERSITY $0(c) \equiv 0$, and the TOTAL PERVERSITY $t(c) = c-2$. Perversities m and n are dual if $m+n=t$.

A k-chain will be said to be P-TRANSVERSE if its intersection with the codimension c stratum of X has dimension $\leq k-c+p(c)$ (and similarly for its *algebraic* boundary). Closed k-chains modulo boundaries as usual form a homology group, denoted $IH^p(X)$. Notice

[6] X is a normal oriented pseudomanifold. Pseudomanifold means that the singular set is of codimension at least two and is nowhere dense and normality means that the link of any simplex of codimension at least two is connected.

that "totally perverse" IH is ordinary homology, and the "totally non-perverse" IH is cohomology. The usual Kronecker duality between these is a special case of the following:

THEOREM ([GM]) IH(X) is a topological invariant, independent of the choice of stratification. If we take field coefficients, $\mathbb{F}$, then the intersection homology groups in dual dimensions with dual perversities are paired perfectly by taking intersections of chains.

More explicitly, chains of complementary dimensions with complementary perversities can be isotoped slightly to intersect in a finite number of points, whose number counted with sign $\in \mathbb{F}$ is well defined, yielding a pairing $IH_k{}^p \otimes IH_l{}^q \longrightarrow \mathbb{F}$ when $k+l=n$, $p+q=t$. Furthermore, this gives an isomorphism $IH_k{}^p \longrightarrow \mathrm{Hom}(IH_l{}^q, \mathbb{F})$.

COROLLARY: If X has only even codimensional strata, then the middle intersection homology groups (with $\mathbb{F}$ coefficients) have a "Poincare" self duality.

By middle, we mean the perversity (0, 0, 1, 1, 2, 2 ...). It is halfway between 0 and t. The dual perversity to the middle perversity only differs from it in on odd codimensional strata, which in the corollary we have posited not to occur.

If we write an intersection homology group without specifying its perversity, then we will always mean with middle perversity.

Remark: It is not true that any X with even codimensional singularities has a $\mathbb{Z}$ self duality. The case of isolated singularities is instructive (for seeing this, as well as many other things about IH). Below some dimension (determined by the perversity) chains and their boundaries are not allowed to touch the singular points, so that the homology is ordinary homology (with compact supports) of the complement. In high dimensions (again determined by the perversity), everything will be allowed through the singular points, so that one obtains the ordinary homology of the space = locally finite homology (=rel ∂ homology) of the (closed) complement of the singularities. In the critical dimension, one has chains in the complement with coboundaries allowed to extend through the singular set, which can be described as the image of an ordinary group into a relative one.

Remark: Cheeger had also been simultaneously and independently led to the conclusion that pseudo-manifolds with even codimensional singularities had a self dual homology associated to them. He gave such polyhedra piecewise flat metrics and then considered L^2 differential forms on the resulting incomplete manifold. The $*$ operator gave a suitable duality. A beautiful implication of this is that IH satisfies a Kunneth formula. Somewhat later Goresky and MacPherson proved the conjecture made by Sullivan that these two theories are the same. (There is an interesting recent paper of Youssin discussing the connection between L^p cohomology and IH for other perversities.)

More careful analysis (performed by Cheeger in his original paper and by Goresky's student Siegel in his thesis) enables one to see a larger class of spaces, the WITT SPACES that have a self dual intersection homology. One allows there to be some odd codimensional strata, but demands that for these, their middle dimensional middle perversity IH vanish. If you think about it, you'll see that the open cone on such a space (thought of as having a new type of isolated singularity) satisfies Poincare duality. (I am really itching to introduce sheaves here, but we'll delay that for a while.) These are then the IH analogs of homology manifolds in the usual theory: they are the spaces which have self duality for a local reason.

THEOREM: Witt spaces have self dual IH.

Combining the isolated point discussion above with Kunneth (even just for one factor Euclidean space, which is MUCH MUCH simpler) one gets a proof of the theorem.

Returning to the theme of the previous lecture, one sees that closed Witt spaces have a signature. (It is natural to use the $\mathbb{Q}$-structure on intersection homology to put the signature in $W(\mathbb{Q})$ where, as computed in the book of Milnor and Husemoller, there is more information than just the usual signature. This reflects the fact that the duality cannot be made nonsingular over $\mathbb{Z}$, as we already remarked.) In fact, with the obvious notion of Witt space with ∂, signature is cobordism invariant.

In fact, we have the following remarkable theorem of Siegel:

THEOREM: Signature is the only cobordism invariant of Witt spaces. More precisely, $\Omega_i^{Witt} = 0$ unless $i = 0 \bmod 4$. $\Omega_{4i}^{Witt} = W(\mathbb{Q})$ unless $i = 0$ when it is $\mathbb{Z}$.

By the way, this theorem is proven entirely geometrically. In odd dimension it is obvious. The cone on an odd dimensional Witt space provides a nullcobodism for that Witt space. The idea is to do some kind of "surgery on middle dimensional cycles" to obtain a cobordism to a Witt space with no middle dimensional homology. So that one can then cone off that space.

Later work of Goresky and Pardon gave cobordism calculations for other classes of singular spaces, with either more refined signatures being definable or other characteristic numbers arising.

In any case, for Witt spaces one can repeat Sullivan's definition of a KO characteristic class. The theorem then has the following corollary:

COROLLARY: $\Delta\colon \Omega_i^{Witt}(X)\otimes\mathbb{Z}[1/2] \longrightarrow KO_i(X)\otimes\mathbb{Z}[1/2]$ is an isomorphism

It is a natural transformation between homology theories which is an isomorphism for X = point.

One immediate application of this is a disproof of the integral Hodge conjecture. Recall that the Hodge conjecture describes which homology classes in a smooth algebraic variety (over $\mathbb{C}$) can be expressed in terms of algebraic cycles. Essentially the only condition is the obvious one suggested by Hodge theory: the dual cohomology class must be of type (p,p) in the Hodge decomposition on the cohomology of any Kahler manifold.

Integrally this cannot be the case, as first argued analytically by Atiyah and Hirzebruch, because the homology class represented by the cycle must have a refinement in KO homology, since all varieties are Witt spaces. Since this needn't be true integrally, the conjecture fails[7].

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[7]Even simpler disproofs follow from resolution of singularities.

A less immediate application[8] concerns the invariant of ATIYAH-PATODI-SINGER, which we must first review. The basic problem that APS were interested in solving was: how can one analytically get a formula for the signature of a manifold with ∂? For simplicity assume W, ∂W is a Riemannian manifold with Riemannian collared boundary. We want a formula of the sort:

$$\mathrm{sign}(W) = \int_W L \, dVol + ?$$

The integral makes sense because the L form can be prescribed in terms of the curvature. Since signature for manifolds with boundary is not multiplicative in covers there is no local expression that one can integrate to give a formula for which ? = 0. In fact, on reflection ? only depends on ∂W, not on W itself because sign(V∪W) = sign(V) + sign(W)[9] (see the analogous discussion of lens spaces in lecture 1).

Theorem (Atiyah-Patodi-Singer index theorem). Let B be the self adjoint operator on even forms on a manifold V^{2l-1} given by $B\varphi = i^l(-1)^{l+1}(*d-d*)\varphi$. One can compute ? above, assuming that 0 is not an eigenvalue of B, from $\eta(0)$ where

$$\eta(s) = \Sigma\, \mathrm{sign}(\lambda)/\lambda^s$$

where the sum is taken over all the eigenvalues (which are necessarily real).

One can now consider $\eta(0)$ as an invariant of a Riemannian manifold which is not necessarily a boundary. More interestingly, APS observed that as a consequence of their index theorem, there are combinations of η's for slightly varying operators which are differential topological invariants[10]: i.e. independent of the metric.

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[8]This application has been subsequently treated by Higson and by Farber and Levine (in different cases) without using singular spaces. Nonetheless, I hope that the original treatment sketched here at least confirms philosophically the contention that it is valuable to enlarge one's perspective to include singular spaces when studying invariants of smooth manifolds.

[9]Using cobordism invariance of intersection signature for Witt spaces give a proof of this (Novikov) additivity property of signature.

[10]In fact with a bit more work, they are topological invariants.

Let $\rho: \pi_1(V) \longrightarrow U(n)$ be a unitary representation that describes a flat bundle. We can twist the signature operator by ρ, and get an invariant

$$\tilde{\eta}_\rho(V) = \eta(0) - 1/n \cdot \eta_\rho(0)$$

Since $\mathrm{sign}_\rho(V\times I) = n\cdot \mathrm{sign}(V\times I) = 0$, by putting a suitable metric on the cylinder one checks that the APS-invariant $\tilde{\eta}_\rho(V)$ is a smooth invariant. It is not a homotopy invariant: indeed, one can distinguish lens spaces using it.

On the other hand, Walter Neumann showed that $\tilde{\eta}_\rho(V)$ is a homotopy invariant when the fundamental group of V is free abelian. He even gave a homotopy invariant formula for it. It seems conceivable that it is always a homotopy invariant for manifolds with torsion free fundamental group.

On the other hand, simple bordism arguments show that for finite groups this difference must be rational. Let me now sketch the proof of the following theorem:

THEOREM . If V and V' are homotopy equivalent then for any representation ρ, $\tilde{\eta}_\rho(V) - \tilde{\eta}_\rho(V') \in \mathbb{Q}$.

Remark: This theorem is in some sense an analogue of Cheeger and Gromov's conjecture on rationality of geometric signatures for complete manifolds with finite volume and bounded geometry (proven in some cases by Rong). It and its proof are (in grosso mode) somewhat analogous to Reznikov's proof of Bloch's conjecture.

The idea of the proof is this. If V and V' were cobordant by a cobordism W whose fundamental group was isomorphic to $\pi_1(V)$, then the APS theorem would show that $\tilde{\eta}_\rho(V) - \tilde{\eta}_\rho(V') = \mathrm{sign}(W) - 1/n \cdot \mathrm{sign}_\rho(W)$ which is certainly rational. (In fact, if π is torsion free than one can hope that the latter is 0 .)

Unfortunately, this need not be true even if V is simply connected. However, $[V] \in \mathrm{Witt}(B\pi)$ has a chance of being rationally a homotopy invariant. Indeed, after Siegel's thesis, this is a restatement of the Novikov conjecture. Cheeger's work would then enable us to perform the above argument using a Witt cobordism in place of a manifold.

Despite all the progress, the Novikov conjecture is not known for all groups, but it is known for enough of them. An argument using the real algebraic variety structure on Hom(π:U(n)) shows that every representation can be deformed to one with real algebraic entries. Farber and Levine computed how η changes as one moves along a curve in this variety by an explicit homotopy invariant formula. So, by a deformation argument (that follows from the Tarski-Seidenberg theorem) for our problem one can suppose that the image of the representation, and for all practical purposes π, is a subgroup of $GL_n(\bar{\mathbb{Q}})$

For finitely generated subgroups of $GL_n(\bar{\mathbb{Q}})$ we do know the Novikov conjecture[11]. The reason is that the current technology knows how to make good use of nonpositive curvature conditions. F.g. subgroups of are discretely embedded in $GL_n(\mathbb{R}) \times \ldots \times GL_n(\mathbb{Z}_p)$ (where $\mathbb{Z}_p$ denotes the p-adics) for a suitable finite set of primes. This gives a proper discontinuous action on a product of a Hadamard manifold and Euclidean Bruhat-Tits buildings, i.e. on a product of metric spaces of nonpositive curvature.

To explain this in more detail would take us rather far afield. However, it is worth noting that the same ideas that control the values of the APS invariant can be used to define a variant of them, "HIGHER APS INVARIANTS" or "HIGHER ρ-INVARIANTS", useful for distinguishing certain homotopy equivalent manifolds. (We have in mind examples like aspherical manifolds crossed with lens spaces. If they have vanishing Euler characteristic and signature then the usual Reidemeister torsion and APS invariant proofs won't work. But, in lecture 1, we saw that at least a circle doesn't make non isometric lens spaces isomorphic.)

Suppose that π is of the form $\Gamma\times\Delta$, where Δ is finite and suppose that V^n is acyclic with respect to $\mathbb{A}\Gamma$ where $\mathbb{A}$ is the augmentation ideal of $\mathbb{Q}\Delta$. (An example to have in mind is a manifold of the form $X\times Y$ where $\pi_1 X \cong \Gamma$ and $\pi_1 Y \cong \Delta$ and the group Δ acts trivially on the homology of the universal cover of Y (i.e. Y is a simple space in the sense of Milnor's article on Whitehead torsion.)

[11]Here the analogy to Reznikov is a bit forced. Whereas we view Novikov as a weak rigidity result proven by the techniques of [FW1,2] or from a different point of view [KS], Reznikov uses the harmonic map techniques that have been also quite successful in proving differential geometric rigidity theorems.

The aim is to define an invariant of V which is an element of $L_{n+1}(\mathbb{A}\Gamma)/L_{n+1}(\Gamma)\otimes\mathbb{Q}$, which can be seen to be quite large for groups for which the Novikov conjecture is known. (e.g. for $\Delta = \mathbb{Z}_p$ one would get, at least, a number of copies, rationally of $K_{n+1}(B\Gamma)\cong K_{n+1}(B\pi)\otimes\mathbb{Q}$.)

To do this, we need to assume the Novikov conjecture anyway. The idea is this. Assuming the Novikov conjecture and using Siegel's theorem, [V] in $\Omega^{Witt}(B\pi)$ can be thought of as a signature. However, the acyclicity assumption implies that V and its Δ cover have the same $\mathbb{Q}\pi$ homotopy type[12], which would violate the multiplicativity of [ ] in $\Omega^{Witt}(B\pi)$ unless [V] = 0, which it therefore must be.

Now, once we have a Witt coboundary with coefficients in $\mathbb{Q}\pi$, we would like to take its signature, etc. Unfortunately, for nontrivial fundamental groups there is no theory of signature for manifolds with boundary. However, by hypothesis, from the point of view of $\mathbb{A}\Gamma$ the boundary is acyclic. So one does get an element of $L_{n+1}(\mathbb{A}\Gamma)$ -- at least assuming that one can assign algebraic Poincare complexes to Witt spaces, for arbitrary $\mathbb{Q}$-algebras. This follows from the material on sheaves I've left to the appendix. Finally one has to get rid of the indeterminacy in the nullcobordism, which is why one has the quotient $L_{n+1}(\mathbb{A}\Gamma)/L_{n+1}(\Gamma)\otimes\mathbb{Q}$.

Remark: One nice theoretical application of these higher-APS type invariants came up in joint work with Alex Nabutovsky: even for manifolds with a given homotopy equivalence between them, one cannot decide whether or not they are (PL, say) homeomorphic.

Remark: John Lott has shown that for groups of polynomial growth the higher APS type invariants defined here can be defined analytically and extend to other elliptic operators. One does not know this more generally (although I would hope that a more K-theoretic approach should exist that would work more easily, although less in the spirit of spectral η invariants).

<u>Appendix</u>: Sheaves

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[12]One of the things one must verify is that the Novikov conjecture for homotopy equivalence implies it for rational equivalence, but it does because of an algebraic localization theorem of Ranicki.

The introduction of sheaf theoretic techniques into intersection homology is very useful. We implicitly used it in describing why Witt spaces should have Poincare duality. The "Deligne construction" of the intersection chain complex applies equally well in characteristic p where our usual geometric models make no sense. Goresky and MacPherson used this and axiomatic characterizations of IC as computational tools enabling proofs of Kunneth and Lefshetz hyperplane section theorems. Finally, IC is an important example of a self dual sheaf, the aggregate of which up to a suitable cobordism relation give a model of $H_*(; L)$ which is also equivalent to the normal invariants in surgery (away from 2) and also equivalent to Witt space bordism., and away from 2, elliptic operators[13].

The derived category forms the appropriate language for the sheaf theoretic construction of IC. Two *bounded* complexes of sheaves $\underline{C}$ and $\underline{D}$ are equivalent in the DERIVED CATEGORY if there is a third complex $\underline{E}$ with maps $\underline{C} \longrightarrow \underline{E} \longleftarrow \underline{D}$ which are QUASIISOMORPHISMS, that is, which induce homotopy equivalences on every stalk. A morphism in the derived category is represented by a morphism of complexes of sheaves up to a quasiisomorphism. In the derived category, it is often convenient to replace complexes of sheaves by injective resolutions, so that the homological algebra becomes nicer. The topologist might find it useful to think about the analogous homotopy category of spaces "over X", i.e. equipped with maps to X.

A very important complex of sheaves, D_X, was introduced by Verdier. The DUALISING COMPLEX has the property that its stalk cohomology at x is the local cohomology of X at x, $H^i(X,X-x)$ and is used roughly to change homology to cohomology. (In fact, D_X is equivalent to the local singular chain complex, for an open U we get $C_p(X,X-U)$.) Global cohomology of X with coefficients in the dualising complex is ordinary homology. If $\underline{E}$ is in the derived category, we define $D\underline{E} = \mathrm{Hom}(\underline{E} : D_X)$ and it is called the DUAL OF $\underline{E}$. Verdier duality has a lot in common with Spanier-Whitehead duality, just performed more locally. (The reader familiar with Zeeman's dihomology will no doubt feel a sense of deja vu.)

[13] with a suitable real structure.

A map $f:X \longrightarrow Y$ induces several functors on derived categories. We can take the derived functors of push forward and pullback Rf_* and Rf^*, and we can take analogous proper analogs $Rf_!$ and $Rf^!$. These latter have nice descriptions in terms of dualising: $Rf_! = DRf_*D$. For proper maps, $Rf_* = Rf_!$ and for closed inclusions $Rf^* = Rf^!$, but in general these notions disagree. VERDIER DUALITY asserts that always

$$Rf_*RHom(\underline{A},Rf^!\underline{B}) \cong RHom(Rf_!\underline{A}, \underline{B}).$$

Verdier duality implies that the induced pairing of hypercohomology groups $H^i(\underline{A})\otimes H^{-i}(D\underline{A}) \longrightarrow H^0(D_X) \longrightarrow \mathbb{F}$ is perfect.

It is quite feasible to do homological algebra in the derived category; indeed it is built for that. The reader should realize that it is not an abelian category, so that one cannot take kernels and cokernels. What replaces this is the DISTINGUISHED TRIANGLE. It is the obvious extension of the situation one gets by considering a mapping cone, where one gets in addition to the usual degree 0 maps of complexes a degree one map from the cone to the domain of the map.

The final ingredient in the construction is the collection of TRUNCATIONS of complexes.

$$(\tau_{\leq p}\underline{A})^n \quad \begin{array}{ll} = \underline{A}^n & \text{if } n\leq p-1 \\ = \underline{\ker}\, d^n & \text{if } n=p \\ = 0 & \text{if } n\geq p+1. \end{array}$$

There is a similar definition for COTRUNCATION. Furthermore, one can truncate over a closed subset by sheafification of the result of truncating only on the open sets that touch the closed set. All of these constructions pass to the derived category.

THEOREM ([GM1 II]) Let X be a stratified space and let $U_k = X-X_k$ be a filtration by open sets, and i_k the inclusion $U_k \longrightarrow U_{k+1}$. Let
$\underline{P} = \tau_{\leq p(n)-n}Ri_n{}^* \ldots \tau_{\leq p(2)-n}Ri_2{}^*\underline{\mathbb{F}}_{X-S}\,[n]$.
The final $[n]$ is a shift by n. Then this is quasiisomorphic to $IC^p(X)$.

The proof, in the spirit of Eilenberg and Steenrod, is axiomatic. There are various axiomatizations of intersection homology with the following ingredients: constructibility, normalization ($\mathbb{F}$ on the nonsingular part, i.e. the analogue of the dimension axiom), a lower bound axiom for the vanishing of homology, an upper bound axiom, and finally one has some choice. One can assume an attaching axiom that tells how the pieces are connected. Alternatively, one can use support and cosupport conditions restricting the dimensions of pieces with high dimensional local homology and cohomology.

I will not explain how one proves these uniqueness theorems in detail. Roughly speaking, one does obstruction theory in the derived category. The basic lemma for producing maps is the following:

LEMMA [GM1II]: Suppose that $\alpha: \underline{A} \longrightarrow \underline{C}$ and $\beta: \underline{B} \longrightarrow \underline{C}$ are morphisms in the derived category, that $H^i(\underline{A}) = 0$ for $i \geq k+1$, and that β is a cohomology isomorphism for $i \leq k$, then α has a unique lift $\underline{A} \longrightarrow \underline{B}$.

The proof is not that hard and resembles in an algebraic language the proof of the analogous results for extending maps between spaces with low dimensional cofibers to spaces with highly connected fibers.

The duality of IH^p and IH^q is now deduced by verifying the q axioms for the dual of IC^p.

The sheaf-theoretic construction of the Poincare duality in IH emphasizes the significance of self duality of IH on the sheaf level. I would like to continue the development by explaining how to associate a characteristic class to any self dual sheaf, so that the L-classes obtained via the Cheeger-Goresky-MacPherson procedures are equal to the (Pontrjagin character of a K-homology) class associated to the IC sheaf.

I should remark that my account is based on the Comptes Rendus note [CSW], which also deals with equivariant sheaves. Steve Hutt, though, has given a very elegant self contained account of the nonequivariant theory (he calls these "Poincare sheaves") in a forthcoming paper.

Definition: A sheaf $\underline{E}$ on X is SELF DUAL if there is the following data: A quasiisomorphism $\varphi:\underline{E} \longrightarrow D\underline{E}$, a homotopy from φ to $D\varphi$ (identifying $DD\underline{E}$ with $\underline{E}$), a homotopy of this homotopy to its dual, etc.

Remark: If the field $\mathbb{F}$ has characteristic $\neq 2$, then one just needs a homotopy from φ to $D\varphi$. We will assume for simplicity that we are in this case.

If X is a Witt space then $\underline{IC}(X)$ is self dual. The higher coherencies are produced using the obstruction theory lemma.

PROPOSITION: There is a functor from self dual $\mathbb{F}$ sheaves on X $\longrightarrow$ "controlled homotopy equivalence classes of controlled algebraic Poincare $\mathbb{F}$ complexes on X", whose Witt group is according to Quinn and Yamasaki $\cong H_*(X ; L(\mathbb{F}))$.

COROLLARY: One can associate a symmetric signature of a Witt space in $L^*(\mathbb{F}\pi)$. It has all of the usual cobordism invariance properties.

We had already invoked this at the end of lecture 2. This discussion also implies the topological invariance of the L-classes and Δ classes produced by IH methods generalizing Novikov's theorem.. (Even more punch comes from the equivariant case, see my book for more discussion.)

We will be discussing controlled topology in more detail in the next lecture.

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[14]This paper was accepted a few years ago, but apparently lost....

LECTURE THREE: Application to classification problems

In lecture one we discussed various sorts of applications of characteristic classes in the nonsingular case: to embedding, to cobordism, and to classification. Now we will turn to classification, but will also indirectly shed light (or darkness) on embedding. (Of course, some amount of cobordism was discussed in the previous lecture.)

Our model theorem is the main result of Browder-Novikov theory which asserts that the closed manifolds homotopy equivalent to a given simply connected M are in a finite to one correspondence with $\oplus H^{4i}(M;\mathbb{Q})$. Other than the relation imposed by the Hirzebruch signature formula a "lattice" in $\oplus H^{4i}(M;\mathbb{Q})$ is realized by such manifolds. (We had seen a bit of this, ad hoc, in the first lecture when we constructed homotopy $S^4 \times S^n$s.)

Of course, as we have already seen, the non-simply connected case is more complicated and invariants not determined by characteristic classes can be infinitely varied within a (simple) homotopy type. A formally complete answer is given by surgery theory -- or better surgery theory tells one how to reduce the general classification problem to problems just involving the fundamental group.

In any case, back to simply connected manifolds. In the PL and Topological categories, there is a nicer theorem due to Sullivan. First a bit of notation:

$$S(M) = \{(N,f) \mid f:N \longrightarrow M \text{ is a homotopy class of homotopy equivalence.}\} \,/\, \text{(PL) homeomorphism}$$

S(M) is called the structure set of M. It turns out that S(M) has a group structure which is useful but not critical in understanding Sullivan's result.

THEOREM[15]: If M^n is simply connected, $n>4$, then $S(M) \cong \widetilde{K}\widetilde{O}^0(M)\otimes\mathbb{Z}[1/2] \cong \widetilde{K}\widetilde{O}_n(M)\otimes\mathbb{Z}[1/2]$.

[15] There is a slight lie here as a computation with spheres shows. One can be off by a $\mathbb{Z}$. This is accounted for by ANR homology manifolds by [BFMW]. Otherwise, one has to be slightly more careful in the statement to guarantee that one only obtains manifolds in the structure set S.

The cohomological version compares well with the Browder-Novikov theorem (via the Pontrjagin character), but the homological one is more suitable to generalization. (The "Poincare" duality implicit in the theorem is implemented by $\Delta(M)$ constructed in lecture one; it is an orientation class.) The reduction in K-theory is a reflection of the Hirzebruch formula -- which boils down to the whole (away from 2) surgery obstruction. $L_i(e) \cong KO_i(\text{point})$. The surgery exact sequence, with this identification, is the exact sequence of the pair (M, point).

There are versions for manifolds with boundary: both rel ∂ and not rel ∂. If one works rel ∂ then one knows that the difference between the Pontrjagin classes is canonically trivialized (by the given homeomorphism) on the boundary, so the classification involves $\widetilde{K}\widetilde{O}^0(M, \partial M)\otimes\mathbb{Z}[1/2] \cong \widetilde{K}\widetilde{O}_n(M)\otimes\mathbb{Z}[1/2]$. For the not rel ∂ classification, there is no boundary condition on the tangent bundle, so one obtains $\widetilde{K}\widetilde{O}^0(M)\otimes\mathbb{Z}[1/2] \cong \widetilde{K}\widetilde{O}_n(M, \partial M)\otimes\mathbb{Z}[1/2]$. For years I have tried to propagandize for the notation $S(M, \partial M)$ to stand for the not rel ∂ structures, because it fits better with both the homological notation and the surgery obstruction theory. One recognizes within the parentheses all of the things one gets to change within a structure. Alas, the ingrained habit of reading pairs as rel coming from first year homology theory is too strong and people always seem to get confused if I try too hard.

Our main goal in this lecture is to sketch the following theorem of Cappell and mine:

THEOREM: Let X be a stratified space with all strata and links of strata simply connected and with all strata of even codimension. Then there is an isomorphism

$$S(X) \cong \bigoplus_{\substack{V \text{ a closed stratum} \\ \text{of } X}} \widetilde{K}\widetilde{O}_V(V) \otimes \mathbb{Z}[1/2]$$

The isomorphism is implemented by the Goresky-MacPherson class discussed in the last section. Elements of S(X) are stratified spaces with stratified maps to X that are homotopy equivalences in the stratified category. A statified map is one that preserves OPEN strata. In various problems one might want to deal with more general maps and homotopy equivalences. In that case one has to do some work: this special problem seems to have a coherent answer. Hopefully whatever other stratified problem one is confronted with can be related to it.

Very evidently, this theorem has applications to circle and torus actions. I will not make these explicit here.

When X is a closed manifold this is Sullivan's theorem. And, as for Sullivan's theorem there are non-simply connected generalizations. It is relatively straightforward to obtain the version where the strata are allowed be non-simply connected; allowing this in links changes the story much more extremely. After all for orbifolds this has the effect of replacing KO by KO^G where the latter is a reflection of the fundamental group of the links of the lower strata in the top one. All of this is discussed in my book.

Another point worth making is that the closed strata that occur in the theorem are *not* necessarily manifolds. Consequently, the fact that KO-*homology* enters is not cosmetic as it might seem in Sullivan's formulation of the classification theorem. Indeed there, one feels like the cohomological formulation is more natural. (Indeeder: for smooth surgery there is no homological formulation so that the surgery exact sequence becomes a sequence of groups and homomorphisms, as it is in Top. There, the cohomological formulation involving a mysterious infinite loopspace F/O is all that we have.) By the time we move on to stratified spaces, we are forced calculationally, rather than just by reasons of mathematical naturality, to homology.

One can weaken the condition of X having only even codimensional strata to being Witt with all strata Witt. (One can even get a theorem without that, if one works rel the non-Witt strata.)

However, one cannot dispense with some such condition. Consider the case of manifolds with boundary. The answer predicted by the theorem would involve $KO(W) \times KO(\partial W)$ which conflicts with what we know to be the case $KO(W, \partial W)$. Of course both of these "answers" fit into exact sequences with $KO(W)$ and $KO(\partial W)$ being the other terms, which is not accidental, and to which the reader might want to return at the end of this lecture.

Another interesting case to think about is $S(W, V)$ where V is a locally flat submanifold of W, i.e. the space we are concerned with is W, but we have stratified it with an extra stratum. The theorem gives $S(W, V) \cong S(V) \times S(W)$ if the codimension of V is at least three. (Actually, as stated the theorem gives this away from the prime 2, but the same reasoning gives a more refined theorem that includes this statement at 2.) This is equivalent to a classical theorem of Browder, Casson, Haefliger, Sullivan, and Wall, (BCHSW) and can be found in Wall's book. It implies that an embedding of V in W (in codimension greater than 2) determines a well-defined isotopy class of embeddings of any manifold homotopy equivalent to V into any manifold homotopy equivalent to W.

In any case it points out the failure of the rational characteristic classes extended to PL or Top to prevent embeddings. Consider $W = S^{2n+4}$ and $V = S^n \times S^n$, where n is a multiple of 4. $S(V) \cong \mathbb{Z}^2$ detected by a Pontrjagin class. None of the nontrivial elements (if n is large) can embed in W smoothly since the normal bundle is not big enough to account for such a high Pontrjagin class, but the PL embeddings are there.

We can view the theorem above on stratified spaces as being the combination of two statements, each of which extend special cases of the theorem.

<u>Claim 1 (Decomposition)</u>: Let X be a stratified space with all links of strata simply connected and with all strata Witt spaces. Then there is an isomorphism

$$S(X) \cong \bigoplus_{\substack{V \text{ a closed stratum} \\ \text{of } X.}} S(V, \text{rel sing}(V)) \otimes \mathbb{Z}[1/2]$$

<u>Claim 2 (Rel sing calculation)</u>: Let X be a stratified space with the links of its singular set in the top stratum simply connected, and with simply connected top stratum, then

$$S(X, \text{rel sing}(X)) \cong \widetilde{K}\widetilde{O}_X(X) \otimes \mathbb{Z}[1/2]$$

For those who know surgery theory, even without the simple connectivity of the stratum, the (integral) result is that $S(X$, rel sing(X)) is the fiber of the assembly map $H_X(X\,;L(e)) \longrightarrow L_X(\pi_1(\text{top}$ stratum of X)) (i.e. fits into a homological surgery exact sequence).

Amazingly enough, all of the detailed geometry of the singularities an the like is irrelevant to the calculation: just the abstract homotopy type of the space (assuming the condition of simply connected links).

We will explain Claim 2 first.

Let us see first what's involved when X has a single isolated singularity. This singularity has the form cM for an appropriate simply connected manifolds M (In the PL case this is by definition; for Top this follows from the theorem of Browder-Livesay-Levine, generalized by Siebenmann in his thesis that puts well defined boundaries on open manifolds with tame simply connected ends.) Let W be the "closed complement" of the singular point.

$$\begin{aligned} S(X \text{ rel sing}) &\cong S(W \text{ not rel } M) \cong KO_X(W, M) \otimes \mathbb{Z}[1/2] \\ &\cong KO_X(X, \text{rel point}) \otimes \mathbb{Z}[1/2]. \cong \widetilde{KO}_X(X) \otimes \mathbb{Z}[1/2] \end{aligned}$$

If there are several components of fixed point the calculation is similar. One has to keep track of the "reductions" in the K-theory. Recall these were due to simply connected surgery obstructions: when W has a single boundary component, the K theory is not reduced. (There is no homotopy invariant bordism invariant signature of manifolds with boundary.) When there are several, these boundaries can have signatures, which are subject to only a single relation: the sum of their signatures is zero. This ends up in changing the guessed answer from KO_X(X, rel all the singular points) to $\widetilde{KO}_X(X)$ (all but one of the boundary components, so to speak, still leave a K(pt) around).

Note that if we did not have simply connected links then there would be an obstruction arising from Siebenmann's thesis and one would have a divergence between Top and PL (not due to the Kirby-Siebenmann obstruction). [Of course, we would also have to use non-simply connected surgery as well, but that would only affect the answer: not the method of analysis.]

The rest of the proof requires with dealing with the case of just a single positive dimensional singular stratum, and then inducting on the number of these. So let us assume that there is just one singular stratum, a manifold we call Σ.

There are three related ways to do this (the benefit of waiting to explain proofs):

i) Do the PL case and compare.
ii) Directly do the analogous geometry in Top
iii) Use "controlled surgery"

Let me explain the last first, since it's quickest, and then the first since it's the most intuitive.

One can lecture forever about controlled topology. There are many variants: The most familiar are controlled or ε-controlled, bounded control, and continuous control at ∞. In all cases one has an old fashioned geometrical problem, say a homotopy equivalence between spaces one want to make into a homeomorphism or an open manifold one wants to put a boundary on or an h-cobordism one wants a product structure on. However, one assumes some extra data here, which one wants to require of the solutions, as well, and that is the "control". One has a control space Z with maps of all of the initial data into Z, which one wants to "almost preserve" when solving the problem. One does not insist that the map to Z be preserved on the nose: that is "fibered topology" and a different story entirely. One allows the fibers over nearby points to intermingle (or maybe points a uniformly bounded distance apart -- interesting for noncompact metric spaces) or something like that. For discussions, see the beginning sections of Quinn's papers, Chapter nine of my book, or Ferry's forthcoming CBMS lectures on homology manifolds.

Since Σ is an ANR we have a map $X-\Sigma \longrightarrow c'\Sigma$ (where $c'\Sigma$ denotes the *open* cone on Σ). The idea is to set up an isomorphism

$$S(X \text{ rel } \Sigma) \cong S^{c}(X-\Sigma \downarrow c'\Sigma). \qquad (*)$$

The control here is over $c'\Sigma$. The whole difficulty is getting control here as one goes to ∞, since one can always rescale the cone. The structure at ∞ reflects the fact that the map $X'-\Sigma \longrightarrow X-\Sigma$ given by a stratified homotopy equivalence rel Σ is not just a proper homotopy equivalence: it extends continuously when one glues Σ on. This is the "continuously controlled at ∞" variant. It turns out that any version of control would work for setting up this isomorphism -- but now that the continuously controlled version is available it is the most efficient for this problem.

Now one invokes the surgery exact sequence[16]:

$$S^c(X-\Sigma \downarrow c'\Sigma) \otimes \mathbb{Z}[1/2] \cong \text{Fiber}\,(KO_X(X-\Sigma) \longrightarrow KO_X(c'\Sigma)) \otimes \mathbb{Z}[1/2]$$

$$\cong \tilde{\tilde{K}}\tilde{O}_X(X) \otimes \mathbb{Z}[1/2]$$

We have used the identification of the controlled surgery group with $H_X(c'\Sigma; L(e))$ and the latter with $KO_X(c'\Sigma))$ away from 2, in the first line. One also has to verify that the surgery obstruction map is the boundary map in the long exact sequence of a pair[17], but this is also quite easy from general machinery. (By the way, (1) it's easy to jimmy this argument to get the correct integral calculation and (2) one only uses the fact that Σ is a finite dimensional ANR, not a stratification of Σ for this argument. However, for not rel Σ classification, one seems to need Σ to be somewhat nice to even say what spaces the Σ''s arising in the structure set should be allowed to run over.)

Also, one should relate the element of $\tilde{\tilde{K}}\tilde{O}_X(X)$ in Witt cases with the Δ-class of the previous lecture. This is also not that hard and follows from tracing carefully through the definitions.

Now let us turn to approach i) and begin by working in the PL category. (Inverting 2, we don't have to worry about the Kirby-Siebenmann obstruction.) One drawback with this approach is that in PL one has to work with codimension at least 6 because of low dimensional topology difficulties: If M and M' are smooth 4-manifolds the cones c'M and c'M' are PL equivalent iff M and M' are diffeomorphic. Moreover, crossing both of these spaces by any other space (to increase dimension) will not produce any new PL isomorphisms. Links are well defined. So really one should work

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16 Following Quinn's thesis, I am always viewing surgery exact sequences "geometrically" i.e. as statements that certain spaces whose homotopy groups we are interested in is the fiber a surgery obstruction map between a space of normal invariants and a surgery space. Similarly, homology groups are homotopy groups of appropriate spaces (e.g. the geometric realization of the singular chain complex). In practice all of these statements boil down to long exact sequences.

17 For noncompact spaces our homology is always locally finite or Borel-Moore homology. Thus, $KO_X(X-\Sigma) \cong KO_X(X,\Sigma) \cong \tilde{K}\tilde{O}_X(X/\Sigma)$.

with PL pretending that low dimensions behave as if surgery held for them, and then compare Top to that thing.

But we'll still oversimplify and assume we're genuinely PL and just that the codimension of Σ in X is high. Now the analog of the isomorphism (*) is the following more complicated description of $S(X \text{ rel } \Sigma)$ (recall the notation: W is the closed complement of Σ and M is its boundary; M can be viewed as a block bundle over Σ with fiber L, the link of a top simplex):

$S(X \text{ rel } \Sigma) \cong$ elements of $S(W \text{ not rel } M)$ together with a lift of the ∂ element in $S(M)$ to the space of sections of a block bundle $\Gamma(E(\tilde{S}(L){\downarrow}\Sigma)$. (**)

The element of $S(W \text{ not rel } M)$ is obvious enough. What is this block bundle and associated section of it? In less fancy words, we are saying that the structure on W must be given on M the structure of a block bundle over Σ. A block bundle over Σ can be thought of as being given over each simplex Δ of Σ by a manifold homotopy equivalent to $\Delta\times\Sigma$ (or better to $\pi^{-1}(\Delta)$, where π is a projection corresponding to the original given block bundle structure on M over Σ), i.e. an element of $S(\Delta\times\Sigma)$. This is exactly a simplex in the Δ-set description of $\tilde{S}(L)$; in other words what one has specifically is a Δ-section of the block bundle over Σ, which over a simplex Δ is $S(\pi^{-1}(\Delta))$ (= Maps(Δ: S(L)) noncanonically), as advertised.

Now the point of blocked surgery (see [Q1, BLR, CW2]) is that blocked structures can be studied just as effectively as ordinary structures. Basically one has (always using K-theory for away from 2 integrally correct L-functors because of our desire to express our final answers in terms susceptible to relating to a priori invariants):

$$\Gamma(E(\tilde{S}(L){\downarrow}\Sigma). \cong \Gamma(E(kO(L){\downarrow}\Sigma). \longrightarrow \Gamma(E(KO_1(\text{point}){\downarrow}\Sigma).$$

Now $\Gamma(E(KO(L){\downarrow}\Sigma) \cong KO(M)$. (A vector bundle over a bundle is a compatible family of vector bundles over the fibers -- which can be expressed as a section of bundle of collections of vector bundles.) We can also identify

$$\Gamma(E(KO_1(\text{point}){\downarrow}\Sigma) \cong KO^{-1}(\Sigma) \cong KO_{S+1}(\Sigma) = KO_{X-1}(\Sigma).$$

What does this correspond to? Recall the projection formula from lecture one:

Proposition: If $f:M \longrightarrow \Sigma$ is a PL block bundle with no monodromy[18] and fiber L, then $f_*\Delta(M) = \mathrm{sign}(L)*\Delta(\Sigma)$.

In other words, the we know what happens to the Δ class on the outside in $KO_X(X-\Sigma)$ when we take ∂ and push it down to $KO_{X-1}(\Sigma)$ -- it is identified with $\mathrm{sign}(L)*\Delta(\Sigma)$. Therefore, if we look at the *difference* $\Delta(X) - \Delta(X')$ for stratified homotopy equivalent spaces rel Σ, we have the vanishing in $KO_{X-1}(\Sigma)$. Thus, we have an element of $KO_X(X)$. We lose a KO(point) since we started with a *structure* on M, so the signature (i.e. push forward of Δ to KO(pt)) is the same for M' and M automatically, i.e. not as a consequence of the block structure.

With this approach, one then must do a careful induction up the strata, even to get the PL result. One has to be careful with the formulations because we do not yet have a formula for the pushforward of Δ to the base of a "stratified block bundle". One just needs though some homotopy invariance properties of such a formula, which are quite easy to get. On the other hand, in the next lecture we will see that when the base and all fibers are even codimensional spaces, there is a formula, due to Cappell and Shaneson.)

The topological result follows from the PL result because one can see that the end of $X-\Sigma$ over Σ is a "surjective tame end with a trivial "local fundamental group", so that one can put a controlled boundary on $X-\Sigma$, and on the inverse image of each Δ of Σ in that controlled ∂, giving a block structure on that ∂. (See Quinn, also Anderson-Hsiang for an earlier more direct but less conceptual approach.)

Finally, we come to approach ii). This is based on the theory of "manifold approximate fibrations", extensively studied by Chapman, Hughes, and Hughes, Taylor, and Williams.

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[18] The deviation of this formulas correctness in the presence of monodromy can be worked out and does not contribute here. In any case, if we assume the pure strata are simply connected, then there is no monodromy.

A map $\varphi: M \longrightarrow N$ is a MAF if M and N are manifolds and the homotopy lifting property for maps of polyhedra into N almost holds: i.e. holds up to arbitrarily small deviation. Thus a CE map is a good example of a MAF. The main result is that MAFs can be classified in terms of sections of a bundle associated to the tangent bundle of structures on a "local germ". (φ^{-1}(any open ball) is the same, and is the "local germ".) This result is a far reaching generalization of Siebenmann's theorem that a CE map between manifolds is approximable by homeomorphisms. These structures can be studied by surgery theoretic methods (especially stably).

The way the MAF theory enters is via the following result:

THEOREM (Teardrop neighborhood theorem [HTWW]): Every neighborhood of Σ in a manifold homotopically stratified space contains a smaller neighborhood that can be given a well-defined structure of a MAF over $\Sigma\times\mathbb{R}$.

In other words Germ neighborhoods including $\Sigma \longleftrightarrow$ MAFs over $\Sigma\times\mathbb{R}$. In the smooth setting one even has a fiber bundle over $\Sigma\times\mathbb{R}$. The tubular neighborhood maps to Σ by projection and to $\mathbb{R}$ by distance to Σ ("radial distance"). The theorem says that in general one can do the same thing, but not with all points having the same inverse image (fiber bundle) but with all little open balls having the same inverse image (approximate fibration).

The map from MAF's over $\Sigma\times\mathbb{R}$ to neighborhoods goes by gluing Σ onto a MAF: the MAF is the deleted neighborhood of Σ. (The teardrops are visible in a picture of convergent sequences of points in a topology for the union MAF $\cup$ Σ: One sees Σ cry.)

This result gives a pretty geometrical picture of the neighborhoods and can be used to recover all sorts of classification results by using MAF classification (which is more in the spirit of Anderson and Hsiang). This is somewhat more clumsy than approach i), but has some advantages. Probably the most important virtue is that this analysis is correct at the space level, i.e. for parametrised families of neighborhoods, etc. and can be used for studying spaces of homeomorphisms of stratified spaces.

Bruce Hughes has extended all of this to a stratified teardrop neighborhood theorem and has developed a theory of SMAF's with some nice applications to stratified space theory such as a very general local contractibility of homeomorphism spaces theorem.

This is all I want to say about claim 2 (the rel sing classification theorem). Now I'll just say a couple of words about claim 1 (the decomposition theorem).

As before, let us just think about the 2 stratum case and also assume we are PL. A little reflection on (**) makes us realize that we should try to study the problem of "base change"

Problem: Suppose one has $M \longrightarrow \Sigma$ a block bundle and a homotopy equivalence $f:\Sigma \longrightarrow \Sigma'$. Can one homotop the composite $M \longrightarrow \Sigma \longrightarrow \Sigma'$ to a block bundle map.

We want to show that under our Wittness conditions this is always possible and that moreover there is a 1-1 correspondence with the block structures over Σ and those over Σ'.

Again, the projection formula gives an obstruction which blocked surgery implies is the only one (assuming no monodromy and simple connectivity[19] of L). According to the projection formula $\mathrm{sign}(L)\Delta(\Sigma)$ can be computed from $\Delta(M)$, and if the composite were homotopic to a block bundle map, so would $\mathrm{sign}(L)\Delta(\Sigma')$. In other words, a necessary condition for base change is that

$$\mathrm{sign}(L)\Delta(\Sigma') = \mathrm{sign}(L)f_*\Delta(\Sigma)$$

Now, if we are in the Witt case then L is either odd dimensional or, if L is even dimensional, there is no middle dimensional homology. In other words, sign(L) is canonically 0. IH allows one to extend this argument over an induction.

Actually, once one has IH, and the Δ-classes, it is not hard to argue more directly. One can always filter S(X) in terms of strata and get relative terms isomorphic to S(V rel sing (V)). This can be formalized as a spectral sequence. Essentially having the Δ classes defined to begin with gives a map $S(X) \longrightarrow S(V \text{ rel sing}(V))$ for each closed stratum. This gives a collapse of the relevant spectral sequence and a calculation of the E^∞ term, e.g. S(X).

[19] The replacement theorems for group actions due to Cappell and me are the results of cases of base change with non-simply connected fiber.

Final Remark: It turns out that for general stratified space X there is a characteristic class living in an appropriate cosheaf homology group $H(X ; L^{BQ})$ where L^{BQ} is a cosheaf of surgery spectra locally generalizing the groups of Browder and Quinn. In terms of these there is a nice surgery theory, etc. Once one has all these, the main result of this lecture could be phrased as a calculation (away from 2) of L^{BQ} of a space with even codimensional strata and simply connected links.

On the other hand, proving the general result ends up using the geometry that we had to do for this particular case anyway. Formally deducing the theorem from the general machinery would have been very efficient, but would have obscured all of the ideas.

References for Lecture Three

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LECTURE FOUR: Toric Varieties and Lattice Point Problems

The motivating problems in this work are extremely concrete. They concern lattice points inside convex lattice polytopes. That is, we take a finite set of lattice points in $\mathbb{Z}^n \subset \mathbb{R}^n$ and consider its convex hull K.

One question we can care about is the number of lattice points inside of K. Our obvious first approximation is vol(K). This is not quite correct even in one dimension: $\#\{v \in \mathbb{Z} \mid a \leq v \leq b\} = b-a+1$. In two dimensions, we have

PICK'S THEOREM: The number of lattice points inside a planar convex lattice polyhedron K = Area(K) + 1/2 "Perimeter" + 1, where the "length" of an edge is 1 less than the number of lattice points on that edge.

In higher dimensions, one wants explicit formulae. In dimension three one would expect there to be contributions from faces and edges. The contribution of an edge can very well involve the angle of the faces coming out of it.... although nothing like that arose in Pick's theorem.

A general theorem of Erhart asserts that if $r \in \mathbb{N}$ then the number of lattice points in rK is a polynomial $p_K(r)$. He also conjectured (and MacDonald proved) that the number in the interior of rK is $(-1)^n p(-r)$. This turns out to be a simple consequence of Serre duality.. In any case, we call p the ERHART POLYNOMIAL of K, and we would like to compute its coefficients.

Another, deeper problem is to find a formula for summing the values of a function f over the lattice points in K. This is of interest even in the case of an interval, and the nontrivial answer is the famous EULER-MACLAURIN FORMULA (valid at least for polynomials):

$$\sum_a^b f(v) = \int_a^b f(x) + 1/2[f(a)+f(b)] + \sum_2^\infty (-1)^r B_r/r![f^{(r-1)}(b) - f^{(r-1)}(a)\}$$

where B_r is the r-th Bernoulli number. In other words the deviation of the integral to be the sum is given by n explicit infinite order partial differential operator evaluated on the boundary faces.

A couple of years ago, by now, Cappell and Shaneson, building on the work of many other mathematicians, especially algebraic geometers, answered these questions. In this lecture, I would like to sketch or motivate some of the ideas that entered this work. Because of the place that this area occupies, as we shall see, at the crossroads of combinatorics, algebraic geometry, symplectic geometry, and topology, it is clear that many more beautiful theorems remain to be proved.

I should point out before beginning that there are several fine expository accounts of algebraic geometric views of this general subject (Danilov, Oda, Fulton) and a recent nice symplectic book by Guillemin. There will necessarily be some overlap with these accounts, but I will also skip over many things that are covered there. I apologize both to those whose work I am slighting (or worse) and to those who in getting their first acquaintance in this very brief lecture will be misled by getting much too small a glimpse at the area.

The first large step in the solution of this problem is to associate to each polytope a projective variety whose Todd class encodes the Erhart polynomial. Thus we introduce the heroes of this story, the TORIC VARIETIES.

Topologically the picture is quite simple. The space $X(K)$ is of the form $T \times K/\sim$. Each face F of K is defined by a hyperplane in $\mathbb{Z}^n$, i.e. by a primitive integer vector $v_F \cdot w \leq C$. We have a circle $\mathbb{R}v_F/\mathbb{Z}v_F$ inside of T. Identify points in $T \times F$ $(t,f) \sim (t'f')$ if $f=f'$ and $t^{-1}t' \in \mathbb{R}v_F/\mathbb{Z}v_F$. Points on codimension two faces will have identifications along T^2s, etc. It is easy to see that this is a manifold outside a set of codimension at most 4, i.e. that the codimension one faces correspond to nonsingular points in the toric variety.

Note that the fixed set of any subtorus is another toric variety. The fixed sets for the whole group correspond exactly to the vertices. Thus, the number of vertices = χ(toric variety).

Clearly an interval in $\mathbb{R}$ corresponds to S^2. A square or rectangle in the plane then corresponds to $S^2 \times S^2$. An isosceles triangle gives $\mathbb{C}P^2$; indeed an n-simplex gives $\mathbb{C}P^n$. On the other hand, your random polygon in the plane gives a nonmanifold (actually an orbifold).

The order of the local fundamental group at the isolated singularities (which must correspond to vertices) is simply $\det(v_F, v_{F'}) = \# (\mathbb{Z}^2/<v_F, v_{F'}>)$ as one sees by playing around with the local picture. In general to get a manifold it is necessary and sufficient that each vertex lies on exactly n faces, and the associated primitive vectors generate $\mathbb{Z}^n$. Guilleman calls such polytopes DELZANT because of Delzant's theorem below.

It is not too hard to see all of these spaces as algebraic varieties (and hence as symplectic spaces, whatever that would mean in the singular case) by an explicit gluing construction: I leave it to the reader to guess how this goes or to check with Danilov, Oda, or Fulton. (Hint: you only need monomials to glue coordinate charts together. As another hint: Explicitly work out the case of $\mathbb{C}P^2$.) These form a nice explicit class of interesting spaces where one should be able to compute everything one ever wants. For instance, it is a worthwhile exercise to compute the cohomology algebra of X(K).

The algebraic geometric view allows us to consider such objects as holomorphic line bundles, their holomorphic sections, and the whole Riemann-Roch game. In the symplectic view, where one has a symplectic manifold with a symplectic automorphism group, there is the apparatus of moment maps, geometric quantization, etc. And, needless to say these ideas have been known to interact with each other.

Remark: Actually, for many purposes it is best to work in the dual lattice, and associate toric varieties to somewhat more general objects known as fans. In the algebraic case the "torus" $(\mathbb{C}^*)^n$ acts. In the simplectic case, the torus action is Hamiltonian. Although not strictly speaking necessary, I should mention a few words about torus actions in symplectic geometry.

If (M, ω) is a symplectic manifold, and G acts symplectically, one can try to build an equivariant map $M \longrightarrow \text{Lie alg}(G)^*$, where G acts on the dual lie algebra by the coadjoint action. The map should have the property that for any one parameter subgroup of G, when you look in TM at the tangent vector to the curve and se ω to get a dual one form that should integrate up to an additive constant to the map. This is always possible by a theorem of Marsden and Weinstein if G is semisimple, and for G a torus if $H^1(X) = 0$. (One needs this for integrals to be path independent.) The map is called the MOMENT MAP.

Note that if T is a torus, the coadjoint action is trivial, so the moment map actually factors through M/T. (This corresponds to the idea in basic physics that, viewing phase space as a symplectic manifold, symmetries give rise to conservation laws. The moment map in the case of translational symmetry gives momentum in $\mathbb{R}^3$, while for rotational symmetry one is led to angular momentum. The moment map tells you about what quantities are preserved.)

The CONVEXITY THEOREM of Atiyah and Guillemin-Sternberg asserts that the image of the moment map for a Hamiltonian torus action on a compact symplectic manifold is necessarily convex.

Note that because of the moment map, the smallest dimensional symplectic action of T^n must be on a 2n-manifold. DELZANT'S THEOREM is that the image of the moment map on a smooth symplectic manifold is always a Delzant polytope. The action is necessarily symplecteomorphic to the action constructed directly from the polytope as above. In any case, we are interested in the singular toric varieties as well.

The connection between lattice points and toric varieties comes via sections of certain holomorphic line bundles. The key point to have in mind is that associated to codimension one subvariaties (called DIVISORS, in algebraic geometrize) one has a complex line bundle. (We topologists know that any codimension two oriented submanifold is the transverse inverse image of $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$ for a suitable map into $\mathbb{C}P^n$ for n large, by the Pontrjagin-Thom construction, and those homotopy classes correspond to complex line bundles. This is a more precise version of this.) Again most elementary algebraic geometry books will explain this: see e.g. Griffiths and Harris.

The divisors in our toric variety that interest up correspond to the faces. Each face in the polytope gives us a line bundle $\mathcal{O}_F$ over the toric variety X(K). If we look at the polytope defined by the equations $\{x | \ x \cdot v_F \leq k_F\}$ we can some polytope somewhat related to K. The theorem of, I believe, Danilov and Khovanskii, is

THEOREM: The number of lattice points inside of $\{x|\ x \cdot v_F \leq k_F\}$ is the number of holomorphic sections of $\otimes \mathcal{O}_F{}^{k_F}$ over $X(K)$.

Essentially a basis for the holomorphic sections can be specified in terms of sections vanishing at appropriate lattice points.

Now the Hirzebruch Riemann-Roch theorem[20] computes the holomorphic Euler characteristic

$$\Sigma(-1)^k \dim H^k(X(K);\otimes \mathcal{O}_F{}^{k_F}) = \int_{X(K)} Td \cup ch(\otimes \mathcal{O}_F{}^{k_F})$$

By definition, H^0 is the holomorphic sections, what we are interested. General algebraic geometry informs us that for k_F's all large the higher cohomology groups vanish. But actually in this case, the other terms drop out automatically.

Corollary: $\#(\mathbb{Z}^n \cap \{x|\ x \cdot v_F \leq k_F\})$ is a polynomial in the k_F's.

This strengthens Erhart's theorem mentioned above. One also sees that computing the Erhart polynomial is essentially equivalent to computing the Todd class of $X(K)$. One can also give a sophisticated proof of Pick's theorem along these lines.

Hirzebruch's classic book tells us a certain amount about the connection between L-classes and Todd classes in the smooth case, and Cappell and Shaneson are definitely inspired by his formulae. Much of their work was devoted to the L-class side. We will turn to some of their ideas in a moment.

However, first as an amusing interlude, let me mention a very nice theorem of Khovanski which gives a solution to the lattice and Euler-MacLaurin problem for the case of smooth toric varieties (for which the Todd class can also be computed inductively quite easily: the faces correspond to subtoric varieties with controlled normal bundles (see [F1, pp109-110] for instance)). One almost sees a Todd class before one's very eyes.

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20 This theorem was the main result of Hirzebruch's book. He proved the signature theorem along the way to proving this. There are other subsequent more perspicacious proofs by Grothendieck (the birth of K-theory) and also using the Atiyah-Singer index theorem.

THEOREM: Let K be a Delzant polytope. Let K_h be the polytope obtained by moving all of the faces outwards by h (h is a vector with one component for each face). If f is a polynomial, then we have

$$\sum_{n \in K \cap \mathbb{Z}^n} f(n) = \tau(\partial/\partial h)\int_{K_h} f(x),$$

evaluated at h = 0, where $\tau = x/(1-e^{-x})$ is applied to each $\partial/\partial h_i$ in turn, and though of as an infinite order differential operator.

τ is of course reminiscent of a Todd class, and the entry of the faces makes sense as they are the relevant divisors. If the dimension is one, this is the usual Euler-MacLaurin formula. It is a nice calculus exercise to see that this boils down to a sum of infinite order differential operators integrated over the faces of K (including K itself as the "dominant terms").

The theorem is proven first for another class of test functions besides the polynomials. One first verifies the theorem for the interval $(-\infty, n]$ and the test function of the form e^{-ax}. Here both sides can be computed explicitly. Differentiating both sides with respect to a, gets one polynomials. The functional equation $\tau(-x) = e^{-x}\tau(x)$ and Taylor's theorem $f(x+1) = e^{d/dx}f(x)$ gives one intervals of the form $[n, \infty)$, and therefore all intervals. From there one used products and changes of variables to get half spaces and quarter spaces, etc. In a Delzant polytope, locally all the faces are faces of a cube by an element of $SL_n(\mathbb{Z})$, which one uses to finish the argument. Details can be found in Guillemin's book.

The Cappell-Shaneson approach to the Todd class problem is based on the idea that it is not so hard to get a formula in the smooth case, so that one can get the formula in general by using a resolution of singularities by a smooth toric variety. If one has suitable projection formulae and has guessed the formula, one can verify it by going downwards. As I understand it, they are not using an explicit resolution of singularities and computing from it. That can sometimes be done, see the work of Pommersheim (which represented a very substantial three dimensional calculation, with interesting connections to number theory, etc.) What they do is obtain projection formulae for L-classes and a number of other algebraic geometric invariants, and play these off one another.

The way they approach integration formulae is to consider families of toric varieties over a given one. To integrate a monomial over something, over each point in the polytope place a parallelpiped with appropriate sides whose lattice point count is the given monomial. These "assemble" to another higher dimensional lattice polytope, which they can analyze. However, probably there is another approach where one is computing an "equivariant Todd class"[21]and a generalized Riemann-Roch theorem would give the result by analysis similar to that in the lattice point count problem.

Before getting to the projection formula in general, let's consider the case of a map $\varphi: M \longrightarrow X$ with one non-regular point, i.e. one has a single point $*$ in X in the complement of which φ is a fiber bundle. Write $M = M_1 \cup M_2$ where M_1 lies over a neighborhood of $*$ and M_2 lies over the complement of $*$. By Novikov additivity, sign(M) = sign(M_1) + sign(M_2). Note sign(M_1) = sign(M_1/∂) where the latter is thought of as a Witt space and can be thought of a local invariant of φ at the singular point $*$. (Note, it is not $\varphi^{-1}(*)$ for your typical nice maps.) If signature were multiplicative for fiber bundles over manifolds with boundary, then one would obtain:

$$\mathrm{sign}(M_2) = \mathrm{sign}(F) \cdot \mathrm{sign}(X - *) = \mathrm{sign}(F)\mathrm{sign}(X).$$

and finally, the formula:

$$\mathrm{sign}(M) = \mathrm{sign}(F)\mathrm{sign}(X) + \mathrm{sign}(\varphi^{-1}(\mathrm{St}(*))/\partial)\mathrm{sign}(*),$$

where St($*$) = star of ($*$) = regular neighborhood. This formula can easily be generalized to the case where Im φ is just a submanifold S:

$$\mathrm{sign}(M) = \mathrm{sign}(F)\mathrm{sign}(X) + \mathrm{sign}(\varphi^{-1}(\mathrm{St}(*))/\partial)\mathrm{sign}(S)$$

In other words, one looks at the "normal bundle" to the lower stratum, and takes the inverse image of a generic fiber, and then makes that into a Witt space, and takes its signature.

The Cappell-Shaneson projection formula asserts that this is correct if one has a stratified map between even codimensional

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[21] this is strongly suggested by comparing the algebra in Shaneson's ICM talk with the calculations if equivariant homology of simplicial toric varieties in [GKM]

spaces and there is "no monodromy over strata". Note, that as above, the strata involved in the map are more refined than the strata involved in the space.

One subtlety is that it is not true that signature is multiplicative for fiber bundles over manifolds with boundary. The unit tangent bundle of S^2 is cobordant to the disjoint union of two Hopf bundles. One can see that the signature of this cobordism is necessarily 1 (with suitable orientation conventions), which is not $\mathrm{sign}(S^1)\mathrm{sign}(\text{Some 3-manifold})$. However, a very small part of their theorem asserts that this is correct when everything is even dimensional[22].

Note that the case of one stratum reduces to the case of a point and inductive arguments if one collapses the boundary of a regular neighborhood of S to a point.

Ultimately their theorem asserts:

THEOREM: Suppose $\varphi\colon X \longrightarrow Y$ is a stratified map between stratified spaces with even codimensional strata[23] and for which all the open strata are simply connected[24] then

$$\mathrm{sign}(X) = \mathrm{sign}(F)\cdot\mathrm{sign}(Y) + \sum_V \mathrm{sign}(\varphi^{-1}(\mathrm{St}(V))/\partial)\cdot\mathrm{sign}(V)$$

where St(V) is the Star of a vertex in a subdivision the top simplex of the open dense stratum in V. V runs over the closed strata of X.

The proof uses intersection homology and algebraic variants of cobordism ideas. Note also in the Cappell-Shaneson formula the key correction terms $\mathrm{sign}(\varphi^{-1}(\mathrm{St}(V))/\partial)\cdot L(V)$ are exactly what came up when we studied "bubble cobordisms" in lecture one. As before, a formula about signatures implies one about L-classes:

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22 If one turns this example into a stratified map there will be a codimension three stratum. The target is still Witt, but not even codimension.

23 They assume these are Whitney stratified, but one can get by with a lot less, if one studies their proofs. On the other hand, this suffices for, say, algebraic maps.

24 Actually no monodromy is sufficient. If there is monodromy, one can correct for it, as in Atiyah's work on nonmultiplicativity of signature.

$$\varphi_*(L(X)) = \mathrm{sign}(F)\cdot L(Y) + \sum_V \mathrm{sign}(\varphi^{-1}(St(V))/\partial)\cdot L(V)$$

in $H_*(Y)$. Also, one can jazz this up to study other signature related invariants, because everything is true at the level of Witt classes of self dual complexes of sheaves.

Remark: This formula is somewhat reminiscent of the decomposition theorem of Beilinson Bernstein and Deligne for *algebraic maps*. That theorem asserts that the push forward of IH decomposes into a sum over strata of IH's with coefficients in things that look a lot like fibers. BBD has very significant implications for calculations of homology, etc. (See [GM] for some examples) One can prove from their work a similar formula $\varphi_*(L(X)) = \mathrm{sign}(F)\cdot L(Y) + \Sigma\, I(V,\varphi)\cdot L(V)$ but t is not clear what the invariant $I(V,\varphi)$ is. We will soon give the more algebraic geometric version of the above decomposition formula.

Just for fun, I'd like to point out that one can combine the Cappell-Shaneson formula with the bubble quotient idea, and get an interesting formula for signature.

PROPOSITION [CSW]: Let X be a space with even codimension strata and suppose S^1 acts nicely on X, then the fixed point set is a Witt space[25] If all monodromies of X are unipotent, the same holds true for the fixed set. In this case, there is a formula: $\mathrm{sign}(X) = \Sigma a_F \cdot \mathrm{sign}(F)$, where a_F is a local multiplier depending on the action near a generic point on the fixed point set.

In fact, a_F is the signature of the local bubble. (The monodromy is said to be unipotent if the characteristic polynomial of all elements are $(t-1)^n$. Unipotence works better for inductive arguments than does trivial monodromy, because a "5-lemma" is true for unipotence, but not for triviality.)

One can use this to redo some of the calculations of signatures of singular hypersurfaces in [CS1].

[25] It actually is a space with a "rational stratification" with even codimensional strata. The typical problem is that one can obtain rational homology manifolds as fixed sets for circle actions on manifolds.

More concretely, if we take the toric variety associated to a polygon in the plane, then for almost any S^1 the fixed set consists of exactly the vertices. The a_F's are all ± 1. We leave to the reader the following interpretation of these signs: If you pick a direction corresponding to a good circle, it'll be transverse to all the faces. One can check that for exactly two vertices will these lines cross through one face and out the other. (It'll be a different pair of vertices for a different toric variety.) This means that we get 2 +'s and (n-2) -'s, and the signature is then 4-n which agrees (at least in the Delzant case for which it is asserted, but correct as well for the case "simplicial" polytopes, i.e. those which satisfy the face-vertex condition in Delzant, but lack the determinant condition -- these correspond to orbifolds) with the formula in Oda p. 132. Indeed, one gets in this way a strange identity for simplicial convex polytopes of any dimension.

NIce as the decomposition formula is, it has the disadvantage that the terms in it, things like $St(V))/\partial$, do not have any algebraic meaning. The way around this it to use a basic idea from intersection theory, called deformation to the normal cone, a construction which has been described as the algebraic geometers substitute for the tubular neighborhood theorem. I recommend Fulton's CBMS lectures [F2, p 13] for a quick account.

Let $\hat{L}(V) = L(V) - \sum_{W<V} \mathrm{sign}(P(V,W))\ L(W)$ be a modified L-class.

THEOREM: If $f: X \longrightarrow Y$ is an algebraic map, then assuming some monodromy hypotheses, we have:

$$f_*L(X) = \mathrm{sign}(F)L(Y) + \sum (\mathrm{sign}(P_{V,f}) - \mathrm{sign}(F)\mathrm{sign}(P_{V,Y}))\ L(W)$$

where $P_{V,f}$ denotes the general fiber for the projectivised cone for $f^{-1}(V)$ upstairs (mapping to V) and $P_{V,Y}$ is the corresponding fiber for V downstairs, and F is the generic fiber of the map.

Moreover, the same is true for other characteristic classes associated to invariants other than signature. These include MacPherson's total Chern class, the Baum-Fulton-MacPherson Todd class. The idea is to use deformation to the normal cone to replace cobordisms.

As an exercise, under some hypothesis about how the normal cone resembles the neighborhood of the subvariety (assumptions which are, in general, false) prove the above formula for Euler characteristic. It is a general "Radon-Hurwitz" formula.

The key formula that Cappell and Shaneson deduce from the interplay between these formulae (together with known facts, like the algebraic genus of any toric variety is 1) vis a vis toric varieties is that $T(X(K))$[26] $= \Sigma L(X(E))$ where $E \leq K$. This formula, in the smooth case, is in Hirzebruch, can now be used to reduce the calculation of Todd classes to L-classes in general[27].

As a last hint for how to do the L-class calculations, let me remind you that everything is known in the smooth case. Sup[pose that K is a simplicial toric variety, then the singularities of X(K) are orbifold singularities and one can approach calculations of L-classes from the point of view of the G-signature theorem as in Hirzebruch-Zagier. In other words, one really studies the group action on a (locally defined, at least) branched cover, branched along faces where the "Delzant minor determinant" is not 1. For the nonsimplicial toric varieties, the story is yet more complicated...

As for the explicit formulae that are obtained this way, I refer the reader to the papers [CS 2,3, Sh]. It seems reasonable to hope that both the specific calculations for toric varieties and the general method of calculating characteristic classes will have many applications in the future.

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[26] T is a slight modiifcation of the Todd class -- one multiplies each coefficient of Td by a suitable power of 2.

[27] This is quite a nontrivial point: the characteristic classes live now in homology so there is no universal way to go from one genus to another. For cohomology classes one just sees that the L-class carries the exact same information (rationally) as the total Pontrjagin class and therefore as any other genus.

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On the cohomological dimensions of Coxeter groups

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Abstract. We give an explicit formula for the virtual cohomological dimension of Coxeter group $vcd_F\Gamma$ in terms of cohomological properties of corresponding panel complex K .
Math. Subj. Class. Primary: 20F32, 55M10, 57S30 Secondary: 16A60, 20J05, 51K10, 54E45

1. Introduction

Let V be a finite set and let $I \subset V \times V$ be a symmetric subset. The group Γ given by the following presentation

$\langle V \mid v^2 = 1, (vw)^{m(v,w)} = 1 \ \forall v \in V \ \forall (v,w) \in I; 0 < m(v,w) = m(w,v) \in \mathbb{Z} \rangle$

is called the Coxeter group. The group Γ together with its presentation is called Coxeter system (Γ, V). Let $\mathcal{F}$ be the set of all non-empty subsets of V that generate a finite subgroup of Γ. Then $\mathcal{F}$ is a partially ordered set (poset) by inclusion. Then [6] $\mathcal{F}$ is isomorphic to the poset of simplices of some simplicial complex $K = K(\Gamma, V)$.

If an Eilenberg MacLane complex $K(\Gamma, 1)$ of a group Γ is finite then the cohomological dimension $cd_F\Gamma$ of the group Γ with respect to an abelian group F can be defined as a maximal number n such that $H^n_c(\bar{X}; F) \neq 0$, where $\bar{X}$ is the universal cover of $K(\Gamma, 1)$. According to the theorem of Serre if $cd_F\Gamma < \infty$ for a subgroup Γ of the finite index of a group G then the number $cd_F\Gamma$ does not depend on a choice of Γ. That number is called the virtual cohomological dimension of G with respect to F and is denoted by $vcd_F\Gamma$.

For every Coxeter group Γ Bestvina has constructed a finite polyhedron $B(\Gamma, F)$ such that $vcd_F\Gamma = dimB(\Gamma, F)$ for sufficiently good F [1]. Using his result we give a formula for $vcd_F\Gamma$ for Coxeter groups in terms of cohomologies of Davis' panel complex. As a consequence we obtain that the logarithmic law $vcd(\Gamma_1 \times \Gamma_2) = vcd\Gamma_1 + vcd\Gamma_2$ generally does not hold for Coxeter groups. Nevertheless the logarithmic law holds if $\Gamma_1 = \Gamma_2$. So this behavior of vcd is analogous to that of the covering dimension dim for ANR-compacta [3].

2. The local cohomological dimension of simplicial complex and the formula for $vcd_F\Gamma$

Let K be a simplicial complex. We recall that the collection of all simplices in K containing a given vertex v forms a complex $St(v, K)$ called the star of v in K and the collection of all simplices in $St(v, K)$ do not containing v forms a complex $Lk(v, K)$, called the link of v in K. We denote by $\beta^1 K$ the first barycentric subdivision of K. For every simplex $\sigma \subset K$ we define the normal star $nst(\sigma, K)$ of σ in K as a subcomplex of the star $St(v(\sigma), \beta^1 K)$ of the barycenter $v(\sigma)$ of the simplex σ in $\beta^1 K$ consisting of those simplices δ whose intersection with $\beta^1\sigma$ consists of $v(\sigma)$. The normal link $nlk(\sigma, K)$ of σ in K is the boundary of $nst(\sigma, K)$

i.e. the subcomplex of $nst(\sigma, K)$ consisting of all simplices having empty intersection with $v(\sigma)$.

We are going to use the same symbol for a simplicial complex and for its geometric realization.

Definition. The *local cohomological dimension* of a simplicial complex K with respect to a coefficient group G is

$lcd_G K = max_{\sigma \in K}\{m \mid H^m(nst(\sigma, K), nlk(\sigma, K); G) \neq 0\}$.

The *global cohomological dimension* [7] of a simplicial complex K over G is $cd_G K = max\{m \mid H^m(K; G) \neq 0\}$. Note that $H^m((nst(\sigma, K), nlk(\sigma, K)); G) = H^{m-1}(nlk(\sigma, K); G)$ and hence $lcd_G K = max_{\sigma \in K}\{m \mid H^{m-1}(nlk(\sigma, K); G) \neq 0\}$.

These dimensions can differ from each other. For example, if K is a triangulation of the join $\mathbb{R}P^2 * \mathbb{R}P^2$ of the projective plane with itself, generated by some triangulation on $\mathbb{R}P^2$, then its dimensions over the integers and the rationals are $cd_{\mathbb{Z}} K = lcd_{\mathbb{Z}} K = 5$ and $lcd_{\mathbb{Q}} K = 4$ and $cd_{\mathbb{Q}} = 0$.

THEOREM 1. *For every abelian group G and every finite-dimensional simplicial complex K there is the inequality $lcd_G K \geq cd_G K$.*

We denote by CK the cone over a simplicial complex K with the natural triangulation.

THEOREM 2. *For every Coxeter group Γ defined by a Coxeter system (Γ, V) there is the equality $vcd_F \Gamma = lcd_F CK$ where $K = K(\Gamma, V)$ and F is an additive group of field or the group of integers.*

Every r-dimensional complex K has a filtration $L_0 \subset L_1 \subset ... \subset L_r = K$ where $L_i = \bigcup_{dim\sigma = r-i} nst(\sigma, K)$. By induction we define the filtration $L'_0 \subset L'_1 \subset ... \subset L'_r = B(K, F)$ for a given abelian group F. We define polyhedra L'_i together with projections $p_i : L'_i \to L_i$ such that p_i is an extension of p_{i-1} for every i. Define $L'_0 = L$ and $p_0 = id$. Assume that $p_{i-1} : L'_{i-1} \to L_{i-1}$ is defined. For every $(r-i)$-simplex σ we consider $A(\sigma) = p_{i-1}^{-1}(nlk(\sigma, K))$. Let $C(\sigma)$ be an F-acyclic polyhedron containing $A(\sigma)$ and of the least possible dimension. We attach $C(\sigma)$ to L'_{i-1} along $A(\sigma)$ and extend the map p_{i-1} over $C(\sigma)$ such that the image of $C(\sigma) - A(\sigma)$ lies in $nst(\sigma, K) - nlk(\sigma, K)$. Do that for all σ to obtain $p_i : L'_i \to L_i$. The resulting space L_r is Bestvina's complex $B(K, F)$ of a complex K with respect to a group F.

Note that the map $p = p_r : B(K, F) \to K$ has the property that the preimage of every normal star $nst(\sigma, K)$ is F-acyclic.

THEOREM3 [1]. *For every Coxeter group Γ defined by a Coxeter system (Γ, V) the following equality holds $vcd_F \Gamma = dim B(CK(\Gamma, V); F)$ if F is a field or the group of integers.*

PROPOSITION 1. *For every simplex $\sigma \subset K$, $H^n(nlk(\sigma, K); F) = H^n(p^{-1}(nlk(\sigma, K); F)$ for all n.*

PROOF: First, we define by induction the notion of a star-subcomplex N of K. The empty set is a star-subcomplex. A union of some normal stars of $(r-i)$-dimensional simplices $nst(\sigma, K)$ form a star-subcomplex of K if all pair wise intersections of that normal stars are star-subcomplexes. Induction, the Mayer-Vietoris sequence and the fact that $\bar{H}^*(p^{-1}(nst(\sigma, K); F) = 0$ imply $H^n(N;F) = H^n(p^{-1}(N);F)$ for all n and for every star-subcomplex $N \subset K$. This argument is well-known and sometimes is called the combinatorial Vietoris-Begle theorem.

PROPOSITION 2. *$dimB(K,F) = lcd_F K$*

PROOF: Let $lcd_F K = n$. Hence $H^{n-1}(nlk(\sigma, K); F) \neq 0$ for some simplex σ in K. By Proposition 1 we have $H^{n-1}(p^{-1}(nlk(\sigma, K)); F) \neq 0$. Therefore, $H^{n-1}(A(\sigma); F) \neq 0$. Then by the construction $C(\sigma)$ is at least n-dimensional. Hence $dimB(K,F) \geq n$.

Now let $dimB(K,F) = n$. then by the construction of $B(K,F)$ there exist a simplex $\sigma \subset K$ with $H^{n-1}(A(\sigma);F) \neq 0$. Then by Proposition 1, $H^{n-1}(nlk(\sigma, K); F) \neq 0$ and hence $lcd_F K \geq n$.

Now the proof of Theorem 2 follows by Proposition 2 and Theorem 3.

PROOF OF THEOREM 1: Assume that $cd_F K = n$. Consider Bestvina's complex $p : B(K,F) \to K$. Proposition 1 implies that $H^n(B(K,F);F) \neq 0$ and, hence, $dimB(K,F) \geq n$. Proposition 2 implies the proof.

PROPOSITION 3. *Let CK be a cone over K then $lcd_F CK = max\{lcd_F K, cd_F K + 1\}$.*

PROOF: For every simplex $\sigma \subset K \subset CK$ the normal link $nlk(\sigma, CK)$ is homeomorphic to the cone over the normal link $nlk(\sigma, K)$ and, hence, is cohomologicaly trivial. The normal link of the cone vertex v is K. The normal link of any other simplex σ containing v is homeomorphic to $nlk(\delta, K)$ for some simplex δ. If $cd_F K < lcd_F K$ then $lcd_F CK = lcd_F K$ by the above and hence the formula is true. If $i = cd_F K \geq lcd_F K$ then $lcd_F CK = i + 1$ and the formula is true again.

3. APPLICATIONS OF THE FORMULA FOR $vcd_F\Gamma$

THEOREM 4. *An every Coxeter group Γ has the following properties:*

a) $vcd_{\mathbb{Q}}\Gamma \leq vcd_G\Gamma$ for any group G,

b) $vcd_{\mathbb{Z}_p}\Gamma = vcd_{\mathbb{Q}}\Gamma$ for almost all primes p,

c) there exists prime p such that $vcd_{\mathbb{Z}_p}\Gamma = vcd\Gamma$

PROOF: a) Let $vcd_{\mathbb{Q}}\Gamma = n$. By Theorem 2 there exists $\sigma \subset CK$ such that $H^{n-1}(nlk(\sigma, CK); \mathbb{Q}) \neq 0$. The Universal Coefficient Formula implies that the group $H^{n-1}(nlk(\sigma, CK); \mathbb{Z})$ has a nontrivial rank. Hence, $H^{n-1}(nlk(\sigma, CK); G) \neq 0$ and therefore, $lcd_G CK \geq n$. Theorem 2 implies that $vcd_G\Gamma \geq n$.

b) Let again $vcd_{\mathbb{Q}}\Gamma = n$. Since we have only finitely many simplices $\delta \subset CK$ and finitely many $m \leq dimCK$, only finitely many prime numbers are involved in the torsions of $H^m(nlk(\delta, CK);\mathbb{Z})$. Hence for almost all prime p, $H^m(nlk(\delta, CK);\mathbb{Z}_p) = 0$ for $m \geq n$. Therefore, $lcd_{\mathbb{Z}_p} CK \leq n$. That together with a) proves b).

c) Let $vcd\Gamma = r$, then by Theorem 2 there is a simplex $\sigma' \subset CK$ such that $H^{r-1}(nlk(\sigma', CK);\mathbb{Z})) \neq 0$. Hence by virtue of the Universal Coefficient Formula, $H^{r-1}(nlk(\sigma', CK);\mathbb{Z}_p) \neq 0$. Hence, $lcd_{\mathbb{Z}_p} CK \geq r$ and hence, $lcd_{\mathbb{Z}_p} CK = r$. The Theorem 2 implies that $vcd_{\mathbb{Z}_p}\Gamma = vcd\Gamma$.

COROLLARY 1. *The equality $vcd(\Gamma \times \Gamma) = 2vcd\Gamma$ holds for every Coxeter group.*

COROLLARY 2. *A Boltyanski's compactum can not be the boundary of a Coxeter group.*

PROOFS: Since the formula $vcd_F\Gamma \times \Gamma' = vcd_F\Gamma + vcd_F\Gamma'$ holds for every field F, Theorem 6 b) implies the proof of Corollary 1. The characterizing property of Boltyanski's compactum B is $dim B \times B = 3$ [4]. Then in view of the following theorem Corollary 2 follows.

THEOREM 5 (BESTVINA-MESS) [5]. *Let $\partial\Gamma$ be the boundary of a Coxeter group Γ then for every abelian group G, $dim_G \partial\Gamma = vcd_G\Gamma - 1$.*

We recall that a compactum X has the cohomological dimension $dim_G X \leq n$ if for every closed subset $A \subset X$ the relative Cech cohomology group $\check{H}^{n+1}(X, A; G)$ is trivial [3],[4].

COROLLARY 3 [2]. *Every Pontryagin surface Π_p is the boundary of some right-angled Coxeter group.*

PROOF: Pontryagin surface Π_p is defined by the properties: $dim_{\mathbb{Q}}\Pi_p = dim_{\mathbb{Z}_q}\Pi_p = 1$ for q relatively prime with p and $dim\Pi_p = 2$. Hence $dim(\Pi_p \times \Pi_q) = 3$ if p and q are relatively prime. If we take the Moore complex $M(\mathbb{Z}_p, 1)$ with an arbitrary triangulation K then by Theorem 2 $vcd_F\Gamma = lcd_F CK$ for the Coxeter group Γ generated by K. By Proposition 3 $lcd_F CK = max\{lcd_F K, cd_F K + 1\}$. It is easy to see that $lcd_F K = 2$ for all F. Note that $cd_F K = 2$ if $F = \mathbb{Z}_p$ and $cd_F K = 1$ for $F = \mathbb{Q}, \mathbb{Z}_q$. Hence $vcd_F\Gamma = 2$ if $F = \mathbb{Z}_q$ or $\mathbb{Q}$ and $vcd\Gamma = 3$. Theorem 7 implies that the boundary $\partial\Gamma$ is a Pontryagin surface Π_p.

We recall that the boundary of the Coxeter system (Γ, K) is the visual sphere at infinity of CAT(0) cubical complex Σ associated with (Γ, K) [6].

Recently M. Davis found a formula for cohomology of Coxeter groups [8] and now the formula for vcd can be derived from his result.

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Transitivity in Negatively Curved Groups

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ABSTRACT. Let G be a negatively curved group. This paper continues the classification of limit points of G. A probability measure is constructed on the space at infinity and with respect to this measure almost every point at infinity is shown to be line transitive.

Introduction

Let M^n be a closed riemannian n-manifold with all sectional curvatures -1. Then $\pi_1(M^n)$ is a discrete Möbius group acting properly discontinuously on the universal cover $\mathbf{H^n}$ and conformally on the boundary $\mathbf{S^{n-1}}$. Since the action of $\pi_1(M^n)$ is cocompact, every $x \in \mathbf{S^{n-1}}$ is a point of approximation (conical limit point) for $\pi_1(M^n)$ (see 2.4.9 of [Nicholls] for instance). Furthermore, Γ (any Cayley graph for $\pi_1(M^n)$) is quasi-isometric to $\mathbf{H^n}$ (see [Cannon]) and so $\partial\Gamma$, the boundary at infinity of Γ, is homeomorphic to $\mathbf{S^{n-1}}$. Therefore every $x \in \partial\Gamma$ is a point of approximation.

Weakening the above hypotheses, assume M^n is any hyperbolic manifold. Let $x \in \mathbf{S^{n-1}}$ and suppose $L \subset \mathbf{H^n}$ is an oriented hyperbolic line with x as one endpoint. If for any $b \in \mathbf{H^n}$ there exists a sequence of distinct deck transformations $\{g_n\} \subset G = \pi_1(M^n)$ such that the images g_nL come arbitrarily close to b, then x is called *point transitive*. If for any oriented hyperbolic line L' there are distinct g_n such that $g_nL \longrightarrow L'$ preserving orientation, then x is *line transitive* or a *Myrberg point*. Clearly, every Myrberg point is also point transitive, but the converse is false in general (see [Sheingorn]). Myrberg [Myrberg] first showed that for $n = 2$ the set of line transitive points has full Lebesgue measure in the boundary at infinity. More recently, Tukia has proved that in all dimensions the collection of conical limit points that are not Myrberg points is a nullset for any conformal G measure [Tukia].

Let G be any negatively curved (Gromov hyperbolic) group with Cayley graph Γ and space at infinity $\partial\Gamma$. In [Freden], the author showed that every $x \in \partial\Gamma$ is a point of approximation. In this paper, almost every $x \in \partial\Gamma$ is shown to be line transitive. The strategy is based on an old idea due to Artin. In [Artin], Artin showed that the line transitive points for the modular group $SL(2,\mathbf{Z})$ are exactly those real numbers whose continued fraction expansion contains each finite sequence of integers. The continued fraction of a real number ξ can be obtained as the cutting sequence of a geodesic ray $R \subset \mathbf{H^n}$ tending to ξ (see 5.4 of [Series]). In the language of geometric group theory, a cutting sequence for R corresponds to reading the label of an equivalent geodesic ray R' in the Cayley graph for $SL(2,\mathbf{Z})$. Since $SL(2,\mathbf{Z}) \simeq \mathbf{Z_2} * \mathbf{Z_3}$ is nearly a free group, cutting sequences and labels of geodesic rays are very similar [Series].

The concept of a geodesic ray containing each finite geodesic subsegment is somewhat similar to the idea of a normal number. Recall that a real number η is *normal to base r* if each block B_k of k digits occurs with frequency $\frac{1}{r^k}$. It is well known that almost all real numbers are normal to every base (chapter 8 [Niven] or 9.3-9.13 in [HW]). Hedlund and Morse generalized a similar concept to strings of symbols in their paper [HM]. Given a finite set of generating symbols and certain concatenation rules they considered infinite sequences containing a copy of every possible finite string of symbols. Such sequences were labelled *transitive*. Yet a third related subject concerns Markov chains, see [Feller]. A state (symbol, outcome, event, etc) is *persistent* if with probability one, that state will recur (infinitely often) within the chain. This paper links all of the above.

Example : Suppose $F = \langle a, b \rangle$ is the free group of rank 2. Embed the associated Cayley graph in $\mathbf{H^2}$. Each geodesic ray represents a unique point at infinity. Let w be a freely reduced word of length k in the generators. Emulating the counting estimate of [Niven], consider the ratio of the number of rays (from 0) of length nk containing w to the total number of rays of length nk. As $n \to \infty$ this ratio tends to 1. The conclusion is that with respect to a certain natural (Cantor) measure, almost every ray contains w, and in fact infinitely often. By a *transitive ray*, I mean a geodesic ray from 0 that contains every finite geodesic word as a subpath. Theorem 1.4 below shows that any point represented by such a ray is line transitive (the proof of theorem 1.4 in the case of a free group is much simpler than what I have written). The set of all geodesic segments form a Markov chain with four states: a, b, a^{-1}, b^{-1}. A given state can be succeeded by any state other than its inverse, with probability $\frac{1}{3}$. In the language of [Feller], each state is aperiodic, persistent, and has finite mean recurrence time, i.e. all states are *ergodic*.

In the case of a generic negatively curved G, relators (of perhaps arbitrarily long length) destroy the Markov aspect and vastly complicate the counting process. Showing that most points at infinity can be represented by an actual geodesic transitive ray seems to be difficult in the general case; in fact it may not be true! The difficulty is avoided by considering quasigeodesic rays.

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http://geom.helsinki.fi/Annales/Anna.html

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Circle Actions on Homotopy Complex Projective Planes

S. Kwasik and R. Schultz

One consequence of M. Freedman's work on topological 4-dimensional surgery is that every closed 4-manifold with the homotopy type of the complex projective plane $\mathbf{CP}^2$ is homeomorphic to either $\mathbf{CP}^2$ or another manifold $\boldsymbol{Ch}$ that is often called the *Chern manifold*. There are several ways of distinguishing these manifolds topologically. For example, the generator of $H_2(\boldsymbol{Ch}; \mathbf{Z}) \approx \mathbf{Z}$ cannot be represented by a locally flat embedding of S^2 and the standard inclusion $S^2 \approx \mathbf{CP}^1 \subseteq \mathbf{CP}^2$ implies the opposite conclusion for $\mathbf{CP}^2$. Another difference is that the unitary group U_3 acts transitively on $\mathbf{CP}^2$ by projective colineations but $\boldsymbol{Ch}$ admits no transitive actions of compact Lie groups.

Although $\boldsymbol{Ch}$ has no transitive compact group of symmetries, it has a rich family of finite symmetries. In particular, every odd order cyclic group acts nontrivially on $\boldsymbol{Ch}$. On the other hand, the main result of this work shows that $\boldsymbol{Ch}$ has no positive dimensional compact Lie groups of symmetries.

THEOREM. $\boldsymbol{Ch}$ *supports no nontrivial circle actions.*

It is not known if $\boldsymbol{Ch}$ has any nontrivial involutions, but if such actions exist they cannot be locally linear.

The approach to proving the theorem involves an analysis of arbitrary circle actions on homotopy complex projective planes, and the proof uses an assortment of techniques including cohomological fixed point theory, surgery theory, and the structure theory of homotopy stratified sets that follows from the machinery of controlled topology.

Notational conventions. For the most part, the balance of this announcement deals with a closed manifold M^4 that is homotopy equivalent to $\mathbf{CP}^2$ and has a given nontrivial circle action.

If M^4 is a closed 4-manifold that is homotopy equivalent to $\mathbf{CP}^2$, then it is well known that cohomological fixed point theory implies that the fixed point set of a nontrivial circle action on M^4 is either a set with three points or the disjoint union of a 2-sphere and a point, in the latter case the fundamental class of the 2-sphere generates $H_2(M;\mathbf{Z}) \approx \mathbf{Z}$. Both possibilities arise for circle actions formed by restricting the transitive U_3 action on $\mathbf{CP}^2$ to a circle subgroup (the so-called *linear actions*). A more detailed analysis shows that *every circle action on M^4 is equivariantly homotopy equivalent to a linear action on* $\mathbf{CP}^2$.

If the fixed point set of the circle action is the disjoint union of a point and a 2-sphere, then the action is *semifree; i.e.*, the action is free on the complement of the fixed point set. The orbit space of a linear semifree action on $\mathbf{CP}^2$ is homomorphic to the disk D^3 such that the fixed point set projects to $S^2 = \partial D^3$ and an interior point; for an arbitrary semifree circle action on M as above, the picture is similar with D^3 replaced by a contractible *generalized* 3-manifold.

Proof of the theorem for semifree actions. Let M^* be the orbit space of a given semifree circle action on M as above, and form N^* from M^* by attaching a collar along the boundary. One can then realize N^* as the orbit space of a semifree circle action on some generalized 4-manifold N^4 such that

(i) N^4 is equivariantly h-cobordant to M^4 in the category of generalized manifolds,

(ii) the generator of $H_2(N^4;\mathbf{Z}) \approx \mathbf{Z}$ is represented by a locally flat 2-sphere.

If N^4 and the h-cobordism were genuine topological manifolds, it would follows that N^4, M^4 and $\mathbf{CP}^2$ were all homomorphic to each other. We circumvent this problem by crossing the h-cobordism theorem with a 2-dimensional torus, applying Quinn's work on resolutions of generalized manifolds to show that the product with the torus are in fact topological manifolds, and using surgery theory to show that crossing with the torus did not introduce additional complications (*e.g.*, one needs to know that $Ch \times T^2$ is not homomorphic to $\mathbf{CP}^2 \times T^2$ and stronger statements of this type). This disposes of the case when the fixed set consists of a point and a 2-sphere, so we shall assume henceforth that the fixed point set consists of three points.

As noted above, the circle action on M is equivariantly homotopy equivalent to a linear model; for the sake of definiteness let V be the 3-dimensional complex representation of S^1 arising from the appropriate faithful homomorphism $S^1 \to U_3$, and let $\mathbf{CP}(V)$ denote the associated linear S^1 action on $\mathbf{CP}^2$. It seems likely that M is in fact isovariantly homotopy equivalent to $\mathbf{CP}(V)$, but for our purpose it suffices to know that there is an equivariant homotopy equivalence that is almost isovariant in an appropriate sense (work of Dula and the second author discusses the appropriate concept for smooth actions), and it is a routine but tedious exercise to show that such equivalences exist.

Given an almost isovariant equivalence $f : M \to \mathbf{CP}(V)$ as above, the balanced product with S^1 defines a nonequivariant homotopy equivalence

$$M \times_{S^1} S^{2N+1} \to \mathbf{CP}(V) \times_{S^1} S^{2N+1}$$

for each positive integer N. If we view this as a representative for a surgery-theoretic homotopy structure on the codomain, then we can compose its normal invariant in

$$[\mathbf{CP}(V) \times_{S^1} S^{2N+1}, F/TOP]$$

with the map induced by the Sullivan class $k_2 \in H^2(G/TOP; \mathbf{Z}_2)$ to obtain an invariant $c_2(f, N) \in H^2(\mathbf{CP}(V) \times_{S^1} S^{2N+1}; \mathbf{Z}_2)$; if $N = 1$ we shall simply write $c_2(f)$. It follows immediately that M is homeomorphic to $\mathbf{CP}^2$ if the fiber restriction of $c_2(f, N)$ to $H^2(\mathbf{CP}(V); \mathbf{Z}_2) \approx \mathbf{Z}_2$ is trivial and M is homeomorhic to $\boldsymbol{Ch}$ if the fiber restriction is nonzero. Thus the proof of the theorem reduces to showing that $c_2(f)$ vanishes. Since the map

$$\rho : \mathbf{CP}(V) \times_{\mathbf{Z}_2} S^3 \to \mathbf{CP}(V) \times_{S^1} S^3$$

determined by the inclusion of $\mathbf{Z}_2$ in S^1 is injective in mod 2 cohomology, it also suffices to show the weaker condition $\rho^* c_2(f) = 0$.

The following result provides a crucial relationship between the algebraic localization in equivariant cohomology and the behavior of the group action near the isolated fixed points.

PROPOSITION. *If y is a fixed point of $\mathbf{Z}_2$ on the S^1-manifold $\mathbf{CP}(V)$ and j_y is the inclusion map from $\mathbf{RP}^3 \approx \{y\} \times_{\mathbf{Z}_2} S^3$ to $\mathbf{CP}(V) \times_{\mathbf{Z}_2} S^3$, then $j_y^* \rho^* c_2(f)$ depends only on the behavior of the $\mathbf{Z}_2$ action near y. In particular, the restricted class vanishes if the action is locally linear at y.*

The proof of this result uses the almost isovariance properties of f and the obstruction-theoretic observation that the restriction of f to a small neighborhood of y is essentially unique up to $\mathbf{Z}_2$-almost isovariant homotopy. This result yields the following strong evidence for the theorem:

COROLLARY. *If the induced involution is locally linear at the nonisolated fixed point of the involution, then $c_2(f) = 0$.*

Here is a sketch of the argument: The proposition implies that $j_y^* \rho^* c_2(f) = 0$ for some y in the 2-dimensional component of the fixed point set, and formal considerations imply that $j^* \rho^* c_2(f) = 0$ for all points x in this component.

By the preceding reductions this reduces the proof of the corollary to showing that $j_y^* \rho^* c_2(f) = 0$ if y is an isolated fixed point of the involution. The proof of this is based on a structure theorem for the local behavior of involutions on 4-manifolds near isolated fixed points. Namely, there is a smooth homology 3-sphere Σ with a free involution and a $\mathbf{Z}_2$ free, simply connected $\mathbf{Z}$ homology h-cobordism W from Σ to itself such that some neighborhood of the fixed point is equivariantly homeomorphic to the one point compactification of the manifold X obtained from the disjoint union $\cup_{k=0}^{\infty} W \times \{k\}$ by identifying $\partial_+ W \times \{k\}$ with $\partial_- W \times \{k+1\}$. A diagram chase shows that $j_y^* \rho^* c_2(f)$ is equal to the normal invariant of the canonical degree map from $\Sigma/\mathbf{Z}_2$ to $\mathbf{RP}^3$, and since

$\Sigma/\mathbf{Z}_2$ has the same mod 2 cohomology as $\mathbf{RP}^3$, it follows that the normal invariant is trivial (*i.e.*, the surgery map obstruction from $[\mathbf{RP}^3, F/Top]$ to $L_3^h(\mathbf{Z}_2)$ is bijective). This completes the proof of the corollary. In fact, the argument yields more. *If* $j_y^*\rho^*c_2(f) = 0$ *for some point* y *in the* 2*-dimensional component of the involution's fixed point set, then* $c_2(f) = 0$ *and* M *is homeomorphic to* $\mathbf{CP}^2$ (the argument involving the isolated fixed point required no assumption of local linearity).

It is well known that the behavior of the involution near the fixed points in the 2-dimensional component can be highly irregular. The first step in dealing with this problem is a stable version of the previous proposition.

EXTENSION OF PREVIOUS PROPOSITION. *Let* Ω *be a linear representation of* $\mathbf{Z}_2$*. Then the class* $j_y^*\rho^*c_2(f)$ *depends only on the behavior of the stabilized involution on* $M \times \Omega$ *near* $(y, 0)$.

This is essentially a formal consequence of standard formulas for normal invariants of product maps where one factor is well behaved.

One specific consequence is that $j_y^*\rho^*c_2(f) = 0$ if the action is *stably locally linear* at y; i.e., the associated involution on $M \times \Omega$ is locally linear at $(y, 0)$. Such considerations and the proof of the previous Corollary also show that $j_y^*\rho^*c_2(f) = 0$ if the structure of the product involution near $(y, 0)$ has the form $\mathbf{R}^q \times U$, where U corresponds to an involution on $\mathbf{R}^4$ with an isolated fixed point. Therefore the main theorem will be a consequence of the following result.

STABLE LOCAL STRUCTURE THEOREM. *Let* Ω *be a linear* $\mathbf{Z}_2$ *representation whose fixed point set has codimension exactly* 2*, and assume* $\dim\Omega$ *is sufficiently large (e.g.,* $\dim\Omega \geq 6$*). Then the local behavior of the involution at* $(y, 0)$ *is equivalent to that of* $\mathbf{R}^q \times U$*, where* U *represents an involution on* $\mathbf{R}^4$ *with an isolated fixed point.*

Here is a summary of the steps in the proof: The fixed point set of the original involution is a manifold, so one can apply the machinery of homotopy stratified sets to the orbit space $(\mathbf{CP}(V) \times \Omega)/\mathbf{Z}_2$ if the dimensions and codimensions of the fixed set components for $\mathbf{CP}(V) \times \Omega$ are sufficiently large. The assumptions on Ω turn out to be adequate (among other things, this uses V. Klee's result that the embedding of the 2-dimensional fixed set in $\mathbf{CP}(V)$ becomes locally flat in $\mathbf{CP}(V) \times \mathbf{R}^2$).

The teardrop construction of Hughes, Taylor, Weinberger and Williams shows that the possibilities for neighborhood germs of $(y, 0)$ in $(\mathbf{CP}(V) \times \Omega)/\mathbf{Z}_2$ are classified by approximate fibrations with homotopy fiber type given by $\mathbf{RP}^3$, and results of Chapman and Hughes show that such approximate fibrations can be described via surgery theory as transfer invariant homotopy structures on some product manifold $T^k \times \mathbf{R}P^3$. The vanishing of $\widetilde{K}_0(\mathbf{Z}[\mathbf{Z}_2])$ and of $K_{-i}(\mathbf{Z}[\mathbf{Z}_2])$ for $i > 0$ combine to show that such structures are given by products for tori with transfer invariant homotopy structures on $S^1 \times \mathbf{RP}^3$, and Freedman's work on 4-dimensional topological surgery shows that all such objects are essentially given by universal coverings of the form $U - \{p\}$, where U represents an involution on $\mathbf{R}^4$ with a single fixed point p. The conclusion of the Stable Local Structure Theorem is an immediate consequence of these considerations.

Link Colorability, Covering Spaces and Isotopy

Ollie Nanyes

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Abstract

In these notes we modify the concept of link diagram colorability and branched cyclic covering spaces to the study of links under the piecewise linear isotopy equivalence relation. We then show that all mod B colorings of a link diagram (B an abelian group) are determined completely by the "nullity corrected" Goeritz matrix developed by Traldi. A quotient space techinque developed by Wada is used to calculate the first homology group of the two fold branched covering space. These notes have been submitted to a journal as a paper.

1. Introduction.

The mod-p colorability of a knot diagram (p usually taken to be a prime integer) is a well known knot diagram (and therefore knot) invariant (e. g. [AND], [LIV], [ADA]). It is often used to describe representations of the knot group onto dihedral groups and to analyze the structure of the first homology of the two fold branched covering space $\widetilde{X}$ of the knot complement (see [HAR], [PER]).

In this paper we extend the notion of mod p colorability of knot diagrams to the notion of mod B colorability of link diagrams where B is taken to be an abelian group. In the first section we give the basic definitions and background. We show that a group B colors a link diagram if and only if there is a representation of the link group onto a generalized dihedral group D which takes the Wirtinger generators of the link group (meridional elements) onto the generator of the non-normal (Z/2Z) factor of D (where D is viewed as a semi-direct product $(Z/2Z)\oslash B$ where B is the abelian group). In the second section we apply colorings and branched covering spaces to study p. l. link isotopy (which is an equivalence relation which ignores local knotting, see [ROL1] and figure 1). In the third section we show that all such B must be homomorphic images of $H_1(\widetilde{X})$ and that $H_1(\widetilde{X})$ indeed colors the diagram of the link group. It follows that the matrix M which presents our possible coloring groups is equivalent via elementary operations to a "corrected" Goeritz matrix (see [TRA]) of the link.

2. Definitions and Colorability.

2.1. Definitions.

We will work in the piecewise linear (p. l.)category throughout this paper. A *link* L will refer to a disjoint union of polygonal simple closed curves in the three sphere S^3. We say that two links L and L' are *equivalent* if one can be deformed by an ambient isotopy into the other. We say that two links L and L' are *isotopic* if there is a p. l. isotopy $H : (\coprod_{i=1}^{k} S_i^1) \times [0,1] \to S^3$ such that the image of H_0 is L and the image of H_1 is L'. Note that the isotopy need not necessarily be *locally flat;* see p. 64 [BIN] or Chapter 4 of [R-S], for a detailed definition of local flatness. Figure 1 shows a non-locally flat p. l. isotopy that cannot be realized by an ambient isotopy. Note that the pictured isotopy is not smooth.

It follows from the work of Rolfsen [ROL1] that two links are *isotopic* if and only if any two diagrams of the respective links can be deformed into one another via the three Reidemeister moves (figure 2) and a fourth move R4 which is suggested in figure 3. The idea behind move R4 is that one can replace the arc cut off in the pictured disk by any other p. l. arc with any number of crossings in it. R4 shows the effect on $D(L)$ of a *cell replacement* in which a local knot can either be tied or deleted.

In this paper we will refer to the corrected Goeritz matrix $\bar{H}(L)$ of a link L. We will give a brief description of how to obtain $\bar{H}(L)$ from $D(L)$. A proof that $\bar{H}(L)$ is an equivalence invariant of a link and that it provides a presentation matrix for the two fold branched cyclic cover of the complement of L can be found in [TRA].

Let $D(L)$ be a diagram on the plane $P = S - \{p\}$ where p is some point on S which misses $D(L)$. Shade in the regions of $P - D(L)$ in a "chessboard" fashion, coloring the disjoint regions black and white alternately. We follow the convention that the unbounded region is colored white. Now label the regions $Y_0, Y_1 \ldots, Y_q$

with the unbounded region being labeled as Y_0. (Note that one can change the shading scheme by deciding to remove a different point p from S.) To each crossing c in $D(L)$ we can assign an incidence number $\eta(c)$ as shown in figure 4(a) and a type number (figure 4(b)); the crossing is of type I if the white surface can be oriented compatibly with the orientation of L (which may be chosen arbitrarily) near c. If not, the crossing is of type II.

Now for $i \neq j \in \{0,1,\ldots,q\}$ let $g_{i,j} = -\sum \eta(c)$, the sum taken over the set of crossings c incident on both Y_i and Y_j. For $i \in \{0,1,\ldots,q\}$ let $g_{i,i} = \sum_{j \neq i} g_{i,j}$. The Goeritz matrix $G(L)$ is then the matrix $[g_{i,j}]$, $i,j \in \{1,\ldots q\}$ (0'th row and column not included). $G(L)$ presents the torsion invariants of the two fold cyclic branched cover [KYL].

To obtain the "signature corrected" Goeritz matrix developed in [G-L] one forms an integral diagonal matrix $A(L)$ with one row and one column for each type II crossing. One enters $-\eta(c)$ in the corresponding diagonal place. Note that if the white surface is orientable then $A(L)$ is the empty matrix. This signature correction will not enter into our discussion.

To obtain a "nullity correction" [TRA], one calculates a square 0 matrix $B(L)$ which is of size $(\beta(L)-1) \times (\beta(L)-1)$ where $\beta(L)$ denotes the number of components of the white surface. For connected white surfaces, $\beta(L)$ is the empty matrix. This nullity correction will be relevant to our discussion.

Finally, the corrected Goeritz matrix $H(L)$ is formed by forming the block matrix:

$$\begin{bmatrix} G(L) & 0 & 0 \\ 0 & A(L) & 0 \\ 0 & 0 & B(L) \end{bmatrix}$$

A *generalized dihedral group* $Dih(B)$ we mean the group

$$(1)\ \{t, b | t \in Z_2, b \in B, tbt = b^{-1}\}$$

with the understanding that t is not the identity in Z_2 ($Z_n = Z/nZ$ in this paper) and B is a finitely generated abelian group. We will sometimes view $Dih(B)$ as a semidirect product $Z_2 \oslash B$ with $B \triangleleft Dih(B)$ (section 1.2 [WEI]). Note that if B is a finitely generated abelian group that B can be uniquely represented as $Z_{d_1} \oplus Z_{d_2} \oplus \ldots \oplus Z_{d_k} \oplus Z \oplus \ldots \oplus Z$ where each $d_i | d_{i+1}$. By the $rank_k(B\ )(k \geq 1)$ we mean the number of d_i that k divides plus the number of infinite cyclic factors.

By $rank_0(B)$ we mean the number of infinite cyclic factors. Let $rank(B)$ denote the total number of cyclic factors in this representation of B.

The *group of the link* L is $\pi_1(S^3 - L)$ and a meridian of L will mean any element of the group of L which is homotopic to a Wirtinger generator of the group of L. By the *n-fold branched cover of* L we mean the standard n-fold cyclic covering space of the complement of L obtained by splitting S^3 along a Seifert surface and appropriately regluing n copies the surfaces together along the Seifert surface (see either [B-Z], [KAU], [H-K], [ROL3] for details; note that these covering spaces are called *strictly-cyclic* in section 11 of [M-M]).

2.2. Coloring and Dihedral Representations.

Let $D(L)$ be a link diagram with stands x_i. A *mod* B coloring of $D(L)$ is the assignment to the x_i elements $b_i \in B$ such that: 1) every x_i is assigned to exactly one element b_i of B, 2) at each crossing in which the elements b_i, b_j, b_k appear (b_j being assigned to the overcrossing), the relation

$$(2)\ 2b_j = b_i + b_k$$

holds in B and 3) $0 \in B$ appears somewhere and 4) the b_i that appear on the diagram generate B. Figure 5 shows the crossing coloring relation and figure 6 shows a colored link diagram. Note that B is completely determined by the coloring of the bridges of $D(L)$.

One can view colorings in another way ([AND], 3.4 of [LIV]): one can form an integral *coloring matrix* M with columns corresponding to the strands x_i and rows corresponding to the crossings and the resulting coefficients of equation (2). That is, if three different strands appear at a crossing, one enters a "2" in the overcrossing (x_j) column and -1 in the other two stand columns (x_i, x_k). If only two strands appear at a crossing (as in the R1 diagram in figure 1) one enters a "1" in the overcrossing column and a -1 in the column of the undercrossing strand. In each case, one fills in the remaining columns with zeros. We then say that B colors $D(L)$ if B is a homomorphic image of the abelian group presented by M (Ch. 6, [JOH]).

Proposition 1. Colorability of a knot diagram is preserved by the Reidemeister moves.

Proof. Suggested by figure 2. □

We can also make the following definition: by $rank(L)$ we mean the largest rank of all abelian groups B that color L. By $rank_k(L)$ we mean the largest $rank_k(B)$ of all groups B that color L. Both ranks are finite as all possible groups B are determined by the colorings of the bridges.

We now relate colorings to specific types of representations of the link group onto dihedral groups. Note that if the link group G represents onto a generalized dihedral group $Z_2 \oslash B$ via a homomorphism φ we can compose φ with the standard projection $\xi : Z_2 \oslash B \longrightarrow Z_2$ to obtain $\xi \circ \varphi : G \longrightarrow Z_2$. Call φ *meridian preserving* if $\xi \circ \varphi$ maps each meridian of the link group onto the generator of Z_2.

Theorem 2. An abelian group B colors a link L if and only if there is a meridian preserving representation of the group G of the link onto $Dih(B)$.

Proof. See figure 7. If B colors L then get a homomorphism $\varphi : G \longrightarrow Dih(B)$ by setting $\varphi(x_i) = tb_i$.

On the other hand suppose the homomorphism φ is given. Because φ is meridian preserving, each Wirtinger generator of G maps to some element of the form tb_i. The Wirtinger relations at each crossing (figure 7) show that the b_i satisfy the color equation (1) at each crossing. Since the map is onto the b_i must generate B. □

The assumption that φ is meridian preserving is essential. For example, consider the following link K (figure 8) studied by Kinoshita ([S-S]). As figure 8 suggests, the group of K represents onto $Z_2 \oslash Z$ but it is easy to see that K is only colored by homomorphic images of Z_4.

We easily obtain the following Corollaries:

Corollary 3. Let L be a link with u components. Then $rank_2(L) = u - 1$.

Corollary 4. Suppose there exists p mutually disjoint p. l. 3-balls B_i^3 such that $L \subset \cup B_i^3$ and $L \cap B_i^3 \neq \emptyset$ for all $i \in \{1, \dots p\}$. Then $rank_0(L) \geq k - 1$.

3. Relation of Coloring and Branched Covering Spaces to the Link Isotopy Equivalence Relation.

We will call an invariant $\mathcal{I}$ a *localized invariant* if the following holds: if a locally unknotted link L has property $\mathcal{I}$ then every link isotopic to L also has property $\mathcal{I}$. The following theorem shows that we can obtain localized invariants from group representations (and hence colorings) and from cyclic branched covering spaces.

Theorem 5. Let L be locally unknotted link and L' any other link isotopic to L. If the group G of L represents onto a group W then the group of L' also represents onto W. The first homology group (integer coefficients) of the k-th cyclic branched covering space of L (denoted $H_1(\widetilde{X}_{k,L})$) then $H_1(\widetilde{X}_{k,L})$ is a direct summand of $H_1(\widetilde{X}_{k,L'})$.

Proof. Proof of the first assertion follows immediately.

To prove the second assertion we employ the Mayer-Vietoris exact sequence as in 7.E.1 of [ROL3]. Let X_1 denote the complement of L, X_2 denote the complement of K which will be the local knot that changes L to L', $X = X_1 \cup X_2$ (which is the complement of L')and W is $\partial B^3 - L = X_1 \cap X_2$ and $\widetilde{X}$, $\widetilde{X}_i$, and $\widetilde{W}$ the respective k-fold cyclic coverings branched over L. Now examine the Mayer-Vietoris sequence in reduced homology (integer coefficients):

$$\ldots \longrightarrow \widetilde{H}_1(\widetilde{W}) \longrightarrow \widetilde{H}_1(\widetilde{X}_1) \oplus \widetilde{H}_1(\widetilde{X}_2) \longrightarrow \widetilde{H}_1(\widetilde{X}) \longrightarrow \widetilde{H}_0(\widetilde{W}) \longrightarrow \ldots$$

Since $\widetilde{W}$ is the k-fold cover of a 2-sphere which is branched over two points, $\widetilde{W}$ itself is a 2-sphere and hence $\widetilde{H}_1(\widetilde{W}) = 0$. But $\widetilde{H}_0(\widetilde{W}) = 0$ also and hence $\widetilde{H}_1(\widetilde{X}_1) \oplus \widetilde{H}_1(\widetilde{X}_2) \longrightarrow \widetilde{H}_1(\widetilde{X})$ is an isomorphism. $\square$

Example 6. Consider the link L_m which was studied by Milnor in [MIL] and is pictured as L_m in figure 6. This figure shows that L_m has a $Z \oplus Z_3 \oplus Z_3$ coloring. Since the two component trivial link has colorings which are homomorphic images of Z and L_m is locally unknotted, it follows that L_m is not (p. l.) isotopic to the unlink. This gives an elementary answer to the question on p. 305. of [MIL] for p. l. isotopies. This question had been answered by Rolfsen by the development of the localized Alexander Invariant in [ROL2].

The pictured coloring was obtained by a "trial and error process" as the coloring matrix in this case is very large. In the next section, we will show how to obtain the coloring matrix via $H(L)$, the "corrected" Goeritz matrix discussed in the previous section.

Example 7. Consider the generalized Whitehead L_k link pictured in figure 9. Here let $\widetilde{X}$ denote the two fold branched cover. We calculate $H_1(\widetilde{X}) = Z_{8k}$ by using a Goeritz matrix with the diagram shaded as shown in the figure. Since L_k is locally unknotted for all k it follows immediately that the L_k are mutually distinct for all positive k (the $\overline{\mu}$ invariants developed in [MIL] fail to detect this).

It should be pointed out that Holmes and Symthe [H-S] developed the concept of F-isotopy and algebraic invariants of F-isotopy and showed that this family of links were mutually distinct (even for negative values of k).

4. Relation of coloring groups to the Two Fold Cyclic Branched Cover.

In this section we relate groups which color L to the two fold cyclic branched covering space of the complement of L (the s-cyclic covering spaces mentioned in [M-M]). We will employ the methods developed by Wada [WAD]. (Another approach would be to use the techniques and results found in [H-K] (Hosokowa polynomial), [M-M], or [SAK].) We then relate the coloring matrix to the Goeritz matrix. Let X denote the complement of L, $\widetilde{X}$ the two fold cyclic branched covering space and $\widetilde{X}_u$ the two fold unbranched covering space. G is the group of L. Homology groups will have integer coefficients.

Theorem 8. The group B colors L if and only if B is a homomorphic image of $H_1(\widetilde{X})$.

Proof. Orient the components of L and derive a Wirtinger presentation for G:

$$(3)\ \{x_0, x_1, \ldots, x_n |\ r_1, r_2 \ldots, r_m\}$$

where the relations r_i are of the following type (figure 10): $x_i x_j = x_j x_l$ for a right handed crossing and $x_j x_i = x_l x_j$ for a left handed crossing. Consider the group G_β given by

$$(4)\ \{x_0, x_1, \ldots, x_n |\ s_1, s_2, \ldots, s_m\}$$

where the relations s_i correspond to the r_i in (2) in the following way: $x_i = x_j x_l^{-1} x_j$ for right handed crossings and $x_l = x_j x_i^{-1} x_j$ for left handed crossings.

Lemma 9. G_β is isomorphic to $Z * \pi_1(\widetilde{X})$.

Proof of the Lemma. The proof can be found in section 5 of [WAD] and is presented here for the convenience of the reader. Let $*$ be the basepoint for $\pi_1(X)$ and let $p : \widetilde{X} \to X$ be the covering projection. Notice that for all Wirtinger generators x_i, (a meridional loop representing) x_i^2 lifts to a loop $\widetilde{x}_i$ in $\widetilde{X}_u$ which pierces each copy of the lift of the splitting Seifert surface exactly once. $\widetilde{x}_i$ can be thought of as being composed of two subarcs, $x_{i,0}$ and $x_{i,1}$ which have the two points of $p^{-1}(*)$ as endpoints where the Z_2 action takes $x_{i,k}$ to $x_{i,k+1}$.

Now consider the quotient space $\widetilde{X}_u/p^{-1}(*)$. Note that $\pi_1(\widetilde{X}_u/p^{-1}(*)) \simeq Z * \pi_1(\widetilde{X}_u)$. The $x_{i,0}$ and $x_{i,1}$ now represent loop in $\widetilde{X}_u/p^{-1}(*)$ and the induced Z_2 action permutes these loops, as before. Now each Wirtinger relation in (3) becomes either $x_{i,k}x_{j,k+1} = x_{j,k}x_{l,k+1}$ or $x_{j,k}x_{i,k+1} = x_{l,k}x_{j,k+1}$ $(k \in Z_2)$ in $\pi_1(\widetilde{X}_u/p^{-1}(*))$ depending upon the type of crossing. Therefore $\pi_1(\widetilde{X}_u/p^{-1}(*))$ has the following presentation:

$$(5)\ \{x_{0,k}, x_{1,k}, \ldots, x_{n,k} |\ r'_{1,k}, r'_{2,k}, , \ldots r'_{m,k},\ k \in Z_2\}$$

where the relations $r'_{j,k}$ correspond to $x_{i,k}x_{j,k+1} = x_{j,k}x_{l,k+1}$ or $x_{j,k}x_{i,k+1} = x_{l,k}x_{j,k+1}$ as appropriate. To obtain $\pi_1(\widetilde{X}/p^{-1}(*))$, which is isomorphic to $Z*\pi_1(\widetilde{X})$, one sews in a copy of L which in turn induces the relations: $x_{i,0}x_{i1} = 1$ for all i. Adding these relations to (5) we then can eliminate the $x_{i,1}$ as well as the $r'_{i,1}$. One can easily check that by substituting $x_{i,0}^{-1}$ for $x_{i,1}$ in the relations $r'_{i,0}$ and then relabeling $x_{i,0}$ as x_i one obtains the group presentation (4).

We can now rewrite (4) by using an automorphism Φ on the free group on n generators which is defined by $\Phi(x_i) = x_0^{-1}x_i$ for $i \neq 0$ and $\Phi(x_0) = x_0$. It is easy to see that $\Phi(s_i) = 1$ in G_β. We then get a new presentation for G_β :

$$(6)\ \{x_0, \Phi(x_1), \Phi(x_2) \ldots, \Phi(x_n) \mid \Phi(s_1), \ldots, \Phi(s_m)\ \}$$

In this presentation of G_β the free Z factor is generated by x_0 and the free $\pi_1(\widetilde{X})$ factor is generated by $\{\Phi(x_i) \mid \Phi(s_i)\ \}$.

We are now ready to prove the Theorem. Suppose B colors L. We then have a representation $\varphi : G \to Z_2 \oslash B$ where $\varphi(x_i) \to tb_i$ (t and b_i as defined in (1)). With no loss of generality, we can assume that $\varphi(x_0) = t$. It follows that φ induces a map $\varphi' : G_\beta \to Z_2 \oslash B$ by defining $\varphi'(x_i) = \varphi(x_i)$. To see this consider a typical G_β relation s_j as in (4): if $x_i = x_j x_l^{-1} x_j$ we see that:

$$\varphi'(x_j)\varphi'(x_l^{-1})\varphi'(x_j) = tb_j tb_l tb_j = tb_j b_l^{-1} b_j = tb_j^2 b_l^{-1} = tb_i = \varphi'(x_i)$$

as required. But if we take the representation of G_β as given by (6) we see that $\varphi'(x_0) = t$ and that $\varphi'(\Phi(x_i)) = b_i \in B$ for $i \neq 0$. So the preimage of B is generated by the $\Phi(x_i)$ and hence B is an abelian homomorphic image of $\pi_1(\widetilde{X})$. It follows that B is a homomorphic image of $\pi_1(\widetilde{X})/[\pi_1(\widetilde{X}), \pi_1(\widetilde{X})]$ which is $H_1(\widetilde{X})$.

We now show that $H_1(\widetilde{X})$ colors L. The abelianization map $\eta : G_\beta \to Z \oplus H_1(\widetilde{X})$ maps $\{\Phi(x_i)|\Phi(s_i)\}$ onto $H_1(\widetilde{X})$. We now show that the $\eta(\Phi(x_i))$ colors L. At a typical crossing we have the relation s_j which is $x_i = x_j x_l^{-1} x_j$ which implies $x_0^{-1}x_i = x_0^{-1}x_j x_l^{-1} x_0 x_0^{-1} x_j$ which is equivalent to $\Phi(x_i) = \Phi(x_j)(\Phi(x_l))^{-1}\Phi(x_j)$. Applying η we get $\eta(\Phi(x_i)) = \eta(\Phi(x_j))(\eta(\Phi(x_l)))^{-1}\eta(\Phi(x_j))$ which is equivalent to (after switching multiplication to addition) $\eta(\Phi(x_i)) = \eta(\Phi(x_j)) - (\eta(\Phi(x_l))) + \eta(\Phi(x_j)) = 2\eta(\Phi(x_j)) - (\eta(\Phi(x_l)))$ which is the desired coloring crossing relation. Because $\eta(\Phi(x_0)) = 0$ and η is onto, it follows that $H_1(\widetilde{X})$ colors L. The Theorem is proved.□

It follows from the previous theorem that a coloring matrix M presents $H_1(\widetilde{X})$ as an abelian group and is therefore equivalent via elementary matrix row and column operations (over the ring of integers) to the corrected Goeritz matrix $H(L)$. The following corollary is immediate:

Corollary 10. $H(L)$ determines the groups which color L.

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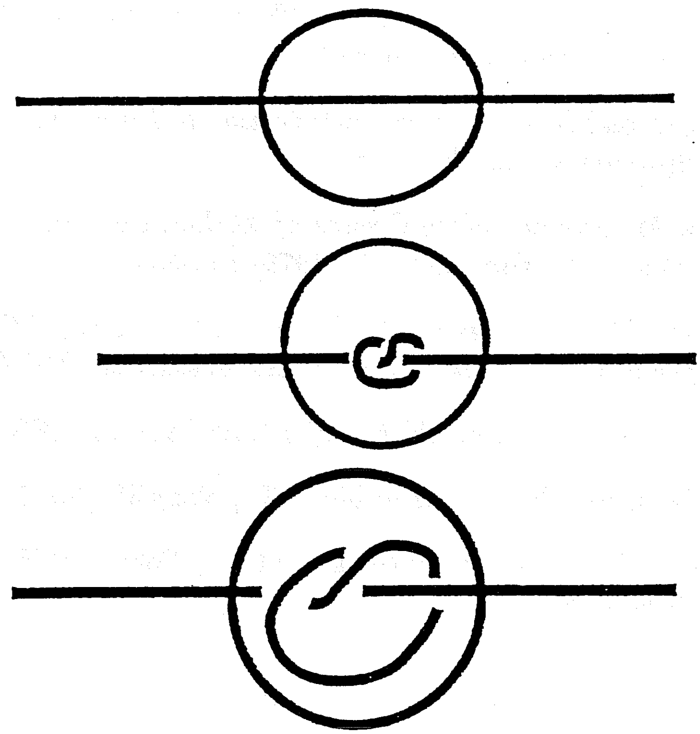

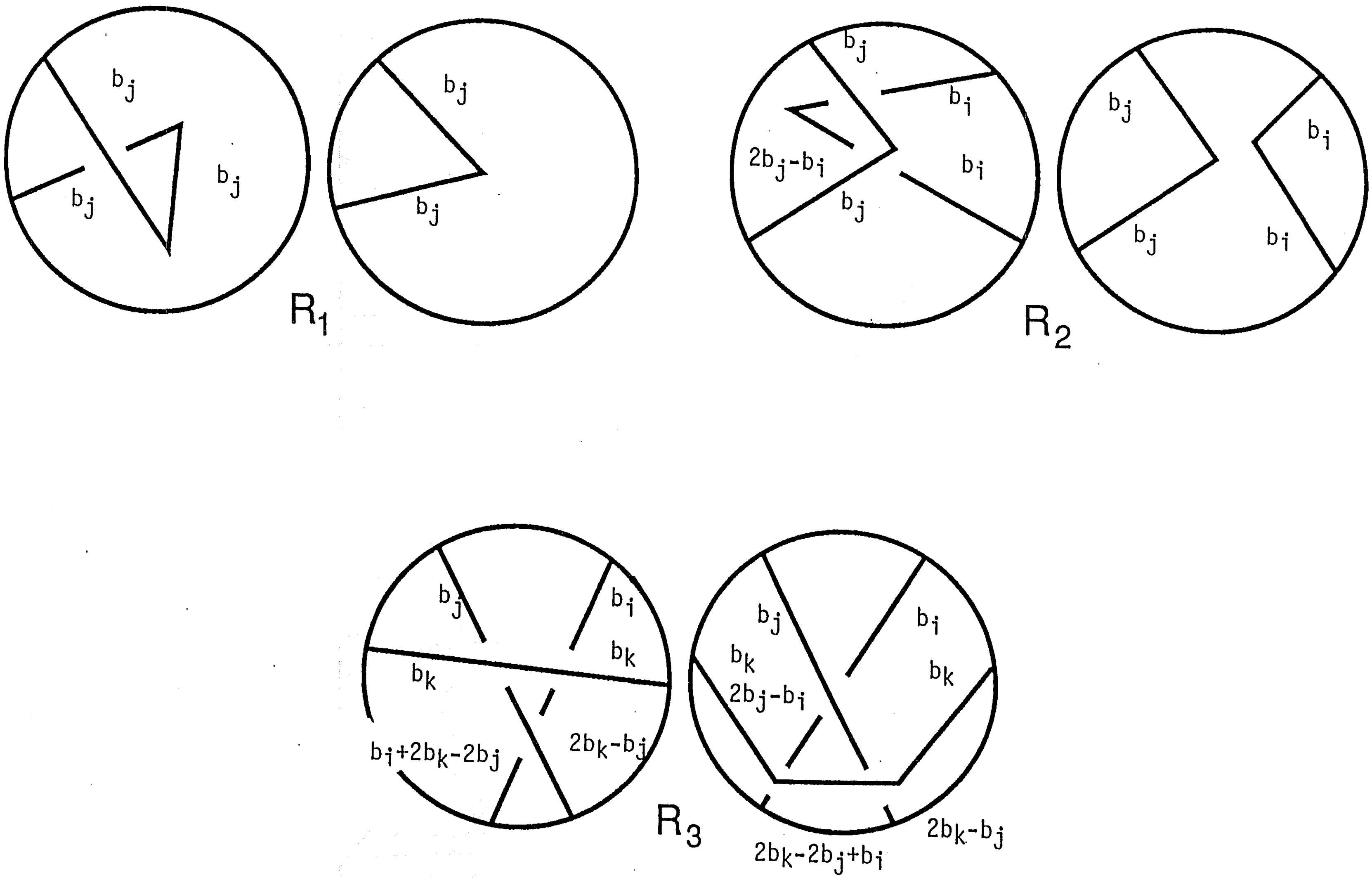

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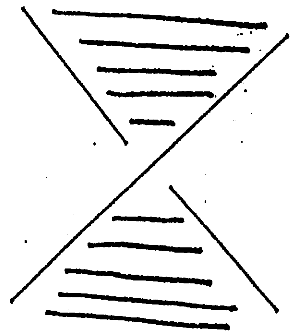

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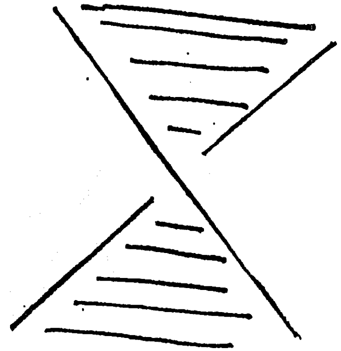

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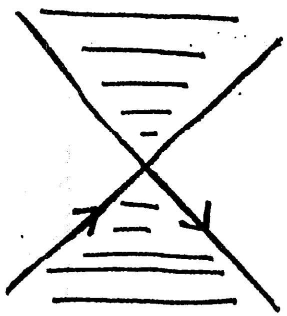

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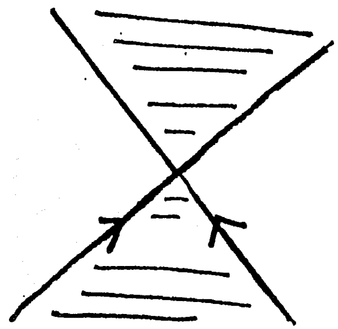
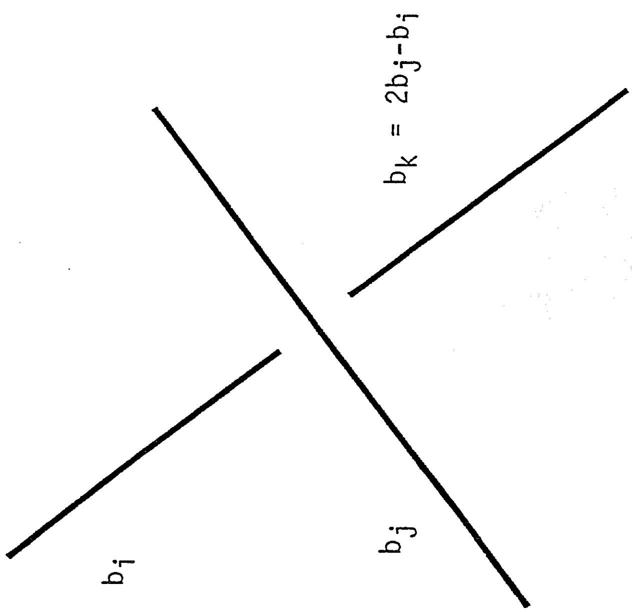


R4 Move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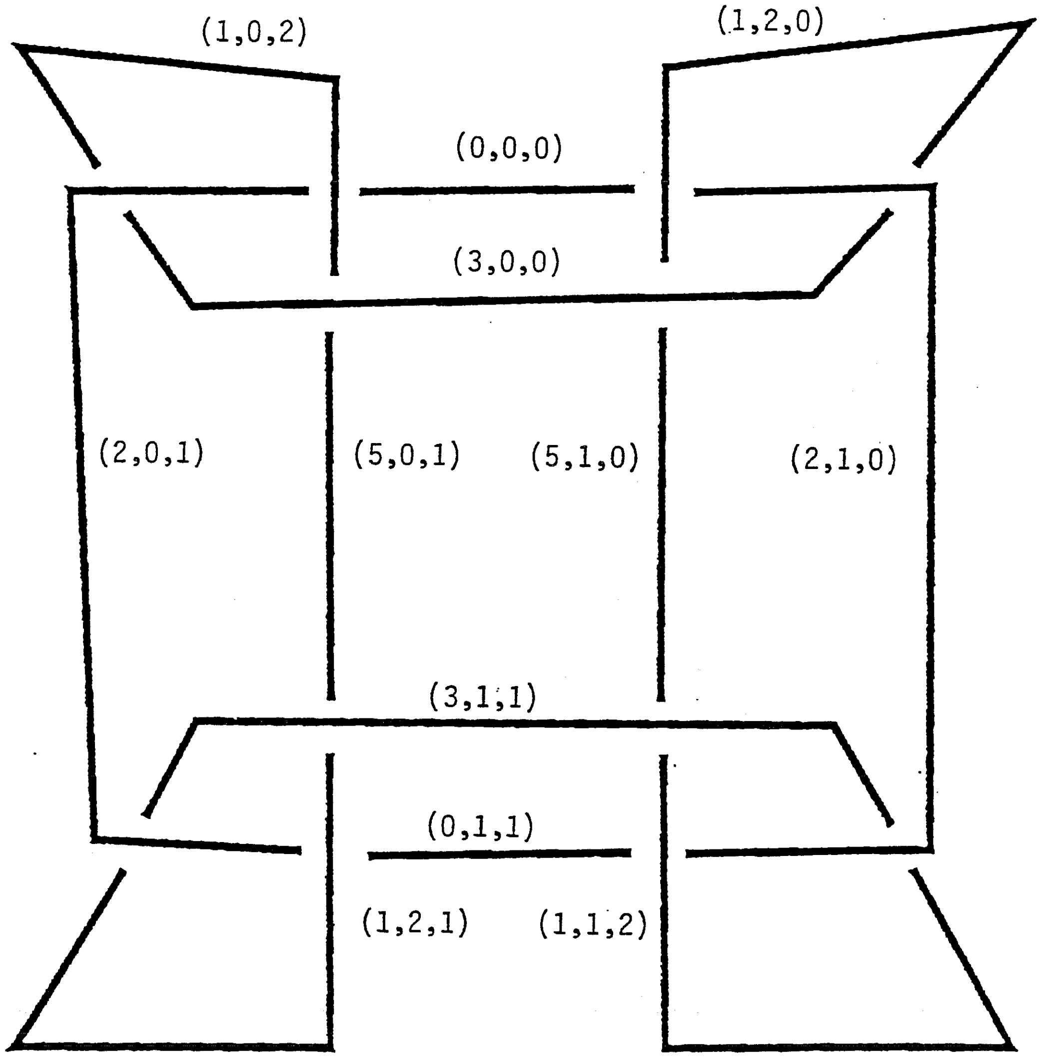
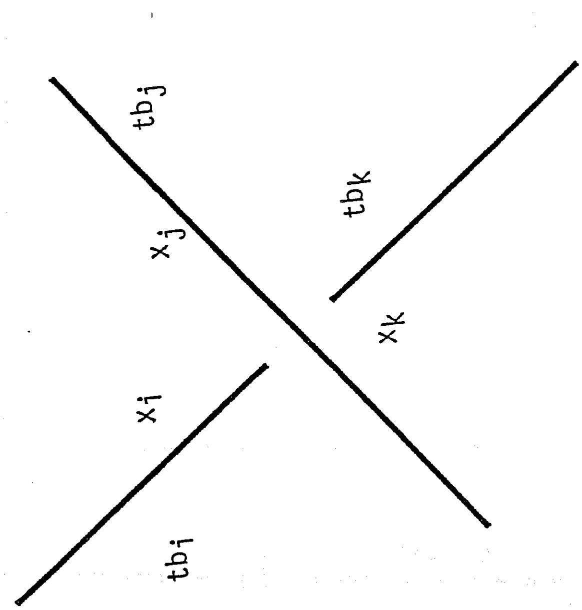
b_i
b_j



A $Z \oplus Z_3 \oplus Z_3$ Coloring of L_M



tb_j
tb_k
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x_k
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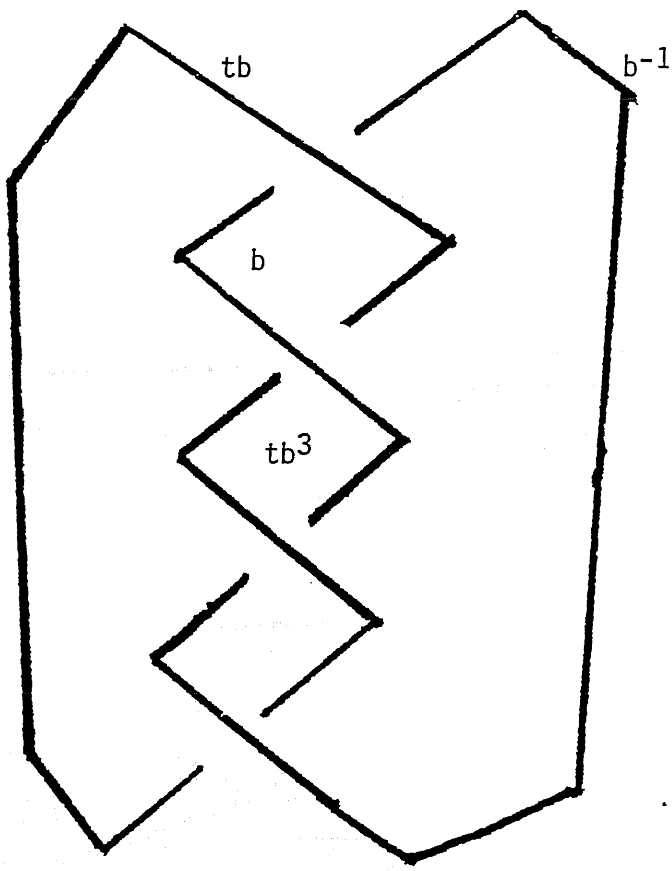
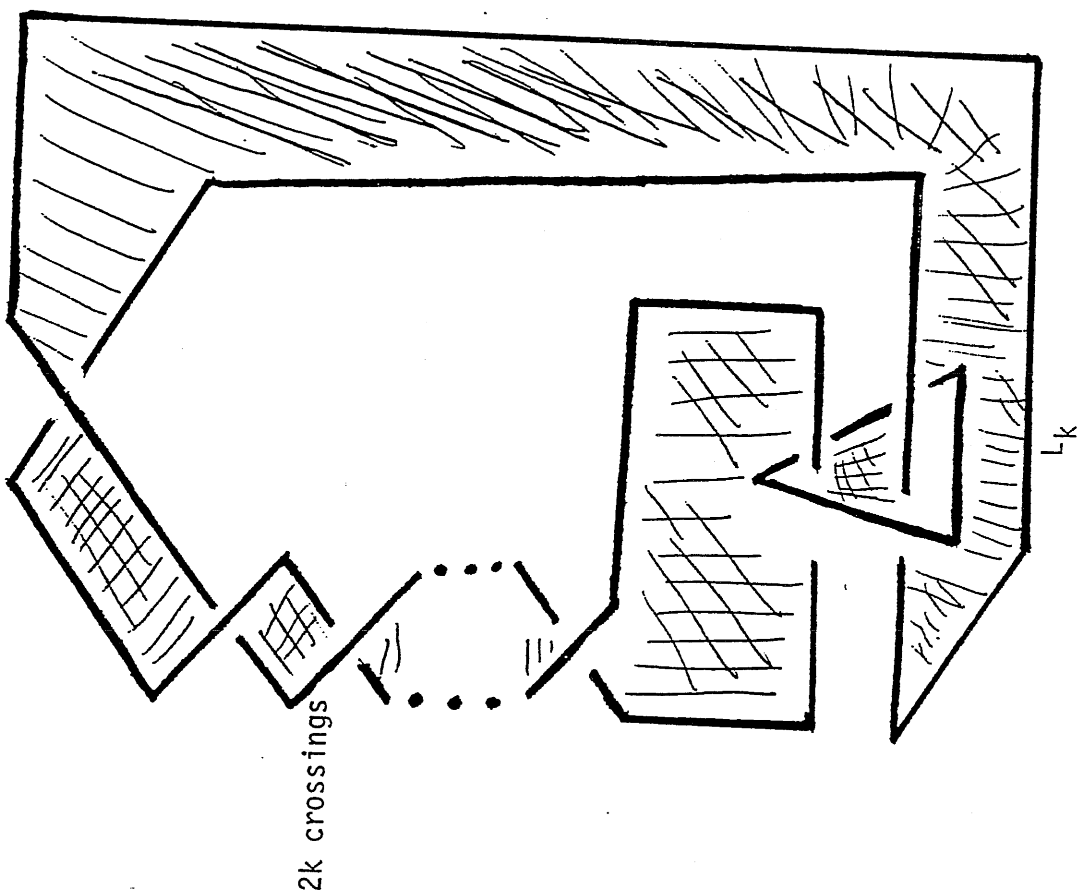
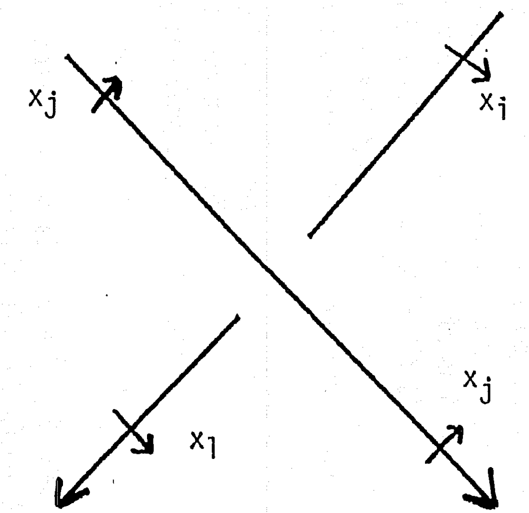
tb_i


tb
b⁻¹
b
tb³


2k crossings
L_k



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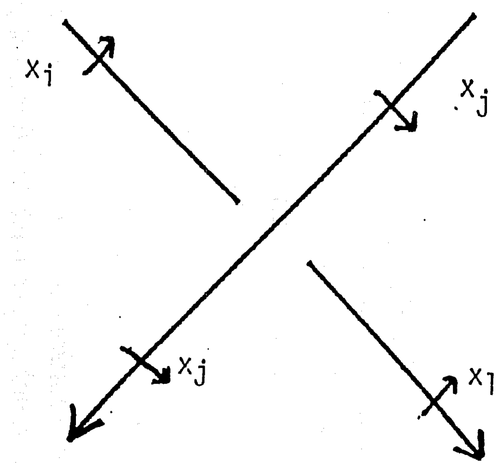

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Codimension two stratified actions on spheres

Monica Nicolau

(Announcement)

Recent developments in the theory of stratified spaces have brought about fundamental new insight into the theory of group actions on manifolds [W2]. Many questions remain unanswered however, and a central one is the codimension two strata problem. This paper concerns codimension two stratifications of actions on spheres, and gives the solution to the isovariant classification and existence of actions problems.

A codimension two *stratification* of S^n by homotopy spheres is a tower of *length* q of embeddings

$$K_0^r \subset K_1^{r+2} \subset \cdots \subset K_q^{r+2q} = S^n$$

where K_i are homotopy spheres. A semifree or free $\mathbf{Z}_m$ action τ on S^n *preserves* the stratification $\{K_i\}$ if K_i are τ-invariant subspaces with the fixed set $S^\tau \subset K_0$. We consider stratifications of arbitrary length q. A stratification of S^n will be called *simple* if all consecutive knots $(K_{i+1}^{r+2i+2}, K_i^{r+2i})$ are spherical simple knots, *i.e.* if the complements of consecutive strata are homotopy circles through dimension $[\frac{r+2i-1}{2}]$, where $[\ \ ]$ denotes the greatest integer function.

Our statements will treat at once the even and odd dimensional cases. These are however quite different in nature, as is witnessed by the difference in the algebraic invariants used in their classification.

Denote by Λ the ring of Laurent polynomials $\mathbf{Z}[t, t^{-1}]$ and recall that a Λ-module M is said to be of *type K* if M is finitely generated over Λ, and $(1-t) : M \to M$ is an isomorphism. Let (A, μ), be a pair: A an abelian group, and $\mu : A \otimes A \to \mathbf{Q}$ an ϵ-symmetric non-degenerate form, $\epsilon = \pm 1$. A Λ-*pair* of parity ϵ is a pair $(Q;t) = ((A,\mu);t)$ where t defines a Λ-module structure on A whereby A is of type K, and such that $\mu(ta \otimes tb) = \mu(a \otimes b)$. An isomorphism $((A,\mu);t) \cong ((A',\mu');t')$ of Λ-pairs is a Λ-module isomorphism $A \cong A'$ which preserves the forms μ and μ'. By [F1] isotopy classes of simple knots (S^{2r+1}, K^{2r-1}), $r \geq 3$ are in one to one correspondence with isomorphism classes of Λ-pairs $(Q;t)$ of parity $(-1)^r$.

Let (A, B, α, l, ψ) be a quintet: A and B abelian groups, $\alpha : A \otimes \mathbf{Z}_2 \to B$ a homomorphism, $l : T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$, and $\psi : B \otimes B \to \mathbf{Z}_4$ ϵ-symmetric forms, with l non-degenerate, where $T(A)$ denotes the torsion subgroup. We further require:

1) $0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} Hom(A, \mathbf{Z}_2) \to 0$ is exact where $\beta(b)(a) = \psi(b \otimes (a \otimes 1))$, and

2) the composition $B \xrightarrow{\gamma} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$ is the $(\times 2)$-map, where π is the projection, and γ is defined by $\psi(b \otimes \alpha(\pi a)) = l(\gamma(b) \otimes a)$.

A Λ-*quintet* of parity ϵ is a pair $(Q;t) = ((A, B, \alpha, l, \psi);t)$ where t defines Λ-module structures on A and B, whereby A is of type K, α is a Λ-homomorphism, and $l(ta \otimes tb) = l(a \otimes b)$, $\psi(ta \otimes tb) = \psi(a \otimes b)$. An isomorphism $((A, B, \alpha, l, \psi);t) \cong ((A', B', \alpha', l', \psi');t')$

of Λ-quintets is a pair of Λ-module isomorphisms $A \cong A'$, $B \cong B'$, which preserve the forms l and l', ψ and ψ', and commute with the homomorphisms α and α'. By [F2] isotopy classes of simple knots (S^{2r+2}, K^{2r}) , $r \geq 4$, are in one to one correspondence with isomorphism classes of Λ-quintets $(Q;t)$ of parity $(-1)^{r+1}$.

A Λ-pair (Λ-quintet) $(Q;t)$ admits a $\Lambda_D(m)$*-structure* if Q admits a $\Lambda = \mathbf{Z}[s, s^{-1}]$ structure $(Q;s)$ such that $(Q;s^m) \cong (Q;t)$.

Theorem 1. (Existence of actions) *Let $\{K_i\}_{i=0}^q$ be a codimension two simple stratification of S^n by homotopy spheres. Let ν be any semifree or free $\mathbf{Z}_m$ action on K_0. Then S^n admits a semifree or free $\mathbf{Z}_m$ action τ preserving $\{K_i\}$ and restricting to ν, if and only if each Λ-invariant (Q_i, t_i) of (K_i, K_{i-1}) admits a $\Lambda_D(m)$-structure (Q_i, s_i) with $s_i^m = t_i$.*

Remark: By taking ν to be the trivial action, we see that *some* action always exists, whenever the algebraic condition on the Λ-invariants is satisfied.

Proof outline: We consider pairs of consecutive strata $k_i = (K_i , K_{i-1})$, with Λ-invariant $(Q_i ; t_i)$. We prove that for each k_i , an action exists, with prescribed semifree $\mathbf{Z}_m$ action on the submanifold K_{i-1} , if and only if $(Q_i ; t_i)$ admits a $\Lambda_D(m)$-structure $(Q_i ; s_i)$, $(Q_i ; s_i^m) \cong (Q_i ; t_i)$. This follows from consideration of the infinite cyclic cover of k_i , together with proving that the quotient by an action τ of the knot complement of k_i is again a knot complement, for a new knot k_{D_i}. The Λ-invariant for k_{D_i} is precisely $(Q_i ; s_i)$. This construction mirrors the case where τ acts on a knot (S, K) with restriction $\tau|_K$ either free or trivial [N1] [N2] [N3] [NS1] [NS2]. In our case the proof necessarily addresses the more general situation where the restriction $\tau|_K$ is any semifree action. □

The collection of Λ-invariants $\{(Q_i ; s_i) \mid i = 0, \ldots, q\}$ is called the *derived* Λ-invariant for the stratified action τ. The knots k_{D_i} are called *derived* knots. Two derived Λ-invariants are isomorphic $\{(Q_i ; s_i) \mid i = 0, \ldots, q\} \cong \{(Q_i' ; s_i') \mid i = 0, \ldots, q\}$ if for all i, $(Q_i ; s_i) \cong (Q_i' ; s_i')$.

Theorem 2. (Isovariant classification) *Let τ be a semifree or free $\mathbf{Z}_m$ action on S^n preserving stratification $\{K_i\}_{i=0}^q$. Then the isovariant isomorphism class of τ is determined, up to finite ambiguity, by the isomorphism class of the derived Λ-invariant, together with the isomorphism class of the restriction $\tau|_{K_0}$ to the lowest stratum.*

Remark: Notice in particular the case where K_0 is precisely the fixed set of the action τ. Then τ is determined, up to finite ambiguity, by the derived Λ-invariant.

Proof outline: Let $k_i = (K_i, K_{i-1})$, and let k_{D_i} be the derived knot. We prove that the isotopy class of k_{D_i}, together with the isomorphism class of the restriction $\tau|_{K_{i-1}}$, determine up to finite ambiguity, the action τ on K_i. The isomorphism of actions is constructed inductively, by building from the lowest stratum to the highest. The finite ambiguity is determined by the rotation by τ in the normal circle for the embedding $K_{i-1} \subset K_i$. □

Let $k = (S, K)$ be a knot, and τ a free or semifree $\mathbf{Z}_m$ action on S, such that: (a) K is a τ-invariant subspace, and (b) the fixed set of τ $S^\tau \subset K$. The we say that τ is a *quasifree* action on (S, K). We prove the following generalization of the unknotting theorem of Sumners [S]:

Unknotting Theorem. *A knot (S, K) admits quasifree $\mathbf{Z}_m$ actions for every m if and only if it is the unknot.*

Remark: We do *not* require that (S, K) be simple, or even 1-simple.

Proof outline: Let (S, K) be a knot which admits a quasifree $\mathbf{Z}_m$ action τ. We prove that (S, K) must then admit another quasifree $\mathbf{Z}_m$ action ν, where K is the fixed set $K = S^{\nu}$, and where ν and τ coincide on the knot complement. The existence, for all m, of quasifree $\mathbf{Z}_m$ actions ν_m where $S^{\nu_m} = K$ then forces (S, K) to be the unknot [S]. The theorem follows. □

Examples

In [CW] examples of codimension two stratified actions are produced, for stratifications of length 2:

$$K_0 \subset K_1 \subset K_2 = S^n$$

where (K_1, K_0) is knotted, and (K_2, K_1) is the unknot. We produce codimension two stratified actions of arbitrary length q:

$$K_0 \subset K_1 \subset \cdots K_{q-1} \subset K_q = S^n$$

where each (K_{i+1}, K_i) is knotted. We give here two classes of examples:

Odd dimensional spheres

Fix $n \geq 3$, and a collection $\{a_i \mid 0 \leq i \leq n\}$ of positive integers such that:

i) a_i are odd,

and

ii) a_i are coprime $(a_i, a_j) = 1$.

Embed $K_r^{2(n+r)-1} \subset K_{r+1}^{2(n+r)+1}$ as the link of singularity for the polynomial:

$$f_r(z_0, \ldots, z_n, \omega_1, \ldots, \omega_r) = \sum_{i=0}^{n} z_i^{a_i} + \sum_{j=1}^{r} \omega_j^2$$

Theorem 4. *The sphere $S^{2(n+q)-1}$ with stratification*

$$K_0 \subset K_1 \subset \cdots K_{q-1} \subset K_q = S^{2(n+q)-1}$$

admits stratified $\mathbf{Z}_m$ actions if and only if m is odd, and $(m, a_i) = 1$. Moreover all $\mathbf{Z}_m$ actions τ_m for the same m agree, up to isomorphism, on the complements of strata $K_r - K_{r-1}$.

Remark: Algebraic knots are not in general spherical. The knots (K_{i+1}, K_i) above are, as can be seen from consideration of the Alexander polynomial.

We also consider questions about the compatibility of $\mathbf{Z}_m$ actions, raised by Weinberger in [W1].

Corollary 5. *With the above stratification, $S^{2(n+q)-1}$ admits a stratified action by an infinite subgroup G of $\mathbf{Q}/\mathbf{Z}$.*

Proof outline: Compatibility on the knot complements follows from the uniqueness statement in Theorem 4. Compatibility on the normal bundle of the embedding $K_r^{2(n+r)-1} \subset K_{r+1}^{2(n+r)+1}$ follows by choosing the generators t_m of τ_m to represent rotations by compatible primitive m^{th} roots of unity. Since (a) rotation by every primitive m^{th} root of unity, together with (b) every free $\mathbf{Z}_m$ action on the complement, and (c) every semifree $\mathbf{Z}_m$ action on the codimension two submanifold K_r define an action τ_m, and since we may choose all actions τ_m to be the trivial action on K_0, the result follows.

Even dimensional spheres

Consider the stratification of $S^{2(n+q)}$

$$K_0^{2n} \subset K_1^{2(n+1)} \subset \ldots \subset K_q^{2(n+q)} = S^{2(n+q)} \qquad n \geq 4$$

where (K_i, K_{i-1}) is the knot with Λ-quintet $((\mathbf{Z}_{r_i}, 0, 0, \frac{d_i}{r_i}, 0); -1)$. Here r_i are odd, and $(d_i, r_i) = 1$ (see [N3]).

Theorem 6. *With the above stratification, $\mathbf{Z}_m$ acts if and only if m is odd. The action on the complements of strata $K_i - K_{i-1}$ is uniquely determined, up to isomorphism, by m. Moreover, for every $\mathbf{Z}_m$ action, the complements of strata in the quotient space are homeomorphic to the complements of strata in the sphere $S^{2(n+q)}$.*

Remark: The last statement implies that the quotient space is made up of the complements of strata in $S^{2(n+q)}$ glued to one another with twists.

Corollary 7. *With the above stratification, $S^{2(n+q)}$ admits a stratified action by an infinite subgroup G of $\mathbf{Q}/\mathbf{Z}$.*

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