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The Eleventh Annual Workshop in Geometric Topology was held at Park City, Utah on June 23-25, 1994. The participants were:

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These proceedings contain a summary of the three one-hour talks delivered by the principal speaker Mike Davis. Summaries of talks given by some of the other participants are also included. The success of the workshop was helped by generous funding from the Department of Mathematics at Brigham Young University.

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NONPOSITIVE CURVATURE AND REFLECTION GROUPS

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(*) These notes essentially are the text of three lectures which I gave at the Eleventh Annual Western Workshop in Geometric Topology in Park City, Utah in June, 1994.

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CONTENTS

I. Nonpositively Curved Spaces.

The notion of "nonpositive curvature" (or more generally of "curvature bounded above by a real number ϵ" makes sense for a more general class of metric spaces than Riemannian manifolds: one need only assume that any two points can be connected by a geodesic segment. For such spaces, the concept of curvature bounded above by ϵ can be defined via comparison triangles and the so-called "$CAT(\epsilon)$-inequality". (This terminology is due to Gromov.) This is explained in §1. In §2 we consider "piecewise constant curvature polyhedra" and give a condition (in terms of links of vertices) for such a polyhedron to have curvature bounded from above. The condition is that each link be $CAT(1)$. In §3 we discuss criteria for such a link to be $CAT(1)$. The two conditions we are most interested in are given in Gromov's Lemma and Moussong's Lemma. These give criteria for piecewise spherical simplicial complexes (with sufficiently big simplices) to be $CAT(1)$. In §4 we discuss a conjecture of H. Hopf concerning the Euler characteristic of a closed, nonpositively curved, even-dimensional manifold. Using the combinatorial version of the Gauss-Bonnet Theorem this leads us to a conjecture concerning a number associated to a piecewise spherical structure on an odd-dimensional sphere.

§1. The $CAT(\epsilon)$-inequality.

Given a smooth Riemannian manifold M one defines its "curvature tensor" and from this its "sectional curvature". The sectional curvature K of M is a real-valued function on the set of all pairs (x, P) where x is a point in M and P is a tangent 2-plane at x. Given a real number ϵ, we say that "M has curvature $\leq \epsilon$" and write $K(M) \leq \epsilon$ if the sectional curvature K is bounded above by ϵ.

It has long been recognized that the condition that the curvature of M is bounded above is equivalent to a condition which can be phrased purely in terms of the underlying metric (i.e., in terms of the distance function) on M. In fact, there are several possible versions of such a condition. We shall focus on one called the "$CAT(\epsilon)$ condition" by Gromov. ("CAT" stands for "Comparison" of "Alexandrov" and "Toponogov"). Once one has such a condition one can define the notion of "curvature $\leq \epsilon$" for many "singular" metric spaces, that is, for a more general class of metric spaces than Riemannian manifolds.

A good reference for this material is [GH].

We begin by stating the following Comparison Theorem of Alexandrov. A proof can be found in the article of M. Troyanov in [GH].

Theorem 1.1. (Alexandrov). *Let M be a simply connected, complete Riemannian manifold and ϵ a real number. Then $K(M) \leq \epsilon$ if and only if each geodesic triangle in M (of perimeter $\leq 2\pi/\sqrt{\epsilon}$) satisfies the $CAT(\epsilon)$-inequality.*

A "geodesic triangle" in M means a configuration in M consisting of three points (the "vertices") and three (minimal) geodesic segments connecting them (the "edges"). The term "$CAT(\epsilon)$" is explained below.

As $\epsilon > 0, = 0$, or < 0, let M^2_ϵ stand for S^2_ϵ (the 2-sphere of constant curvature ϵ) $\mathbb{E}^2$ (the Euclidean plane) or $\mathbb{H}^2_\epsilon$ (the hyperbolic plane of curvature ϵ).

Let T be a geodesic triangle in M. A *comparison triangle* for T is a geodesic triangle T^* in M^2_ϵ with the same edge lengths as T. Choose a vertex x of T and a point y on the opposite edge. Let x^* and y^* denote the corresponding points in T^*. (See Figure 1).

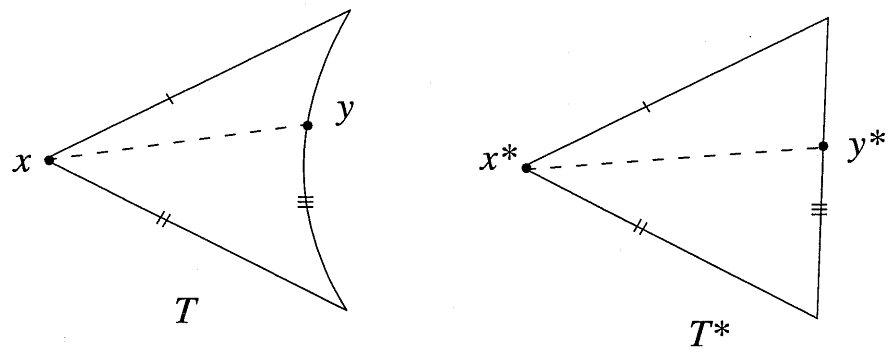


Figure 1

The $CAT(\epsilon)$-inequality is

$$d(x,y) \leq d^*(x^*,y^*)$$

where d and d^* denote distance in M and M^2_ϵ, respectively.

Remark. S^2_ϵ is the sphere of radius $1/\sqrt{\epsilon}$. Since any geodesic triangle $T^* \subset S^2_\epsilon$, must lie in some hemisphere, we see that the perimeter of T^* can be no larger than $2\pi/\sqrt{\epsilon}$ (the circumference of the equator). So, when $\epsilon > 0$, the $CAT(\epsilon)$-inequality only makes sense for triangles of perimeter $\leq 2\pi/\sqrt{\epsilon}$.

Now let (X,d) be a metric space. A path $\alpha : [a,b] \to X$ is a *geodesic* if it is an isometric embedding, i.e., if $d(\alpha(t),\alpha(s)) = |t-s|$ for all s,t in $[a,b]$.

Definition 1.2. A metric space X is a *geodesic space* (or a "length space") if an two points can be connected by a geodesic segment.

We shall also assume that X is complete and locally compact. (The hypothesis of local compactness could be replaced by local convexity.)

The notion of a geodesic triangle clearly makes sense in a geodesic space as does the $CAT(\epsilon)$-inequality.

Definition 1.3. A geodesic space X is $CAT(\epsilon)$ if the $CAT(\epsilon)$-inequality holds for all geodesic triangles T of perimeter $\leq 2\pi/\sqrt{\epsilon}$ and for all choices of vertex x and point y on the opposite edge. (The condition that the perimeter be $\leq 2\pi/\sqrt{\epsilon}$ is interpreted to be vacuous if $\epsilon \leq 0$.) X has *curvature* $\leq \epsilon$, written $K(X) \leq \epsilon$, if it satisfies $CAT(\epsilon)$ locally.

Remarks. (i) If $\epsilon' < \epsilon$, then $CAT(\epsilon')$ implies $CAT(\epsilon)$ and $K(X) \leq \epsilon'$ implies $K(X) \leq \epsilon$.

(ii) There is a completely analogous definition of curvature bounded from below: one simply reverses the $CAT(\epsilon)$-inequality. (See [ABN].)

Some consequences of $CAT(\epsilon)$.
(i) *There are no digons* in X of perimeter $< 2\pi/\sqrt{\epsilon}$. (A *digon* is a configuration consisting of two distinct geodesic segments between points x and y.) The reason is that we could

introduce a third vertex in the interior of one segment and obtain a triangle for which the $CAT(\epsilon)$-inequality clearly fails. As special cases of this principle, we have the following.

a) If X is $CAT(1)$, then a geodesic between two points of distance $< \pi$ is unique.

b) If X is $CAT(1)$, then there is no closed geodesic of length $< 2\pi$. (A *closed geodesic* is an isometric embedding of a circle.)

c) If X is $CAT(0)$, then any two points are connected by a unique geodesic.

(ii) *If X is $CAT(0)$, then the distance function $d : X \times X \longrightarrow [0, \infty]$ is convex.* (In general, a function $\varphi : Y \to \mathbb{R}$ on a geodesic space Y is *convex*, if given any geodesic path $\alpha : [a, b] \to Y$ the function $\varphi \circ \alpha : [a, b] \to \mathbb{R}$ is a convex function. In particular, $X \times X$ is a geodesic space and the statement that $d : X \times X \longrightarrow [0, \infty]$ is convex means that given geodesic paths $\alpha : [a, b] \longrightarrow X$ and $\beta : [c, d] \longrightarrow X$ the function $(s, t) \longrightarrow d(\alpha(s), \beta(t))$ is a convex function on $[a, b] \times [c, d]$.)

There is the following generalization of the Cartan-Hadamard Theorem for nonpositively curved manifolds.

Proposition 1.4. *If X is a geodesic space with convex distance function (e.g. if X is $CAT(0)$), then X is contractible.*

Proof. The convexity of the distance function implies that X has no digons. Hence, any two points of X are connected by a unique geodesic. Choose a base point x_0 and define the contraction $H : X \times I \to X$ by contracting along the geodesic to x_0. The proof that H is continuous follows easily from the convexity of d.

Remark. Suppose $K(X) \leq \epsilon$. Then since $CAT(\epsilon)$ holds locally, X is locally convex (i.e., in any sufficiently small open set, any two points are connected by a unique geodesic). Therefore, X is locally contractible. In particular, any such X has a universal cover.

Theorem 1.5. *Let $\epsilon \leq 0$. If X is a geodesic space with $K(X) \leq \epsilon$, then its universal cover $\tilde{X}$ is $CAT(\epsilon)$. (In particular, $\tilde{X}$ is contractible.)*

This theorem stated by Gromov in [G, p. 119] and proved in W. Ballman's article in [GH, p. 193]. (Quite possibly, it was known to Alexandrov.)

Remark. Theorem 1.5 is not true for $\epsilon > 0$. There is an analogous result for $\epsilon > 0$: the hypothesis of simple connectivity is unimportant, but one needs to rule out the possibility of closed geodesics of length $< 2\pi/\sqrt{\epsilon}$. A version of this is stated as Lemma 3.1, below.

Corollary 1.6. *If $K(X) \leq 0$, then X is a $K(\pi, 1)$-space (i.e., X is aspherical).*

2. Piecewise constant curvature polyhedra.

Let M^n_ϵ stand for $S^n_\epsilon, \mathbb{E}^n$ or $\mathbb{H}^n_\epsilon$ as ϵ is greater than, equal to, or less than 0, respectively. A "half-space" in S^n_ϵ is a hemisphere; a "half-space" in $\mathbb{E}^n$ or $\mathbb{H}^n_\epsilon$ has its usual meaning.

Definition 2.1. A (convex) *cell* in M^n_ϵ is a compact intersection of a finite number of half-spaces. (When $\epsilon > 0$, one can also require that the cell does not contain a pair of antipodal points.)

Definition 2.2. An M_ϵ *cell complex* X is a cell complex formed by gluing together cells in M^n_ϵ via isometries of their faces. (ϵ is fixed, n can vary.) If $\epsilon = 0, X$ is called *piecewise Euclidean* (abbreviated PE). If $\epsilon = 1, X$ is *piecewise spherical* (abbreviated PS).

Example 2.3. The surface of a cube is a PE complex.

If X is an M_ϵ cell complex, then we can measure the length ℓ of a path in X: the length of the portion of the path within a given cell is defined using arc length in M^n_ϵ. The *intrinsic metric* ℓ on X is defined as follows:

$$d(x,y) = inf\{\ell(w)|\alpha \text{ is a path from } x \text{ to } y\}$$

(If X is not path connected, the d may take ∞ as a value.)

Does the intrinsic metric give X the structures of a geodesic space? The issue is whether the infimum occurring in the definition of d can actually be realized by a minimal path. If X is locally finite and if there is a $\delta > 0$ so that all closed δ-balls in X are compact (e.g., if X is a finite complex), then the Arzela-Ascoli Theorem implies that X is a complete geodesic space.

Links. Suppose that σ is an n-cell in M^n_ϵ and that v is a vertex of σ. The Riemannian metric on M^n_ϵ gives an inner product on its tangent space $T_v(M^n_\epsilon)$ at v. The set of inward pointing directions at v is subset of the unit sphere in $T_v(M^n_\epsilon)$. In fact, this subset is a spherical cell, which we denote by $Lk(v,\sigma)$. We think of it as a cell in S^{n-1}, well-defined up to isometry.

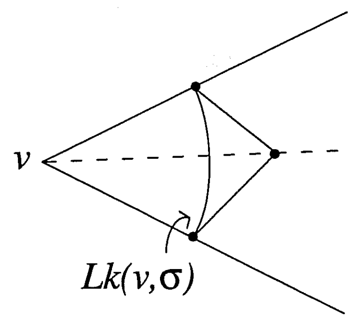


Figure 2

If X is an M_ϵ-complex and v is a vertex of X, then the *link of* v *in* X is defined by

$$Lk(v,X) = \bigcup_{\substack{v \\ v\subset\sigma}} Lk(v,\sigma)$$

This is a PS cell complex. Thus, the link of a vertex in any M_ϵ cell complex has a natural piecewise spherical structure.

Example 2.4. As in 2.3, let X be the surface of a cube and v, a vertex. Then the link of v in each square is a circular arc of length $\pi/2$; hence, the link of v in X is a circle of length $3\pi/2$.

In [G, p. 120] Gromov gave the following "infinitesimal" condition for deciding if a piecewise constant curvature polyhedron has curvature bounded from above.

Theorem 2.5. (Alexandrov, Gromov, Ballman) *Suppose X is an M_ϵ-cell complex. Then $K(X) \leq \epsilon$ if and only if for each vertex $v, Lk(v, X)$ is $CAT(1)$.*

A proof of this can be found in Ballman's article in [GH; p. 197]. The result must have also been known to Alexandrov's school, since they knew that an "M_ϵ cone" on a $CAT(1)$ space was $CAT(\epsilon)$.

Example 2.6. A PS structure on a circle S is $CAT(1)$ if and only if $\ell(S) \geq 2\pi$. Therefore, a PE structure on a surface has $K \leq 0$ if and only if at each vertex the sum of the angles is $\geq 2\pi$. For example, the surface of a cube does not have nonpositive curvature.

§3. The $CAT(1)$ condition for links.

In order to use Theorem 2.5 we need to be able to decide when the link of a vertex is $CAT(1)$. So, suppose L is some PS cell complex. We need to be able to answer the following.

Question. How do you tell if L is $CAT(1)$?

The following lemma gives an inductive procedure for studying this question.

Lemma 3.1. *A PS complex L is $CAT(1)$ if and only if*
(i) $K(L) \leq 1$, and
(ii) every closed geodesic in L has length $\geq 2\pi$.

By Theorem 2.5, condition (i) can be checked by looking at links of vertices in L. Thus, (ii) is the crucial condition.

We next would like to explain several situations in which we have a satisfactory answer to our question. These will be grouped under the following headings:

a) Gromov's Lemma
b) Moussong's Lemma
c) Orthogonal joins
d) Spherical buildings
e) Polar duals of hyperbolic cells
f) Branched covers of round spheres

In these notes we will mostly be concerned with the first two headings, (and we will confine ourselves to a few brief comments about the other four).

a) Gromov's Lemma. Let $\square^n$ denote a regular n-cube in $\mathbb{E}^n$ and let v be a vertex of $\square^n$. Then $Lk(v, \square^n)$ is the regular spherical $(n-1)$-simplex Δ^{n-1} spanned by the standard basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$. (So, Δ^{n-1} is the intersection of S^{n-1} with the positive "quadrant" $[0, \infty)^n$ in $\mathbb{R}^n$).

A spherical $(n-1)$-simplex isometric to Δ^{n-1} will be called an *all right* simplex.

An all right simplex is characterized by the fact that all its edge lengths are $\pi/2$. Alternatively, it can be characterized by the fact that all its dihedral angles are $\pi/2$.

Definition 3.2. A PS simplicial cell complex is *all right* if each of its simplices is all right.

Example 3.3. If X is a PE cubical complex, then each of its links is an all right simplicial cell complex.

Definition 3.4. A simplicial complex K is a *flag complex* if any finite set of vertices, which are pairwise joined by edges, span a simplex in K.

Combinatorialists use "clique complex" instead of "flag complex". Alternative terminology, which has been used elsewhere, is that K is "determined by its 1-skeleton", or K has "no empty simplices", or K satisfies the "no Δ-condition". (The last is Gromov's terminology). The term "flag complex" is taken from [Br].

Remark 3.5. Let V be a set with a symmetric, anti-reflexive, transitive relation (an "incidence relation"). Let K be the abstract simplicial complex whose simplices are the finite subsets of V which are pairwise related. Then K is a flag complex. Conversely, given a flag complex K, one defines a relation on its vertex set V by saying that two vertices are related if they are joined by an edge. This relation gives back K as its associated complex.

Example 3.6. Let $\mathcal{P}$ be a poset. Then (if we make the order relation symmetric) and take the associated simplicial complex, we get a flag complex. Its poset of simplices is denoted by $\mathcal{P}'$, and called the *derived complex* of $\mathcal{P}$. The elements of $\mathcal{P}'$ are finite chains $(v_0 < \cdots < v_k)$ in $\mathcal{P}$.

Example 3.7. If $\mathcal{P}$ is the poset of cells in a cell complex, then $\mathcal{P}'$ can be identified with the posets of simplices in its barycentric subdivision. Thus, *the barycentric subdivision of any cell complex is a flag complex.*

Example 3.8. If K is the boundary of an m-gon (i.e., K is a circle with m edges) then K is a flag complex if and only if $m > 3$.

Lemma 3.9. (Gromov's Lemma) *Let L be an all right, PS simplicial complex. Then L is $CAT(1)$ if and only if it is a flag complex.*

Corollary 3.10. (Berestovskii [Ber]) *Any polyhedron has a PS structure which is $CAT(1)$.*

Proof. Let L be a cell complex. By taking the barycentric subdivision we may assume that L is a flag complex. Then give L a piecewise spherical structure by declaring each simplex to be all right.

Corollary 3.11. *Let X be a PE cubical complex. Then $K(X) \leq 0$ if and only if the link of each vertex is a flag complex.*

Application 3.12. *(Hyperbolization).* In [G] Gromov described several functorial procedures for converting a cell complex J (usually a simplicial or cubical complex) into a PE cubical complex $\mathcal{H}(J)$ with nonpositive curvature. (See also [DJ] and [CD2].) $\mathcal{H}(J)$ is called a "hyperbolization" of J. Since $\mathcal{H}(J)$ is aspherical it cannot, in general, be homeomorphic to J. However, there is a natural surjection $\mathcal{H}(J) \to J$ Also, $\mathcal{H}(J)$ should have the same local structure as J in the following sense: the link of each "hyperbolized cell" is PL homeomorphic to the link of the corresponding cell in J. Usually, the new link will be the barycentric subdivision of the old one (or else the suspension of a barycentric subdivision of an old link). Thus, the new links will be flag complexes and Gromov's Lemma can be used to prove that $\mathcal{H}(J)$ is nonpositively curved. (A different argument is given in [DJ]] and [G].)

The proof of Gromov's Lemma is based on the following.

Sublemma 3.13. ([G, p.122]). *Let v be a vertex in an all right, PS simplicial complex and let B be the closed ball of radius $\pi/2$ about v (i.e., B is the closed star of v). Let x, y be points in ∂B (the sphere of radius $\pi/2$ about v) and let γ be a geodesic segment from x to y such that γ intersects the interior of B. Then $\ell(\gamma) \geq \pi$.*

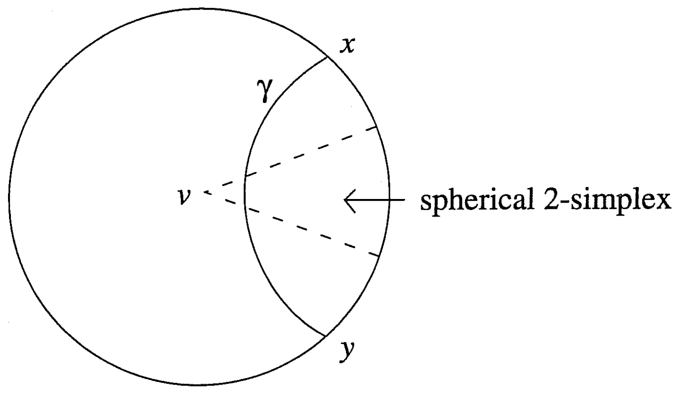


Figure 3

Proof. Let Δ be an all right simplex in B with one vertex at v and suppose that γ intersects the interior of Δ. Consider the union of all geodesic segment which start at v, pass through a point in $\gamma \cap \Delta$ and end on the face of Δ opposite to v. It is an isoceles spherical 2-simplex with two edges of length $\pi/2$. (Think of a spherical 2-simplex with one vertex at the north pole and the other two on the equator.) Let Ω be the union of all these 2-simplices. Then Ω can be "developed" onto the northern hemisphere of S^2. If $\ell(\gamma) < \pi$, then Ω is isometric to a region of S^2 so that v maps to the north pole and x and y to points on the equator. But if two points on the equator of S^2 are of distance $< \pi$, then the geodesic between them is a segment of the equator. This contradicts the hypothesis that the image of γ intersects the open northern hemisphere.

Proof of Gromov's Lemma. Let L be an all right, PS simplicial cell complex. First suppose that L is not a flag complex. Then either L or the link of some simplex of L contains an "empty" triangle. Such a triangle is a closed geodesic of length $3\pi/2$ (which is $< 2\pi$). Hence, L is not $CAT(1)$.

Conversely, suppose that L is a flag complex. Then the link of each vertex is also a flag complex and by induction on dimension we may assume that $K(L) \leq 1$. Hence, it suffices to show every closed geodesic in L has length $\geq 2\pi$. Suppose, to the contrary, that α is a closed geodesic with $\ell(\alpha) < 2\pi$. Let L' be the full subcomplex of L spanned by the set of vertices v such that $\alpha \cap \mathrm{Star}(v) \neq \emptyset$. (Here $\mathrm{Star}(v)$ denotes the open star of v.) By Sublemma 3.13, α cannot intersect two disjoint open stars. Hence any two vertices of L' must be connected by an edge. Since L is a flag complex, this implies L' is an all right simplex. But this is impossible since a simplex contains no closed geodesic.

b) Moussong's Lemma.

Definition 3.14. A spherical simplex has *size* $\geq \pi/2$ if all of its edge lengths are $\geq \pi/2$.

Let L be a PS simplicial complex with simplices of size $\geq \pi/2$.

Definition 3.15. L is a *metric flag complex* if given a set of vertices $\{v_0, \cdots, v_k\}$, which are pairwise joined by edges, such that there exists a spherical k-simplex with these edge lengths, then $\{v_0, \cdots, v_k\}$ spans a k-simplex in L.

Lemma 3.16. (Moussong's Lemma) *Let L be a PS simplicial complex with simplices of size $\geq \pi/2$. Then L is $CAT(1)$ if and only if L is a metric flag complex.*

This generalization of Gromov's Lemma is the main technical result in the Ph.D. thesis of G. Moussong [M]. Its proof is quite a bit more difficult than that of Gromov's Lemma and we will not try to explain it here. We will use it in the next chapter to show that a certain PE complex associated to any Coxeter group is $CAT(0)$.

Definition 3.17. A cell is *simple* if the link of each vertex is a simplex.

The edge lengths of such a simplex are interior angles in the 2-dimensional faces. Thus, such a simplex has size $\geq \pi/2$ if all such angles in the 2-cells are $\geq \pi/2$.

Corollary 3.18. *Let X be a PE complex with simple cells and with 2-cells having nonacute angles. Then $K(S) \leq 0$ if and only if the link of each vertex is a metric flag complex.*

c) Orthogonal joins. Suppose that $\sigma_1 \subset S^{k_1}$ and $\sigma_2 \subset S^{k_2}$ are spherical cells. Regard S^{k_1} and S^{k_2} as a pair of orthogonal great subspheres in $S^{k_1+k_2+1} \subset \mathbb{R}^{k_1+1} \times \mathbb{R}^{k_2+1}$. Then the *orthogonal join* $\sigma_1 * \sigma_2$ of σ_1 and σ_2 is the union of all geodesic segments from σ_1 to σ_2 in $S^{k_1+k_2+1}$. It is naturally a spherical cell of dimension equal to $\dim \sigma_1 + \dim \sigma_2 + 1$. If L_1 and L_2 are PS cell complexes, then their orthogonal join $L_1 * L_2$ is defined to be the union of all cells $\sigma_1 * \sigma_2$ where σ_1 is a cell in L_1 and σ_2 is a cell in L_2. It is naturally a PS cell complex, homeomorphic to the usual topological join of the underlying polyhedra.

The following result is proved in the Appendix of [CD1].

Proposition 3.19. *If L_1 and L_2 are $CAT(1)$, PS cell complexes, then $L_1 * L_2$ is $CAT(1)$.*

Remark 3.20. For example, taking L_2 to be a point we see if L_1 is $CAT(1)$, then so is the "spherical cone" on L_1. Similarly, taking $L_2 = S^0$, we see that the "spherical suspension" of L_1 is $CAT(1)$.

d) Spherical buildings. Tits has defined a certain remarkable class of simplicial complexes called "buildings", e.g., see [Br] and [R]. Associated to a building B there is a Coxeter group W. (This will be defined in Chapter II.) The building B can be written as a union of apartments A_α,

$$B = \bigcup A_\alpha,$$

where each A_α is isomorphic to the Coxeter complex for W. If W is a finite group, then this Coxeter complex can naturally be thought of as a triangulation of S^n, the round n-sphere, for some n. The building is called *spherical* if its associated Coxeter group is finite (so that each apartment is a round sphere). Thus, a spherical building has a natural structure of a PS simplicial complex.

The axioms for buildings imply that any two points in B lie in a common apartment. Furthermore, (at least when B is spherical) the geodesic between them also lies in this apartment. From this we can immediately deduce the following.

Theorem 3.21. *Any spherical building is $CAT(1)$.*

Example 3.22. A *generalized m-gon* is a connected, bipartite graph of diameter m and girth $2m$. (A graph is *bipartite* if its vertices can be partitioned into two sets so that no two vertices in different sets span an edge. The *diameter* of a graph is the maximum distance between two vertices, its *girth* is the minimum length of a circuit.) A 1-dimensional spherical building is the same thing as a generalized m-gon ($m \neq \infty$). The piecewise spherical structure is defined by declaring each edge to have length π/m.

e) Polar duals of hyperbolic cells. Suppose that C^n is a convex n-cell in hyperbolic n-space $\mathbb{H}^n$. Let F be a face of codimension k in $C^n, k \geq 1$. Choose a point x in the relative interior of F and consider the unit sphere S^{n-1} in the tangent space $T_x\mathbb{H}^n$. The set of outward-pointing unit normals to the codimension one faces which contain F span a spherical $(k-1)$-cell in S^{n-1} which we denote by σ_F. (Roughly, σ_F is the set of all outward pointing unit normals at F.) The *polar dual* of C^n is defined as

$$P(C^n) = \bigcup_F \sigma_F.$$

It is a PS cell complex, which, it is not difficult to see, is homeomorphic to S^{n-1}. For further details, see [CD4].

Remark 3.23. (i) The same construction can be carried out in $\mathbb{E}^n$ or S^n. For a cell in $\mathbb{E}^n$ its polar dual is a PS cell complex which is isometric to the round $(n-1)$-sphere. For a cell C^n in S^n, its polar dual is just the boundary of the dual cell C^*, where $C^* = \{x \in S^n | d(x, C^n) \geq \pi/2\}$. In all three cases, the cell structure on $P(C^n)$ is combinatorically equivalent to the boundary to the dual polytope to C^n.

(ii) If we use the quadratic form model for $\mathbb{H}^n$, and $C^n \subset \mathbb{H}^n$, then $P(C^n)$ is naturally a subset of the unit pseudosphere, $S^n_1 = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle = 1\}$, where $\langle x, x\rangle = -(x_1)^2 + (x_2)^2 + \cdots + (x_{n+1})^2$.

Theorem 3.24. *Suppose C^n is a convex cell in $\mathbb{H}^n$. Then $P(C)$ is $CAT(1)$.*

When $n = 2$, $P(C^2)$ is a circle, the length of which is the sum of the exterior angles of C^2. By the Gauss-Bonnet Theorem, this sum is $2\pi + \text{Area}(C^2)$. This completes the proof for $n = 2$. When $n = 3$, the theorem is due to Rivin and Hodgsen [RH]. For $n > 3$, it appears in [CD4].

Further Remarks 3.25. (i) A stronger result is actually true. The length of any closed geodesic in $P(C^n)$ is strictly greater than 2π. (As we saw, for $n = 2$, this follows from the Gauss-Bonnet Theorem.) Furthermore, the same is true for the link of every cell in $P(C^n)$ (since such a link is, in fact, the polar dual of some face of C^n.) Sometimes I have defined a PS cell complex to be "large" if it is $CAT(1)$. Perhaps PS complexes satisfying the above stronger condition should be "extra large".

(ii) The definition of polar dual makes sense for any intersection of half-spaces in $\mathbb{H}^n$ (compact or not) and it is proved in [CD 4] that these polar duals are also $CAT(1)$.

(iii) The main argument of [RH] is in the converse direction. They show that any PS structure on S^2 which is extra large arises as the polar dual of a 3-cell in $\mathbb{H}^3$ (unique up to isometry). An analogous result, relating metrics with $K \geq 1$ on S^2 to convex surfaces in $\mathbb{E}^3$ had been proved much earlier by Alexandrov.

(iv) My main interest in Theorem 3.24 is that it provides a method for constructing a large number of examples of $CAT(1)$, PS structures on S^{n-1}, which are not covered by Moussong's Lemma. Moreover, if we deform a convex cell in $\mathbb{H}^n$ we obtain a large family of deformations of its polar dual through $CAT(1)$ structures.

f) Branched covers of round spheres.

Suppose that M^n is a smooth Riemannian manifold and that $p : \tilde{M}^n \to M^n$ is a branched covering by some other manifold $\tilde{M}^n$.

Question. If M^n has sectional curvature $\leq \epsilon$, then when is $K(\tilde{M}^n) \leq \epsilon$?

We further suppose that the branching is locally modeled on $\mathbb{R}^n \to \mathbb{R}^n/G$ where G is some finite linear group. (Since M^n is a manifold we must therefore have that $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$.)

The following two conditions are easily seen to be necessary for $K(\tilde{M}) \leq \epsilon$:

(i) $K(M) \leq \epsilon$

(ii) locally, the closure of each stratum of the branched set is a convex subset of M.

Let $x \in M$ be a branch point and let S_x be the unit sphere in T_xM. There is an induced finite sheeted branched cover $\tilde{S}_x \to S_x$. Since the branched set in S_x must satisfy (ii), it follows that the metric on $\tilde{S}_x$ (induced from the round metric on S_x) is piecewise spherical. We think to $\tilde{S}_x$ as the "link" at a point $\tilde{x} \in p^{-1}(x)$. It turns out ([CD1, Theorem 5.3]) that together with (i) and (ii) the following condition is necessary and sufficient for $K\tilde{M})$ to be $\leq \epsilon$:

(iii) $\tilde{S}_x$ is $CAT(1)$, for all branch points x.

Therefore, the answer to our question is closely tied to the question of when the branched cover of a round sphere is $CAT(1)$. A detailed study of this question is made in [CD 1].

For example, suppose that G is a finite, noncyclic subgroup of $SO(3)$ (so that G is either dihedral or the group of orientation-preserving symmetries of a regular solid). Then S^2/G is homeomorphic to S^2 and $S^2 \to S^2/G$ has three branch points. Choose three points x_1, x_2 and x_3 in the round 2-sphere S^2 and assign x_i a branching order of m_i, where $\Sigma(1/m_i) > 1$. Let $\tilde{S}^2 \to S^2$ be the corresponding $|G|$-fold branched cover. In [CD1] we prove the following result.

Proposition 3.25. *$\tilde{S}^2$ is $CAT(1)$ if and only if*

(i) x_1, x_2, and x_3 lie on a great circle in S^2, but are not contained in any semi-circle, and

(ii) $d(x_i, x_j) \geq \pi/m_k$, where (i, j, k) is some permutation of $(1, 2, 3)$.

4. Euler characteristics and the Combinatorial Gauss-Bonnet Theorem.

Hopf's Conjecture. *Suppose M^{2n} is a closed Riemannian manifold, with $K(M) \leq 0$. Then $(-1)^n\chi(M^{2n}) \geq 0$. (Here χ denotes the Euler characteristic.)*

Remark 4.1. (i) There is an analogous version of this conjecture for nonnegative curvature: the Euler characteristic should be nonnegative.

(ii) Thurston has conjectured that Hopf's Conjecture should hold for any closed, aspherical $2n$-manifold.

The reason for believing this is the Gauss-Bonnet Theorem (proved by Chern in dimensions > 2). Recall that this is the following theorem.

Gauss-Bonnet Theorem.

$$\chi(M^{2n}) = \int P$$

Here P is a certain $2n$-form called the "Pfaffian" or the "Euler form". This leads to the following.

Question 4.2. Does $K(M^{2n}) \leq 0$ imply that $(-1)^n P \geq 0$? (In other words, is $(-1)^n P$ equal to the volume form multiplied by a nonnegative function?)

In dimension 2 the answer is, of course, yes, since P is then just the volume form times the curvature. The answer is also yes in dimension 4. A proof is given by Chern in [C], where the result is attributed to Milnor.

Hopf's Conjecture holds in higher dimensions under the hypothesis that the "curvature operator" is negative semi-definite (which is stronger than assuming that the sectional curvature is nonpositive). This is actually easy to see. The curvature tensor can be viewed as a self-adjoint (= symmetric) linear endomorphism $R : \Lambda^2 T_x M \to \Lambda^2 T_x M$, called the curvature operator. This is negative semi-definite if $\langle Rv, v\rangle \leq 0$ for all $v \in \Lambda^2 T_x M$. On the other hand, the sectional curvature is ≤ 0 if and only if $\langle Rv, v\rangle \leq 0$ for all primitive 2-vectors v (i.e., for v of the form $e \wedge f$). If the eigenvalues of R are all ≤ 0, then $(-1)^n P \geq 0$.

On the other hand, in dimensions ≥ 6, Geroch [Ge] showed in 1976 that the answer to Question 4.2 is no. Geroch's argument comes down to an example in linear algebra. He writes down a symmetric matrix $R : \Lambda^2\mathbb{R}^6 \to \Lambda^2\mathbb{R}^6$ such that $\langle Rv, v\rangle < 0$ or all primitive 2-vectors v and such that $P = \Lambda^3(R) > 0$.

The following combinatorial version of the Gauss-Bonnet Theorem is a classical result. A proof can be found in [CMS], where one can also find a convincing argument that it is the correct analog of the smooth Gauss-Bonnet Theorem.

Theorem 4.3. *Suppose X is a finite, PE cell complex. Then*

$$\chi(X) = \sum_v P(Lk(v, X)).$$

Here P is a certain function which assigns a real number to any finite, PS cell complex. We define it below.

Let $\sigma \subset S^k$ be a spherical k-cell. Its dual cell σ^* is defined by $\sigma^* = \{x \in S^k | d(x, \sigma) \geq \pi/2\}$. Let $a^*(\sigma)$ be the volume of σ^* normalized so that volume of S^k is 1, i.e.,

$$a^*(\sigma) = \frac{\mathrm{vol}(\sigma^*)}{\mathrm{vol}(S^k)}$$

If L is a finite, PS cell complex then $P(L)$ is defined by

$$P(L) = 1 + \sum_{\sigma} (-1)^{\dim \sigma + 1} a^*(\sigma),$$

where the summation is over all cells σ in L.

Example 4.4. . Suppose that σ is an all right k-simplex. Then σ^* is also an all right k-simplex. Since S^k is tessellated by 2^{k+1} copies of σ^* we see that $a^*(\sigma) = (\frac{1}{2})^{k+1}$. Now let L be an all right PS simplicial complex and f_i the number of i-simplices in L. Then

$$\begin{aligned} P(L) &= 1 + \sum_{\sigma} (-\frac{1}{2})^{\dim \sigma + 1} \\ &= 1 + \sum_{i} (-\frac{1}{2})^{i+1} f_i. \end{aligned}$$

The following conjecture asserts that the answer to the combinatorial version of Question 4.2 should always be yes.

Conjecture 4.5. *Suppose that L^{2n-1} is a PS cell complex homeomorphic to S^{2n-1}. If L^{2n-1} is $CAT(1)$, then $(-1)^n P(L^{2n-1}) \geq 0$.*

Thus, this conjecture implies Hopf's Conjecture for PE manifolds.

If L is a flag complex, then, by Gromov's Lemma and Example 4.4, we have the following special case.

Conjecture 4.6. *Suppose that L^{2n-1} is a flag complex which triangulates S^{2n-1}. Then*

$$(-1)^n (1 + \sum (-\frac{1}{2})^{i+1} f_i) \geq 0.$$

So, this conjecture implies Hopf's Conjecture for PE cubical complexes which are closed manifolds.

Remark 4.7. Conjecture 4.6 is analogous to the Lower Bound Theorem in the combinatorics (a result concerning inequalities among the f_i for simplicial polytopes). For example, the Lower Bound Theorem of [W] asserts that for any simplicial complex L which triangulates S^3, we have $f_1 \geq 4f_0 - 10$. Conjecture 4.6 asserts that, if, in addition, L is a flag complex, then $f_1 \geq 5f_0 - 16$. Some evidence for these conjectures is provided by the following two propositions. The first result follows from recent work of R. Stanley [St] as was observed by Eric Babson.

Proposition 4.8. *Suppose that L^{2n-1} is the barycentric subdivision of the boundary complex of a convex $2n$-cell (so that L is a flag complex). Then Conjecture 4.6 hold for L.*

Proposition 4.9. *Suppose that L^{2n-1} is the polar dual of a convex cell C^{2n} in $\mathbb{H}^n$. Then Conjecture 4.5 holds for L.*

This proposition follows from a formula of Hopf (predating the general Gauss-Bonnet Theorem) which asserts the $(-1)^n P(L^{2n-1})$ is one-half the hyperbolic volume of C^{2n} (suitably normalized).

Further details about these conjectures and further evidence for them can be found in [CD3].

Remark 4.10. A natural reaction to Conjecture 4.5 is that it might contradict Geroch's result. One could try to obtain such a contradiction as follows. Take a smooth Riemannian manifold M^{2n} whose curvature operator at a point x is as in Geroch's result. Then try to approximate M^{2n} near x by a PE cell complex with nonpositive curvature. By the main result of [CMS] the numbers $P(L)$ for L a link in the complex, should approximate the Pfaffian at x and hence, have the wrong sign. However, it is not clear that such a approximation exists. Thus, we are led to ask the following.

Question 4.11. Suppose M is a Riemannian manifold with $K(M) \leq 0$ (we could even assume the inequality is strict). Is M homeomorphic to a PE cell complex X with $K(X) \leq 0$?

For our conjectures to be correct, the answer should be no.

II Coxeter Groups.

Coxeter groups and Coxeter systems are defined in §5. Associated to a Coxeter system there is a simplicial complex called its "nerve". The basic result of §5 is Lemma 5.6, which asserts that any finite polyhedron can occur as the nerve of some Coxeter system. Eventually, this will be used to show that Coxeter groups provide a rich and very flexible source of examples.

In §6 and §7 we discuss a beautiful, piecewise Euclidean cell complex Σ which is naturally associated to a Coxeter system (W, S). From the results of Chapter I, we get the important result of Moussong (generalizing an earlier observation of Gromov), that Σ is nonpositively curved and hence, contractible (since it is simply connected).

In §8 we briefly discuss some important special cases of this construction.

§5. Coxeter systems.

Let S be a finite set.

Definition 5.1. A *Coxeter matrix* $M = (m_{ss'})$ is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{ss'} = \begin{cases} 1 & : \text{ if } s = s' \\ \geq 2 & ; \text{ if } s \neq s' \end{cases}$$

Definition 5.2. Given a Coxeter matrix M, define a group W with presentation:

$$W = \langle S | (ss')^{m_{ss'}} = 1, \forall (s, s') \in S \times S \rangle$$

W is called a *Coxeter group.*

If all the off-diagonal entries of M are 2 or ∞, then W is called *right-angled.*

Coxeter groups are intimately connected to the theory of reflection groups. This connection will not be emphasized in these notes. For now it should suffice to mention that if a group W acts properly on a connected manifold and if W is generated by reflections (where a *reflection* is an involution whose fixed point set separates the manifold), then W is a Coxeter group (cf. [D1, Theorem 4.1]).

Given M, it is proved in Ch. V §4.3 of [B, pp. 91-92] that one can find a family $(\rho_s)_{s \in S}$ of linear reflections $\rho_s : \mathbb{R}^S \to \mathbb{R}^S$ so that $\rho_s \circ \rho_{s'}$ has order $m_{ss'}$ for all $(s, s') \in S \times S$. It follows that the map $s \to \rho_s$ extends to a representation $\rho : W \to GL(\mathbb{R}^S)$ called the *canonical representation.* The existence of this representation immediately implies the following:

a) the natural map $S \to W$ is an injection (and henceforth, we shall identify S with its image in W),

b) order $(s) = 2$, for all $s \in S$

c) order $(ss') = m_{ss'}$, for all $(s, s') \in S \times S$.

The pair (W, S) is called a *Coxeter system.*

Remark 5.3. It is proved in Ch. V §4.4 of [B, pp. 92-94] that the dual $\rho^* : W \to G((\mathbb{R}^S)^*)$ is faithful and has discrete image. Moreover, as explained in §9, W acts properly on certain open convex subset of $(\mathbb{R}^S)^*$. (These results are due to Tits.)

If T is a subset of S, then let W_T denote the subgroup of W generated by T.

Lemma 5.4. ([B, p. 20]) *For any $T \subset S$, the pair (W_T, T) is a Coxeter system (i.e., its Coxeter matrix is $M|T$).*

Let (W, S) be a Coxeter system. We define a poset, denoted $\mathcal{S}^f(W, S)$ (or simply $\mathcal{S}^f$) by

$$\mathcal{S}^f = \{T | T \subset S \text{ and } W_T \text{ is finite}\}$$

It is partially ordered by inclusion. Consider $\mathcal{S}^f - \{\emptyset\}$. It is isomorphic to the poset of simplices of an abstract simplicial complex which we shall denote by $N(W, S)$ (or simply N). N is called the *nerve* of (W, S).

In other words, the vertex set of N is S and a subset T of S spans a simplex if and only if W_T is finite.

Example 5.5. If W is finite, then N is the simplex on S.

Which finite polyhedra occur as the nerve of some Coxeter system? The next two results show that they all do.

Lemma 5.6. *Let L be any flag complex. Then there is a right-angled Coxeter system (W, S) with $N(W, S) = L$.*

Proof. Let S be the vertex set of L and define a Coxeter matrix $(m_{ss'})$ by

$$m_{ss'} = \begin{cases} 1 & , \text{ if } s = s' \\ 2 & , \text{ if } \{s, s'\} \text{ spans an edge in } L \\ \infty & , \text{ otherwise} \end{cases}$$

If W is the associated right-angled Coxeter group, then $N(W, S) = L$.

In particular, since the barycentric subdivision of any (regular) cell complex is a flag complex, we have the following corollary.

Corollary 5.7. *For any finite polyhedron P, there is a right-angled Coxeter system (W, S) with $N(W, S)$ homeomorphic to P.*

The main result of this chapter is the following theorem.

Theorem 5.8. (Gromov, Moussong) *Associated to a Coxeter system (W, S) there is a PE cell complex $\Sigma(W, S)(= \Sigma)$ with the following properties.*

i) The poset of cells in Σ is

$$\coprod_{T \in \mathcal{S}^f} W/W_T.$$

(ii) W acts by isometries on Σ with finite stabilizers and with compact quotient.

(iii) Each cell in Σ is simple (so that for each vertex v, $Lk(v, \Sigma)$ is a simplicial cell complex). In fact, this complex is just $N(W, S)$.

(iv) Σ is $CAT(0)$.

§6. Coxeter cells.

Throughout this section we suppose that W *is finite.*

In this case, we will show that Σ can be identified as a convex cell in $\mathbb{R}^n$ ($n =$ Card (S)).

The dual of the canonical representation shows that W can be represented as an orthogonal linear reflection group on $\mathbb{R}^n$. They hyperplanes of reflection divide $\mathbb{R}^n$ into "chambers", each of which is a simplicial cone. (See p. 85 in [B].)

Choose a point x in the interior of some chamber. Define Σ to be the convex hull of Wx (the orbit of x). Σ is called a *Coxeter cell* of *type* W.

The proof of the next lemma is an easy exercise.

Lemma 6.1. *Suppose* W *is finite.*

(i) The vertex set of the Coxeter cell Σ *is* Wx.

(ii) Each face F *of* Σ *is the convex hull of a set of vertices of the form* $(wW_T)x$ *for some* $T \subset S$ *and some coset* wW_T *of* W_T. *(So,* F *is a Coxeter cell of type* W_T.*)*

(iii) The poset of faces of Σ *is therefore,*

$$\coprod_{T \subset S} W/W_T.$$

(iv) Σ *is simple cell.* $Lk(x, \Sigma)$ *is the spherical* $(n-1)$*-simplex spanned by the outward pointing unit normals to the supporting hyperplanes of a chamber.*

Remark 6.2. If x lies in a chamber with supporting hyperplanes indexed by S, then the vertex set of $Lk(x, \Sigma)$ is naturally identified with S. Moreover, the length of the edge from s to s' is $\pi - \pi/m_{ss'}$. (In other words, the corresponding angle in a 2-cell in Σ is $\pi - \pi/m_{ss'}$.)

Example 6.3. (i) If $W = \mathbb{Z}/2$, then Σ is an interval.

(ii) If $W = D_m$ (the dihedral group of order $2m$), then Σ is a $2m$-gon.

(iii) If (W, S) is the direct product of two Coxeter systems (W_1, S_1) and (W_2, S_2) (so that $W = W_1 \times W_2$ and $S = S_1 \coprod S_2$), then $\Sigma(W, S) = \Sigma(W_1, S_1) \times \Sigma(W_2, S_2)$. In particular, if $W = (\mathbb{Z}/2)^n$, then Σ is a n-dimensional box ($=$ the product of n intervals).

(iv) If $W = S_n$, the symmetric group on n letters, then Σ is the $(n-1)$-cell called the permutahedron. The picture for $n = 4$ is given below.

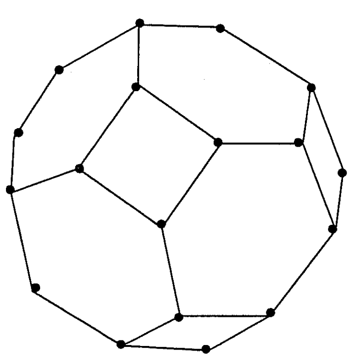


Figure 4

Remark 6.4. By choosing x to be of distance 1 from each supporting hyperplane we can normalize each Coxeter cell so that every edge length is 2.

§7. The cell complex Σ (in the case where W is infinite).

There is an obvious way to generalize the material of the previous section to the case where W is infinite. The cell complex Σ is defined as follows. The vertex set of Σ is W. Take a Coxeter cell of type W_T for each coset $wW_T, T \in \mathcal{S}^f$. Identify the vertices of this Coxeter cell with the elements of wW_T. Identify two faces of two Coxeter cells if they have the same set of vertices. This completes the definition of Σ as a cell complex.

If we normalize each Coxeter cell as in Remark 6.4, then the faces of the cells are identified isometrically and hence, Σ has the structure of a PE cell complex.

Remark 7.1. Let $\lambda : S \longrightarrow (0, \infty)$ be a function. If, in the definition of each Coxeter cell, we choose the point x to be of distance $\lambda(s)$ from the hyperplane corresponding to s, then the Coxeter cells again fit together to give a PE structure on Σ. We have arbitrarily chosen λ to be the constant function.

Remark 7.2. By construction, the poset

$$W\mathcal{S}^f = \coprod_{T \in \mathcal{S}^f} W/W_T$$

is the poset of cells in Σ. If $\mathcal{P}$ is any poset, then let $\mathcal{P}'$ be its derived complex defined as in Example 3.8, (i.e., $\mathcal{P}'$ is the poset of finite chains in $\mathcal{P}$). $\mathcal{P}'$ is the poset of simplices in an abstract simplicial complex. Moreover, if $\mathcal{P}$ is the poset of cells in a (regular) cell complex, then $\mathcal{P}'$ is the poset of simplices in its barycentric subdivision. Applying these remarks to the case at hand, we see that the barycentric subdivision Σ' of Σ is just the geometric realization of $(W\mathcal{S}^f)'$. Alternatively, we could have defined Σ' as the geometric realization of $(W\mathcal{S}^f)'$ and then remarked that the poset of chains which terminate in wW_T can naturally be identified with the set of simplices in the barycentric subdivision of a Coxeter cell of type W_T. Hence, the cellulation of Σ by Coxeter cells could be recovered from Σ' by collecting together the appropriate simplices.

The link of a vertex. The group W acts isometrically on Σ and freely and transitively on its vertex set. Thus, there is an isometry of Σ which takes any vertex onto the element $1 \in W$. What is $Lk(1, \Sigma)$ (as a simplicial complex)? A cell contains the element 1 if and only if it corresponds to some identity coset W_T. Hence, the poset of simplices in $Lk(1, \Sigma)$ is just $\mathcal{S}^f - \{\emptyset\}$, i.e.,

$$Lk(1, \Sigma) = N(W, S).$$

What is the induced PS structures on N? Two distinct vertices s and s' of N are connected by an edge $e_{ss'}$ if and only if $m_{ss'} \neq \infty$. By Remark 6.2, $\ell(e_{ss'}) = \pi - \pi/m_{ss'}$ (where ℓ stands for length). Since a spherical simplex is determined by its edge lengths this determines PS structure on N.

A proof of the following lemma can be found in [B, p.98].

Lemma 7.3. *Let T be a subset of S. Consider the $T \times T$ matrix, $c_{ss'}$, where*

$$c_{ss'} = cos(\pi - \pi/m_{ss'}).$$

Then W_T is finite if any only if $(c_{ss'})$ is positive definite.

Proof. Consider the dual of the canonical representation of W_T into $GL(n)$. Suppose W_T is finite. Then we may assume that the image of this representation is contained in $O(n)$. For each $s \in T$, let u_s be the outward-pointing unit normal to the hyperplane corresponding to s. Then $(u_s \cdot u_{s'}) = (c_{ss'})$ and hence, this matrix is positive definite since $(u_s)_{s\in T}$ is a basis for $\mathbb{R}^n$. Conversely, suppose that $(c_{ss'})$ is positive definite. Since the dual of the canonical representation preserves the corresponding bilinear form, we get that the image of this representation is contained in $O(n)$. Since this representation is also discrete and faithful, W_T is a discrete subgroup of $O(n)$; hence, finite.

Corollary 7.4. *$N(W, S)$ is a metric flag complex.*

We can now prove the main result.

Proof of Theorem 5.8. We have already demonstrated the required properties of Σ except for (iv). Thus, it suffices to prove that Σ is $CAT(0)$. First of all, it is easy to see that Σ is simply connected. (One argument is to observe that the 2-skeleton of Σ is just the universal cover of the 2-complex associated to standard presentation of W.) Hence, by Theorem 1.5, it suffices to show $K(\Sigma) \leq 0$. The link of any vertex is isometric to N. By Corollary 7.4 and Moussong's Lemma (Lemma 3.16) N is $CAT(1)$. Therefore, $K(\Sigma) \leq 0$ (by Theorem 2.5).

Remark 7.5. Theorem 5.8 was proved by Gromov [G, pp. 131-132] in the special case where W is right-angled. The general case was proved in [M]. The point is that in the right-angled case Σ is a cubical complex so we can use Gromov's Lemma (Lemma 3.9) rather than Moussong's generalization of it.

§8. Applications.

a) Two dimensional complexes. Let L be a finite graph and m and integer ≥ 2. Let k be the girth of L (the length of the shortest circuit). If $m = 2$, then we assume $k \geq 4$. Let $S = Vert(L)$ (the vertex set of L) and define a Coxeter matrix by

$$m_{ss'} = \begin{cases} 1 & ;\ \text{if } s = s' \\ m & ;\ \text{if}\{s, s'\} \text{ spans an edge} \\ \infty & ;\ \text{otherwise.} \end{cases}$$

Let W be the resulting Coxeter group. Our assumption implies that $N(W, S) = L$. Thus, $\Sigma(W, S)$ is a $CAT(0)$, PE 2-complex such that each 2-cell is a regular $2m$-gon and such that the link of each vertex is L.

Here the condition that L was $CAT(1)$ was just that $k(\pi - \pi/m) \geq 2\pi$ (which holds provided $k \geq 4$ is $m = 2$).

We can give Σ a piecewise hyperbolic (abbreviated PH) structure by declaring each 2-cell to be a small regular $2m$-gon in $\mathbb{H}^2$. Since the angles of such a $2m$-gon will be slightly less than in the Euclidean case, we will be able to do this so that links are $CAT(1)$ provided that $k(\pi - \pi/m) > 2\pi$. This holds provided $k > 4$ if $m = 2$ and $k > 3$ if $m = 3$. Thus, provided the condition holds, Σ can be given a PH structure which is $CAT(-1)$.

These "regular" 2-complexes can be thought of as generalization of well known examples of regular tessellations of $\mathbb{E}^2$ and $\mathbb{H}^2$.

Nadia Benakli has made a detailed study of these 2-complexes in her thesis [Be]. For example, she shows that the "visual sphere" (or "boundary" of the compactification of Σ discussed in [DJ, §2b]) is usually a Menger curve. Benakli also has another construction of such 2-complexes where the 2-cells are n-gons, with n odd, provided that there is a group G of automorphisms of L such that L'/G is an interval (L' is the barycentric subdivision.)

b) Word hyperbolic Coxeter groups. In [M] Moussong also analyzed when the idea of the previous subsection of replacing the Euclidean Coxeter cells of Σ by hyperbolic Coxeter cells works in higher dimensions.

Consider the following condition $(*)$ on a Coxeter system (W, S).

$(*)$ For any subset T of S neither of the following holds:
(1) $W_T = W_{T_1} \times W_{T_2}$ with both factors infinite,
(2) W_T is a Euclidean Coxeter group with $\operatorname{Card}(T) \geq 3$.

Here a "Euclidean Coxeter group" means the Coxeter group of an orthogonal affine reflection group on $\mathbb{E}^n$ with compact quotient. The "Coxeter diagrams" of these groups are listed in [B, pp. 133 and 199].

Theorem 8.1. (Moussong) *The following conditions are equivalent.*
(i) (W, S) satisfies $(*)$
(ii) Σ can be given a PH, $CAT(-1)$ structure,
(iii) W is word hyperbolic,
(iv) W does not contain a subgroup isomorphic to $\mathbb{Z} + \mathbb{Z}$.

To show (i) $\Rightarrow$ (ii) one wants to replace the cells of Σ by Coxeter cells in $\mathbb{H}^n$. In order for the links to remain $CAT(1)$, one needs to know that the length of every closed geodesic in $N(W, S)$ is *strictly* greater than 2π and that the same condition holds for the link of each simplex in $N(W, S)$. In his proof of Lemma 3.16, Moussong analyzed exactly when a metric flag complex has closed geodesics of length equal to 2π. In the case at hand, it was only when conditions (1) and (2) of $(*)$ hold. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are all either well-known or obvious.

c) Buildings. As we mentioned in §3 d) associated to each building B there is a Coxeter system (W, S) so that each "apartment" is isomorphic to a complex associated to (W, S). Traditionally, this complex is the Coxeter complex (where each chamber is a simplex). For general Coxeter groups, however, it seems more appropriate to use the barycentric subdivision Σ' of Σ. (In [CD5] w call this the "modified Coxeter complex"). This point is made by Ronan in [R, p. 184]. The buildings which arise in nature (in algebra or geometry) are usually of spherical or Euclidean type. This means that the Coxeter group W is either finite or Euclidean. In the case where W is Euclidean and irreducible these two definitions essentially agree.

In the general case let us agree that the correct definition of a building should be as a simplicial complex such that each apartment is isomorphic to Σ'. Since, by Theorem 5.8, Σ' can be given the structure of a $CAT(0)$, PE cell complex, we get an induced PE

structure on the building B. As explained in §3c) , the axioms for buildings imply this structure is $CAT(0)$. Thus, we have shown the following result.

Theorem 8.2. *Any building (correctly defined) has the structure of a PE simplicial complex which is $CAT(0)$.*

d) The Eilenberg-Ganea Problem. Let Γ be a torsion-free group. The cohomological dimension of Γ is denoted by $cd(\Gamma)$. The *geometric dimension* of Γ, denoted $gd(\Gamma)$, is the smallest dimension of a $K(\Gamma, 1)$ complex.

The following result is proved in [EG] except for the case when $cd(\Gamma) = 1$ which follows from subsequent work of Stallings.

Theorem 8.3. (Eilenberg-Ganea, Stallings)
(i) If $cd(\Gamma) \neq 2$, then $cd(\Gamma) = gd(\Gamma)$.
(ii) If $cd(\Gamma) = 2$, then $2 \leq gd(\Gamma) \leq 3$.

Problem. Does $cd(\Gamma) = 2$ imply $gd(\Gamma) = 2$?

In [Bes], Bestvina computed the virtual cohomological dimension, denoted *vcd*, of any Coxeter group W. (By definition, the *vcd* is the cohomological dimension of any torsion-free subgroup of finite index.) Usually, $vcd(W) = \dim \Sigma$, but no always. For example, if $N(W, S)$ is an acyclic simplicial complex, then there is a W-stable subcomplex $\Sigma_0 \subset \Sigma'$ (the barycentric subdivision of Σ) such that

(i) Σ_0 is acyclic, and
(ii) $\dim(\Sigma_0) = \dim N = \dim \Sigma - 1$.

So, in this case, $vcd(W) \leq \dim \Sigma - 1$. The subcomplex Σ_0 consists of all those simplices in Σ' whose minimal vertex is not an element of W (i.e., the minimal vertex is not vertex of a Coxeter cell).

Now, choose (W, S) so that $N(W, S)$ is an acyclic 2-complex and so that $\pi_1(N(W, S))$ is not trivial. (For example, if N is a given acyclic 2-dimensional flag complex, then W could be the associated right-angled Coxeter group.) By the above remarks, $vcd(W) = \dim \Sigma_0 = 2$.

Conjecture 8.4. (Bestvina) Let W be as above and let Γ be a torsion-free subgroup of finite index in W. Then $gd(\Gamma) = 3$.

Since W can be represented as a subgroup of some $GL(n)$, such as subgroup Γ always exists (Selberg's Lemma).

The reason for believing this conjecture is that it seems that N (a subcomplex of Σ_0) should embed in the universal cover $E\Gamma$ of a $K(\Gamma, 1)$. On the other hand, if $\pi_1(N)$ is not trivial, then it should not be possible to embed it in any contractible 2-complex, since it should not be possible to kill $\pi_1(N)$ by adding the same number of 1 and 2 cells.

e) When is Σ a manifold ? Since the link of every vertex in Σ is isomorphic to N, this question can be answered as follows.

Proposition 8.5.
(i) Σ is a homology n-manifold if and only if N is a homology $(n-1)$-manifold with the homology of S^{n-1}.

(ii) For $n \geq 5, \Sigma$ is a topological n-manifold if and only if N is as in (i) and N is simply connected.

(iii) Σ is a PL n-manifold if and only if N is PL homeomorphic to S^{n-1}.

Statements (i) and (iii) are just restatements of the definitions. Statement (ii) follows from the celebrated work of Cannon [Ca] and Edwards on the Double Suspension Theorem. In particular, (ii) follows from a theorem of Edwards in [E], wich states that a polyhedral homology n-manifold, $n \geq 5$, is a topological manifold if and only if the link of each vertex is simply connected.

Let us begin by discussing some examples of (iii) when N is a PL triangulation of S^{n-1}.

Example 8.6. (Lanner [L]). Suppose that $N(W, S)$ is isomorphic to the boundary complex of an n-simplex. Then $n = \text{Card}(S) - 1$, $W$is infinite, and for every proper subset T of S, the group W_T is finite. Such groups were classified in 1950 by Lanner: they are either irreducible Euclidean reflection groups or hyperbolic reflection groups. In both cases a fundamental chamber is an n-simplex. In the Euclidean case there are a few families in each dimension n and a few exceptional cases in dimensions ≤ 8. In the hyperbolic case, in dimension 2, we have the hyperbolic triangle groups: these are the groups such that $\text{Card}(S) = 3$ and the 3 entries p, q, r of the Coxeter matrix above the diagonal satisfy $(1/p) + (1/q) + (1/r) < 1$. Furthermore, there are 9 hyperbolic examples in dimension 3, 5 more in dimension 4, and none in dimensions > 4. Complete lists can be found on pages 133 and 199 of [B].

Example 8.7. (Andreev [A]) Suppose that N is a triangulation of S^2 and that condition $(*)$ of subsection b) holds. Then Andreev proved that W can be realized (uniquely, up to conjugation by an isometry) as a reflection group on $\mathbb{H}^3$. In fact, he shows that there is a convex cell C^3 in $\mathbb{H}^3$, the polar dual of which is N. (Thus, the faces of C^3 corresponding to s and s' make a dihedral angle of $\pi/m_{ss'}$.) W is the group generated by reflections across the faces of C^3.

Example 8.8. ([T], [D2] and [CD2]). Suppose L is the boundary complex of a n-dimensional octahedron (i.e., L is the n-fold join of S^0 with itself). Let W' be a finite Coxeter group of rank n. Use W' to label the edges of one $(n-1)$-simplex in L. Label the other edges 2. This defines a Coxeter group W, with $N(W, S) = L$. Each chamber of W on Σ is combinatorially equivalent to the cone on the dual cellulation of N, i.e., each chamber is a combinatorial cube. If $W' = (\mathbb{Z}/2)^n$, then each chamber actually is an n-cube.

In the case where $W' = S_{n+1}$, the symmetric group, the W-manifolds Σ actually arise in nature. For example, there is an obvious homomorphism $W \longrightarrow W' \times (\mathbb{Z}/2)^n$ with kernel Γ_0. Tomei showed in [T], the n-manifold Σ/Γ_0 can be identified with the manifold of all tridiagonal, symmetric $(n+1) \times (n+1)$ matrices with constant spectrum (of distinct eigenvalues). This is also explained in [D2]. From a completely direction, it is shown in [CD2] that for a certain torsion-free, finite-index, normal subgroup Γ_1 of $W, \Sigma/\Gamma_1$ can be identified with Gromov's "Moebius band" hyperbolization construction applied to boundary of a $(n+1)$-cube, and that for a different subgroup $\Gamma_2, \Sigma/\Gamma_2$ can be identified with the "product with interval" hyperbolization construction applied to the boundary of a simple regular $(n+1)$-cell.

Example 8.9. Let L be any flag complex which is a PL triangulation of S^{n-1}, let (W, S) be the corresponding right-angled Coxeter system with $N = L$.

Proposition 8.10. *([CD3, §6]). Let L be as above. Then there is a closed, PE cubical manifold M^n with $K(M^n) \leq 0$ such that the link of each vertex is isomorphic to L.*

Proof. Let Γ be any torsion-free, finite index subgroup of W, where W is as in Example 8.9. Set $M^n = \Sigma/\Gamma$.

Corollary 8.11. *Hopf's Conjecture (from §4) for PE cubical manifolds is equivalent to Conjecture 4.6.*

Proof. We saw in §4 that Conjecture 4.6 implied Hopf's Conjecture for PE cubical manifolds. Let L be an arbitrary triangulation of S^{2n-1} by a flag complex and let M^{2n} be as in the previous proposition. By the Combinatorial Gauss-Bonnet Theorem (Theorem 4.3),

$$\begin{aligned} \chi(M^{2n} &= \sum P(L) \\ &= [W : \Gamma] P(L). \end{aligned}$$

Hence, $\chi(M^{2n})$ and $P(L)$ have the same sign.

f) Is Σ homeomorphic to $\mathbb{R}^n$?

Proposition 8.12. *(i) If N is a PL triangulation of S^{n-1}, then Σ is PL homeomorphic to $\mathbb{R}^n$.*

(ii) If N is a PL homology sphere (i.e., N is a PL manifold) and $\pi_1(N)$ is not trivial, then Σ is not simply connected at infinity. (However, Σ is not a manifold, rather it is only a homology manifold.)

(iii) If N is a simply connected homology manifold with the homology of S^{n-1}, then, for $n \geq 5$, Σ is a contractible manifold and it may happen that Σ is not simply connected at infinity (and hence, not homeomorphic to $\mathbb{R}^n$).

Comments on the proof. (i) In [Sto], D. Stone proved that if M^n is a $CAT(0)$, PE cell complex and if the underlying cell structure is that of a PL manifold, then M^n is PL homeomorphic to $\mathbb{R}^n$. (A different proof is given in [DJ].)

(ii) It is proved in [D1] that the fundamental group at infinity of Σ is the inverse limit of free products of an increasing number of copies of $\pi_1(N)$. (The "visual sphere" of Σ the inverse limit of connected sums of an increasing number of copies of N.)

(iii) Let A^{n-1} be an acyclic PL manifold with boundary. Suppose that $\pi_1(\partial A) \to \pi_1(A)$ is onto. Take a triangulation of A as a flag complex so that ∂A is a subcomplex and a flag complex. Let N be the simplicial complex resulting from attaching the cone on ∂A. Finally, let (W, S) be a Coxeter system with $N(W, S) = N$. Then $\pi_1(N) = 0$, hence, as in 8.5 (iii), Σ is a topological n-manifold, provided $n \geq 5$. It is proved in [DJ, §5b] that if the double of A along ∂A is not simply connected, then Σ is not simply connected at infinity.

As a corollary of (iii) of the previous proposition we have the following result (which answers a question of Gromov about whether such examples could exist).

Corollary 8.13. ([DJ, p. 383]) *There a Coxeter system (W, S) so that the corresponding PE polyhedron Σ is*
(a) $CAT(0)$,
(b) a topological n-manifold, $n \geq 5$, and
(c) not homeomorphic to $\mathbb{R}^n$.

In [ADG] we show that a modified version of the Σ in part (ii) of Proposition 8.12 can be taken to be a topological manifold. The rough idea is to blow up the PL singularities of Σ from isolated vertices into intervals. More precisely, we prove the following result.

Proposition 8.14. ([ADG]) *Let L^{n-1} be a PL homology sphere. Then there is a right-angled Coxeter system (W, S), with $N(W, S) = L$, and a PE cubical complex Σ_1 with W action such that (a) Σ_1 is $CAT(0)$ and (b) Σ_1 is a topological n-manifold.*

Idea of Proof. We can find a codimension-one homology sphere $L_0 \subset L$ such that 1) L_0 divides L into two pieces L_1 and L_2 (each of which is an acyclic manifold with boundary) and 2) $\pi_1(L_0) \to \pi_1(L_i)$ is onto, for $i = 1, 2$. Triangulate L as a flag complex so that L_0 is a full subcomplex and let (W, S) be the right-angled Coxeter system such that $N(W, S) = L$. For $i = 1, 2$, let N_i denote the union of L_i with the cone over L_0. Then N_i is simply connected. In the construction of Σ a fundamental chamber is essentially a cubical cone over L. As explained in [ADG], to construct Σ_1, one uses as a chamber the union of two cubical cones: one over N_1 and the other over N_2. These cones are glued together along the cone on L_0. Such a chamber now has PL singularities along an interval which connects the two cone points c_1 and c_2. The link of c_i is N_i, which is simply connected; hence, by Edwards' Theorem [E], Σ_1 is a topological manifold.

III. Artin Groups.

As explained in §9 any Coxeter group (finite or not) has a representation as a reflection group on a real vector space. Take the complexification of this vector space. It contains a certain convex open subset such that after deleting the reflection hyperplanes, we obtain an open manifold M on which the Coxeter group W acts freely. The fundamental group of M/W is the "Artin group" A associated to W. When W is finite, Deligne proved that M/W is a $K(A,1)$-space. The conjecture that this should always be the case, is here called the "Main Conjecture". The purpose of this chapter is to outline some work on this conjecture in [CD5] and [CD6].

Associated to the Artin group there is a cell complex Φ (which is very similar to Σ). It turns out (Corollary 12.2) that proving the Main Conjecture for W is equivalent to showing Φ is contractible. The complex Φ has a natural PE structure, which we conjecture is always $CAT(0)$. We do not know how to prove this; however, in §13 we show that there is a (less natural) cubical structure on Φ and that in "most cases" it is $CAT(0)$. Hence, the Main Conjecture holds in most cases.

§9. Hyperplane Complements.

Let S_n denote the symmetric group on n letters. It acts on $\mathbb{R}^n$ by permutation of coordinates. In fact, this action is as an orthogonal reflection group: the reflections are the transpositions (ij), $1 \leq i < j \leq n$, the corresponding reflection hyperplanes are the $H_{ij} = \{x \in \mathbb{R}^n | x_i = x_j\}$. Complexifiying we get an action of S_n on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$ such that S_n acts freely on

$$M = \mathbb{C}^n - \bigcup (H_{ij} \otimes \mathbb{C}).$$

Thus, M/S_n is the configuration space of unordered sets of n distinct points in $\mathbb{C}$.

It is a classical fact that the fundamental group of M/S_n is B_n, the braid group on n strands. The following result is also classical.

Theorem 9.1. (Fox-Neuwirth [FN])

1) M is a $K(\pi,1)$ space, where π is PB_n the pure braid group (i.e., PB_n is the kernel of $B_n \to S_n$).

2) M/S_n is a $K(\pi,1)$ space, for $\pi = B_n$.

Next suppose that W is a finite reflection group on $\mathbb{R}^n$ and that

$$M = \mathbb{C}^n - \bigcup_r H_r \otimes \mathbb{C}$$

where the union is over all reflections r in W (i.e., all conjugates of elements in S) and where H_r is the hyperplane fixed by r. Arnold and Brieskorn asked if the analogous result to Theorem 9.1 holds in this context. In [De], Deligne proved that this was indeed the case.

Theorem 9.2. (Deligne [De]) *Suppose that W is a finite reflection group. Then M/W is a $K(\pi,1)$ space, where π is the "Artin group" associated to W (as defined below).*

Artin groups. Suppose that (W, S) is a Coxeter system and that $M = (m_{ss'})$ is the associated Coxeter matrix. Introduce a new set of symbols $X = \{x_s | s \in S\}$ one for each element of s.

Notation. If m is an integer ≥ 2, then let $\mathrm{prod}(x, y; m)$ denote the word: $xyx\cdots$, where there are a total of m in letters in a word.

Definition 9.3. The *Artin group* associated to (W, S) (or to M) is the group generated by X and with relations:

$$\mathrm{prod}(x_s, x_{s'}; m_{ss'}) = \mathrm{prod}(x_{s'}, x_s; m_{ss'})$$

where (ss') range over all elements of $S \times S$ such that $s \neq s'$ and $m_{ss'} \neq \infty$.

Remark 9.4. If we add the relations $(x_s)^2 = 1$, then the relation appearing in the previous definition can be rewritten as $(x_s x_{s'})^{m_{ss'}} = 1$; hence, we recover the standard presentation of W. Thus, if A is the Artin group associated to (W, S), we see that there is a canonical surjection $p : A \to W$ which send x_s to s.

Example 9.5. If W is S_n, then the associated Artin group is B_n.

It is natural to ask if Theorem 9.2 holds in the case where W is infinite. In order to make sense of this question we first need to discuss what is meant by a "linear reflection group" in the infinite case.

Linear reflection groups. Let V be a finite dimensional real vector space. A *linear reflection* on V means a linear involution with fixed space a hyperplane.

Suppose that C is a convex polyhedral cone in V (Figure 5) and that S is a finite set which indexes the set of codimension-one faces of C. Thus, $(C_s)_{s\in S}$ will be the family of codimension-one faces of C. Let H_s denote the linear hyperplane spanned by C_s.

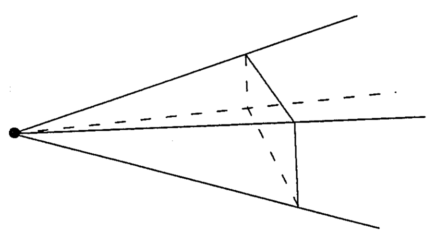


Figure 5

For each $s \in S$, choose a reflection ρ_s with fixed subspace H_s. Let W denote the subgroup of $GL(V)$ generated by $\{\rho_s | s \in S\}$.

Definition 9.6. W is a *linear reflection group* if $w\mathring{C} \cap \mathring{C} = \emptyset$ or all $w \in W, w \neq 1$. (Here $\mathring{C}$ denotes the interior of C.)

Definition 9.7. Let

$$\overline{I} = \bigcup_{w \subset W} wC$$

and let I denote the interior of $\overline{I}$. $\overline{I}$ is called the *Tits cone.*

Example 9.8. Consider the quadratic form model of $\mathbb{H}^2$: the hyperbolic plane is identified with one sheet of a hyperboloid in $\mathbb{R}^{2,1}$ (3-dimensional Minkowski space). An isometric reflection on $\mathbb{H}^2$ extends to a linear reflection on $\mathbb{R}^{2,1}$ preserving the indefinite quadratic form. Now suppose that W is the reflection group on $\mathbb{H}^2$ generated by the reflections across the edges of a hyperbolic polygon with angles of the form π/m. Then W can be regarded as a linear reflection group on $\mathbb{R}^{2,1}$. (See Figure 6.)

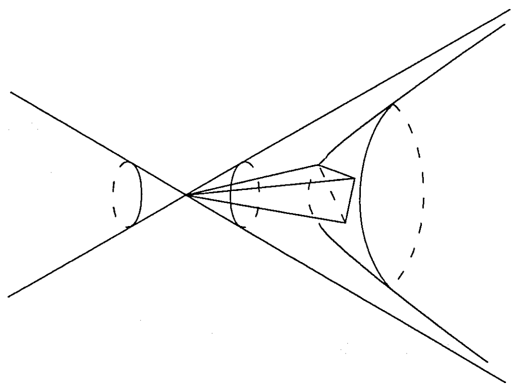


Figure 6

In this case the interior I of the Tits cone is just the interior of the light cone.

Theorem 9.10. (Vinberg [V]) *Suppose that $W \subset GL(V)$ is a linear reflection group with fundamental polyhedral cone C. Put $C^f = \{x \in C | W_x \text{ is finite}\}$. Then*

(i) (W, S) is a Coxeter system,
(ii) $\overline{I}$ is a convex cone,
(iii) K is W-stable and W acts properly on it,
(iv) $I \cap C = C^f$,
(v) the poset of face of C^f is S^f (where $S^f = \{T \subset S | W_T \text{ is finite}\}$.

Let W be as in Vinberg's Theorem and consider the domain $V + iI$ in $V \otimes \mathbb{C}$ ($V + iI$ denotes the set of vectors whose imaginary part lies in I). Set

$$M = (V + iI) - \bigcup(H_r \otimes \mathbb{C})$$

The following is the main conjecture which we shall be concerned with in this chapter. According to [Lek] it is due to Arnold, Pham and Thom.

Conjecture 9.11. (the "Main Conjecture") *M/W is a $K(\pi, 1)$ space, where $\pi = A_W$, the Artin group associated to (W, S).*

Some progress on this was made in the thesis of H. van der Lek [Lek], where the following result is proved. (Another proof can be found in [CD5].)

Proposition 9.12. (van der Lek) $\pi_1(M/W) = A_W$.

The Main Conjecture can also be formulated in terms of the cell complex Σ which was introduced in the previous chapter in §7. In fact, in view of Theorem 9.10 (v), the following lemma is not surprising.

Lemma 9.13. ([CD5, §2]) *There are W-isovariant homotopy equivalences:*

$$I \sim \Sigma \quad \textit{and}$$

$$V + iI \sim I \times I \sim \Sigma \times \Sigma.$$

Set

$$Y = (\Sigma \times \Sigma) - \bigcup_r \Sigma_r \times \Sigma_r$$

where Σ_r denotes the subcomplex of Σ' fixed by r. Then

$$Y/W \sim M/W.$$

Hence, we have the following

Conjecture 9.14. (reformulation of the Main Conjecture) *Y/W is a $K(A_W, 1)$.*

§10. The Salvetti complex.

In this section, which is independent of the last three sections of this chapter, we describe a PE cell complex $\tilde{\Sigma}$ homotopy equivalent to M. The quotient space $\tilde{\Sigma}/W$ is a finite CW complex. Hence, when the Main Conjecture holds, $\tilde{\Sigma}/W$ will be a $K(A_W, 1)$ space. The complete details of this construction are given in [CD6].

In §5, we considered two posets:

$$\mathcal{S}^f = \{T \subset S | W_T \text{ is finite}\} \text{and}$$
$$W\mathcal{S}^f = \coprod_{T \in \mathcal{S}^f} W/W_T.$$

Here we consider a third poset $W \times \mathcal{S}^f$. The partial ordering on $W \times \mathcal{S}^f$ is defined as follows: $(w, T) < (w', T')$ if and only if

(i) $T < T'$
(ii) $w^{-1}w' \in W_{T'}$, and
(iii) for all $t \in T, \ell(w^{-1}w') < \ell(tw^{-1}w')$

(where ℓ denotes word length in W). There is a natural projection $\pi : W \times \mathcal{S}^f \to W\mathcal{S}^f$ defined by $(w, T) \to wW_T$. Conditions (i) and (ii) just mean that π is order-preserving. Condition (iii) comes out of the proof of Proposition 10.1, below.

The quickest way to define $\tilde{\Sigma}$ is to first define its barycentric subdivision $\tilde{\Sigma}'$; it is the geometric realization of the derived complex of $W \times \mathcal{S}^f$. One then observes that the union of simplices with maximal vertex is (w, T) can be identified with a Coxeter cell of type W_T.

If Z is a cell complex then $\mathcal{P}(Z)$ denotes the poset of cells in Z. For example, $\mathcal{P}(\Sigma) = W\mathcal{S}^f$.

Proposition 10.1. (Salvetti [S] and [CD6]). There is a PE cell complex $\tilde{\Sigma}$ such that
(i) $\mathcal{P}(\tilde{\Sigma}) = W \times \mathcal{S}^f$,
(ii) each cell of $\tilde{\Sigma}$ is a Coxeter cell,
(iii) W acts freely on $\tilde{\Sigma}$,
(iv) $\tilde{\Sigma}$ is W-equivariantly homotopy equivalent to M (or to Y) and hence, $\tilde{\Sigma}/W \sim M/W$.

Sketch of Proof. First, for each (w,T) in $W \times \mathcal{S}^f$ we will describe two open sets in Σ'. Let $U'_{(w,T)}$ denote the open star of the vertex corresponding to wW_T in Σ'. Let $U''_{(w,T)}$ denote the intersection of the open "half spaces" in Σ' which are bounded by the Σ_r with r a reflection in wW_Tw^{-1} and which contain the vertex w. ($U''_{(w,T)}$ is an open "sector".) We note that $U''_{(w,T)}$ contains no point in $U'_{(w,T)}$ with nontrivial isotropy group.

Consider $Y = (\Sigma \times \Sigma) - \cup(\Sigma_r \times \Sigma_r)$. Let $U_{(w,T)} = U'_{(w,T)} \times U''_{(w,T)}$. One checks easily that a) $U_{(w,T)} \subset Y$, b) $\{U_{(w,T)}\}$ is an open cover of Y, c) each nonempty intersection of elements in this cover is contractible, and d) the nerve of this cover is $\tilde{\Sigma}'$ (the geometric realization of $(W \times \mathcal{S}^f)'$). The proposition follows.

Remark. In [S] Salvetti carries out the above construction for arbitrary hyperplane complements. The special case above is done in [CD6]. When W is the symmetric group, the result was known earlier (for example, to J. Milgran, C. Squier, K. Tatsuoka and L. Paris).

The CW complex $\tilde{\Sigma}/W$ has one cell of dimension $\mathrm{Card}\,(T)$ for each $T \in \mathcal{S}^f$. In particular, when W is finite, $\tilde{\Sigma}/W$ is the CW complex formed by identifying faces of a single Coxeter cell: the precise identifications can be worked out using conditions (i), (ii) and (iii) in the definition of the partial order on $W \times \mathcal{S}^f$.

Corollary 10.2. *The Main Conjecture holds if and only if $\tilde{\Sigma}/W$ is a $K(A_W, 1)$ space.*

Example 10.3. The Main Conjecture holds when W is finite. In particular, $\tilde{\Sigma}/W$ is a $K(A,1)$ when A is a braid group. For example, when $A = B_3$, a $K(B_3,1)$ can be constructed by identifying edges of a hexagon in the following pattern:

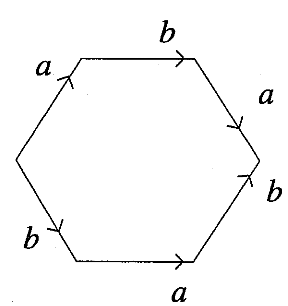
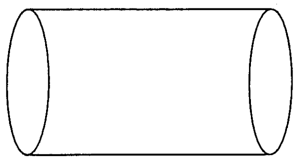


Figure 7

Corollary 10.4. ([CD6]) *Suppose the Main Conjecture holds for (W,S). Then*
(i) $cd(A_W) = \dim \tilde{\Sigma} = \dim \Sigma$.
(ii) $\chi(A_W) = \chi(\tilde{\Sigma}/W) = 1 - \chi(N(W,S))$.

A naive idea for proving the Main Conjecture would be to show that $K(\tilde{\Sigma}) \leq 0$. This actually works when W is right-angled (in [CD6]); however, it does not work in general.

For example, when $A = B_3$, the link of a vertex in the complex in Example 10.3 is the following graph.



$Lk(v, \tilde{\Sigma})$

Figure 8

Each edge length is $2\pi/3$, but then the digons have length $4\pi/3$ which is $< 2\pi$.

§11. Complexes of groups.

Graphs of groups.

We begin by recalling some well-known results from the theory of graphs of groups (cf. [Se]).

Let Ω be a graph, $\mathcal{P}(\Omega)$ the poset of cells in Ω and $\mathcal{P}(\Omega)^{op}$ the dual poset, thought of as a category. A *graph of groups over* Ω is a functor $\mathcal{G}$ from $\mathcal{P}(\Omega)^{op}$ to the category of groups and monomorphisms. Thus, to each vertex v of Ω we are given a group $\mathcal{G}(v)$ and similarly a group $\mathcal{G}(e)$ for each edge e. Moreover, if v is a vertex of e, then there is a monomorphism $\mathcal{G}(e) \to \mathcal{G}(v)$.

¿From these data one can construct a group G, called the *fundamental group of* G and denoted by $\pi_1(\mathcal{G})$. The basic result in the theory is the following.

Theorem 11.1. ([Se]). *Given a graph of groups $\mathcal{G}$ over Ω, there exists a tree T with G-action ($G = \pi_1(\mathcal{G})$) so that the following hold.*

(i) $T/G = \Omega$

(ii) Suppose e is an edge of Ω, v a vertex of e, $\tilde{e}$ a lift of e to T and $\tilde{v}$ the corresponding vertex of $\tilde{e}$. Then there is an isomorphism $G_{\tilde{v}} \cong \mathcal{G}(v)$ taking $G_{\tilde{e}}$ onto the image of $\mathcal{G}(e)$.

One consequence of (ii) is that the natural map $\mathcal{G}(v) \to \pi_1(\mathcal{G})$ is injective (since it is isomorphic to the inclusion $G_{\tilde{v}} \subset G$). In the language of [H1] this means that $\mathcal{G}$ is *developable.* The tree T is called the *universal cover* of $\mathcal{G}$. It is unique up to G-isomorphism. The other feature of a graph of groups is that this universal cover is not only simply connected, it is contractible.

An important application of this theory is to the problem of gluing together various $K(\pi, 1)$ spaces and then being able to decide if the result is also aspherical.

Definition 11.2. An *aspherical realization* of a graph of groups $\mathcal{G}$ is a CW complex B and a map $p : B \to \Omega'$ so that for each vertex v of Ω, $p^{-1}(\text{Star}(v))$ is a $K(\mathcal{G}(v), 1)$. Here "Star" refers to the open star of a vertex in the barycentric subdivision Ω'. (Actually, this is only an approximation of the correct definition which can be found i [H2, §3.3 and 3.4].)

Remark 11.3. It is proved in [H2], in the more general context of complexes of groups, that aspherical realizations exist and are unique up to homotopy.

We will usually denote an aspherical realization of $\mathcal{G}$ by $B\mathcal{G}$ and call it the *classifying space* of $\mathcal{G}$. As a definition of $\pi_1(\mathcal{G})$ we could take the usual fundamental group of $B\mathcal{G}$.

The following result is classical; its proof probably goes back to J. H. C. Whitehead.

Theorem 11.4. *Let $\mathcal{G}$ be a graph of groups, $B\mathcal{G}$ an aspherical realization, and $G = \pi_1(\mathcal{G})$. Then $B\mathcal{G}$ is a $K(G,1)$.*

Proof. Let EG be the universal cover of $K(G,1)$. Consider the diagonal G-action on $EG \times T$. Projection on the second factor $EG \times T \to T$ induces a map of quotient spaces $EG \times_G T \to \Omega$ which is clearly an aspherical realization. Hence, we can take $B\mathcal{G} = EG \times_G T$. On the other hand, the universal cover of $EG \times_G T$ is $EG \times T$ which is contractible.

Complexes of groups. Here we present a simplified version of the theory developed in [H1] and [H2]. (For the applications we have in mind we do not need the most general version of the theory.)

Definition 11.5. Let $\mathcal{P}$ be a poset. A *simple complex of groups over* $\mathcal{P}$ is a functor $\mathcal{G}$ from $\mathcal{P}$ to the category of groups and monomorphisms.

Remark. In the general situation of [H2], $\mathcal{P}$ need not be a poset but only a "category without loop". More importantly, $\mathcal{G}$ need not be a functor. The appropriate triangular diagrams relating compositions of morphisms need not commute on the nose but only up to conjugation by some elements in the target group; furthermore, these elements must be kept track of.

The concept of an "aspherical realization" is defined as before. Such an aspherical realization is devoted by $B\mathcal{G}$ and called the *classifying space of* $\mathcal{G}$. By definition, $\pi_1(\mathcal{G}) = \pi_1(B\mathcal{G})$. It can also be defined via generators and relations [H2, §12.8]. A complex of groups need not be developable. Moreover, even if it is developable its universal cover need not be contractible.

Example 11.6. Suppose $\mathcal{P} = \mathcal{P}(\Delta^2)^{op}$ where Δ^2 is a 2-simplex. Define a complex of groups $\mathcal{G}$ by the following picture.

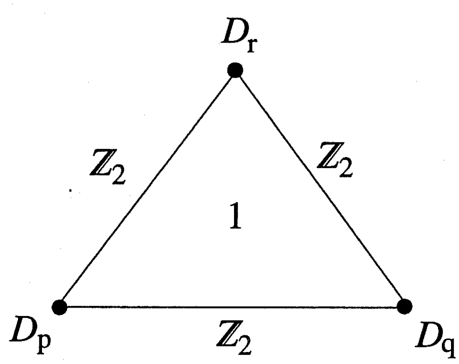


Figure 9

where D_m denotes the dihedral groups of order $2m$. Then $\pi_1(\mathcal{G})$ is a Coxeter group W on three generators. $\mathcal{G}$ is developable. Assume that $(1/\pi) + (1/q) + (1/r) > 1$. Then W

is finite. The universal cover of $\mathcal{G}$ is homeomorphic to S^2 (triangulated as the Coxeter complex) and $B\mathcal{G} = EW \times_W S^2$ which is *not* a $K(W,1)$ (its universal cover is homotopy equivalent to S^2).

What is missing in higher dimensions is a hypothesis of nonpositive curvature (which is automatic in the case of a graph of groups). The proof of the following basic result is outlined by Haefliger in [H1] and the details are supplied in the thesis of B. Spieler [Sp].

Theorem 11.7. (Haefliger, Spieler) *Let $\mathcal{G}$ be a complex of groups. Suppose that $\mathcal{G}$ admits "a metric with $K \leq 0$". Then*

(i) $\mathcal{G}$ is developable,

(ii) its universal cover is $CAT(0)$ (and hence, contractible),

(iii) $B\mathcal{G}$ is a $K(G,1)$ space.

Remark. Suppose $\mathcal{G}$ is a complex of groups over a poset $\mathcal{P}$. The hypothesis of nonpositive curvature means that the geometric realization of $\mathcal{P}'$ admits a PE "orbihedral structure" with $K \leq 0$. In other words, the local models of a universal cover must be PE and $CAT(0)$.

Example 11.8. Suppose (W,S) is a Coxeter system and that $\mathcal{S}^f$ is the poset defined in §5. Define a simple complex of groups $\mathcal{W}$ over $\mathcal{S}^f$ to be the functor $\mathcal{W}(T) = W_T$. Then it is easily seen that $\mathcal{W}$ is developable and that $\pi_1(\mathcal{W}) = W$. Moreover, its universal cover is just the geometric realization of $(W\mathcal{S}^f)'$, i.e., it is Σ' (the barycentric subdivision of Σ). Moussong's Theorem 5.8 shows that the natural PE structure on Σ' is $CAT(0)$, i.e., we are in the situation of Theorem 11.7.

Example 11.9. Let A be the Artin group associated to (W,S). Let $\mathcal{S}^f$ be as above. Define a simple complex of groups $\mathcal{A}$ over $\mathcal{S}^f$ to be the functor $\mathcal{A}(T) = A_T$, where A_T is the Artin group corresponding to (W_T,T) (A_T is an Artin group of "finite type"). It follows easily that $\pi_1(\mathcal{A}) = A$. Moreover, it can be shown that for each $T \in \mathcal{S}^f, A_T \to A$ is injective, i.e., $\mathcal{A}$ is developable.

Consider the set,

$$A\mathcal{S}^f = \coprod_{T \in \mathcal{S}^f} A/A_T,$$

partially ordered by inclusion. The geometric realization Φ of the derived complex of $A\mathcal{S}^f$ will be called the *modified Deligne complex.* It will be our principal object of interest in the remaining two sections. As is in the case of Σ', it is easy to see that Φ is simply connected and therefore, that it is the universal cover of $\mathcal{A}$.

§12. Reinterpretation of the Main Conjecture.

Recall that

$$Y = (\Sigma \times \Sigma) - \bigcup \Sigma_r \times \Sigma_r$$

Let X denote the geometric realization of $(\mathcal{S}^f)'$. Let $\pi : Y/W \to \Sigma'/W = X$ be the map induced by projection on the first factor.

Proposition 12.1. ([CD6]). *$\pi : Y/W \to X$ is an aspherical realization of $\mathcal{A}$ (from Example 11.9).*

Proof. Basically, this is just what Deligne's Theorem (Theorem 9.2) tells us. Indeed, if T is a vertex of X, then

$$\pi^{-1}(\mathrm{Star}(T)) = [(\mathrm{Star}(1,T) \times \Sigma - \bigcup \text{hyperplanes}]/W_T$$

which is homotopy equivalent the orbit space of the hyperplane complement for the finite Coxeter group W_T. By Deligne's result, this is a $K(A_T, 1)$. Thus, in the general situation, Y/W is homotopy equivalent to $B\mathcal{A}$.

Corollary 12.2. *The Main Conjecture holds for (W, S) if and only if Φ is contractible.*

Proof. We are using the form of the Main Conjecture in 9.14. By Proposition 12.1, Y/W is homotopy equivalent to $EA \times_A \Phi$. Therefore, the universal cover of Y/W is contractible if and only if Φ is contractible.

Φ is "building-like". As explained in §9, there is a natural epimorphism $p : A \to W$. We can define a section $\varphi : W \to A$ of p as follows. Given w in W, write $w = s_1 \cdots s_m$ where $s_1 \cdots s_m$ is a word of minimum length for w. Set $\varphi(w) = x_{s_1} \cdots x_{s_m} \in A$. It can be shown that φ is well-defined (i.e., the value of $\varphi(w)$ does not depend on the choice of minimal word). Of course, φ is not a homomorphism. The map φ induces a embedding of posets $WS^f \to AS^f$ and therefore, a simplicial embedding $\Sigma' \to \Phi$. The translates of Σ' by elements of A are the *apartment-like* subcomplexes. We have that

$$\Phi = \bigcup_{a \in A} a\Sigma'$$

and in this sense, Φ is "building-like". Φ is not in fact a building: two points need to lie in a common apartment.

Nevertheless, there is an obvious idea for trying to prove Φ is contractible. Give Φ a *PE* structure by declaring each apartment-like complex to be isometric to Σ' with its natural *PE* structure (described in §6 and §7). Then prove Φ is $CAT(0)$. To attack this we must study links of vertices in Φ.

The simplicial complexes $\hat{\Sigma}$ and $\hat{\Phi}$. Suppose for the moment that W is a finite Coxeter group. Let Δ be a simplex, the codimension-one faces of which are indexed by S: if $s \in S$, then Δ_s denotes the corresponding face. Given $x \in \Delta$, put $S(x) = \{s \in S | x \in \Delta_s\}$. Define

$$\hat{\Sigma}_W = (W \times \Delta)/\sim$$

where the equivalence relation $\sim$ is defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. $\hat{\Sigma}_W$ is the usual *Coxeter complex* of W. Its poset of simplices is

$$(\coprod_{T \neq S} W/W_T)^{op}$$

It can be identified with the triangulation of the unit sphere in $\mathbb{R}^n$ where the $(n-1)$ simplices are the intersection of S^{n-1} with the translates of a fundamental simplicial cone. In other words, if we identify Δ with a spherical simplex so that the dihedral angle along $\Delta_s \cap \Delta_{s'}$ is $\pi/m_{ss'}$, then the induced PS structure on $\hat{\Sigma}_W$ is that of a round sphere.

Similarly, define

$$\hat{\Phi}_W = (A \times \Delta)/\sim$$

where $\sim$ is defined by $(a, x) \sim (a', x')$ if and only if $x = s'$ and $a^{-1}a' \in A_{S(x)}$. The simplicial complex $\hat{\Phi}_W$ is called the *Deligne complex* of (W, S). (It was studied in [De].) In particular, Deligne proved that $\hat{\Phi}_W$ is homotopy equivalent to a wedge of $(n-1)$-spheres (where $n-1 = \dim \Delta$).

As in the previous subsection,

$$\Phi_W = \bigcup_{a \in A} a\hat{\Sigma}_W$$

So, $\hat{\Phi}_W$ is spherical building-like. If we identify Δ with a spherical simplex as above, then each apartment-like subcomplex $a\hat{\Sigma}_W$ is isometric to a round sphere.

Links of vertices in Φ. The vertices of Φ correspond to elements of $A\mathcal{S}^f$, i.e., to cosets of the form $aA_T, T \in \mathcal{S}^f$. We classify these into three types.

Type 1. $T = \emptyset$. In this case $Lk(v, \Phi) = N(W, S)$ (the same link as for a vertex of Σ).

Type 2. T is maximal in $\mathcal{S}^f$. In this case, $Lk(v, \Phi) = \hat{\Phi}_{W_T}$. (In the analogous case for Σ' the link would be a round sphere.)

Type 3. In the general case, $Lk(v, \Phi)$ is the orthogonal join of a link of a simplex in $N(W, S)$ and a link of type 2.

Thus, if we give Φ its natural PE structure every link is of the form $N(W, S)$ (which is $CAT(1)$ by Moussong's Lemma and Cor. 7.4), or a Deligne complex associated to a finite Coxeter group, or a join of these two types. Thus, Φ will be $CAT(0)$ provided the following holds.

Conjecture 12.3. *Let W be a finite Coxeter group. Then the Deligne complex $\hat{\Phi}_W$ with its round metric is $CAT(1)$.*

Theorem 12.4. *Conjecture 12.3 implies the Main Conjecture 9.11.*

It follows from a lemma of [AS, p.210] that Conjecture 12.3 holds when $W = D_m$, the dihedral group of order $2m$. (The lemma asserts that $\hat{\Phi}_{D_m}$ has no circuits of length $\leq 2m$). This yields the following.

Corollary 12.5. ([CD6]) *The Main Conjecture holds whenever* $\dim \Phi = 2$.

§13. A cubical structure on Φ.

A Coxeter cell of type W_T can be subdivided into combinatorial cubes, so that cubes containing the barycenter of the cell correspond to the simplices of $\hat{\Sigma}_{W_T}$.

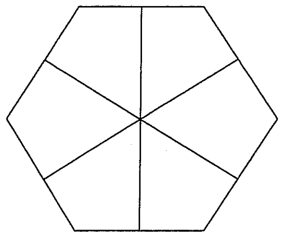


Figure 10

Given an arbitrary Coxeter system (W, S), we could give $\Sigma'(W, S)$ a cubical structure by declaring each such combinatorial cube to be regular Euclidean cube. As in the previous section, there are three types of vertices to consider. For those of Type 2 (where T is maximal), $Lk(v, \Sigma')$ is $\hat{\Sigma}_{W_T}$ with its all right PS structure. By [Br, p. 29], the Coxeter complex of a finite Coxeter group is a flag complex; hence, by Gromov's Lemma, the all right structure on $\hat{\Sigma}_{W_T}$ is $CAT(1)$. Consider a vertex of Type 1 (where $T = \emptyset$). The link is $N(W, S)$ with an all right PS structure. Hence, this link is $CAT(1)$ if and only if $N(W, S)$ is a flag complex. Since the links of Type 3 are orthogonal joins of versions of Type 1 and 2, we see that the cubical structure on Σ is $CAT(0)$ if and only if $N(W, S)$ is a flag complex.

In exactly the same way, we can put a cubical structure on Φ. In the case of vertices of Type 2, we have the following key lemma of [CD 5, Lemma 4.3.2].

Lemma 13.1. (CD6]) *Let W be a finite Coxeter group. Then the Deligne complex $\hat{\Phi}_W$ is a flag complex.*

For vertices of Type 1 the link is again $N(W, S)$. The conclusion is that the cubical structure on Φ is $CAT(0)$ exactly when the cubical structure on Σ is $CAT(0)$.

Theorem 13.2. ([CD6]) *The Main Conjecture is true when $N(W, S)$ is a flag complex.*

Remark 13.3. Taken together, Corollary 12.5 and Theorem 13.2 constitute a proof of the Main Conjecture in most cases.

Remark 13.4. In [L], Lanner showed that $N(W_T, T)$ is the boundary of a simplex (an "empty simplex" in $N(W, S)$) if and only if W_T is a reflection group on either hyperbolic space or Euclidean space with fundamental chamber a simplex. (See Example 8.6.) Hence, $N(W, S)$ is a flag complex if and only if for all subsets T of S, with Card $(T) \geq 3$, neither of the following conditions hold:

a) W_T is an irreducible, affine Euclidean reflection group,
b) W_T is a hyperbolic reflection group with fundamental chamber a simplex.

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UNKNOTTING NUMBERS OF MINIMAL PROJECTIONS OF SOME ALTERNATING RATIONAL KNOTS

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Section 1. Introduction.

This paper contains of summary of results presented in a talk at the Western Workshop in Geometric Topology held at Park City, Utah in June, 1994. These results were obtained during the summer of 1992 and the summer of 1993 at the Research Experiences for Undergraduates Program in Mathematics at Oregon State University. The results in this paper represent joint work with James Bernhard, Cassandra McGee and Eva Wailes. These results appear in more detail in the proceedings of the summer REU programs ([REU1], [REU2]).

In the early 1980s, Bleiler [Bl] and Nakanishi [Na] independently produced an example of a knot whose unknotting number was not realized in a minimal projection. The example was the knot (5, 1, 4) in Conway notation [Co]. The method of proof involved showing that the ten crossing projection given by the Conway notation (5, 1, 4) was the minimal projection and had unknotting number 3, whereas the alternate projection with 14 crossings given by the Conway notation $(-2, -2, -2, -2, -2, 4)$ has unknotting number 2.

More recent results make it easier to determine when a given projection of a knot or link is the minimal projection, and thus make it easier to analyze when gaps between the unknotting number of a minimal projection of a knot or link and the actual unknotting number of the knot or link occur. The needed definitions and background material are presented in the next section. The new results from the past two summers are presented in the third section.

Section 2. Definitions and Background Material.

We use the word link to represent a knot or link in R^3. The unknotting number of a projection of a link is the minimal number of crossing changes necessary to change the link into the trivial link. The unknotting number of a link is the minimum, taken over all projections of the link, of the unknotting number of the projections of that link.

Key words and phrases. Rational knots, unknotting number.

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We use Conway notation [Co] to represent projections of rational knots and links. The continued fraction associated with a link given by Conway notation $(a_1,\ a_2,\ \dots\ ,a_n)$ is the continued fraction:

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\dots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}$$

A main result from [Co] is the following.

Theorem [Co].
If the continued fractions associated with two links given in Conway notation are the same , then the links are equivalent.

The following result of Kauffman, Thistlethwaite, and Murasugi makes it easier to check for minimal projections.

Theorem (see [Ka], [Mu]).
Any reduced alternating projection of a link is a minimal projection.

It is easy to check that any link given in Conway notation by a sequence of length at least two of all positive, or all negative integers is reduced alternating, and thus the corresponding projection is minimal and the link is nontrivial.

The following result of Menasco and Thistlethwaite makes it possible to just check the unknotting number of a single reduced alternating projection of a link to find the unknotting number of minimal projections of that link.

Theorem [MT].
Any two reduced alternating projections of a link differ by a series of flypes, and so have the same unknotting number.

Finally, the unknotting gap of a link is the difference between the unknotting number of a minimal projection of the link and the actual unknotting number of the link. The strategy for finding rational links with positive unknotting gap should now be clear. One analyzes the unknotting number of links given in Conway notation by a sequence of all positive or all negative numbers. The associated projection is necessarily minimal, and the result of changes in various positions can be analyzed. If a sequence of changes results in a link that has an alternate Conway expression involving all positive or all negative integers, then that particular sequence of changes does not result in the trivial link. Analyzing all possible changes gives the unknotting number of the minimal projection of the link. Finally, one must determine if there are alternate projections that have a smaller unknotting number. If one can find such a projection, then one has a proof that the given link has a positive unknotting gap.

Much of the work involves finding a systematic procedure for analyzing all possible changes, and finding a method for producing alternate projections with possibly lower unknotting numbers.

Section 3.Main results.

During the summer REU program in 1992, James Bernhard, working with the author, generalized the examples of Bleiler and Nakanishi to produce an infinite class of knots with unknotting gap at least one. This result has been published [Be]. The main result is the following.

Theorem 1 [REU1].

The knots given in Conway notation by $(2k+1,\ 1,\ 2k)$, for $k \geq 2$, have unknotting number of any minimal projection greater than k, and have a projection with unknotting number less than or equal to k. Thus each knot of this infinite class of knots has unknotting gap greater than or equal to 1.

The proof of this result proceeded by induction, starting with the base case that Bleiler and Nakanishi had already established.

During the summer of 1993, Cassandra McGee, Eva Wailes and the author investigated other rational knots and links given by Conway notation $(a,\ b,\ c)$ and found other knots and links with positive unknotting gap. The main results of the investigation are contained in the following two theorems. Eva Wailes continued the investigation of knots of the form $(a,\ b,\ c)$ during the following year and has submitted a paper for publication containing further results [Wa].

Theorem 2 [REU2].

The unknotting number of the minimal projection of the knot $(2n+3,\ 2m+1,\ 2n+2)$, for $k \geq 1$ and $m \geq 0$ is greater than $2n+1$ if $n \leq m$ and is greater than $n+m+1$ if $n \geq m$. In addition, if $n \geq m$, the knot $(2n+3,\ 2m+1,\ 2n+2)$ has a nonminimal projection with unknotting number $\leq n+m+1$. As a consequence, any knot of the form $(2n+3,\ 2m+1,\ 2n+2)$ with $n \geq m$ has unknotting gap at least 1.

The proof of this theorem proceeded by a more complicated induction than the proof of theorem 1 above. The nonminimal projections that were found to produce the unknotting gap were variations on the nonminimal projections in theorem 1.

Theorem 3 [REU2].

The unknotting number of the two component link of the form $(2k,\ 1,\ 2j)$, for $k \geq j \geq 2$ is $k+j-1$. These links have a nonminimal projection with unknotting number less than or equal to k. As a consequence, the unknotting gap of these links is at least $j-1$.

Section 4. Question.

The last theorem produced examples of two component links with arbitrarily large unknotting gaps. By taking connected sums of knots with unknotting gap one, it is possible to produce examples of non–prime knots with arbitrarily large unknotting gaps. The question as to whether there are prime knots with unknotting gap greater than one remains open.

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Some Theorems on Approximate Fibrations

In this note we present several theorems that help to recognize approximate fibrations among certain type of proper piecewise linear(PL) maps between manifolds.

Definition (Approximate Fibration)

A surjective map $P : E \to B$ between metric ANR's is called an **approximate fibration** if for any given $f : X \to E$ and a homotopy $G : X \times I \to B$ with $G_0 = p \circ f$ and for any $\varepsilon > 0$ there exists a lifted homotopy $\tilde{G} : X \times I \to E$ such that $\tilde{G}_o = f$ and $p \circ \tilde{G}$ and G are ε -close.

Definition(N^n-type Map)

Let N^n be a closed, orientable, connected n-manifold. A proper map p:$M^{n+k} \to B^k$ is said to be an **N^n-type map** if $p^{-1}b \simeq N^n$ for all b in B. Furthermore we say that p is N^n-like if $p^{-1}b$ collapses to a homotopy type of N^n for all b in B.

It is known that if an N^n-like map p:$M^{n+k} \to B^k$ is an approximate fibration, then B is a k-manifolds.[D2] But there is an example of N^n-type PL approximate fibration p: M $\to$ B where B fails to be a manifold[D:private communication]. Our goal is to identify closed, orientable, connected n-manifold N^n such that any N^n-type PL map p: M $\to$ B between PL manifolds is an approximate fibration. Our development is similar to the case of N^n-like PL map which is extensively investigated by R.J. Daverman.[D1,D2,D3]

Definition(h-fibrator)

A closed, orientable, connected n-manifold N^n is called a **codimension-k h-fibrator** if any N^n-type map p:$M^{n+k} \to B^k$ between manifolds is an approximate fibration. N^n is called an **h-fibrator** if it is a codimension-j h-fibrator for all positive integer j.

Examples

S^n is a codimension-n h-fibrator but not a codimension-(n+1) h-fibrator. The torus $T^2 = S^1 \times S^1$ is not a codimension-2 h-fibrator but $T^2 \sharp T^2$ is an h-fibrator.

Fact If N^n is a codimension-k fibrator, then it is a codimension-j fibrator for all j $\leq$ k-1.

In the PL setting we have two fundamental results concerning PL map and recognition of approximate fibrations.[D]

Codimension Reduction Lemma[D2]

Let p:$M^{n+k} \to B^k$ be a PL map. Then each b in B has a PL neighborhood S=b$*$L in B such that $p^{-1}S$ is a regular neighborhood of $p^{-1}b$ in M and p^{-1}L $= \partial\ p^{-1}$S is an (n+k-1)-manifold.

We write L'=p^{-1}L and S'=p^{-1}S.

Fundamental Recognition Theorem [D2]

A PL map p: M $\rightarrow$ B is an approximate fibration if and only if each v in B has a stellar neighborhood S=v*L whose preimage collapses to $p^{-1}v$ via a map r:$p^{-1}S \rightarrow p^{-1}v$ such that for all b in L, r$|p^{-1}b$: $p^{-1}b \rightarrow p^{-1}v$ is a homotopy equivalence. Here r=D_1 where D_1 is a strong deformation retraction of $p^{-1}S$ onto $p^{-1}v$.

The following results and definitions are indispensable to establish our theorems .

Basic Lemma (Notation as in the Codimension Reduction Lemma)

For any N^n-type map p:$M^{n+k} \rightarrow B^k$, k $\geq$ 2, and for any v in B, the inclusion j:$L' \rightarrow S'$ induces an epimorphism $j_\sharp$: π_1 $(L') \rightarrow \pi_1(S')$.

Homotopy Exact Sequence [D2,D3] ,[C]

For an approximate fibration p:E $\rightarrow$ B, there is a following homotopy exact sequence:

$$\rightarrow \pi_{i+1}(\mathrm{B}) \rightarrow \pi_i(p^{-1}b) \rightarrow \pi_i(\mathrm{M}) \rightarrow \pi_i(\mathrm{B}) \rightarrow$$

Wang Exact Sequence [D3]

For an approximate fibration p:E $\rightarrow$ S^m, there exists following exact sequences:

$$\rightarrow H^j(\mathrm{E}) \rightarrow H^j(\mathrm{F}) \rightarrow H^{j-m+1}(\mathrm{F}) \rightarrow H^{j+1}(\mathrm{E}) \rightarrow$$

$$\rightarrow H_j(\mathrm{F}) \rightarrow H_j(\mathrm{E}) \rightarrow H_{j-m}(\mathrm{F}) \rightarrow H_{j-1}(\mathrm{F}) \rightarrow$$

Here F is a fiber of the map p.

Definition (Hopfian Manifold)

A closed, orientable, connected manifold N is called hopfian if every degree one map f:N $\rightarrow$ N which induces an isomorphism on the fundamental group is a homotopy equivalence.

Example Any closed, orientable, connected manifold with a finite fundamental group is hopfian.

Also we say that a group G is hopfian if any epimorphism of G onto itself is an isomorphism. So every finite group is hopfian as well as the infinite cyclic group.

Now we state several theorems concerning h-fibrators.

Theorem

Let N^n be a hopfian manifold with a hopfian fundamental group. Suppose N^n is a codimension-(n+1) PL h-fibrator. Then N^n is a PL h-fibrator.

Note that we are assuming that the base space B is a manifold and the above theorem without this assumption may not be true. A proof of above theorem uses an induction on the codimensions utilizing homotopy exact sequence, Basic Lemma, Wang exact sequence.

Theorem

Let N^n be a closed, orientable, connected, aspherical n-manifold with a hopfian fundamental group. Then N^n is a PL h-fibrator if it is a codimension-2 PL h-fibrator.

Theorem

Let N^n be a closed, orientable, connected, hopfian n-manifold with a hopfian fundamental group. Suppose H^j(N)=0 for 0 < j < m and H^n(N) is in the subring of H^*(N), generated by H^m(N). Then N^n is a PL h-fibrator if it is a codimension-(m+1) PL h-fibrator.

Theorem

The n-dimensional quarternionic projective space QP^n is a codimension-5 PL h-fibrator. Hence it is a PL h-fibrator.

This work was done while I was at University of Tennessee under the supervision of Prof. R. J. Daverman. I am thankful to him and Prof. Dennis Garity to provide the arrangement and guidance.

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Some New Wild Embeddings of Codimension 1 Manifolds
by F. C. Tinsley (joint with R. J. Daverman)

Introduction:

During the 1950's and 60's a significant portion of geometric topology focused on the study of embeddings of S^2 in E^3. Many examples were constructed in which the image of S^2 could not be bicollared. These were called *wild embeddings.*

One way of detecting wildness was to look locally at the fundamental group of the complement of the surface. For example, Figure 1 shows the wild part of the Alexander Horned Sphere and a loop in the complement. The loop bounds homologically but not homotopically. Therefore, it represents a non-trivial element of the fundamental group of the complement yet the first homology group of the complement is trivial. Homology theory reveals that these local π_1 groups are always *perfect*, ie, equal to their commutator subgroups.

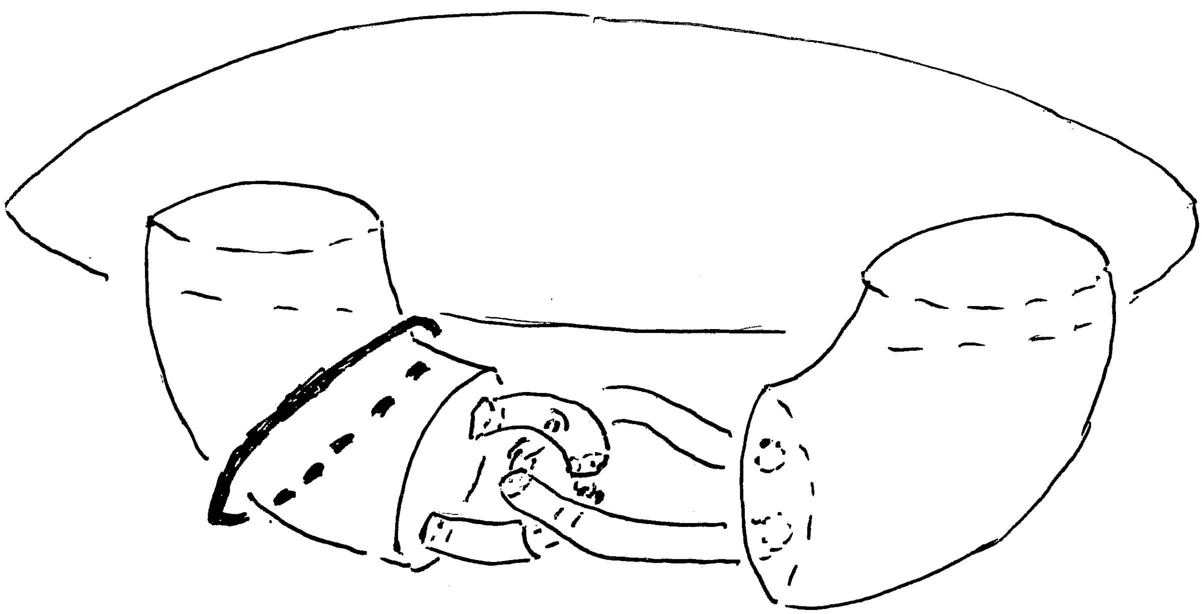

Figure 1

Naturally, topologists asked similar questions about embeddings of S^n in E^{n+1} for $n > 2$. Local homotopy and perfect groups are important in the same way as for $n = 2$. J. W. Cannon gives a beautiful explication of this relationship in [Ca[1]]. However, early examples of wild embeddings in high dimensions mimicked constructions for $n = 2$.

In the early 1980's R. J. Daverman ([Da]) constructed explicit examples of "crumpled" cobordisms (W, Σ^n, S^n) for $n \geq 4$ where $W - S^n \cong \Sigma^n \times [0, 1)$, Σ^n is a non-simplyconnected homology sphere, S^n is wildly embedded in W, W is an acyclic mapping cylinder, and W plus a collar attach to S^n is an $(n+1)$ – manifold. Because the fundamental group of the

$W - S^n$ is finitely generated, these examples cannot occur in low dimensions.

The results of this work completely characterize wildness of the type alluded to in the previous paragraph. All the examples are intrinsically high dimensional.

The Construction:

The basic construction uses the notion of a grope, due to J. W. Cannon. In [Ca[1]] he observes that an element, g, of a group G lies in a perfect subgroup of G if and only if g is equal to a product of commutators each of which is again equal to a product of commutators each of which is again equal to a product of commutators . . . ad infinitum. Geometrically, a loop which bounds a disk-with-handles is a product of commutators. So, any loop in a space which represents an element of the fundamental group of that space bounds a singular disk-with-handles. Similarly, each handle curve itself bounds a singular disk-with-handles . . . ad infinitum. In a high dimensional manifold, all these stages can be embedded. Cannon called the resulting infinite 2-complex a grope, identified the interior of a regular neighborhood of this object as a crumpled cube, showed how to replace the interior of this crumpled cube by the interior of a cell, and showed how the two spaces were related by an arc decomposition [Ca[2]].

We are indebted to John Walsh for the following strategy. Suppose M^n is an n-manifold ($n \geq 5$) with $[l] \in \pi_1(M)$ in a perfect subgroup of $\pi_1(M)$. Properly embed a grope in $M \times [0,1)$ with l as its boundary and then replace the interior of a regular neighborhood by the interior of the cell. Using end theory put an end on the resulting cobordism. Finally, collapse out the arc decomposition.

The problem stems from the fact that in many cases the half-open arcs resulting from the grope replacement cannot possibly converge to points at the end. As a result standard decomposition theory cannot collapse out these arcs.

The solution is to add extra structure at the end. In particular, after making the grope replacement, identify the new end as $N \times [0,1)$, the new cobordism as (W, M, N), and "spin" about this new end. More precisely, consider

$$(W - N) \times S^1 \bigcup N$$

where N is attached according to the spin topology

$$N \times B^2 \cong N \times (0,1] \times S^1 \bigcup N$$

In this setting, the wild codimension one manifold, W^*, will correspond to

$$W \times 0 \bigcup W \times \pi \bigcup N$$

Each half-open arc, α, from the replacement corresponds to a punctured disk, $\alpha \times S^1$, with

$$\alpha \times S^1 \bigcap W^* = \alpha \times 0 \bigcup \alpha \times \pi$$

We collapse the punctured disks $\alpha \times S^1$ back to W^* in two steps. First, collapse out the tame half-open arcs $\alpha \times \left(\frac{\pi}{2}\right)$ and $\alpha \times \left(\frac{3\pi}{2}\right)$. Finally, we identify an usc, shrinkable arc decomposition transverse to W^* where each decomposition element is close to one of the original arcs $pt \times \left[\frac{-\pi}{2} \times \frac{\pi}{2}\right]$ or $pt \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ in the spin structure.

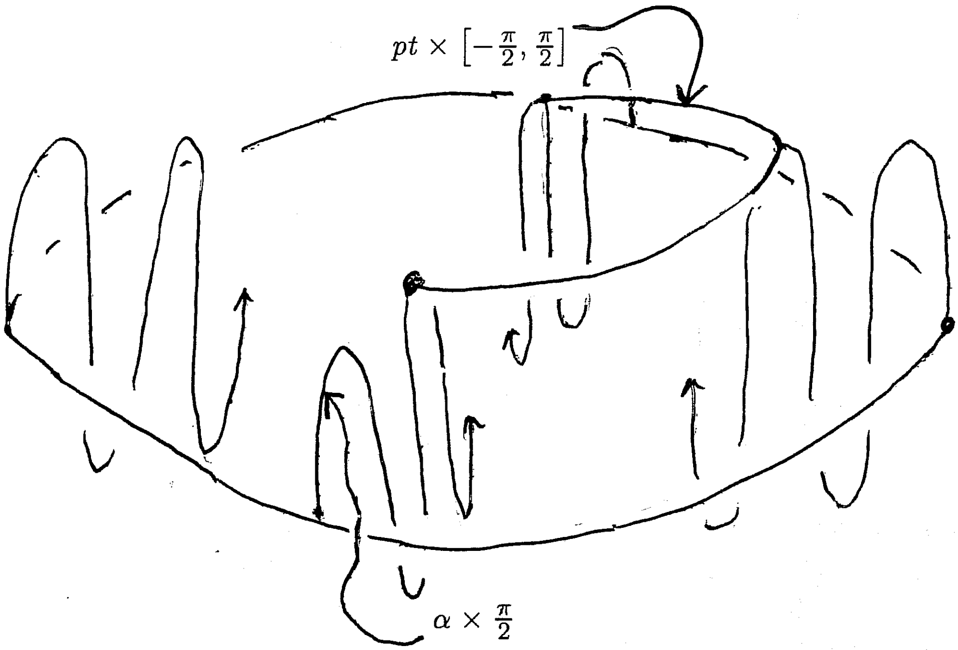

Figure 2 — $\alpha \times S^1$

The Example:

Consider the group $G = \left\langle y, t \,\middle|\, y = \left[y^t, y^{t^2}\right] \right\rangle$. Note that $[G,G]$ is perfect and $[G,G] = Ncl\langle y, G\rangle$. Let $G = \pi_1(M)$. Now, replace the interior of a regular neighborhood in $M \times [0,1)$ of a grope bounded by a loop reprenting y by that of a cell. The end of this manifold is homeomorphic to $N \times [0,1)$ where $\pi_1(N) \cong \mathbb{Z}$ with generator t. However, the half open arcs resulting from the replacement cannot possibly converge to points so we must use the above construction.

Main Results:

By working in a neighborhood of a 2 complex carrying G, the funcamental group of M, we obtain the following general results.

Theorem: Let G be a finitely presented group with perfect, normal subgroup P such that P is finitely generated as a normal subgroup[1]. Let M^n be an n-manifold ($n \geq 5$) with $\pi_1(M^n) \cong G$. Then there is a crumpled cobordism (W, M, N) with

(1) $W - N \cong M \times [0,1)$
(2) N wildly embedded in W
(3) $ker(i_\#) = P$ where $i : M \mapsto W$ is inclusion
(4) $\pi_1(N) \cong G/P$
(5) the inclusion $j : N \mapsto W$ a homotopy equivalence.

Corollary: Given (G, P) as in the theorem and $n \geq 5$ then there is a wild embedding $i : S^n \mapsto S^{n+1}$ such that

(1) each component of $S^{n+1} - S^n$ is homeomorphic to E^{n+1}
(2) the local π_1 – kernel of $S^{n+1} - S^n \mapsto S^{n+1}$ is isomorphic to P.

[1]For all known pairs (G, P) with G finitely presented, P perfect, and $P \triangleleft G$, P is finitely generated as a normal subgroup. The existence of an example in which P is not finitely generated as a normal subgroup appears to be an extremely difficult question.

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The Higson Compactification and its Corona

by

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Introduction. Let X be a locally compact metric space. A proper metric on X is a metric d having the property that all bounded closed sets are compact. The Higson compactification of X is a Hausdorff compactification $\overline{X}^d$ characterized by the property that if $f{:}X \to [0,1]$ is a continuous map, then f has a continuous extension to $\overline{X}^d$ if and only if for every $r>0$, $\mathrm{diam}(f(B_r(x))) \to 0$ as $x \to \infty$ in X. A good introduction to this compactification is given by Roe in [10]. This compactification is used in the study of group graphs and Riemannian manifolds as well as other locally compact metric spaces. It has also been used to settle the Novikov conjecture for coarse (or exotic) cohomology for hyperbolic metric spaces. There is hope to settle this conjecture for more general spaces using the Higson compactification. It is hoped that these results will help to provide a framework to carry this out.

In this paper topological properties of the Higson compactification and its corona are outlined. The results are obtained by using homotopy properties of maps from $\overline{X}^d$ to various polyhedra. The techniques are reminiscent of those developed by the author to study the Stone-Čech compactification [5], [6], and [7]. In [8] the author answered a question of Higson concerning the Čech cohomology of this compactification. The question was posed in [10].

1. Basic Properties of the Higson Compactification. Here are the basic definitions and basic results concerning the Higson compactification and its corona. Our approach is equivalent to [10].

Suppose that X is noncompact with d a proper metric. Let $f{:}X \to Y$ be a continuous function into a metric space Y with specific metric. We say that the function f satisfies (*) provided that

$$(*) \qquad \lim_{x\to\infty} \mathrm{diam}(f(B_r(x))) = 0 \quad \forall r > 0.$$

Property (*) means that for each $r>0$ and each $\varepsilon>0$, there is a compact set $K = K_{r,\varepsilon}$ in X such that for all $x \notin K$, $\mathrm{diam}(f(B_r(x))) < \varepsilon$. Recall the standard notation of [3] that $C(X)$ ($C^*(X)$) denotes the set of all (bounded) real-valued continuous functions on X. $C(X)$ and $C^*(X)$ are rings under pointwise addition and multiplication with $C^*(X)$ a subring of $C(X)$. By analogy we define $C_d(X) = \{f \in C(X) \mid f \text{ satisfies } (*)\}$ and $C_d^*(X) = \{f \in C^*(X) \mid f \text{ satisfies } (*)\}$.

With the supremum norm on $C^*(X)$, $C_d^*(X)$ is a closed subring of $C^*(X)$ containing all the constant functions. Because the metric d on X is proper, $C_d^*(X)$ generates the topology of X. It is well-known that the compactifications of X are in one-to-one correspondence with the closed subrings F of $C^*(X)$ which contain the constants and generate the topology of X. One way to construct the compactification associated with F is to let the points of the compactification be the maximal ideals of the ring F with the hull-kernel topology with the point x being identified with the maximal ideal $M_x = \{f \in F \mid f(x) = 0\}$.

The *Higson compactification* is the compactification associated with the closed subring $F = C_d^*(X) \subset C^*(X)$. It is characterized as the compactification $\overline{X}^d$ such that the real-valued continuous functions on X that extend to $\overline{X}^d$ are precisely the ones in $C_d^*(X)$. In [8] we give another characterization of $\overline{X}^d$ which we repeat here.

Proposition 1. *Suppose that X is noncompact and that d is a proper metric on X. The Higson compactification $\overline{X}^d$ is the unique compactification of X such that if Y is any compact metric space and $f{:}X \to Y$ is continuous, then f has a continuous extension to $\overline{X}^d$ if and only if f has property (*).*

The Higson corona is simply the difference set, $v_d X = \overline{X}^d \setminus X$. The letter d is used in the designation of the Higson compactification and its corona to emphasize the dependence on the proper metric d.

2. One-Dimensional Čech Cohomology. In [8] it was shown that the one-dimensional Čech cohomology is nontrivial for the Higson compactification of any noncompact, connected X. The result is given precisely in the following theorem.

Theorem 1. *Suppose that X is a noncompact connnected metric space and suppose that d is a proper metric on X. Then the following exact sequence holds.*

$$0 \to C_d^*(X) \to C_d(X) \to \check{H}^1(\overline{X}^d) \to \check{H}_d^1(X) \to 0$$

It was also shown in [8] that the group $C_d(X) / C_d^*(X)$ is isomorphic as a group to the additive group of real numbers R under the hypotheses of Theorem 1. So, whenever X is a locally compact noncompact connected metric space with proper metric d, the one-dimensional Čech cohomology is nontrivial. It was conjectured that if X was uniformly contractible in the metric d, then all Čech cohomology groups would be trivial. Theorem 1 shows that this is false.

3. Maps onto Tori. Here we sketch results we have obtained concerning the Higson compactification and its corona using maps onto tori. Let T^n be the n-dimensional torus and let $e{:}R^n \to T^n$ be the natural covering map with R^n the universal covering space. Let $f{:}R^n \to R^n$ be defined by $f(x_1,\ldots,x_n) = \left(\sqrt{|x_1|},\ldots,\sqrt{|x_n|}\right)$. One can easily check that f has property (*) for the usual Euclidean metric on R^n. The map $e \circ f{:}R^n \to T^n$ will also satisfy (*) for the usual metric on T^n. Thus, there will be an extension $\overline{e \circ f}{:}\overline{R^n}^d \to T^n$ of $e \circ f$. This map may seem innocent enough on its face, but there are serious implications. This map is onto T^n and any map homotopic to it is also onto T^n. We say that a map is *homotopically onto* when it is onto and each map homotopic to it is also onto [5].

Theorem 2. *The map* $\overline{e \circ f}: \overline{R^n}^d \to T^n$ *is homotopically onto. Also, this map restricted to the corona,* $\overline{e \circ f}: v_d R^n \to T^n$, *is also homotopically onto.*

Among the implications of this theorem is that the dimension and the shape dimension of $v_d R^n$ is greater than or equal to n.

The above theorem is the simplest application of maps onto tori. There are much more general circumstances when one can show that there is a map $f: X \to R^n$ so that f has property (*) and has the property that the consequent extension $\overline{e \circ f}: \overline{X}^d \to R^n$ is homotopically onto as well as $\overline{e \circ f}: v_d X \to R^n$. If one has a continuum $K \subset v_d X$, then often one can construct a $g: K \to T^n$ which is homotopically onto. In [5] an analogous construction was used to produce maps $g: K \to T^n$ which were homotopically onto whenever K is a continuum in $\beta X \setminus X$ with X Lindelöf and $\dim K \geq n$.

4. Maps of Subcontinua of $v_d X$. The next theorem is analogous to results that can be found in [6] concerning subcontinua of $\beta X \setminus X$. In some sense the one-dimensional Čech cohomology of subcontinua of $v_d X$ completely determines these subcontinua.

Theorem 3. *Let K be a subcontinuum of $v_d X$ and suppose that* $f: K \to Y$ *has the property that* $f^*: \check{H}^1(Y) \to \check{H}^1(K)$ *is an isomorphism. Then f is one-to-one. If f is also onto, then f is a homeomorphism.*

5. Closures of Subsets of $v_d X$. The closures of subsets of $v_d X$ are often the Stone-Čech compactifications of those subsets.

Theorem 4. *Let A be a σ-compact subset of $v_d X$. Then the closure of A in $v_d X$ is equivalent to the Stone-Čech compactification of A,* $Cl_{\overline{X}^d} A \approx \beta A$.

This result is analogous to the theorem concerning the Stone-Čech compactification that if A is a subset of $\beta X \setminus X$ with X and A both Lindelöf and with A closed in $X \cup A$, then $Cl_{\beta X} A \approx \beta A$.

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A Topological Characterization of Finitely Generated Free Abelian Groups

Derek J. S. Robinson and Mathew Timm

1. Introduction.

The topological characterization of finitely generated free abelian groups that is reported here is part of a longer and more detailed work by the authors[3]. The result reported in this note is part of a general attempt to understand topological spaces that have the property that all of their finite sheeted covering spaces have total space of the covering projection homeomorphic to the base space. The interested reader should consult [3] as well as [4] and [5]. All of [3], [4], and [5] contain more extensive bibliographies and motivations.

The second author originally became interested in these sorts of questions after reading Jungck[J] where he investigated developed of connected first countable Hausdorff spaces whose only connected finite sheeted covering spaces are the trivial covers of the spaces by themselves. We thank Nigel Boston for the contact that led to this collaboration. Bob Daverman is thanked for originally suggesting the particular question whose answer is reported in this manuscript.

2. Definitions and elementary examples.

Assume all topological spaces are connected and all groups are finitely generated (denoted f.g.). Let M be a topological space. Then M is *h-connected* if and only if whenever $p:X \longrightarrow M$ is a finite sheeted connected covering it follows that X is homeomorphic to M via some homeomorphism. Based on the correspondence between subgroups of the fundamental group of a connected manifold and finite sheeted coverings of the manifold there is the corresponding group theoretic definition. A f.g. group G is *hc* if and only if every finite index subgroup of G is isomorphic to G. Note that an h-connected manifold has hc fundamental group.

Examples 2.1. The trivial group G=1 is hc. Any f.g. free abelian group is hc. Any f.g group that has no proper finite index subgroups is (trivially) hc. See for example the Higman groups in [1] or any f.g. infinite simple group. $G=Z\times \ldots \times Z\times H$ where H is a f.g. group with no proper subgroups of finite index is hc. Topological examples of h-connected spaces include any simply connected manifold, your favorite n-manifold whose fundamental group is a finitely presented group that has no proper subgroups of finite order, the n-tori $T^n=S^1\times \ldots \times S^1$, $T^n\times D^m$, where D^m is the m-disc, and $T^n\times M$ where M has no proper finite sheeted covering spaces.

2. A structure theorem for f.g. hc groups

Definition 2.1. Let G be a f.g. group. The finite residual is the (normal) subgroup of G given by

$$F = \bigcap \{ H : G/H \text{ is a finite group} \}.$$

Theorem 2.2. (A partial statement of a theorem in [3]) If G is a f.g., hc group with finite residual F then G/F is f.g. free abelian.

The above theorem is only the part of a much more detailed result given in [3] that applies to the task at hand. The result in [3] gives a very detailed description of the structure of hc groups.

3. A Topological consequence

Corollary 3.1. Let G be f.g. Then G is free abelian if and only if there is a compact, finite dimensional, h-connected manifold M with fundamental group isomorphic to G that satisfies the following condition on covering spaces:

(*) There is a unique covering space $q:X \longrightarrow M$ such that, given any finite sheeted covering space $p:N \longrightarrow M$ there is a covering projection $r:X \longrightarrow N$ such that $q = p \circ r$.

Proof. Clearly if G is f.g. free abelian of rank n then the n-torus T^n satisfies the theorem. To prove the converse assume M is an h-connected manifold with $\pi_1(M) = G$ that satisfies (*). If the finite residual F of G is trivial we are clearly done since the space X is then the universal cover of M. So assume that F is not trivial. Let $q : X_F \longrightarrow M$ denote the (regular) covering space that corresponds to F. By construction X_F covers all of the finite sheeted covering spaces as is required in (*). However the uniqueness requirement of (*) is not fulfilled since M has a universal covering space and X_F is not simply connected. Contradiction. So F is trivial and we are done.

Question 3.2. In the light of Corollary 3.1 one wonders if, modulo the Poincare Conjecture, the only h-connected 3-manifolds are S^3, $S^1 \times D^2$, $S^1 \times S^2$, $T^2 \times I$, and T^3? Note that there is an example of a 3-generator group G that is hc, has a proper subgroup of finite index, and is irreducible with respect to direct product. See [3]. It is not finitely related so can't be a 3-manifold group. These facts, and the relationship of properties of the fundamental group of a 3-manifold to its topology, cause one to wonder if there is any finitely presented hc group that is irreducible with respect to direct product.

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Stretching Rubberbands Inside Planar Domains

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Abstract

We develop a notion of "shortest" arc between a pair of points in a closed topological planar disk.

Given a closed topological disk $D \subset R^2$ and distinct points $\{a, b\} \subset \partial D$ there is not necessarily a closed arc in D of finite length which connects a to b. However, we show that if F consists of those points $x \in D$ such that no chord of D separates x and$\{a, b\}$ then F is a closed arc in D which connects a to b. Furthermore F is continuous in the data D, a, b, and all closed subarcs of $\text{int}(F)$ are uniquely of minimal length. We have thus obtained in the topological category a natural extension of the notion of the shortest arc between two points in a closed disk.

Definition 1 *Suppose $D \subset R^2$ is homeomorphic to the closed unit disk D. The straight line segment $[c, d]$ is said to be a* **chord** *of D if $\{c, d\} \subset \partial D$ and $(c, d) \subset int(D)$.*

Definition 2 *If $A \subset D$ and $B \subset D$ then the chord $[c, d]$ is said to* **separate** A **and** B *if A and B are contained in different components of $D \setminus [c, d]$.*

Definition 3 *To say that a collection of mutually homeomorphic subsets of the plane is endowed with the* **uniform topology** *means that a sequence $A_n \to A$ iff there is a sequence of homeomorphisms $h_n : R^2 \to R^2$ such that $h_n(A_n) = A$ and $h_n \overset{unif}{\to} id$.*

Definition 4 *Let X denote the collection of all subsets of R^2 which are homeomorphic to the closed unit disk. Let Y denote the collection of all subsets of R^2 which are homeomorphic to the closed unit interval $[0, 1] \subset R^1$. We endow X and Y with the uniform topology.*

Definition 5 *Let Z denote the subspace of $X \times R^2 \times R^2$ consisting of all triples (D, a, b) such that $a \neq b$ and $\{a, b\} \subset D$.*

Theorem 1 *If* $(D, a, b) \in Z$ *and* $F = \{x \in D \mid nochordofDseparatesxand\{a, b\}\}$ *then*

- F is a closed arc with endpoints $\{a, b\}$.
- F admits a parameterization by arclength by exactly one of $[0, 1]$, $[0, \infty]$, or $[-\infty, \infty]$.
- If $\{c, d\} \subset F \setminus \{a, b\}$ and E is the unique closed subarc of F with endpoints $\{c, d\}$, then E has finite arclength $\| E \|$, and the arclength $\| E \|$ is uniquely minimal over all arcs in D which contain $\{c, d\}$
- F is unique in the sense that F is the only subset of D which enjoys the preceding properties.

Theorem 2 *The function* $\mathcal{F} : Z \rightarrow Y$ *is* **continuous** *where*

$$\mathcal{F}(D, a, b) = \{x \in D \mid nochordofDseparatesxand\{a, b\}\}.$$

Proof of Theorem 1 (sketch) . To verify that F is indeed a closed arc let α denote the closure of either component of $\partial D \setminus \{a, b\}$. We will construct a homeomorphism $\phi : \alpha \rightarrow F$ as follows. Index the components $\alpha \setminus (\alpha \cap F)$ and let α_i denote the closure of some component of $\alpha \setminus (\alpha \cap F)$. We will map α_i homeomorphically onto the closure of a corresponding component, F_i, of $F \setminus (\alpha \cap F)$ via ϕ_i, a homeomorphism which fixes the endpoints of α_i. Finally we let $\phi = id_{\alpha \cap F} \cup_i \phi_i$. Continuity of ϕ will follow from local connectivity of α, and the fact that $\mathrm{diam}(F_i) \leq \mathrm{diam}(\alpha_i)$. One must also verify that ϕ is $1-1$ and surjective. To see that F admits a parameterization by arclength one first verifies that $F \setminus \{a, b\}$ is locally isometric to the boundary of a convex disk. Hence $F \setminus \{a, b\}$ can be realized locally as the graph of a convex function, and therefore $F \setminus \{a, b\}$ admits a parameterization which is locally absolutely continuous. This also shows that each closed subarc $E \subset F \setminus \{a, b\}$ has finite arclength. If the endpoints of E are $\{c, d\}$ then given any other arc $\beta \subset D$ with endpoints $\{c, d\}$, one can construct a sequence of closed arcs β_n such that $\beta_n \rightarrow E$ in the Hausdorf metric, and such that $\| \beta \| > \| \beta_1 \|$, $\| \beta_n \| \geq \| \beta_{n+1} \|$ and $\| \beta_n \| \rightarrow \| E \|$. This establishes the unique minimality of $\| E \|$.

Proof of Theorem 2 (sketch) . Suppose $(D_n, a_n, b_n) \rightarrow (D, a, b)$. To verify that $\phi(D_n, a_n, b_n) \rightarrow \phi(D, a, b)$, we first check convergence in the Hausdorf metric. We then verify that $\phi(D_n, a_n, b_n)$ is **uniformly locally connected**, and conclude that $\phi(D_n, a_n, b_n)$ converges uniformly to $\phi(D, a, b)$.

Problem Session

1. (Daverman) Is there a hereditarily aspherical generalized n-manifold ($n>4$)? X is *hereditarily aspherical* if for any open subset U of X, $\pi_i(U) = 0$ for $i>1$.

2. (Swenson) Suppose G acts by homeomorphisms on a dendrite (or $\mathbb{R}$-tree) T.

   a) Under what conditions is there a path metric on T such that G acts by isometries? What if $T = \mathbb{R}$ and no closed interval of $\mathbb{R}$ is mapped to a proper subset of itself?

   b) Under what conditions is G a non-trivial free product?

3. (Plaut) Let G be a locally compact, locally connected, first countable topological group. Put a Haar measure on G. Then $L = L^2(G)$ is a Hilbert space and G acts on L via isometries: $g \cdot F(x) = F(g^{-1}x)$. Do there exist functions $F \in L$ such that their orbits (with induced inner metric) have curvature $\geq K$ for some K?

4. (Wright) Let X be a contractible n-manifold which covers something non-trivially. Determine conditions on the fundamental group at infinity of X (such as *finitely generated at infinity*, *stable at infinity*, or *semi-stable at infinity*) which imply that X is simply connected at infinity.

5. (Guilbault) Under what conditions do the metric spheres in a $CAT(0)$ space X satisfy $CAT(K)$ for some K when given the induced inner metric? Under what conditions do they support any $CAT(K)$ metric? In particular, what if X is a manifold?

6. (suggested by Davis, orginally posed by Gromov in all dimensions) Must a simply connected $CAT(0)$ 4-manifold be homeomorphic to $\mathbb{R}^4$? Note. For $n \leq 3$, a simply connected $CAT(0)$ n-manifold is known to be homeomorphic to $\mathbb{R}^n$. For $n \geq 5$, there are counterexamples. P. Thurston has given partial results for $n=4$.

7. (attributed to Bruce Kleiner) Does the space of spaces of curvature bounded above contain a dense set of manifold points?